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Giacomo Ziglio

SPDEs with Random Dynamic Boundary Conditions

Supervisor

Prof. Stefano Bonaccorsi

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Contents

Preface

In this Ph.D. thesis we study some examples of stochastic partial differential equations on bounded domains which are subject to different kinds of boundary conditions, such as equations arising in heat diffusion problems in material with thick boundary (see e.g. [**36, 37, 38**]) or in mathematical biology, for the potential diffusion on neuronal cells and networks (see e.g. [**51**]).

In our approach, it is useful to rewrite the concrete problem in abstract form, where by-now classical results of existence and regularity for the solution are known, see e.g. [**22, 23**]. We define "classical" homogeneous boundary conditions which are either of Dirichlet type (zero boundary value for the unknown function) or of Neumann type (zero boundary value for the normal derivative). Boundary value problems with "classical" boundary conditions, are frequently converted into an abstract Cauchy problem; let us cite as references for instance [**31**] or [**40**]. The abstract problem can be seen as an evolution equation in a suitable Banach space, driven by a given operator, and boundary values are necessary for defining the domain of this operator.

On the other hand, in this thesis we shall be concerned mainly with non-classical boundary conditions, where, for instance, the boundary value is a non-constant function of time. Further, this function can also depend on the function itself, and can also involve time-derivatives, thus modelling a different dynamics which can act on the boundary. Similar problems are already solved in the literature with different techniques: the case of boundary white-noise perturbation were studied, among others, by [**2, 24, 73**], the case of stochastic dynamical boundary conditions is treated, e.g., in [**18**].

A different approach to evolution problem with dynamical boundary conditions were recently proposed in functional analysis, motivated by the study on matrix operator theory [**30**]. Starting from the paper [**15**], this approach has shown useful to translate non-classical boundary value problems in an abstract setting, where the semigroup techniques are available. We will now give an overview of the contents of the different chapters, additional information may be found at the beginning of every chapter.

In CHAPTER 1 we discuss in some details the main examples of stochastic boundary value problems with time-inhomogeneous or dynamical conditions, comparing some of the approaches found in the literature and we explain the motivations that drive toward a matrix theory for unbounded matrices operator. Then we give a short abstract setting and some generation results that will permit us to treat many examples those follow later. The results presented here are mainly taken from [**15**] and [**69**]. In the last part of the chapter we present some evolution problems on domains derived by physical models, which involve dynamical boundary conditions. They arise from models for impulse propagation through dendritic spine with random potential's source, diffusion problem on bounded region with mixed boundary conditions and applications to optimal control problems. The last argument is taken mainly from [**11**] and [**9**].

CHAPTER 2 collects the relevant results, connected with our work, that are present thorough the thesis. In the first part of the chapter we state some generation properties and spectral theory for matrix operators obtained in recent articles [**30, 66, 68**]. In the second part, motivated by the theory of stochastic differential equations in infinite dimensions, we introduce shortly the basic tools of stochastic integration with respect to Lévy processes developed for example by [**63**] and [**43**].

CHAPTERS 3 and 4 are applications of the techniques developed before to some problems in neurophysiology and in heat diffusions in material with memory. In the former we prove global well-posedness in the mild sense for a stochastic partial differential equation with a power-type nonlinearity and driven by an additive Lévy noise. Such a system of nonlinear diffusion equations on a finite network in the presence of noise arises in various models of neurophysiology; as an example we consider the FitzHugh-Nagumo equations (see the monograph [**51**] for more details).

CHAPTER 4 is devoted to study existence, uniqueness and asymptotic behaviour of the solution for a class of stochastic partial differential equations arising in the theory of heat conduction, in presence of a nonlinear, temperature dependent heat source located on the boundary, perturbed by a Wiener noise, to cover the case of rapidly varying, highly irregular random perturbation. An existence and uniqueness result in pathwise sense is achieved. Further, we provide the existence of a random attracting set, according to the definition arising in the theory of random attractors.

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CHAPTER 1

Introduction

When one studies a space-time partial differential equation on a bounded domain, has to specify, in addiction to the initial data, a set of additional restraints on the boundary. In case of the behaviour along the boundary curve is independent from time, usually one fixes the boundary value of the function (the well known DIRICHLET BOUNDARY CONDITIONS) or the boundary value of the normal derivative (NEUMANN B.C.) or a combination of the two (ROBIN OR MIXED B.C.). Also more unusual choices are possible, like GENERALIZED WENTZELL B.C. (involving the second space derivative). As an example, let us consider a reaction-diffusion problem in a bounded domain $\mathcal{O}$ with boundary layer $\partial\mathcal{O}$, endowed with inhomogeneous Robin b.c., given in the form:

$$\begin{cases} u_t(t,x) = \sum_{i,j=1}^n a_{i,j}(x) u_{x_i x_j}(t,x) + f(t,u(t,x)) \\ \alpha u(t,y) + \beta u_\nu(t,y) = g(y) \\ u(0,x) = u_0(x) \end{cases}$$

where $t \in [0,T]$, $x \in \mathcal{O}$, $y \in \partial\mathcal{O}$, $\alpha, \beta \in \mathbb{R}$ and $\frac{\partial}{\partial \nu}$ is the normal derivative.

In some cases, it would be natural to assign the boundary value in dependence of time, such as

$$\alpha u(t,y) + \beta u_\nu(t,y) = g(t,y).$$

Such boundary conditions clearly extend the static case mentioned above. Notice that, from an abstract point of view, the evolution of the system can be defined in term of a differential operator $A(t)$ (defined, for instance, on the Hilbert space $L^2(\mathcal{O})$), with time dependent domain:

$$\begin{aligned} A(t)u &= \sum_{i,j=1}^n a_{i,j}(x) u_{x_i x_j}(x) \\ D(A(t)) &= \{u \in H^2(\mathcal{O}) : \alpha u + \beta u_\nu = g(t)\}. \end{aligned}$$

More generally, we can consider the same reaction-diffusion problem, but further we suppose that the region $\mathcal{O}$ is taken by a heat conductor and the boundary material on $\partial\mathcal{O}$ has a certain thickness and sufficiently large thermal conductivity so that to permit heat exchange between internal and boundary environment. Then there exists a different diffusion process on the boundary, surely connected to, and maybe influenced by, the interior process. We point out as examples, free boundary problems like multiphase Stefan problem or two-phase systems like Cahn-Hilliard equation, where the border has a proper evolution. The mathematical model at this time can be assumed of the form

$$\begin{cases} u_t(t,x) = \sum_{i,j=1}^n a_{i,j}(x) u_{x_i x_j}(t,x) + f(t,u(t,x)) \\ u(t,y) = v(t,y) \\ \alpha v_t(t,y) + \beta u_\nu(t,y) = \sum_{i,j=1}^n c_{i,j}(y) v_{y_i y_j}(t,y) + \sum_{i=1}^n d_i(y) v_{y_i}(t,y) \\ u(0,x) = u_0(x) \\ v(0,y) = v_0(y) \end{cases}$$

where $t \in [0,T]$, $x \in \mathcal{O}$, $y \in \partial\mathcal{O}$.

This case of boundary conditions are called DYNAMICAL BOUNDARY CONDITIONS and are often motivated in literature by control problems. These boundary conditions are qualitative different from the static ones, because they contain the time derivative of the state function on the boundary, in order to describe the physical evolution of the system in the boundary $\partial\mathcal{O}$. Similar models arise in hydrodynamics, chemical kinetics, heat transfer theory and have been studied by many authors, see for instance [**46, 65**].

In this work we examine a stochastic version of these problems. Actually, in several applications, external forces enter the system through the boundary of the region where the system evolves; in some cases, we may assume that these forces are of random type. Working at different time scales, we can approximate this random influence either with Gaussian noise, at a fast time scale, or with a Poisson noise, working at a slower time scale. So in general we will consider stochastic partial differential equations driven by a Lévy noise.

1. The stochastic boundary value problems

In this section, we give a brief account of the literature concerning stochastic problems with boundary noise, that is motivated as a comparison with the approach that we will use here. In many interesting examples, the external forces – modelled as a white noise– are introduced as dynamical boundary conditions of the problem.

Da Prato and Zabczyk consider in [**24**] the nonlinear evolution equation, with white-noise boundary conditions, on a Hilbert space H

$$\begin{aligned} \frac{\partial X}{\partial t}(t) &= \mathcal{A}X(t) + F(X(t)), \quad t > 0 \\ \tau X(t) &= Q^{1/2}\frac{\partial W}{\partial t}, \quad t > 0 \\ X(0) &= X_0 \end{aligned} \tag{1.1}$$

with different boundary conditions.

The semigroup approach followed in [**24, 25**] allows to study problem (1.1) with both Dirichlet and Neumann boundary conditions, depending on the choice of the boundary operator τ. Here, $W(t)$ is a cylindrical Wiener process with values in a Hilbert space U and Q is a symmetric nonnegative bounded operator on U. Using the so called Dirichlet or Neumann map $\mathcal{D}_\lambda$, they study the associated problem with homogeneous boundary conditions. On account of that, they define the operator A

$$\begin{aligned} Ax &= \mathcal{A}x, \\ D(A) &= \{x : \mathcal{A}x \in H;\ \tau x = 0\} \end{aligned}$$

and assume that A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \geq 0$.

In case of Neumann b.c., an H-valued, H-continuous, adapted process X is the mild solution of the problem (1.1) if it is a solution of the following integral equation:

$$\begin{aligned} X(t) &= S(t)X_0 + (\lambda - A)\int_0^t S(t-s)\mathcal{D}_\lambda Q^{1/2}\,\mathrm{d}W(s) \\ &\quad + \int_0^t S(t-s)F(X(s))\,\mathrm{d}s \end{aligned} \tag{1.2}$$

Under dissipative conditions on the operator A and on the nonlinearity F the authors obtain a result of global existence and uniqueness of the mild solution. Moreover they give sufficient conditions for continuity of the Ornstein-Uhlenbeck process

$$Z(t) = (\lambda - A) \int_0^t S(t-s) \mathcal{D}_\lambda Q^{1/2} \, \mathrm{d}W(s)$$

in the Sobolev spaces of fractional order $H^\alpha = D((-A)^\alpha)$.

More interesting is the case of Dirichlet b.c., where the solution takes value in a larger space than H. In fact the problem (1.1) does not have a solution in the original state space H, but has a H^α continuous version for $\alpha < -\frac{1}{4}$.

A similar approach is used in Maslowski [**62**] to study stochastic nonlinear boundary value parabolic problems with boundary or pointwise noise on a bounded domain $G \subset \mathbb{R}^n$. The kind of problem which he deals is given, for example, by the second order stochastic parabolic equation

$$\frac{\partial y}{\partial t}(t,x) = -A(x,D)y(t,x) + F(y(t,x)) + \Gamma(y(t,x))\eta_1(t,x),$$

$(t,x) \in \mathbb{R}^+ \times G$, with initial and boundary conditions

$$\begin{aligned} y(0,x) &= y_0(x) \qquad x \in G, \\ By(t,x) &= h(y(t,\cdot))(x) + K(y(t,\cdot))(x)\eta_2(t,x) \quad (t,x) \in \mathbb{R}^+ \times \partial G, \end{aligned}$$

where η_1 and η_2 stand for mutually stochastically independent, space dependent Gaussian noises on G and ∂G, respectively. Following the previous approach he reduces the boundary value problem to a semilinear equation of the form

$$\mathrm{d}X_t = [AX_t + f(X_t) + Bh(X_t)] \, \mathrm{d}t + g(X_t) \, \mathrm{d}W_t + Bk(X_t) \, \mathrm{d}V_t \quad t \geq 0 \qquad (1.3)$$

where W_t and V_t are independent cylindrical Wiener process on different Hilbert spaces H and U. The author obtains an existence and uniqueness statement for the mild solution of the equation by a standard fixed point argument, as well as basic results on their asymptotic behaviour, such as exponential stability in the mean and the existence and uniqueness of an invariant measure.

Sowers in [**73**] studies a stochastic reaction-diffusion equation on some n-dimensional ($n \geq 2$) Riemannian manifold M with smooth boundary, in the case of mixed boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + (b, \nabla u) + cu + f(x,u) \qquad (1.4)\\ u(\cdot, 0) &= u_0 \\ (\nu, \nabla u) + \beta(x) u|_{\mathbf{R}_+ \times \partial M} &= \sigma(x)\zeta, \end{aligned}$$

where x takes values on M and ζ is a space-time white noise on $\mathbf{R}_+ \times \partial M$. Existence and uniqueness of the solution are obtained by the integration of the fundamental solution with respect to the boundary data $\sigma(x)\zeta$. The solution is degenerate near the boundary, and using some estimates for the fundamental solution and its derivatives, one can study the boundary behaviour of $u(t,x)$ when x is close to ∂M. The main result is that $u(t,x)$ has a continuous version in $[0,T] \times (M \setminus \partial M)$ which satisfies

$$\limsup_{x \to \partial M, t \in [0,T]} (\mathrm{dist}(x, \partial M))^\gamma |u(t,x)| = 0,$$

for each $\gamma > (n-1)/2$.

Alòs and Bonaccorsi in [**2**] treat the case of a stochastic evolution equation

$$\mathrm{d}u = (\partial_x^2 u + b(x)\partial_x u)\,\mathrm{d}t + g(u)\,\mathrm{d}W(t) \qquad (1.5)$$

on the real positive half-line with inhomogeneous time dependent Dirichlet boundary condition $u(t,0) = \partial V(t)/\partial t$ for a real standard Brownian motion $V(t)$ independent from $W(t)$. The mild solution of the equation is given in a certain weighted space $L^p_\gamma(\mathbf{R}_+)$, in terms of the fundamental solution of the corresponding linear stochastic parabolic equation on the half-line

$$\begin{aligned} \mathrm{d}p(t,x) &= \frac{\partial^2}{\partial x^2} p(t,x)\,\mathrm{d}t + b(x)\frac{\partial}{\partial x} p(t,x)\,\mathrm{d}W(t), \qquad t > 0, x \in \mathbf{R}_+, \\ p(t,0) &= 0 \qquad (1.6) \end{aligned}$$

Using techniques of the Malliavin calculus, the authors prove the existence of the solution $u \in L^p(\Omega \times [0,T]; L^p_\gamma)$ for every $\gamma \in \,]0,1[$. Moreover, if some additional estimate holds, then the function $u(t,\cdot)$ is continuous on $]0,\infty[$ and $x^{1+\alpha}u(t,x) \to 0$ as $x \searrow 0$ almost surely for any $\alpha > 0$. In [**3**] the asymptotic behaviour of the solution is studied, and it is proved that the equation has an unique invariant measure that is exponentially mean-square stable.

Yang and Duan in [**28**] study the impact of stochastic dynamic boundary conditions on the long term dynamics of a stochastic viscous Cahn-Hilliard equation of the form:

$$\begin{aligned} (\varepsilon + A^{-1})\,\mathrm{d}\phi &= (\Delta\phi - \langle \partial_\nu \phi\rangle - f(\phi))\,\mathrm{d}t + \sigma_1 A^{-1}\,\mathrm{d}W^1 \quad \text{on } G \qquad (1.7)\\ \phi(0) &= \phi_0 \\ \mathrm{d}\psi &= (\Delta_{\|}\psi - \lambda\psi - \partial_n u\phi)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W^2 \quad \text{on } \Gamma \\ \psi(0) &= \psi_0 \\ \phi|_\Gamma &= \psi \end{aligned}$$

where $G := \prod_{i=1}^n (0, L_i),\ L_i > 0,\ n \in \{1,2,3\}$ with boundary Γ and A coincides with the operator $-\Delta$ with homogeneous Neumann boundary conditions and zero average on the domain. This problem is a system of Itô parabolic stochastic partial differential equations. Following a classical method in stochastic analysis, they compute the unique stationary solution for the linear stochastic equations with additive noise

$$\begin{aligned} (\varepsilon + A^{-1})\,\mathrm{d}z^1 &= Az^1\,\mathrm{d}t + \sigma_1 A^{-1}\,\mathrm{d}W^1 \\ \mathrm{d}z^2 &= (\Delta_{\|} z^2 - \lambda z^2)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W^2 \end{aligned}$$

As known, the stochastic convolutions

$$z^1(t) = \sigma_1 \int_{-\infty}^t e^{-A_\varepsilon(t-s)}(I + \varepsilon A)^{-1}\,\mathrm{d}W^1(s)$$

and

$$z^2(t) = \sigma_2 \int_{-\infty}^t e^{A_\lambda(t-s)}\,\mathrm{d}W^2(s)$$

are the unique stationary mild solutions for these problems with $A_\varepsilon = A(\varepsilon + A^{-1})^{-1}$ and $A_\lambda = (\Delta_{\|} - \lambda)$. Setting $(u, v) = (\phi - z^1, \psi - z^2)$ they obtain a family of deterministic systems which depends on parameter $\omega \in \Omega$ and solve it pathwise. By a result contained in [**65**] for deterministic Cahn-Hilliard equation with dynamic boundary conditions, there exists a continuous operator

$S_\varepsilon(t,s;\omega)$ on a suitable Banach space that describes the solution for every initial value. The corresponding stochastic flow can be defined by

$$\varphi(t,\omega)(\phi(0),\psi(0)) = S_\varepsilon(t,0;\omega)(\phi(0),\psi(0))$$

where $\mathbb{P}$-a.s.

$$S_\varepsilon(t,s;\omega) = S_\varepsilon(t-s,0;\theta_s\omega)$$

and

$$\theta_s(\omega)(t) = \omega(t+s) - \omega(s).$$

In order to examine the impact of stochastic boundary conditions on the system, the authors study how the random attractor varies with the dynamic intensity parameter ε_0 in the equation:

$$\frac{1}{\varepsilon_0}\,\mathrm{d}\psi = (\Delta_{\|}\psi - \lambda\psi - \partial_n u\phi)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W^2 \quad \text{on } \Gamma$$

They show that the dimension's estimation of the random attractor increases as the coefficient $\frac{1}{\varepsilon_0}$ for the dynamic term in the stochastic dynamic boundary condition decreases. However, in the limiting case $\varepsilon_0 = \infty$, that is the stochastic dynamic boundary conditions reduce to the stochastic static boundary conditions

$$0 = (\Delta_{\|}\psi - \lambda\psi - \partial_n u\phi)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W^2 \quad \text{on } \Gamma \tag{1.8}$$

the dimension does not tend to infinity. Instead, (1.8) does not have impact on the dimension.

Chueschov and Schmalfuss in [**18**] consider a system of quasi-linear Itô parabolic SPDEs, whose coefficients for the spatial differential operators depending on space and time, of the form

$$\begin{aligned}
\mathrm{d}u &= [-A(t)u + f(t,u,\nabla u)]\,\mathrm{d}t + g(t,u)\,\mathrm{d}W^0 \quad \text{on } \mathcal{O}\times\mathbb{R}_+ \\
\varepsilon^2\,\mathrm{d}u &= [-B(t)u + h(t,u)]\,\mathrm{d}t + \varepsilon\sigma(t,u)\,\mathrm{d}W^1 \quad \text{on } \Gamma_1\times\mathbb{R}_+ \\
B(t)u &= 0 \quad \text{on } \Gamma_2\times\mathbb{R}_+ \\
u|_{t=0} &= u_0
\end{aligned}$$

where $\mathcal{O}\subset\mathbb{R}^n$ is a bounded domain with boundary $\Gamma_0 = \bar{\Gamma}_1\cup\bar{\Gamma}_2$ and Γ_1, Γ_2 are open subset of Γ_0 such that $\Gamma_1\cap\Gamma_2 = \emptyset$. They study existence and uniqueness for these partial differential equations under the influence of a white noise as the temporal derivative of an infinite-dimensional Wiener process. The presence of the parameter $\varepsilon \in (0,1]$ emphasizes that this problem is treated as a perturbation of a parabolic SPDE with classical boundary conditions. At first they consider the linear deterministic partial differential equation with dynamical boundary conditions obtained by removing nonlinearities and noise

$$\begin{aligned} \partial_t u + \mathcal{A}(t)u &= 0 \quad \text{on } \mathcal{O} \times (0,T] \\ \varepsilon^2 \partial_t(\gamma_1 u) + \mathcal{B}_1(t)u &= 0 \quad \text{on } \Gamma_1 \times (0,T] \\ \mathcal{B}_2(t)u &= 0 \quad \text{on } \Gamma_2 \times (0,T] \end{aligned}$$

The authors prove, under some assumptions, that this problem generates a fundamental solution or equivalently a family of evolution operators $U(t,s)$. Then they investigate mild solution of very general stochastic evolution equations with time dependent coefficients, i.e. solutions of the following stochastic integral equation

$$\begin{aligned} u(t) = U(t,0)u_0 \quad + & \int_0^t U(t,s)F(s,\omega,u(s))\,\mathrm{d}s \\ & + \int_0^t U(t,s)G(s,\omega,u(s))\,\mathrm{d}W(s,\omega) \end{aligned}$$

assuming existence of a fundamental solution $U(t,s)$. At end they use this general result to show that the nonlinear stochastic dynamical boundary problem has a unique mild solution if the nonlinear drift and diffusion coefficients are Lipschitz continuous with respect to the unknown function on both the domain and the boundary.

Remark 1.1. *In the next section we explain the technical motivations that lead toward a semigroup approach. In particular we introduce the concept of matrix operator in order to convert an evolution problem to an abstract Cauchy problem on a product space.*

2. Operator matrices and generation of semigroups

The abstract semigroup theory, developed during last century, provides a natural setting, where it is possible, to study the existence and uniqueness of linear dynamical systems, starting with the simple case of systems that evolve in time with a constant law: "semigroups everywhere" as stated in [**31**]. Writing the problem in an abstract setting it is possible to give general conditions of existence and uniqueness of the solution in suitable classes of functions. Recently this approach has been extended to cover dynamical systems described by a set of unknows, see the paper [**68**] by R. Nagel. Linear evolution equations can be in fact naturally embedded in an abstract framework introducing a suitable Banach space $\mathcal{X}$ and a (possibly unbounded) infinitesimal operator $\mathcal{A}$ on $\mathcal{X}$ that describes the rate of change of the abstract dynamical system. In this functional setting, the evolution of the system can be written formally as an Abstract Cauchy Problem in $\mathcal{X}$:

$$\begin{cases} \dot{X}(t) &= \mathcal{A}X(t) \\ X(0) &= X_0. \end{cases} \tag{1.9}$$

The generation properties of the operator $\mathcal{A}$ allow to solve the question of well posedness of the problem (existence, uniqueness and continuous dependence on the data for solution) and to characterize the regularity of the solution.

2.1. Derivation of an abstract setting. Let $u(t,x)$ be a physical quantity that verifies a differential problem in a bounded domain $\mathcal{O}$. To describe the phenomenon we have to specify its behaviour on the boundary. We shall see in Chapter 4 that, when we incorporate the boundary conditions into the formulation of the problem, different physical assumptions lead to all of standard boundary conditions as well as general Wentzell and dynamical boundary conditions. Classical applications of semigroup requires homogeneous (possibly zero) boundary conditions, normally given by fixing the value of the function (Dirichlet b.c.) or of the normal derivative (Neumann b.c.). The evolution operator $\mathcal{A}$ then incorporates these conditions into the definition of its domain. A direct extension of this approach is possible even when Wentzell or dynamic boundary conditions are given; however in this case we obtain a time dependent family of operators $\mathcal{A}(t)$ which generate a two parameter family of evolution operators. An application of this framework is given in the paper [**18, 28**]. On the other hand, in the last ten years, several papers proposed to set the problem in the space $L^p(\bar{\mathcal{O}}, \mathrm{d}\mu)$ with $\mathrm{d}\mu = \mathrm{d}x|_{\mathcal{O}} \otimes \mathrm{d}\sigma|_{\partial\mathcal{O}}$, where $\mathrm{d}x$ denotes the Lebesgue measure on the domain $\mathcal{O}$ and $\mathrm{d}\sigma$ the natural surface measure on the boundary $\partial\mathcal{O}$. This space is isometric to the product space $L^p(\mathcal{O}, \mathrm{d}x) \times L^p(\partial\mathcal{O}, \mathrm{d}\sigma)$ and the problem is written as a system of two equations, which define, respectively, the internal and the boundary dynamics. On this space we define a matrix-valued operator $\mathcal{A}$, whose components describe the evolution of the system, while its domain should contain the relationship between the two dynamics and it is of non-diagonal type. Differential equations with dynamical b.c. are considered, for instance, in [**15, 54**], see also the references therein; also Wentzell b.c. enter in this setting and were the object of a series of papers (see e.g. [**32**]).

In this thesis we apply the techniques of product spaces and operator matrices to solve stochastic evolution equations with randomly perturbed dynamic boundary conditions. Moreover, we combine these tools with stochastic analysis in infinite dimensions to obtain existence and uniqueness for mild solutions. By spectral methods for the matrix operators we derive some properties of stability like existence of invariant measure and random attractors. We will see in the next chapter several results on generation and spectral properties for matrix operators and a short introduction to stochastic integration in infinite dimension with respect to a Lévy process.

2.2. Setting of the problem. We give as a general reference for this introduction the following STOCHASTIC DYNAMIC BOUNDARY VALUE PROBLEM, given in the form

$$\begin{cases} \mathrm{d}u(t) = [A_m u(t) + F(u(t))]\,\mathrm{d}t + G(u(t))\,\mathrm{d}W(t), & t > 0 \\ x(t) := Lu(t), & \\ \mathrm{d}x(t) = [Bx(t) + \Phi u(t)]\,\mathrm{d}t + \Gamma(x(t))\,\mathrm{d}V(t), & t > 0 \\ u(0) = u_0, \quad x(0) = x_0. & \end{cases} \tag{1.10}$$

When W, V are infinite dimensional Brownian motion, this approach leads us to models studied in [**23, 24, 25**] for stochastic differential equations in infinite dimension, while in the case that are general Lévy noises we refer to [**5, 4**] or [**61**]. In the following, we present an appropriate setting for our problem.

Let X and ∂X be two Hilbert spaces, called the STATE SPACE and BOUNDARY SPACE, respectively, and $\mathcal{X} = X \times \partial X$ their product space. Then $u(t) \in X$ describes the state of the system at time t while $x(t) \in \partial X$ is its boundary value.

We consider the following linear operators:

- $A_m : D(A_m) \subset X \to X$, called MAXIMAL OPERATOR, describes the internal dynamics on X,

- $B : D(B) \subset \partial X \to \partial X$ describes a different dynamics on ∂X;
- $L : D(A_m) \to \partial X$, called BOUNDARY OPERATOR is the map which associates the internal and the boundary dynamics;
- $\Phi : D(\Phi) \subset X \to \partial X$, called FEEDBACK OPERATOR defines the effects of the internal dynamics on the boundary; we assume that $D(A_m) \subset D(\Phi)$.

Finally, we define on the product space $\mathcal{X}$ the matrix operator

$$\mathcal{A} = \begin{pmatrix} A_m & 0 \\ \Phi & B \end{pmatrix}, \qquad D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A_m) \times D(B) \big| Lu = x \right\}.$$

In the next section, following [**15**], we introduce the main assumptions and results concerning the generation property of the matrix operator $\mathcal{A}$, and discuss sufficient conditions in order to solve (1.10); then in section 3 we discuss some examples that fit into our framework.

2.3. Notation and main results. Given Hilbert spaces U, V, we shall denote $\mathcal{L}(U;V)$ (resp. $\mathcal{L}(U)$) the space of linear bounded operators from U into V (resp. into U itself) and $\mathcal{L}_2(U;V)$ the space of Hilbert-Schmidt operators from U into V. In order to consider the evolution of the system with dynamic boundary conditions, we start by introducing the operator A_0 on X, defined by

$$\begin{cases} D(A_0) = \{f \in D(A_m) \mid Lf = 0\} \\ A_0 f = Af \text{ for all } f \in D(A_0), \end{cases}$$

i.e. the internal evolution operator with homogeneous boundary conditions.

We are in the position to formulate the main set of assumptions on the deterministic dynamic of the system.

ASSUMPTION 1.1.

(1) A_0 is the generator of a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on the space X;

(2) B is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on the space ∂X;

(3) $L : D(A_m) \subset X \to \partial X$ is a surjective mapping;

(4) the operator $\begin{pmatrix} A_m \\ L \end{pmatrix} : D(A_m) \subset X \to \mathcal{X} = X \times \partial X$ is closed.

2.4. No boundary feedback. In order to separate difficulties, in this section we consider the case $\Phi \equiv 0$. In order to treat (1.10) using semigroup theory, we consider the operator matrix $\mathcal{A}$ on the product space $\mathcal{X}$ given by

$$\mathcal{A} = \begin{pmatrix} A_m & 0 \\ 0 & B \end{pmatrix}, \qquad D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A_m) \times D(B) \middle| Lu = x \right\}$$

Our first step is to introduce the DIRICHLET OPERATOR D_μ. This construction is justified by [**41**, Lemma 1.2]. For given $\mu \in \rho(A_0)$, assume that the stationary boundary value problem

$$\mu w - A_m w = 0, \qquad Lw = x$$

has a unique solution $D_\mu x := w \in D(A_m)$ for arbitrary $x \in \partial X$. Then D_μ is the Green (or Dirichlet) mapping associated with A_m and L. Let us define the operator matrix

$$\mathcal{D}_\mu = \begin{pmatrix} I_X & -D_\mu \\ 0 & I_{\partial X} \end{pmatrix};$$

from [**69**, Lemma 2] we obtain the representation

$$(\mu - \mathcal{A}) = (\mu - \mathcal{A}_0)\mathcal{D}_\mu$$

where $\mathcal{A}_0$ is the diagonal operator matrix

$$\mathcal{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & B \end{pmatrix}$$

on the diagonal domain $D(\mathcal{A}_0) = D(A_0) \times D(B)$.

Using [**15**, Theorem 2.7 and Corollary 2.8], we are in the position to characterize the generation property of $(\mathcal{A}, D(\mathcal{A}))$.

LEMMA 1.2. *Assume that A_0 is invertible. Then $\mathcal{A}$ generates a C_0 semigroup on $\mathcal{X}$ if and only if the operator $Q_0(t) : D(B) \subset \partial X \to X$,*

$$Q_0(t)y := -A_0 \int_0^t T_0(t-s) D_0 S(s) y \, \mathrm{d}s, \qquad y \in \partial X \tag{1.11}$$

has an extension to a bounded operator on ∂X, satisfying

$$\limsup_{t \searrow 0} \|Q_0(t)\| < +\infty. \tag{1.12}$$

Moreover, in this case we can also give a representation of $\mathcal{T}(t)$*:*

$$\mathcal{T}(t) = \begin{pmatrix} T_0(t) & Q_0(t) \\ 0 & S(t) \end{pmatrix}. \tag{1.13}$$

Note that this result can be extended to the case $\lambda \in \rho(A_0)$, defining

$$Q_\lambda(t)y := (\lambda - A_0)\int_0^t T_0(t-s)D_\lambda S(s)y\,\mathrm{d}s, \qquad y \in \partial X$$

This lemma explains the fact that in lack of feedback term, it is possible to solve the two equations separately, considering the internal equation with homogeneous boundary conditions and providing to add the further term $-D_0S(t)y_0$.

Corollary 1.3. *Assume that* A_0 *and* B *generate analytic semigroups on* X *and* ∂X*, respectively; then* $\mathcal{A}$ *generates an analytic semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *on* $\mathcal{X}$*.*

Corollary 1.4. *If* $B \in \mathcal{L}(\partial X)$ *is bounded, then* $\mathcal{A}$ *generates a* C_0 *semigroup on* $\mathcal{X}$*; in particular, if* A_0 *is invertible and* $B = 0$*, then* $Q_0(t) = (I - T_0(t))D_0$*.*

We are now in position to write the original problem (1.10) in a equivalent problem as a stochastic abstract Cauchy problem

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = [\mathcal{A}\mathbf{x}(t) + \mathcal{F}(\mathbf{x}(t))]\,\mathrm{d}t + \mathcal{G}(\mathbf{x}(t))\,\mathrm{d}\mathcal{W}(t), \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{1.14}$$

In order to solve this stochastic problem, we introduce some assumptions on the nonlinear and stochastic terms which appear in (1.10); these assumptions, in turn, will be reflected to the operators $\mathcal{F}$ and $\mathcal{G}$ in (1.14).

Assumption 1.5.

We are given a stochastic basis $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$*;*

(1) W *and* V *are Wiener noises on* X *and* ∂X*, respectively, and* $\mathcal{W} = (W, V)$ *is a* Q*-Wiener process on* $\mathcal{X}$*, with trace class covariance operator* Q*;*

(2) the mappings $F : X \to X$ *and* $G : X \to \mathcal{L}_2(X, X)$ *are Lipschitz continuous, i.e. exists a constant* $C > 0$ *such that*

$$|F(x) - F(y)| + \|G(x) - G(y)\| \leq C|x - y|, \qquad \forall x, y \in X$$

with linear growth bound

$$|F(x)|^2 + \|G(x)\|^2 \le C(1 + |x|^2); \qquad \forall x \in X$$

(3) the mapping $\Gamma : \partial X \to \mathcal{L}_2(\partial X, \partial X)$ is Lipschitz continuous with linear growth bound:

$$\|\Gamma(x) - \Gamma(y)\| \le C|x - y|, \qquad \|\Gamma(x)\|^2 \le C(1 + |x|^2) \qquad \forall x \in \partial X.$$

Let us recall the relevant result from [**23**].

LEMMA 1.6. *Assume that $\mathcal{A}$ is the generator of a C_0 semigroup $(\mathcal{T}(t))_{t\ge0}$ on $\mathcal{X}$ and Assumption 1.5 holds. Then for every $\mathbf{x}_0 \in \mathcal{X}$, there exists a unique mild solution X to (1.14) defined by*

$$X(t) = \mathcal{T}(t)\mathbf{x}_0 + \int_0^t \mathcal{T}(t-s)\mathcal{F}(X(s))\,\mathrm{d}s + \int_0^t \mathcal{T}(t-s)\mathcal{G}(s)\,\mathrm{d}\mathcal{W}(s) \quad \mathbb{P}-a.s.$$

Moreover, it has a continuous modification.

2.5. Boundary feedback. We are now in the position to include the feedback operator Φ into our discussion. In order to simplify the exposition, and in view of the examples below, we choose to concentrate on two cases, which are far from being general.

We shall prove some generation results for the operator matrix

$$\mathcal{A} = \begin{pmatrix} A_m & 0 \\ \Phi & B \end{pmatrix}, \qquad D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A_m) \times D(B) \mid Lu = x \right\},$$

where we assume that A_0 is the generator of a C_0 analytic semigroup with $0 \in \rho(A_0)$.

As in Section 2.4 we write

$$\mathcal{A} = \mathcal{A}_0 \mathcal{D}_0^\Phi,$$

where the operator matrix $\mathcal{D}_0^\Phi$ is given by

$$\mathcal{D}_0^\Phi = \begin{pmatrix} I_X & -D_0 \\ B^{-1}\Phi & I_{\partial X} \end{pmatrix} = I_{\mathcal{X}} + \begin{pmatrix} 0 & -D_0 \\ B^{-1}\Phi & 0 \end{pmatrix}.$$

The first result can be proved as in [**15**, Section 4].

LEMMA 1.7. *Assume that the feedback operator $\Phi : D(A_m) \to \partial X$ is bounded. Then the matrix operator $\mathcal{A}$ is the generator of a C_0 semigroup.*

Next, we consider a generation result in case Φ is unbounded, which is the case in several applications (see for instance [**18**]). This case may be treated using the techniques of ONE-SIDED COUPLED OPERATORS, compare [**30**, Theorem 3.13 and Corollary 3.17].

LEMMA 1.8. *Assume that* $B \in \mathcal{L}(\partial X)$ *and* $D_0\Phi$ *is a compact operator; then the matrix operator* $\mathcal{A}$ *is the generator of an analytic semigroup.*

REMARK 1.2. *Assume that the boundary space* ∂X *is finite dimensional; this is the case, for instance, when the boundary consists of a finite number of points. Then* $B \in \mathcal{L}(\partial X)$ *is bounded and* $D_0\Phi$ *is a finite rank operator, hence it is compact, so that* $\mathcal{A}$ *is the generator of an analytic semigroup thanks to previous lemma.*

3. Motivating examples

We are concerned with the following examples from [**11**]. The first two are special cases of the paper [**18**]; notice that the first one has some applications in mathematical biology (for instance, to study impulse propagation along a neuron). The third example was considered in the paper [**24**] and (in the special case discussed here) in the paper [**26**].

3.1. Impulse propagation with boundary feedback. A widely accepted model for a dendritic spine with passive spine activity can be described by means of the following equation for the potential

$$\frac{\partial u}{\partial t}(t,\xi) = \frac{\partial^2 u}{\partial \xi^2}(t,\xi) + f(\xi, u(t,\xi)), \qquad t > 0, \quad \xi > 0;$$

the extremal point $\xi = 0$ denotes the cellular SOMA, where the potential evolves with a different dynamic; setting $x(t) = u(t,0)$, the following equation is a possible model for this dynamic

$$\mathrm{d}x(t) = [-bx(t) + cu'(t,0)]\,\mathrm{d}t + \sigma(x(t))\,\mathrm{d}W(t),$$

where $W(t)$ is a real standard Brownian motion.

In order to set the problem in an abstract setting, we consider the spaces $X = L^2(\mathbf{R}_+)$ and $\partial X = \mathbf{R}$; the matrix operator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ \\ c\frac{\partial}{\partial x}\big|_{x=0} & -b \end{pmatrix}.$$

Since the boundary space ∂X is finite dimensional and the leading operator $\frac{\partial^2}{\partial x^2}$ on $\mathbf{R}_+$ with Dirichlet boundary condition generates an analytic semigroup, then so does $\mathcal{A}$ on $\mathcal{X} = X \times \partial X$. Therefore, we write our problem in the equivalent form

$$\begin{aligned} \mathrm{d}\mathbf{x}(t) &= [\mathcal{A}\mathbf{x}(t) + \mathcal{F}(\mathbf{x}(t))]\,\mathrm{d}t + \mathcal{G}(\mathbf{x}(t))\,\mathrm{d}\mathcal{W}(t) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

and we obtain existence and uniqueness of the solution thanks to Lemma 1.6.

3.2. Dynamic on a domain with mixed boundary conditions. In previous example, the boundary space was finite dimensional. Here, we shall be concerned with a dynamical system which evolves in a bounded region $\mathcal{O} \subset \mathbf{R}^d$, with smooth boundary $\Gamma = \partial\mathcal{O}$. We assume that $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, where Γ_i are open subsets of Γ with $\Gamma_1 \cap \Gamma_2 = \emptyset$.

We are given the second order differential operator

$$\begin{aligned} A(x,\partial)u(x) &= \sum_{k,j=1}^{d} \frac{\partial}{\partial x_k}\left(a_{kj}(x)\frac{\partial}{\partial x_j}u(x)\right) \\ &\qquad + \sum_{k=1}^{d} a_k(x)\frac{\partial}{\partial x_k}u(x) + a_0(x)u(x), \end{aligned}$$

uniformly elliptic, with regular coefficients:

$$a_{kj}(x), a_k(x) \text{ and } a_0(x) \in C^2(\bar{\mathcal{O}}) \qquad j,k = 1,\dots,d.$$

We are concerned with the Sobolev spaces $H^s(\mathcal{O})$, $s > 0$ (see for instance [**59**] for the definition). The construction of the Sobolev spaces $H^s(\Gamma)$ for functions defined on the boundary $\Gamma = \partial\mathcal{O}$ is given in terms of the Laplace-Beltrami operator $B := \Delta_\Gamma$ on Γ; indeed we have

$$H^s(\Gamma) = \text{ domain of } (-\Delta_\Gamma)^s.$$

Denote $B^s(\Gamma) = H^{s-\frac{1}{2}}(\Gamma)$, for $s > \frac{1}{2}$, and similarly for $B^s(\Gamma_i)$. Then the trace mapping γ (and similarly for γ_i) is continuous from $H^s(\mathcal{O})$ into $B^s(\Gamma)$, for $s > \frac{1}{2}$.

In order to state the equation in an abstract setting, we introduce, on the Sobolev space $X = L^2(\mathcal{O})$, the operator

$$\begin{aligned} &A_m u := A(x,\partial)u(x), \qquad \text{with domain} \\ &D(A_m) = \{\varphi \in H^{1/2}(\mathcal{O}) \cap H^2_{loc}(\mathcal{O}) \mid A_m\varphi \in X\}. \end{aligned}$$

We also consider the normal boundary derivative

$$\mathcal{B}_2(x,\partial) = \sum_{k,j=1}^d a_{kj}(x)\nu_k\gamma_2\frac{\partial}{\partial x_j}, \qquad x \in \Gamma_2,$$

where $\nu = (\nu_1,\dots,\nu_d)$ is the outward normal vector field to Γ. Then we consider the following linear equation

$$\begin{cases} \mathrm{d}u(t) = A_m u(t)\,\mathrm{d}t + G(u(t))\,\mathrm{d}W(t), \\ x_1 = \gamma_1 u, \qquad \mathrm{d}x_1(t) = Bx_1(t)\,\mathrm{d}t + \Gamma(x_1(t))\,\mathrm{d}V(t), \\ x_2 = \mathcal{B}_2 u, \qquad \mathrm{d}x_2(t) = 0, \end{cases} \tag{1.15}$$

with the initial conditions

$$u(0) = u_0, \qquad x_1(0) = \gamma_1 u_0, \qquad x_2(0) = 0.$$

We shall transform our problem in an abstract Cauchy problem in a larger space. We define the Hilbert space $\mathcal{X} = L^2(\mathcal{O}) \times L^2(\Gamma_1) \times L^2(\Gamma_2)$, and we denote $\mathbf{x} \in \mathcal{X}$ the column vector with components (u, x_1, x_2). On the product space $\mathcal{X}$ we introduce the matrix operator $\mathcal{A}$, defined as

$$\mathcal{A} := \begin{pmatrix} A_m & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

on

$$D(\mathcal{A}) = \Big\{\mathbf{x} = (u, x_1, x_2) \mid u \in H^2(\mathcal{O}),\ x_1 = \gamma_1 u,\ x_1 \in D(B), \\ x_2 = \mathcal{B}_2 u,\ x_2 = 0\Big\}.$$

Then $\mathcal{A}$ satisfies the assumptions of Corollary 1.3, hence it is the generator of an analytic semigroup; we solve problem (1.15) in the equivalent form

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = \mathcal{A}\mathbf{x}(t)\,\mathrm{d}t + \mathcal{G}(\mathbf{x}(t))\,\mathrm{d}\mathcal{W}(t) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

using assumption (1.5) and Lemma 1.6.

3.3. Inhomogeneous boundary conditions. Let us consider, with the notation of previous example, the case when the boundary conditions on Γ_2 are given by $x_2(t) = f(t)$ for a function $f : \mathbf{R}_+ \to \Gamma_2$ that is continuously differentiable in time; in order to separate the difficulties we consider the following form of (1.10)

$$\begin{cases} \mathrm{d}u(t) = A_m u(t)\,\mathrm{d}t, & t > 0, \\ x(t) = Lu(t), & t \geq 0, \\ \mathrm{d}x(t) = [Bx(t) + \Pi_2 f'(t)]\,\mathrm{d}t + R\,\mathrm{d}W(t), & t > 0, \\ u(0) = u_0, \quad x(0) = x_0, \quad f(0) = f_0. \end{cases} \tag{1.16}$$

We define the abstract problem

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = [\mathcal{A}\mathbf{x}(t) + \underline{f}'(t)]\,\mathrm{d}t + \mathcal{R}\,\mathrm{d}W(t), \\ \mathbf{x}(0) = \zeta, \end{cases} \tag{1.17}$$

where $\mathcal{R} \in L(U, \mathcal{X})$ is defined by

$$\mathcal{R} \cdot h := \begin{pmatrix} 0 \\ R \cdot h \end{pmatrix} \qquad \text{for all } h \text{ in the Hilbert space } U,$$

$\underline{f}(t) = \begin{pmatrix} 0 \\ \Pi_2 f(t) \end{pmatrix}$ and $\zeta = (u_0, x_0, f_0)^*$.

The solution in MILD FORM is given by the formula

$$\mathbf{x}(t) = \mathbf{x}(t, \tau, \zeta) = \mathcal{T}(t-\tau)\zeta + \int_\tau^t \mathcal{T}(t-s) \begin{pmatrix} 0 \\ \Pi_2 f'(s) \end{pmatrix} \mathrm{d}s + \int_\tau^t \mathcal{T}(t-s)\mathcal{R}\,\mathrm{d}W(s). \tag{1.18}$$

We consider first the middle integral in (1.18); using the representation of the semigroup $\mathcal{T}(t)$ given by formula (1.13) we obtain

$$\int_\tau^t \mathcal{T}(t-s)\begin{pmatrix}0\\ \Pi_2 f'(s)\end{pmatrix}\,\mathrm{d}s = -\mathcal{T}(t-\tau)\underline{f}(\tau)+\underline{f}(t) - A_0\int_\tau^t T_0(t-\sigma)D_0\Pi_2 f(\sigma)\,\mathrm{d}\sigma.$$

We then write (1.18) in the form

$$X(t) = \mathcal{T}(t-\tau)\zeta + \begin{pmatrix}0\\ \Pi_2 f(t)\end{pmatrix} - A_0\int_\tau^t T_0(t-\sigma)D_0\Pi_2 f(\sigma)\,\mathrm{d}\sigma + \int_\tau^t \mathcal{T}(t-s)\mathcal{R}\,\mathrm{d}W(s);$$

notice that we do not need anymore the differentiability condition on f.

In the next statement, we are concerned with the properties of the stochastic convolution process

$$W_{\mathcal{A}}^{\mathcal{R}}(t) = \int_0^t \mathcal{T}(t-s)\mathcal{R}\,\mathrm{d}W(s). \tag{1.19}$$

Corollary 1.9. *Under the assumptions of Proposition 1.3, assume*

$$\int_0^t \|\mathcal{T}(s)\mathcal{R}\|_{HS}^2\,\mathrm{d}s < +\infty \qquad \forall\, t\in[0,T]. \tag{1.20}$$

Then $W_{\mathcal{A}}^{\mathcal{R}}$ is a gaussian process, centered, with covariance operator defined by

$$\mathrm{Cov}\; W_{\mathcal{A}}^{\mathcal{R}}(t) = Q_t := \int_0^t [\mathcal{T}(s)\mathcal{R}\mathcal{R}^*\mathcal{T}^*(s)]\mathrm{d}s \tag{1.21}$$

and $Q_t \in \mathcal{L}_2(H)$ for every $t\in[0,T]$.

Remark 1.3. *Condition (1.20) is verified whenever $R\in\mathcal{L}_2(U,\partial X)$ and, in particular, in case $R\in\mathcal{L}(U,\partial X)$ and U is finite dimensional.*

3.4. Stochastic boundary conditions. In several papers, the case of a white-noise perturbation on the boundary $f(t) = \dot V(t)$ is considered. Following [**24**], we shall define as mild solution of (1.18) in the interior of the domain the process $u(t)$ given by

$$u(t) = T_0(t)u_0 + Q_0(t)x_0 - A_0 \int_0^t T_0(t-\sigma) D_0(\Pi_2 1)\,\mathrm{d}V(\sigma) \tag{1.22}$$

We study this example with state space $\mathcal{O} = [0,1]$ and mixed boundary condition on $\partial\mathcal{O}$, that is, $D(A_0) = \{u \in X \ :\ u(0) = 0,\ u'(1) = 0\}$, $A_0 u(\xi) = u''(\xi)$, $U = \mathbf{R}$, $Q = I$, $R = I$ and we choose

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad S(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix},$$

which determine the boundary condition at 0, while there exists a continuous function $f(t)$ which determines the boundary condition $\frac{\mathrm{d}}{\mathrm{d}x}u(t,1) = f(t)$.

In this case we can explicitly work out the solution. At first, notice that $(D_0\Pi_2\alpha)(x) = \alpha x$; next, we construct the orthogonal basis $\{g_k \ :\ k \in \mathbf{N}\}$, setting $g_k(x) = \sin((\pi/2 + k\pi)x)$, to which it correspond the eigenvalues $\lambda_k = -(\pi/2 + k\pi)^2$. Since

$$T_0(t)(x) = \sum_{k=1}^{\infty} \langle x, g_k(x)\rangle e^{\lambda_k t} g_k(x)$$

and

$$\int_0^1 x \sin((\pi/2 + k\pi)x)\,\mathrm{d}x = \frac{(-1)^k}{|\lambda_k|},$$

we obtain

$$-A_0 \int_0^t T_0(t-\sigma) D_0(\Pi_2 1)\,\mathrm{d}V(\sigma) = -\sum_{k=1}^{\infty} (-1)^{k+1} g_k(x) \int_0^t e^{\lambda_k(t-\sigma)}\,\mathrm{d}V(\sigma). \tag{1.23}$$

In this setting estimate (1.20) is verified due to the choice of $\partial X = \mathbf{R}^2$, see Remark 1.3. Also, the new stochastic term is well defined for every $t \geq 0$, since

$$\mathbf{E}\left| A_0 \int_0^t T_0(t-\sigma) D_0(\Pi_2 1)\,\mathrm{d}V(\sigma)\right|^2 \leq C \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < +\infty, \tag{1.24}$$

with a constant C independent from t, compare (1.23).

3.5. Application to an optimal control problem. In the paper [**9**], the authors are concerned with a one dimensional semilinear diffusion equation in a bounded interval, where interactions with extremal points cannot be disregarded. The extremal points have a mass and the boundary potential evolves with a specific dynamic. The overall dynamic of the system is controlled through a control process acting on the boundary; stochasticity enters through fluctuations and random perturbations both in the interior as on the boundaries; in particular, this means that the control process is perturbed by a noisy term.

As reference interval is taken $D = [0,1]$. Then, the internal dynamic is described by the following stochastic evolution equation

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + f(t,x,u(t,x)) + g(t,x,u(t,x))\dot{W}(t,x) \tag{1.25}$$

where f and g are real valued mappings, defined on $[0,T] \times [0,1] \times \mathbb{R}$, which verify some boundedness and Lipschitz continuity assumptions and $W(t,x)$ is a real-valued space time Wiener process.

This equation must be supplied with initial and boundary conditions. As mentioned above, the latter are nonstandard in the mathematical literature, although of some interest in the applications.

The boundary dynamic is governed by a finite dimensional system which follows a (ordinary, two dimensional) stochastic differential equation

$$\partial_t v_i(t) = -b_i v_i(t) - \partial_\nu u(t,i) + h_i(t)[z_i(t) + \dot{V}_i(t)], \qquad i = 0,1 \tag{1.26}$$

where b_i are positive numbers and $h_i(t)$ are bounded, measurable functions, $V(t) = (V_1(t), V_2(t))$ is a $\mathbb{R}^2$-valued Wiener process , that is independent from $W(t,x)$.; ∂_ν is the normal derivative on the boundary, and coincides with $(-1)^i\partial_x$ for $i = 0,1$; $z(t) = (z_0(t), z_1(t))$ is the control process and takes values in a given subset of $\mathbb{R}^2$.

The initial condition is given both for the state variable and the boundary condition by

$$u(0,x) = u_0(x), \qquad 0 < x < 1; \qquad v_i(0) = v_0(i), \qquad i = 0,1. \tag{1.27}$$

The optimal control problem consists in minimizing, as the control process z varies within a set of admissible controls, a cost functional of the form

$$J(t_0, u_0, z) = \mathbb{E} \int_{t_0}^{T} \lambda(s, u^z_s, v^z_s, z_s) \, \mathrm{d}s + \mathbb{E}\phi(u^z_T, v^z_T) \tag{1.28}$$

where λ and ϕ are given real functions.

The problem is formulated in an abstract setting, where it will be possible to use results, already known in the literature, concerning the existence of an optimal control for this problem. Introducing the vector $\mathbf{u} = \begin{pmatrix} u(\cdot) \\ v(\cdot) \end{pmatrix}$ on the space $\mathcal{X} = L^2(0,1) \times \mathbb{R}^2$ the original problem (1.25)–(1.26)–(1.27) can be written in the form

$$\begin{cases} \mathrm{d}\mathbf{u}^z_t = \mathbb{A}\mathbf{u}^z_t \, \mathrm{d}t + \mathbb{F}(t, \mathbf{u}^z_t) \, \mathrm{d}t + \mathbb{G}(t, \mathbf{u}^z_t)[P z_t \, \mathrm{d}t + \mathrm{d}\mathbf{W}_t] \\ \mathbf{u}(0) = \mathbf{u}_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \end{cases} \tag{1.29}$$

Here, $P : \mathbb{R}^2 \to \mathcal{X}$ denotes the immersion of the boundary space in the product space $\mathcal{X} = L^2(0,1) \times \mathbb{R}^2$: $Pb := \begin{pmatrix} 0 \\ b \end{pmatrix}$.

The first concern is to study existence and uniqueness for the solution of (1.29). To this end, it is considered the uncontrolled equation

$$\begin{cases} \mathrm{d}\mathbf{u} = \mathbb{A}\mathbf{u}(t) + \mathbb{F}(t, \mathbf{u}(t)) \, \mathrm{d}t + \mathbb{G}(t, \mathbf{u}(t)) \, \mathrm{d}\mathbb{W}(t) \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{1.30}$$

where the operators $\mathbb{A}$, $\mathbb{F}$ and $\mathbb{G}$ are defined in terms of the coefficients of the original problem. Then they prove that the operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous, analytic semigroup of contractions $\mathrm{e}^{t\mathbb{A}}$, self-adjoint and compact. Further, they obtain that $\mathbb{A}$ is a self-adjoint operator with compact resolvent, which implies that the semigroup $\mathrm{e}^{t\mathbb{A}}$ is Hilbert-Schmidt. Moreover, they can characterize the complete orthonormal system of eigenfunctions associated to $\mathbb{A}$. Following this approach, the authors characterize existence and uniqueness of the solution to problem (1.30), by means of standard results on stochastic evolution equations in infinite dimensions.

It is necessary to mention that different examples of stochastic problems with boundary control are already present in the literature, see for instance

Maslowski [**62**] and the references therein, or the paper [**26**] for a one dimensional case where the boundary values are given by the control perturbed by an additive white noise mapping.

Similarly to this paper, it is introduced the control process $z = \{z(t),\ t \in [0,T]\}$, that is a square integrable process taking values in a bounded domain in $\mathbb{R}^2$. Using the abstract setting delineated above, although the control lives in a finite dimensional space, they obtain an abstract optimal control problem in infinite dimensions. Such type of problems has been exhaustively studied by Fuhrman and Tessitore in [**34**]. The control problem is understood in the usual weak sense (see [**33**]). Finally they prove that, under a set of hypothesis, the abstract control problem can be solved and characterize the optimal controls by a feedback law.

CHAPTER 2

Matrix operator and stochastic integration

In section 3 we have shown how is possible to convert an initial stochastic boundary value problem into an abstract Cauchy stochastic problem on some product space, provided to introduce a suitable functional setting which involves the concept of operator matrices. In the last thirty years, this approach has been applied to a wide class of systems of deterministic evolution equations as well as fluid dynamics [**74**], Volterra integro-differential equations [**16, 27, 64**], wave equations [**40**], delay equations [**49, 50**] or diffusion problems in viscoelastic materials [**38, 37, 55, 57**]. Motivated by the abstract setting introduced in section 2, we want to define an appropriate framework that permits to treat such problems also in presence of randomness. In the first part of this chapter, we present some results from [**30, 66, 68, 69**] about generation and spectral properties of matrix operators in order to characterize either the well posedness of deterministic problems given in the form

$$\begin{cases} \dot{X}(t) & = & \mathcal{A}X(t) \\ X(0) & = & X_0. \end{cases} \tag{2.1}$$

or asymptotic behaviour and regularity of the solution. In case of stochastic perturbations of the system, we obtain a linear stochastic evolution equation driven, in general, by a Lévy process, given in the form

$$\begin{cases} \mathrm{d}X(t) & = & [\mathcal{A}X(t) + \mathcal{F}(X(t))]\,\mathrm{d}t + \mathcal{G}(X(t))\,\mathrm{d}\mathcal{L}(t) \\ X(0) & = & X_0. \end{cases} \tag{2.2}$$

So, following the semigroup approach given, for example in [**23, 25**], we have to be able to define a solution of mild type for the stochastic problem (2.2) defined $\mathbb{P}$-a.s., formally, by

$$X(t) = \mathcal{T}(t)X_0 + \int_0^t \mathcal{T}(t-s)\mathcal{F}(X(s))\,\mathrm{d}s + \int_0^t \mathcal{T}(t-s)\mathcal{G}(X(s))\,\mathrm{d}\mathcal{L}(s)$$

where $(\mathcal{T}(t))_{t\geq 0}$ is the semigroup generated by $\mathcal{A}$. Then, in section 2, we give a short introduction to infinite stochastic integration for operator valued functions with respect to Banach valued Lévy process.

1. Matrix theory for operator matrices

Suppose that we are interested for an evolution differential problem on a certain domain, where the internal and boundary dynamics are connected by some evaluation conditions and possibly a feedback term. Rewriting the differential system as an unique evolution equation in some product Banach space, we obtain a matrix operator that describes the rate of change of the vector whose components are the internal and boundary state functions of the problem. Then we have to work with generally unbounded operator matrices of this form:

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.3}$$

defined on $D(\mathcal{A}) \subset \mathcal{E} = E \times F$, where E and F are Banach spaces. At first we consider the simple case in which the domain of the operator is diagonal, i.e. $D(\mathcal{A}) = D(A) \times D(D)$. The lack of an evaluation operator on the boundary and the fact that the degree of unboundedness of the matrix operator is given by the diagonal entries simplifies the characterization of spectral and generation properties. Afterward, introducing some kind of evaluation on the boundary, we need to define the so called one-sided coupled operators and then to explain analogous results for this type of operators. Note that for both cases the diagonal operators A and D give the degree of unboundedness of the operator matrices, i.e. $D(D) \subset D(B)$ and $D(A) \subset D(C)$. If this condition is not satisfied, we have to verify directly the hypothesis of some generation result for the whole operator $\mathcal{A}$ (e.g. Hille-Yosida, Feller-Miyadera-Phillips, Lumer-Phillips...).

1.1. Operator matrices with diagonal domain.

SPECTRAL PROPERTIES

At beginning of his paper [**68**], Nagel examines the simple case when the operator (2.3) has all bounded entries (and everywhere defined). Under this assumption he introduces a non-commutative characteristic function yielding an efficient way of computing the spectrum of $\mathcal{A}$.

LEMMA 2.1 ([**68**]). *Let E, F Banach spaces and $A \in \mathcal{L}(E)$, $D \in \mathcal{L}(F)$, $B \in \mathcal{L}(F, E)$, $C \in \mathcal{L}(E, F)$ and consider the matrix operator $\mathcal{A} \in \mathcal{L}(E \times F)$ (2.3). If A and D are invertible then the following assertions are equivalent:*

- **(a):** *$\mathcal{A}$ is invertible in $\mathcal{L}(E \times F)$.*
- **(b):** *$A - BD^{-1}C$, hence $Id - BD^{-1}CA^{-1}$ is invertible in $\mathcal{L}(E)$.*
- **(c):** *$D - CA^{-1}B$, hence $Id - CA^{-1}BD^{-1}$ is invertible in $\mathcal{L}(F)$.*

This Lemma indicates that, under suitable assumptions, one could take either the linear operators $A - BD^{-1}C$ or $D - CA^{-1}B$ as a substitute for the determinant of $\mathcal{A}$.

In the case of unbounded entries the author needs some assumptions on the domains of the operators involved.

ASSUMPTION 2.2 ([**68**]). *Let E, F be Banach spaces and assume that*

(1) the operator A with domain $D(A)$ has non empty resolvent set $\rho(A)$ in E,

(2) the operator D with domain $D(D)$ has non empty resolvent set $\rho(D)$ in F,

(3) the operator B with domain $D(B)$ is relatively D-bounded, i.e. $D(D) \subset D(B)$ and $BR(\lambda, D) \in \mathcal{L}(F, E)$ for $\lambda \in \rho(D)$,

(4) the operator C with domain $D(C)$ is relatively A-bounded, i.e. $D(A) \subset D(C)$ and $CR(\lambda, A) \in \mathcal{L}(E, F)$ for $\lambda \in \rho(A)$,

(5) the operator $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with domain $D(\mathcal{A}) = D(A) \times D(D)$ is closed in $E \times F$.

The conditions $(1) - (4)$ express the fact that the diagonal elements determine the "degree of unboundedness" of the operator $\mathcal{A}$. Under this assumptions it is possible to define an operator theoretical analogue to the characteristic polynomial of a scalar 2×2 -matrix.

DEFINITION 2.3 ([**68**]). *Under the Assumption 2.2 and for $\lambda \notin \sigma(A) \cup \sigma(D)$ we consider the operators*

$$\Delta_E(\lambda) := \lambda - A - BR(\lambda, D)C \tag{2.4}$$

and

$$\Delta_F(\lambda) := \lambda - D - CR(\lambda, A)B \tag{2.5}$$

with domain $D(A)$ in E, respectively domain $D(D)$ in F. The function $\lambda \mapsto \Delta_E(\lambda)$, resp. $\lambda \mapsto \Delta_F(\lambda)$ will be called the E-CHARACTERISTIC, *resp.* F-CHARACTERISTIC OPERATOR FUNCTION *of the matrix operator*

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

With these characteristic operator functions we are able to determine those spectral values of $\mathcal{A}$ which are not contained in $\sigma(A) \cup \sigma(D)$.

THEOREM 2.4 ([**68**]). *Consider an operator matrix $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on $E \times F$ for which the Assumption 2.2 holds. For $\lambda \notin \sigma(A) \cup \sigma(D)$ the following assertions are equivalent:*

(a): $\lambda \in \sigma(\mathcal{A})$,
(b): $0 \in \sigma(\Delta_E)$,
(c): $0 \in \sigma(\Delta_F)$

Moreover, for $\lambda \notin \sigma(A) \cup \sigma(D) \cup \sigma(\mathcal{A})$ the resolvent of $\mathcal{A}$ is given by

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} \Delta_E^{-1}(\lambda) & \Delta_E^{-1}(\lambda) B R(\lambda, D) \\ R(\lambda, D) C \Delta_E^{-1}(\lambda) & R(\lambda, D)(Id + C\Delta_E^{-1}(\lambda) B R(\lambda, D)) \end{pmatrix} \tag{2.6}$$

or the analogous expression using $\Delta_F(\lambda)$.

REMARK 2.1. *If E and F are of different size we clearly compute $\sigma(\mathcal{A})$ through the characteristic operator function in the smaller space. In particular if $\dim E < \infty$ then $\Delta_E(\lambda)$ becomes a matrix and $0 \in \sigma(\Delta_E(\lambda))$ becomes the characteristic equation* $\det(\Delta_E(\lambda)) = 0$.

GENERATION PROPERTIES

By standard semigroup theory it is well known that systems of linear evolution equations are well-posed if and only if the corresponding operator matrices are the generator of a strongly continuous semigroup on the product space. Let us first deal with operator matrices of the form

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

with diagonal domain $D(\mathcal{A}) = D(A) \times D(D)$. It is easy to check that A and D generate a semigroup on E and F, respectively, if and only if the operator matrix $\mathcal{A}$ generates a semigroup on the product space $E \times F$. By standard perturbation results one can extend this result to more interesting matrix operator.

PROPOSITION 2.5 ([**68**]). *Assume that A and D generate strongly continuous semigroups $(e^{tA})_{t\geq 0}$ on E and $(e^{tD})_{t\geq 0}$ on F. For a D-bounded operator $B : D(D) \to E$ the following assertions are equivalent:*

(a): *The matrix $\mathcal{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with domain $D(\mathcal{A}) = D(A) \times D(D)$ is a generator on $E \times F$.*

(b): *The operators*

$$R(t) := \int_0^t e^{(t-s)A} B e^{sD} \, \mathrm{d}s \tag{2.7}$$

are uniformly bounded as $t \downarrow 0$.

In that case, the semigroup generated by $\mathcal{A}$ is given by the matrices

$$\mathcal{T}(t) := \begin{pmatrix} e^{tA} & R(t) \\ 0 & e^{tD} \end{pmatrix}, \qquad t \geq 0 \tag{2.8}$$

COROLLARY 2.6 ([**68**]). *Assume that A and D generate semigroups on E, resp. on F. If B is bounded from $D(D)$ into $D(A)$ then $\mathcal{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is a generator on $E \times F$.*

COROLLARY 2.7 ([**68**]). *Assume that A and D generate analytic semigroups on E, resp. on F. If B is D-bounded then $\mathcal{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is the generator of an analytic semigroup.*

Using basic properties of the convolution of operator-valued mappings, Mugnolo in [**66**] obtains new stability results for semigroups generated by operator matrices with diagonal domain.

REMARK 2.2. *We define by $[D(A)]$ the Banach space obtained by endowing the domain of a closed operator A on a Banach space by its graph norm.*

THEOREM 2.8 ([**66**]). *Under the Assumptions 2.2 the following assertions hold for the operator matrix $\mathcal{A}$ defined in (2.3) with diagonal domain $D(\mathcal{A}) = D(A) \times D(D)$ on the product space $\mathcal{E} = E \times F$.*

(1) Let

- $B \in \mathcal{L}([D(D)], [D(A)])$, *or else* $B \in \mathcal{L}(F, E)$, *and moreover*
- $C \in \mathcal{L}([D(A)], [D(D)])$, *or else* $C \in \mathcal{L}(E, F)$.

Then A and D both generate C_0-semigroups $(e^{tA})_{t\geq 0}$ on E and $(e^{tD})_{t\geq 0}$ on F , respectively, if and only if $\mathcal{A}$ generates a C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ on $\mathcal{E}$.

(2) Let

- $C \in \mathcal{L}([D(A)], F)$ *and* $B \in \mathcal{L}(F, E)$, *or else*
- $B \in \mathcal{L}([D(D)], E)$ *and* $C \in \mathcal{L}(E, F)$.

Then A and D both generate analytic semigroups $(e^{tA})_{t\geq 0}$ on E and $(e^{tD})_{t\geq 0}$ on F, respectively, if and only if $\mathcal{A}$ generates an analytic semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ on $\mathcal{E}$.

(3) Let A and D both generate analytic semigroup $(e^{tA})_{t\geq 0}$ on E and $(e^{tD})_{t\geq 0}$ on F, respectively. Let both these semigroup have analyticity angle $\delta \in (0, \frac{\pi}{2}]$. If there exists $\alpha \in (0,1)$ such that

- $B \in \mathcal{L}([D(A)], [D(D), F]_\alpha)$ *and*
- $C \in \mathcal{L}([D(D)], [D(A), E]_\alpha)$,

then $\mathcal{A}$ generates an analytic semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ of angle $\delta \in (0, \frac{\pi}{2}]$ on $\mathcal{E}$. Conversely, if $\mathcal{A}$ generates an analytic semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ of angle $\delta \in (0, \frac{\pi}{2}]$ on $\mathcal{E}$ and

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \mathcal{L}([D(\mathcal{A})], [D(\mathcal{A}), \mathcal{X}]_\alpha)$$

for some $\alpha \in (0,1)$, then also A and D generate semigroups of angle δ on E and F, respectively.

If the above assertion holds with $B = 0$, then

$$R(t) := \int_0^t e^{(t-s)D} C e^{sA}\, \mathrm{d}s$$

is well-defined as a bounded operator from E to F for all $t \geq 0$ and there holds

$$e^{t\mathcal{A}} = \begin{pmatrix} e^{tA} & 0 \\ R(t) & e^{tD} \end{pmatrix}, \qquad t \geq 0.$$

Likewise, if instead $C = 0$, then the semigroup generated by $\mathcal{A}$ has the form

$$e^{t\mathcal{A}} = \begin{pmatrix} e^{tA} & S(t) \\ 0 & e^{tD} \end{pmatrix}, \qquad t \geq 0,$$

where

$$S(t) := \int_0^t e^{(t-s)A} B e^{sD} \, \mathrm{d}s \in \mathcal{L}(F, E), \qquad t \geq 0.$$

1.2. Operator matrices with non-diagonal domain. Motivated by initial-boundary value problems we present some results for operator matrices with non diagonal domain, more precisely with coupled domain. Such operators have been introduced in [**30**].

ASSUMPTION 2.9 ([**30**]). *Let E and F be Banach spaces and assume that*

(1) $A : D(A) \subseteq E \to E$ and $D : D(D) \subseteq F \to F$ are densely defined, invertible operators.

(2) X and Y are Banach spaces such that $[D(A)] \hookrightarrow X \hookrightarrow E$ and $[D(D)] \hookrightarrow Y \hookrightarrow F$.

(3) $K \in \mathcal{L}(Y, X)$, $L \in \mathcal{L}(X, Y)$ are bounded linear operators.

Note that even in this assumptions is required that the degree of unboundedness is characterized by the diagonal operators A and D.

Engel in [**30**] defines an operator matrix $\mathcal{A}$ on $\mathcal{E} := E \times F$ in the following way, where $\mathcal{X} := X \times Y$.

DEFINITION 2.10 ([**30**]). *If A, D, K, L satisfy Assumption 2.9 we consider*

$$\mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad D(\mathcal{A}_0) := D(A) \times D(D),$$

$$\mathcal{K} := \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{X})$$

and define the operator $\mathcal{A}$ on $\mathcal{E} = E \times F$ by

$$\mathcal{A} := \mathcal{A}_0(Id + \mathcal{K}), \quad D(\mathcal{A}) := \{X \in \mathcal{X} : (Id + \mathcal{K})X \in D(\mathcal{A}_0)\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times Y : \begin{array}{c} x + Ky \in D(A) \\ Lx + y \in D(D) \end{array} \right\}. \tag{2.9}$$

If the matrix $\mathcal{A}$ defined by (2.9) satisfies

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times D(D) : x + Ky \in D(A) \right\}$$

it is called ONE-SIDED K-COUPLED.

Let us consider an evolution problem on a product Banach space $\mathcal{E} = E \times F$, in the form

$$\begin{cases} \dot{X}(t) &= \mathcal{A}X(t) \\ X(0) &= X_0 \end{cases}$$

where $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ is a linear operator (possibly unbounded) with domain $D(\mathcal{A}) = \{D(A) \times D(D) : Lu = v\}$. Let us suppose, for simplicity, that $0 \in \rho(A_0)$. Then, by the Dirichlet map D_0 associated to A and L, we can decompose the operator $\mathcal{A}$ as

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & -D_0 \\ 0 & I \end{pmatrix}$$

where A_0 is the operator with homogeneous boundary conditions $Lu = 0$. So $\mathcal{A} = \mathcal{A}_0(I - D_0)$ and $D(\mathcal{A}) = \{X \times D(D) : x - D_0 y \in D(A_0)\}$. This explains why the theory of one-sided coupled operators can be used to study problems with dynamical boundary conditions.

PROPOSITION 2.11 ([**30**]). *Let $\mathcal{A}$ be defined by (2.9).*

- **(a):** *if $K : Y \subseteq F \to E$ and $L : X \subseteq E \to F$ are closed then $\mathcal{A}$ is closed in $\mathcal{E}$.*
- **(b):** *if $KY \subseteq D(A)$ or $LX \subseteq D(D)$ then $\mathcal{A}$ is densely defined.*

LEMMA 2.12 ([**30**]). *Let $\mathcal{K} := \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{X})$. Then we have*

$$\begin{aligned} Id + \mathcal{K} &= \begin{pmatrix} Id & 0 \\ L & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ 0 & Id - LK \end{pmatrix} \begin{pmatrix} Id & K \\ 0 & Id \end{pmatrix} \\ &= \begin{pmatrix} Id & K \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id - KL & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ L & Id \end{pmatrix} \end{aligned}$$

In particular, $Id + \mathcal{K} \in \mathcal{L}(\mathcal{X})$ is invertible if and only if $Id - KL \in \mathcal{L}(X)$ is invertible if and only if $Id - LK \in \mathcal{L}(Y)$ is invertible.

PROPOSITION 2.13 ([**30**]). *For a one-sided K-coupled operator matrix $\mathcal{A}$ on $\mathcal{E}$ we have*

$$\mathcal{A} = \begin{pmatrix} Id & 0 \\ DLA^{-1} & Id - DLKD^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & KD^{-1} \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ 0 & D \end{pmatrix} \tag{2.10}$$

PROPOSITION 2.14 ([**30**]). *For a one-sided K-coupled operator matrix $\mathcal{A}$ we have for all $\lambda \in \rho(A) \cap \rho(D)$*

$$\begin{aligned} \lambda - \mathcal{A} &= \mathcal{B}_\lambda \mathcal{U}_\lambda \qquad (2.11) \\ \mathcal{B}_\lambda &:= \begin{pmatrix} Id & KD^{-1} \\ (\lambda - D)L_\lambda R(\lambda, A) & Id - (\lambda - D)L_\lambda K_\lambda R(\lambda, D) \end{pmatrix} \\ \mathcal{U}_\lambda &:= \begin{pmatrix} \lambda - A & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & K_\lambda R(\lambda, D) \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ 0 & \lambda - D \end{pmatrix} \end{aligned}$$

where the operators $K_\lambda \in \mathcal{L}(Y, X)$ and $L_\lambda \in \mathcal{L}(X, Y)$ are defined by

$$\begin{aligned} K_\lambda &:= -AR(\lambda, A)K \\ L_\lambda &:= -DR(\lambda, D)L \end{aligned}$$

Next we state the main result of spectral theory for one-sided K coupled operators, using the notation

$$\Delta_F(\lambda - \mathcal{A}) := Id - (\lambda - D)L_\lambda K_\lambda R(\lambda, D) \in \mathcal{L}(F) \qquad \text{for } \lambda \in \rho(A) \cap \rho(D)$$

THEOREM 2.15 ([**30**]). *Every one-sided K-coupled operator matrix $\mathcal{A}$ is densely defined. Moreover, the following assertions hold true.*

(a): *For $\lambda \in \rho(A) \cap \rho(D)$ we have*

$$\begin{aligned} \lambda \in \sigma(\mathcal{A}) &\Leftrightarrow 0 \in \sigma(\Delta_F(\lambda - \mathcal{A})), \\ \lambda \in \sigma_p(\mathcal{A}) &\Leftrightarrow 0 \in \sigma_p(\Delta_F(\lambda - \mathcal{A})), \\ \lambda \in \sigma_{ess}(\mathcal{A}) &\Leftrightarrow 0 \in \sigma_{ess}(\Delta_F(\lambda - \mathcal{A})). \end{aligned}$$

(b): *For $\lambda \in \rho(\mathcal{A})$ the resolvent $R(\lambda, \mathcal{A})$ of $\mathcal{A}$ is given by*

$$\begin{pmatrix} (Id - K_\lambda R(\lambda, D)\Delta_F(\lambda - \mathcal{A})^{-1}DL)R(\lambda, A) & -K_\lambda R(\lambda, D)\Delta_F(\lambda - \mathcal{A})^{-1} \\ R(\lambda, D)\Delta_F(\lambda - \mathcal{A})^{-1}DLR(\lambda, A) & R(\lambda, D)\Delta_F(\lambda - \mathcal{A})^{-1} \end{pmatrix}.$$

(c): *The adjoint of $\mathcal{A}$ is given by*

$$\mathcal{A}' = \begin{pmatrix} Id & 0 \\ 0 & D' \end{pmatrix} \begin{pmatrix} Id & 0 \\ (KD^{-1})' & Id \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & (DLA^{-1})' \\ 0 & Id - (DLKD^{-1})' \end{pmatrix}.$$

Let us consider triangular operator matrices, i.e., matrices $\mathcal{A}$ defined by (2.9) for $L = 0$. In view of previous Proposition the operator $\mathcal{A}$ is given by

$$\begin{aligned} \mathcal{A} &:= \begin{pmatrix} A & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & Q \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ 0 & D \end{pmatrix} \\ D(\mathcal{A}) &:= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in E \times D(D) : x + QDy \in D(A) \right\}, \end{aligned} \tag{2.12}$$

i.e.,

$$\mathcal{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(x + QDy) \\ Dy \end{pmatrix},$$

where for simplicity we set $Q := KD^{-1} \in \mathcal{L}(F, E)$.

Analogously to the diagonal domain case we study the problem when an operator matrix $\mathcal{A}$ as defined in (2.9) is a generator.

Theorem 2.16 ([**30**]). *Let $\mathcal{A}$ be defined by (2.12). If there exists $w \in \mathbb{R}$ such that $(w, \infty) \subset \rho(A) \cap \rho(D)$ then the following assertions are equivalent.*

(a): *$\mathcal{A}$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t \geq 0}$ on $\mathcal{E}$.*

(b): **(i):** *A and D are generators of strongly continuous semigroups $(T(t))_{t\geq 0}$ on E and $(S(t))_{t \geq 0}$ on F, respectively.*

(ii): *For all $t \geq 0$ the operators $\tilde{R}(t) : D(D^2) \subseteq F \to E$ are bounded and satisfy $\limsup_{t \downarrow 0} |\tilde{R}(t)| < \infty$.*

In case these conditions hold true, the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is given by

$$\mathcal{T}(t) = \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix}$$

where $R(t) \in \mathcal{L}(E, F)$ is the unique bounded extension of $\tilde{R}(t)$.

Corollary 2.17 ([**30**]). *Let $\mathcal{A}$ satisfy the assumptions of the previous theorem. If one of the following conditions $(a) - (c)$ is satisfied then $\mathcal{A}$ is a generator on $\mathcal{E}$ if and only if A and D are generators on E and F, respectively.*

(a): $A^2 Q \in \mathcal{L}(F, E)$.

(b): *$QD : D(D) \to E$ has a bounded extension $\bar{Q}D \in \mathcal{L}(F, E)$ and $A\bar{Q}D \in \mathcal{L}(F, E)$.*

(c): *$QD^2 : D(D^2) \to E$ has a bounded extension in $\mathcal{L}(F, E)$.*

Corollary 2.18 ([**30**]). *Let $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} Id & K \\ L & Id \end{pmatrix}$ be an one-sided K-coupled operator matrix with $D \in \mathcal{L}(F)$. Then the following assertions are equivalent*

(a): *$\mathcal{A}$ is a generator on $\mathcal{E}$.*

(b): *$(A + KDL, D(A))$ is a generator on E.*

Corollary 2.19 ([**30**]). *Let $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} Id & K \\ L & Id \end{pmatrix}$ be an one-sided K-coupled operator matrix with $D \in \mathcal{L}(F)$. If A generates an analytic semigroup and $KDL \in \mathcal{L}([D(A), E])$ is compact, then $\mathcal{A}$ generates an analytic semigroup.*

Remark 2.3 ([**30**]). *In case $dim(F) < \infty$ every operator matrix $\mathcal{A}$ on $\mathcal{E}$ given by definition 2.9 is one-sided K-coupled and every linear operator D on F is bounded. Moreover, KDL is an operator of finite rank and therefore compact. Hence, $\mathcal{A}$ is the generator of an analytic semigroup provided A generates an analytic semigroup on E.*

1.3. Other cases. In this section we want to exhibit an example where the previous results on generation of semigroup by matrix operator can not be applied.

As in [**38**] let us consider a bounded domain $\mathcal{O} \subset \mathbb{R}^N$ taken by an isotropic, rigid and homogeneous heat conductor with linear memory (viscoelastic). On this we want to examine a heat flux equation in presence of a source term given in the form

$$\begin{cases} u_t(t,x) &= \int_0^\infty \mu(s)\Delta\eta(s,t,x)ds + f(t,x) \quad \text{on } \mathbb{R}^+ \times \mathcal{O} \\ \partial_t\eta(s,t,x) &= u(t,x) - \partial_s\eta(s,t,x) \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{O} \\ \eta(s,t,x) &= 0 \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^+ \times \partial\mathcal{O} \\ u(0,x) &= u_0(x) \qquad x \in \mathcal{O} \\ \eta(s,0) &= \eta_0(s) \qquad x, s \in \mathcal{O} \times \mathbb{R}^+ \end{cases} \tag{2.13}$$

where u is the temperature variation function, η is the summed past history of u and μ denotes the heat flux memory kernel. So we are in the situation described before: a system of two evolution equations, that can be studied in abstract manner defining opportunely a product Banach space and some operators.

Now we introduce the product Hilbert space

$$\mathcal{H} = L^2(\mathcal{O}) \times L^2_\mu(\mathbb{R}^+; H^1_0(\mathcal{O}))$$

endowed with the inner product

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} := \langle \cdot, \cdot \rangle_{L^2} + \int_0^\infty \langle \cdot, \cdot \rangle_{H^1_0} \mu(s)\, ds.$$

Let us introduce the matrix operator $\mathcal{A}$ defined as

$$\mathcal{A} = \begin{pmatrix} 0 & \int_0^\infty \mu(s)\Delta \cdot \; ds \\ I & -\frac{\partial}{\partial s} \end{pmatrix}$$

with domain

$$\begin{aligned} D(\mathcal{A}) &= \Big\{ X \in \mathcal{H} : u \in H^1_0(\mathcal{O}), \frac{\partial \eta}{\partial s} \in L^2_\mu(\mathbb{R}^+; H^1_0(\mathcal{O})), \\ &\qquad \int_0^\infty \mu(s)\Delta\eta(s)\, ds \in L^2(\mathcal{O}), \eta(0) = 0 \Big\} \\ &= H^1_0(\mathcal{O}) \times \left\{ L^2_\mu(\mathbb{R}^+; H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})) \cap H^1_\mu(\mathbb{R}^+; H^1_0(\mathcal{O})) : \right. \\ &\qquad \left. \eta(0) = 0 \right\}. \end{aligned}$$

Setting $X(t) = (u(t), \eta(t,s))^T \in \mathcal{H}$, system (2.13) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases} \dot{X}(t) &= \mathcal{A}X(t) + \mathcal{F}(t) \\ X(0) &= X_0 \end{cases}$$

In this case the degree of unboundedness of the matrix operator $\mathcal{A}$ in not characterized by the diagonal entries, because the element B and C are not bounded, not even on $D(D)$ and $D(A)$ respectively. We can return to a similar framework before, restricting the domain of A and D as:

$$D(A) = H^1_0(\mathcal{O}), \quad D(D) = \{ H^1_\mu(\mathbb{R}^+; H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})) : \eta(0) = 0 \}$$

and then we obtain the diagonal domain

$$D(\mathcal{A}) = D(A) \times D(D).$$

But the operators A and D generate no more strongly continuous semigroups on their new domains. We can prove to modify the Hilbert spaces to verify one of the two conditions imposed in theorem (2.8), for example, for the operator C:

(1) Suppose $C \in \mathcal{L}(D(A), D(D))$. But u is constant with respect to the variable s, so $Cu = u \in D(A)$ can not verify $\eta(0) = 0$ unless $u \equiv 0$.
(2) Suppose $C \in \mathcal{L}(E, L^2_\mu(\mathbb{R}^+; F))$, for some Banach spaces E, F. Then the spaces E and F must have the same space regularity. But in this case the operator B verifies neither $B \notin \mathcal{L}(Y, X)$ because of the presence of Laplacian nor $B \notin \mathcal{L}(D(D), D(A))$ for the same reason.

If we restrict the domain of D, it could not generate a strongly continuous semigroup. So we can not apply the results contained in [**30, 66, 68**].

It is rather easy, instead, prove that the whole matrix operator $\mathcal{A}$ on $\mathcal{H}$ is the infinitesimal generator of a C_0-semigroup of contractions, verifying the maximal dissipativity of the operator (see [**38**]) and applying the Lumer-Phillips theorem.

2. Stochastic Integration

In chapters 3 and 4 we shall study existence, uniqueness and longtime behaviour of mild and weak solutions for stochastic nonlinear equations with additive Lévy noise on some Hilbert space H. In general, this class of problems can be written in the form

$$\begin{cases} \mathrm{d}X(t) &= [AX(t) + F(X(t))]\,\mathrm{d}t + B(t)\,\mathrm{d}M(t) \qquad t \in [0,T] \\ X(0) &= \xi \end{cases} \tag{2.14}$$

where ξ is an H-valued random variable $\mathscr{F}_0$-measurable, $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $(e^{tA})_{t\in\mathbb{R}^+}$, $F : D(F) \subset H \to H$ is a mapping, $B(\cdot) : [0,T] \to L(U,H)$ is a linear operators valued process and $M(\cdot)$ is a locally square integrable càdlàg martingale taking values in a Hilbert space U.

Comparing with [**23**], where the Wiener case is treated, we recall the definition of weak and mild solution of problem (2.14).

Definition 2.20. *An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* weak solution *to (2.14) if $\mathbb{P}$-a.s. for all $t \in [0,T]$ and for all*

$\zeta \in D(A^*)$

$$\begin{aligned}\langle X(t), \zeta\rangle &= \langle \xi, \zeta\rangle + \int_0^t \langle X(s), A^*\zeta\rangle \,\mathrm{d}s + \\ &\quad + \int_0^t \langle F(X(s)), \zeta\rangle \,\mathrm{d}s + \int_0^t \langle \zeta, B(s)\,\mathrm{d}M(s)\rangle. \qquad (2.15)\end{aligned}$$

DEFINITION 2.21. *An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* MILD SOLUTION *to (2.14) if $\mathbb{P}$-a.s. for all $t \in [0,T]$*

$$X(t) = \xi + \int_0^t e^{(t-s)A} F(X(s))\,\mathrm{d}s + \int_0^t e^{(t-s)A} B(s)\,\mathrm{d}M(s). \qquad (2.16)$$

These objects are well defined if and only if the integrals that appear in (2.15) and (2.16) are well defined. While those with respect to Lebesgue measure can be defined in a pathwise sense as Bochner integrals, this is not possible in the martingale case. Then we make a short digression on concept of stochastic integrals

$$\int_0^t X(s)\,\mathrm{d}M(s),$$

when X is an operator valued process from U into another Hilbert space H and M is an U-valued square integrable martingale.

We state some results taken from [**79**] on stochastic integration and from [**63**] for the corresponding Itô formula. For a complete treatment of stochastic integration with respect to semimartingales we refer to [**63**] and [**23**] in the case of infinite dimensional Wiener process. An Itô formula for Banach valued functions acting on stochastic processes with jumps, the martingale part given by stochastic integrals of time dependent Banach valued random functions w.r.t. the compensated Poisson random measure is proved in [**72**]. At first we illustrate the theory developed for square integrable martingales, then we apply these results for the case of Lévy martingale. In the whole next section we consider implicitly a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in T}, \mathbb{P})$ which is assumed to fulfill the usual hypotheses (right continuity and completeness).

2.1. Square integrable martingale. If we consider as integrator an Hilbert-valued Wiener process with covariance operator Q, the stochastic integral is defined by an isometry given by

$$\mathbb{E}\left|\int_0^t X(s)\,\mathrm{d}W(s)\right|^2 = \mathbb{E}\int_0^t |X(s)Q^{1/2}|^2_{HS}\,\mathrm{d}s. \qquad (2.17)$$

In this section we show this isometric formula in the general case, when the integrator is a square integrable martingale.

DEFINITION 2.22. *Let U be a Hilbert space. We say that a martingale $(M_t)_{t\in T}$ with values in U is* SQUARE INTEGRABLE *if $\mathbb{E}|M_t|^2 < \infty$ for all $t \in [0,T]$. We call $\mathscr{M}^2(U)$ the vector space of right continuous U-valued square integrable martingale.*

Now we introduce a classical decomposition theorem for submartingale that permits to characterize a fundamental process in stochastic integration.

THEOREM 2.23 (Doob-Meyer decomposition). *For arbitrary càdlàg real valued submartingale X there exists a unique predictable and right continuous increasing process $A(t)$, $t \in [0,T]$ such that $A(0) = 0$ and*

$$M(t) = X(t) - A(t), \qquad t \in [0,T]$$

is a martingale.

If $M \in \mathscr{M}^2(U)$, then $|M|_U^2$ is a real càdlàg valued submartingale. The corresponding predictable process, denoted by $\langle M \rangle_t$, is called the MEYER or BRACKET PROCESS associated to M. In the case of Lévy martingales one has an explicit expression for the Meyer process.

PROPOSITION 2.24. *Assume that $M(t)$, $t \in [0,T]$ is a square integrable, right continuous process, having independent, time homogeneous increments and starting from 0. Then*

$$\langle M \rangle_t = t\, Tr\, Q \tag{2.18}$$

where Q is a trace class, non negative operator on U such that

$$\langle Qa, b\rangle_U = \mathbb{E}[\langle M(1), a\rangle_U \langle M(1), b\rangle_U] \qquad a, b \in U$$

For a generalization of the isometric formula (2.17) we need to study the evolution in time of the covariance operator. Then we introduce the operator valued angle bracket process $\langle\langle M \rangle\rangle_t$.

EXAMPLE 2.4. *Let $M(t) = W(t)$, $t \geq 0$ be an U-valued Q-Wiener process. Then*

$$\langle W \rangle_t = t\, Tr\, Q \quad \text{and } \langle\langle W \rangle\rangle_t = t\, Q$$

Denote by $L_1(U)$ the space of all nuclear operators on U and by $L_1^+(U)$ the subspace of $L_1(U)$ consisting of all self-adjoint, non-negative, nuclear operators; for all $a, b \in U$ we denote by $a \otimes b$ the operator on U given by $a \otimes b(u) = a\langle b, u\rangle_U$ for all $u \in U$. If $M(t)$, $t \in [0,T]$ is a right continuous, square integrable process then the process

$$M(t) \otimes M(t) \qquad t \in [0,T]$$

is an $L_1(U)$ valued, right continuous process.

THEOREM 2.25. *There exists a unique right continuous, $L_1^+(U)$ valued, increasing, predictable process $\langle\langle M\rangle\rangle_t$, $t \in [0,T]$, $\langle\langle M\rangle\rangle_0 = 0$ such that the process*

$$M(t) \otimes M(t) - \langle\langle M\rangle\rangle_t, \qquad t \in [0,T]$$

is an $L_1(U)$ martingale. Moreover there exists a predictable $L_1^+(U)$ valued process Q_t, $t \in [0,T]$ such that

$$\langle\langle M\rangle\rangle_t = \int_0^t Q_s \,\mathrm{d}\langle M\rangle_s$$

Defining the angle bracket process $\langle\langle M\rangle\rangle_t$ we have completed the basic tool to extend the identity (2.17). The isometric formula in the general case was discovered by Métivier and Pistone and is given in the form

$$\mathbb{E}\left|\int_0^t X(s)\,\mathrm{d}M(s)\right|^2 = \mathbb{E}\int_0^t |X(s)Q_s^{1/2}|_{HS}^2 \,\mathrm{d}\langle M\rangle_s. \tag{2.19}$$

Then an adapted process X is stochastically integrable with respect to a square integrable martingale M if

$$\mathbb{E}\int_0^t |X(s)Q_s^{1/2}|_{HS}^2 \,\mathrm{d}\langle M\rangle_s < \infty.$$

By a localization procedure the concept of the stochastic integral can be extended to all adapted processes X for which

$$\mathbb{P}\left(\int_0^t |X(s)Q_s^{1/2}|_{HS}^2 \,\mathrm{d}\langle M\rangle_s < \infty\right) = 1.$$

Now we treat the particular case when $M(t) = L(t)$ is a Lévy martingale.

2.2. Lévy Processes. Let X be a Lévy process taking values in U, so that X has stationary and independent increments, is stochastically continuous and satisfies $X(0) = 0$ (a.s.). We have the Lévy-Khinchine formula which yields for all $t \geq 0$, $\psi \in U$,

$$\mathbb{E}(e^{i\langle \psi, X(t)\rangle_U}) = e^{ta(\psi)},$$

where

$$a(\psi) = i\langle \varsigma, \psi\rangle_U - \frac{1}{2}\langle \psi, Q\psi\rangle_U + \int_{U\setminus\{0\}} (e^{i\langle u,\psi\rangle_U} - 1 - i\langle u, \psi\rangle_U \mathbf{1}_{\{|u|<1\}})\nu(\,\mathrm{d}u), \tag{2.20}$$

where $\varsigma \in U$, Q is a positive, self-adjoint trace class operator on U and ν is a LÉVY MEASURE on $U \setminus \{0\}$, i.e. $\int_{U\setminus\{0\}}(|x|^2 \wedge 1)\nu(\,\mathrm{d}x) < \infty$. We call the triple (ς, Q, ν) the CHARACTERISTICS of the process X and the mapping a, the characteristic exponent of X. From now on we will always assume that Lévy processes have strongly càdlàg paths. We also strengthen the independent increments requirement on X by assuming that $X(t) - X(s)$ is independent of $\mathscr{F}_s$ for all $0 \leq s < t < \infty$. If X is a Lévy process, we write $\Delta X(t) = X(t) - X(t-)$, for all $t > 0$. We obtain a Poisson random measure N on $\mathbb{R}^+ \times (U \setminus \{0\})$ by the prescription:

$$N(t, E) = \#\{0 \leq s \leq t : \Delta X(s) \in E\},$$

for each $t \geq 0$, $E \in \mathscr{B}(U \setminus \{0\})$. The associated compensated Poisson random measure $\tilde{N}$ is defined by

$$\tilde{N}(\,\mathrm{d}t, \,\mathrm{d}x) = N(\,\mathrm{d}t, \,\mathrm{d}x) - \,\mathrm{d}t\; \nu(\,\mathrm{d}x).$$

Let $A \in \mathscr{B}(U \setminus \{0\})$ with $0 \notin \bar{A}$. If $f : A \to U$ is measurable, we may define

$$\int_A f(x)N(t, \,\mathrm{d}x) = \sum_{0\leq s\leq t} f(\Delta X(s))\mathbf{1}_A(\Delta X(s))$$

as a random finite sum. Let ν_A denote the restriction of the measure ν to A, so that ν_A is finite. If $f \in L^2(A, \nu_A; U)$, we define

$$\int_A f(x)\tilde{N}(t, \,\mathrm{d}x) = \int_A f(x)N(t, \,\mathrm{d}x) - t\int_A f(x)\nu(\,\mathrm{d}x),$$

then by standard arguments we see that $(\int_A f(x)\tilde{N}(t,\,\mathrm{d}x), t \geq 0)$ is a centered square-integrable martingale with

$$\mathbb{E}\left(\left|\int_A f(x)\tilde{N}(t,\,\mathrm{d}x)\right|_U^2\right) = t\int_A |f(x)|_U^2 \nu(\,\mathrm{d}x), \tag{2.21}$$

for each $t \geq 0$.

THEOREM 2.26 (Lévy-Itô decomposition). *If U is a separable Hilbert space and $X = (X(t), t \geq 0)$ is a càdlàg U-valued Lévy process with characteristic exponent given by (2.20), then for each $t \geq 0$,*

$$X(t) = t\varsigma + W_Q(t) + \int_{|x|<1} x\tilde{N}(t,\,\mathrm{d}x) + \int_{|x|\geq 1} xN(t,\,\mathrm{d}x), \tag{2.22}$$

where W_Q is a Q-Wiener process which is independent of N.

In (2.22),

$$\int_{|x|<1} x\tilde{N}(t,\,\mathrm{d}x) = \lim_{n\to\infty}\int_{\frac{1}{n}<|x|<1} x\tilde{N}(t,\,\mathrm{d}x),$$

where the limit is taken in the L^2-sense, and it is a square-integrable martingale.

A symmetric Lévy martingale L_t can be represented as

$$L(t) = W_Q(t) + \int_0^t\int_{|x|<1} x\tilde{N}(\,\mathrm{d}t,\,\mathrm{d}x).$$

L is a square integrable, right continuous martingale, having independent, time homogeneous increments and starting from 0. Then by (2.18) $\langle L\rangle_t = t\;Tr\;Q_L$ where Q_L is uniquely defined by the relation

$$\langle Q_L a, b\rangle_U = \mathbb{E}[\langle L(1), a\rangle_U \langle L(1), b\rangle_U] \qquad a, b \in U.$$

So

$$\begin{aligned}
&\mathbb{E}\left[\left\langle W_Q(1) + \int_0^t\int_{|x|<1} x\,\mathrm{d}\tilde{N}, a\right\rangle\left\langle W_Q(1) + \int_0^t\int_{|x|<1} x\,\mathrm{d}\tilde{N}, b\right\rangle\right] = \\
&= \mathbb{E}\left[\langle W_Q(1), a\rangle\,\langle W_Q(1), b\rangle + \left\langle\int_0^t\int_{|x|<1} x\,\mathrm{d}\tilde{N}, a\right\rangle\left\langle\int_0^t\int_{|x|<1} x\,\mathrm{d}\tilde{N}, b\right\rangle\right] \\
&= \langle Qa, b\rangle + \int_{|x|<1}\langle x, a\rangle\langle x, b\rangle\nu(\,\mathrm{d}x).
\end{aligned}$$

Then

$$Q_L = Q + \int_{|x|<1} \langle \cdot, x \rangle \; x \; \nu(\,\mathrm{d}x).$$

and

$$\langle L \rangle_t = t \cdot Tr\; Q_L = t \left(Tr\; Q + \int_{|x|<1} |x|^2 \nu(\,\mathrm{d}x) \right) \tag{2.23}$$

Let us look for an isometric formula for Lévy martingales. By (2.17), (2.21) and the independence of W_Q and $\tilde{N}$ we obtain for an adapted process X

$$\begin{aligned} \mathbb{E} \left| \int_0^t X(s)\,\mathrm{d}L(s) \right|^2 &= \mathbb{E} \int_0^t |X(s)Q^{1/2}|^2_{HS}\,\mathrm{d}s \\ &\quad + \mathbb{E} \int_0^t |X(s)|^2 \int_{|x|<1} |x|^2 \nu(\,\mathrm{d}x)\,\mathrm{d}s \end{aligned}$$

Then

$$\langle\langle L \rangle\rangle_t = \frac{t \cdot Q_L}{Tr Q_L} \tag{2.24}$$

Then an adapted process X is stochastically integrable with respect to a Lévy martingale L if

$$\begin{aligned} \mathbb{E} \int_0^t |X(s)Q_L^{1/2}|^2_{HS}\,\mathrm{d}\langle L \rangle_s &= \mathbb{E} \int_0^t |X(s)Q^{1/2}|^2_{HS}\,\mathrm{d}s \\ &\quad + \mathbb{E} \int_0^t |X(s)|^2 \int_{|x|<1} |x|^2 \nu(\,\mathrm{d}x)\,\mathrm{d}s < \infty. \end{aligned}$$

By a localization procedure the concept of the stochastic integral can be extended to all adapted processes X for which

$$\mathbb{P} \left(\int_0^t |X(s)Q^{1/2}|^2_{HS}\,\mathrm{d}s < \infty \right) = 1 \quad \text{and}$$
$$\mathbb{P} \left(\int_0^t |X(s)|^2 \int_{|x|<1} |x|^2 \nu(\,\mathrm{d}x)\,\mathrm{d}s < \infty \right) = 1.$$

2.3. Itô formula.

Theorem 2.27. *Let $M \in \mathscr{M}^2$. There exists an increasing càdlàg process which is uniquely defined up to $\mathbb{P}$-equality denoted by $[M]$ and called the* QUADRATIC VARIATION *of M with the following properties:*

(1): *For every increasing sequence* (Π_n) *of increasing subsequences of* $\mathbb{R}^+$*:* $\Pi_n := \{0 < t_0 < t_1 < ... < t_n < ...\}$ *such that* $\lim_{k\uparrow\infty} t_k = +\infty$ *and* $\lim_{n\to\infty} \delta(\Pi_n) = 0$*, where* $\delta(\Pi_n)$ *is the mesh of* Π_n *defined by* $\delta(\Pi_n) := \sup_{t_i\in\Pi_n}(t_{i+1} - t_i)$*, one has*

$$[M]_t = \lim_n (L^1) \sum_{t_i\in\Pi_n} (M_{t_{i+1}\wedge t} - M_{t_i\wedge t})^2.$$

(2): $|M|^2 - [M]$ *is a martingale.*

(3): *If* M *is continuous* $[M] = \langle M\rangle$*.*

(4): $\mathbb{E}\int_0^t X(s)\,\mathrm{d}[M]_s = \mathbb{E}\int_0^t X(s)\,\mathrm{d}\langle M\rangle_s$*.*

If M is a real square integrable martingale this definition coincide with

$$[M]_t := M_t^2 - M_0^2 - 2\int_0^t M_{s-}\,\mathrm{d}M_s$$

Theorem 2.28 (Itô formula). *Let* M *be a real square integrable martingale and* φ *a twice continuously differentiable function on* $\mathbb{R}$*. Then the process* $(\varphi(M_t))_{t\in\mathbb{R}}$ *is a semimartingale which is* $\mathbb{P}$*-equal to the process* $T^\varphi(M)$ *defined by*

$$\begin{aligned} T^\varphi(M)_t &:= \varphi(M_0) + \int_0^t \varphi'(M_{s-})\,\mathrm{d}M_s + \frac{1}{2}\int_0^t \varphi''(M_{s-})\,\mathrm{d}[M]_s \\ &+ \sum_{s\le t}[\varphi(M_s) - \varphi(M_{s-}) - \Delta M_s\varphi'(M_{s-}) - \frac{1}{2}|\Delta M_s|^2\varphi''(M_{s-})] \end{aligned}$$

where the family

$$[\varphi(M_s(\omega)) - \varphi(M_{s-}(\omega)) - \Delta M_s(\omega)\varphi'(M_{s-}(\omega)) - \frac{1}{2}|\Delta M_s(\omega)|^2\varphi''(M_{s-}(\omega))]_{s\le t}$$

of real numbers occurring in the latter expression is summable for all t*,* $\mathbb{P}$*-a.s..*

CHAPTER 3

SPDEs driven by Lévy noise

This chapter is mainly taken from the paper [**10**].

The electrical behaviour of neuronal membranes and the role of ion currents have been studied and understood since the landmark 1952 papers by Hodgkin and Huxley [**45**] for the diffusion of the transmembrane electrical potential in a neuronal cell. This model consists of a system of four equations describing the diffusion of the electrical potential and the behaviour of various ion channels. Since then, there was a dramatic increase in experimental progress; contemporary to this, there has been a certain success in the efforts to develop mathematical models capable of reflecting and predicting neuronal activity. It is the goal of our research to provide one such model, in order to illuminate the connection between the complex physical structure of the dendritic tree in a neuron, or the tree-like structure of a neural network, and the overall behaviour.

Successive simplifications of the model, trying to capture the key phenomena of the Hodgkin-Huxley model, lead to the reduced FitzHugh-Nagumo equation, which is a scalar equation with three stable states (see e.g. [**71**]). Among other papers dealing with the case of a whole neuronal network (usually modelled as a graph with m edges and n nodes), which is intended to be a simplified model for a large region of the brain, let us mention a series of recent papers [**53, 67**], where the well-posedness of the isolated system is studied.

Note that, for a diffusion on a network, other conditions must be imposed in order to define the behaviour at the nodes. We impose a continuity condition, that is, given any node in the network, the electrical potentials of all its incident edges are equal. Each node represents an active soma, and in this part of the cell the potential evolves following a generalized Kirchhoff condition that we model with stochastic dynamical boundary conditions for the internal dynamics.

Since the classical work of Walsh [**77**], stochastic partial differential equations have been an important modelling tools in neurophysiology, where a random forcing is introduced to model several external perturbations acting on the system. In our neural network, we model the electrical activity of background neurons with a stochastic input with an impulsive component, to take into account the stream of excitatory and inhibitory action potentials coming from the neighbours of the network. The need to use models based on impulsive noise was already pointed out in several papers by Kallianpur and co-authors – see e.g. [**47, 48**].

> *Models designed to reflect the interesting behaviour of neurons must be stochastic, infinite-dimensional and highly non-linear*
> (G. Kallianpur)

Why Lévy noise? Stochastic disturbances to a quiescent neuron arise from several different sources, each with different characteristics. Starting from a slow time scale and going toward a faster one, we can list

- action potentials in the neighbours (arrivals of vesicles - each contains some of chemicals called neurotransmitters) are unpredictable and chaotic,
- arrivals of neurotransmitters: arrival sites and times are unpredictable, since they depend on the composition of the vescicle and the state of the neuron,
- opening of ionic channels: it occurs several times a second, in an apparently random manner; each time a gate opens, a tiny current flows in a direction and at a rate depending on the state of the neuron, and this current gives its contribution to the membrane voltage potential,
- finally, the passage of single ions causes extremely rapid current impulses.

Motivated by our interest in the Hodgkin-Huxley model, we are interested both in the second and also in the third source of randomness listed above. If we scale time in a suitable way, then

- the passage of individual ions will seem to proceed at such a furious rate that random fluctuations are not apparent. Only the smooth mean behaviour, predicted by the central limit theorem, would be observable. It gives rise to a GAUSSIAN component.
- On such a time scale action potential will occur so infrequently that we can safely focus on the behaviour of the neuron in ABSENCE of arriving action potentials.
- The remaining two random sources give rise to a GENERALIZED POISSON IMPULSE process.

Hence we can assume that the stochastic perturbation is a Lévy process with decomposition

$$L(t) = mt + QW(t) + \int_{|x|<1} x[N(t, \mathrm{d}x) - t\nu(\mathrm{d}x)] + t\int_{|x|\geq 1} xN(t, \mathrm{d}x).$$

Following the approach of [**11**], we use the abstract setting of stochastic PDEs by semigroup techniques (see e.g. [**22, 23**]) to prove existence and uniqueness of solutions to the system of stochastic equations on a network. In particular, the specific stochastic dynamics is rewritten in terms of a stochastic evolution equation driven by an additive Lévy noise on a certain class of Hilbert spaces. Even though there is a growing interest in stochastic PDEs driven by jump noise (let us just mention [**42, 52, 56**]), it seems like the case we are interested in, i.e. with a power-type nonlinearity, is not covered by existing results.

The rest of the chapter is organized as follows: in section 1 we first introduce the problem and we motivate our assumptions in connection with the applications to neuronal networks. Then in section 2, we provide a suitable abstract setting and we prove, following [**67**], that the linear operator appearing as leading drift term in the stochastic PDE generates an analytic semigroup of contractions. Section 3 contains our main results. First we prove existence and uniqueness of mild solution for the problem under Lipschitz conditions on the nonlinear term (theorem 3.8). This result (essentially already known) is used to obtain existence and uniqueness in the mild sense for the SPDE with a locally Lipschitz continuous dissipative drift of FitzHugh-Nagumo type by techniques of monotone operators.

1. Setting of the problem

The physical structure of a neuron is well known; the following description is taken from [**51**]. A typical neuron consists of three parts: the dendrites, the cell body (also called soma), and the axon. Dendrites are the input stage of a neuron: they receive synaptic input from other neurons and carry the informations to the soma. The soma elaborates these inputs and the axon is the output stage. However, in a first stage we shall not introduce the axon: the main reason for this choice is that the ACTIVE impulse propagation, which is the correct model for its behaviour, requires a different construction, and this will be carried on in the second part. The dendritic network is identified with the underlying graph G, described by a set of n vertices $\mathsf{v}_1, \dots, \mathsf{v}_n$ and m oriented edges $\mathsf{e}_1, \dots, \mathsf{e}_m$ which we assume to be normalized, i.e., $\mathsf{e}_j = [0,1]$ for $j = 1, ..., m$. The soma is assumed to be isopotential and can therefore be identified with a point. The graph is described by the INCIDENCE MATRIX $\Phi = \Phi^+ - \Phi^-$, where $\Phi^+ = (\phi^+_{ij})_{n\times m}$ and $\Phi^- = (\phi^-_{ij})_{n\times m}$ are given by

$$\phi^-_{ij} = \begin{cases} 1, & \mathsf{v}_i = \mathsf{e}_j(1) \\ 0, & \text{otherwise} \end{cases} \qquad \phi^+_{ij} = \begin{cases} 1, & \mathsf{v}_i = \mathsf{e}_j(0) \\ 0, & \text{otherwise.} \end{cases}$$

The degree of a vertex is the number of edges entering or leaving the node. We denote

$$\Gamma(\mathsf{v}_i) = \{ j \in \{1, \dots, m\} \ : \ \mathsf{e}_j(0) = \mathsf{v}_i \text{ or } \mathsf{e}_j(1) = \mathsf{v}_i \}$$

hence the degree of the vertex v_i is the cardinality $|\Gamma(\mathsf{v}_i)|$. The electrical voltage will be a function both of the position x along the network and of time t. We will measure the voltage as a DEVIATION from the quiescent potential; we shall assume that the extracellular fluid is large enough to have a negligible resistance and to be virtually isopotential. The electrical potential in the network shall be denoted by $\bar{u}(t,x)$ where $\bar{u} \in (L^2(0,1))^m$ is the vector $(u_1(t,x), \dots, u_m(t,x))$ and $u_j(t,\cdot)$ is the electrical potential on the edge e_j.

Diffusion's model. We impose a general diffusion equation on every edge

$$\frac{\partial}{\partial t} u_j(t,x) = \frac{\partial}{\partial x}\left(c_j(x) \frac{\partial}{\partial x} u_j(t,x) \right) + f_j(u_j(t,x)), \tag{3.1}$$

for all $(t, x) \in \mathbb{R}_+ \times (0, 1)$ and all $j = 1, ..., m$. The generality of the above diffusion is motivated by the discussion in the biological literature, see for example [**51**], who remark, in discussing some concrete biological models, that the basic cable properties are not constant throughout the dendritic tree. The above equation shall be endowed with suitable boundary and initial conditions. Initial conditions are given for simplicity at time $t = 0$ of the form

$$u_j(0, x) = u_{j0}(x) \in C([0, 1]), \qquad j = 1, ..., m. \tag{3.2}$$

Since we are dealing with a diffusion in a network, we require first a continuity assumption on every node

$$p_i(t) := u_j(t, \mathsf{v}_i) = u_k(t, \mathsf{v}_i), \qquad t > 0,\ j, k \in \Gamma(\mathsf{v}_i),\ i = 1, ..., n \tag{3.3}$$

and a stochastic generalized Kirchhoff law in the nodes

$$\frac{\partial}{\partial t} p_i(t) = -b_i p_i(t) + \sum_{j \in \Gamma(\mathsf{v}_i)} \phi_{ij} \mu_j c_j(\mathsf{v}_i) \frac{\partial}{\partial x} u_j(t, \mathsf{v}_i) + \sigma_i \frac{\partial}{\partial t} L(t, \mathsf{v}_i), \tag{3.4}$$

for all $t > 0$ and $i = 1, \ldots, n$. Observe that the positive sign of the Kirchhoff term in the above condition is consistent with a model of purely excitatory node conditions, i.e. a model of a neuronal tissue where all synapses depolarize the postsynaptic cell. Postsynaptic potentials can have graded amplitudes modelled by the constants $\mu_j > 0$ for all $j = 1, ..., m$.

Finally, $L(t, \mathsf{v}_i),\ i = 1, ..., n$, represent the stochastic perturbation acting on each node, due to the external surrounding, and $\frac{\partial}{\partial t} L(t, \mathsf{v}_i)$ is the formal time derivative of the process L, which takes a meaning only in integral sense. As seen in the introduction of this chapter, biological motivations lead us to model this term by a Lévy-type process. In fact, the evolution of the electrical potential on the molecular membrane can be perturbed by different types of random terms, each modelling the influence, at different time scale, of the surrounding medium. On a fast time scale, vesicles of neurotransmitters released by external neurons cause electrical impulses which arrive randomly at the soma causing a sudden change in the membrane voltage potential of an amount, either positive or negative, depending on the composition of the vesicle and possibly even on the state of the neuron. We model this behaviour perturbing the equation by an additive term driven by a n-dimensional Lèvy noise of the form

$$L(t) = mt + QW(t) + \int_{\mathbb{R}^n} x\tilde{N}(t, \mathrm{d}x), \tag{3.5}$$

see Hypothesis 3.2 below for a complete description of the process. See also [**48**] for a related model.

Although many of the above reasonings remain true also when considering the diffusion process on the fibers, we shall not pursue such generality and assume that the random perturbation acts only on the boundary of the system, i.e. on the nodes of the network.

Let us state the main assumptions on the data of the problem.

Hypothesis 3.1.

(1) In (3.1), we assume that $c_j(\cdot)$ belongs to $C^1([0,1])$, for $j = 1, \dots, m$ and $c_j(x) > 0$ for every $x \in [0,1]$.

(2) There exists constants $\eta \in \mathbb{R}$, $c > 0$ and $s \geq 1$ such that, for $j = 1, \dots, m$, the functions $f_j(u)$ satisfy $f_j(u) + \eta u$ is continuous and decreasing, and $|f_j(u)| \leq c(1 + |u|^s)$.

(3) In (3.4), we assume that $b_i \geq 0$ for every $i = 1, \dots, n$ and at least one of the coefficients b_i is strictly positive.

(4) $\{\mu_j\}_{j=1,\dots,m}$ and $\{\sigma_i\}_{i=1,\dots,n}$ are real positive numbers.

Stochastic setting. Given a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual hypotheses

(i) $\mathscr{F}_0$ contains all the $\mathbb{P}$-null set of $\mathscr{F}$,

(ii) $(\mathscr{F}_t)_{t\geq 0}$ is right continuous,

and a Hilbert space $\mathcal{H}$, let us define the space $L^2_{\mathscr{F}}(\Omega \times [0,T]; \mathcal{H})$ of adapted processes $Y : [0,T] \to \mathcal{H}$ endowed with the natural norm

$$|Y|_{L^2_{\mathscr{F}}} = \left(\mathbb{E} \int_0^T |Y(t)|^2_{\mathcal{H}} \, \mathrm{d}t \right)^{1/2}.$$

We shall consider a Lévy process $\{L(t),\ t \geq 0\}$ on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with values in $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$, i.e., a stochastically continuous, adapted process starting almost surely from 0, with stationary and independent increments and càdlàg trajectories, hence with discontinuities of jump type. By the classical

Lévy-Itô decomposition theorem, the n-dimensional Lévy process $L(t)$ has a decomposition

$$L(t) = mt + QW(t) + \int_{|x|\le 1} x[N(t,\mathrm{d}x) - t\nu(\mathrm{d}x)] + \int_{|x|>1} xN(t,\mathrm{d}x), \qquad t \ge 0 \tag{3.6}$$

where $m \in \mathbb{R}^n$, $Q \in M_{n\times n}(\mathbb{R})$ is a symmetric, positive defined matrix, $\{W_t,\ t \ge 0\}$ is an n-dimensional centered Brownian motion, for any set $\Lambda \in \mathscr{B}(\mathbb{R}^n)$ such that $0 \notin \bar{\Lambda}$, $N_t^\Lambda = \int_\Lambda N(t,\,\mathrm{d}x)$ is a Poisson process independent of W and the Lévy measure $\nu(\,\mathrm{d}x)$ is σ-finite on $\mathbb{R}^n \setminus \{0\}$ and such that $\int \min(1,x^2)\nu(\,\mathrm{d}x) < \infty$. We denote by $\tilde{N}(\,\mathrm{d}t,\,\mathrm{d}x) := N(\,\mathrm{d}t,\,\mathrm{d}x) -\,\mathrm{d}t\nu(\,\mathrm{d}x)$ the compensated Poisson measure.

Hypothesis 3.2. *We suppose that the measure ν has finite second order moment, i.e.*

$$\int_{\mathbb{R}^n} |x|^2\nu(\mathrm{d}x) < \infty. \tag{3.7}$$

Condition (3.7) implies that the generalized compound Poisson process $\int_{|x|>1} x\,N(t,\mathrm{d}x)$ has finite moments of first and second order. Then, with no loss of generality, we assume that

$$\int_{|x|>1} x\nu(\mathrm{d}x) = 0. \tag{3.8}$$

Remark 3.1. *In view of assumptions (3.7) and (3.8) the Lévy process (3.6) can be represented as*

$$L(t) = mt + QW(t) + \int_{\mathbb{R}^n} x\tilde{N}(t,\mathrm{d}x), \qquad t \ge 0.$$

2. Well-posedness of the linear deterministic problem

We consider the product space $\mathbb{H} = (L^2(0,1))^m$. A general vector $\bar{u} \in \mathbb{H}$ is a collection of functions $\{u_j(x),\ x \in [0,1],\ j = 1,\dots,m\}$ which represents the electrical potential inside the network.

Remark 3.2. *For any real number $s \ge 0$ we define the Sobolev spaces*

$$\mathbb{H}^s = (H^s(0,1))^m,$$

*where $H^s(0,1)$ is the fractional Sobolev space defined for instance in [**59**]. In particular we have that $\mathbb{H}^1 \subset (C[0,1])^m$. Hence we are allowed to define the boundary evaluation operator $\Pi : \mathbb{H}^1 \to \mathbb{R}^n$ defined by*

$$\Pi \bar{u} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \quad \textit{where } p_i = \bar{u}(\mathsf{v}_i) = u_k(\mathsf{v}_i) \quad \textit{for } k \in \Gamma(\mathsf{v}_i),\ i = 1, ..., n.$$

On the space $\mathbb{H}$ we introduce the linear operator $(A, D(A))$ defined by

$$D(A) = \{\bar{u} \in \mathbb{H}^2 \mid \exists\, p \in \mathbb{R}^n \text{ such that } \Pi\bar{u} = p\}$$

$$A\bar{u} = \left(\frac{\partial}{\partial x}\left(c_j(x)\frac{\partial}{\partial x}u_j(t,x)\right)\right)_{j=1,\dots,m}$$

As discussed in [**67**], the diffusion operator A on a network, endowed with active nodes, fits the abstract mathematical theory of parabolic equations with dynamic boundary conditions and in particular it can be discussed in an efficient way by means of sesquilinear forms. Here, we shall follow the same approach.

First, notice that no other condition except continuity on the nodes is imposed on the elements of $D(A)$. This is often stated by saying that the domain is MAXIMAL.

The so called feedback operator, denoted by C, is a linear operator from $D(A)$ to $\mathbb{R}^n$ defined as

$$C\bar{u} = \left(\sum_{j\in\Gamma(\mathsf{v}_i)} \phi_{ij}\mu_j c_j(\mathsf{v}_i)\frac{\partial}{\partial x}u_j(t,\mathsf{v}_i)\right)_{i=1,\dots,n}.$$

On the vectorial space $\mathbb{R}^n$ we define also the diagonal matrix

$$B = \begin{pmatrix} -b_1 & & \\ & \ddots & \\ & & -b_n \end{pmatrix}.$$

With the above notation, problem (3.1)–(3.4) can be written as an abstract Cauchy problem on the product space $\mathcal{H} = \mathbb{H} \times \mathbb{R}^n$ endowed with the natural inner product

$$\langle X, Y\rangle_{\mathcal{H}} = \langle \bar{u}, \bar{v}\rangle_{\mathbb{H}} + \langle p, q\rangle_{\mathbb{R}^n}, \qquad \text{where } X, Y \in \mathcal{H} \text{ and } X = \begin{pmatrix}\bar{u}\\ p\end{pmatrix},\ Y = \begin{pmatrix}\bar{v}\\ q\end{pmatrix}$$

We introduce the matrix operator $\mathcal{A}$ on the space $\mathcal{H}$, given in the form

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \{X = (\bar{u}, p) \in \mathcal{H} \,:\, \bar{u} \in D(A), u_j(\mathsf{v}_i) = p_i \text{ for every } j \in \Gamma(\mathsf{v}_i)\}.$$

Then the linear deterministic part of problem (3.1)–(3.4) becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} X(t) &= \mathcal{A} X(t) \\ X(0) &= x_0 \end{aligned} \tag{3.9}$$

where $x_0 = (u_j(0, x))_{j=1,\dots,m} \in C([0,1])^m$ is the m-vector of initial conditions. This problem is well posed, as the following result shows.

Proposition 3.3. *Under Hypothesis 3.1.1 and 3.1.2 the operator* $(\mathcal{A}, D(\mathcal{A}))$ *is self-adjoint, dissipative and has compact resolvent. In particular, it generates a* C_0 *analytic semigroup of contractions.*

Proof. For the sake of completeness, we provide a sketch of the proof following [**67**]. The idea is simply to associate the operator $(\mathcal{A}, D(\mathcal{A}))$ with a suitable form $\mathfrak{a}(X, Y)$ having dense domain $\mathcal{V} \subset \mathcal{H}$.

The space $\mathcal{V}$ is defined as

$$\mathcal{V} = \left\{ X = \begin{pmatrix} \bar{u} \\ p \end{pmatrix} \mid \bar{u} \in (H^1(0,1))^m,\ u_k(\mathsf{v}_i) = p_i \text{ for } i = 1, \dots, n,\ k \in \Gamma(\mathsf{v}_i) \right\}$$

and the form $\mathfrak{a}$ is defined as

$$\mathfrak{a}(X, Y) = \sum_{j=1}^{m} \int_0^1 \mu_j c_j(x) u_j'(x) v_j'(x) \,\mathrm{d}x + \sum_{l=1}^{n} b_l p_l q_l, \qquad X = \begin{pmatrix} \bar{u} \\ p \end{pmatrix},\ Y = \begin{pmatrix} \bar{v} \\ q \end{pmatrix}.$$

The form $\mathfrak{a}$ is clearly positive and symmetric; furthermore it is closed and continuous. Then a little computation shows that the operator associated with the form $\mathfrak{a}$ is $(\mathcal{A}, D(\mathcal{A}))$ defined above. Classical results in Dirichlet forms theory, see for instance [**70**], lead to the desired result. □

The assumption that $b_l > 0$ for some $l \in \{1, ..., n\}$ is a dissipativity condition on $\mathcal{A}$. In particular it implies the following result (for a proof see [**67**]).

PROPOSITION 3.4. *Under Hypothesis 3.1.1 and 3.1.3, the operator $\mathcal{A}$ is invertible and the semigroup $\{\mathcal{T}(t),\ t \geq 0\}$ generated by $\mathcal{A}$ is exponentially bounded, with growth bound given by the strictly negative spectral bound of the operator $\mathcal{A}$.*

3. The stochastic Cauchy problem

We can now solve the system of stochastic differential equations (3.1)–(3.4). The functions $f_j(u)$ which appear in (3.1) are assumed to have a polynomial growth.

REMARK 3.3. *We remark that the classical FitzHugh-Nagumo problem requires a nonlinearity term of the form*

$$f_j(u) = u(u-1)(a_j - u) \qquad j = 1, ..., m$$

for some $a_j \in (0,1)$. Hence they satisfy Hypothesis 3.1.2 with

$$\eta \leq -\max_j \frac{(a_j^3+1)}{3(a_j+1)} \qquad \textit{and} \qquad s = 3.$$

We set

$$F(\bar{u}) = \Big(f_j(u_j)\Big)_{j=1,\dots,m} \qquad \text{and} \qquad \mathcal{F}(X) = \begin{pmatrix} -F(\bar{u}) \\ 0 \end{pmatrix} \qquad \text{for } X = \begin{pmatrix} \bar{u} \\ p \end{pmatrix} \tag{3.10}$$

and we write our problem in abstract form as a nonlinear stochastic differential equation on the infinite dimensional product space $\mathcal{H}$

$$\begin{aligned} \mathrm{d}X(t) &= [\mathcal{A}X(t) - \mathcal{F}(X(t))]\,\mathrm{d}t + \Sigma\,\mathrm{d}\mathcal{L}(t) \\ X(0) &= x_0 \end{aligned} \tag{3.11}$$

where Σ is the matrix defined by

$$\Sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{diag}(\sigma_1, \dots, \sigma_n) \end{pmatrix},$$

and $\mathcal{L}(t)$ is the natural embedding in $\mathcal{H}$ of the n-dimensional Lévy process $L(t)$, i.e.

$$\mathcal{L}(t) = \begin{pmatrix} 0 \\ L(t) \end{pmatrix}.$$

Remark 3.4. *Note that in general the nonlinearity $\mathcal{F}$ can be defined only on its domain $D(\mathcal{F})$, (possibly) strictly smaller than $\mathcal{H}$.*

The aim of this section is to prove existence, uniqueness and regularity for mild solution of (3.11). Let us recall the definition of mild solution for the stochastic Cauchy problem (3.11).

Definition 3.5. *An $\mathcal{H}$-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* mild solution *of (3.11) if*

$$\mathbb{P}\left(\int_0^T |\mathcal{F}(X(s))| \, \mathrm{d}s < +\infty\right) = 1 \tag{3.12}$$

and

$$X(t) = \mathcal{T}(t)x_0 - \int_0^t \mathcal{T}(t-s)\mathcal{F}(X(s)) \, \mathrm{d}s + \int_0^t \mathcal{T}(t-s)\Sigma \, \mathrm{d}\mathcal{L}(s) \tag{3.13}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$, where $\mathcal{T}(t)$ is the semigroup generated by $\mathcal{A}$.

Condition (3.12) implies that the first integral in (3.13) is well defined. The second integral, which we shall refer to as stochastic convolution, is well defined as will be shown in the following subsection.

3.1. The stochastic convolution process. In our case the stochastic convolution $Z(t) := \int_0^t \mathcal{T}(t-s)\Sigma \, \mathrm{d}\mathcal{L}(s)$ can be written as

$$\begin{aligned} Z(t) &= \int_0^t \mathcal{T}(t-s)\begin{pmatrix} 0 \\ \sigma m \end{pmatrix} \mathrm{d}s + \int_0^t \mathcal{T}(t-s)\Sigma Q \, \mathrm{d}W(s) \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \mathcal{T}(t-s)\begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \tilde{N}(\mathrm{d}s, \mathrm{d}x). \end{aligned}$$

The first integral is well defined because $|\mathcal{T}(s)\Sigma m|_{\mathcal{H}}$ is bounded on $[0,T]$ and then $\mathcal{T}(s)\Sigma m$ is a Bochner integrable function (see e.g. [**78**]) with respect to the Lebesgue measure $\mathrm{d}s$ on $[0,T]$, it follows

$$\int_0^T \left|\mathcal{T}(t-s)\begin{pmatrix} 0 \\ \sigma m \end{pmatrix}\right|_{\mathcal{H}} \mathrm{d}s < \infty.$$

The second one is a stochastic integral of a deterministic, bounded function with respect to a finite dimensional Wiener process. Hence

$$\int_0^T \|\mathcal{T}(s)\Sigma Q^{1/2}\|_{HS}^2 \mathrm{d}s < \infty$$

where $\|\cdot\|_{HS}$ stands for the Hilbert-Schmidt norm on $L_2(\mathcal{H})$. A such condition is sufficient to define the Wiener stochastic integral by an isometric transformation (see e.g. [**23**]). The third term involves as integrator a compensated Poisson process. The definition of stochastic integral with respect to a compensated Poisson measure has been discussed by many authors, see for instance [**1, 4, 5, 17, 35, 44**]. Here we limit ourselves to briefly recall some conditions for the existence of such integrals. In particular, in this chapter we only integrate deterministic functions, such as $\mathcal{T}(\cdot)\Sigma$, taking values in (a subspace of) $L(\mathcal{H})$, the space of linear operators from $\mathcal{H}$ to $\mathcal{H}$. We follow the approach by martingale-valued measures developed in [**5**] to investigate weak and strong integration w.r.t. a such class of measures. A MARTINGALE-VALUED MEASURE is a set function $M : \mathbb{R}^+ \times \mathscr{B}(\mathbb{R}^n) \times \Omega \to \mathcal{H}$ which satisfies the following:

1. $M(0, A) = M(t, \emptyset) = 0$ (a.s.), for all $A \in \mathscr{B}(\mathbb{R}^n)$, $t \geq 0$.
2. $M(t, A \cup B) = M(t, A) + M(t, B)$ (a.s.), for all $t \geq 0$ and all disjoint $A, B \in \mathscr{B}(\mathbb{R}^n)$.
3. $(M(t, A), t \geq 0)$ is a square-integrable martingale for each $A \in \mathscr{B}(\mathbb{R}^n)$ and is orthogonal to $(M(t, B), t \geq 0)$,whenever $A, B \in \mathscr{B}(\mathbb{R}^n)$ are disjoint.
4. $\sup\{\mathbb{E}(|M(t, A)|^2), A \in \mathscr{B}(\mathbb{R}^n);\ 0 \notin \bar{A}\} < \infty$, for all, $t > 0$.

M is also called σ-FINITE L^2-VALUED ORTHOGONAL MARTINGALE MEASURE. Whenever $0 \leq s \leq t < \infty$, $M((s, t], \cdot) := M(t, \cdot) - M(s, \cdot)$. M is said to have independent increments if $M((s, t], A)$ is independent of $\mathscr{F}$ for all $A \in \mathscr{B}(\mathbb{R}^n)$, $0 \leq s \leq t < \infty$. In order to define the stochastic integral of this class of processes with respect to the Lévy martingale-valued measure

$$M(t, B) = \int_B x\, \tilde{N}(t, \mathrm{d}x), \tag{3.14}$$

one requires that the mapping $\mathcal{T}(\cdot)\Sigma : [0, T] \times \mathbb{R}^n \ni (t, x) \mapsto \mathcal{T}(t)(0, \sigma x)$ belongs to the space $L^2((0, T) \times B; \langle M(\mathrm{d}t, \mathrm{d}x)\rangle)$ for every $B \in \mathscr{B}(\mathbb{R}^n)$, i.e. that

$$\left(\int_0^T \int_B \left| \mathcal{T}(s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\mathcal{H}}^2 \nu(\mathrm{d}x)\, \mathrm{d}s \right)^{1/2} < \infty. \tag{3.15}$$

Thanks to (3.7), one has

$$\begin{aligned}\int_0^T \int_B \left| \mathcal{T}(s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\mathcal{H}}^2 \nu(\mathrm{d}x)\,\mathrm{d}s \\ \leq |\sigma|^2 \left(\int_0^T |\mathcal{T}(s)|_{L(\mathcal{H})}^2 \,\mathrm{d}s \right) \left(\int_B |x|^2 \,\nu(\mathrm{d}x) \right) < \infty,\end{aligned}$$

thus the stochastic convolution $Z(t)$ is well defined for all $t \in [0, T]$.

Space regularity. We shall now prove a regularity property (in space) of the stochastic convolution. In theorem 3.8 below we will also prove a time regularity result, i.e. we show that the stochastic convolution has càdlàg paths. Let us define the product spaces $\mathcal{E} := (C[0, 1])^m \times \mathbb{R}^n$ and $C_{\mathscr{F}}([0, T]; L^2(\Omega; \mathcal{E}))$, the space of $\mathcal{E}$-valued, adapted mean square continuous processes Y on the time interval $[0, T]$ such that

$$|Y|_{C_{\mathscr{F}}}^2 := \sup_{t \in [0,T]} \mathbb{E}|Y(t)|_{\mathcal{E}}^2 < \infty.$$

LEMMA 3.6. *For all $t \in [0, T]$, the stochastic convolution $\{Z(t),\ t \in [0, T]\}$ belongs to the space $C_{\mathscr{F}}([0, T]; L^2(\Omega; \mathcal{E}))$. In particular, $Z(t)$ is predictable.*

PROOF. Let us recall that the unbounded matrix operator $\mathcal{A}$ on $\mathcal{H}$ is similar to this one, still denoted by $\mathcal{A}$ and defined by

$$\mathcal{A} = \begin{pmatrix} \partial_x^2 & 0 \\ -\partial_\nu & B \end{pmatrix},$$

with domain

$$D(\mathcal{A}) = \{X = (\bar{u}, p) \in \mathcal{H} \,:\, \bar{u} \in D(A), u_l(\mathsf{v}_i) = p_i \text{ for every } l \in \Gamma(\mathsf{v}_i)\}$$

and, by proposition 3.3, it generates a C_0-analytic semigroup of contractions on $\mathcal{H}$.

Let us introduce the interpolation spaces $\mathcal{H}_\theta = (\mathcal{H}, D(\mathcal{A}))_{\theta,2}$ for $\theta \in (0, 1)$. By classical interpolation theory (see e.g. [**60**]) it results that, for $\theta < 1/4$, $\mathcal{H}_\theta = \mathbb{H}^{2\theta} \times \mathbb{R}^n$ while for $\theta > 1/4$ the definition of $\mathcal{H}_\theta$ involves boundary conditions, that is

$$\mathcal{H}_\theta = \left\{ \begin{pmatrix} \bar{u} \\ p \end{pmatrix} \in \mathbb{H}^{2\theta} \times \mathbb{R}^n : \Pi\bar{u} = p \right\}.$$

Therefore, one has

- $(0, \sigma x), (0, \sigma m) \in \mathcal{H}_\theta$ for all $\theta < 1/4$ and
- $\Sigma Q W(s) \in \mathcal{H}_\theta$ $\mathbb{P}$-a.s. for all $\theta < 1/4$, $s \in [0, T]$.

Furthermore, for $\theta > 1/2$, one also has $\mathcal{H}_\theta \subset \mathbb{H}^1 \times \mathbb{R}^n \subset (C[0,1])^m \times \mathbb{R}^n$ by Sobolev embedding theorem. Moreover, for all $x \in \mathcal{H}_\theta$ and $\theta + \gamma \in (0,1)$, it holds

$$|\mathcal{T}(t)x|_{\theta+\gamma} \leq t^{-\gamma} |x|_\theta e^{\omega_{\mathcal{A}} t}, \tag{3.16}$$

where $\omega_{\mathcal{A}}$ is the spectral bound of the operator $\mathcal{A}$.

Let θ, γ be real numbers such that $\theta \in (0, 1/4)$, $\gamma \in (0, 1/2)$ and $\theta + \gamma \in (1/2, 1)$. Then for all $t \in [0, T]$

$$\begin{aligned} |Z(t)|_{\theta+\gamma} \leq & \int_0^t \left| \mathcal{T}(t-s) \begin{pmatrix} 0 \\ \sigma m \end{pmatrix} \right|_{\theta+\gamma} \mathrm{d}s + \left| \int_0^t \mathcal{T}(t-s) \Sigma \, \mathrm{d}W(s) \right|_{\theta+\gamma} \\ & + \int_0^t \int_{\mathbb{R}^n} \left| \mathcal{T}(t-s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\theta+\gamma} \tilde{N}(\mathrm{d}x, \mathrm{d}s) \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

The right hand side of the above inequality is well defined if and only if the following integrals are finite

- $\int_0^T s^{-\gamma} |\Sigma m|_\theta \, \mathrm{d}s$,
- $\mathbb{E} \left| \int_0^T \mathcal{T}(s) \Sigma Q \, \mathrm{d}W(s) \right|_{\theta+\gamma}^2 = \int_0^T \left\| \mathcal{T}(s) \Sigma Q^{1/2} \right\|_{HS}^2 \mathrm{d}s$,
- $\mathbb{E} \left| \int_0^T \int_{\mathbb{R}^n} |\mathcal{T}(s) \Sigma x|_{\theta+\gamma} \tilde{N}(\mathrm{d}x, \mathrm{d}s) \right|^2 = \int_0^T \int_{\mathbb{R}^n} |\mathcal{T}(s)\Sigma x|_{\theta+\gamma}^2 \nu(\mathrm{d}x) \, \mathrm{d}s$,

where the last two identities follow by the classical isometries for Wiener and Poisson integrals. The first condition is obviously verified for $0 < \gamma < 1$. On the other hand, one has by (3.16)

$$\begin{aligned} \int_0^T \left\| \mathcal{T}(s) \Sigma Q^{1/2} \right\|_{HS}^2 \mathrm{d}s &= \int_0^T Tr[\mathcal{T}(s) \Sigma Q \Sigma^* \mathcal{T}^*(s)] \, \mathrm{d}s \\ &\leq \int_0^T s^{-2\gamma} Tr[\Sigma Q \Sigma^*] \, \mathrm{d}s < \infty \end{aligned}$$

and

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \left| \mathcal{T}(s) \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_{\theta+\gamma}^2 \nu(\mathrm{d}x) \, \mathrm{d}s &\leq \int_0^T \int_{\mathbb{R}^n} s^{-2\gamma} \left| \begin{pmatrix} 0 \\ \sigma x \end{pmatrix} \right|_\theta^2 e^{2\omega_{\mathcal{A}} s} \nu(\mathrm{d}x) \, \mathrm{d}s \\ &\leq |\sigma|^2 \int_0^T s^{-2\gamma} e^{2\omega_{\mathcal{A}} s} \, \mathrm{d}s \int_{\mathbb{R}^n} |x|^2 \nu(\mathrm{d}x) < \infty \end{aligned}$$

using $\gamma \in (0, 1/2)$ and assumption (3.7). So $Z(t) \in \mathcal{H}_{\theta+\gamma}$ for $\theta + \gamma > 1/2$ and then $Z(t) \in (C[0,1])^m \times \mathbb{R}^n = \mathcal{E}$. It remains to prove that $Z(t)$ is mean square continuous as $\mathcal{E}$-valued process. For $0 \le s < t \le T$ we can write

$$\begin{aligned}
\mathbb{E}|Z(t) - Z(s)|^2_{\mathcal{E}} &= \mathbb{E}\left|\int_0^t \mathcal{T}(t-r)\Sigma \, \mathrm{d}\mathcal{L}(r) - \int_0^s \mathcal{T}(s-r)\Sigma \, \mathrm{d}\mathcal{L}(r)\right|^2_{\mathcal{E}} \\
&\le 2\mathbb{E}\left|\int_0^s \int_{\mathbb{R}^n} [\mathcal{T}(t-r) - \mathcal{T}(s-r)]\Sigma \, \mathrm{d}\mathcal{L}(r)\right|^2_{\mathcal{E}} \\
&\quad +2\mathbb{E}\left|\int_s^t \int_{\mathbb{R}^n} \mathcal{T}(t-r)\Sigma \, \mathrm{d}\mathcal{L}(r)\right|^2_{\mathcal{E}} \\
&\le 8\int_0^s \left|[\mathcal{T}(t-r) - \mathcal{T}(s-r)]\begin{pmatrix} 0 \\ \sigma m \end{pmatrix}\right|^2_{\mathcal{E}} \mathrm{d}r \\
&\quad +8\int_0^s \|[\mathcal{T}(t-r) - \mathcal{T}(s-r)]\Sigma Q^{1/2}\|^2_{HS} \, \mathrm{d}r \\
&\quad +8\int_0^s \int_{\mathbb{R}^n} \left|[\mathcal{T}(t-r) - \mathcal{T}(s-r)]\begin{pmatrix} 0 \\ \sigma x \end{pmatrix}\right|^2_{\mathcal{E}} \nu(\mathrm{d}x)\,\mathrm{d}r \\
&\quad +8\int_s^t \left|\mathcal{T}(t-r)\begin{pmatrix} 0 \\ \sigma m \end{pmatrix}\right|^2_{\mathcal{E}} \mathrm{d}r \\
&\quad +8\int_s^t \|\mathcal{T}(t-r)\Sigma Q^{1/2}\|^2_{HS}\,\mathrm{d}r \\
&\quad +8\int_s^t \int_{\mathbb{R}^n} \left|\mathcal{T}(t-r)\begin{pmatrix} 0 \\ \sigma x \end{pmatrix}\right|^2_{\mathcal{E}} \nu(\mathrm{d}x)\,\mathrm{d}r \longrightarrow 0
\end{aligned}$$

for the strong continuity of the semigroup $\mathcal{T}(t)$.

Since the stochastic convolution $Z(t)$ is adapted and mean square continuous, it is predictable. □

4. Existence and uniqueness in the Lipschitz case

We consider as a preliminary step the case of Lipschitz continuous nonlinear term and we prove existence and uniqueness of solutions in the space $C_{\mathscr{F}}$ of adapted mean square continuous processes taking values in the product space $\mathcal{H}$. We would like to mention that this result is included only for the sake of completeness and for the simplicity of its proof (which is essentially based only on the isometry defining the stochastic integral). In fact, a much more general existence and uniqueness result was proved by Kotelenez in [**52**].

THEOREM 3.7. *Assume Hypothesis 3.2 and let x_0 be an $\mathscr{F}_0$-measurable $\mathcal{H}$-valued random variable such that $\mathbb{E}|x_0|^2 < \infty$. Let $G : \mathcal{H} \to \mathcal{H}$ be a function satisfying Lipschitz and linear growth conditions:*

$$|G(x)| \leq c_0(1 + |x|), \qquad |G(x) - G(y)| \leq c_0|x - y|, \qquad x, y \in \mathcal{H}. \tag{3.17}$$

for some constant $c_0 > 0$. Then there exists a unique mild solution $X : [0, T] \to L^2(\Omega, \mathcal{H})$ to equation (3.11) with $-\mathcal{F}$ replaced by G, which is continuous as $L^2(\Omega, \mathcal{H})$-valued function. Moreover, the solution map $x_0 \mapsto X(t)$ is Lipschitz continuous.

PROOF. We follow the semigroup approach of [**23**, Theorem 7.4] where the case of Wiener noise is treated. We emphasize only the main differences in the proof.

The uniqueness of solutions reduces to a simple application of Gronwall's inequality. To prove existence we use the classical Banach's fixed point theorem in the space $C_{\mathscr{F}}([0, T]; L^2(\Omega; \mathcal{H}))$. Let $\mathcal{K}$ be the mapping

$$\mathcal{K}(Y)(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t - s)G(Y(s))\,\mathrm{d}s + Z(t)$$

where $Y \in C_{\mathscr{F}}([0, T]; L^2(\Omega; \mathcal{H}))$ and $Z(t)$ is the stochastic convolution. $Z(\cdot)$ and $\mathcal{T}(\cdot)x_0$ belong to $C_{\mathscr{F}}([0, T]; L^2(\Omega; \mathcal{H}))$ respectively in view of Lemma 3.6 and initial assumption. Moreover, setting

$$\mathcal{K}_1(Y)(t) = \int_0^t \mathcal{T}(t - s)G(Y(s))\,\mathrm{d}s,$$

it is sufficient to note that

$$|\mathcal{K}_1(Y)|^2_{C_{\mathscr{F}}} \leq (Tc_0)^2(1 + |Y|^2_{C_{\mathscr{F}}})$$

by the linear growth of G and the contractivity of $\mathcal{T}(t)$. Then we obtain that $\mathcal{K}$ maps the space $C_{\mathscr{F}}([0, T]; L^2(\Omega; \mathcal{H}))$ to itself. Further, using the Lipschitz continuity of G, it follows that for arbitrary processes Y_1 and Y_2 in $C_{\mathscr{F}}([0, T]; L^2(\Omega; \mathcal{H}))$ we have

$$|\mathcal{K}(Y_1) - \mathcal{K}(Y_2)|^2_{C_{\mathscr{F}}} = |\mathcal{K}_1(Y_1) - \mathcal{K}_1(Y_2)|^2_{C_{\mathscr{F}}} \leq (c_0T)^2|Y_1 - Y_2|^2_{C_{\mathscr{F}}}.$$

If we choose an interval $[0,\tilde{T}]$ such that $\tilde{T} < c_0^{-1}$, it follows that the mapping $\mathcal{K}$ has a unique fixed point $X \in C_{\mathscr{F}}([0,\tilde{T}]; L^2(\Omega;\mathcal{H}))$. The extension to arbitrary interval $[0,T]$ follows by patching together the solutions in successive time intervals of length $\tilde{T}$.

The Lipschitz continuity of the solution map $x_0 \mapsto X$ is again a consequence of Banach's fixed point theorem, and the proof is exactly as in the case of Wiener noise.

It remains to prove the mean square continuity of X. Observe that $\mathcal{T}(\cdot)x_0$ is a deterministic continuous function and it follows, again from Lemma 3.6, that the stochastic convolution $Z(t)$ is mean square continuous. Hence it is sufficient to note that the same holds for the term $\int_0^t \mathcal{T}(t-s)G(X(s))\,\mathrm{d}s$, that is $\mathbb{P}$-a.s. a continuous Bochner integral and then continuous as the composition of continuous functions on $[0,T]$. □

REMARK 3.5. *By standard stopping time arguments one actually shows that existence and uniqueness of a mild solution holds assuming only that x_0 is $\mathscr{F}_0$-measurable.*

In order to prove that the solution constructed above has càdlàg paths, unfortunately one cannot adapt the factorization technique developed for Wiener integrals (see e.g. [**23**]). However, the càdlàg property of the solution was proved by Kotelenez [**52**], under the assumption that $\mathcal{A}$ is dissipative. Therefore, thanks to proposition 3.3, the solution constructed above has càdlàg paths. One could also obtain this property proving the following a priori estimate, which might be interesting in its own right.

THEOREM 3.8. *Under the assumptions of theorem 3.7 the unique mild solution of problem* (3.11) *verifies*

$$\mathbb{E} \sup_{t\in[0,T]} |X(t)|^2_{\mathcal{H}} < \infty.$$

PROOF. Let us consider the Itô formula for the function $|\cdot|^2_{\mathcal{H}}$, applied to the process X. Although our computations are only formal, they can be justified using a classical approximation argument. We obtain

$$\mathrm{d}|X(t)|^2_{\mathcal{H}} = 2\langle X(t-), \mathrm{d}X(t)\rangle_{\mathcal{H}} + \mathrm{d}[X](t).$$

By the dissipativity of the operator $\mathcal{A}$ and the Lipschitz continuity of G, we obtain

$$\begin{aligned}\langle X(t-), \mathrm{d}X(t)\rangle_{\mathcal{H}} =& \langle \mathcal{A}X(t), X(t)\rangle_{\mathcal{H}}\,\mathrm{d}t + \langle G(X(t)), X(t)\rangle_{\mathcal{H}}\,\mathrm{d}t \\ &+ \langle X(t-), \Sigma\,\mathrm{d}\mathcal{L}(t)\rangle_{\mathcal{H}} \\ &\leq c_0|X(t)|^2_{\mathcal{H}} + \langle X(t-), \Sigma\,\mathrm{d}\mathcal{L}(t)\rangle_{\mathcal{H}}.\end{aligned}$$

Therefore

$$|X(t)|^2_{\mathcal{H}} \leq |x_0|^2_{\mathcal{H}} + 2c_0\int_0^t |X(s)|^2_{\mathcal{H}}\,\mathrm{d}s + 2\int_0^t \langle X(s-), \Sigma\,\mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}} + \int_0^t |\Sigma|^2\,\mathrm{d}[\mathcal{L}](s)$$

and

$$\begin{aligned}\mathbb{E}\sup_{t\leq T}|X(t)|^2_{\mathcal{H}} \leq& \mathbb{E}|x_0|^2_{\mathcal{H}} + 2c_0T\mathbb{E}\sup_{t\leq T}|X(t)|^2_{\mathcal{H}} \\ &+ 2\mathbb{E}\sup_{t\leq T}\left|\int_0^t \langle X(s-), \Sigma\,\mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}}\right| + T\int_{\mathbb{R}^n} |\Sigma|^2|x|^2\,\nu(dx),\end{aligned} \tag{3.18}$$

where we have used the relation

$$\begin{aligned}\mathbb{E}\sup_{t\leq T}[X](t) \leq \mathbb{E}\int_0^T |\Sigma|^2\,d[\mathcal{L}](t) &= \mathbb{E}\int_0^T |\Sigma|^2\,d\langle\mathcal{L}\rangle(t) \\ &= \int T\int_{\mathbb{R}^n} \left|\Sigma\begin{pmatrix}0\\x\end{pmatrix}\right|^2\,\nu(dx).\end{aligned}$$

By the Burkholder-Davis-Gundy inequality applied to the martingale $M_t = \int_0^t \langle X(s-), \Sigma\,\mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}}$, for $p=1$, there exists a constant c_1 such that

$$\begin{aligned}\mathbb{E}\sup_{t\leq T}\left|\int_0^t \langle X(s-), \Sigma\,\mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}}\right| &\leq c_1\mathbb{E}\left(\left[\int_0^\cdot \langle X(s-), \Sigma\,\mathrm{d}\mathcal{L}(s)\rangle_{\mathcal{H}}\right](T)\right)^{1/2} \\ &\leq c_1\mathbb{E}\left(\sup_{t\leq T}|X(t)|^2_{\mathcal{H}}\int_0^T |\Sigma|^2\,\mathrm{d}[\mathcal{L}](s)\right)^{1/2} \\ &\leq c_1\left(\varepsilon\mathbb{E}\sup_{t\leq T}|X(t)|^2_{\mathcal{H}} + \frac{1}{4\varepsilon}\mathbb{E}\int_0^T |\Sigma|^2\,\mathrm{d}[\mathcal{L}](s)\right) \\ &= c_1\varepsilon\mathbb{E}\sup_{t\leq T}|X(t)|^2_{\mathcal{H}} \\ &\quad + \frac{c_1T}{4\varepsilon}\int_{\mathbb{R}^n} |\Sigma|^2|x|^2\nu(\,\mathrm{d}x),\end{aligned} \tag{3.19}$$

where we have used Young's inequality. Then, by (3.18) and (3.19)

$$\begin{aligned}\mathbb{E}\sup_{t\le T}|X(t)|^2_{\mathcal{H}} \;\le\;& \mathbb{E}|x_0|^2_{\mathcal{H}} + 2c_0T\mathbb{E}\sup_{t\le T}|X(t)|^2_{\mathcal{H}} + 2c_1\varepsilon\mathbb{E}\sup_{t\le T}|X(t)|^2_{\mathcal{H}}\\ &+\Big(\frac{c_1}{2\varepsilon}+1\Big)\,T\int_{\mathbb{R}^n}|\Sigma|^2|x|^2\,\nu(dx),\end{aligned}$$

hence

$$\mathbb{E}\sup_{t\le T}|X(t)|^2_{\mathcal{H}} \le N\left[\mathbb{E}|x_0|^2_{\mathcal{H}} + T\left(1+\frac{c_1}{2\varepsilon}\right)\right] < +\infty,$$

where

$$N = N(c_0,c_1,T,\varepsilon) = \frac{1}{1-2c_0T-2c_1\varepsilon}.$$

Choosing $\varepsilon > 0$ and $T > 0$ such that $N < 1$, one obtains the claim for a small time interval. The extension to arbitrary time interval follows by classical extension arguments. □

5. FitzHugh-Nagumo type nonlinearity

Let us now consider the general case of a nonlinear drift term $\mathcal{F}$ which is a dissipative mapping with domain $D(\mathcal{F})$ strictly contained in $\mathcal{H}$. A method to solve equations such as (3.11) driven by Wiener noise is given in [**25**]: in that approach it is necessary to find a (reflexive Banach) space $\mathcal{V}$, continuously embedded in $\mathcal{H}$, which is large enough to contain the paths of the stochastic convolution, and, on the other hand, not too large so that it is contained in the domain of the nonlinearity $\mathcal{F}$. As discussed in section 3.1, in our setting the natural candidates for this space are $\mathcal{V} = (\mathbb{H}^1(0,1))^m\times\mathbb{R}^n$ and $\mathcal{E} = (C[0,1])^m\times\mathbb{R}^n$. Unfortunately, it is not possible to give a direct application of the results in [**25**, Section 5.5], as we do not have continuity in time of the stochastic convolution, but only a càdlàg property. Hence, we need a different approach to the problem, based on regularizations and weak convergence techniques.

Theorem 3.9. *Let us consider the equation*

$$\begin{aligned} \mathrm{d}X(t) &= [\mathcal{A}X(t) - \mathcal{F}(X(t))]\,\mathrm{d}t + \Sigma\,\mathrm{d}\mathcal{L}(t)\\ X(0) &= x_0 \end{aligned} \tag{3.20}$$

where $\mathcal{F}$ is a polynomial nonlinearity only defined on its domain $D(\mathcal{F})$, which is strictly smaller than $\mathcal{H}$. Then (3.20) admits a unique mild solution, denoted *by $X(t,x_0)$, which satisfies the estimate*

$$\mathbb{E}|X(t,x)-X(t,y)|^2 \le \mathbb{E}|x-y|^2.$$

for all initial data $x, y \in \mathcal{H}$.

Proof. As observed in section 3 above, there exists $\eta > 0$ such that $F+\eta I$ is m-dissipative. By a standard argument one can reduce to the case of $\eta = 0$ (see e.g. [**7**]), which we shall assume from now on, without loss of generality. Let us set, for $\lambda > 0$, $F_\lambda(u) = F((1+\lambda F)^{-1}(u))$ (Yosida regularization). $\mathcal{F}_\lambda$ is then defined in the obvious way.

Let $\mathcal{G}y = -\mathcal{A}y + \mathcal{F}(y)$. Then $\mathcal{G}$ is maximal monotone on $\mathcal{H}$. In fact, since $\mathcal{A}$ is self-adjoint, setting

$$\varphi(u) = \begin{cases} |\mathcal{A}^{1/2}u|^2, & u \in D(\mathcal{A}^{1/2}) \\ +\infty, & \text{otherwise,} \end{cases}$$

one has that φ is convex and $\mathcal{A} = \partial\varphi$.

Let us also set $F = \partial g$, where $g : \mathbb{R}^m \to \mathbb{R}$ is a convex function, the construction of which is straightforward. Well-known results on convex integrals (see e.g. [**7**, sec. 2.2]) imply that F on H is equivalently defined as $F = \partial I_g$, where

$$I_g(u) = \begin{cases} \displaystyle\int_{[0,1]^m} g(u(x))\,\mathrm{d}x, & \text{if } g(u) \in L^1([0,1]^m), \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us recall that

$$\mathcal{F} = \begin{pmatrix} -F \\ 0 \end{pmatrix}.$$

Since $D(\mathcal{F}) \cap D(\mathcal{A})$ is not empty, $\mathcal{G}$ is maximal monotone if $\varphi((I+\lambda\mathcal{F})^{-1}(u)) \le \varphi(u)$ (see e.g. [**13**, Thm. 9]), which is verified by a direct (but tedious) calculation using the explicit form of $\mathcal{A}$, since $(I+\lambda f_j)^{-1}$ is a contraction on $\mathbb{R}$ for each $j = 1, \dots, m$.

Let us consider the regularized equation

$$\begin{aligned} &\mathrm{d}X_\lambda(t) + \mathcal{G}_\lambda X_\lambda(t)\,\mathrm{d}t = \Sigma\,\mathrm{d}\mathcal{L}(t) \\ &X_\lambda(0) = x_0. \end{aligned}$$

where $\mathcal{G}_\lambda := -\mathcal{A} + \mathcal{F}_\lambda$. Appealing to Itô's formula for the square of the norm one obtains

$$|X_\lambda(t)|^2 + 2\int_0^t \langle \mathcal{G}_\lambda X_\lambda(s), X_\lambda(s)\rangle \,\mathrm{d}s = |x_0|^2 + 2\int_0^t \langle X_\lambda(s-), \Sigma\,\mathrm{d}\mathcal{L}(s)\rangle + [X_\lambda](t)$$

for all $t \in [0,T]$. Taking expectation on both sides yields

$$\begin{aligned}\mathbb{E}|X_\lambda(t)|^2 + 2\mathbb{E}\int_0^t \langle \mathcal{G}_\lambda X_\lambda(s), X_\lambda(s)\rangle \,\mathrm{d}s &= \mathbb{E}|x_0|^2 + \int_0^t \|\Sigma Q^{1/2}\|_{HS}^2 \,\mathrm{d}s \\ &\quad + t\int_{\mathbb{R}^n} |\Sigma|^2\,|z|^2\,\nu(\mathrm{d}z), \quad (3.21)\end{aligned}$$

where we have used the identity

$$[X_\lambda](t) = \Big[\int_0^\cdot \Sigma\,\mathrm{d}\mathcal{L}(s)\Big](t) = \int_0^t \|\Sigma Q^{1/2}\|_{HS}^2\,\mathrm{d}s + t\int_{\mathbb{R}^n} |\Sigma|^2\,|z|^2\,\nu(\mathrm{d}z).$$

Let us define the space $\mathbb{L}^p$ as the set of $\mathcal{H}$ valued random variables with finite p-th moment. Therefore, since by (3.21) we have that $\{X_\lambda\}$ is a bounded subset of $L^\infty([0,T],\mathbb{L}^2)$, and $\mathbb{L}^2$ is separable, Banach-Alaoglu's theorem implies that

$$X_\lambda \overset{*}{\rightharpoonup} X \qquad \text{in } L^\infty([0,T],\mathbb{L}^2),$$

on a subsequence still denoted by λ. Thanks to the assumptions on f_j, one can easily prove that $\langle F(u), u\rangle \le c|u|^{p+1}$ for some $c > 0$ and $p \ge 1$, hence (3.21) also gives

$$\mathbb{E}\int_0^T |X_\lambda(s)|_{p+1}^{p+1}\,\mathrm{d}s < C,$$

which implies that

$$X_\lambda \rightharpoonup X \qquad \text{in } L^{p+1}(\Omega \times [0,T] \times D, \mathbb{P} \times \mathrm{d}t \times \mathrm{d}\xi), \tag{3.22}$$

where $D = [0,1]^m \times \mathbb{R}^n$. Furthermore, (3.21) and (3.22) also imply

$$\mathcal{G}_\lambda(X_\lambda) \rightharpoonup \eta \qquad \text{in } L^{\frac{p+1}{p}}(\Omega \times [0,T] \times D, \mathbb{P} \times \mathrm{d}t \times \mathrm{d}\xi).$$

The above convergences immediately imply that X and η are predictable, then in order to complete the proof of existence, we have to show that $\eta(\omega,t,\xi) = \mathcal{G}(X(\omega,t,\xi))$, $\mathbb{P} \times \mathrm{d}t \times \mathrm{d}\xi$-a.e.. For this it is enough to show that

$$\limsup_{\lambda\to 0} \mathbb{E}\int_0^T \langle \mathcal{G}_\lambda X_\lambda(s), X_\lambda(s)\rangle\,\mathrm{d}s \le \mathbb{E}\int_0^T \langle \eta(s), X(s)\rangle\,\mathrm{d}s.$$

Using again Itô's formula we get

$$\mathbb{E}|X(T)|^2 + 2\mathbb{E}\int_0^T \langle \eta(s), X(s)\rangle \, \mathrm{d}s = \mathbb{E}|x_0|^2 + T\int_{\mathbb{R}^n} |\Sigma|^2\,|z|^2\,\nu(\,\mathrm{d}z). \quad (3.23)$$

However, (3.22) implies that

$$\liminf_{\lambda\to 0} \mathbb{E}|X_\lambda(T)|^2 \geq \mathbb{E}|X(T)|^2$$

(see e.g. [**14**, Prop. 3.5]), from which the claim follows comparing (3.21) and (3.23).

The Lipschitz dependence on the initial datum as well as (as a consequence) uniqueness of the solution is proved by observing that $X(t,x)-X(t,y)$ satisfies $\mathbb{P}$-a.s. the deterministic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(X(t,x)-X(t,y)) = \mathcal{A}(X(t,x)-X(t,y)) - \mathcal{F}(X(t,x)) + \mathcal{F}(X(t,y)),$$

hence

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|X(t,x)-X(t,y)|^2 &= \big\langle \mathcal{A}(X(t,x)-X(t,y)), X(t,x)-X(t,y)\big\rangle \\ &\quad -\big\langle \mathcal{F}(X(t,x)-\mathcal{F}(X(t,y)), X(t,x)-X(t,y)\big\rangle \\ &\leq \eta|X(t,x)-X(t,y)|^2,\end{aligned}$$

where $X(\cdot,x)$ stands for the mild solution with initial datum x. By the Gronwall lemma

$$\mathbb{E}|X(t,x)-X(t,y)|^2 \leq e^{2\eta t}\mathbb{E}|x-y|^2$$

which concludes the proof of the theorem. □

REMARK 3.6. *By arguments similar to those used in the proof of theorem 3.8 one can also obtain that*

$$\mathbb{E}\sup_{t\leq T}|X_\lambda(t)|^2 < C,$$

i.e. that $\{X_\lambda\}$ *is bounded in* $\mathbb{L}^2(\Omega; L^\infty([0,T];\mathcal{H}))$*. By means of Banach-Alaoglu's theorem, one can only conclude that* $X_\lambda \overset{*}{\rightharpoonup} X$ *in* $\mathbb{L}^2(\Omega; L^1([0,T];\mathcal{H}))'$*, which is larger than* $\mathbb{L}^2(\Omega; L^\infty([0,T];\mathcal{H}))$*. In fact, from* [**29**, Thm. 8.20.3]*, being* $L^1([0,T];\mathcal{H})$ *a separable Banach space, one can only prove that if* F *is a continuous linear form on* $\mathbb{L}^2(\Omega; L^1([0,T];\mathcal{H}))$*, then there exists a function* f *mapping* Ω *into* $L^\infty([0,T];\mathcal{H})$ *that is weakly measurable and such that*

$$F(g) = \mathbb{E}\langle f, g\rangle$$

for each $g \in \mathbb{L}^2(\Omega; L^1([0,T]; \mathcal{H}))$.

CHAPTER 4

Longtime behaviour for SPDEs with R.D.B.C.

This chapter is mainly taken from the paper [**12**].

The purpose of this chapter is to investigate existence, uniqueness and asymptotic behaviour in time of solutions to a stochastic linear problem of heat conduction in the interior of materials and dynamical conditions on the boundary. In particular we prove the existence of an invariant measure for the weak solution (see e.g. [**23**]). Consider a rigid, isotropic, homogeneous heat conductor which occupies a bounded domain $\mathcal{O} \subset \mathbb{R}^3$, for $t \geq 0$, in presence of a random source of heat supply acting on the boundary. In this model we suppose that the boundary has a certain thickness and a sufficiently large thermal conductivity, so that to permit heat exchange between internal and boundary material and to consider on this one a different diffusion process of its own.

We will show how the resulting system of stochastic differential equations can be treated in the framework of using techniques arising in the theory of matrix operators on Hilbert spaces and of stochastic equations in infinite dimensions. In particular we shall determine under which conditions the deterministic system becomes exponentially stable.

Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded set with smooth boundary $\partial\mathcal{O}$ of class C^2. We assume that $\mathcal{O}$ is the region occupied by a rigid heat conductor, such as a metallic body, embedded in a moving fluid.

A pure diffusion mechanism acts for the temperature $u(t,x)$ inside $\mathcal{O}$ according to Fourier's law

$$\partial_t u(t,x) = \Delta u(t,x), \qquad t > s,\ x \in \mathcal{O}. \tag{4.1}$$

for some initial time $s \in \mathbb{R}$.

In the traditional approach, equation (4.1) is endowed with boundary conditions either in the form of Dirichlet boundary conditions (when the temperature is given at the boundary) or of Neumann boundary conditions (when the heat flow is given at the boundary). However, neither of these conditions is suitable in order to model the case of a heat source located on the boundary, since the amount of heat in the region $\mathcal{O}$ shall take into account the action of the heat source on the boundary. For instance, we are taking into account the case when the outer surface of the region is covered with a conducting material which contains sensors and heat sources. According to [**39**], depending on the physical assumptions given on the system, all kind of standard, as well as dynamical boundary conditions arise naturally in the formulation of the problem. Let us denote $\phi(t, \xi, u, \Delta u)$ the heat source acting on the boundary: then an application of Fourier's law implies that the identity

$$\partial_t u(\xi, t) = \phi(t, \xi, u, \partial_\nu u) \qquad t > s, \ \xi \in \partial\mathcal{O}$$

represents the boundary condition associated to (4.1).

Further, we assume that heat flow can occur across the boundary together with a dissipative condition and a heat source. Denote by ∂_ν the outward normal derivative on the boundary; the heat source thus takes the form

$$\phi(t, \xi, u, \partial_\nu u) = \partial_\nu u(t, \xi) - \mu u(t, \xi) - g(u(t, \xi)) + F(t, \xi)$$

for a rapidly varying heat source $F(t, \xi)$ that we model by setting $F(t, \cdot) = Q^{1/2} \dfrac{\mathrm{d}W(t)}{\mathrm{d}t}$, where $\{W(t), \ t \geq 0\}$ is a cylindrical Wiener process on $L^2(\partial\mathcal{O})$ and Q is a linear, bounded, symmetric, trace class operator on $L^2(\partial\mathcal{O})$.

HYPOTHESIS 4.1. *In our model, we assume that there exists a finite number of heat sources, located at points $\{\xi_k, \ k = 1, \dots, N\}$ on $\partial\mathcal{O}$, such that*

$$F(\xi, t) = \sum_{k=1}^{N} d(\xi, \xi_k) \dot\beta_k(t)$$

for given functions $d_k(\xi) = d(\xi, \xi_k)$ in $L^2(\partial\mathcal{O}) \cap L^\infty(\partial\mathcal{O})$, and the random processes $\{\beta_k, \ k = 1, \dots, N\}$ are independent, two-sided real-valued Brownian motions on a probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$.

HYPOTHESIS 4.2. *We assume that μ is a strictly positive constant while the nonlinear part of the heat supply $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying*

(1) $|g(u)| \leq k_1(1 + |u|^{2p-1})$,
(2) $u \cdot g(u + v) \geq -k_2|v|^{2p} + k_3|u|^{2p}$,

(3) $g'(u) \geq -k_4$

for some $p \in \mathbb{N}$, $k_j \geq 0$, $j = 1, 2, 3, 4$ *and* $k_4 < \mu$.

REMARK 4.1. *As a particular case of nonlinearity satisfying Hypothesis 4.2 we mention the case that g is a polynomial of odd degree with positive leading coefficient*

$$g(v) = \sum_{k=1}^{2p-1} g_k v^k, \qquad g_{2p-1} > 0, \quad p \in \mathbb{N}. \tag{4.2}$$

The initial-boundary value problem that we take into account connects the internal and the boundary dynamics, as well as the random heat source acting on the system. Setting for simplicity $v(t,\xi) = u(t,\xi)$ the value of the temperature on the boundary, we get:

$$\begin{aligned} &\partial_t u(t,x) = \Delta u(t,x) && \text{on } \mathcal{O} \times [s,+\infty) \\ &\partial_t v(t,\xi) = \partial_\nu u(t,\xi) - \mu v(t,\xi) - g(v(t,\xi)) + F(t,\xi) && \text{on } \partial\mathcal{O} \times [s,\infty) \\ &u(s,x) = u_s(x), \quad v(s,\xi) = v_s(\xi). \end{aligned} \tag{4.3}$$

Our concern is to convert system (4.3) in an abstract Cauchy problem in a suitable product space $\mathcal{H}$ and prove that the linear part of the system is well posed (i.e., prove generation properties for the associated matrix operator). Then, our main result Theorem 4.9 will provide the existence of a unique solution $\{w(t) = (u(t), v(t)),\ t \geq s\}$ for problem (4.3) as a continuous and adapted process in $\mathcal{H}$.

The structure of problem (4.3) allows the introduction on the Hilbert space $\mathcal{H}$ of a family of mappings $S(t,s,\omega) : \mathcal{H} \to \mathcal{H}$, $s \leq t$, $\omega \in \Omega$, which defines the random dynamical system for the problem (see e.g. [**6**]), in the sense that $S(t,s,\omega)\bar{z}$ is the unique solution of (4.3) with initial condition $\bar{z}$ at time $s \in \mathbb{R}$. We obtain that the family S satisfies $\mathbb{P}$-a.s. the following properties:

(1) $S(t,r,\omega)S(r,s,\omega)\bar{z} = S(t,s,\omega)\bar{z}$, for all $s < r < t$ and $\bar{z} \in \mathcal{H}$,

(2) $t \mapsto S(t,s,\omega)\bar{z}$ is continuous in $\mathcal{H}$ for all $t > s$.

The basic definition of attracting set arise naturally in the theory of random attractors, as introduced in [**20, 21**]. For given $t \in \mathbb{R}$ and $\omega \in \Omega$, we say that $K(t,\omega)$ is an ATTRACTING SET if, for all bounded subsets $B \subset \mathcal{H}$,

$$\|K(t,\omega), S(t,s,\omega)B\|_{\mathcal{H}} \to 0 \quad \text{as } s \to -\infty. \tag{4.4}$$

Moreover, a generalized *invariance* property is required:

$$S(t,s,\omega)K(s,\omega)=K(t,\omega). \tag{4.5}$$

Relation (4.4) is called the PULL-BACK convergence to $K(t,\omega)$. Unfortunately, a random absorbing set in the sense of (4.4) does not have a straightforward physical interpretation as in the deterministic case. We shall remark that classical theory of random attractors requires a compactness assumption for $K(t,\omega)$ which is not achieved here.

If there exists a random set $A(\omega)$ which is a compact measurable forward invariant set for the random dynamical system $S(t,s,\omega)$, then there exist invariant measures for φ which are supported by A. The compactness of the set is needed to apply a fixed point result (Markov-Kakutani) on the family of measures supported by the random set.

So universally or globally attracting sets, that are compact and invariant, can be considered the natural link between the theory of random attractors and invariant measures. These sets attract $\mathbb{P}$-a.s. every bounded deterministic set and are the support of some invariant measure.

In our problem we prove existence of an invariant random set absorbing every bounded deterministic set, but unfortunately it is not compact in the topology of $\mathcal{H}$. So we are not in presence of a globally attracting set and the lack of compactness compels us to show existence of an invariant measure in another way.

Longtime behaviour of the solutions of stochastic partial differential equations has been widely investigated using procedures arising from the general theory of semigroups applied to stochastic systems. In our case, such approach leads to prove the existence of a unique invariant measure ν for Eq. (4.3) in the space $\mathcal{H}=L^2(\mathcal{O})\times L^2(\partial\mathcal{O})$ which is strongly mixing, i.e., the Markov transition semigroup P_t associated with the solution $w(t)$ of Eq. (4.3), defined by

$$P_t\phi(x)=\mathbb{E}[\phi(w(t;s,x))],$$

converges weakly in $L^2(\mathcal{H},\nu)$ to the space average of ϕ:

$$\lim_{t\to\infty}P_t\phi=\int_{\mathcal{H}}\phi\,\mathrm{d}\nu,\qquad \phi\in L^2(\mathcal{H},\nu).$$

1. Setting of the problem

For the reader's convenience, we briefly recall some results on Sobolev spaces of functions on $\mathcal{O}$ and $\partial\mathcal{O}$.

For $\mathcal{O} \subset \mathbb{R}^3$ being a bounded domain with smooth boundary $\partial\mathcal{O}$, we fix as reference space $L^2(\mathcal{O})$, the set of square integrable functions acting on $\mathcal{O}$ and similarly $L^2(\partial\mathcal{O})$ is the space of square integrable functions acting on the boundary $\partial\mathcal{O}$. Recall that for every $s > 1/2$, the function $u \mapsto u_{|\partial\mathcal{O}}$ from $C(\bar{\mathcal{O}})$ to $C(\partial\mathcal{O})$ has a unique extension to a linear bounded operator $L : H^s(\mathcal{O}) \to H^{s-1/2}(\mathcal{O})$. Also, for every $s > 3/2$, the function $u \mapsto \partial_\nu u$ from $C^1(\bar{\mathcal{O}})$ to $C(\partial\mathcal{O})$ has a unique extension to a linear bounded operator $N : H^s(\mathcal{O}) \to H^{s-3/2}(\partial\mathcal{O})$.

Next result connects the norm in the interior and the trace on the boundary. As a general reference, we quote [**76**].

Proposition 4.3 (Friedrichs). *Let $\mathcal{O}$ be a nonempty, open and bounded subset in $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{O}$ of class C^1. Then there exists $k \geq 1$ such that*

$$\frac{1}{k}\|u\|^2_{H^1(\mathcal{O})} \leq \|\nabla u\|^2_{L^2(\mathcal{O})} + \|u_{|\partial\mathcal{O}}\|^2_{L^2(\partial\mathcal{O})} \leq k\|u\|^2_{H^1(\mathcal{O})}$$

for each $u \in H^1(\mathcal{O})$.

We shall also use the following result, compare [[**58**], pg.135]. Define for all $\phi \in H^{1/2}(\partial\mathcal{O})$ the function $D_0\phi = \psi \in H^1(\mathcal{O})$ as the solution of the boundary problem

$$\begin{aligned} &\Delta\psi = 0 \qquad \text{inside } \mathcal{O}, \\ &\psi|_{\partial\mathcal{O}} = \phi. \end{aligned}$$

Then it is possible to define $\partial_\nu\psi \in H^{-1/2}(\partial\mathcal{O})$ and the following holds.

Lemma 4.4. *For every $\lambda > 0$ there exists $\alpha > 0$ such that the following estimate holds for every $\phi \in H^{1/2}(\partial\mathcal{O})$:*

$$\langle \partial_\nu D_0\phi, \phi\rangle_{L^2(\partial\mathcal{O})} + \lambda\|\phi\|^2_{L^2(\partial\mathcal{O})} \geq \alpha\|\phi\|^2_{H^{1/2}(\partial\mathcal{O})}. \tag{4.6}$$

1.1. A generation result. On the space $\mathcal{H} = L^2(\mathcal{O}) \times L^2(\partial\mathcal{O})$, the system of internal and boundary dynamics defines a linear operator $A : D(A) \subset$

$\mathcal{H} \to \mathcal{H}$,

$$A = \begin{pmatrix} \Delta & 0 \\ -\partial_\nu & -\mu I \end{pmatrix} \tag{4.7}$$

with domain

$$D(A) = \{(u,v) \in H^{3/2}(\mathcal{O}) \times H^1(\partial\mathcal{O}) \,:\, \Delta u \in L^2(\mathcal{O}),\ \partial_\nu u \in L^2(\partial\mathcal{O}),\ u_{|\partial\mathcal{O}} = v\}. \tag{4.8}$$

For the sake of completeness we recall the regularity properties for the evolution equation $\partial_t u = Au$.

Proposition 4.5. *The matrix operator $(A, D(A))$ defined in* (4.7)–(4.8) *is the infinitesimal generator of a compact and analytic C_0-semigroup of contraction.*

Remark 4.2. *We sketch the proof given in Vrabie* [**76**], *see also* [**8**]. *Other approaches are given in the literature, see for instance the papers* [**32**], [**75**] *and* [**66**].

We begin with a simple lemma, whose proof is based on Lax-Milgram theorem, see [**76**, Lemma 7.4.1].

Lemma 4.6. *For any $\gamma \geq 0$, $\lambda > 0$, consider the elliptic problem for $f \in L^2(\mathcal{O})$, $g \in L^2(\partial\mathcal{O})$*

$$\begin{aligned} &\gamma u - \Delta u = f, \\ &\lambda v + \partial_\nu u = g, \\ &u_{|\partial\mathcal{O}} = v. \end{aligned} \tag{4.9}$$

Then there exists a unique solution $u \in H^{3/2}(\mathcal{O})$, $v \in H^1(\partial\mathcal{O})$, $\Delta u \in L^2(\mathcal{O})$ and $\partial_\nu u \in L^2(\partial\mathcal{O})$.

Proof of Proposition 4.5. We first notice that $D(A)$ is dense in $\mathcal{H} = L^2(\mathcal{O}) \times L^2(\partial\mathcal{O})$.

For every $\lambda > 0$, $(f,g) \in \mathcal{H}$, the equation

$$(\lambda I - A) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

has a unique solution, by Lemma 4.6; taking the scalar product in both lines of (4.9), respectively with u and v, implies that

$$\lambda \|u\|^2_{L^2(\mathcal{O})} + \|\nabla u\|^2_{L^2(\mathcal{O})} - \langle \partial_\nu u, u \rangle_{L^2(\partial\mathcal{O})} \le \langle f, u \rangle_{L^2(\mathcal{O})}$$
$$\lambda \|v\|^2_{L^2(\partial\mathcal{O})} + \langle \partial_\nu u, u \rangle_{L^2(\partial\mathcal{O})} \le \langle g, u \rangle_{L^2(\partial\mathcal{O})}$$

which leads to

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \le \frac{1}{\lambda}$$

for every $\lambda > 0$, and also implies that $(0, +\infty) \subset \rho(A)$. An application of Hille-Yosida theorem implies that A is the generator of a C_0-semigroup of contractions $\{S(t),\ t \ge 0\}$.

Further, a simple computation shows that A is symmetric and, since it is $(I - A)^{-1} \in \mathcal{L}(\mathcal{H})$, it follows also that A is self-adjoint, hence the semigroup $S(t)$ is analytic. Finally, it is easily seen that $D(A)$, endowed with the graph norm, is compactly embedded in $\mathcal{H}$, hence the semigroup $S(t)$ is compact. □

2. The stochastic problem

Let $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^N) \mid \omega(0) = 0\}$ be the Wiener space and $\mathbb{P}$ is the product measure of two Wiener measures on the negative and the positive parts of Ω. On Ω we define the time-shift θ setting

$$\theta_s \omega(t) = \omega(t + s) - \omega(s), \qquad t, s \in \mathbb{R}.$$

$\{\theta_t,\ t \in \mathbb{R}\}$ is a family of ergodic transformations.

We now rewrite system (4.3) as a stochastic differential equation in $\mathcal{H} = L^2(\mathcal{O}) \times L^2(\partial\mathcal{O})$ for the unknown $z = (u, v)$:

$$\begin{aligned} \mathrm{d}z &= [Az(t) + G(z(t))]\,\mathrm{d}t + \sum_{k=1}^N D_k \mathrm{d}\beta_k(t), \qquad t \ge s \\ z(s) &= \bar{z}. \end{aligned} \tag{4.10}$$

The mapping G is defined as $G(u, v) = (0, -g(v))$ and we assume that $D_k = (0, d_k)$, for $k = 1, \dots, N$, belongs to $D(A_m)$, where A_m is the matrix operator

$$A_m = \begin{pmatrix} \Delta & 0 \\ -\partial_\nu & -\mu I \end{pmatrix}$$

with maximal domain

$$D(A_m) = \{(u, v) \in H^{3/2}(\mathcal{O}) \times H^1(\partial\mathcal{O}) \,:\, \Delta u \in L^2(\mathcal{O}),\ \partial_\nu u \in L^2(\partial\mathcal{O})\}.$$

We introduce the stationary Ornstein-Uhlenbeck process $\zeta = \{\zeta(t),\ t \in T\}$ in $L^2(\partial\mathcal{O})$ setting

$$\zeta(t) = \sum_{k=1}^{N} d_k \zeta_k(s) = \sum_{k=1}^{N} d_k \int_{-\infty}^{t} e^{-\mu(t-s)} \,\mathrm{d}\beta_k(s), \qquad t \in \mathbb{R}, \tag{4.11}$$

where $\{d_k,\ k = 1, \dots, N\}$ are given functions in $L^2(\partial\mathcal{O})$. When no confusion can arise, we write ζ also the process in $\mathcal{H} = L^2(\mathcal{O}) \times L^2(\partial\mathcal{O})$ defined by the vector $(0, \zeta)$. Notice also that $A_m \zeta = -\mu\zeta$. For further reference, we state the following result on the pathwise behaviour of the process $\zeta(t)$.

Lemma 4.7. *Assume that $\{d_k,\ k = 1, \dots, N\}$ is a bounded sequence in $L^2(\partial\mathcal{O}) \cap L^\infty(\partial\mathcal{O})$. For every $T > 0$ and $p \in \mathbb{N}$ it holds*

$$\int_0^T \|\zeta(s)\|_{L^{2p}(\partial\mathcal{O})}^{2p} \,\mathrm{d}s < \infty \qquad \mathbb{P}\text{-}a.s.$$

Proof. It is a standard computation to show that

$$\|\zeta(s)\|_{L^{2p}(\partial\mathcal{O})}^{2p} \le c_1 \sum_{k=1}^{N} |\zeta_k(s)|^{2p}$$

for a constant c_1 depending on meas$(\partial\mathcal{O})$, N, n and the sup-norm of the family $\{d_k\}$. Then it follows

$$\mathbb{E} \int_0^T \|\zeta(s)\|_{L^{2p}(\partial\mathcal{O})}^{2p} \,\mathrm{d}s < c_2 T$$

with $c_2 = c_1(2p-1)!$, which implies that $\mathbb{P}$-a.s. the random variable is finite, as required. □

In order to study equation (4.10) we apply a random change of variables, setting $w = z - \zeta$: the unknown w satisfies a differential equation with random coefficients

$$\frac{\mathrm{d}w}{\mathrm{d}t} = Aw(t) + G(w(t) + \zeta(t)) \tag{4.12}$$

and initial condition (given at time $s = 0$ for simplicity)

$$w(0) = \bar{z} - \zeta(0), \qquad \mathbb{P}\text{-a.s.}$$

Theorem 4.8. *For $s = 0$ and $\bar{z} \in \mathcal{H}$, $\mathbb{P}$-a.s., there exists a unique weak solution $w(t) = w(t, 0, \omega)$, $t \in [0, T]$, for equation (4.12) such that:*

(i) *$w(t) = (u(t), v(t))$ belongs to $L^2(0, T; \mathcal{H})$, and*

(ii) *for every vector* $(\sigma, \xi) \in D(A)$ *it holds*

$$\langle w(t), (\sigma, \xi) \rangle_{\mathcal{H}} = \langle w(0), (\sigma, \xi) \rangle_{\mathcal{H}} + \int_0^t \langle w(s), A(\sigma, \xi) \rangle_{\mathcal{H}} \, \mathrm{d}s \\ + \int_0^t \langle (\sigma, \xi), G(w(s) + \zeta(s)) \rangle_{\mathcal{H}} \, \mathrm{d}s.$$

Proof. *Existence.* We follow a standard Faedo-Galerkin scheme. Let $\{0 = x_0,\ x_j,\ j \geq 1\}$ be an orthonormal basis of $L^2(\mathcal{O})$ where the vectors $x_j,\ j \geq 1$ are eigenvalues of the Dirichlet Laplacian operator on $\mathcal{O}$, hence $\langle \Delta x_j, x_j \rangle = -\nu_j < 0$ and $\langle \Delta x_j, x_k \rangle = 0$ whenever $k \neq j$. Let further $\{0 = y_0,\ y_j,\ j \geq 1\}$ be an orthonormal basis of $L^2(\partial\mathcal{O})$.

For every given $n \in \mathbb{N}$ we denote by P_n and Q_n the projections on the subspaces

$$\mathrm{Span}\{x_1, \dots, x_n\} \subset L^2(\mathcal{O}) \quad \text{and} \quad \mathrm{Span}\{y_1, \dots, y_n\} \subset L^2(\partial\mathcal{O}).$$

Then we search for a function $w_n = (u_n, v_n) \in \mathcal{H}$ having the form

$$u_n(t) = \sum_{k=1}^n \alpha_k(t) x_k, \qquad v_n(t) = \sum_{k=1}^n \beta_k(t) y_k$$

that solves the system

$$\begin{aligned} \langle \tfrac{\mathrm{d}}{\mathrm{d}t} w_n(t), (\sigma, \xi) \rangle_{\mathcal{H}} &= \langle w_n(t), A(\sigma, \xi) \rangle_{\mathcal{H}} + \langle G(w_n(t) + \zeta(t)), (\sigma, \xi) \rangle_{\mathcal{H}} \\ w_n(0) &= (P_n(\bar{z} - \zeta(0)), Q_n(\bar{z} - \zeta(0))) \end{aligned} \tag{4.13}$$

for every $(\sigma, \xi) \in D(A)$. Choosing repeatedly $(x_k, 0)$ and $(D_0 y_k, y_k)$ in (4.13), for $k = 1, \dots, n$, we obtain the system of equations

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \alpha_k(t) &= -\nu_k \alpha_k(t) \\ \tfrac{\mathrm{d}}{\mathrm{d}t} \beta_k(t) &= -\mu \beta_k(t) - \sum_{j=1}^n \beta_j(t) \langle y_j, \partial_\nu D_0 y_k \rangle_{L^2(\partial\mathcal{O})} \\ &\quad - \langle g(v_n(t) + \zeta(t)), y_k \rangle_{L^2(\partial\mathcal{O})} \end{aligned} \tag{4.14}$$

with initial conditions

$$\alpha_k(0) = \langle \bar{z} - \zeta(0), x_k \rangle, \qquad \beta_k(0) = \langle \bar{z} - \zeta(0), y_k \rangle. \tag{4.15}$$

According to standard theory for ordinary differential equations there exists a unique continuous solution for the system (4.14) in some interval $[0, T_n]$.

Next step is to give a priori estimates for the solutions of (4.14). We multiply both sides of first equality in (4.14) by $\alpha_k(t)$ and second equality by $\beta_k(t)$; taking the sum in both equations as $k = 1, \dots, n$ we obtain

$$\begin{aligned} &\frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|^2 \le -\nu_1 \|u_n(t)\|^2 \\ &\frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|v_n(t)\|^2 = -\langle v_n(t), \partial_\nu D_0 v_n(t)\rangle_{L^2(\partial\mathcal{O})} - \mu\|v_n(t)\|^2 \\ &\qquad\qquad - \langle g(v_n(t)+\zeta(t)), v_n(t)\rangle_{L^2(\partial\mathcal{O})}. \end{aligned} \tag{4.16}$$

where $-\nu_1$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions. Recall from Hypothesis 4.2 that $g(v_n(t)+\zeta(t))v_n(t) \ge -k_2|\zeta(t)|^{2p} + k_3|v_n(t)|^{2p}$ for all $p \in \mathbb{N}$; Poincaré inequality implies that for some $\lambda_0 > 0$ it holds

$$\|x_k\|_{L^2(\mathcal{O})} \le \lambda_0 \|\nabla x_k\|^2_{L^2(\mathcal{O})}$$

and choosing suitable λ_1, $\lambda = \min\{\frac{1}{2}\nu_1, \frac{1}{2}\mu - 2\lambda_1\}$, α from estimate (4.6),

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}\|w_n(t)\|^2}{\mathrm{d}t} + \lambda\|w_n(t)\|^2 + \frac{\nu_1}{2\lambda_0}\|\nabla u_n(t)\|^2 + \alpha\|v_n(t)\|^2_{H^{1/2}(\partial\mathcal{O})} \\ + k_3\|v_n(t)\|^{2p}_{L^{2p}(\partial\mathcal{O})} \le \|\zeta(t)\|^{2p}_{L^{2p}(\partial\mathcal{O})}. \end{aligned} \tag{4.17}$$

Application of Gronwall's lemma leads to the estimates

$$\|w_n(t)\|^2_{\mathcal{H}} \le \mathrm{e}^{-\lambda t}\|w_n(0)\|^2_{\mathcal{H}} + \int_0^t \mathrm{e}^{-\lambda(t-s)}\|\zeta(s)\|^{2p}_{L^{2p}(\partial\mathcal{O})}\,\mathrm{d}s$$

as well as

$$\begin{aligned} \frac{\nu_1}{2\lambda_0}\int_0^t \|\nabla u_n(s)\|^2_{L^2(\mathcal{O})}\,\mathrm{d}s + \alpha\int_0^t \|v_n(s)\|^2_{H^{1/2}(\partial\mathcal{O})}\,\mathrm{d}s + k_3\int_0^t \|v_n(s)\|^{2p}_{L^{2p}(\mathcal{O})}\,\mathrm{d}s \\ \le \|w_n(0)\|^2_{\mathcal{H}} + \int_0^t \|\zeta(s)\|^{2p}_{L^{2p}(\partial\mathcal{O})}\,\mathrm{d}s. \end{aligned}$$

In particular we see that

$$\begin{aligned} &\{w_n,\ n\in\mathbb{N}\} \quad \text{is bounded in} \quad L^\infty(0,T;\mathcal{H}) \\ &\{u_n,\ n\in\mathbb{N}\} \quad \text{is bounded in} \quad L^2(0,T;H^1(\mathcal{O})) \\ &\{v_n,\ n\in\mathbb{N}\} \quad \text{is bounded in} \quad L^2(0,T;H^{1/2}(\partial\mathcal{O})) \cap L^{2p}(0,T;L^{2p}(\partial\mathcal{O})). \end{aligned} \tag{4.18}$$

Upon passing to a subsequence there exists a function $w = (u,v)$ such that $(u_n, v_n) \stackrel{*}{\rightharpoonup} (u,v)$ in $L^\infty(0,T;\mathcal{H})$.

In order to apply a compactness argument for v_n it is necessary to find an estimate for the time derivative v_n'. From Eq. (4.13) we obtain

$$\langle v_n', y\rangle = \langle -\partial_\nu D_0 v_n, y\rangle - \mu\langle v_n, y\rangle - \langle g(v_n+\zeta), y\rangle$$

and taking y in the subdifferential of $|v_n'|$ we deduce that

$$|v_n'|_{L^2(0,T;H^{-1/2}(\partial\mathcal{O}))} \le c_1\,|\partial_\nu D_0 v_n|_{L^2(0,T;H^{-1/2}(\partial\mathcal{O}))} + |v_n|_{L^2(0,T;L^2(\partial\mathcal{O}))} + c_2|g(v_n+\zeta)|_{L^{2q}(0,T;L^{2q}(\partial\mathcal{O}))}$$

where $2q$ is the conjugate exponent of $2p$, $2q = 2p/(2p-1)$. Notice that $|g(x)|^{2q} \le c(1+|x|^{2p})$ by Hypothesis 4.2(1.), hence

$$|g(v_n+\zeta)|_{L^{2q}(0,T;L^{2q}(\partial\mathcal{O}))} \le c(1+|v_n|^{2p}_{L^{2p}(0,T;L^{2p}(\partial\mathcal{O}))} + |\zeta|^{2p}_{L^{2p}(0,T;L^{2p}(\partial\mathcal{O}))})$$

which is bounded, uniformly in n, thanks to Lemma 4.7 and (4.18). Further, the operator $\partial_\nu D_0$ is a linear bounded operator between $H^{1/2}(\partial\mathcal{O})$ and $H^{-1/2}(\partial\mathcal{O})$ hence, again by (4.18), we have that

$$\{v_n,\ n\in\mathbb{N}\}\quad\text{is bounded in}\quad L^2(0,T;H^{-1/2}(\partial\mathcal{O})).$$

Now choose $s \ge \frac{1}{2}$ such that $H^s(\partial\mathcal{O}) \subset L^{2p}(\partial\mathcal{O})$ and notice that the injections $H^{-1/2}(\partial\mathcal{O}) \to H^{-s}(\partial\mathcal{O})$ and $L^{2q}(\partial\mathcal{O}) \to H^{-s}(\partial\mathcal{O})$ are bounded. We use then the compactness theorem of Lions [**58**, Théorème I.5.1], see also [**58**, Théorème I.11.1], to get the existence of a subsequence (again denoted by $\{v_n\}$) that converges strongly in $L^2(0,T;L^2(\partial\mathcal{O}))$ and almost everywhere. Further, directly from (4.18), since the injection $H^1(\mathcal{O}) \to L^2(\mathcal{O})$ is compact, we get that there exists a subsequence $\{u_n\}$ that converges strongly in $L^2(0,T;L^2(\mathcal{O}))$. All told, this means that (upon passing to a subsequence) the sequence $\{w_n = (u_n, v_n)\}$ converges strongly to $w = (u,v)$ in $L^2(0,T;\mathcal{H})$.

Finally, the thesis follows by a standard passage to the limit. Choose a smooth function $(\sigma,\xi) \in L^2(0,T;D(A))$ and substitute (σ,ξ) to (x_k, y_j) in (4.13) to get

$$\langle \frac{\mathrm{d}w_n(t)}{\mathrm{d}t}, (\sigma(t),\xi(t))\rangle_{\mathcal{H}} = \langle Aw_n(t), (\sigma(t),\xi(t))\rangle_{\mathcal{H}} + G(w_n(t)+\zeta(t)), (\sigma(t),\xi(t))\rangle_{\mathcal{H}}$$

Integrating over $(0,T)$, using the symmetry of A, we have

$$\langle w_n(t),(\sigma,\xi)\rangle_{\mathcal{H}} = \langle w_n(0),(\sigma,\xi)\rangle_{\mathcal{H}} + \int_0^t \langle w_n(s), A(\sigma,\xi)\rangle_{\mathcal{H}}\,\mathrm{d}s + \int_0^t \langle (\sigma,\xi), G(w_n(s)+\zeta(s))\rangle_{\mathcal{H}}\,\mathrm{d}s$$

and passing to the limit as $n\to\infty$, in force of (4.18) we get

$$\langle w(t),(\sigma,\xi)\rangle_{\mathcal{H}} = \langle w(0),(\sigma,\xi)\rangle_{\mathcal{H}} + \int_0^t \langle w(s), A(\sigma,\xi)\rangle_{\mathcal{H}}\,\mathrm{d}s - \lim_{n\to\infty}\int_0^t \langle \xi, g(v_n(s)+\zeta(s))\rangle_{L^2(\partial\mathcal{O})}\,\mathrm{d}s. \tag{4.19}$$

We finally claim that

$$\lim_{n\to\infty}\int_0^t \langle \xi, g(v_n(s)+\zeta(s))\rangle_{L^2(\partial\mathcal{O})}\,\mathrm{d}s = \int_0^t \langle \xi, g(v(s)+\zeta(s))\rangle_{L^2(\partial\mathcal{O})}\,\mathrm{d}s \tag{4.20}$$

which in view of (4.19) implies the existence of a weak solution for (4.12).

The claim follows from an application of the dominated convergence theorem, using the above proved convergence of $v_n\to v$ in $L^2(0,T;L^2(\partial\mathcal{O}))$ and almost everywhere, as well as the uniform bound for the norm of $|v_n|^{2p}$ in (4.18).

Uniqueness. Assume that w_1 and w_2 are two solutions of (4.12) with initial conditions $z_0^1-\zeta(0)$ and $z_0^2-\zeta(0)$ respectively. Set $\bar{w}=w_1-w_2$ and $\bar{w}_0=z_0^1-z_0^2$ its initial condition. Then proceeding formally as above, using dissipativity of the operator A, we obtain

$$\frac{1}{2}\frac{\mathrm{d}\|w(t)\|^2}{\mathrm{d}t} \le -\langle g(v_1(t)+\zeta(t)) - g(v_2(t)+\zeta(t)), v_1(t)-v_2(t)\rangle_{L^2(\partial\mathcal{O})}. \tag{4.21}$$

Our assumptions on g imply that $(a-b)[g(a)-g(b)]\ge -k_4|a-b|^2$ hence

$$\frac{1}{2}\frac{\mathrm{d}\|w(t)\|^2}{\mathrm{d}t} \le k_4\|w(t)\|^2$$

and Gronwall's lemma yields

$$\|w(t)\|^2 \le \mathrm{e}^{2k_4 t}\|z_0^1-z_0^2\|^2$$

which implies uniqueness and continuous dependence on initial data. □

3. Longtime behaviour

We introduce the *stochastic dynamical system* $\{S(t,s,\omega),\ t \geq s, \omega \in \Omega\}$ associated to problem (4.10) by setting

$$S(t,s,\omega)\bar{z} = w(t,\omega) + \zeta(t,\omega), \qquad \bar{z} \in \mathcal{H},$$

where $w(t,\omega)$ is the solution of equation (4.12) with initial condition $w(s,\omega) = \bar{z} - \zeta(s,\omega)$. It is possible to check that $S(t,s,\omega)$ actually is a stochastic dynamical system. The proof follows easily by translating the assertions of Theorem 4.8 in terms of S.

Corollary 4.9. *Setting* $z(t,s,\bar{z})(\omega) = S(t,s,\omega)\bar{z}$ *then* z *is a* mild solution *of equation* (4.10) *in the sense of* [**25**] *and it verifies*

(1) $S(t,r,\omega)S(r,s,\omega)\bar{z} = S(t,s,\omega)\bar{z}$, *for all* $s < r < t$ *and* $\bar{z} \in \mathcal{H}$,
(2) $t \mapsto S(t,s,\omega)\bar{z}$ *is continuous in* $\mathcal{H}$ *for* $t > s$,
(3) for all $t \in \mathbb{R}$, $\bar{z} \in \mathcal{H}$, *the mapping*

$$\begin{aligned}&(s,\omega) \mapsto S(t,s,\omega)\bar{z}\\ &((-\infty,t] \times \Omega, \mathscr{B}((-\infty,t]) \times \mathscr{F}) \longrightarrow (\mathcal{H}, \mathscr{B}(\mathcal{H}))\end{aligned}$$

is measurable.

3.1. Invariant Measure. In this section we discuss the presence of an invariant measure for the stochastic problem (4.10).

Theorem 4.10. *The transition semigroup* $\{P_t\}$ *associated to the solution of* (4.10) *admits a unique invariant measure in* $\mathcal{H}$ *that is strongly mixing.*

Proof. We know from Theorem 4.8 that there exists a unique solution $z(t,s,\bar{z})(\omega) = S(t,s,\omega)\bar{z}$ of equation (4.10).

We now show that there exists $c_1 > 0$ such that

$$\mathbb{E}|z(t,s,\bar{z})| \leq c_1(1 + |\bar{z}|)$$

for all $t > s$, $\bar{z} \in \mathcal{H}$, (see [**25**, Theorem 6.3.3]). Notice that it holds

$$\tfrac{\mathrm{d}}{\mathrm{d}t}|w(t)| \leq -\omega_A|w(t)| + |g(\zeta(t))| \tag{4.22}$$

(compare with the analog situation in the proof of Theorem 4.9), but notice that here we take a pointwise estimate in time whereas there it was in the space of square integrable mappings). Then from the estimate (4.22) we have

$$|w(t)| \leq e^{-\omega_A(t-s)}|x| + \int_s^t e^{-\omega_A(t-\sigma)}|g(\zeta(\sigma))|\,\mathrm{d}\sigma, \qquad t \geq s$$

and taking the mean in both sides, using Hölder inequality and the bound 4.2(1.), one gets

$$\mathbb{E}|w(t)| \leq e^{-\omega_A(t-s)}|x| + \frac{1}{\omega_A}\mathbb{E}\left[\sup_{t\geq s}\sum_{k=1}^{N}|\zeta_k(t)|^{2p-1}\right]$$

and the last quantity is bounded, using the assumptions on the coefficients $\{d_k\}$ and Burkholder-Davis-Gundy inequality.

In a similar manner we prove that there exists $c_2 > 0$ such that

$$\mathbb{E}|z(t,s,\bar{z}) - z(t,\sigma,\bar{z})| \leq c_2 e^{\omega(s+\sigma)}(1+|\bar{z}|).$$

Therefore there exists a random variable η, the same for all $\bar{z} \in \mathcal{H}$, such that

$$\lim_{s\to-\infty}\mathbb{E}|z(t,s,\bar{z}) - \eta| = 0.$$

Then a standard argument implies that the law $\mu = \mathcal{L}(\eta)$ is the unique invariant measure for P_t. □

4. Existence of an attracting set

The aim of this section is prove the existence of an attracting set $K(\omega)$ at time $t = 0$.

Let $B \subset \mathcal{H}$ be a bounded subset of $\mathcal{H}$ and, for any $s \in \mathbb{R}$ and $\bar{z} \in B$, let $w(t,s,\bar{z})$ be the solution of problem (4.12) with initial condition $\bar{z} - \zeta(s)$. We take the scalar product, in $\mathcal{H}$, of (4.12) with $w(t)$ and we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{\mathcal{H}} = \langle Aw(t), w(t)\rangle_{\mathcal{H}} + \langle G(w(t)+\zeta(t)), w(t)\rangle_{\mathcal{H}} + \langle A\zeta(t), w(t)\rangle_{\mathcal{H}}$$

which is equal, by our assumptions, to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{\mathcal{H}} = -\|\nabla u\|^2_{L^2(\mathcal{O})} - \mu\|v\|^2_{L^2(\partial\mathcal{O})} - \langle g(v(t)+\zeta(t)), v(t)\rangle_{L^2(\partial\mathcal{O})} - \mu\langle\zeta(t), v(t)\rangle_{L^2(\partial\mathcal{O})}$$

Recall from Hypothesis 4.2 that $\mu > k_4$; since then $\bar{g}(v) = -g(v) - k_4 v$ is a decreasing mapping, it is dissipative and we estimate

$$\begin{aligned}
&-\langle g(v(t)+\zeta(t)), v(t)\rangle_{L^2(\partial\mathcal{O})} \\
&\quad = \langle -g(v(t)+\zeta(t)) - k_4(v(t)+\zeta(t)), v(t)\rangle_{L^2(\partial\mathcal{O})} + k_4\langle v(t)+\zeta(t), v(t)\rangle_{L^2(\partial\mathcal{O})} \\
&\quad = \underbrace{\langle \bar{g}(v+\zeta) - \bar{g}(\zeta), v\rangle}_{\leq 0} + \langle -g(\zeta) - k_4\zeta, v\rangle + k_4\langle v(t)+\zeta(t), v(t)\rangle \\
&\quad \leq k_4\|v(t)\|_{L^2(\partial\mathcal{O})} - \langle g(\zeta(t)), v(t)\rangle_{L^2(\partial\mathcal{O})}
\end{aligned}$$

hence we get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{\mathcal{H}} \leq -\|\nabla u(t)\|^2_{L^2(\mathcal{O})} - (\mu - k_4)\|v(t)\|^2_{L^2(\partial\mathcal{O})} \\
- \langle g(\zeta(t)), v(t)\rangle_{L^2(\partial\mathcal{O})} + (\alpha-\mu)\langle \zeta(t), v(t)\rangle_{L^2(\partial\mathcal{O})}.
\end{aligned}$$

For λ large enough, using Cauchy-Schwartz inequality, Hypothesis 4.2 and setting $k_\lambda = k_1 + \lambda|\alpha-\mu|$, $\mu_\lambda = \mu - k_4 - \frac{|\alpha-\mu|+1}{4\lambda} > 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{\mathcal{H}} + \|\nabla u(t)\|^2_{L^2(\mathcal{O})} + \mu_\lambda\|v(t)\|^2_{L^2(\partial\mathcal{O})} \leq k_\lambda(1 + \|\zeta(t)\|^{2n-1}).$$

According to Friedrichs' theorem Proposition 4.3 we obtain that there exist constants $0 < c_1 < 1$, $0 < c_2 < \min(1, \mu_\lambda)$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{\mathcal{H}} + c_1\|\nabla u(t)\|^2_{L^2(\mathcal{O})} + c_2\|w(t)\|^2_{\mathcal{H}} \leq k_\lambda(1 + \|\zeta(t)\|^{2n-1}). \tag{4.23}$$

By Gronwall's lemma, for $s < -1$ and $t \in [-1, 0]$ we get

$$\begin{aligned}
\|w(t)\|^2_{\mathcal{H}} &\leq \|w(s)\|^2_{\mathcal{H}}\mathrm{e}^{-2c_2(t-s)} + 2k_\lambda\int_s^t \mathrm{e}^{-2c_2(t-\sigma)}(1 + \|\zeta(\sigma)\|^{2n-1})\,\mathrm{d}\sigma \\
&\leq \mathrm{e}^{2c_2}\|w(s)\|^2_{\mathcal{H}}\mathrm{e}^{2c_2 s} + 2k_\lambda\mathrm{e}^{2c_2}\int_{-\infty}^0 \mathrm{e}^{2c_2\sigma}(1 + \|\zeta(\sigma)\|^{2n-1})\,\mathrm{d}\sigma
\end{aligned}$$

and recalling the initial condition $w(s) = \bar{z} + \zeta(s)$, it is clear that there exists $s_1 = s_1(B)$ such that for all $s < s_1(B)$:

$$\|w(t,s,\bar{z})\|^2_{\mathcal{H}} \leq 1 + 2\mathrm{e}^{2c_2}\sup_{s\in(-\infty,1]}\left\{\|\zeta(s)\|^2\mathrm{e}^{2c_2 s}\right\} + \int_{-\infty}^0 \mathrm{e}^{2c_2\sigma}(1 + \|\zeta(\sigma)\|^{2n-1})\,\mathrm{d}\sigma$$

The right hand side of the above inequality is an almost surely finite random variable $r_1(\omega)$, independent from s and t.

Conclusions

In this thesis we study different equations arising in the field of stochastic boundary value problems with non homogeneous boundary conditions in one (network models) and more dimensions. Similar problems were studied in the last years using other techniques; the abstract approach by matrix operators allows a unified treatment of these problems. Notice that we introduce a new boundary variable and we read the problem as a system of (coupled) stochastic evolution equations, which is then written in an abstract form. Similarly to the case of standard boundary conditions, the theory of semigroup generation by matrix operators, and their spectral properties, can be used to characterize well posedness and longtime behaviour for such systems. Also, our approach is suitable to study other problems which are not present in the thesis. We mention among them:

1. in [**24, 25**] the case of evolution equations with white noise boundary conditions is treated by using the Dirichlet or Neumann map to convert the stochastic problem into a pathwise deterministic evolution equation with homogeneous conditions on the boundary. The approach adopted in [**62**] is similar to the previous one and it is applied to stochastic semilinear equations with boundary noise. Note that in these models the behaviour on the boundary not involves an influence from the internal dynamics,

2. a typical problem which presents a connection between internal and external dynamics, is the stochastic Cahn-Hilliard equation with stochastic dynamic boundary conditions treated in [**28**]. The authors reduce the stochastic system to a pathwise deterministic system which is known to possess a unique solution,

3. motivating by [**36, 37, 38**], it seems natural to extend the results in Chapter 4 to solve stochastically perturbed problems of heat conduction in materials with memory effects in the interior and dynamical conditions on the boundary. In particular the aim shall be to prove the existence of an invariant measure for the weak solution of the problem,

4. in the framework of biological applications, studied in Chapter 3, in the case of neuronal networks, it can be interesting to consider a more precise model for the electrical potential diffusion through neuronal cells. In particular

- we can introduce a recovery variable having a linear dynamics, that provides a slower negative feedback such that both depolarisation and repolarisation can be modelled,
- we can consider a passive behaviour on a subset of the nodes, e.g. a conservative law for the electrical flux: $\sum_{j\in\Gamma(\mathsf{v}_i)} \phi_{ij}\mu_j c_j(\mathsf{v}_i)\frac{\partial}{\partial x}u_j(t,\mathsf{v}_i) = 0$,
- we can also consider a passive behaviour on the edges (e.g. the term f is linear) which models the dendritic spines.

Remark 4.3. *In the light of a talk of Schmalfuss, which took place in the* 8th International Meeting on Stochastic Partial Differential Equations and Applications (January 2008) *and which referred to the recent paper* [**19**] *in collaboration with Chueshov, we can state the existence of a random (pullback) attractor for the stochastic problem in Chapter* 4*. On the other hand, it seems hard to extend this result to the case of diffusion in materials with memory, where lack of compactness for absorbing sets not permits to define a random attractor (see* [**37**]*).*

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