A CLASS OF ONE-DIMENSIONAL GEOMETRIC VARIATIONAL PROBLEMS WITH ENERGY INVOLVING THE CURVATURE AND ITS DERIVATIVES. SOME RESULTS AIMED TO A GEOMETRIC MEASURE-THEORETIC APPROACH.

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ABSTRACT. The notions of Legendrian and Gaussian towers are defined and indagated. Then applications in the context of one-dimensional geometric variational problems with the energy involving the curvature and its derivatives are provided. Particular attention is paid to the case when the functional is defined on smooth boundaries of plane sets.

1. INTRODUCTION

Since [11] a great deal of our work has been focused on looking for and indagating sufficient conditions for the high-order rectifiability of a rectifiable set. In the recent paper [13] we proved the following result which can be considered at some extent as a satisfactory conclusive step in the special case of the one-dimensional sets.

Theorem 1.1. *Let D, H be a couple of integer numbers satisfying $D \geq 2$ and $H \geq 1$. Then, for a given a Lipschitz map $\gamma : [a, b] \to \mathbf{R}^D$, the set $\gamma([a, b])$ is C^{H+1}-rectifiable provided the following condition is met:*

There are a family of 2^H Lipschitz maps

$$\gamma_\alpha : [a, b] \to \mathbf{R}^D, \qquad \alpha \in \{0, 1\}^H$$

and a family of H bounded functions

$$c_n : [a, b] \to \mathbf{R}, \qquad n = 0, \dots , H - 1$$

such that

$$\gamma_{0^H} = \gamma$$

and

$$\gamma'_{0^{H-n}\beta} = c_n \, \gamma_{0^{H-1-n}1\beta} \qquad \textit{(almost everywhere)} \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary 49Q15, 49Q20, 49J45; Secondary 28A75, 53A04.

Key words and phrases. Rectifiable sets, Geometric measure theory, Geometric variational problems, Functionals involving the curvature and its derivatives.

for all $n = 0, \dots, H-1$ and $\beta \in \{0,1\}^n$ (where $0^0 := \emptyset$ and $\{0,1\}^0 := \{\emptyset\}$).

The present paper provides applications of Theorem 1.1 in the setting of geometric variational problems, via a geometric measure-theoretic approach. This is done by developing a suitable machinery based on the topics of Legendrian and Gaussian towers, which extend the notion of one-dimensional generalized Gauss graph (first introduced in [3]).

Former achievements in the context of the applications of generalized Gauss graphs include [10] (a somehow surprising application to differential geometry context), [2] (an application to Willmore problem) and [9] (an application to a problem introduced in [4]). In particular, the papers [2, 9] (and[4]) follow the idea by De Giorgi of relaxing the functional with respect to the L^1-convergence of the domains of integration.

In order to explain the notion of H-storey Gaussian tower, given in §3 below, let us consider the particular case $H = 3$. In such a case, let X_3 denote the Euclidean space of dimension $8D$ and consider an orthonormal basis of X_3

$$\{e^j_{000}\}_{j=1}^D, \{e^j_{001}\}_{j=1}^D, \{e^j_{010}\}_{j=1}^D, \{e^j_{011}\}_{j=1}^D, \{e^j_{100}\}_{j=1}^D, \{e^j_{101}\}_{j=1}^D, \{e^j_{110}\}_{j=1}^D, \{e^j_{111}\}_{j=1}^D.$$

Also, for $h = 0, 1, 2$, let X_h be the space spanned by the first $2^h D$ vectors of such a basis, namely

$$X_0 := \operatorname{span}\left\{\{e^j_{000}\}_{j=1}^D\right\}, \qquad X_1 := \operatorname{span}\left\{\{e^j_{000}\}_{j=1}^D, \{e^j_{001}\}_{j=1}^D\right\}$$

and

$$X_2 := \operatorname{span}\left\{\{e^j_{000}\}_{j=1}^D, \{e^j_{001}\}_{j=1}^D, \{e^j_{010}\}_{j=1}^D, \{e^j_{011}\}_{j=1}^D\right\}.$$

In X_0 let us consider a smooth curve C_0, oriented by the smooth unit tangent vector field τ_0. Then the graph of τ_0

$$C_1 = \{(x_0, \tau_0(x_0)) \,|\, x_0 \in C_0\}$$

can be naturally viewed as an oriented smooth curve in X_1, whose orientation is induced by τ_0. Let τ_1 be the corresponding smooth unit tangent vector field. Analogously, we can view

$$C_2 = \{(x_1, \tau_1(x_1)) \,|\, x_1 \in C_1\}$$

as an oriented smooth curve in X_2 and we can denote by τ_2 the related smooth unit tangent vector field. Finally, we can consider the smooth oriented curve in X_3

$$C_3 = \{(x_2, \tau_2(x_2)) \,|\, x_2 \in C_2\}.$$

and the smooth tangent vector field τ_3. Then the rectifiable current in X_3

$$T := [\![C_3, \tau_3, 1]\!]$$

is an example of 3-storey Gaussian tower. Roughly speaking, a general 3-storey (hence H-storey) Gaussian tower is defined by axiomatizing, in the framework of rectifiable currents, the properties of T concerning tangentiality and orientation. This is done in Definition 3.1 where the further notion of H-storey Legendrian tower is also given by only requiring the tangentiality condition.

Also we provide the notion of “special H-storey Legendrian (Gaussian) tower” which extends that of special generalized Gauss graph, fruitfully introduced in [9]: it consists of a H-storey Legendrian (Gaussian) tower $[\![R,\eta,\theta]\!]$ such that

$$\mathcal{H}^1(R_0)=0, \qquad R_0 := \{P\in R \,|\, X_0\eta(P)=0\}$$

compare Definition 3.2. In rough terms, this equality means that the “purely $X_0^\perp$-directed part” of the carrier R has measure zero.

In §4 we consider a suitable class of functions $F : X_H \times X_H \to [0,+\infty]$ and define the following integral functionals over the one-dimensional integral currents in X_H

$$\mathcal{F}_F([\![R,\eta,\theta]\!]) := \int_{R\setminus R_0} F\left(P, \frac{\eta(P)}{\|X_0\eta(P)\|}\right) \|X_0\eta(P)\|\theta(P)\, d\mathcal{H}^1(P).$$

Then we get some results related to the implementation of the direct method of the calculus of variations in the context of Legendrian and Gaussian towers. In particular we prove that:

- (Theorem 4.1). For all constants c, the set Σ_c of special H-storey Gaussian towers T such that $\mathcal{F}_F(T) \le c$ has to be closed (with respect to the weak topology of currents). Moreover, the restriction $\mathcal{F}_F|\Sigma_c$ is lower semicontinuous. Finally, under further assumptions about coherciveness of $\mathcal{F}_F$ and boundedness of the boundary masses, the set Σ_c has to be compact.
- (Theorem 5.1(1)). If $D=2$ and A is a “regular” plane set (i.e. an open subset of $\mathbf{R}^2$ whose boundary has a regular parametrization of class C^{H+1}), let T_A denote the special H-storey Gaussian tower naturally associated to ∂A. Then the functional $A \mapsto \mathcal{F}_F(T_A)$ is $L^1_{\rm loc}$-lower semicontinuous on the family of regular plane sets.

We also mention Proposition 6.3 which, in the case $D=2$ and for any given 2-storey Gaussian tower $[\![G,\eta,\theta]\!]$, states a formula for the representation of η in terms of the absolute curvature of the carrier R and its approximate derivative (compare [12] where this notion of curvature has been defined or §2 below, where it has been recalled).

As a simple application of Theorem 5.1(1) and Proposition 6.3, we get Theorem 6.1(3) which concludes the paper. It states that if A_h $(h=1,2,\dots)$ and A are regular plane sets such that $A_h \to A$ in $L^1_{\rm loc}(\mathbf{R}^2)$, then

$$\int_{\partial A} 1+2\alpha^2+\frac{\|D^{\partial A}\alpha\|^2}{(1+\alpha^2)^2}\, d\mathcal{H}^1 \le \liminf_h \int_{\partial A_h} 1+2\alpha_h^2+\frac{\|D^{\partial A_h}\alpha_h\|^2}{(1+\alpha_h^2)^2}\, d\mathcal{H}^1$$

where α and α_h are the absolute curvatures of ∂A and ∂A_h, respectively.

2. Preliminaries

In this section we collect some well-known results which will be useful below. For the definition of integral current we refer the reader to the general literature about geometric measure theory [15, 18, 19].

Let us begin by recalling from [7] the notion of generalized Gauss graph, in the special case of dimension one. To this aim, consider an Euclidean space E of even dimension $2D$, with $D \geq 2$, and let

$$\left\{ \{e_i^j\}_{j=1}^D \right\}_{i \in \{0,1\}}$$

be an orthonormal basis of E. The coordinate with respect to the direction e_i^j will be denoted by x_i^j. Then define X and Y as the D-dimensional linear subspaces of E spanned by

$$\{e_0^j\}_{j=1}^D \qquad \text{and} \qquad \{e_1^j\}_{j=1}^D$$

respectively. Also, set

$$S := \{y \in Y \mid \|y\| = 1\}$$

and

$$\varphi := \sum_{j=1}^D x_1^j \, dx_0^j.$$

Observe that $\varphi|(X \times S)$ coincides with the usual contact form on $X \times S$. Finally let

$$P : Y \to X$$

be the isomorphism mapping e_1^j to e_0^j, for all $j = 1, \ldots, D$.

Definition 2.1 ([3, 7]). *A "one-dimensional generalized Gauss graph (in E)" is a one-dimensional integral current T in E such that:*

(i) *If G is the rectifiable carrier of T, then the projection of G to Y is (in measure) a subset of S i.e.*
$$\mathcal{H}^1(YG \backslash S) = 0;$$
(ii) *One has*
$$T(*\varphi \llcorner \omega) = 0$$
for all smooth $(2D-2)$-forms ω with compact support in E, where $*$ denotes the usual Hodge star operator in E;
(iii) *The inequality*
$$T(g\varphi) \geq 0$$
holds for all nonnegative smooth functions g with compact support in E.

The following result provides a geometric interpretation of the assumptions in Definition 2.1 above.

Proposition 2.1 ([7]). *For a one-dimensional rectifiable current $T = [\![G, \eta, \theta]\!]$ in E the following facts hold.*

(1) *If T satisfy (i) and (ii) in Definition 2.1, then there exists a measurable sign function $\sigma : G \to \{\pm 1\}$ such that*

$$X\eta(z) = \sigma(z)\,\|X\eta(z)\|\,P(Yz) \tag{2.1}$$

for $\mathcal{H}^1 \llcorner G$-a.e. z;

(2) *Let T be a one-dimensional generalized Gauss graph. Then (2.1) holds with σ identically equal to* 1.

We say that a Borel subset R of E is C^{H+1}-rectifiable if there exist countably many curves C_j of class C^{H+1}, embedded in E and such that

$$\mathcal{H}^1(R \setminus \cup_j C_j) = 0 \tag{2.2}$$

compare [1, Definition 1.1]. Observe that for $H = 0$ this is equivalent to say that R is countably 1-rectifiable, e.g. by [19, Lemma 11.1].

According to [12], for a one-dimensional C^2-rectifiable subset R of E, a notion of absolute curvature can be provided as follows. First of all, given a countable family $\mathcal{A} = \{C_j\}$ (referred in the sequel as "C^2-covering of R") of curves of class C^2 embedded in E and such that (2.2) holds, at each density point x of the sets $R \cap C_j$ one can define

$$\alpha_R^{\mathcal{A}}(x) := \text{absolute curvature of } C_j \text{ at } x$$

where j is just any index such that $R \cap C_j$ has density one at x. The function $\alpha_R^{\mathcal{A}}$ is well-defined. Moreover, if $\mathcal{B}$ is another C^2-covering of R, then $\alpha_R^{\mathcal{A}}$ and $\alpha_R^{\mathcal{B}}$ are representatives of the same measurable function, with domain R. Such a measurable function is called "absolute curvature of R" and is denoted by α_R.

Proposition 2.2 ([12]). *If R if C^3-rectifiable, then α_R is approximately differentiable, namely:*

(1) *For any given C^3-covering $\mathcal{A} = \{C_i\}$ of R, the function $\alpha_R^{\mathcal{A}}$ is approximately differentiable at every point in $(R \cap C_i)^*$, for all i;*
(2) *If $\mathcal{A}$ and $\mathcal{B}$ are C^3-coverings of R, then one has*

$$apD\alpha_R^{\mathcal{A}} = apD\alpha_R^{\mathcal{B}}, \text{ a.e. in } R.$$

3. H-storey Legendrian and Gaussian towers

Consider an Euclidean space X_H of dimension $2^H D$. Let

$$\left\{\{e_\beta^j\}_{j=1}^D\right\}_{\beta \in \{0,1\}^H}$$

be an orthonormal basis of X_H and x_β^j be the coordinate with respect to the direction e_β^j. Then, for $n = 0, \dots, H-1$, define X_n and Y_n as the $(2^n D)$-dimensional linear subspaces of

X_H respectively spanned by

$$\left\{\{e^j_{0^{H-n}\beta}\}_{j=1}^D\right\}_{\beta\in\{0,1\}^n} \qquad \text{and} \qquad \left\{\{e^j_{0^{H-n-1}1\beta}\}_{j=1}^D\right\}_{\beta\in\{0,1\}^n}$$

where the following notation is conventionally assumed

$$0^0 := \emptyset, \qquad \{0,1\}^0 := \{\emptyset\}.$$

Example 3.1. If $D = 2$ and $H = 3$, then X_2 and Y_2 are the spaces spanned by

$$\{e^1_{000}, e^2_{000}, e^1_{001}, e^2_{001}, e^1_{010}, e^2_{010}, e^1_{011}, e^2_{011}\}, \qquad \{e^1_{100}, e^2_{100}, e^1_{101}, e^2_{101}, e^1_{110}, e^2_{110}, e^1_{111}, e^2_{111}\}$$

respectively.

Observe that one has

$$X_{n+1} = X_n \oplus Y_n, \qquad n = 0, \dots, H-1$$

hence

$$X_H = X_0 \oplus \left(\bigoplus_{n=0}^{H-1} Y_n\right).$$

Also, for $n = 0, \dots, H-1$, we set

$$S_n := \{y \in Y_n \mid \|y\| = 1\}$$

and

$$\varphi_n := \sum_{\substack{\beta\in\{0,1\}^n \\ j=1,\dots,D}} x^j_{0^{H-n-1}1\beta}\, dx^j_{0^{H-n}\beta}. \tag{3.1}$$

Then $\varphi_n|(X_n \times S_n)$ coincides with the usual contact form on $X_n \times S_n$. Finally, for $n = 0, \dots, H-1$, let

$$P_n : Y_n \to X_n$$

be the isomorphism mapping $e^j_{0^{H-n-1}1\beta}$ to $e^j_{0^{H-n}\beta}$, for all $\beta \in \{0,1\}^n$ and $j = 1, \dots, D$.

Now we are ready to give the notions of H-storey Legendrian tower and H-storey Gaussian tower.

Definition 3.1. *A "H-storey Legendrian tower (in X_H)" is a one-dimensional integral current T in X_H which verifies this couple of conditions:*

(i) *Let G denote the rectifiable carrier of T. Then, for $n = 0, \dots, H-1$, the projection of G to Y_n is (in measure) a subset of S_n i.e.*

$$\mathcal{H}^1(Y_n G \backslash S_n) = 0, \qquad n = 0, \dots, H-1;$$

(ii) *One has*

$$T(*\varphi_n \llcorner \omega) = 0, \qquad n = 0, \dots, H-1$$

for all smooth $(2^H D - 2)$-forms ω with compact support in X_H, where $*$ denotes the usual Hodge star operator in X_H.

If in addition to (i) and (ii), the following assumption is satisfied, then T is said to be a "H-storey Gaussian tower":

(iii) *The inequalities*

$$T(g\varphi_n) \geq 0, \qquad n = 0, \dots, H-1$$

hold for all nonnegative smooth functions g with compact support in X_H.

Definition 3.2. *A H-storey (Gaussian) Legendrian tower $T = [\![G, \eta, \theta]\!]$ is said to be "of the special type", or simply "special", if one has*

$$|T| << |T_{0^H}|$$

where

$$|T| := \theta\,\mathcal{H}^1 \llcorner G, \qquad |T_{0^H}| := \theta \|\eta_{0^H}\|\,\mathcal{H}^1 \llcorner G \qquad (\eta_{0^H} := X_0\eta).$$

Remark 3.1. One-dimensional (special) generalized Gauss graphs and (special) 1-storey Gaussian towers are just the same items, compare [3, 7, 9]. It follows easily that T is a H-storey Gaussian tower if and only if, for all $n = 1, \dots, H$, the current $(X_n)_\# T$ is a one-dimensional generalized Gauss graph in X_n.

Remark 3.2. A H-storey (Gaussian) Legendrian tower $T = [\![G, \eta, \theta]\!]$ is of the special type if and only if $\mathcal{H}^1(\{P \in G \,|\, \eta_{0^H}(P) = 0\}) = 0$.

Remark 3.3. Let T_j $(j = 1, 2, \dots)$ be H-storey (Gaussian) Legendrian towers and let T be a one-dimensional integral current in X_H such that $T_j \rightharpoonup T$. Then T is a H-storey (Gaussian) Legendrian tower too. In the particular case when the T_j are of the special type, the limit current T has not necessarily to be of the special type itself (e.g. the generalized Gauss graphs associated to plane circles shrinking to a point converge to a non-trivial current T with carrier G such that $|T_0|(G) = 0$). A closure condition for special H-storey Gaussian towers will be provided in Theorem 4.1 below.

Example 3.2 (smooth case, $H = 2$). A situation to keep in mind, in order to understand the meaning of Definition 3.1, is the following one. Given a regular 1-1 curve of class C^3

$$\gamma : [a, b] \to \mathbf{R}^D$$

we can consider the maps

$$\gamma_\alpha = (\gamma_\alpha^1, \dots, \gamma_\alpha^D) : [a, b] \to \mathbf{R}^D, \qquad \alpha \in \{0, 1\}^2$$

defined as

$$\gamma_{00} := \gamma, \qquad \gamma_{01} := \frac{\gamma_{00}'}{\|\gamma_{00}'\|} = \frac{\gamma'}{\|\gamma'\|}$$

and

$$(\gamma_{10}, \gamma_{11}) := \frac{(\gamma_{00}', \gamma_{01}')}{\|(\gamma_{00}', \gamma_{01}')\|} = \frac{\big(\gamma', (\gamma'/\|\gamma'\|)'\big)}{\big\|\big(\gamma', (\gamma'/\|\gamma'\|)'\big)\big\|}. \tag{3.2}$$

Then the multiplicity-one current

$$T := [\![G, \eta, 1]\!]$$

with

$$G := \Gamma([a,b]), \qquad \Gamma := \sum_{\alpha\in\{0,1\}^2}\sum_{i=1}^{D}\gamma_\alpha^i e_\alpha^i : [a,b]\to X_2$$

and

$$\eta : G \to X_2, \qquad \eta(Q) := \Gamma'\Big(\Gamma^{-1}(Q)\Big)\Big/\Big\|\Gamma'\Big(\Gamma^{-1}(Q)\Big)\Big\|.$$

is a special 2-storey Gaussian tower.

Observe that, since $(*\gamma')\llcorner\gamma' = 0$, one has

$$(*\gamma_{01})\llcorner\frac{\gamma_{11}}{\|\gamma_{10}\|} = \frac{1}{\|\gamma'\|^2}(*\gamma')\llcorner(\gamma'/\|\gamma'\|)' = \frac{1}{\|\gamma'\|^3}(*\gamma')\llcorner\gamma'' \tag{3.3}$$

where $*$ denotes the usual Hodge star operator in $\mathbf{R}^D$. Also, if

$$u, v \in \mathbf{R}^D, \quad \|u\| = 1$$

and $\{e_j\}$ is an orthonormal basis of $\mathbf{R}^D$, then

$$(*u)\llcorner v = (e_2\wedge\cdots\wedge e_D)\llcorner\sum_{j=1}^{D}v_je_j = \sum_{j=2}^{D}v_j(e_2\wedge\cdots\wedge e_D)\llcorner e_j$$

where $v_j := v\cdot e_j$. Hence

$$\|(*u)\llcorner v\|^2 = \sum_{j=2}^{D}v_j^2 = \|v\|^2 - (v\cdot u)^2 = \|v\wedge u\|^2. \tag{3.4}$$

By recalling the formula (8.4.13.1) of [5], we then obtain the following expression for the absolute curvature α_γ of γ

$$\alpha_\gamma = \frac{\|\gamma'\wedge\gamma''\|}{\|\gamma'\|^3} = \frac{\|(*\gamma')\llcorner\gamma''\|}{\|\gamma'\|^3} \tag{3.5}$$

which can be written in terms of γ_{01}, γ_{10} and γ_{11}, as it follows

$$\alpha_\gamma = \frac{\|(*\gamma_{01})\llcorner\gamma_{11}\|}{\|\gamma_{10}\|}$$

by (3.3). In the particular case when $D = 2$, (3.3) provides the following formula for the signed curvature κ_γ of γ (compare [14, §1-5, Exercise 12]):

$$\kappa_\gamma = \frac{\gamma''\cdot(*\gamma')}{\|\gamma'\|^3} = \frac{\gamma_{11}\cdot(*\gamma_{01})}{\|\gamma_{10}\|}.$$

The following result summarizes some properties of a H-storey Legendrian (Gaussian) tower. In particular it proves that the carrier is projected to the space X_0 into a C^{H+1}-rectifiable set. Recall from [15, §4.2.25] that an indecomposable one-dimensional integral current has always multiplicity one.

Theorem 3.1. *Let $T = [\![G,\eta,\theta]\!]$ be a H-storey Legendrian tower in X_H. Then the following facts hold.*

(1) *There exist countably many indecomposable H-storey Legendrian towers $T_j = [\![G_j, \eta_j, 1]\!]$ such that*

$$T = \sum_j T_j$$

and

$$\mathbf{M}(T) = \sum_j \mathbf{M}(T_j), \qquad \mathbf{M}(\partial T) = \sum_j \mathbf{M}(\partial T_j). \tag{3.6}$$

Moreover one has

$$G = \cup_j G_j, \qquad \eta | G_j = \eta_j, \qquad \theta(x) = \#\{j \,|\, x \in G_j\} \tag{3.7}$$

where the equality sign has to be intended "modulo null-measure sets". The T_j are Gaussian provided T is Gaussian. The T_j are special provided T is special.

(2) *The projection of G to the space X_0 is a C^{H+1}-rectifiable one-dimensional set.*

(3) *If T is indecomposable, then there exists a Lipschitz map*

$$\Gamma : [0, \mathbf{M}(T)] \to X_H$$

such that

(i) $\Gamma | [0, \mathbf{M}(T))$ *is injective,* $\Gamma_\# [\![0, \mathbf{M}(T)]\!] = T$ *and* $\|\Gamma'(t)\| = 1$ *for a.e.* $t \in [0, \mathbf{M}(T)]$.

(ii) *There exists a family of measurable sign functions*

$$\sigma_n : [0, \mathbf{M}(T)] \to \{\pm 1\}, \qquad n = 0, \dots, H-1$$

such that

$$X_n \Gamma' = \sigma_n \|X_n \Gamma'\| \, (P_n \circ Y_n) \Gamma \qquad \textit{(almost everywhere)} \tag{3.8}$$

for all $n = 0, \dots, H-1$. If T is Gaussian then the functions σ_n are identically equal to 1.

Proof. (1) From [15, §4.2.25] we can find a sequence of indecomposable integral currents T_j in X_H such that

$$T = \sum_j T_j, \qquad \mathbf{N}(T) = \sum_j \mathbf{N}(T_j).$$

Since

$$\mathbf{M}(T) \le \sum_j \mathbf{M}(T_j), \qquad \mathbf{M}(\partial T) \le \sum_j \mathbf{M}(\partial T_j) \tag{3.9}$$

we obtain

$$\sum_j \mathbf{N}(T_j) = \mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T) \le \sum_j \mathbf{M}(T_j) + \sum_j \mathbf{M}(\partial T_j) = \sum_j \mathbf{N}(T_j)$$

hence (3.6) follows by recalling again (3.9). Now [8, Proposition 4.2] yields the equalities (3.7). As a consequence of such equalities, T_j inherits from T the geometric properties characterizing a H-storey Legendrian (Gaussian) tower, compare [7, Proposition 4.1]. In particular, each T_j has to be itself a H-storey Legendrian (Gaussian) tower. The last assertion follows from (3.7) too, by recalling Remark 3.2.

(2) Let $\{T_j\}$ be as in (1). Then, since

$$X_0 G = \cup_j X_0 G_j$$

by the first equality in (3.7), it will be enough to prove that $X_0 G_j$ is a C^{H+1}-rectifiable set, for all j. To this aim, let us fix j and apply (3) below to T_j. Then define

$$\gamma_\alpha = (\gamma_\alpha^1, \dots, \gamma_\alpha^D) : [0, \mathbf{M}(T_j)] \to \mathbf{R}^D, \qquad \alpha \in \{0,1\}^H$$

by

$$\gamma_\alpha^i := \Gamma \cdot e_\alpha^i, \qquad i = 1, \dots, D$$

and

$$\gamma := \gamma_{0^H}.$$

From (3.8), for all $n = 0, \dots, H-1$, $\beta \in \{0,1\}^n$ and $i = 1, \dots, D$, we get

$$\begin{aligned}
\frac{d}{dt}\gamma^i_{0^{H-n}\beta} &= \Gamma' \cdot e^i_{0^{H-n}\beta} = \sigma_n \|X_n\Gamma'\| \, P_n(Y_n\Gamma) \cdot P_n(e^i_{0^{H-n-1}1\beta}) \\
&= \sigma_n \|X_n\Gamma'\| (Y_n\Gamma) \cdot e^i_{0^{H-n-1}1\beta} = \sigma_n \|X_n\Gamma'\| \Gamma \cdot e^i_{0^{H-n-1}1\beta} \\
&= \sigma_n \|X_n\Gamma'\| \gamma^i_{0^{H-n-1}1\beta}
\end{aligned}$$

namely (1.1) holds with $c_n := \sigma_n \|X_n\Gamma'\|$. The C^{H+1}-rectifiability of G_j follows now from Theorem 1.1.

(3) Assertion (i) follows from the structure theorem in [15, §4.2.25], while (ii) is a consequence of Proposition 2.1 and Remark 3.1. $\square$

Remark 3.4. If Γ is the map in Theorem 3.1(3), then one has

$$(Y_n\Gamma)' \cdot Y_n\Gamma = 0 \qquad (n = 0, \dots, H-1) \tag{3.10}$$

almost everywhere in $[0, \mathbf{M}(T)]$. Such a fact follows at once applying the following simple claim to

$$\varphi := \|Y_n\Gamma\|^2$$

and recalling that $\varphi \equiv 1$, by Definition 3.1(i).

Claim 3.1. Let $\varphi : [a,b] \to \mathbf{R}$ be differentiable at s and assume that it is constant on some sequence $\{t_h\} \subset [a,b]$ converging to s, with $t_h \neq s$ (for all h). Then $\varphi'(s) = 0$.

Remark 3.5. In the very special case when $D = 2$ and $H = 1$, the number of the T_j in claim (1) of Theorem 3.1 has to be finite, in that

$$\mathbf{N}(T_j) \geq \begin{cases} \mathbf{M}(T_j) \geq 2\pi & \text{if } \partial T_j = 0, \\ \mathbf{M}(\partial T_j) = 2 & \text{if } \partial T_j \neq 0 \end{cases}$$

by [6, Theorem 4.1]. In all the other cases, namely when $D \geq 3$ or $H \geq 2$, there could exist infinitely many indecomposable components T_j as it is shown in the following example where $H = 1$ and $D \geq 3$. Consider a point $P \in X_0$ and infinitely many closed regular curves C_j included in the $(D-1)$-dimensional unit sphere S_0, such that

$$\sum_j \mathcal{H}^1(C_j) < +\infty.$$

For every j, we can define T_j as the one-dimensional rectifiable current of multiplicity one naturally carried by $P \times C_j$. Then $T := \sum_j T_j$ is a 1-storey Legendrian tower, having the T_j as indecomposable components. This example can be easily extended to the case when $H \geq 2$.

We finally state a result which will be invoked in the next section. It extends to H-storey Gaussian towers some facts valid for generalized Gauss graphs, compare [3, 7, 8].

Proposition 3.1. *The following facts hold.*

(1) *If $T = [\![G, \eta, \theta]\!]$ is a H-storey Gaussian tower, then*

$$\eta_{0^H} = \|\eta_{0^H}\| \, P_0 Y_0 \tag{3.11}$$

almost everywhere with respect to $\mathcal{H}^1 \llcorner G$, hence

$$T(g\varphi_0) = \int g \, d|T_{0^H}| \tag{3.12}$$

for all $g \in C_c(X_H)$, where φ_0 is the 1-form defined as in (3.1) with $n = 0$.

(2) *Let $T_j = [\![G_j, \eta_j, \theta_j]\!]$ ($j = 1, 2, \ldots$) and $T = [\![G, \eta, \theta]\!]$ be H-storey Gaussian towers such that $T_j \rightharpoonup T$. Then*

$$|(T_j)_{0^H}| \to |T_{0^H}|$$

as $j \to \infty$, in the weak sense of measures. Moreover, if M is a one-dimensional rectifiable subset of X_0 such that $M \subset X_0 G_j$, for all j, then one also has $M \subset X_0 G$.*

Proof. (1) Since (3.12) follows immediately from (3.11), we have only to prove that (3.11) holds. To this aim observe that, by virtue of (3.7), we are reduced to prove the case when T is indecomposable. In such a case, by Theorem 3.1(3), there exists a Lipschitz map

$$\Gamma : [0, \mathbf{M}(T)] \to X_H$$

such that

$$\eta \circ \Gamma = \Gamma'.$$

By recalling (3.8), we get

$$\eta_{0^H} \circ \Gamma = X_0(\eta \circ \Gamma) = X_0 \Gamma' = \|X_0 \Gamma'\| (P_0 \circ Y_0)\Gamma = \|\eta_{0^H} \circ \Gamma\| (P_0 \circ Y_0)\Gamma$$

which obviously yields (3.11).

(2) The first claim is a trivial consequence of the formula (3.12) and implies the second one by the same argument used to prove [8, Proposition 5.1]. □

4. A class of integral functionals on H-storey Legendrian towers

Let $\mathcal{I}_{H,D}$, $\mathcal{L}^*_{H,D}$ and $\mathcal{G}^*_{H,D}$ denote, respectively, the set of one-dimensional integral currents in X_H, the set of special H-storey Legendrian towers in X_H and the set of special H-storey Gaussian towers in X_H. Then, given a measurable function

$$F(P,Q) : X_H \times X_H \to [0,+\infty]$$

and tracing the path in [9], we can define a functional

$$\mathcal{F}_F : \mathcal{I}_{H,D} \to [0,+\infty]$$

as follows:

$$\mathcal{F}_F([\![R,\eta,\theta]\!]) := \int_{R^*} F\left(P, \frac{\eta(P)}{\|\eta_{0^H}(P)\|}\right) \|\eta_{0^H}(P)\| \theta(P)\, d\mathcal{H}^1(P)$$

where

$$R^* := \{P \in R \,|\, \eta_{0^H}(P) \neq 0\}. \tag{4.1}$$

According to [16], the integrand F is said to be "standard" when:

(i) it is continuous;
(ii) it is convex with respect to its second argument;
(iii) there exists a continuous function

$$f(P,t) : X_H \times [0,+\infty) \to [0,+\infty)$$

which is nondecreasing with respect to t and satisfies

$$F(P,Q) \geq f(P, \|Q\|)\|Q\|$$

for all $P, Q \in X_H$;
(iv) one has $f(P,t) \to +\infty$ locally uniformly in P, as $t \to +\infty$.

Example 4.1. Consider $p \geq 1$. Then the function

$$F_p(P,Q) := \|Q\|^p, \qquad P,Q \in X_H$$

verifies the assumption (iii) above, with $f(P,t) := t^{p-1}$. Moreover F_p is a standard integrand, provided $p > 1$.

Remark 4.1. If F satisfies (iii) above and

$$m := \inf\{f(P,1) \,|\, P \in X_H\} > 0 \tag{4.2}$$

then, for all $T = [\![R,\eta,\theta]\!] \in \mathcal{I}_{H,D}$, one has

$$\begin{aligned}
\mathcal{F}_F(T) &\geq \int_{R^*} f\left(P, \frac{1}{\|\eta_{0^H}(P)\|}\right) \frac{1}{\|\eta_{0^H}(P)\|} \|\eta_{0^H}(P)\|\, \theta(P)\, d\mathcal{H}^1(P) \\
&\geq m \int_{R^*} \theta(P)\, d\mathcal{H}^1(P) \\
&= m\, |T|(R^*).
\end{aligned} \tag{4.3}$$

It follows that the restriction of $\mathcal{F}_F$ to $\mathcal{L}^*_{H,D}$ is cohercive with respect to the mass of currents. Indeed (4.3) and Remark 3.2 imply

$$\mathcal{F}_F(T) \geq m\,\mathbf{M}(T)$$

for all $T \in \mathcal{L}^*_{H,D}$. We finally observe that, for $p \geq 1$, the function F_p defined in Example 4.1 satisfies (4.2) with $m = 1$. Hence, if F is a standard integrand satisfying $F \geq c\,F_p$ with $p \geq 1$ and $c > 0$, then $\mathcal{F}_F$ is cohercive in $\mathcal{L}^*_{H,D}$. Indeed in such a case one has

$$\mathcal{F}_F(T) \geq c\,\mathcal{F}_{F_p}(T) \geq c\,|T|(R^*)$$

for all $T \in \mathcal{I}_{H,D}$, where R denotes the rectifiable carrier of T.

Remark 4.2. Let $[\![R, \eta, \theta]\!] \in \mathcal{I}_{H,D}$. Then [15, Theorem 3.2.22] implies that $X_0^{-1}(x) \cap R^*$ is countable for $\mathcal{H}^1$-a.e. $x \in X_0 R$ and

$$+\infty > \mathbf{M}([\![R, \eta, \theta]\!]) = \int_R \theta\, d\mathcal{H}^1 \geq \int_{R^*} \theta\, d\mathcal{H}^1 = \int_{X_0 R} \sum_{P \in X_0^{-1}(x) \cap R^*} \frac{\theta(P)}{\|\eta_{0^H}(P)\|}\, d\mathcal{H}^1(x).$$

Hence

$$\# \left(X_0^{-1}(x) \cap R^* \right) \leq \sum_{P \in X_0^{-1}(x) \cap R^*} \frac{\theta(P)}{\|\eta_{0^H}(P)\|} < +\infty \tag{4.4}$$

for $\mathcal{H}^1$-a.e. $x \in X_0 R$.

We have the following result about closure, semicontinuity and compactness in $\mathcal{G}^*_{H,D}$. In particular, the proof of the first claim is based on [16, Theorem 4.4.2].

Theorem 4.1. *Let F be a standard integrand, $\{T_h\}_{h=1}^{\infty}$ be in $\mathcal{G}^*_{H,D}$ and assume that*

$$\sup_h \mathcal{F}_F(T_h) < +\infty. \tag{4.5}$$

The following claims hold:

(1) *If $T_h \rightharpoonup T$ as $h \to \infty$, where T is a one-dimensional integral current in X_H, then one has*

$$T \in \mathcal{G}^*_{H,D}, \qquad \mathcal{F}_F(T) \leq \liminf_h \mathcal{F}_F(T_h).$$

(2) *Assume that the functional $\mathcal{F}_F$ is cohercive in $\mathcal{G}^*_{H,D}$ with respect to the mass of currents (e.g. $F \geq c\,F_p$, with $p \geq 1$ and $c > 0$) and also that*

$$\sup_h \mathbf{M}(\partial T_h) < +\infty.$$

Then there exists a subsequence $\{T_{h_j}\}_{j=1}^{\infty}$ and $T \in \mathcal{G}^*_{H,D}$ such that

$$T_{h_j} \rightharpoonup T$$

and

$$\mathcal{F}_F(T) \leq \lim_j \mathcal{F}_F(T_{h_j}) = \liminf_h \mathcal{F}_F(T_h).$$

Proof. (1) Let η_h denote the orientation vector field of T_h and consider the sequence of measure-function pairs (μ_h, f_h) with

$$\mu_h := |(T_h)_{0^H}|, \qquad f_h := \frac{\eta_h}{\|(\eta_h)_{0^H}\|}.$$

Then one has

$$\mu_h \to \mu := |T_{0^H}| \qquad (\text{as } h \to \infty)$$

in the weak* sense of measures, by Proposition 3.1(2). Moreover

$$\sup_h \int F(P, f_h(P))\, d\mu_h(P) < +\infty$$

by the assumption (4.5).

From now on the proof strictly follows the same lines as that of [9, Theorem 4.1].

(2) It is an immediate consequence of the claim (1) and of the Federer-Fleming Compactness Theorem [19, 27.3] (also by recalling Remark 4.1 above). □

Using Theorem 4.1(2), we can now apply the direct method of the calculus of variations in order to minimize functionals in suitable classes of Gaussian towers. An example is provided by the following easy corollary, the proof of which also needs Proposition 3.1(2).

Corollary 4.1. *Let a one-dimensional rectifiable subset M of X_0 and a finite mass zero dimensional current S in X_H be given in such a way that*

$$\mathcal{D} := \left\{ T = [\![G, \eta, \theta]\!] \in \mathcal{G}^*_{H,D} \mid \partial T = S,\, M \subset X_0 G \right\}$$

is nonempty. Moreover, let F be a standard integrand such that $\mathcal{F}_F$ is cohercive in $\mathcal{G}^*_{H,D}$ (e.g. $F \geq c\, F_p$, with $p \geq 1$ and $c > 0$) and

$$\inf_{\mathcal{D}} \mathcal{F}_F < +\infty.$$

Then $\mathcal{F}_F|\mathcal{D}$ has a minimizer.

5. H-storey Legendrian towers over boundaries of regular plane sets

So, in the remainder of this section we shall assume $D = 2$. Moreover, a "regular set" will be an open subset of X_0 such that its boundary has a regular parametrization of class C^{H+1}. In particular (if A is regular) such a parametrization, denoted in the remainder by

$$\gamma : [a, b] \to X_0,$$

can be chosen in such a way that it induces the positive orientation of ∂A. Then a family of C^1-maps

$$\gamma_\alpha = (\gamma^1_\alpha, \gamma^2_\alpha) : [a, b] \to \mathbf{R}^2, \qquad \alpha \in \{0, 1\}^H$$

can be defined by setting

$$\gamma^1_{0^H} := \gamma \cdot e^1_{0^H}, \qquad \gamma^2_{0^H} := \gamma \cdot e^2_{0^H}$$

and

$$\gamma_{0^{H-1-n}1\beta} := \frac{\gamma'_{0^{H-n}\beta}}{\left[\sum_{\mu\in\{0,1\}^n}(\gamma'_{0^{H-n}\mu})^2\right]^{1/2}}$$

for all $n = 0, \dots, H-1$ and $\beta \in \{0,1\}^n$. Consider the X_H-valued map

$$\Gamma(t) := \sum_{\alpha\in\{0,1\}^H}\sum_{i=1}^{2}\gamma^i_\alpha(t)\, e^i_\alpha, \qquad t \in [a,b] \tag{5.1}$$

and define the one-dimensional current

$$T_A := [\![G_A, \eta_A, 1]\!]$$

where

$$G_A := \Gamma([a,b])$$

and η_A is the unit vector field orienting G_A such that

$$\eta_A \circ \Gamma(t) = \frac{\Gamma'(t)}{\|\Gamma'(t)\|}, \qquad t \in [a,b].$$

As one can easiliy verify, the current T_A does not depend on the choice of γ and

$$T_A \in \mathcal{G}^*_{H,1}.$$

It is called "the special H-storey Gaussian tower associated to A".

Remark 5.1. Observe that obviously, if $\gamma : [a,b] \to X_0$ is a regular C^H parametrization for ∂A oriented positively, then

$$t \mapsto \gamma(-t), \qquad t \in [-b,-a]$$

provides a regular C^H parametrization for $\partial(\mathbf{R}^2\backslash A)$ oriented positively. Hence A is a regular set if and only if $\mathbf{R}^2\backslash A$ is a regular set. In such a case, one has

$$T_{\mathbf{R}^2\backslash A} = \Psi_\# T_A$$

namely

$$G_{\mathbf{R}^2\backslash A} = \Psi G_A, \qquad \eta_{\mathbf{R}^2\backslash A} = \Psi \circ \eta_A$$

where $\Psi : X_H \to X_H$ is defined by

$$\Psi|X_0 = \mathrm{Id}_{X_0}, \qquad \Psi|Y_n = (-1)^{n+1}\mathrm{Id}_{Y_n} \quad (n = 0, \dots, H-1).$$

Continuing to trace the path in [9], we now pass to study the lower semicontinuity properties of the integral functionals defined in Section 4, with respect to the L^1-convergence of open subsets in $\mathbf{R}^2$.

First of all we will give some results about the weak limit of a sequence of towers over boundaries of regular sets converging in measure to a regular set.

Proposition 5.1. *Let A_j $(j = 1, 2, \dots)$ and A be regular sets such that*

$$A_j \to A \text{ in } L^1_{loc}(X_0)$$

and

$$T_{A_j} \rightharpoonup T \in \mathcal{I}_{H,1}.$$

Then

1. *The current T is a H-storey Gaussian tower;*
2. *If $[\![\partial A]\!]$ denotes the one-dimensional current of multiplicity one in X_0 carried by ∂A, equipped with the positive orientation, one has*

$$(X_0)_\# T = [\![\partial A]\!].$$

Proof. (1) It follows from Remark 3.3.

(2) By (i) in Definition 3.1 the carriers of the T_{A_j} and of T have equibounded projections in each space Y_n. Hence $T_{A_j}(X_0^\# \omega)$ and $T(X_0^\# \omega)$ make sense for all $\omega \in \mathcal{D}^1(X_0)$ and one has

$$\lim_j T_{A_j}(X_0^\# \omega) = T(X_0^\# \omega) = (X_0)_\# T\,(\omega). \tag{5.2}$$

Moreover, by proceeding similarly as in the proof of [2, Proposition 4.3], we find

$$T_{A_j}(X_0^\# \omega) = (X_0)_\# T_{A_j}\,(\omega) = [\![\partial A_j]\!](\omega) = \int_{A_j} d\omega$$

for all $\omega \in \mathcal{D}^1(X_0)$. By letting $j \to \infty$ and recalling (5.2), we get

$$(X_0)_\# T\,(\omega) = \int_A d\omega = [\![\partial A]\!](\omega)$$

for all $\omega \in \mathcal{D}^1(X_0)$. □

Proposition 5.2. *Let A be a regular set, let $T = [\![R, \eta, \theta]\!]$ be a H-storey Legendrian tower such that*

$$(X_0)_\# T = [\![\partial A]\!]$$

and let R^* be defined as in (4.1). One has

1. *$\mathcal{H}^1(\partial A \backslash X_0 R) = 0$;*
2. *Let τ denote the orientation of $[\![\partial A]\!]$ and consider the measurable map*

$$\zeta : X_0 R \to X_0, \qquad \zeta(x) := \sum_{P \in X_0^{-1}(x) \cap R^*} \frac{\eta_{0^H}(P)}{\|\eta_{0^H}(P)\|} \theta(P)$$

which is well-defined by (4.4). Then

$$\zeta | \partial A = \tau, \qquad \zeta | (X_0 R \backslash \partial A) = 0.$$

Proof. First of all observe that all the projections $Y_n R$ are bounded, hence $T(X_0^{\#}\omega)$ makes sense for all $\omega \in \mathcal{D}^1(X_0)$ and

$$\begin{aligned}(X_0)_{\#}T(\omega) &= T(X_0^{\#}\omega)) \\ &= \int_R \langle \eta\,,\, X_0^{\#}\omega\rangle \theta\, d\mathcal{H}^1 \\ &= \int_{R^*} \langle \eta_{0^H}\,, X_0^{\#}\omega\rangle \theta\, d\mathcal{H}^1 \\ &= \int_{X_0 R} \langle \zeta\,,\, \omega\rangle\, d\mathcal{H}^1.\end{aligned}$$

On the other hand, one has

$$(X_0)_{\#}T(\omega) = [\![\partial A]\!] = \int_{\partial A} \langle \tau\,,\, \omega\rangle\, d\mathcal{H}^1.$$

Finally (1) and (2) follow at once by equating the two formulas. $\square$

Proposition 5.3. *Let A be a regular set and $T = [\![R, \eta, \theta]\!]$ be a H-storey Gaussian tower such that*

$$(X_0)_{\#}T = [\![\partial A]\!].$$

Let τ denote the orientation of $[\![\partial A]\!]$, let R^* be defined as in (4.1) and consider the measurable function

$$\sigma : R \to \{0, \pm 1\}, \qquad P \mapsto sign\Big[\eta_{0^H}(P) \cdot \tau(X_0 P)\Big].$$

Then

(1) $\eta_{0^H} = \sigma \|\eta_{0^H}\|\, \tau \circ X_0$;
(2) $\sigma|(R^* \cap G_A) = 1$ *and* $\sigma|(R^* \cap G_{\mathbf{R}^2\setminus A}) = -1$;
(3) *Except for null-measure sets, one has*

$$G_A \;\subset\; R^* \cap X_0^{-1}(\partial A) \;\subset\; G_A \cup G_{\mathbf{R}^2\setminus A}$$

namely:

$$\mathcal{H}^1(G_A \backslash (R^* \cap X_0^{-1}(\partial A))) = 0 \tag{5.3}$$

and

$$\mathcal{H}^1(R^* \cap X_0^{-1}(\partial A) \setminus (G_A \cup G_{\mathbf{R}^2\setminus A})) = 0. \tag{5.4}$$

Proof. Let $\{T_j\}$ be as in Theorem 3.1(1). Then let us fix j arbitrarily and consider

$$\Gamma : [0, \mathbf{M}(T_j)] \to X_H$$

as in Theorem 3.1(3). Define

$$E := \{t \in [0, \mathbf{M}(T_j)]\,|\, \Gamma'(t) \text{ exists and (3.8) holds at } t,\ X_0\Gamma'(t) \neq 0,\ X_0\Gamma(t) \in \partial A\}$$

and consider $\varepsilon > 0$ (arbitrary). By the Lusin Theorem there exists a closed subset E_ε of E such that

$$(X_0\Gamma')|E_\varepsilon \text{ is continuous} \tag{5.5}$$

and

$$\mathcal{L}^1(E \backslash E_\varepsilon) \leq \varepsilon.$$

Observe that if E_ε^* denotes the set of the density points of E_ε then

$$E_\varepsilon^* \subset E_\varepsilon, \qquad \mathcal{L}^1(E_\varepsilon \backslash E_\varepsilon^*) = 0.$$

Now consider a positively oriented parametrization

$$\lambda : (a, b) \to X_0$$

of a connected arc of ∂A. We can assume $\|\lambda'\| \equiv 1$, hence λ has to be of class C^{H+1}. Also we can choose the interval (a, b) small enough so that

$$\|\lambda'(s_2) - \lambda'(s_1)\| < \frac{1}{2}, \qquad s_1, s_2 \in (a, b).$$

Thus, for all $s_1, s_2 \in (a, b)$, one has

$$\begin{aligned}
\|\lambda(s_2) - \lambda(s_1)\| &= \left\| \int_{s_1}^{s_2} \lambda' \right\| = \left\| \int_{s_1}^{s_2} \lambda'(s_1) - \int_{s_1}^{s_2} [\lambda'(s_1) - \lambda'] \right\| \\
&\geq \left\| \int_{s_1}^{s_2} \lambda'(s_1) \right\| - \left\| \int_{s_1}^{s_2} [\lambda'(s_1) - \lambda'] \right\| \\
&\geq \|\lambda'(s_1)\| \, |s_2 - s_1| - \left| \int_{s_1}^{s_2} \|\lambda'(s_1) - \lambda'\| \right| \qquad (5.6) \\
&> |s_2 - s_1| - \frac{1}{2}|s_2 - s_1| \\
&= \frac{1}{2}|s_2 - s_1|.
\end{aligned}$$

In particular it follows that λ is injective.

Set

$$J_\varepsilon := \{t \in E_\varepsilon^* \mid X_0\Gamma(t) \in \mathrm{Im}(\lambda)\} = E_\varepsilon^* \cap (X_0\Gamma)^{-1}(\mathrm{Im}(\lambda)).$$

and

$$s(t) := \lambda^{-1}\left(X_0\Gamma(t)\right).$$

Observe that

$$X_0\Gamma'(t) = [X_0\Gamma'(t) \cdot \lambda'(s(t))] \; \lambda'(s(t)) = \sigma(\Gamma(t))\|X_0\Gamma'(t)\| \, \lambda'(s(t)) \qquad (5.7)$$

for all $t \in J_\varepsilon$, hence

$$\eta_{j,0^H} \circ \Gamma = \sigma \circ \Gamma \, \|\eta_{j,0^H} \circ \Gamma\| \, \tau \circ X_0 \circ \Gamma$$

a.e. in E_ε^*, by Theorem 3.1(3). Then the equality (1) follows from the arbitrariness of ε and j, by also recalling 3.1(1).

For $t_0, t \in J_\varepsilon$, one has

$$X_0\Gamma(t) = \lambda(s(t)) = \lambda(s(t_0)) + (s(t) - s(t_0))\lambda'(s(t_0)) + o(s(t) - s(t_0))$$

whereby

$$\frac{X_0\Gamma(t) - X_0\Gamma(t_0)}{t - t_0} = \frac{s(t) - s(t_0)}{t - t_0}\lambda'(s(t_0)) + \frac{o(s(t) - s(t_0))}{t - t_0}$$

namely (recalling that λ is a unit speed parametrization)

$$\frac{s(t)-s(t_0)}{t-t_0} = \lambda'(s(t_0)) \cdot \frac{X_0\Gamma(t) - X_0\Gamma(t_0)}{t-t_0} + \frac{o(s(t)-s(t_0))}{t-t_0}$$

i.e.

$$\frac{s(t)-s(t_0)}{t-t_0}\left(1+\frac{o(s(t)-s(t_0))}{s(t)-s(t_0)}\right) = \lambda'(s(t_0)) \cdot \frac{X_0\Gamma(t) - X_0\Gamma(t_0)}{t-t_0}$$

for all $t_0, t \in J_\varepsilon$. It follows that

$$\lim_{\substack{t\to t_0\\ t\in J_\varepsilon}} \frac{s(t)-s(t_0)}{t-t_0} = \lambda'(s(t_0)) \cdot X_0\Gamma'(t_0) = \|X_0\Gamma'(t_0)\|\sigma(\Gamma(t_0)) \tag{5.8}$$

for all $t_0 \in J_\varepsilon$.

Now we are in position to show that if $\Lambda : (a,b) \to X_H$ denotes the "tower parametrization" induced by λ (i.e. as in (5.1), with λ in place of γ), then one has

$$Y_n\Gamma(t) = \sigma(\Gamma(t))^{n+1} Y_n\Lambda(s(t)), \qquad n = 0, \dots, H-1 \tag{5.9}$$

for all $t \in J_\varepsilon$. We will prove it by induction.

First of all, by invoking (5.7) and (3.8) with $n = 0$ (and also recalling that $\sigma_0 \equiv 1$), we get

$$P_0\left(Y_0\Gamma(t)\right) = \sigma(\Gamma(t))\, P_0\left(Y_0\Lambda(s(t))\right)$$

for all $t \in J_\varepsilon$. The equality (5.9) for $n = 0$ follows.

Now let us assume (5.9) be true for $n = 0, 1, \dots, h$ and show that

$$Y_{h+1}\Gamma(t) = \sigma(\Gamma(t))^{h+2} Y_{h+1}\Lambda(s(t)) \tag{5.10}$$

for all $t \in J_\varepsilon$. Indeed , if $t \in J_\varepsilon$ and consider $\{t_i\} \subset J_\varepsilon$ converging to t, then

$$\lim_i \sigma(\Gamma(t_i)) = \sigma(\Gamma(t))$$

by (5.5). Hence, without loss of generality, we can suppose

$$\sigma(\Gamma(t_i)) = \sigma(\Gamma(t)) \tag{5.11}$$

for all i. It follows that

$$\frac{Y_h\Gamma(t_i) - Y_h\Gamma(t)}{t_i - t} = \sigma(\Gamma(t))^{h+1} \frac{Y_h\Lambda(s(t_i)) - Y_h\Lambda(s(t))}{t_i - t}.$$

Letting $i \to \infty$ and invoking (5.8), we obtain

$$Y_h\Gamma'(t) = \sigma(\Gamma(t))^h \|X_0\Gamma'(t)\| Y_h\Lambda'(s(t))$$

namely

$$\|X_{h+1}\Gamma'(t)\| Y_h(P_{h+1}(Y_{h+1}\Gamma(t))) = \sigma(\Gamma(t))^h \|X_0\Gamma'(t)\| \|X_{h+1}\Lambda'(s(t))\| Y_h(P_{h+1}(Y_{h+1}\Lambda(s(t))).$$

by (3.8). Thus we are reduced to verify that

$$\|X_{h+1}\Gamma'(t)\| = \|X_{h+1}\Lambda'(s(t))\| \|X_0\Gamma'(t)\|. \tag{5.12}$$

In order to prove such an equality, observe that (by assumption!)

$$X_{h+1}\Gamma(\tau) = X_0\Gamma(\tau) + \sum_{n=0}^{h} Y_n\Gamma(\tau) = X_0\Lambda(s(\tau)) + \sum_{n=0}^{h} \sigma(\Gamma(\tau))^{n+1} Y_n\Lambda(s(\tau))$$

for all $\tau \in J_\varepsilon$. By also recalling (5.11), we get

$$X_{h+1}\left(\frac{\Gamma(t_i)-\Gamma(t)}{t_i-t}\right) = X_0\left(\frac{\Lambda(s(t_i))-\Lambda(s(t))}{t_i-t}\right) + \sum_{n=0}^{h} \sigma(\Gamma(t))^{n+1} Y_n\left(\frac{\Lambda(s(t_i))-\Lambda(s(t))}{t_i-t}\right)$$

thus, as usual letting $i \to \infty$ and recalling (5.8), it follows that

$$X_{h+1}\Gamma'(t) = \left[X_0\Lambda'(s(t)) + \sum_{n=0}^{h} \sigma(\Gamma(t))^{n+1} Y_n\Lambda'(s(t))\right] \sigma(\Gamma(t))\|X_0\Gamma'(t)\|$$

which implies at once (5.12), hence the formula (5.9).

Let us set

$$E^* := \cup_{\varepsilon>0} E^*_\varepsilon.$$

Then (5.9) and Remark 5.1 imply that:

- If $t \in E^*$ and $\sigma(\Gamma(t)) = 1$, then $\Gamma(t) \in G_A$;
- If $t \in E^*$ and $\sigma(\Gamma(t)) = -1$, then $\Gamma(t) \in G_{\mathbf{R}^2\backslash A}$.

Moreover

- For a.e. $t \in [0, \mathbf{M}(T_j)]\backslash E^*$ one has $X_0\Gamma(t) \notin \partial A$ or $\sigma(\Gamma(t)) = 0$.

Hence we find (denoting the carrier of T_j by R_j)

- $\sigma = 1$ a.e. in $R_j^* \cap G_A$;
- $\sigma = -1$ a.e. in $R_j^* \cap G_{\mathbf{R}^2\backslash A}$.

Now the assertion (2) follows at once from the arbitrariness of j, taking into account Theorem 3.1(1).

In order to prove (3), let us again recall (5.9) and Remark 5.1. We get

$$\Gamma(J_\varepsilon) \subset G_A \cup G_{\mathbf{R}^2\backslash A}$$

for all $\varepsilon > 0$, hence

$$\Gamma(E^*) \subset G_A \cup G_{\mathbf{R}^2\backslash A}.$$

Then

$$\begin{aligned}\mathcal{H}^1\left(\Gamma(E)\backslash(G_A\cup G_{\mathbf{R}^2\backslash A})\right) &\leq \mathcal{H}^1\left(\Gamma(E)\backslash\Gamma(E^*)\right)\\ &\leq \mathcal{H}^1\left(\Gamma(E\backslash E^*)\right)\\ &\leq (\mathrm{Lip}\ \Gamma)\,\mathcal{L}^1(E\backslash E^*)\\ &= 0\end{aligned}$$

which proves (5.4).

It remains to verify (5.3). To this aim, observe that Proposition 5.2(2) and the assertion (1) above imply

$$1=\zeta(x)\cdot\tau(x)=\sum_{P\in X_0^{-1}(x)\cap R^*}\frac{\eta_{0^H}(P)\cdot\tau(x)}{\|\eta_{0^H}(P)\|}\theta(P)=\sum_{P\in X_0^{-1}(x)\cap R^*}\sigma(P)\theta(P)$$

for a.e. $x\in\partial A$. By recalling the assertion (2) and the equality (5.4), we obtain

$$1=\varphi_{R^*}(X_0^{-1}(x)\cap G_A)\theta(X_0^{-1}(x)\cap G_A)\ -\ \varphi_{R^*}(X_0^{-1}(x)\cap G_{\mathbf{R}^2\backslash A})\theta(X_0^{-1}(x)\cap G_{\mathbf{R}^2\backslash A})$$

for a.e. $x\in\partial A$. Hence it follows

$$X_0^{-1}(x)\cap G_A\in R^*$$

for a.e. $x\in\partial A$, which is equivalent to (5.3). □

We are finally ready to prove the following result extending [9, Theorem 5.1] to the context of H-storey Gaussian towers. For the convenience of the reader we provide the complete proof, even if it follows strictly the lines of [9].

Theorem 5.1. *Let F be a standard integrand such that $F\geq c\,F_p$, with $p\geq 1$ and $c>0$. Consider the functional on the class of Lebesgue measurable subsets of $\mathbf{R}^2$, defined as follows*

$$\mathcal{E}_F(A):=\begin{cases}\mathcal{F}_F(T_A) & \text{if } A \text{ is a regular set,}\\ +\infty & \text{otherwise}\end{cases}$$

and let $\overline{\mathcal{E}}_F$ denote the lower semicontinuous envelope of $\mathcal{E}_F$ with respect to the $L^1_{loc}(\mathbf{R}^2)$ topology, namely

$$\overline{\mathcal{E}}_F(E):=\inf\left\{\liminf_h\mathcal{E}_F(E_h)\,\Big|\,E_h\to E\ \text{in}\ L^1_{loc}\right\}.$$

Then

(1) *One has*

$$\overline{\mathcal{E}}_F(A)=\mathcal{E}_F(A)$$

whenever A is a regular set. In other words, the functional $\mathcal{E}_F$ is L^1_{loc}-lower semicontinuous on the family of regular sets, i.e. if A and A_h ($h=1,\dots,\infty$) are regular sets such that $A_h\to A$ in L^1_{loc} then one has

$$\mathcal{E}_F(A)\leq\liminf_h\mathcal{E}_F(A_h). \tag{5.13}$$

(2) *If the equality in (5.13) holds and its members are finite, i.e. if there exists a subsequence* $\{A_{h'}\}$ *such that*

$$\mathcal{E}_F(A) = \lim_{h'} \mathcal{E}_F(A_{h'}) < +\infty,$$

then one has

$$T_{A_{h'}} \to T_A.$$

Moreover, if $Q \mapsto F(P,Q)$ *is strictly convex (for all* $P \in X_H$*) then one also has*

$$\mathcal{E}_\Phi(A) = \lim_{h'} \mathcal{E}_\Phi(A_{h'})$$

for all $\Phi \in C_c(X_H \times X_H)$*.*

Proof. (1) Let A and A_h ($h = 1, \ldots, \infty$) be regular sets such that $A_h \to A$ in $L^1_{\rm loc}$. Without loss of generality we can assume

$$\liminf_h \mathcal{E}_F(A_h) < +\infty$$

hence a subsequence $\{A_{h'}\}$ of $\{A_h\}$ has to exist such that

$$\lim_{h'} \mathcal{F}_F(T_{A_{h'}}) = \lim_{h'} \mathcal{E}_F(A_{h'}) = \liminf_h \mathcal{E}_F(A_h). \tag{5.14}$$

From Theorem 4.1(2) it follows that for every sequence $\{h''\} \subset \{h'\}$ there exist $\{h'''\} \subset \{h''\}$ and a null-boundary current

$$T = [\![R, \eta, \theta]\!] \in \mathcal{G}^*_{H,1}$$

such that

$$T_{A_{h'''}} \rightharpoonup T, \qquad \mathcal{F}_F(T) \le \lim_{h'''} \mathcal{F}_F(T_{A_{h'''}}). \tag{5.15}$$

By recalling Proposition 5.1 and (5.3), we obtain

$$\mathcal{H}^1(G_A \backslash (R^* \cap X_0^{-1}(\partial A))) = 0$$

hence

$$\mathcal{H}^1(G_A \backslash R) = 0. \tag{5.16}$$

Moreover, by recalling

- the assertions (1) and (3) of Theorem 3.1,
- the assertion (2) of Proposition 5.3,
- the formulas (5.8) and (5.9),

we easily obtain that

$$\eta | G_A = \eta_A. \tag{5.17}$$

Invoking (5.14), (5.15), (5.16) and (5.17), we finally get the semicontinuity inequality (5.13):

$$
\begin{aligned}
\liminf_h \mathcal{E}_F(A_h) &= \lim_{h'''} \mathcal{E}_F(A_{h'''}) = \lim_{h'''} \mathcal{F}_F(T_{A_{h'''}}) \geq \mathcal{F}_F(T) \\
&= \int_R F\left(P, \frac{\eta(P)}{\|\eta_{0^H}(P)\|}\right) \|\eta_{0^H}(P)\| \theta(P)\, d\mathcal{H}^1(P) \\
&= \int_{R\setminus G_A} F\left(P, \frac{\eta(P)}{\|\eta_{0^H}(P)\|}\right) \|\eta_{0^H}(P)\| \theta(P)\, d\mathcal{H}^1(P) \ + \\
&\qquad + \int_{G_A} F\left(P, \frac{\eta(P)}{\|\eta_{0^H}(P)\|}\right) \|\eta_{0^H}(P)\| \theta(P)\, d\mathcal{H}^1(P) \\
&\geq \int_{G_A} F\left(P, \frac{\eta_A(P)}{\|(\eta_A)_{0^H}(P)\|}\right) \|(\eta_A)_{0^H}(P)\|\, d\mathcal{H}^1(P) \\
&= \mathcal{E}_F(A).
\end{aligned} \tag{5.18}
$$

In order to prove (2), let us assume that the equality holds in (5.13). Then (5.18) yields

$$
\begin{aligned}
0 &= \int_{R\setminus G_A} F\left(P, \frac{\eta(P)}{\|\eta_{0^H}(P)\|}\right) \|\eta_{0^H}(P)\| \theta(P)\, d\mathcal{H}^1(P) \\
&\geq c \int_{R\setminus G_A} \frac{\theta}{\|\eta_{0^H}\|^{p-1}}\, d\mathcal{H}^1 \\
&\geq c\, \mathcal{H}^1(R\setminus G_A).
\end{aligned}
$$

By also recalling (5.16), it follows that (except for a null-measure set)

$$R = G_A.$$

Then, recalling again (5.18), we find

$$
\begin{aligned}
0 &= \int_R F\left(P, \frac{\eta(P)}{\|\eta_{0^H}(P)\|}\right) \|\eta_{0^H}(P)\| \big(\theta(P) - 1\big)\, d\mathcal{H}^1(P) \\
&\geq c \int_R \frac{\theta - 1}{\|\eta_{0^H}\|^{p-1}}\, d\mathcal{H}^1 \geq \int_R (\theta - 1)\, d\mathcal{H}^1
\end{aligned}
$$

hence $\theta \equiv 1$.

Thus we have proved that $T = T_A$. In particular the limit current T does not depend on the choice of the subsequence $\{h''\}$, whereby we conclude that

$$T_{A_{h'}} \rightharpoonup T = T_A.$$

The last statement in (2) follows at once from [16, Theorem 4.4.2], by setting

$$(\mu_{h'}, f_{h'}) := \left(|(T_{A_{h'}})_{0^H}|\,,\ \frac{\eta_{A_{h'}}}{\|(\eta_{A_{h'}})_{0^H}\|}\right), \qquad (\mu, f) := \left(|(T_A)_{0^H}|\,,\ \frac{\eta_A}{\|(\eta_A)_{0^H}\|}\right)$$

and recalling Proposition 3.1(2). □

6. Further results in the particular case of 2-storey Legendrian towers

Proposition 6.1. *Let T be a 2-storey Legendrian tower in X_2 and let T_j be as in claim (1) of Theorem 3.1. Denote with R (resp. R_j) the projection to X_0 of the carrier of T (resp. T_j) and consider a C^3-covering $\mathcal{A}$ of R (it exists by Theorem 3.1(2)!). Then one has*

(1) *$R_j \subset R$ (modulo null-measure sets) and $\alpha_R^{\mathcal{A}}|R_j = \alpha_{R_j}^{\mathcal{A}}$.*

Moreover, if

$$\Gamma : [0, \mathbf{M}(T_j)] \to X_2$$

is a Lipschitz parametrization of T_j with the properties stated in Theorem 3.1(3) and also define

$$\gamma_\alpha = (\gamma_\alpha^1, \dots, \gamma_\alpha^D) : [0, \mathbf{M}(T_j)] \to \mathbf{R}^D, \qquad \alpha \in \{0,1\}^2$$

by

$$\gamma_\alpha^i := \Gamma \cdot e_\alpha^i, \qquad i = 1, \dots, D$$

then the following equalities (where $*$ is the usual Hodge star operator in $\mathbf{R}^D$)

(2) $\alpha_R^{\mathcal{A}} \circ (X_0\Gamma) = \frac{\|(*\gamma_{01}) \llcorner \gamma_{11}\|}{\|\gamma_{10}\|}$

(3) $\langle (apD\alpha_R^{\mathcal{A}}) \circ (X_0\Gamma)\, ,\, \gamma_{00}' \rangle = \left(\frac{\|(*\gamma_{01}) \llcorner \gamma_{11}\|}{\|\gamma_{10}\|}\right)'$

hold almost everywhere in

$$E := \Big\{ t \in [0, \mathbf{M}(T_j)] \,\Big|\, \gamma_{00}'(t) \text{ exists and } \gamma_{00}'(t) \neq 0 \Big\}.$$

Proof. (1) The inclusion $R_j \subset R$ (modulo null-measure sets) follows trivially from (3.7). Hence $\mathcal{A}$ covers R_j too and the conclusion follows from the definition of absolute curvature given in [12] and summarized in §2.

Now on, we will concentrate on a (arbitrarily chosen) curve C of $\mathcal{A}$. Without affecting the generality of our argument, we will assume that

$$C = G_f := \Big\{ xu + f(x) \,\Big|\, x \in \mathbf{R} \Big\}$$

where u is a unit vector in X_0 and

$$f : \mathbf{R} \to (\mathbf{R}u)^\perp$$

is a function of class C^3 (with $(\cdot)^\perp$ we denote the orthogonal complement in X_0).

For the sake of simplicity, without loss of generality, we will identify

$$X_0\Gamma,\ Y_0\Gamma,\ X_1\Gamma,\ Y_1\Gamma$$

with

$$\gamma_{00},\ \gamma_{01},\ (\gamma_{00},\gamma_{01}),\ (\gamma_{10},\gamma_{11}),$$

respectively. Then the identities (3.8), which have to hold almost everywhere with $n = 0, 1$, assume the form

$$\gamma_{00}' = \sigma_0\|\gamma_{00}'\|\gamma_{01}, \qquad (\gamma_{00}',\gamma_{01}') = \sigma_1\|(\gamma_{00}',\gamma_{01}')\|(\gamma_{10},\gamma_{11}). \tag{6.1}$$

Let us define

$$L := \gamma_{00}^{-1}(G_f) \cap \{t \,|\, \gamma_{00}'(t), \gamma_{01}'(t) \text{ exist}, \gamma_{00}'(t) \neq 0 \text{ and (6.1) holds }\}.$$

We can assume

$$\mathcal{L}^1(L) > 0 \tag{6.2}$$

the null-measure case being trivial, as we shall understand below. Then, also invoking the regularity of $\mathcal{L}^1$, we can find $\varepsilon_0 > 0$ such that a closed subset L_ε of L satisfying

$$\mathcal{L}^1(L\backslash L_\varepsilon) \leq \varepsilon, \qquad \mathcal{L}^1(L_\varepsilon) > 0.$$

has to exist for all $\varepsilon \in (0, \varepsilon_0]$. If L_ε^* denotes the set of the density points of L_ε, one has

$$L_\varepsilon^* \subset L_\varepsilon, \qquad \mathcal{L}^1(L_\varepsilon\backslash L_\varepsilon^*) = 0$$

the first one due to the fact that L_ε is closed. If set

$$L^* := \cup_{\varepsilon\in(0,\varepsilon_0]} L_\varepsilon^*.$$

then

$$L^* \subset L, \qquad \mathcal{L}^1(L\backslash L^*) = 0. \tag{6.3}$$

Since

$$L \subset \gamma_{00}^{-1}(G_f) \cap E, \qquad \mathcal{H}^1\left(\gamma_{00}^{-1}(G_f) \cap E\backslash L\right) = 0$$

by definition, it follows from (6.3) that one also has

$$L^* \subset \gamma_{00}^{-1}(G_f) \cap E, \qquad \mathcal{H}^1\left(\gamma_{00}^{-1}(G_f) \cap E\backslash L^*\right) = 0. \tag{6.4}$$

Observe that $\gamma_{00}(L^*) \subset \gamma_{00}(L)$ and

$$\mathcal{H}^1\left(\gamma_{00}(L)\backslash\gamma_{00}(L^*)\right) \leq \mathcal{H}^1\left(\gamma_{00}(L\backslash L^*)\right) = \int_{L\backslash L^*} \|\gamma_{00}'\| = 0.$$

Moreover, one obviously has

$$\gamma_{00}(L) \subset G_f \cap \gamma_{00}([0,\mathbf{M}(T_j)]), \qquad \mathcal{H}^1\left(G_f \cap \gamma_{00}([0,\mathbf{M}(T_j)]) \setminus \gamma_{00}(L)\right) = 0$$

hence also

$$\gamma_{00}(L^*) \subset G_f \cap \gamma_{00}([0,\mathbf{M}(T_j)]), \qquad \mathcal{H}^1\left(G_f \cap \gamma_{00}([0,\mathbf{M}(T_j)]) \setminus \gamma_{00}(L^*)\right) = 0.$$

Recalling [17, Theorem 16.2], we conclude that

$$\gamma_{00}(L^*) \sim \{\text{points of density of } G_f \cap \gamma_{00}([0,\mathbf{M}(T_j)])\} \tag{6.5}$$

where $\sim$ means "is equivalent with respect to the measure $\mathcal{H}^1$ to".

Preliminary to proving (2) and (3), we further need the following version of the obvious Claim 3.1 stated above.

Claim 6.1. Let $\varphi, \psi : [a,b] \to \mathbf{R}$ be differentiable at s and assume that $\varphi(t_h) = \psi(t_h)$ for some sequence $\{t_h\} \subset [a,b]$ converging to s, with $t_h \neq s$ (for all h). Then $\varphi'(s) = \psi'(s)$.

Consider $s \in L^*$ and observe that if define

$$x(t) := \gamma_{00}(t) \cdot u, \qquad t \in [0, \mathbf{M}(T_j)]$$

then one has

$$f(x(t)) = \gamma_{00}(t) - x(t)u$$

for all $t \in \gamma_{00}^{-1}(G_f)$. Now a sequence $\{t_h\}$ with

$$t_h \in L^* \subset \gamma_{00}^{-1}(G_f), \qquad t_h \neq s \qquad \text{(for all } h\text{)}$$

and

$$t_h \to s \qquad \text{(as } h \uparrow \infty\text{)}$$

has to to exist, hence Claim 6.1 yields

$$f'(x(s))x'(s) = \gamma_{00}'(s) - x'(s)u.$$

As a consequence, for all $s \in L^*$, one has

$$x'(s) = \gamma_{00}'(s) \cdot u \neq 0$$

hence

$$\gamma_{01}(s) \cdot u \neq 0 \tag{6.6}$$

and

$$f'(x(s)) = \frac{\gamma_{01}(s)}{\gamma_{01}(s) \cdot u} - u. \tag{6.7}$$

by (6.1). Now, just the same argument can be invoked to differentiate (6.7) at any $s \in L^*$. We get

$$\begin{aligned} f''(x(s))x'(s) &= \frac{[\gamma_{01}(s) \cdot u]\gamma_{01}'(s) - [\gamma_{01}'(s) \cdot u]\gamma_{01}(s)}{[\gamma_{01}(s) \cdot u]^2} \\ &= \frac{[\gamma_{01}(s) \wedge \gamma_{01}'(s)] \llcorner u}{[\gamma_{01}(s) \cdot u]^2} \end{aligned}$$

namely

$$f''(x(s)) = \frac{[\gamma_{01}(s) \wedge \gamma_{11}(s)] \llcorner u}{[\gamma_{10}(s) \cdot u][\gamma_{01}(s) \cdot u]^2} \tag{6.8}$$

by (6.1).

We can finally proceed to prove (2) and (3).

Proof of (2). We shall prove that one has

$$\left. \frac{\|f''\|^2(1 + \|f'\|^2) - (f' \cdot f'')^2}{(1 + \|f'\|^2)^3} \right|_{x(s)} = \frac{\|(*\gamma_{01}(s)) \llcorner \gamma_{11}(s)\|^2}{\|\gamma_{10}(s)\|^2}, \qquad s \in L^*. \tag{6.9}$$

Then (2) will follow at once, by recalling the statement (6.5) and observing that the left hand side of (6.9) is just the square of the curvature of G_f at $(\gamma_{00}\cdot u)u + f(\gamma_{00}\cdot u)$ (as one can easily infer from (3.5) with $\gamma(x) = xu + f(x)$, by also recalling (3.4)).

In order to prove (6.9), first of all observe that the right hand side member makes sense, in that γ_{10} does not vanish in L (hence in L^*), by the second equality in (6.1). Also observe that

$$\left.\frac{\|f''\|^2(1+\|f'\|^2)-(f'\cdot f'')^2}{(1+\|f'\|^2)^3}\right|_{x(\cdot)} = \frac{\|(\gamma_{01}\wedge\gamma_{11})\llcorner u\|^2 - \left([(\gamma_{01}\wedge\gamma_{11})\llcorner u]\cdot\gamma_{01}\right)^2}{(\gamma_{01}\cdot u)^2\|\gamma_{10}\|^2}.$$

holds in L^*, by (6.7) and (6.8). Therefore, by also recalling (3.4), we remain to show that the equality

$$\frac{\|[(\gamma_{01}\wedge\gamma_{11})\llcorner u]\wedge\gamma_{01}\|}{|\gamma_{01}\cdot u|} = \|\gamma_{01}\wedge\gamma_{11}\| \tag{6.10}$$

holds in L^*.

To this aim consider $s\in L^*$, assume $\gamma_{01}(s)\wedge\gamma_{11}(s)\neq 0$ (otherwise there is nothing to prove!), denote by S the span of $\{\gamma_{01}(s),\gamma_{11}(s)\}$ and by $\tilde u$ the projection of u to S. Observe that $\tilde u\neq 0$, by (6.6), hence we can find an orthonormal basis $\{\varepsilon_1,\varepsilon_2\}$ of S such that

$$\varepsilon_1 = \frac{\tilde u}{\|\tilde u\|}.$$

Then one has

$$\begin{aligned}\|[(\gamma_{01}(s)\wedge\gamma_{11}(s))\llcorner u]\wedge\gamma_{01}(s)\| &= \|[(\gamma_{01}(s)\wedge\gamma_{11}(s))\llcorner \tilde u]\wedge\gamma_{01}(s)\| \\ &= \|\tilde u\|\|\gamma_{01}(s)\wedge\gamma_{11}(s)\|\|[(\varepsilon_1\wedge\varepsilon_2)\llcorner\varepsilon_1]\wedge\gamma_{01}(s)\|\end{aligned}$$

where

$$[(\varepsilon_1\wedge\varepsilon_2)\llcorner\varepsilon_1]\wedge\gamma_{01}(s) = \varepsilon_2\wedge\gamma_{01}(s) = (\gamma_{01}(s)\cdot\varepsilon_1)\,\varepsilon_2\wedge\varepsilon_1.$$

It follows that

$$\begin{aligned}\|[(\gamma_{01}(s)\wedge\gamma_{11}(s))\llcorner u]\wedge\gamma_{01}(s)\| &= \|\tilde u\|\|\gamma_{01}(s)\wedge\gamma_{11}(s)\||\gamma_{01}(s)\cdot\varepsilon_1| \\ &= \|\gamma_{01}(s)\wedge\gamma_{11}(s)\||\gamma_{01}(s)\cdot\tilde u|\end{aligned}$$

hence (6.10).

Proof of (3). Define

$$\rho(s) := \frac{\|(*\gamma_{01}(s))\llcorner\gamma_{11}(s)\|}{\|\gamma_{10}(s)\|},\qquad s\in L^*$$

and let Ω be the set of the $s\in L^*$ such that $\rho'(s)$ exists and $\gamma_{00}(s)$ is a point of density of $G_f\cap\gamma_{00}([0,\mathbf{M}(T_j)])$.

At almost every point of $\gamma_{00}(L^*\backslash\Omega)$, the set $G_f\cap\gamma_{00}([0,\mathbf{M}(T_j)])$ has not density one. It follows that

$$0 = \mathcal{H}^1\left(\gamma_{00}(L^*\backslash\Omega)\right) = \int_{L^*\backslash\Omega}\|\gamma_{00}'\|$$

by (6.5), hence

$$\mathcal{H}^1(L^*\backslash\Omega) = 0. \tag{6.11}$$

Then consider

$$s \in \Omega \subset L^*$$

and observe that L^* has density one at s. By (6.11) also Ω has density one at s, hence a sequence $\{s_n\}_{n=1}^{\infty}$ has to exist such that

$$\Omega \ni s_n \to s, \qquad s_n \neq s.$$

Since (6.9) holds, we get

$$\rho'(s) = \lim_n \frac{\rho(s_n) - \rho(s)}{s_n - s} = \lim_n \frac{\alpha_R^{\mathcal{A}} \circ \gamma_{00}(s_n) - \alpha_R^{\mathcal{A}} \circ \gamma_{00}(s)}{s_n - s}.$$

But $\alpha_R^{\mathcal{A}}$ is approximately differentiable at $\gamma_{00}(s)$ and one has

$$\alpha_R^{\mathcal{A}} \circ \gamma_{00}(s_n) - \alpha_R^{\mathcal{A}} \circ \gamma_{00}(s) = \Big\langle \mathrm{ap}D\alpha_R^{\mathcal{A}}(\gamma_{00}(s)), \gamma_{00}(s_n) - \gamma_{00}(s) \Big\rangle + o\Big(\gamma_{00}(s_n) - \gamma_{00}(s)\Big)$$

by Proposition 2.2 and by definition of $\mathrm{ap}D\alpha_R^{\mathcal{A}}$, hence the formula (3) holds in Ω.

The conclusion follows at once recalling that Ω is equivalent in measure to $\gamma_{00}^{-1}(G_f) \cap E$, by (6.4) and (6.11). $\square$

Remark 6.1. In the special case of a smooth plane curve, the following representation formula holds.

Proposition 6.2. *Assume $D = 2$ and let γ, Γ, η be as in Example 3.2. Then one has*

$$\frac{\eta}{\|\eta_{00}\|} \circ \Gamma = \Bigg(\gamma_{01}, \kappa_\gamma(*\gamma_{01}), \frac{\kappa_\gamma}{(1+\kappa_\gamma^2)^{3/2}}\left[(1+\kappa_\gamma^2)(*\gamma_{01}) - \frac{\kappa_\gamma'}{\|\gamma'\|}\gamma_{01}\right], \\ \frac{1}{(1+\kappa_\gamma^2)^{3/2}}\left[\frac{\kappa_\gamma'}{\|\gamma'\|}(*\gamma_{01}) - (1+\kappa_\gamma^2)\kappa_\gamma^2\gamma_{01}\right]\Bigg).$$

Proof. Denote by λ the reparametrization of $\gamma([a,b])$ by arc length satisfying $\lambda(0) = \gamma(a)$. By adopting the same notation as in Example 3.2, we have

$$\lambda_{00}' = \lambda_{01}, \qquad \lambda_{01}' = \kappa_\lambda(*\lambda_{01}).$$

Moreover

$$(\lambda_{10}, \lambda_{11}) = \frac{(\lambda', \lambda'')}{\|(\lambda', \lambda'')\|} = \frac{(\lambda_{01}, \kappa_\lambda(*\lambda_{01}))}{(1+\kappa_\lambda^2)^{1/2}}$$

by (3.2), hence

$$\begin{aligned}(\lambda_{10}, \lambda_{11})' &= \frac{(1+\kappa_\lambda^2)^{1/2}(\lambda_{01}', \kappa_\lambda'(*\lambda_{01}) + \kappa_\lambda(*\lambda_{01}')) - (1+\kappa_\lambda^2)^{-1/2}\kappa_\lambda\kappa_\lambda'(\lambda_{01}, \kappa_\lambda(*\lambda_{01}))}{1+\kappa_\lambda^2} \\ &= \frac{(\kappa_\lambda(1+\kappa_\lambda^2)(*\lambda_{01}) - \kappa_\lambda\kappa_\lambda'\lambda_{01}\,,\,(1+\kappa_\lambda^2)\kappa_\lambda'(*\lambda_{01}) - (1+\kappa_\lambda^2)\kappa_\lambda^2\lambda_{01} - \kappa_\lambda^2\kappa_\lambda'(*\lambda_{01}))}{(1+\kappa_\lambda^2)^{3/2}}\end{aligned}$$

namely

$$\lambda'_{10} = \frac{\kappa_\lambda}{(1+\kappa_\lambda^2)^{3/2}} \Big((1+\kappa_\lambda^2)(*\lambda_{01}) - \kappa'_\lambda \lambda_{01} \Big)$$

and

$$\lambda'_{11} = \frac{1}{(1+\kappa_\lambda^2)^{3/2}} \Big(\kappa'_\lambda (*\lambda_{01}) - (1+\kappa_\lambda^2)\kappa_\lambda^2 \lambda_{01} \Big) .$$

Recalling the equalities

$$\lambda = \gamma \circ \tau, \qquad \kappa_\lambda = \kappa_\gamma \circ \tau$$

where τ denotes the inverse function of $t \mapsto \int_a^t \|\gamma'\|$, it follows immediately that

$$\lambda'_{00} = \frac{\gamma'}{\|\gamma'\|} \circ \tau = \gamma_{01} \circ \tau, \qquad \lambda'_{01} = \kappa_\lambda(*\lambda_{01}) = [\kappa_\gamma(*\gamma_{01})] \circ \tau,$$

$$\lambda'_{10} = \left[\frac{\kappa_\gamma}{(1+\kappa_\gamma^2)^{3/2}} \left((1+\kappa_\gamma^2)(*\gamma_{01}) - \frac{\kappa'_\gamma}{\|\gamma'\|}\gamma_{01} \right) \right] \circ \tau$$

and

$$\lambda'_{11} = \left[\frac{1}{(1+\kappa_\gamma^2)^{3/2}} \left(\frac{\kappa'_\gamma}{\|\gamma'\|}(*\gamma_{01}) - (1+\kappa_\gamma^2)\kappa_\gamma^2 \gamma_{01} \right) \right] \circ \tau.$$

From

$$\frac{\eta}{\|\eta_{00}\|} \circ \Gamma \circ \tau = (\lambda'_{00}, \lambda'_{01}, \lambda'_{10}, \lambda'_{11})$$

we finally get the conclusion. □

The next result shows that the representation formula given in Proposition 6.2 holds for a general tower.

Proposition 6.3. *Let $D = 2$ and consider a 2-storey Gaussian tower $T = [\![G, \eta, \theta]\!]$ in X_2. For $\alpha \in \{0,1\}^2$, let us set*

$$\eta^i_\alpha := \eta \cdot e^i_\alpha, \qquad i = 1, 2$$

and

$$\eta_\alpha := (\eta^1_\alpha, \eta^2_\alpha).$$

The following facts hold

(1) *If T is indecomposable, let*

$$\Gamma : [0, \mathbf{M}(T)] \to X_2$$

be as in Theorem 3.1(3). For $\alpha \in \{0,1\}^2$, define

$$\gamma^i_\alpha := \Gamma \cdot e^i_\alpha, \qquad i = 1, 2$$

and

$$\gamma_\alpha = (\gamma^1_\alpha, \gamma^2_\alpha).$$

Moreover set

$$\kappa_\Gamma := \frac{\gamma_{11} \cdot (*\gamma_{01})}{\|\gamma_{10}\|}$$

where $*$ denotes the usual Hodge star operator in $\mathbf{R}^2$. Then almost everywhere in

$$E := \Big\{ t \in [0, \mathbf{M}(T)] \,\Big|\, \gamma'_{00}(t) \text{ exists and } \gamma'_{00}(t) \neq 0 \Big\}$$

one has $\eta_{00} \circ \Gamma \neq 0$ and

$$\frac{\eta_{00}}{\|\eta_{00}\|} \circ \Gamma = \gamma_{01}$$

$$\frac{\eta_{01}}{\|\eta_{00}\|} \circ \Gamma = \kappa_\Gamma (*\gamma_{01})$$

$$\frac{\eta_{10}}{\|\eta_{00}\|} \circ \Gamma = \frac{\kappa_\Gamma}{(1+\kappa_\Gamma^2)^{3/2}} \left[(1+\kappa_\Gamma^2)(*\gamma_{01}) - \frac{\kappa'_\Gamma}{\|\gamma'_{00}\|} \gamma_{01} \right]$$

$$\frac{\eta_{11}}{\|\eta_{00}\|} \circ \Gamma = \frac{1}{(1+\kappa_\Gamma^2)^{3/2}} \left[\frac{\kappa'_\Gamma}{\|\gamma'_{00}\|} (*\gamma_{01}) - (1+\kappa_\Gamma^2)\kappa_\Gamma^2 \gamma_{01} \right].$$

(2) *Given a C^3-covering $\mathcal{A}$ of*

$$R := X_0 G$$

(existing by Theorem 3.1(2)), for $Q \in X_2$ let us define

$$Q^i_\alpha := Q \cdot e^i_\alpha, \qquad \alpha \in \{0,1\}^2;\ i = 1, 2$$

and

$$Q_\alpha := (Q^1_\alpha, Q^2_\alpha), \qquad \alpha \in \{0,1\}^2.$$

Moreover set

$$\sigma(x,y) := \operatorname{sign}\big(y \cdot (*x)\big), \qquad x, y \in \mathbf{R}^2.$$

Then the following formulae hold at almost every Q in G, with respect to the measure $\mathcal{H}^1 \llcorner \|\eta_{00}\|$:

$$\frac{\eta_{00}}{\|\eta_{00}\|}(Q) = Q_{01},$$

$$\frac{\eta_{01}}{\|\eta_{00}\|}(Q) = \sigma(Q_{01}, Q_{11}) \alpha^{\mathcal{A}}_R(Q_{00})\, (*Q_{01}),$$

$$\frac{\eta_{10}}{\|\eta_{00}\|}(Q) = \frac{\sigma(Q_{01}, Q_{11})\alpha^{\mathcal{A}}_R(Q_{00})}{\left[1+\alpha^{\mathcal{A}}_R(Q_{00})^2\right]^{3/2}} \Bigg(\left[1+\alpha^{\mathcal{A}}_R(Q_{00})^2\right] (*Q_{01}) + \\ - \sigma(Q_{01}, Q_{11}) \langle apD\alpha^{\mathcal{A}}_R(Q_{00}), Q_{01} \rangle Q_{01} \Bigg),$$

$$\frac{\eta_{11}}{\|\eta_{00}\|}(Q) = \frac{1}{\left[1+\alpha^{\mathcal{A}}_R(Q_{00})^2\right]^{3/2}} \Bigg(\sigma(Q_{01}, Q_{11}) \langle apD\alpha^{\mathcal{A}}_R(Q_{00}), Q_{01} \rangle (*Q_{01}) + \\ - \left[1+\alpha^{\mathcal{A}}_R(Q_{00})^2\right] \alpha^{\mathcal{A}}_R(Q_{00})^2\, Q_{01} \Bigg).$$

Proof. (1) First of all, observe that

$$\eta_\alpha \circ \Gamma = \gamma'_\alpha, \qquad \alpha \in \{0,1\}^2$$

holds a.e. in $[0,\mathbf{M}(T)]$, by (i) of Theorem 3.1(3). Hence

$$\frac{\eta_\alpha}{\|\eta_{00}\|}\circ\Gamma=\frac{\gamma'_\alpha}{\|\gamma'_{00}\|},\qquad \alpha\in\{0,1\}^2$$

holds a.e. in E. In particular, invoking (3.8) with $n=0$, we get at once

$$\frac{\eta_{00}}{\|\eta_{00}\|}\circ\Gamma=\frac{\gamma'_{00}}{\|\gamma'_{00}\|}=\gamma_{01}$$

almost everywhere in E, namely just the first equality.

We are reduced to prove that the following formulas

$$\frac{\gamma'_{01}}{\|\gamma'_{00}\|}=\kappa_\Gamma(*\gamma_{01}) \tag{6.12}$$

$$\frac{\gamma'_{10}}{\|\gamma'_{00}\|}=\frac{\kappa_\Gamma}{(1+\kappa_\Gamma^2)^{3/2}}\left[(1+\kappa_\Gamma^2)(*\gamma_{01})-\frac{\kappa'_\Gamma}{\|\gamma'_{00}\|}\gamma_{01}\right] \tag{6.13}$$

$$\frac{\gamma'_{11}}{\|\gamma'_{00}\|}=\frac{1}{(1+\kappa_\Gamma^2)^{3/2}}\left[\frac{\kappa'_\Gamma}{\|\gamma'_{00}\|}(*\gamma_{01})-(1+\kappa_\Gamma^2)\kappa_\Gamma^2\gamma_{01}\right] \tag{6.14}$$

hold almost everywhere in E.

Proof of (6.12). By (3.8) with $n=1$ and (3.10) with $n=0$, one has respectively

$$(\gamma'_{00},\gamma'_{01})=\|(\gamma'_{00},\gamma'_{01})\|(\gamma_{10},\gamma_{11}) \tag{6.15}$$

and

$$\gamma'_{01}\cdot\gamma_{01}=0 \tag{6.16}$$

almost everywhere in $[0,\mathbf{M}(T)]$. Hence

$$\gamma_{11}\cdot\gamma_{01}=0 \tag{6.17}$$

almost everywhere in E. Also recalling the definition of κ_Γ and that (3.8) with $n=0$, i.e.

$$\gamma'_{00}=\|\gamma'_{00}\|\gamma_{01}, \tag{6.18}$$

holds almost everywhere in $[0,\mathbf{M}(T)]$, we get

$$\frac{\gamma'_{01}}{\|\gamma'_{00}\|}=\frac{\gamma_{11}}{\|\gamma_{10}\|}=\left[\frac{\gamma_{11}}{\|\gamma_{10}\|}\cdot(*\gamma_{01})\right](*\gamma_{01})\ =\kappa_\Gamma\,(*\gamma_{01}) \tag{6.19}$$

almost everywhere in E.

Proof of (6.13). From (6.18) and (6.19), it follows that

$$|\kappa_\Gamma|=\frac{\|\gamma_{11}\|}{\|\gamma_{10}\|} \tag{6.20}$$

almost everywhere in E, hence

$$1 + \kappa_\Gamma^2 = \frac{1}{\|\gamma_{10}\|^2} \tag{6.21}$$

almost everywhere in E, by (6.15). As a consequence, we easily obtain

$$\kappa_\Gamma \kappa_\Gamma' = -\frac{\gamma_{10} \cdot \gamma_{10}'}{\|\gamma_{10}\|^4} = -\frac{\gamma_{01} \cdot \gamma_{10}'}{\|\gamma_{10}\|^3} \tag{6.22}$$

almost everywhere in E, by Claim 6.1 and (6.15) , (6.18).

By (6.21), (6.22) and recalling the definition of κ_Γ, we conclude that the following formulae hold almost everywhere in E:

$$\frac{\kappa_\Gamma}{(1+\kappa_\Gamma^2)^{1/2}} = \frac{\gamma_{11} \cdot (*\gamma_{01})}{\|\gamma_{10}\|} \|\gamma_{10}\| = \gamma_{11} \cdot (*\gamma_{01}) \tag{6.23}$$

and

$$-\frac{\kappa_\Gamma \kappa_\Gamma'}{(1+\kappa_\Gamma^2)^{3/2}} = \frac{\gamma_{01} \cdot \gamma_{10}'}{\|\gamma_{10}\|^3} \|\gamma_{10}\|^3 = \gamma_{01} \cdot \gamma_{10}'. \tag{6.24}$$

Hence

$$\frac{\kappa_\Gamma}{(1+\kappa_\Gamma^2)^{3/2}} \left[(1+\kappa_\Gamma^2)(*\gamma_{01}) - \frac{\kappa_\Gamma'}{\|\gamma_{00}'\|} \gamma_{01} \right] = [\gamma_{11} \cdot (*\gamma_{01})](*\gamma_{01}) + \left(\frac{\gamma_{10}'}{\|\gamma_{00}'\|} \cdot \gamma_{01} \right) \gamma_{01}.$$

It just remains to prove that

$$\gamma_{11} \cdot (*\gamma_{01}) = \frac{\gamma_{10}'}{\|\gamma_{00}'\|} \cdot (*\gamma_{01})$$

almost everywhere in E. But such an equality is an easy consequence of (6.15) and (6.18), as the following computation (which holds almost everywhere in E) shows:

$$\begin{aligned} \frac{\gamma_{10}'}{\|\gamma_{00}'\|} \cdot (*\gamma_{01}) &= \frac{[\gamma_{10} \cdot (*\gamma_{01})]' - \gamma_{10} \cdot (*\gamma_{01}')}{\|\gamma_{00}'\|} = -\gamma_{10} \cdot \left[* \left(\frac{\gamma_{01}'}{\|\gamma_{00}'\|} \right) \right] \\ &= -\gamma_{10} \cdot \left[* \left(\frac{\gamma_{11}}{\|\gamma_{10}\|} \right) \right] = -\gamma_{01} \cdot (*\gamma_{11}) \\ &= \gamma_{11} \cdot (*\gamma_{01}). \end{aligned}$$

Proof of (6.14). We will prove separately that the formula holds almost everywhere in Z and in $E \backslash Z$, where

$$Z := \left\{ t \in E \,|\, \kappa_\Gamma(t) = 0 \right\} \subset E.$$

Observe that

$$\kappa_\Gamma' = 0$$

almost everywhere in Z. Also one has $\gamma_{11} \equiv 0$ in Z, by (6.20), hence

$$\gamma_{11}' = 0$$

almost everywhere in Z. It follows that (6.14) holds almost everywhere in Z.

On the other hand, almost everywhere in $E\backslash Z$, one has

$$\frac{\kappa_\Gamma'}{(1+\kappa_\Gamma^2)^{3/2}} = -\frac{(\gamma_{01}\cdot\gamma_{10}')\|\gamma_{10}\|}{\gamma_{11}\cdot(*\gamma_{01})} = -\frac{\gamma_{10}\cdot\gamma_{10}'}{\gamma_{11}\cdot(*\gamma_{01})}$$

and

$$\frac{\kappa_\Gamma^2}{(1+\kappa_\Gamma^2)^{1/2}} = \frac{[\gamma_{11}\cdot(*\gamma_{01})]^2}{\|\gamma_{10}\|} = \frac{\|\gamma_{11}\|^2}{\|\gamma_{10}\|}$$

by (6.15), (6.17), (6.18), (6.23), (6.24) and by recalling the definition of κ_Γ.

Then all we have to prove is that the following two equalities hold almost everywhere in $E\backslash Z$:

$$\gamma_{11}'\cdot(*\gamma_{01}) = -\frac{\gamma_{10}\cdot\gamma_{10}'}{\gamma_{11}\cdot(*\gamma_{01})} \tag{6.25}$$

and

$$\frac{\gamma_{11}'\cdot\gamma_{01}}{\|\gamma_{00}'\|} = -\frac{\|\gamma_{11}\|^2}{\|\gamma_{10}\|}. \tag{6.26}$$

First one has

$$\|\gamma_{10}\|^2 + \|\gamma_{11}\|^2 = 1$$

almost everywhere in $[0,\mathbf{M}(T)]$. Recalling (6.17), we get

$$\begin{aligned}-\gamma_{10}\cdot\gamma_{10}' = \gamma_{11}\cdot\gamma_{11}' &= [(\gamma_{11}\cdot\gamma_{01})\gamma_{01} + [\gamma_{11}\cdot(*\gamma_{01})](*\gamma_{01})]\cdot\gamma_{11}' \\ &= [\gamma_{11}\cdot(*\gamma_{01})][\gamma_{11}'\cdot(*\gamma_{01})]\end{aligned}$$

almost everywhere in E. Hence, in particular, (6.25) follows.

Since (6.17) and (6.18) imply

$$\gamma_{11} = (\gamma_{11}\cdot\gamma_{01})\gamma_{01} + [\gamma_{11}\cdot(*\gamma_{01})](*\gamma_{01}) = [\gamma_{11}\cdot(*\gamma_{01})](*\gamma_{01})$$

almost everywhere in E, one also has

$$\begin{aligned}\gamma_{11}' &= [\gamma_{11}'\cdot(*\gamma_{01}) + \gamma_{11}\cdot(*\gamma_{01}')](*\gamma_{01}) + [\gamma_{11}\cdot(*\gamma_{01})](*\gamma_{01}') \\ &= [\gamma_{11}'\cdot(*\gamma_{01})](*\gamma_{01}) + [\gamma_{11}\cdot(*\gamma_{01})](*\gamma_{01}')\end{aligned}$$

almost everywhere in E, by (6.15). Hence, invoking again (6.15), (6.18) and (6.17), we finally obtain

$$\begin{aligned}\gamma_{11}'\cdot\gamma_{01} &= [\gamma_{11}\cdot(*\gamma_{01})][(*\gamma_{01}')\cdot\gamma_{01}] \\ &= -\|\gamma_{00}'\|[\gamma_{11}\cdot(*\gamma_{01})]\left[\frac{\gamma_{01}'}{\|\gamma_{00}'\|}\cdot(*\gamma_{01})\right] \\ &= -\frac{\|\gamma_{00}'\|}{\|\gamma_{10}\|}[\gamma_{11}\cdot(*\gamma_{01})]^2 \\ &= -\frac{\|\gamma_{00}'\|\|\gamma_{11}\|^2}{\|\gamma_{10}\|}\end{aligned}$$

almost everywhere in E, namely (6.26).

(2) It can be easily derived from the statement (1), Theorem 3.1(1) and Proposition 6.1. □

As an example of application of the machinery developed throughout the paper, we provide the following result.

Theorem 6.1. (1) *Let* $[\![G, \eta, \theta]\!]$ *be a 2-storey Gaussian tower in* X_2. *Then, at* $\mathcal{H}^1 \llcorner G^*$ *a.e.* Q, *one has*

$$F_2\left(\frac{\eta(Q)}{\|\eta_{00}(Q)\|}\right) = 1 + 2\alpha_{X_0G}(Q_{00})^2 + \frac{\langle apD\alpha_{X_0G}(Q_{00})\,,\,Q_{01}\rangle^2}{[1+\alpha_{X_0G}(Q_{00})^2]^2}.$$

(2) *If* A *is a regular set, then*

$$\mathcal{E}_{F_2}(A) = \int_{\partial A} 1 + 2\alpha^2 + \frac{\|D^{\partial A}\alpha\|^2}{(1+\alpha^2)^2}\, d\mathcal{H}^1$$

where α *is the absolute curvature of* ∂A *and* $D^{\partial A}$ *denotes the tangential differentiation operator in* ∂A.

(3) *Let* A_h $(h = 1, 2, \dots)$ *and* A *be regular sets such that* $A_h \to A$ *in* $L^1_{loc}(X_0)$. *Then*

$$\int_{\partial A} 1 + 2\alpha^2 + \frac{\|D^{\partial A}\alpha\|^2}{(1+\alpha^2)^2}\, d\mathcal{H}^1 \le \liminf_h \int_{\partial A_h} 1 + 2\alpha_h^2 + \frac{\|D^{\partial A_h}\alpha_h\|^2}{(1+\alpha_h^2)^2}\, d\mathcal{H}^1$$

where α *and* α_h *are the absolute curvatures of* ∂A *and* ∂A_h, *respectively.*

Proof. (1) follows from the second assertion in Proposition 6.3, by a standard computation. Hence we get (2), also by recalling the area formula. Finally Theorem 5.1(1) yields (3). □

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