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INTEGRAL NON-LINEARITY IN MEMORYLESS A/D CONVERTERS

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April 2004

Technical Report # DIT-04-048

INTEGRAL NON-LINEARITY IN MEMORYLESS A/D CONVERTERS

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Abstract – This paper investigates the statistical properties of quantization noise. In particular, a theoretical model is discussed, which evaluates the power of quantization noise introduced by a memoryless Analog to Digital Converter (ADC) as a function of both the converted signal distribution and the ADC thresholds positioning. Expressions have also been derived to express the Integral Non-Linearity (INL) contribution to quantization noise power as an additive term, and to evaluate such a term with a simple formula. Simulation results that validate the proposed expression are provided.

Keywords: Quantization Noise, Integral Non-Linearity.

1. INTRODUCTION

Analog to Digital (A/D) and Digital to Analog (D/A) converters are widely used in many modern fields of application, allowing to replace analog systems with digital high performance implementations. Modeling the behavior of A/D and D/A converters is important both for characterizing the performance of the converters themselves and for embedding such devices in real systems. The matter has been subject of several investigations, however the effects of ADC and DAC unidealities upon the properties of quantization noise have not been deeply investigated yet [1]. In particular, an additive, white, and uniformly distributed quantization noise is usually considered, whose power does not depend on the statistical properties of the input signal.

However, such assumptions are not verified when the converter is affected by INL. This paper is focused on the effects of INL on the noise power of a memoryless A/D converter, fed with various kinds of stochastic signals. An exact model is discussed, which describes the quantization noise power as a function of both the input signal probability density function (pdf) and the transition levels of the quantizer [2][3]. The model is then extended to evaluate the effects of INL and input signal statistical properties. In particular, the INL effect is modeled as an additive contribution, extending the results presented in [4]. It is worth of notice that the noise power of a quantizer affected by INL may noticeably depend on the amplitude of the quantizer stimulus. As an exact model may lead to very complicated expressions, a simplified formula has been derived, which accurately describes the effects of INL on the noise power when the stimulus covers enough quantizer levels. This model has been applied to various stochastic stimuli, showing a very good agreement with simulation results. In particular, uniform, Gaussian and noisy sinewave inputs have been considered.

2. ANALYSIS RESULTS

2a. Quantizer model

The theoretical model, whose detailed derivation is shown in Appendix A**,** has been obtained by assuming that the quantizer thresholds define a partition of the real axis, and by evaluating the conditional error pdf in each of the partition subsets. Notice that such an approach does not need any particular hypothesis about the threshold positioning, and can be applied indifferently to uniform or non-uniform converters. In particular, it can be shown that quantization noise pdf can be expressed as:

$$\begin{aligned} f_e(e) &= \sum_k f_x(y_{k-1}-e)\cdot i(A_k), \\ A_k &= [y_{k-1}-s_k, y_{k-1}-s_{k-1}], \end{aligned} \tag{1}$$

where s_k is the *k-th* quantizer decision threshold, y_k is the *k-th* output level, $f_x(\cdot)$ is the input signal pdf, and i(.) is the indicator function [2].

In this paper, an ideal uniform quantizer is considered, with infinite quantizer thresholds and quantization levels. Quantization noise power can be estimated by calculating the variance of the error, according to:

$$\sigma_e^2 = \int_{-\infty}^{\infty} e^2 f_e(e)de, \tag{2}$$

which leads to the following:

$$\sigma_e^2 = \sum_{k=-\infty}^{\infty} \int_{y_{k-1}-s_k}^{y_{k-1}-s_{k-1}} e^2 f_x(y_{k-1}-e)de, \tag{3}$$

Figg. 1(a)-(b) report quantization noise power as a function of signal standard deviation σ_{IN}, normalized to Δ, for uniform and Gaussian input signals respectively. The error power is normalized to the error power σ_0^2 of an uniform ADC fed with an uniformly distributed stimulus**.** As it is well known, with very high accuracy we can assume $\sigma_0^2 \cong \Delta^2/12$. In all the considered situations the model shows a good agreement with simulation results.

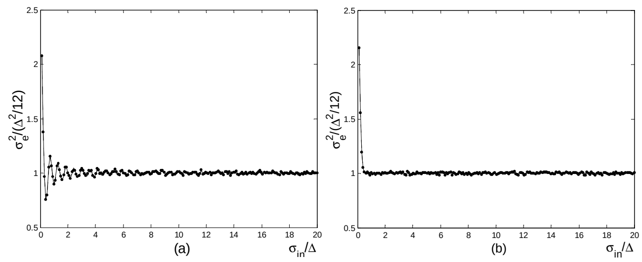

Fig 1: Quantization noise power, normalized to $\Delta^2/12$, obtained in absence of INL for an uniformly distributed input signal (a) and for a Gaussian distributed one (b). Dots are simulation results, while the continuous line is obtained by means of the theoretical model expressed by (3).

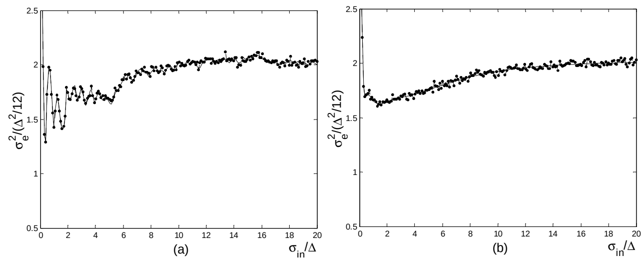

Fig 2: Quantization noise power, normalized to $\Delta^2/12$, obtained in presence of INL for an uniformly distributed input signal (a) and for a Gaussian distributed one (b). Dots are simulation results, while the continuous line is obtained by means of the theoretical model expressed by (3) and (5).

2b. Uniform quantizers affected by Integral Non-Linearity

The presented model can easily keep into account INL, by replacing the ideal decision threshold values in (3) with the ones affected by INL. In particular, the quantizer thresholds may be expressed as follows:

$$s_k = s_{0k} + inl_k, \qquad (4)$$

where s_{0k} is the k-th ideal threshold and inl_k is the offset caused by INL.

By using (4), Eq. (3) can be further refined, obtaining an equivalent form in which the ideal noise power and the INL contributions appear as two distinct additive terms. In fact, the properties of the integral operator allow to obtain the following (see appendix B):

$$\begin{aligned} \sigma_e^2 &= \sigma_0^2 + \delta_{inl}, \\ \delta_{inl} &= 2\Delta \sum_{k=-\infty}^{\infty} \delta_k, \\ \delta_k &= \int_{s_{0k}}^{s_{0k}+inl_k} (x - s_{0k}) f_x(x)dx \end{aligned} \qquad (5)$$

where N is the number of quantizer levels, σ_0^2 is the noise power generated in absence of INL, and δ_{inl} is the INL contribution to quantization noise. This term has been evaluated for uniform, sinusoidal and Gaussian input signals. In particular, for a Gaussian stimulus, it results:

$$\begin{aligned} f_x(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \\ \delta_k &= \frac{\sigma}{\sqrt{2\pi}} \left(e^{-\frac{s_{0k}^2}{2\sigma^2}} - e^{-\frac{(s_{0k}+inl_k)^2}{2\sigma^2}} \right) + \\ &\quad - s_{0k} \left(erf\left(\frac{s_{0k}+inl_k}{\sigma} \right) - erf\left(\frac{s_{0k}}{\sigma} \right) \right), \end{aligned} \qquad (5a)$$

where erf(·) is the error function [5]. For a zero mean uniformly distributed stimulus, we have

$$\begin{aligned} f_x(x) &= \frac{1}{2A} i([-A, A]), \\ \delta_k &= \frac{1}{2A} \left[\frac{\beta_k^2 - \alpha_k^2}{2} - s_{0k}(\beta_k - \alpha_k) \right], \\ [\alpha_k, \beta_k] &= [s_{0k}, s_{0k} + inl_k] \cap [-A, A] \end{aligned} \qquad (5b)$$

where $\cap$ is the intersection operator. Finally, for a sinusoidal stimulus, it results

$$\begin{aligned} f_x(x) &= \frac{1}{\pi\sqrt{A^2 - x^2}} i(]-A, A[), \\ \delta_k &= \frac{1}{\pi} \left(\sqrt{(A^2 - \alpha_k^2)} + s_{0k} \arcsin\left(\frac{\alpha_k}{A} \right) \right) + \\ &\quad - \frac{1}{\pi} \left(\sqrt{(A^2 - \beta_k^2)} + s_{0k} \arcsin\left(\frac{\beta_k}{A} \right) \right), \\ [\alpha_k, \beta_k] &= [s_{0k}, s_{0k} + inl_k] \cap]-A, A[\end{aligned} \qquad (5c)$$

Figg. 2(a)-(b) report the noise power curves obtained in presence of INL for the considered input signals, normalized to $\Delta^2/12$, as a function of σ_{IN}/Δ. In particular, both stimuli have been applied to an ADC affected by deterministic INL, where each inl_k has been taken from a set of values uniformly distributed between $-\Delta/2$ and $\Delta/2$. Again, it can be seen that the model shows a good agreement with simulation results.

While the theoretical model provide very accurate results, it's derivation and application may lead to very complicated closed form expressions, depending on the expression of the input signal pdf. This may easily happen in practical applications, where the quantizer stimulus is usually distorted or corrupted by noise. A typical case is the usage of dither [4][6]. Thus, by assuming that the input signal pdf $f_x(\cdot)$ is almost constant in $[s_{0k}, s_{0k} + inl_k]$, a simplified model has been introduced, expressed as follows:

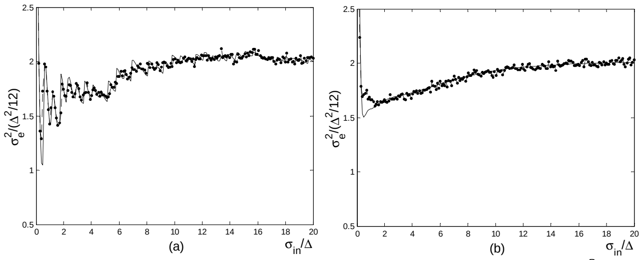

Fig 3: Quantization noise power, normalized to $\Delta^2/12$, obtained in presence of INL for an uniformly distributed input signal (a) and for a Gaussian distributed one (b). Dots are simulation results, while the continuous line is obtained by means of the simplified theoretical model expressed by (6).

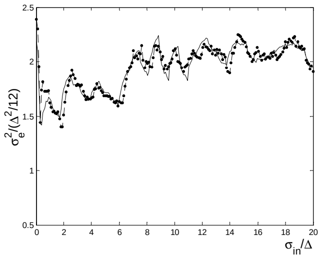

Fig 4: Quantization noise power, normalized to $\Delta^2/12$, obtained in presence of INL for an input signal consisting in a sinewave added to a dither, uniform in $[-\Delta/2,\Delta/2]$. Dots are simulation results, while the continuous line is obtained by means of the simplified theoretical model expressed by (6).

$$\delta_{inl} \cong \Delta \cdot \sum_k f_x(s_{ok}) \cdot inl_k^2 \quad , \tag{6}$$

By substituting Eq. (6) in Eq. (5), the noise power of a quantizer affected by INL may be expressed as:

$$\sigma_e^2 \cong \frac{\Delta^2}{12} + \Delta \sum_k f_x(s_{0k}) \cdot inl_k^2 \tag{7}$$

It should be noticed that, for an uniformly distributed input signal whose dynamic range equals the ADC one, (5) and (6) reduce to the INL contribution reported in Eq. (A.8) of [4]. Figg. 3(a)-(b) show the results obtained by using (7) for both uniform and Gaussian input signals. It can be seen that, as far as the input signal excites a few quantizer levels, Eq.(7) shows a good agreement with simulation results. In order to analyze another situation of practical interest, a sinewave signal, affected by an additive dither uniformly distributed in $[-\Delta/2, \Delta/2]$, has been considered. Fig. 4 reports both simulation results and the value returned by Eq.(7), in which the signal pdf derived in [6] has been used. It can be seen that also in this case the simplified model (7) provide a very good accuracy.

3. CONCLUSIONS

A theoretical exact model has been presented, which describes the quantization noise distribution and power of memoryless A/D converters as a function of both threshold spacing and input signal pdf. The model has been extended to keep into account the INL contribution to quantization noise power, which has been expressed as an additive term. A simplified model has been also proposed, which provides good results as far as the dynamic range is sufficiently larger than the quantizer step. The model has been applied to various input signals, including a noisy sinewave.

APPENDIX A

Quantizer error model

Let us assume that s_k is the *k-th* quantizer decision threshold, and y_k is the *k-th* output level, such that

$$s_k < s_{k+1}, \qquad y_k < y_{k+1}, \qquad s_k < y_k < s_{k+1}$$

The quantizer decision thresholds define a partition of the real axis, given by

$$\Re = \bigcup_k X_k, \qquad X_k =]s_{k-1}, s_k]. \tag{A.1}$$

The quantizer error can be obtained as the difference between the quantizer input x, which may me modeled as a random variable with pdf $f_x(x)$, and the quantizer output y. By keeping in mind that the value of y depends on which interval X_k the input x belongs to, $e|X_k$ that is the quantizer error conditioned to X_k, is given by:

$$e \mid X_k = x \mid X_k - y_k \,, \tag{A.2}$$

where $x|X_k$ is the input x conditioned to X_k, that is, x such that $x \in X_k$ Hence, the error pdf, conditioned to X_k, can be obtained as:

$$f_{e|X_k}(e \mid X_k) = \begin{cases} \dfrac{f_x(y_{k-1} - e \mid X_k)}{P(X_k)}, & e \mid X_k \in A_k \\ 0, & elsewhere \end{cases}$$

$$A_k = [y_{k-1} - s_k, y_{k-1} - s_{k-1}] \tag{A.3}$$

where $P(X_k)=P(x \in X_k)$. The unconditioned error pdf $f_e(e)$ can be obtained as [5]

$$f_e(e) = \sum_{k=-\infty}^{+\infty} f_{e|X_k}(e) P(X_k) i(A_k), \tag{A.4}$$

where $i(\cdot)$ is the indicator function, which equals 1 if $e \in A_k$ and equals 0 if $e \notin A_k$. By inserting (A.3) in (A.4), Eq. (1) results. In particular, if the considered quantizer is uniform, it results that $A_k=[-\Delta/2, \Delta/2]$, regardless of k. Hence, (1) reduces to:

$$f_e(e) = \begin{cases} \sum_k f_x(y_{k-1} - e), & |e| < \Delta/2 \\ 0, & elsewhere \end{cases} \quad \text{(A.5)}$$

APPENDIX B

Effects of Integral Non-Linearity on quantizer error power

Let us express the quantizer thresholds affected by INL as:

$$s_k = s_{0k} + inl_k, \quad \text{(B.1)}$$

where s_{0k} is the k-th ideal threshold and inl_k is the offset caused by INL. The effect of INL is to change the decision intervals X_k, which causes the quantizer to produce incorrect output levels. By assuming that $|inl_k|<\Delta/2$, it can be shown that when INL is introduced, the quantization error can be expressed as

$$e = e_0 + e_{INL}, \quad \text{(B.2)}$$

$$e_{INL} = -\sum_k \Delta sign(inl_k) i(I_k), \quad \text{(B.3)}$$

$$e_0(x) = \sum_k (x - s_{0k} - \Delta/2) I(X_{k+1}) \quad \text{(B.4)}$$

where e_0 is the quantization error of a quantizer not affected by INL, e_{INL} is the INL contribution to the error, sign($\cdot$) is the sign function and I_k=[min(s_k,s_{0k}), max(s_k,s_{0k})] is the input interval where INL causes the quantizer to produce an incorrect output level. In fact, when inl_k is positive, if x belongs to [s_{0k},s_k], the quantizer output equals y_{k-1}, rather than the correct value y_k. Conversely, when inl_k is negative, the quantizer output equals y_k, rather than the correct value y_{k-1}, only if x belongs to [s_k,s_{0k}]. Eq. (B.3) shows that the sign of e_{INL} is always opposite to the sign of e_0, and its magnitude is always Δ. The error power can be evaluated according to (2), which, by applying a change of variable, can be expressed in terms of the input x, obtaining

$$\sigma_e^2 = \sum_k \int_{s_k}^{s_{k+1}} (e_0(x) + e_{INL}(x))^2 f_x(x) dx \quad \text{(B.5)}$$

Eq. (B.5) can be expanded into

$$\sigma_e^2 = \sigma_{e0}^2 + \sigma_{INL}^2 + C_{INL}, \quad \text{(B.6)}$$

where σ_{e0}^2 is the error power of an ideal quantizer, given by (3), σ_{INL}^2 is the power of the INL error, given by

$$\sigma_{INL}^2 = \sum_k \int_{I_k} e_{INL}(x)^2 f_x(x) dx = \Delta^2 \sum_k P(I_k), \quad \text{(B.7)}$$

and C_{INL} is a cross power term, expressed as:

$$C_{INL} = 2\sum_k \int_{I_k} e_0(x) e_{INL}(x) f_x(x) dx \quad \text{(B.8)}$$

Notice that C_{INL} is the correlation between the ideal error and the INL error. This term can be further developed by substituting the expressions for e_0 and e_{INL}, thus obtaining

$$C_{INL} = -2\Delta \sum_k \int_{I_k} (x - s_k - \Delta/2) sign(inl_k) i(I_k) \quad \text{(B.9)}$$

Eq. (B.9) can be expanded into

$$C_{INL} = -\sum_k \Delta^2 P(I_k) + 2\Delta \sum_k \int_{s_{0k}}^{s_{0k}+inl_k} (x - s_{0k}) f_x(x) dx \quad \text{(B.10)}$$

where the properties of I_k and the sign function have been used to simplify the integral in (B.9). Finally, by inserting (B.7) and (B.10) in (B.6), the first addendum of the right part of (B.10) cancels σ_{INL}^2, and Eq. (5) is obtained.

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