

One-loop correction to primordial tensor modes during radiation era

Markus B. Fröb^a, Dražen Glavan^b, Paolo Meda^{c,d} and Ignacy Sawicki^b

^a*Institut für Theoretische Physik, Universität Leipzig,
 Brüderstraße 16, 04103 Leipzig, Germany*

^b*CEICO, FZU — Institute of Physics of the Czech Academy of Sciences,
 Na Slovance 1999/2, 182 21 Prague 8, Czech Republic*

^c*Dipartimento di Fisica, Università di Pavia, and
 Istituto Nazionale di Fisica Nucleare, Gruppo IV,
 Via Bassi 6, 27100 Pavia, Italy*

^d*Dipartimento di Matematica, Università di Trento,
 via Sommarive 14, I-38123 Povo, Trento, Italy*

E-mail: mfroeb@itp.uni-leipzig.de, glavan@fzu.cz, paolo.meda@unitn.it,
sawicki@fzu.cz

ABSTRACT: The ability to infer properties of primordial inflation relies on the conservation of the superhorizon perturbations between their exit during inflation, and their re-entry during radiation era. Any considerable departure from this property would require reinterpreting the data. This is why it is important to understand how superhorizon perturbations interact with the thermal plasma driving the radiation dominated Universe. We model the plasma by free photons in a thermal state and compute the one-loop correction to the power spectrum of primordial tensor perturbations. This correction grows in time and is not suppressed by any small parameter. While one-loop result is not reliable because it invalidates perturbation theory, it signals potentially interesting effects that should be investigated further.

KEYWORDS: Early Universe Particle Physics, Non-Equilibrium Field Theory, Thermal Field Theory, Cosmological models

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1 Introduction

The leading paradigm for the earliest stage of our Universe is primordial inflation, that posits our Universe began with a very brief period of very rapid accelerated expansion. Our current ability to infer precise properties of this period, and to constrain a plethora of proposed models [1] relies on interpreting how signals emitted much later are connected to processes that took place during inflation. These signals come in the form of a cosmic microwave background (CMB) emitted some 380,000 years after the end of inflation. The fluctuations in the CMB carry the imprints of density perturbations and gravitational wave perturbations from the last scattering surface. These perturbations are supposed to have originated from primordial inflationary perturbations, that have exited the horizon during inflation, with their amplitude frozen until re-entering the horizon in late radiation-dominated period. Therefore, our ability to infer the properties of primordial inflation hinges on assuming the conservation of primordial perturbations on superhorizon scales. Any considerable departure from this property would alter the way we interpret data, and would necessitate revisiting the constraints on models of inflation delivered in [2].

The conservation of linearized superhorizon tensor modes indeed is valid for cosmological perturbations around backgrounds the expansion of which is driven by homogeneous and isotropic condensates of matter fields [3]. However, this has not been established for backgrounds driven by fluctuating media. The thermal plasma driving the expansion of the universe during radiation domination is an instance of such a medium. Given that most

observationally relevant modes re-enter the horizon only by the end of radiation-dominated era, understanding the details of the evolution of superhorizon modes in a thermal medium is of paramount importance.

Only recently has this question been considered more closely in [4], with staggering conclusions: in the simplest one-loop approximation the primordial tensor power spectrum shows large secular enhancement, unsuppressed by a small parameter. We find this result to be of utmost importance as, taken at face value, it challenges our current ability to quantify the properties of the primordial Universe. That is why we set out to revisit independently the same question by considering a different system where the thermal radiation bath is composed out of photons, rather than excitations of a massless minimally coupled scalar as in [4].

We find the same large secular one-loop enhancement as [4], confirming the expectation that the effect does not depend on the number of degrees of freedom in the plasma. This amplification has recently been reinterpreted in [5] as the effect of stimulated emission of gravitational waves. This enhancement is not suppressed by a small parameter naively expected at one loop, and signals the breakdown of perturbative loop expansion and/or the breakdown of linearity of evolution of superhorizon primordial gravitational waves. While perturbative results obtained in [4], and the ones we report here should not be trusted as predictions because of this breakdown, they point to the pressing need to better understand the evolution of long-wavelength primordial fluctuations during the radiation era when they propagate on what are essentially stochastic backgrounds. The need for such an understanding has been advocated recently in [6], motivated by the question of whether gravitational waves remain massless when propagating on such backgrounds. Our findings conform with the tensor perturbations remaining massless, which at one loop level we find to be captured by the Hartree approximation.

The computation considered here pertains to the radiation-dominated era following the post-inflationary reheating period, during which the expansion is dominated by a thermalized plasma of relativistic particle species. The standard assumptions about this period are collected in section 2. We set up the computation in section 3 starting from considering the Einstein and Maxwell equations to describe the evolution of a priori stochastic variables. These equations are then turned into linearized equations for metric perturbations evolving on a background whose expansion is driven by thermal fluctuations. The interaction between metric perturbations and thermal fluctuations of the plasma¹ are then truncated at the one-loop level when computing the two-point function for the tensor perturbations. We find this approximation to correspond to the equations of stochastic gravity [7, 8] when linearized in metric perturbations.

The real-time, thermal, non-local self-energy diagrams are computed in the vanishing external spatial momentum limit in section 4, with the more complicated integrals involved relegated to the appendix. These are then used to compute the connected diagrams contributing to the tensor power spectrum. There are two qualitatively different contributions: (i) thermally induced gravitational wave production [9, 10] that does not depend on the amplitude of the primordial perturbations and is rather sourced by the plasma, and (ii) a radiation exchange effect that takes the form of a secular, multiplicative enhancement of

¹We refer to the bath of noninteracting photons with the thermal spectrum as plasma for short.

the primordial tensor power spectrum. While we find a different result compared to [4] for the first blue-tilted contribution, this is of little consequence as it is negligible for the observationally relevant scales. It is only the second effect that is relevant, for which we agree with [4]. Given that this effect is captured by a linear equation, it is possible to solve for its evolution exactly, and thus to resum the infinite series of self-energy insertions. We do this in section 5 to reveal that the growing logarithmic secular correction becomes a power-law growing secular enhancement. Such a rapid growth indicates the breakdown of the loop expansion and/or the breakdown of the linear evolution of primordial tensor perturbations on superhorizon scales. The relevance of these results is discussed in the concluding section, where we also outline what we think should be the next steps in understanding this problem.

2 Standard results for tensors in the early universe

One of the key predictions of inflationary cosmology, though as of yet unobserved, is the existence of a stochastic background of primordial gravitational waves. These are defined as perturbations $h_{\mu\nu}$ of the conformally rescaled Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$g_{\mu\nu} = a(\eta)^2 (\eta_{\mu\nu} + \kappa h_{\mu\nu}), \quad (2.1)$$

where $\eta_{\mu\nu}$ is the Minkowski metric, η is the conformal time, $a(\eta)$ is the scale factor, and where we find convenient to factor out $\kappa = \sqrt{16\pi G_N} = \sqrt{2}/M_P$ from the definition of the fluctuation. The two propagating tensor polarisations are contained in the transverse traceless (TT) part of the metric fluctuations and we will find it convenient to use as our variable

$$\gamma_{ij} = (h_{ij})^{\text{TT}}, \quad (2.2)$$

so that $\gamma_{ii} = 0$, and $\partial_i \gamma_{ij} = 0$. They evolve according to the equation,

$$\left(\partial_0^2 + 2\mathcal{H}\partial_0 - \nabla^2 \right) \gamma_{ij} = 0, \quad (2.3)$$

where ∂_0 denotes a derivative with respect to conformal time, where $\mathcal{H} = \partial_0 a/a$ is the conformal Hubble rate related to the physical Hubble rate as $\mathcal{H} = aH$, and where $\nabla^2 = \partial_i \partial_i$ is the Laplacian.

The initially small subhorizon fluctuations are amplified by the inflationary background as their wavelengths stretch to superhorizon scales. This property of tensor fluctuations is described by the coincident limit of the two-point correlation function expressed in momentum space,

$$\langle \gamma_{ij}(\eta, \mathbf{x}) \gamma_{ij}(\eta, \mathbf{x}') \rangle = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} P_t^0(\eta, k), \quad (2.4)$$

where we have already assumed spatial homogeneity and isotropy of the background during inflation, so that the dimensionful power spectrum P_t is a function of momentum modulus only, $k = \|\mathbf{k}\|$. The dimensionful power spectrum is often expressed in terms of its dimensionless counterpart,

$$\mathcal{P}_t^0(\eta, k) = \frac{\kappa^2 k^3}{2\pi^2} P_t^0(\eta, k), \quad (2.5)$$

that is the main object of our discussion. The inflationary spacetime is a quasi-de Sitter period during which $\mathcal{H} = -1/(H_{\text{inf}}\eta)$, with η negative, and H_{inf} the nearly constant physical Hubble parameter. Normalising the modes to the Chernikov-Tagirov-Bunch-Davies [11, 12] vacuum deep inside the cosmological horizon, one obtains as the solution a key prediction of inflation: the power spectrum obtains an amplitude given by the curvature scale of the inflating background at the moment the mode crosses the cosmological horizon [13–16] (for a now-textbook presentation see [17]),

$$\mathcal{P}_t^0 = \frac{\kappa^2 H_{\text{inf}}^2}{\pi^2} \Big|_{k=\mathcal{H}}. \quad (2.6)$$

Since the expansion is accelerating during inflation, these modes continue being stretched, but their amplitude remains constant once they have crossed the horizon. Inflation thus predicts a primordial tensor power spectrum which is homogeneous, isotropic, with the only deviation from scale invariance resulting from the slow evolution of H_{inf} during the quasi-de Sitter phase. This is usually described by the tensor tilt,

$$n_t \equiv \frac{d \ln \mathcal{P}_t^0}{d \ln k}. \quad (2.7)$$

Rather than giving the amplitude of tensor directly (since it is not yet observed), one usually constrains the ratio r of the amplitude of the power spectrum of tensor fluctuations to the already observed scalar fluctuations. In the simplest single-scalar field models of inflation the tensor tilt is simply related to r through, $n_t = r/8$, although this relation is modified in alternative models.

Eventually, the slow-roll phase ends, the universe reheats and the radiation-domination era begins. At this point the universe is filled with an extremely hot and interacting plasma with temperature T_{rh} , while the Hubble parameter is given by

$$H_{\text{rh}}^2 = \frac{\pi^2 \kappa^2}{180} g_{\text{eff}} T_{\text{rh}}^4, \quad (2.8)$$

where g_{eff} is the effective number of relativistic degrees of freedom fixed by the high-energy physics at the reheating temperature. The temperature then scales as a^{-1} giving $\mathcal{H} = H_{\text{rh}}(a_{\text{rh}}/a)$ — the expansion decelerates. The equation of motion for the tensor fluctuations (2.3) then has as the solution in the superhorizon limit, $k \ll \mathcal{H}$, a constant mode and a decaying one. This means that during radiation domination, the evolution of the tensor fluctuations is effectively frozen on superhorizon scales. In the concordance scenario, there is no evolution of the amplitude superhorizon between reheating and when the mode shrinks sufficiently to re-enter the cosmological horizon.

This simple picture is slightly modified by the damping of the amplitude squared of gravitational waves by around one third by neutrinos when their wavelengths are comparable to the cosmological horizon, first described in ref. [20]. Such damping was later investigated much more generally in ref. [21], showing that no significant additional modification of gravitational wave amplitude can be caused by the matter sources of the type present in the universe in approximately equilibrium distributions. Other sources of gravitational waves, such as topological defects or black-hole inspirals, are only active at subhorizon scales and therefore do not contribute to the horizon-scale observations in the CMB.

The tensor fluctuations that re-enter the horizon close to the time of matter-radiation equality contribute to the power spectrum of temperature fluctuations in the CMB at the largest scales, and are a unique source of primary B-modes in the polarisation power spectrum. Since the details of the radiation era are completely transparent to the dynamics of superhorizon tensor fluctuations (and the scalar ones for similar reasons), the Einstein-Boltzmann codes used for predicting the power spectra of fluctuations at the last scattering surface (e.g. CAMB [18] and CLASS [19]), do not consider the bulk of the radiation era at all, but rather start the modes' evolution when their wavelengths are some order of magnitude larger than the Hubble scale. A measurement of the tensor amplitude from the last cosmic microwave background is thus really a constraint on the amplitude as it was close to the time of matter-radiation equality. We are belabouring this point, since the main result of this paper is the statement that there is a superhorizon correction to the evolution equation (2.3) during the radiation domination era which challenges the superhorizon conservation law.

Let us fix some orders of magnitude and connect to parameters being constrained by observations. The headline Planck constraint [2] on tensor modes, $r_{0.002} < 0.010$, comes from assuming the single-scalar-field prediction for the tensor tilt and a pivot in r at the scale $k = 0.002 \text{ Mpc}^{-1}$ and is obtained from the temperature power spectrum. Alternative analyses leave the tensor tilt free (although constant), constraining the tensor modes at a second scale $k = 0.02 \text{ Mpc}^{-1}$, giving $r_{0.002} < 0.044$ and $r_{0.02} < 0.184$. Finally, BICEP/KECK [22] constrains the polarisation B-modes directly at the pivot scale $k = 0.05 \text{ Mpc}^{-1}$ assuming $n_t = 0$, giving $r < 0.036$. For us, the most relevant information is that the smallest of these scales re-enters the horizon when non-relativistic matter provides a significant 10% of the energy budget of the universe, with this fraction larger for the longer modes. As a first-order approximation, we will assume that all these modes are affected by the physics presented in this paper equally and will not discuss the potential additional scale dependence induced at the end of the radiation-domination epoch.

Of key importance in our computation is the duration of the radiation domination era. This is not known, beyond the fact that the Universe must have reheated to a temperature high enough to allow for Big Bang Nucleosynthesis, $T_{\text{BBN}} \sim 1 \text{ MeV}$. Assuming particular classes of inflation models, one can obtain constraints on the reheating temperature from Planck observations [23], but they are not strong and the range of possibilities is wide; for example $T_{\text{rh}} > 400 \text{ TeV}$ for small-field models, $T_{\text{rh}} > 18 \text{ PeV}$ for supergravity scenarios, but $T_{\text{rh}} \sim \mathcal{O}(10^{13} \text{ GeV})$ [2] for Higgs inflation [24]. Reheating is thought to be short and to occur soon after inflation ends (see e.g. ref. [25] for a review of reheating mechanisms), although for some mechanisms this can be extended, adding to the uncertainty (e.g. reheating through primordial black hole evaporation [26]). We take as a fiducial temperature scale of reheating $T_{\text{rh}} \sim 10^{13} \text{ GeV}$. We wish to compute the ratio of the scale factor at reheating a_{rh} and at the end of the radiation-domination era a_* . This is given by

$$\exp(\mathcal{N}_{\text{rad}}) = \frac{a_*}{a_{\text{rh}}} = \left(\frac{g_s(T_{\text{rh}})}{g_s(T_0)} \right)^{1/3} \frac{T_{\text{rh}}}{T_0} a_* \quad (2.9)$$

with T_0 the CMB temperature today, and g_s the effective number of entropy degrees of freedom, with $g_s(T_0) = 3.91$ while the value at reheating depends on the particle content of the universe at that energy scale. Usually one takes $g_s(100 \text{ GeV}) \sim 100$ (see e.g. [27]), but at

the large energy scales of reheating this is unknown. Nonetheless, unless it is very large, it does not significantly contribute to $\mathcal{N}_{\text{rad}}$. For a fiducial duration of the radiation era we thus have,

$$\mathcal{N}_{\text{rad}} = 52 + \ln \left(\frac{T_{\text{rh}}}{10^{13} \text{ GeV}} \right) + \frac{1}{3} \ln \left(\frac{g_S(T_{\text{rh}})}{100} \right). \quad (2.10)$$

This is long enough that even logarithmic corrections to the standard expectations of superhorizon conservation could be a significant correction. In what follows we extend the standard formalism to account for the fluctuating nature of the radiation-era background.

3 Setting up one-loop equation for tensor perturbations

The expansion during the radiation-dominated period of cosmology is driven by the thermal plasma, rather than classical condensates of matter fields as during inflation. For that reason we should consider the matter fields sourcing the Einstein equation as stochastic random variables. For consistency, the left-hand side of the Einstein equation should have a stochastic character as well, and thus the metric should be considered to be stochastic as well. This is why we start by considering the radiation dominated period to be described by the Einstein and Maxwell equations,

$$G_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu}, \quad \nabla^\mu F_{\mu\nu} = 0, \quad (3.1)$$

with the photon energy-momentum tensor,

$$T_{\mu\nu} = \left(\delta_\mu^\rho \delta_\nu^\sigma - \frac{1}{4} g_{\mu\nu} g^{\rho\sigma} \right) g^{\alpha\beta} F_{\rho\alpha} F_{\sigma\beta}, \quad (3.2)$$

where all the quantities are in principle stochastic random variables. We assume for simplicity that the plasma driving the expansion can be modeled by free-streaming photons with a thermal distribution. Starting from these equations we shall derive one-loop equations for the metric perturbations around homogeneous and isotropic backgrounds. In particular, in the following we derive the equation of motion for the two-point function of traceless transverse perturbations.

3.1 Perturbing equations

The assumption of the setup is that the matter sector has thermalized efficiently during the reheating period, without disrupting the homogeneity and isotropy of the gravitational background. That is why we treat the stochastic photon field as the background, while treating the stochastic feature of the metric perturbatively. Consequently we want to derive a linear equation for the metric perturbation $h_{\mu\nu}$, defined in (2.1) as the perturbation of the conformally rescaled FLRW metric. This metric perturbation induces perturbations of all the quantities appearing in equations of motion (3.1), and we expand to linear order,

$$G_{\mu\nu} = G_{\mu\nu}^{(0)} + \kappa G_{\mu\nu}^{(1)}, \quad T_{\mu\nu} = T_{\mu\nu}^{(0)} + \kappa T_{\mu\nu}^{(1)}, \quad F_{\mu\nu} = F_{\mu\nu}^{(0)} + \kappa F_{\mu\nu}^{(1)}. \quad (3.3)$$

Note that the power of κ is explicitly factored out of the perturbed quantities, and that the parenthesized superscripts denote the number of metric perturbations $h_{\mu\nu}$ the quantity contains.

The background part of the Einstein tensor is just

$$G_{\mu\nu}^{(0)} = \left(3\delta_\mu^0 \delta_\nu^0 + (2\epsilon - 3)\bar{\eta}_{\mu\nu} \right) \mathcal{H}^2, \quad (3.4)$$

where $\bar{\eta} = \eta_{\mu\nu} + \delta_\mu^0 \delta_\nu^0$, and where, $\epsilon = 1 - \mathcal{H}'/\mathcal{H}^2 = -\dot{H}/H^2$ is the so-called principal slow-roll parameter (in radiation-dominated period $\epsilon = 2$). The perturbation of the Einstein tensor reads

$$G_{\mu\nu}^{(1)} = -\frac{1}{2} \left[\partial^2 h_{\mu\nu} - 2\partial^\rho \partial_{(\mu} h_{\nu)\rho} + \partial_\mu \partial_\nu h + \eta_{\mu\nu} (\partial^\rho \partial^\sigma h_{\rho\sigma} - \partial^2 h) \right] \\ - \mathcal{H} \left[2\partial_{(\mu} h_{\nu)0} - \partial_0 h_{\mu\nu} - \eta_{\mu\nu} (2\partial^\rho h_{\rho 0} - \partial_0 h) \right] + (2\epsilon - 3)\mathcal{H}^2 (h_{\mu\nu} + \eta_{\mu\nu} h_{00}), \quad (3.5)$$

where $h = h^\mu{}_\mu$ denotes the trace, and where $\partial^2 = \partial^\mu \partial_\mu = -\partial_0^2 + \nabla^2$ is the flat space d'Alembertian operator. The background energy-momentum tensor matches the flat space expression on account of conformal coupling of photons to gravity,

$$T_{\mu\nu}^{(0)} = \frac{1}{2a^2} \left(\delta_{(\mu}^\rho \delta_{\nu)}^\sigma - \frac{1}{4}\eta_{\mu\nu} \eta^{\rho\sigma} \right) \eta^{\alpha\beta} \{ F_{\rho\alpha}^{(0)}, F_{\sigma\beta}^{(0)} \}, \quad (3.6)$$

where $\{A, B\} = AB + BA$ denotes the anticommutator. The perturbation of the energy-momentum tensor contains two contributions: (i) perturbation of the explicit metric dependence, and (ii) implicit perturbation contained in the perturbation of the photon field strength tensor,

$$T_{\mu\nu}^{(1)} = \frac{1}{2a^2} W_{\mu\nu}{}^{\rho\sigma\alpha\beta\gamma\delta} \{ F_{\rho\alpha}^{(0)}, F_{\sigma\beta}^{(0)} \} h_{\gamma\delta} + \frac{1}{a^2} \left(\delta_{(\mu}^\rho \delta_{\nu)}^\sigma - \frac{1}{4}\eta_{\mu\nu} \eta^{\rho\sigma} \right) \eta^{\alpha\beta} \{ F_{\rho\alpha}^{(0)}, F_{\sigma\beta}^{(1)} \}, \quad (3.7)$$

where the tensor structure is

$$W_{\mu\nu}{}^{\rho\sigma\alpha\beta\gamma\delta} = -\delta_{(\mu}^\rho \delta_{\nu)}^\sigma \eta^{\gamma(\alpha} \eta^{\beta)\delta} + \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} \eta^{\gamma(\alpha} \eta^{\beta)\delta} - \frac{1}{4} \delta_{(\mu}^\gamma \delta_{\nu)}^\delta \eta^{\rho\sigma} \eta^{\alpha\beta}. \quad (3.8)$$

The Maxwell equation allows us to express the implicit perturbation explicitly in terms of the metric perturbation. The background stochastic photon field strength satisfies the flat space equation, $\partial_\nu F_{(1)}^{\nu\mu} = 0$, while its first perturbation satisfies the equation sourced by the metric perturbation,

$$\partial_\nu F_{(1)}^{\nu\mu} = \frac{1}{2} V^{\mu\rho\nu\sigma\alpha\beta} \partial_\nu (F_{\rho\sigma}^{(0)} h_{\alpha\beta}) \equiv J_{(1)}^\mu, \quad (3.9)$$

where the tensor structure is

$$V^{\mu\rho\nu\sigma\alpha\beta} = 4\eta^{(\alpha} \eta^{\mu)} [\eta^{\nu]} [\eta^{\rho} \eta^{\sigma}]^{(\beta} + \eta^{\mu[\rho} \eta^{\sigma]\nu} \eta^{\alpha\beta}. \quad (3.10)$$

The source of eq. (3.9) is necessarily conserved, $\partial_\mu J_{(1)}^\mu = 0$. With the help of Bianchi identity the solution of this equation is found to be

$$F_{\mu\nu}^{(1)}(x) = \int d^4 x' G_{\text{R}}(x - x') \times 2\partial'_{[\mu} J_{\nu]}^{(1)}(x'), \quad (3.11)$$

where G_{R} is the retarded Green's function that satisfies the equation

$$\partial^2 G_{\text{R}}(x - x') = \delta^4(x - x'). \quad (3.12)$$

This retarded Green's function is conveniently expressed in terms of its spatial Fourier transform,

$$G_{\text{R}}(x - x') = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \tilde{G}_{\text{R}}(\eta - \eta' | k), \quad (3.13)$$

where $k = \|\mathbf{k}\|$ is the momentum vector modulus, and where the momentum space retarded Green's function reads

$$\tilde{G}_{\text{R}}(\eta - \eta' | k) = -\theta(\eta - \eta') \frac{\sin[k(\eta - \eta')]}{k}. \quad (3.14)$$

Therefore, the perturbation of the photon field strength tensor is found to be

$$F_{\mu\nu}^{(1)}(x) = \int d^4 x' G_{\text{R}}(x - x') \partial'_\lambda \partial'_{[\mu} [V_{\nu]}^{\rho\lambda\sigma\alpha\beta} F_{\rho\sigma}^{(0)}(x') h_{\alpha\beta}(x')]. \quad (3.15)$$

Implicit in the expression above is the lower limit of temporal integration that is assumed to be the reheating time η_{rh} . Thus the photon field strength perturbation vanishes at that time, which accounts for the assumption of rapidly thermalized plasma during reheating in a process not much influenced by gravitational interactions.

Having solved for the photon field strength perturbation, we can express the perturbed Einstein equation solely in terms of the metric perturbation. The energy-momentum tensor now takes a nonlocal form,

$$\begin{aligned} T_{\mu\nu}^{(1)}(x) = & \frac{1}{2a^2} W_{\mu\nu}^{\rho\sigma\alpha\beta\gamma\delta} \{F_{\rho\alpha}^{(0)}(x), F_{\sigma\beta}^{(0)}(x)\} h_{\gamma\delta}(x) + \frac{1}{a^2} \left(\delta_{(\mu}^{\rho} \delta_{\nu)}^{\sigma} - \frac{1}{4} \eta_{\mu\nu} \eta^{\rho\sigma} \right) \eta^{\alpha\beta} \\ & \times \int d^4 x' G_{\text{R}}(x - x') \partial'_\kappa \partial'_{[\sigma} [V_{\beta]}^{\theta\kappa\lambda\gamma\delta} \{F_{\rho\alpha}^{(0)}(x), F_{\theta\lambda}^{(0)}(x')\} h_{\gamma\delta}(x')], \end{aligned} \quad (3.16)$$

owing to solving the Maxwell equation. The Einstein equation is now written in terms of the thermally fluctuating $F_{\mu\nu}^{(0)}$ on an FLRW background characterized by the scale factor a^2 , and linearized in the metric perturbation $h_{\mu\nu}$.

We proceed by introducing an additional assumptions about the system, namely that the background metric is driven by the expectation value of the background energy-momentum tensor,

$$G_{\mu\nu}^{(0)} = \frac{\kappa^2}{2} \langle T_{\mu\nu}^{(0)} \rangle, \quad (3.17)$$

which is given by

$$\langle T_{\mu\nu}^{(0)} \rangle = \frac{1}{2a^2} \left(\delta_{(\mu}^{\rho} \delta_{\nu)}^{\sigma} - \frac{1}{4} \eta_{\mu\nu} \eta^{\rho\sigma} \right) \eta^{\alpha\beta} \langle \{F_{\rho\alpha}^{(0)}, F_{\sigma\beta}^{(0)}\} \rangle. \quad (3.18)$$

This corresponds to the assumption that the thermal plasma is the dominant matter component during radiation-dominated period. This allows to write the linearized Einstein equation as

$$G_{\mu\nu}^{(1)} = \frac{\kappa}{2} \left(T_{\mu\nu}^{(0)} - \langle T_{\mu\nu}^{(0)} \rangle \right) + \frac{\kappa^2}{2} T_{\mu\nu}^{(1)}. \quad (3.19)$$

We make another reorganization of this linearized equation by subtracting the Hartree term from both sides of the equation, so that we write it in the form

$$G_{\mu\nu}^{(1)} - \frac{\kappa^2}{4a^2} W_{\mu\nu}^{\rho\sigma\alpha\beta\gamma\delta} \langle \{F_{\rho\alpha}^{(0)}, F_{\sigma\beta}^{(0)}\} \rangle h_{\gamma\delta} = \frac{S_{\mu\nu}}{2a^2}, \quad (3.20)$$

where we split the right-hand side into three pieces, $S_{\mu\nu} = S_{\mu\nu}^I + S_{\mu\nu}^{II} + S_{\mu\nu}^{III}$. The first of these pieces does not depend on the metric perturbation, but rather on the fluctuation of the background energy-momentum tensor,

$$S_{\mu\nu}^I(x) = \kappa a^2 \left(T_{\mu\nu}^{(0)}(x) - \langle T_{\mu\nu}^{(0)}(x) \rangle \right). \quad (3.21)$$

The other two sources depend on the metric perturbation: the second one locally,

$$S_{\mu\nu}^{II}(x) = \frac{\kappa^2}{2} W_{\mu\nu}{}^{\rho\sigma\alpha\beta\gamma\delta} \left(\{ F_{\rho\alpha}^{(0)}(x), F_{\sigma\beta}^{(0)}(x) \} - \langle \{ F_{\rho\alpha}^{(0)}(x), F_{\sigma\beta}^{(0)}(x) \} \rangle \right) h_{\gamma\delta}(x), \quad (3.22)$$

and the third one nonlocally,

$$S_{\mu\nu}^{III}(x) = \kappa^2 \left(\delta_{(\mu}^{\rho} \delta_{\nu)}^{\sigma} - \frac{1}{4} \eta_{\mu\nu} \eta^{\rho\sigma} \right) \eta^{\alpha\beta} \times \int d^4 x' G_R(x - x') \partial'_\kappa \partial'_{[\sigma} \left[V_{\beta]}{}^{\theta\kappa\lambda\gamma\delta} \{ F_{\rho\alpha}^{(0)}(x), F_{\theta\lambda}^{(0)}(x') \} h_{\gamma\delta}(x') \right]. \quad (3.23)$$

This step of subtracting the Hartree term will ensure that the transverse traceless component of the metric perturbation remains massless when propagating on the fluctuating background.

We note that the background equation of motion (3.17) and the first-order one (3.19) can alternatively be derived in the stochastic gravity formalism [7, 8]. The stochastic Langevin equation that is considered there reads²

$$G_{\mu\nu}[g + \kappa h] = \frac{\kappa^2}{2} \left(\langle T_{\mu\nu}[g + \kappa h] \rangle_A + \xi_{\mu\nu}[g] \right), \quad (3.24)$$

where $\langle \cdot \rangle_A$ denotes the expectation value over the vector matter field only. Linearizing this equation in the metric perturbation $h_{\mu\nu}$, and considering the stochastic field $\xi_{\mu\nu}$ to be of the same order as $h_{\mu\nu}$, we obtain eq. (3.17) at zeroth order and

$$G_{\mu\nu}^{(1)} = \frac{\kappa^2}{2} \langle T_{\mu\nu}^{(1)} \rangle_A + \frac{\kappa}{2} \xi_{\mu\nu}[g] \quad (3.25)$$

at first order. Since $\xi_{\mu\nu}$ is a stochastic variable with vanishing mean $\langle \xi_{\mu\nu}[g] \rangle = 0$ and covariance $\langle \xi_{\mu\nu}(x) \xi_{\rho\sigma}(x') \rangle = \langle [T_{\mu\nu}^{(0)}(x) - \langle T_{\mu\nu}^{(0)}(x) \rangle] [T_{\rho\sigma}^{(0)}(x') - \langle T_{\rho\sigma}^{(0)}(x') \rangle] \rangle$, this is exactly equivalent to eq. (3.19).

3.2 Vector field strength correlator

The photon field does not see directly the expansion of the cosmological background due to its conformal coupling to gravity. Therefore, its thermal correlator takes the same form as in flat space,

$$\langle \{ F_{\mu\nu}^{(0)}(x), F_{\rho\sigma}^{(0)}(x') \} \rangle = 8 \partial_{[\mu} \eta_{\nu][\sigma} \partial'_{\rho]} F(x - x'), \quad (3.26)$$

²Note that in the general stochastic gravity equation also tensors that arise from (the variation of) finite parts of counterterms appear. Since the renormalization of perturbative quantum gravity as an effective field theory is well understood (e.g. [33]), and the corresponding terms do not contribute to the leading corrections for large temperature, we may ignore them for our purposes. In general those tensors result in higher-derivative terms, which can pose problems for the solution of the equations [28]. However, if those higher derivative terms are treated perturbatively, in the same spirit in which they arise, the problems also disappear (e.g. [29, 30]).

where the thermal statistical two-point function [32] of the scalar field in flat space

$$F(x - x') = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \tilde{F}(\eta - \eta' | k) \quad (3.27)$$

is most conveniently expressed in terms of its spatial Fourier transform

$$\tilde{F}(\eta - \eta' | k) = \frac{\cos[k(\eta - \eta')]}{k} \left[\frac{1}{2} + \frac{1}{e^{k/T_{\text{rh}}} - 1} \right], \quad (3.28)$$

where T_{rh} is the reheating temperature. We can thus write (3.26) in Fourier space,

$$\langle \{ F_{\mu\nu}^{(0)}(x), F_{\rho\sigma}^{(0)}(x') \} \rangle = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} (8k_{[\mu} \eta_{\nu][\sigma} k_{\rho]}) \tilde{F}(\eta - \eta' | k), \quad (3.29)$$

where $k^\mu = (k^0, \mathbf{k}) = (\|\mathbf{k}\|, \mathbf{k})$.

The coincidence limit of the field strength correlator determines the expectation value of the energy-momentum tensor (3.18), that sources the background expansion via (3.17). In taking the coincidence limit we keep only the thermal contribution, assuming the large temperature limit which we verify a posteriori to be consistent. In particular, we discard all ultraviolet divergences which we assume have been taken care of by suitable counterterms that cannot depend on the temperature. This yields

$$\langle \{ F_{\mu\nu}^{(0)}(x), F_{\rho\sigma}^{(0)}(x) \} \rangle \xrightarrow{T_{\text{rh}} \gg H_{\text{rh}}} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{8k_{[\mu} \eta_{\nu][\sigma} k_{\rho]}}{k} \frac{1}{e^{k/T_{\text{rh}}} - 1} = \left(2\delta_{[\mu}^0 \bar{\eta}_{\nu][\sigma} \delta_{\rho]}^0 + \bar{\eta}_{\mu[\rho} \bar{\eta}_{\sigma]\nu} \right) \frac{4\pi^2 T_{\text{rh}}^4}{45}, \quad (3.30)$$

where $\bar{\eta}_{\mu\nu} = \eta_{\mu\nu} + \delta_\mu^0 \delta_\nu^0$. Then we can compute the expectation value of the background energy-momentum tensor (3.18),

$$\langle T_{\mu\nu}^{(0)} \rangle = \left(3\delta_\mu^0 \delta_\nu^0 + \bar{\eta}_{\mu\nu} \right) \frac{\pi^2 T_{\text{rh}}^4}{45a^2} = \left(4u_\mu u_\nu + g_{\mu\nu} \right) \frac{\pi^2 T_{\text{rh}}^4}{45}, \quad (3.31)$$

where in the second equality we have defined the normalized four-velocity $u_\mu = -a\delta_\mu^0$. Combining this with the background Einstein tensor in (3.4) gives the background solution,

$$\epsilon = 2 \quad \implies \quad \mathcal{H} = \frac{H_{\text{rh}}}{a} \quad \implies \quad H_{\text{rh}}^2 = \frac{\pi^2 \kappa^2 T_{\text{rh}}^4}{90}. \quad (3.32)$$

From here we see that the assumption $T_{\text{rh}} \gg H_{\text{rh}}$ is justified as long as $(\kappa H_{\text{rh}})^2 \ll 1$, which indeed is the case for the radiation dominated universe, as explained in section 2.³

The coincidence limit of the field strength correlator also appears in the Hartree term of the equation of motion (3.20) for the metric fluctuations. It is computed by appropriately contracting the tensor structure (3.8) and the metric perturbation into (3.30),

$$-\frac{\kappa^2}{4a^2} W_{\mu\nu}{}^{\rho\sigma\alpha\beta\gamma\delta} \langle \{ F_{\rho\alpha}^{(0)}, F_{\sigma\beta}^{(0)} \} \rangle h_{\gamma\delta} = -\mathcal{H}^2 \left(h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h - 2\delta_\mu^0 \delta_\nu^0 h - 4\delta_{(\mu}^0 h_{\nu)0} \right). \quad (3.33)$$

where we have used the last two relations from (3.32) to recognize that this contribution, corresponding to the 4-vertex diagram given in figure 1, shows no κ^2 suppression expected by naive power counting.

³The large temperature limit also allows us to neglect any contribution coming from the trace anomaly (e.g. [31, 34]), whose contribution to the energy-momentum tensor (3.31) is proportional to $\eta_{\mu\nu} H_{\text{rh}}^4 a^{-6}$.

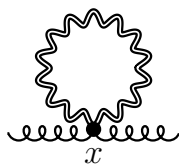


Figure 1. 1PI 4-vertex diagram contributing to the graviton equation of motion. The double wavy line stands for the statistical two-point function of the photon, while curly lines denote amputated gravitons. This diagram is conventionally defined as the expectation value of the second variation of the Maxwell action with respect to the metric, which does not correspond directly to the contribution in (3.33), that is defined by perturbing the Einstein equation with the square root of the metric stripped off. This detail is immaterial in practice as the same additional term not included (3.33) is also omitted in the left-hand side when perturbing the Einstein tensor.

3.3 Transverse traceless projection

We are primarily interested in the propagation of transverse traceless perturbations, i.e. tensor perturbations. We isolate these from the general equation (3.20) with the help of the transverse traceless projector,

$$\Pi_{ijkl} = \Pi_{i(k}\Pi_{l)j} - \frac{1}{2}\Pi_{ij}\Pi_{kl}, \quad (3.34)$$

defined in terms of the idempotent transverse projector,

$$\Pi_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}, \quad \Pi^{ij} \partial_j = \partial_j \Pi^{ij} = 0, \quad \Pi_{ij} \Pi^{jk} = \Pi_i^k, \quad \Pi_i^i = 2. \quad (3.35)$$

The tensor perturbation is then defined as

$$\gamma_{ij} = \Pi_{ij}{}^{kl} h_{kl}. \quad (3.36)$$

Acting with the projector (3.34) onto the equation of motion (3.20) we get the equation of motion for the tensor perturbation,

$$\left(\partial_0^2 + 2\mathcal{H}\partial_0 - \nabla^2\right)\gamma_{ij} = \frac{1}{a^2}\Pi_{ij}{}^{kl} S_{kl}. \quad (3.37)$$

We should note here that the left-hand side of the equation reveals the graviton to be massless. This is due to the Hartree term (3.33), depicted in figure 1, canceling the would-be mass term from the perturbation of the Einstein tensor (3.5).

We aim to solve eq. (3.37) treating the source as a perturbation. That is facilitated by introducing another retarded scalar Green's function,

$$\left(\partial_0^2 + 2\mathcal{H}\partial_0 - \nabla^2\right)G_{\text{R}}^{\text{TT}}(x; x') = \frac{\delta^4(x - x')}{a^2}, \quad (3.38)$$

that had been worked out in [4]. Here we choose the scale factor normalization on the right-hand side for convenience, so that the Green's function absorbs all the explicit powers of the scale factors in loop computations, and so that the Green's function is symmetric in

time arguments, apart from the overall step function. The solution for this Green's function is best expressed in spatial momentum space,

$$G_{\text{R}}^{\text{TT}}(x; x') = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \tilde{G}_{\text{R}}^{\text{TT}}(\eta; \eta' | k), \quad (3.39)$$

where the Green's function reads

$$\tilde{G}_{\text{R}}^{\text{TT}}(\eta; \eta' | k) = \frac{\theta(\eta - \eta') \sin[k(\eta - \eta')]}{a(\eta)a(\eta')} \frac{1}{k}. \quad (3.40)$$

This allows us to cast eq. (3.37) in the form of the Yang-Feldman equation,

$$\gamma_{ij}(x) = \gamma_{ij}^0(x) + \int d^4 x' G_{\text{R}}^{\text{TT}}(x; x') \Pi_{ij}{}^{kl}(\mathbf{x}') S_{kl}(x'), \quad (3.41)$$

where γ_{ij}^0 is the tree-level solution. This is now an integral equation that is well adapted to a perturbative expansion, that is obtained by iterating the equation.

3.4 Tensor perturbation correlator

We want to compute the one-loop correction to the statistical two-point functions of tensor perturbations,

$$\langle \{ \gamma_{ij}(x), \gamma_{kl}(x') \} \rangle = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \mathbb{P}_{ijkl}(\mathbf{k}) \times P(\eta; \eta' | k), \quad (3.42)$$

where the momentum space transverse traceless projector is

$$\mathbb{P}_{ijkl}(\mathbf{k}) = \mathbb{P}_{i(k}(\mathbf{k}) \mathbb{P}_{l)j}(\mathbf{k}) - \frac{1}{2} \mathbb{P}_{ij}(\mathbf{k}) \mathbb{P}_{kl}(\mathbf{k}), \quad \mathbb{P}_{ij}(\mathbf{k}) = \delta_{ij} - \frac{\mathbf{k}_i \mathbf{k}_j}{k^2}, \quad (3.43)$$

In particular, we are interested in the dimensionless counterpart of the momentum space two-point function, expressed in terms of the dimensionful one introduced in (3.42) as

$$\mathcal{P}(\eta; \eta' | k) = \frac{\kappa^2 k^3}{2\pi^2} P(\eta; \eta' | k), \quad (3.44)$$

whose coincident limit reduces to the definition of the dimensionless tensor power spectrum defined in section 2, $\mathcal{P}(\eta; \eta | k) = \mathcal{P}_t(\eta, k)$.

By plugging in the Yang-Feldman equation (3.41) into the definition (3.42), and applying Wick's theorem, we can collect all the one-loop contributions,

$$\langle \{ \gamma_{ij}(x), \gamma_{kl}(x') \} \rangle = \langle \{ \gamma_{ij}^0(x), \gamma_{kl}^0(x') \} \rangle \quad (3.45)$$

$$+ \int d^4 y G_{\text{R}}^{\text{TT}}(x; y) \int d^4 y' G_{\text{R}}^{\text{TT}}(x'; y') \Pi_{ij}{}^{mn}(\mathbf{y}) \Pi_{kl}{}^{ab}(\mathbf{y}') \langle \{ S_{mn}^{\text{I}}(y), S_{ab}^{\text{I}}(y') \} \rangle \quad (3.46)$$

$$+ \int d^4 y G_{\text{R}}^{\text{TT}}(x; y) \Pi_{ij}{}^{mn}(\mathbf{y}) \langle \{ S_{mn}^{\text{III}}(y), \gamma_{kl}^0(x') \} \rangle \quad (3.47)$$

$$+ \int d^4 y' G_{\text{R}}^{\text{TT}}(x'; y') \Pi_{kl}{}^{ab}(\mathbf{y}') \langle \{ \gamma_{ij}^0(x), S_{ab}^{\text{III}}(y') \} \rangle, \quad (3.48)$$

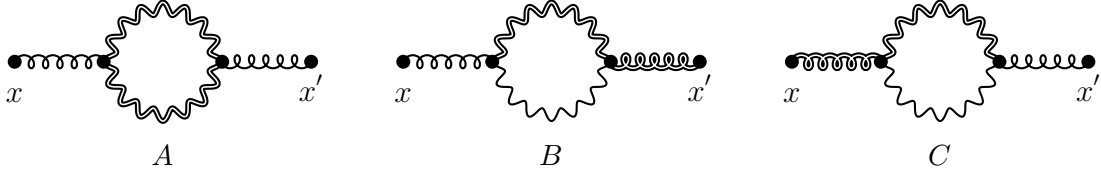


Figure 2. One-loop corrections to the graviton two-point function. Curly represent gravitons, while wavy lines represent photons. Double lines stand for statistical two-point functions, while single lines stand for retarded propagators. Diagram *A* depicts the so-called thermally induced gravitational wave production. Diagrams *B* and *C* depict contributions dubbed radiation exchange in [4].

where the correlators involving sources S_{ij}^{I-III} are given by:

$$\begin{aligned} \langle \{S_{mn}^I(y), S_{ab}^I(y')\} \rangle &= \kappa^2 (a_y a_{y'})^2 \langle \{T_{mn}^{(0)}(y) - \langle T_{mn}^{(0)}(y) \rangle, T_{ab}^{(0)}(y') - \langle T_{ab}^{(0)}(y') \rangle\} \rangle \\ &= \kappa^2 \left(\delta_{(m}^\mu \delta_{n)}^\nu - \frac{1}{4} \eta_{mn} \eta^{\mu\nu} \right) \eta^{\alpha\beta} \left(\delta_{(a}^\rho \delta_{b)}^\sigma - \frac{1}{4} \eta_{ab} \eta^{\rho\sigma} \right) \eta^{\omega\lambda} \\ &\quad \times \left[\langle \{F_{\mu\alpha}^{(0)}(y), F_{\rho\omega}^{(0)}(y')\} \rangle \langle \{F_{\nu\beta}^{(0)}(y), F_{\sigma\lambda}^{(0)}(y')\} \rangle \right. \\ &\quad \left. + \langle [F_{\mu\alpha}^{(0)}(y), F_{\rho\omega}^{(0)}(y')] \rangle \langle [F_{\nu\beta}^{(0)}(y), F_{\sigma\lambda}^{(0)}(y')] \rangle \right], \end{aligned} \quad (3.49)$$

$$\begin{aligned} \langle \{S_{mn}^{III}(y), \gamma_{kl}^0(x')\} \rangle &= \kappa^2 \left(\delta_{(m}^\mu \delta_{n)}^\nu - \frac{1}{4} \eta_{mn} \eta^{\mu\nu} \right) \eta^{\alpha\beta} \int d^4 y' G_R(y - y') \\ &\quad \times \partial_\omega^{y'} \partial_{[\nu}^{y'} [V_{\beta]}^{\rho\omega\sigma ab} \langle \{F_{\mu\alpha}^{(0)}(y), F_{\rho\sigma}^{(0)}(y')\} \rangle \langle \{\gamma_{ab}^0(y'), \gamma_{kl}^0(x')\} \rangle], \end{aligned} \quad (3.50)$$

$$\begin{aligned} \langle \{\gamma_{ij}^0(x), S_{ab}^{III}(y')\} \rangle &= \kappa^2 \left(\delta_{(a}^\rho \delta_{b)}^\sigma - \frac{1}{4} \eta_{ab} \eta^{\rho\sigma} \right) \eta^{\omega\lambda} \int d^4 y G_R(y' - y) \\ &\quad \times \partial_\alpha^y \partial_{[\sigma}^y [V_\lambda]^{\mu\alpha\nu mn} \langle \{F_{\rho\omega}^{(0)}(y'), F_{\mu\nu}^{(0)}(y)\} \rangle \langle \{\gamma_{ij}^0(x), \gamma_{mn}^0(y)\} \rangle]. \end{aligned} \quad (3.51)$$

Note that of the three contributions S_{mn}^I , S_{mn}^{II} and S_{mn}^{III} only the first and the last contribute to order κ^2 . Namely, terms involving two of S_{mn}^{II} and/or S_{mn}^{III} are of order κ^4 , while the mixed terms $\langle \{S_{mn}^{II}(y), \gamma_{kl}^0(x')\} \rangle = 0$ vanish.

The contribution in (3.45) is just the tree-level two-point function. The three contributions of order κ^2 in (3.46)–(3.48) are one-loop contributions given diagrammatically in figure 2, and labeled respectively as *A*, *B*, and *C*. Contribution *A* accounts for the so-called thermally induced gravitational wave production [9, 10]. This contribution does not depend on primordial tensor perturbations, and is there regardless of them, purely because of the thermal nature of the photon thermal medium. Contributions *B* and *C* depend linearly on the primordial tensor perturbation and represent the modulation of the primordial signal. In the following section we compute these three one-loop diagrams.

4 Computing diagrams

This section is devoted to computing the three diagrams in figure 2. They are computed in momentum space, in the superhorizon limits, and in the late time limit. For a more detailed computation and a discussion of various limits see [35].

4.1 Thermal GW production

Type A contribution in (3.46) written out explicitly reads

$$\begin{aligned} \langle \{ \gamma_{ij}(x), \gamma_{kl}(x') \} \rangle^A &= \kappa^2 \int d^4 y G_R^{\text{TT}}(x; y) \int d^4 y' G_R^{\text{TT}}(x'; y') \Pi_{ij}^{mn}(\mathbf{y}) \Pi_{kl}^{ab}(\mathbf{y}') \\ &\quad \times \eta^{\mu\nu} \eta^{\rho\sigma} \langle \{ F_{m\mu}^{(0)}(y), F_{a\rho}^{(0)}(y') \} \rangle \langle \{ F_{n\nu}^{(0)}(y), F_{b\sigma}^{(0)}(y') \} \rangle. \end{aligned} \quad (4.1)$$

where we have discarded the term in the last line of (3.49) on account of it not containing any temperature dependence. Expressed in momentum space, in terms of the dimensionless two-point function (3.44), this contribution reads,

$$\begin{aligned} \Delta \mathcal{P}^A(\eta; \eta' | k) &= \frac{2\kappa^4 k^3}{\pi^2} \int_{\eta_{\text{rh}}}^{\eta} d\eta_y \tilde{G}_R^{\text{TT}}(\eta; \eta_y | k) \int_{\eta_{\text{rh}}}^{\eta'} d\eta_{y'} \tilde{G}_R^{\text{TT}}(\eta'; \eta_{y'} | k) \\ &\quad \times \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \tilde{F}(\eta_y - \eta_{y'} | q) \tilde{F}(\eta_y - \eta_{y'} | \|\mathbf{k} - \mathbf{q}\|) \times \mathbb{P}_{mnab}(\mathbf{k}) \\ &\quad \times \left[q^m q^n q^a q^b + 2q^{(m} \eta^{n)(a} q^{b)} (k - q)^\mu q_\mu + \frac{1}{2} (k - q)^\mu q_\mu (k - q)^\nu q_\nu \eta^{m(a} \eta^{b)n} \right]. \end{aligned} \quad (4.2)$$

where $(k - q)^\mu q_\mu = \|\mathbf{q}\| \|\mathbf{k} - \mathbf{q}\| + \mathbf{q} \cdot (\mathbf{k} - \mathbf{q})$. We are interested in the small momentum limit $k \ll \mathcal{H}$, corresponding to superhorizon modes, for which the expression reduces to

$$\begin{aligned} \Delta \mathcal{P}^A(\eta; \eta' | k) &\stackrel{k \ll \mathcal{H}}{\simeq} \frac{2\kappa^4 k^3}{\pi^2} \int_{\eta_{\text{rh}}}^{\eta} d\eta_y \tilde{G}_R^{\text{TT}}(\eta; \eta_y | 0) \int_{\eta_{\text{rh}}}^{\eta'} d\eta_{y'} \tilde{G}_R^{\text{TT}}(\eta'; \eta_{y'} | 0) \\ &\quad \times \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \left[\tilde{F}(\eta_y - \eta_{y'} | q) \right]^2 \times \left[\mathbb{P}_{mnab}(\mathbf{k}) q^m q^n q^a q^b \right]_{\mathbf{k} \rightarrow 0}. \end{aligned} \quad (4.3)$$

This limit is assumed in the remainder of this subsection. The loop integral over $\mathbf{q}$ is computed in the appendix A, which leaves the two integrals over retarded Green's functions to evaluate. However, since we are not interested in the details of the transient contributions that quickly become irrelevant, but only in the leading contributions, it is more convenient to write the problem as a double differential equation making use of the equation of motion (3.38),

$$\left(\partial_0^2 + 2\mathcal{H}\partial_0 \right) \left(\partial_0'^2 + 2\mathcal{H}'\partial_0' \right) \Delta \mathcal{P}^A(\eta; \eta' | k) = \frac{2\kappa^4 k^3}{\pi^2 (aa')^2} I_1(T_{\text{rh}}, \Delta\eta), \quad (4.4)$$

where the I_1 is given in eq. (A.14), and where $\Delta\eta = \eta - \eta'$. Ultimately we are interested in the time coincidence limit of the tensor two-point function, for which it is sufficient to consider the equation above close to time coincidence (adopting the strategy from [36]),

$$\left(\partial_0^2 + 2\mathcal{H}\partial_0 \right) \left(\partial_0'^2 + 2\mathcal{H}'\partial_0' \right) \Delta \mathcal{P}^A(\eta; \eta' | k) = \frac{32\kappa^2 k^3 T_{\text{rh}} H_{\text{rh}}^2}{5\pi^2 (aa')^2} \left[1 + \mathcal{O}(\Delta\eta^2) \right]. \quad (4.5)$$

The solution at late times is then simply found to be

$$\Delta \mathcal{P}^A(\eta; \eta' | k) = \frac{32\kappa^2 k^3 T_{\text{rh}}}{5\pi^2 H_{\text{rh}}^2} \times \ln \left(\frac{a}{a_{\text{rh}}} \right) \ln \left(\frac{a'}{a_{\text{rh}}} \right), \quad (4.6)$$

while the coincidence limit gives the desired correction to the superhorizon tensor power spectrum,

$$\frac{\Delta \mathcal{P}_t^A}{\mathcal{P}_t^0} = \frac{32}{5} \times \left(\frac{k^3}{H_{\text{rh}}^3} \right) \times \left(\frac{T_{\text{rh}}}{H_{\text{inf}}} \right) \times \left(\frac{H_{\text{rh}}}{H_{\text{inf}}} \right) \times \ln^2 \left(\frac{a}{a_{\text{rh}}} \right). \quad (4.7)$$

While we obtain a different result for this contribution compared to the respective result in [4], this is of little consequence as this contribution is negligible for observable scales compared to the tree-level spectrum (cf. the discussion in section 2), on account of the $(k/H_{\text{inf}})^3$ suppression, despite the enhancement due to $T_{\text{rh}}/H_{\text{inf}}$.

4.2 Radiation exchange

Plugging in the tensor structure (3.10) into contribution B in (3.47), using Maxwell's equation, and partially integrating some spatial derivatives gives

$$\begin{aligned} \langle \{ \gamma_{ij}(x), \gamma_{kl}(x') \} \rangle^B &= \kappa^2 \int d^4 y G_{\text{R}}^{\text{TT}}(x; y) \Pi_{ij}{}^{mn}(\mathbf{y}) \int d^4 y' \\ &\times \left[\partial_n^{y'} \partial_{y'}^a G_{\text{R}}(y - y') \times \langle \{ F_{m\omega}(y), F^{b\omega}(y') \} \rangle^{(0)} \times \langle \{ \gamma_{ab}^0(y'), \gamma_{kl}^0(x') \} \rangle \right. \\ &\quad + \partial_n^{y'} G_{\text{R}}(y - y') \times \langle \{ F_m{}^a(y), F^{b\omega}(y') \} \rangle^{(0)} \times \partial_\omega^{y'} \langle \{ \gamma_{ab}^0(y'), \gamma_{kl}^0(x') \} \rangle \\ &\quad + \partial_{y'}^a G_{\text{R}}(y - y') \times \langle \{ F_m{}^\omega(y), F^b{}_n(y') \} \rangle^{(0)} \times \partial_\omega^{y'} \langle \{ \gamma_{ab}^0(y'), \gamma_{kl}^0(x') \} \rangle \\ &\quad \left. + G_{\text{R}}(y - y') \times \langle \{ F_m{}^\omega(y), F^{b\lambda}(y') \} \rangle^{(0)} \times \partial_\omega^{y'} \partial_\lambda^{y'} \langle \{ \gamma_{nb}^0(y'), \gamma_{kl}^0(x') \} \rangle \right]. \end{aligned} \quad (4.8)$$

Taking the spatial Fourier transform then gives

$$\begin{aligned} \Delta \mathcal{P}^B(\eta; \eta' | k) &= -\kappa^2 \int_{\eta_{\text{rh}}}^\eta d\eta_y \tilde{G}_{\text{R}}^{\text{TT}}(\eta; \eta_y | k) \int_{\eta_{\text{rh}}}^{\eta_y} d\eta_{y'} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \tilde{G}_{\text{R}}(\eta_y - \eta_{y'} | q) \\ &\times \left\{ \tilde{F}(\eta_y - \eta_{y'} | \|\mathbf{k} - \mathbf{q}\|) \left[2q^i q^j q^k q^l \mathbb{P}_{ijkl}(\mathbf{k}) + q^i q^j \mathbb{P}_{ij}(\mathbf{k}) [(\partial_0^{y'})^2 + k^2] \right. \right. \\ &\quad \left. \left. + (k - q)^i k_i (k - q)^j k_j + 2\|\mathbf{k} - \mathbf{q}\|^2 (\partial_0^{y'})^2 \right] \mathcal{P}(\eta_{y'}; \eta' | k) \right. \\ &\quad \left. - \partial_0^y \tilde{F}(\eta_y - \eta_{y'} | \|\mathbf{k} - \mathbf{q}\|) \times 2q^i q^j \mathbb{P}_{ij}(\mathbf{k}) \times \partial_0^{y'} \mathcal{P}^0(\eta_{y'}; \eta' | k) \right\}, \end{aligned} \quad (4.9)$$

where $\mathcal{P}^0$ is the tree-level dimensionless two-point function. Finally, we are interested in the superhorizon limit $k \ll \mathcal{H}$, henceforth assumed until the end of the subsection, in which the expression reduces to,

$$\begin{aligned} \Delta \mathcal{P}^B(\eta; \eta' | k) &\stackrel{k \ll \mathcal{H}}{\simeq} -2\kappa^2 \int_{\eta_{\text{rh}}}^\eta d\eta_y \tilde{G}_{\text{R}}^{\text{TT}}(\eta; \eta_y | k) \mathcal{P}_0(\eta_y; \eta' | 0) \\ &\times \int_{\eta_{\text{rh}}}^{\eta_y} d\eta_{y'} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \tilde{G}_{\text{R}}(\eta_y - \eta_{y'} | q) \tilde{F}(\eta_y - \eta_{y'} | q) \left[\mathbb{P}_{ijkl}(\mathbf{k}) q^i q^j q^k q^l \right]_{\mathbf{k} \rightarrow 0}. \end{aligned} \quad (4.10)$$

where we used the fact that the kernel of the integral over $\eta_{y'}$ has effectively a narrow support in order to extract the tree-level power spectrum outside of that integral. The loop integral over $\mathbf{q}$ is the integral I_2 evaluated in the appendix in eq. (A.15). Instead of evaluating explicitly the remaining integral over the retarded Green's function, it is again advantageous to rewrite the expression as a differential equation,

$$\Delta \mathcal{P}^B(\eta; \eta' | 0) = \mathcal{P}_0(\eta; \eta' | 0) \times \int_{\eta_{\text{rh}}}^\eta d\eta_y \tilde{G}_{\text{R}}^{\text{TT}}(\eta; \eta_y | k) \times \frac{\kappa^2}{120\pi^2} \left[\left(\frac{\partial}{\partial \Delta \eta} \right)^3 \mathcal{I}(T, \Delta \eta) \right]_0^{\eta_y - \eta_{\text{rh}}}, \quad (4.11)$$

where we have evaluated the temporal integral over I_2 using its derivative representation (A.9), that close to coincidence reads

$$\left(\partial_0^2 + 2\mathcal{H}\partial_0\right)\Delta\mathcal{P}^B(\eta; \eta'|k) = \left[\frac{\pi^2\kappa^2 T_{\text{rh}}^4}{225a^2} + \mathcal{O}(\Delta\eta^2)\right] \times \mathcal{P}_0(\eta; \eta|0) = \frac{2H_0^2}{5a^2} \times \mathcal{P}_0(\eta; \eta|0). \quad (4.12)$$

Inverting this equation at late times is straightforward,

$$\Delta\mathcal{P}^B(\eta; \eta'|0) = \frac{2}{5} \ln\left(\frac{a}{a_{\text{rh}}}\right) \times \mathcal{P}_0(\eta; \eta'|0). \quad (4.13)$$

and the C type contribution is obtained by simply exchanging time arguments,

$$\Delta\mathcal{P}^C(\eta; \eta'|0) = \frac{2}{5} \ln\left(\frac{a'}{a_{\text{rh}}}\right) \times \mathcal{P}_0(\eta; \eta|0). \quad (4.14)$$

Adding the two contributions together and taking the coincidence limit gives the full correction to tensor power spectrum,

$$\frac{\Delta\mathcal{P}_t^{B+C}}{\mathcal{P}_t^0} = \frac{4}{5} \ln\left(\frac{a}{a_{\text{rh}}}\right). \quad (4.15)$$

This result matches that reported in [4]. The most striking feature of this result is the absence of any suppression compared to the tree-level result, which signals the breakdown of perturbation theory, and points to potentially interesting physics, provided that a reliable way of quantifying the correction is found.

5 Resummed computation

The perturbative results for the corrections found in the preceding section show two features for B and C type corrections: (i) they are not suppressed by a small parameter, and (ii) they grow in time. This points to the fact that treating these corrections perturbatively is not justified. Rather, these corrections are parametrically of the same order as the terms in the part of the linear equation that determines the tree-level solution. That is why here we explore the limits of the linear approximation for the evolution of tensor modes, by treating B and C type corrections on the same footing. This is facilitated by the A type contribution being negligible, which allows to write a homogeneous linear equation for the tensor two-point function,

$$\begin{aligned} & \left(\partial_0^2 + 2\mathcal{H}\partial_0 - \nabla^2\right) \langle \{ \gamma_{ij}(x) \gamma_{kl}(x') \} \rangle - \frac{2\kappa^2}{a^2} \Pi_{ij}{}^{mn} \eta^{\alpha\beta} \\ & \times \int d^4x'' G_{\text{R}}(x - x'') \partial_\kappa'' \partial_{[m}'' \left[V_{\beta]}{}^{\theta\kappa\lambda rs} \langle \{ F_{n\alpha}^{(0)}(x), F_{\theta\lambda}^{(0)}(x'') \} \rangle \langle \{ \gamma_{rs}(x'') \gamma_{kl}(x') \} \rangle \right] = 0. \end{aligned} \quad (5.1)$$

By writing this equation in momentum space, and applying the same approximations for the superhorizon limit ($k \ll \mathcal{H}$) made in section 4.2 we find the equation to reduce to a local one,

$$\left(\partial_0^2 + 2\mathcal{H}\partial_0 + \frac{m_{\text{eff}}^2}{a^2}\right) \mathcal{P}(\eta; \eta'|k) = 0, \quad (5.2)$$

where the original nonlocality is captured by the effective mass term,

$$m_{\text{eff}}^2 = \frac{2\pi^3 \kappa^2 T_{\text{rh}}^5}{15} \int_0^{\eta - \eta_{\text{rh}}} d\Delta\eta \left[\frac{3}{8(\pi T_{\text{rh}} \Delta\eta)^5} - \frac{11 \operatorname{ch}(2\pi T_{\text{rh}} \Delta\eta) + \operatorname{ch}(6\pi T_{\text{rh}} \Delta\eta)}{\operatorname{sh}^5(2\pi T_{\text{rh}} \Delta\eta)} \right] \\ = \frac{\pi^2 \kappa^2 T_{\text{rh}}^4}{15} \left[\frac{2}{\operatorname{sh}^2(2\pi T_{\text{rh}} \Delta\eta_{\text{rh}})} + \frac{3}{\operatorname{sh}^4(2\pi T_{\text{rh}} \Delta\eta_{\text{rh}})} - \frac{3}{16(\pi T_{\text{rh}} \Delta\eta_{\text{rh}})^4} - \frac{1}{15} \right], \quad (5.3)$$

with $\Delta\eta_{\text{rh}} = \eta - \eta_{\text{rh}}$. We should emphasize that this mass term is not a local mass term that is canceled by the Hartree approximation [6], but rather an approximation for the nonlocal correction to the linear equation of motion for the tensor perturbation. Most of the terms in this effective mass term decay quickly, so that the late time approximation is warranted,

$$m_{\text{eff}}^2 \xrightarrow{T_{\text{rh}} \Delta\eta_{\text{rh}} \gg 1} -\frac{\pi^2 \kappa^2 T_{\text{rh}}^4}{225} = -\frac{2}{5} H_{\text{rh}}^2. \quad (5.4)$$

In general the effective mass squared is proportional to the 4-dimensional momentum space retarded self-energy in momentum space.⁴ Finally, the simple equation governing the evolution of the tensor perturbation two-point function on superhorizon scales reads

$$\left(\partial_0^2 + 2\mathcal{H}\partial_0 - \lambda\mathcal{H}^2 \right) \mathcal{P}(\eta; \eta'|k) = 0, \quad \lambda = \frac{2}{5}. \quad (5.5)$$

It is evident from this form of the equation that treating parts of it perturbatively is not warranted, as there is no naive κ^2 suppression that remains. The late time solution is found to be (remembering that another equation with respect to the primed coordinate must also hold),

$$\mathcal{P}(\eta; \eta'|k) = \mathcal{P}_0(\eta; \eta'|k) \times \left(\frac{a a'}{a_{\text{rh}}^2} \right)^{\frac{-1 + \sqrt{1 + 4\lambda}}{2}}, \quad (5.6)$$

where the tree-level two-point function is essentially time-independent. This means that the growth of the tensor power spectrum turns into a power-law⁵

$$\mathcal{P}_t = \mathcal{P}_t^0 \times \left(\frac{a}{a_{\text{rh}}} \right)^{-1 + \sqrt{1 + 4\lambda}}, \quad (5.7)$$

with the exponent is $(-1 + \sqrt{1 + 4\lambda}) \approx 0.612$. This correctly reproduces the perturbative result in (4.15) when formally expanded for small λ to linear order, but the resummation shows a much faster growth than the logarithmic one obtained by the one-loop approximation.

6 Discussion

Interpreting the fluctuations of the cosmic microwave background as having their origin in primordial inflation hinges on the assumption of conservation of primordial fluctuations on superhorizon scales, before they re-enter the horizon during the late radiation-dominated era. Motivated by the work of ref. [4], we examined whether this remains true when we

⁴The Fourier transforms need to be performed in the correct sequence: first the spatial one, and then the temporal one. In general the two transforms are known not to commute in thermal field theory [37].

⁵This result seems to have been confirmed numerically in [38].

account for the fluctuating nature of thermal plasma driving the expansion. We consider a simplified model for the plasma that is made up of free streaming photons in a thermal state. We have computed a perturbative one-loop correction to the superhorizon tensor power spectrum, showing a multiplicative growing secular correction (4.15) that is unsuppressed by any small parameter. The absence of a suppression factor is a consequence of the Friedmann equation (2.8) marrying the physical scales that multiply the one-loop correction, $\pi^2 \kappa^2 T_{\text{rh}}^4 / (90 H_{\text{rh}}^2) = 1$. Despite the weakness of the interaction between individual photons in the plasma and the tensor modes, the occupation number of photons is high enough to overcome the suppression.

Despite the photon's having two physical polarizations, this enhancement is the same as the one found in ref. [4], that considered plasma modeled by a scalar field. This suggests the amplification might be independent of the number of relativistic degrees of freedom making up the non-interacting plasma. Superficially the effect seems additive for the relativistic species, via the factor $g_{\text{eff}} T_{\text{rh}}^4$. However, it is precisely this combination that is constant for a given model of inflation and a given reheating mechanism, as evident from the Friedmann equation (2.8), so that the reheating temperature is proportional to $g_{\text{eff}}^{-1/4}$.

Because of the conformal coupling of photons to gravity we were able to treat the plasma at leading order as a thermal state of photons that is unchanged by the expansion of the Universe apart from the redshifting of its temperature, $T = T_{\text{rh}}/a$. One-loop corrections to the superhorizon tensor perturbations descend from interactions with the thermal fluctuations of the photons. The effect of the local one-loop diagram 1 is to ensure that tensor perturbations are massless at the linearized level, as captured by the Hartree approximation, while the enhancement effect derives from the nonlocal diagrams of figure 2. The first of these nonlocal diagrams represents the effect of thermally induced gravitational wave production [9, 10], that is negligible for observationally relevant scales owing to its blue tilt and thus suppression on superhorizon scales. It is only the B and C type contributions that are relevant here. They represent the effect of the tensor perturbation disturbing the thermal distribution of photons, which in turn backreacts back onto the tensor perturbations. This is seen from the fact that these diagrams arise from the perturbation of the Maxwell equation induced by tensor modes. Similar mechanism at one-loop was found to be relevant when considering corrections to the tensor power spectrum during inflation coming from an excited state of scalar fluctuations [39] (however, see [40] for a recent criticism of this work).

We were also able to explore the limitations of the late-time one-loop corrected linearized equation for tensor perturbations. By solving eq. (5.2) we have resummed an infinite number of self-energy insertions depicted in parts B and C of figure 2. This resummation shows a growing power-law secular correction (5.7), which, if taken at face value, would imply an enormous correction to primordial tensor perturbations. Given that the correction grows in time, the effective representation of the effect in terms of shear viscosity [20] that applies at early times, is invalidated at late times, and rather eq. (5.2) should be used to capture the effect at late times. However, we should be very careful when analyzing this result, as it is clearly very soon driven outside of the range of validity of the approximations made. Furthermore, the absence of a suppression factor at one-loop order implies that some of the higher 1-particle-irreducible loop corrections might be equally unsuppressed, making the problem essentially non-perturbative. In that sense the results we report should be

taken as an indicator of potentially interesting effects, but not as a prediction. Further work is necessary in order to understand how to quantify the effect reliably, and there are several aspects to explore.

One should be aware of the limitations of our description for the plasma: we are treating it as thermal state of non-interacting photons, that is thermalized on a homogeneous and isotropic cosmological background at time of reheating. In reality, the primordial plasma is in addition made up of a number charged relativistic baryonic species that interact with the photons. This interaction is able to maintain thermal equilibrium efficiently, at least over causal length scales. This begs the question whether the interacting plasma in the presence of an external tensor fluctuation would re-equilibrate more quickly than in our model where plasma interactions are neglected, leading to dampening of the one-loop amplification we have found. One possible way of gaining insight into the effects of plasma interactions, while maintaining a degree of technical simplicity, would be to consider a conformally coupled but interacting model for the plasma. However, addressing the question of equilibrating upon being perturbed by superhorizon tensor modes would still require methods beyond equilibrium thermal field theory as we are dealing with an inherently time-dependent system.

Of course, another relevant question is what happens to scalar perturbations during the radiation period. Depending on this, interpretation of the upper bound for tensor-to-scalar ratio might require revisiting, and the way we connect the measured quantity to the inflationary ratio might require changing. This would also call for a reassessment of the exclusions bounds of inflationary models. However, at this stage, the only thing we can safely claim is that the issue of large secular enhancements of tensor modes found originally in [4], and also obtained here should be taken as pointing to an important open problem that needs resolving. In any case, whatever is necessary to obtain a reliable result is likely to update our understanding of the propagation of superhorizon modes through the primordial Universe.

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A Loop integrals

The two loop integrals in the vanishing external momentum limit needed in section 4 are given by

$$I_1(T, \Delta\eta) = \int \frac{d^3q}{(2\pi)^3} \left[\tilde{F}(\Delta\eta|q) \right]^2 \left[\mathbb{P}^{ijkl}(\mathbf{k}) q_i q_j q_k q_l \right]_{\mathbf{k} \rightarrow 0}, \quad (\text{A.1})$$

$$I_2(T, \Delta\eta) = \int \frac{d^3q}{(2\pi)^3} \tilde{G}_R(\Delta\eta|q) \tilde{F}(\Delta\eta|q) \left[\mathbb{P}_{ijkl}(\mathbf{k}) q^i q^j q^k q^l \right]_{\mathbf{k} \rightarrow 0}, \quad (\text{A.2})$$

where brevity in this appendix we omit the subscript on the reheating temperature. The vanishing momentum limit of the contracted tensor structures is best expressed in spherical integration coordinates in which $\mathbf{k} \cdot \mathbf{q} = kq \cos(\vartheta)$,

$$\mathbb{P}^{ijkl}(\mathbf{k}) q_i q_j q_k q_l = \frac{q^4}{2} [1 - \cos^2(\vartheta)]^2. \quad (\text{A.3})$$

Integrals over both angular coordinates can then be performed,

$$I_1(T, \Delta\eta) = \frac{2}{15\pi^2} \int_0^\infty dq q^6 [\tilde{F}(\Delta\eta|q)]^2, \quad (\text{A.4})$$

$$I_2(T, \Delta\eta) = \frac{2}{15\pi^2} \int_0^\infty dq q^6 \tilde{G}_R(\Delta\eta|q) \tilde{F}(\Delta\eta|q). \quad (\text{A.5})$$

In the limit where temperature is much larger than the Hubble rate these integrals are finite,

$$I_1(T, \Delta\eta) \xrightarrow{T \gg H} \frac{2}{15\pi^2} \int_0^\infty dq \frac{q^4 \cos^2(q\Delta\eta)}{(e^{q/T} - 1)(1 - e^{-q/T})}, \quad (\text{A.6})$$

$$I_2(T, \Delta\eta) \xrightarrow{T \gg H} -\frac{\theta(\Delta\eta)}{15\pi^2} \int_0^\infty dq \frac{q^4 \sin(2q\Delta\eta)}{e^{q/T} - 1}. \quad (\text{A.7})$$

They can both be written in terms of integral 3.911.2 from the table of definite integrals from [41],

$$\mathcal{I}(T, \Delta\eta) = \int_0^\infty dq \frac{\sin(2q\Delta\eta)}{e^{q/T} - 1} = \frac{\pi T}{2} \left[\text{cth}(2\pi T \Delta\eta) - \frac{1}{2\pi T \Delta\eta} \right]. \quad (\text{A.8})$$

For the second integral this is accomplished by simply extracting derivatives,

$$I_2(T, \Delta\eta) = -\frac{\theta(\Delta\eta)}{240\pi^2} \left(\frac{\partial}{\partial \Delta\eta} \right)^4 \mathcal{I}(T, \Delta\eta). \quad (\text{A.9})$$

For the first integral this is only a little more involved. It requires recognizing that,

$$\frac{1}{(e^{q/T} - 1)(1 - e^{-q/T})} = -T \frac{\partial}{\partial q} \frac{1}{(e^{q/T} - 1)}, \quad (\text{A.10})$$

then partially integrating the q -derivative,

$$I_1(T, \Delta\eta) = \frac{2T}{15\pi^2} \int_0^\infty dq \frac{q^3 [4 \cos^2(q\Delta\eta) - 2q\Delta\eta \sin(q\Delta\eta) \cos(q\Delta\eta)]}{e^{q/T} - 1}. \quad (\text{A.11})$$

Then after using simple trigonometric identities $2 \sin(\alpha) \cos(\alpha) = \sin(2\alpha)$ and $2 \cos^2(\alpha) = 1 + \cos(2\alpha)$ it becomes clear how to extract derivatives to arrive at the desired form,

$$I_1(T, \Delta\eta) = \frac{2T}{15\pi^2} \left[2T^4 \Gamma(4) \zeta(4) - \frac{1}{4} \left(1 + \frac{\Delta\eta}{4} \frac{\partial}{\partial \Delta\eta} \right) \left(\frac{\partial}{\partial \Delta\eta} \right)^3 \mathcal{I}(T, \Delta\eta) \right], \quad (\text{A.12})$$

where we recognized the definition of the zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dz \frac{z^{s-1}}{e^z - 1}. \quad (\text{A.13})$$

Finally, we have closed-form expressions for both integrals,

$$I_1(T, \Delta\eta) = \frac{2\pi^2 T^5}{15} \left[\frac{2}{15} + \frac{3 \operatorname{sh}(2\pi T \Delta\eta) + \operatorname{sh}(6\pi T \Delta\eta)}{\operatorname{sh}^5(2\pi T \Delta\eta)} - \frac{\pi T \Delta\eta [11 \operatorname{ch}(2\pi T \Delta\eta) + \operatorname{ch}(6\pi T \Delta\eta)]}{\operatorname{sh}^5(2\pi T \Delta\eta)} \right], \quad (\text{A.14})$$

$$I_2(T, \Delta\eta) = \theta(\Delta\eta) \frac{\pi^3 T^5}{15} \left[\frac{3}{8(\pi T \Delta\eta)^5} - \frac{11 \operatorname{ch}(2\pi T \Delta\eta) + \operatorname{ch}(6\pi T \Delta\eta)}{\operatorname{sh}^5(2\pi T \Delta\eta)} \right]. \quad (\text{A.15})$$

Data Availability Statement. This article has no associated data or the data will not be deposited.

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