Injectivity of non-singular planar maps with one convex component

Marco Sabatini *

Abstract

We prove that if a non-singular planar map $\Lambda \in \mathbb{C}^2(\mathbb{R}^2, \mathbb{R}^2)$ has a convex component, then Λ is injective. We do not assume strict convexity.

Keywords: Local invertibility, global injectivity, non-strict convexity, Jacobian Conjecture.

1 Introduction

Let Ω be an open connected subset of $\mathbb{R}^n$. We say that $\Lambda : \Omega \to \mathbb{R}^n$ is *locally injective (invertible)* at $X \in \Omega$ if there exists a neighbourhoods $U_X \subset \Omega$ of X and $V_{\Lambda(X)}$ of $\Lambda(X)$ such that the restriction $\Lambda : U_X \to V_{\Lambda(X)}$ is injective (invertible). If $\Lambda \in C^1(\Omega, \mathbb{R}^n)$, we denote by $J(X)$ the Jacobian matrix of Λ at X. By the inverse function theorem, if $J(X)$ is non-singular then Λ is locally injective at X. It is well-known that locally injective maps need not be globally injective, even if $J(X)$ is non-singular for all $X \in \Omega$, as in the case of the exponential map $\Lambda(x, y) = (e^x \cos y, e^x \sin y)$. Injectivity (invertibility) of locally injective (invertible) maps under suitable additional assumptions has been studied for a long time. In [14] it was conjectured that every polynomial map $\Lambda : \mathbb{C}^n \to \mathbb{C}^n$ with constant non-zero Jacobian determinant is globally invertible, with polynomial inverse. Such a problem, known as *Jacobian Conjecture*, was widely studied and inserted in a list of relevant problems in [20]. The Jacobian Conjecture was studied in several settings, even replacing $\mathbb{C}$ with other fields, but still remains unsolved for $n \geq 2$, [1, 4, 8, 24]. In [17] it was proved that asking for the determinant of $J(X)$ not to vanish is not sufficient to guarantee Λ injectivity. After Pinchuk's counterexample several papers appeared, dealing with injectivity of non-singular polynomial maps under suitable additional assumptions [5, 6, 7].

Injectivity appears also in connection to a global stability problem formulated in [15]. In this paper it was conjectured that if at any point $J(X)$ has

*Dipartimento di Matematica, Univ. di Trento, I-38123 Povo (TN) - Italy; email: marco.sabatini@unitn.it. This paper has been partially supported by GNAMPA. Preprint published at https://arxiv.org/abs/2107.12043.

eigenvalues with negative real parts then a critical point O of the differential system

$$\dot{X} = \Lambda(X) \tag{1}$$

is globally asymptotically stable. Global asymptotic stability of (1) implies Λ injectivity. In [16] it was proved that if $n = 2$, then the vice-versa is true, i. e. injectivity implies global asymptotical stability. Using such a result the conjecture was proved to be true for $n = 2$ [9, 10, 11]. On the other hand the conjecture does not hold in higher dimension, even for polynomial vector fields [2, 3].

Other additional conditions to get injectivity are growth conditions. A classical result in this field is Hadamard theorem [13], which states that if Λ is proper, i. e. if $\Lambda^{-1}(K)$ is compact for every compact set $K \subset \mathbb{R}^n$, then Λ is a bijection. Properness is ensured if Λ is norm-coercive, that is if

$$\lim_{|X|\to+\infty} |\Lambda(X)| = +\infty. \tag{2}$$

Coerciveness requires all the component of Λ to grow enough for (2) to hold. On the other hand coerciveness is not necessary in order to have injectivity, as the real map $x \mapsto \arctan x$ shows. In [18], studying planar maps $\Lambda(z) = (P(z), Q(z))$, injectivity was proved under a growth condition on just one component of Λ. In fact, if

$$\int_0^{+\infty} \inf_{|z|=r} |\nabla P(z)| dr = +\infty, \tag{3}$$

then Λ is injective. As a consequence, if there exists $k > 0$ such that $|\nabla P(z)| \geq k$, then Λ is injective.

Also in this paper, studying planar maps, we prove injectivity imposing a suitable condition on just one component. In fact, we prove that if one of the components $\Lambda(z) = (P(z), Q(z))$ is a non-strictly convex function, then $\Lambda(z)$ is injective. One of the steps in the proof is the same as in [18], since we prove the parallelizability of the Hamiltonian system

$$\begin{cases} \dot{x} = P_y \\ \dot{y} = -P_x \end{cases} . \tag{4}$$

That is equivalent to prove the connectedness of the level sets of $P(z)$. The connectedness has been used in order to study maps injectivity or non-injectivity in [5, 7, 12, 17, 18]. We observe that the non-strict convexity of the function $P(z)$ implies the non-strict convexity of the orbits of (4), but the vice-versa is not true, as the exponential map shows. Hence injectivity cannot be proved assuming only the non-strict convexity of the orbits of (4).

For the special case of planar polynomial maps with constant, non-zero Jacobian determinant, the level set connectedness was considered in [21, 22, 23]. In [22, 23] was proposed an approach based on the commutativity of the hamiltonian flows having $P(x, y)$ and $Q(x, y)$ as hamiltonian functions, similarly to what done in [19].

2 Maps having one convex component

In order to introduce the proof of next theorem, we recall some properties of convex functions.

Proposition 1. *Let $f \in C^2(\mathbb{R},\mathbb{R})$, $H \in C^2(\mathbb{R}^2,\mathbb{R}^2)$ be (non strictly) convex funtions. Then:*

i) if f is non-constant then it is unbounded from above;

ii) if there exist $u_1 < u_2 < u_3 \in \mathbb{R}$ such that $f(u_1) = f(u_2) = f(u_3)$, then f is constant on the interval $[u_1, u_3]$;

iii) the restriction of H to every line is a convex one-variable function;

iv) sub-level sets of f and H are convex;

v) every level set of H at every point has a tangent line and lies entirely on one side of such a tangent.

vi) the intersection of a level set of H with any of its tangent lines is connected (a closed interval, in generalized sense).

In the proof of next theorem we consider the family of orbits of the differential system (4). A regular C^1 curve σ is said to be a *section* of (4) if it is transversal to (4) at every point of σ. If γ is a non-trivial orbit, then for every $z \in \gamma$ there exists a neighbourhood U_z of z and two open disjoint connected subsets $U_z^\pm \subset U_z$ lying on different sides of γ, such that $U_z = U_z^- \cup (\gamma \cap U) \cup U_z^+$. If σ is a section of γ and $\sigma \cap \gamma = \{z\}$, then there exist a neighbourhood U_z of z and two sub-curves $\sigma^\pm$, called *half-sections*, such that $\sigma^\pm = \sigma \cap U_z^\pm$.

Given a planar differential system without critical points, two orbits γ_1 and γ_2 are said to be *inseparable* if and only if there exist two half-sections σ_1 and σ_2 such that every orbit meeting σ_1 meets also σ_2 and vice-versa. It can be proved that if γ_1 and γ_2 are inseparable, then for every couple of points $z_1 \in \gamma_1$ and $z_2 \in \gamma_2$ there exist half-sections such that every orbit meeting σ_1 meets also σ_2 and vice-versa. In other words, the definition of inseparability does not depend on the choice of z_1 and z_2.

We denote by $\phi(t,z)$ the local flow of (4). Since we deal with non-singular maps, such a system has no critical points. Its orbits are positively and negatively unbounded and separate the plane into two connected components. Every orbit is contained in a level set of $P(z)$, even if in general level sets of $P(z)$ do not reduce to a single orbit. In what follows we denote by A^o the interior of a set A and by $\overline{A}$ its closure.

Theorem 1. *Let $\Lambda \in C^2(\mathbb{R}^2,\mathbb{R}^2)$ be a non-singular map. If one of its components is convex, then Λ is injective.*

Proof. Possibly exchanging the components, we may assume $P(z)$ to be convex. By lemma 2.2 and theorem 2.1 in [18], it is sufficient to prove that the level sets of $P(z)$ are connected. By absurd, let us assume that a level set of $P(z) = h$ is disconnected. As a consequence by lemma 2.2 in [18] the system (4) has a couple $\gamma_1 \neq \gamma_2$ of inseparable orbits. By continuity, $P(z)$ assumes the same value on γ_1 and γ_2, say $P(\gamma_1) = P(\gamma_2) = k$.

Let us consider two cases.

1) One among γ_1 and γ_2 is not a line. Assume γ_1 is not a line. Let Γ_1 be the closed convex set having γ_1 as boundary.

1.1) If $\gamma_2 \subset \Gamma_1$, then it is not a line, otherwise it would meet γ_1, contradicting uniqueness of solutions. Let z_1 be an arbitrary point of γ_1 and τ_{12} be the line passing through z_1 and tangent to γ_2, existing by the convexity of Γ_2. Since γ_2 is not a line one can rotate τ_{12} around z_1 until it meets γ_2 at two points $z_2^1 \neq z_2^2$. Let us call τ^* such a line. Then τ^* meets the level set $P(z) = k$ at three distinct points, z_1, z_2^1, z_2^2. By proposition 1, *ii*), $P(z)$ is constant on the smallest segment Σ containing z_1, z_2^1, z_2^2. The set $\gamma_1 \cup \Sigma \cup \gamma_2$ is connected and contained in $P(z) = k$, contradicting the fact that γ_1 and γ_2 are distinct connected components of $P(z) = k$.

1.2) Let $\gamma_2 \subset \Gamma_1^c$. If $\gamma_1 \subset \Gamma_2$, then one can reply the argument of point 1.2), exchanging the role of γ_1 and γ_2.

1.3) Assume $\gamma_1 \not\subset \Gamma_2$ and $\gamma_2 \not\subset \Gamma_1$. Let D_1 be the subset of γ_1 consisting of its linear parts, i.e. half-lines and line segments. Since γ_1 is not a line, one has $D_1 \neq \gamma_1$. Let us choose arbitrarily $z_1 \in \gamma_1 \setminus D_1$ and let τ_1 be the tangent line of γ_1 at z_1. By point *v*) of Proposition 1 γ_1 lies on one side of τ_1. One has $\gamma_1 \cap \tau_1 = \{z_1\}$. Let $\tau_1^{\pm}$ be the half-lines contained in τ_1 having z_1 as extreme point, τ_1^+ tangent to the positive semi-orbit of z_1, τ_1^- tangent to the negative semi-orbit of z_1. Let Π_1 the closed half-plane having τ_1 as boundary and containing γ_1. For all $\epsilon > 0$ one has $\phi(\pm\epsilon, z_1) \in \Pi_1^o$. Every such orbit meets τ_1 at least at two points lying on distinct half-lines. As a consequence, z_1 is an isolated point of minimum of the restriction of $P(z)$ to the line τ_1. Hence γ_2 does not meet τ_1.
By the inseparability of γ_1 and γ_2 there are half-sections σ_1 of γ_1 at z_1 and σ_2 of γ_2 at z_2 such that every orbit meeting σ_1 meets also σ_2 and vice-versa. One can take σ_1 and σ_2 small enough to have $\overline{\sigma_1}$ and $\overline{\sigma_2}$ compact, disjoint and such that $\sigma_2 \cap \Pi_1 = \emptyset$.

There exist neighbourhoods $U_\epsilon^{\pm}$ of $\gamma_1(\pm\epsilon)$ such that $U_\epsilon^{\pm} \subset \Pi^o$. By the continuous dependance on initial data there exists a neighbourhood U_1 of z_1 such that $\phi(\pm\epsilon, U_1) \subset U_\epsilon^{\pm}$. This holds in particular for the points of $\delta_1 = \sigma_1 \cap U_1$, so that $\phi(\pm\epsilon, \delta_1) \subset U_\epsilon^{\pm} \subset \Pi_1^o$. δ_1 is itself a half-section at z_1. For all $z \in \delta_1$ the orbit $\phi(t, z)$ meets both τ_1^- and τ_1^+, hence both half-lines contain points $z^{\pm}$ such that $P(z^-) = P(z^+) > P(z_1)$. Moreover, $\phi(t, z)$ does not meet τ_1 at a third point, since in that case, by point *ii*) of Proposition 1, $P(z)$ would be constant on a segment of τ_1 containing z_1, contradiction. Hence, for all $z \in \delta_1$, both semi-orbits starting at z are definitively (resp. for $t \to \pm\infty$) contained in Π_1^o.

The set $W = \phi([-\epsilon, \epsilon], \overline{\delta_1})$ is compact. It is possible to take $\overline{\delta_1}$ small enough in order to have $z_2 \notin W \cup \Pi_1$ (otherwise $z_2 = z_1$). By construction, every orbit starting at a point of $\overline{\delta_1}$ is contained in the closed set $W \cup \Pi_1$. Let us denote by δ_2 the part of σ_2 met by orbits starting at points of δ_1. Since every point of δ_2 lies on an orbit starting at δ_1, the half-section δ_2 is contained in $W \cup \Pi_1$. As a consequence, one has

$$z_2 \in \overline{\delta_2} \subset W \cup \Pi,$$

contradiction.

2) Assume both γ_1 and γ_2 to be lines. They are parallel, since otherwise they should meet at a point z_0 which should be a fixed point of (4), contradicting the nonsingularity of Λ. Let Σ_{12} be the closed strip having boundary $\gamma_1 \cup \gamma_2$. Let σ be a line orthogonal to γ_1 and γ_2, and let us set $z_1 = \gamma_1 \cap \sigma$, $z_2 = \gamma_2 \cap \sigma$, $\sigma_{12} = \Sigma_{12} \cap \sigma$. The orbits γ_1 and γ_2 are inseparable, hence there exist open sub-segments σ_1 and σ_2 of σ_{12} such that $z_1 \in \overline{\sigma_1}$, $z_2 \in \overline{\sigma_2}$, $\overline{\sigma_1} \cap \overline{\sigma_2} = \emptyset$ and every orbit meeting σ_1 meets σ_2, and vice-versa. Let Φ_{12} be the union of the orbits meeting σ_1 and σ_2. Both γ_1 and γ_2 are contained in $\partial\,\Phi_{12}$. The restriction of $P(z)$ to the compact set σ_{12} is convex and non constant (because if it was constant γ_1, γ_2 and σ_{12} would be in $P(z) = k$, contradiction). One has

$$\max\{P(z) : z \in \sigma_{12}\} = P(z_1) = P(z_2) = k.$$

Let z_m a point of σ_{12} such that

$$P(z_m) = \min\{P(z) : z \in \sigma_{12}\} < P(z_1) = P(z_2) = k.$$

The orbit starting at z_m is tangent to σ_{12} and lies entirely on one side of σ_{12}. One has $\nabla P(z_m) \perp \sigma_{12}$, with the vector $\nabla P(z_m)$ pointing towards the half-strip Σ_{12}^+ not containing $\phi(t, z_m)$. Let η be the line parallel to γ_1 and γ_2 passing through z_m. The line η meets all the orbits passing through σ_1 and σ_2, hence the restriction of $P(z)$ to η assumes every value belonging to $[P(z_m), k)$. On the other hand, by proposition 1, i), $P(z)$ is unbounded from above on η, hence there exists a point in $z \in \eta$ such that $P(z) = k$. Let z_{12} the point such that $P(z_{12}) = k$, closest to z_m. Then the orbit $\phi(t, z_{12})$ is inseparable from γ_1 and γ_2, since every orbit meeting σ_1 and σ_2 also meets η in a neighbourhood of z_{12}. In other words, a suitable sub-segment η_{12} of η is a half-section of $\phi(t, z_{12})$ such that every orbit meeting σ_1 and σ_2 meets also η_{12}, and vice-versa.

The orbit $\phi(t, z_{12})$ cannot be a line because in such a case either it would be parallel to γ_1 and γ_2, contradicting their inseparability, or transversal to them, implying the existence of two critical points, $\gamma_1 \cap \gamma_{12}$ and $\gamma_2 \cap \gamma_{12}$. Since γ_{12} is not a line point 1) applies.

♣

A simple example of non-linear non-singular map with both non-strictly convex components is

$$\Lambda(x, y) = (x + y + e^x, x + y + e^y).$$

The Hamiltonian system of a non-strictly convex two-variables function has non-strictly convex orbits. The vice-versa is not true, as the function $e^x \cos y$ shows.

In fact, the connected components of $e^x \cos y = 0$ are lines, and the connected components of $e^x \cos y = k \neq 0$ are strictly convex, since they are graphs of the one-variable functions

$$x = \ln\left(\frac{k}{\cos y}\right),$$

whose second derivative does not vanish. On the other hand the hessian matrix of $e^x \cos y$ is:

$$\begin{pmatrix} e^x \cos y & -e^x \sin y \\ -e^x \sin y & -e^x \cos y \end{pmatrix},$$

whose Jacobian determinant is $-e^{2x} < 0$. In fact, the map $\Lambda(x, y) = (e^x \cos y, e^x \sin y)$ is not injective, even if both Hamiltonian systems of its components have non-strictly convex orbits.

3 Relationship to previous results

The key point in the proof of our main result is the level sets connectedness. This property has been already used in previous papers

References

[1] H. Bass, E. H. Connell, D. Wright, *The Jacobian Conjecture, reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N.S.) **7, 2** (1982), 287–330.

[2] J. Bernat, J. Llibre, *Counterexample to Kalman and Markus-Yamabe conjectures in dimension larger than 3*, Dynam. Contin. Discrete Impuls. Systems **2, 3** (1996), 337–379.

[3] A. Cima, A. van den Essen, A. Gasull , E. Hubbers, F. Mañosas, *A polynomial counterexample to the Markus-Yamabe conjecture*, Adv. Math. **131, 2** (1997), 453–457.

[4] M. de Bondt, A. van den Essen, *Recent progress on the Jacobian conjecture* Ann. Polon. Math. **87** (2005), 1–11.

[5] F. Braun, B. Oréfice-Okamoto *On polynomial submersions of degree 4 and the real Jacobian conjecture in* $\mathbb{R}^2$, J. Math. Anal. Appl. **443, 2** (2016), 688–706.

[6] F. Braun, J. Giné. J. Llibre *A sufficient condition in order that the real Jacobian conjecture in* $\mathbb{R}^2$ *holds*, J. Differential Equations **260, 6** (2016), 5250–5258.

[7] F. Braun, J. R. dos Santos Filho *The real Jacobian conjecture on* $\mathbb{R}^2$ *is true when one of the components has degree 3*, Discrete Contin. Dyn. Syst. **26, 1** (2010), 75–87.

[8] A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, 190. Birkhäuser Verlag, Basel, 2000.

[9] R. Fessler, *A proof of the two-dimensional Markus-Yamabe stability conjecture and a generalization*, Ann. Pol. Math. **62, 1** (1995), 45-74.

[10] A. A. Glutsyuk, *Complete solution of the Jacobian problem for planar vector fields (Russian)*, Uspekhi Mat. Nauk **49, 3** (1994), 179–180, translation in Russian Math. Surveys **49, 3** (1994)185–186.

[11] C. Gutierrez, *A solution to the bidimensional global asymptotic stability conjecture*, Ann. Inst. H. Poincaré Anal. Non Lineaire **12, 6** (1995), 627–671.

[12] C. Gutierrez, X. Jarque, J. Llibre, M. A. Teixeira, *Global Injectivity of* C^1 *Maps of the Real Plane, Inseparable Leaves and the Palais-Smale Condition*, Canad. Math. Bull. **50, 3** (2007), 377–389.

[13] J. M. Hadamard *Sur le correspondances ponctuelles*, Oeuvres, Editions du Centre Nationale de la Recherche Scientifique, Paris, (1968), 383–384.

[14] O. H. Keller, *Ganze Cremona-Transformationen*, Monats. Math. Physik. **47** (1939), 299–306.

[15] L. Markus, H. Yamabe, *Global stability criteria for differential systems*, Osaka Math. Jour. **12** (1960), 305–317.

[16] C. Olech, *On the global stability of an autonomous system on the plane*, Contributions to Differential Equations **1** (1963), 389–400.

[17] S. Pinchuk, *A counterexample to the strong real Jacobian conjecture*, Math. Z. **217, 1** (1994), 1–4.

[18] M. Sabatini, *An extension to Hadamard global inverse function theorem in the plane*, Nonlinear Anal. **34, 6** (1998), 829–838.

[19] M. Sabatini, *Commutativity of flows and injectivity of nonsingular mappings*, Polynomial automorphisms and related topics (Krakòw, 1999). Ann. Polon. Math. **76, 1-2** (2001), 159–168.

[20] S. Smale, *Mathematical problems for the next century*, Math. Intelligencer **20, 2** (1998), 7–15.

[21] N. Van Chau, C. Gutierrez, *Properness and the Jacobian conjecture in* $\mathbb{R}^2$, Vietnam J. Math. **31, 4** (2003), 421–427.

[22] X. Zhang, *Jacobian conjecture in* $\mathbb{R}^2$, arXiv:2011.12701 (2020).

[23] X. Zhang, *Jacobian conjecture in* $\mathbb{R}^2$, arXiv:2011.12701v2 (2021).

[24] A. V. Yagzhev, *Keller's problem*, Siberian Math. J. **21** (1980), 747–754.