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Quaternionic slice regular functions on domains without real points

Author:
Amedeo Altavilla
XXVII Cicle

Supervisor:
Prof. Alessandro Perotti

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Contents

Introduction

After the introduction of quaternions, the 4-dimensional real skew algebra $\mathbb{H}$ generated by $1, i, j, k$ with the following relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k,$$

by Sir William Hamilton, there were several attempts to introduce a satisfactory notion of regularity for functions defined over these numbers. The particular structure of $\mathbb{H}$, that is non-commutative and containing a whole sphere of imaginary units, namely

$$\mathbb{S} := \{I \in \mathbb{H} \mid I^2 = -1\} = \{x = x_1 i + x_2 j + x_3 k \,|\, x_1^2 + x_2^2 + x_3^2 = 1\},$$

implies that, in this context, all the equivalent notions of complex analysis to define holomorphicity, cease to be equivalent and produce different sets of functions. In fact, several definitions of regularity for a quaternionic function of one quaternionic variable were given in the last century, but none of them seemed to be satisfactory enough: some definitions contemplate too few functions, some too many (for more details see [**48**] and the references therein).

The most explored way to generalize the concept of regularity was settled down and developed by R. Fueter in the 1930's and 1940's, who considered the kernel of the following differential operator:

$$\overline{\mathcal{D}} = \frac{1}{4}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right),$$

where $x = x_0 + x_1 i + x_2 j + x_3 k$ is a generic quaternion.

The function theory based on this notion of regularity (called *Fueter-regularity*), is now well understood and generalized to other frameworks. A good starting point to approach this notion is again the survey [**48**], in which are stated the main properties and features of this kind of functions. Even if this notion is very important and seems to be very rich[1], it does not contemplates polynomial functions: even the identity map, $f(q) = q$, fails to be

[1]We point out that is available, for Fueter-regular functions, a Cauchy theorem and formula; moreover, Fueter-regular functions are harmonic functions.

regular. The issue of finding a theory of quaternionic regular functions that contemplates polynomials was partially solved by Fueter himself who introduced the class of *quaternionic holomorphic* functions as solutions of the equation

$$\overline{\mathcal{D}}\Delta f(x) = 0,$$

where Δ denotes the Laplacian in the four real variables $x_0, \ldots, x_3$. Even if this theory contains polynomial functions, it is not of much interest from our point of view, because is extremely large. It in fact includes the class of harmonic functions of four real variables, which includes the class of Fueter-regular functions. For other information about this theory and its generalizations we refer to [**10, 18, 39, 40**].

So, the main object of this thesis is the latest theory, in chronological order, of regular quaternionic functions, called *slice regularity*. This theory, based on a definition of regularity for quaternionic-valued functions of one quaternionic variable given by C. G. Cullen in 1965 (see [**13**]), was reintroduced and developed, in the last years, by G. Gentili, D. C. Struppa and others (see [**12, 25, 27, 11**] and their bibliography), and independently by S. De Leo S and P.P. Rotelli in [**15**]. More precisely the main concept is the following. A point $x = x_0 + x_1 i + x_2 j + x_3 k$ in $\mathbb{H}\setminus\mathbb{R}$ can be written as[2] $x = \alpha + I_x\beta$, where $\alpha = Re(x) = x_0 \in \mathbb{R}$, $\beta = ||Im(x)|| = \sqrt{x_1^2 + x_2^2 + x_3^2} \in \mathbb{R}$ and $I_x = Im(x)/||Im(x)|| \in \mathbb{S}$. Therefore, putting $\mathbb{C}_I$ to be the real subspace of $\mathbb{H}$ generated by 1 and I, we give the following definition.

DEFINITION 0.1. Let Ω be a domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a quaternion-valued function defined on Ω. Let $\Omega_I = \Omega \cap \mathbb{C}_I$ and let $f_I = f|_{\mathbb{C}_I}$. The restriction f_I is called *holomorphic* if it has continuous partial derivatives and

$$\overline{\partial}_I f(\alpha + I\beta) = \frac{1}{2}\left(\frac{\partial}{\partial\alpha} + I\frac{\partial}{\partial\beta}\right) f_I(\alpha + I\beta)$$

vanishes identically. The function is called *Cullen-regular* if, for all $I \in \mathbb{S}$, f_I is holomorphic.

Examples of such functions are convergent quaternionic power series defined on a ball centered in the origin

$$f : B(0, R) \to \mathbb{H}, \quad f(x) = \sum_{n\in\mathbb{N}} x^n a_n,$$

where $R > 0$ denote the radius of convergence of the sum. Moreover it was proven in [**27**] that if $f : B(0, R) \to \mathbb{H}$ is a regular function, then there exist a sequence of quaternions $\{a_n\}_{n\in\mathbb{N}}$, such that,

$$f(x) = \sum_{n\in\mathbb{N}} x^n a_n$$

for all $x \in B(0, R)$. In particular, $f \in \mathcal{C}^\infty(B(0, R))$.

Anyway, this definition is not enough to obtain a satisfactory theory. In fact, examples of "bad" quaternionic functions which are regular, are the following two:

[2]When there will not be ambiguity we will denote $x = \alpha + I_x\beta$ by $x = \alpha + I\beta$.

(i) $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ defined as

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{H} \setminus \mathbb{C}_i \\ 0, & \text{if } x \in \mathbb{C}_i \setminus \mathbb{R}. \end{cases} \tag{1}$$

This function is of course regular but is not even continuous. So regularity, by itself, do not implies even continuity. However this example is quite meaningless since we could restrict to functions which are already differentiable. In this view the next example is more meaningful.

(ii) Fix a $J \in \mathbb{S}$ and a real number $\lambda \notin \{-1, 0, 1\}$. Let $x = \alpha + I_x\beta$ be a non-real quaternion and define $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ as

$$f(x) = I_x + \lambda J I_x J. \tag{2}$$

This function, which is of class $\mathcal{C}^\infty$ and regular, sends $\mathbb{H}\setminus\mathbb{R}$ into an ellipsoidal surface. The reason why we don't want this kind of examples will be more clear later. For now let's just say that, while the behavior over any complex plane $\mathbb{C}_K$, for any $K \in \mathbb{S}$, is the desired one, in the remaining directions it is not "under control".

These examples are constructed removing the real line from the domain of definition. To solve this issue one can choose (1) to study regular functions defined over domains that do intersect the real axis or (2) to add some hypothesis to the set of functions, to avoid examples such as the previous two.

The first solution is very well studied by several authors and the main elements can be found in the book [**25**]. Among the several achieved results, one can found: • the possibility to expand a regular function in series of particular polynomials, • the structure of zeros set (which gives, together with the right notion of multiplicity, a good version of the Fundamental theorem of Algebra), • a classification of singularities, • a Cauchy integral representation formula, • a Maximum and Minimum modulus principle, • an Open Mapping theorem and • an analogous of the fractional and Möbius transformations.

Of course these attainments do not exhaust the whole theory (which is still very fruitful). In particular, we point out that several progresses were made, for instance, in the study of functional spaces of regular functions such as Bergman, Hardy and Fock spaces (see, respectively, [**8**], [**14, 6**], [**2**]).

The second solution, which is the one adopted in this thesis, is the following. In theorem 3.1 of [**9**] it is stated, implicitly, that a regular function f defined over a domain that intersects the real line results to be of the form $f(\alpha+I\beta) = f_1(\alpha, \beta)+If_2(\alpha, \beta)$ with f_1 and f_2 quaternionic valued functions, i.e.: f is quaternionic left-affine w.r.t. the imaginary unit I. As the reader can see, the functions defined in the previous examples do not satisfies this requirement. Indeed, while imposing this hypothesis to a regular function does not affects the theory in the case in which the domain intersects the real axis, at the same time this gives new and satisfactory results in the other case. To be more precise we remember definitions 4 and 5 of [**30**].

Definition 0.2. Let D be a domain in $\mathbb{C}$. A function $F : D \to \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}(=: \mathbb{H}_{\mathbb{C}})$ is called a *stem function* if it is complex intrinsec, i.e.: $F(\bar{z}) = \overline{F(z)}$, for each $z \in D$ such that $\bar{z} \in D$, where, if $w \in \mathbb{H}_{\mathbb{C}}$, then $w = x + \sqrt{-1}y$ with $x, y \in \mathbb{H}$ and we define $\bar{w} = x - \sqrt{-1}y$.

Given a domain D in $\mathbb{C}$ we denote by Ω_D its *circularization* in $\mathbb{H}$, that is,

$$\Omega_D := \{\alpha + I\beta \in \mathbb{H} \mid \alpha + i\beta \in D,\, I \in \mathbb{S}\}.$$

Definition 0.3. Let Ω_D be a circular domain in $\mathbb{H}$. A *(left)*[3] *slice function* is a function $f : \Omega_D \to \mathbb{H}$ that is induced by a stem function $F = F_1 + \sqrt{-1}F_2$, in the following way: let $x = \alpha + I\beta \in \Omega_D$, let $z = \alpha + i\beta \in D$, then

$$f(x) = F_1(z) + IF_2(z).$$

Usually, when f is a slice function induced by F, we denote it by $f = \mathcal{I}(F)$, furthermore we will denote by $\mathcal{S}(\Omega_D)$ and by $\mathcal{S}^1(\Omega_D)$ the spaces of slice functions on Ω_D induced respectively by continuous and differentiable with continuous partial derivatives stem functions.

Note that examples in equations 1 and 2 are not slice functions.

A function F is a stem function if and only if the $\mathbb{H}$-valued components F_1, F_2 of $F = F_1 + \sqrt{-1}F_2$ form an even-odd pair with respect to the imaginary part of z, i.e.:

$$F_1(\overline{z}) = F_1(z),\ F_2(\overline{z}) = -F_2(z),\ \forall z \in D \,\text{s.t.}\ \overline{z} \in D.$$

Thanks to this fact, the slice function f is well defined, in fact $f(\alpha + (-I)(-\beta)) = F_1(\overline{z}) + (-I)F_2(\overline{z}) = F_1(z) + IF_2(z)$.

So, morally, a slice function is a quaternionic function of one quaternionic variable that is quaternionic left-affine w.r.t. the imaginary unit.

With this in mind it is not difficult to understand part of the behavior of such a function. In particular given a slice function $f : \Omega_D \to \mathbb{H}$ it is possible to show that, having the values of f over two different half planes (or *semislices*) of the form $\mathbb{C}_J^+ := \{\alpha + J\beta \mid \alpha \in \mathbb{R},\, \beta \geq 0\}$ and $\mathbb{C}_K^+$, one can reconstruct the whole function with the following *representation formula* (see proposition 6 of [**30**]):

$$f(x) = (I - K)(J - K)^{-1}f(\alpha + J\beta) - (I - J)(J - K)^{-1}f(\alpha + K\beta)$$

for all $I \in \mathbb{S}$ and for all $x = \alpha + I\beta \in D_I := \Omega_D \cap \mathbb{C}_I$.

Now, thanks to "sliceness", one can control the "spherical behavior" of f using the following notion stated in definition 6 of [**30**].

Definition 0.4. Let $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$ be a slice function. We define the *spherical derivative* of f as the slice function $\partial_s f = \mathcal{I}\left(\frac{F_2(z)}{Im(z)}\right) : \Omega_D \setminus \mathbb{R} \to \mathbb{H}$, i.e.: $\partial_s f(x) := \frac{1}{2}Im(x)^{-1}(f(x) - f(x^c))$, where, as before, if $x = \alpha + I\beta$, then $x^c = \alpha - I\beta$.

[3]Of course, there is an analogous definition for right slice functions when the element $J \in \mathbb{S}$ is placed on the right of $F_2(z)$.

Obviously, this function is constant on every sphere $\mathbb{S}_x = \{y \in \mathbb{H} \mid y = \alpha + J\beta, J \in \mathbb{S}\}$, for $x = \alpha + I\beta \in \Omega_D$. Moreover $\partial_s f = 0$ if and only if f is constant on $\mathbb{S}_x$, in other terms:

$$\partial_s(\partial_s(f)) = 0.$$

Let now F be a differentiable stem function over a domain D. The two functions $\frac{\partial F}{\partial z}, \frac{\partial F}{\partial \bar{z}} : D \to \mathbb{H}_{\mathbb{C}}$, are stem functions that induce the continuous *slice derivatives*

$$\frac{\partial f}{\partial x} = \mathcal{I}\left(\frac{\partial F}{\partial z}\right), \ \frac{\partial f}{\partial x^c} = \mathcal{I}\left(\frac{\partial F}{\partial \bar{z}}\right).$$

Left multiplication by $\sqrt{-1}$ defines a complex structure on $\mathbb{H}_{\mathbb{C}}$ and, with respect to this structure, a C^1 stem function $F = F_1 + \sqrt{-1}F_2 : D \to \mathbb{H}_{\mathbb{C}}$ is holomorphic if and only if it satisfies the Cauchy-Riemann equations

$$\begin{cases} \frac{\partial F_1}{\partial \alpha} = \frac{\partial F_2}{\partial \beta}, \\ \frac{\partial F_1}{\partial \beta} = -\frac{\partial F_2}{\partial \alpha} \end{cases} \quad \Leftrightarrow \frac{\partial F}{\partial \bar{z}} \equiv 0 \Leftrightarrow \frac{\partial f}{\partial x^c} \equiv 0.$$

We are now in position to remember the definition of slice regular functions given in definition 8 of [**30**].

DEFINITION 0.5. A function $f \in \mathcal{S}^1(\Omega_D)$ is *(left) slice regular* if its stem function F is holomorphic. The space of slice regular function defined on Ω_D will be denoted by $\mathcal{SR}(\Omega_D)$.

It is possible to show, as already mentioned, that *if D is such that $D \cap \mathbb{R} \neq \emptyset$ (and so $\Omega_D \cap \mathbb{R} \neq \emptyset$), then $f : \Omega_D \to \mathbb{H}$ is Cullen regular if and only if is slice regular.*

We have now a theory which actually extends the theory of Cullen regular functions, when the domain is free of real points, and for which it might be useful and interesting to understand its properties.

The elements of this theory can be found in [**30, 32, 33**], where the authors show many results in the more general setting of real alternative $*$-algebras (among the others there is a Cauchy integral formula for slice functions of class $\mathcal{C}^1$).

Based on this groundwork, in the present thesis we study properties of slice regular functions with domains that do not contain real points, with the following aim: to extend, as much as possible, the very interesting correspondence, settled down in [**20**], between quaternionic analysis of slice regular functions and twistor geometry of $\mathbb{R}^4$.

To enter into details, let (Ω^{2n}, g) be a $2n$-dimensional oriented Riemannian manifold. An *almost complex structure* over Ω is an endomorphism of the tangent bundle $J : T\Omega \to T\Omega$, such that $J^2 = -id$. An almost complex structure is said to be a *complex structure* if J is *integrable*, meaning, for instance, that the associate *Nijenhuis tensor*

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY],$$

vanishes everywhere for each couple of tangent vectors X and Y; it is said to be *orthogonal* if it preserves the orientation and the Euclidean product, i.e. $g(JX, JY) = g(X, Y)$ for each couple of tangent vectors X and Y. An *orthogonal complex structure (OCS)* is an almost complex structure which is integrable and orthogonal.

If the manifold Ω is a 4-dimensional open subset of $\mathbb{R}^4$ endowed with the standard Euclidean metric, then it is possible to construct standard OCSes, called *constant*, in the following way: think $\mathbb{R}^4 \simeq \mathbb{H}$ as the space of real quaternions and fix an element $q \in \mathbb{S}$ (e.g.: $q = i$). Identifying each tangent space $T_p\Omega$ with $\mathbb{H}$ himself, we define the complex structure everywhere by left multiplication by q, i.e. $J(p)v = qv$. Any OCS defined globally on $\mathbb{H}$ is known to be constant, moreover it was proven in [**43**] that any OCS J of class $\mathcal{C}^1$ on $\mathbb{R}^4 \setminus \Lambda$, where Λ is a closed set of zero 1-dimensional Hausdorff measure, is either constant or can be maximally extended to the complement of a point $\mathbb{R}^4 \setminus \{p\}$. In both cases, J is the push-forward of the standard OCS on $\mathbb{R}^4$ under a conformal transformation.

In the same paper it was proven that if Λ is a *round circle* or a *straight line* and $\mathbb{J}$ is an OCS of class $\mathcal{C}^1$ on $\mathbb{R}^4 \setminus \Lambda$ that is not conformally equivalent to a constant OCS, then $\mathbb{J}$ is unique up to sign, and $\mathbb{R}^4 \setminus \Lambda$ is a maximal domain for $\mathbb{J}$.

In this particular context it is possible to construct explicitly the OCS $\mathbb{J}$ as follows. For each $x = \alpha + I_x\beta \in \mathbb{H} \setminus \mathbb{R}$ we define $\mathbb{J}$ such as $\mathbb{J}(x)v = I_xv$, for each $v \in T_x(\mathbb{H} \setminus \mathbb{R})$. Since the last is an OCS over $\mathbb{H} \setminus \mathbb{R}$, then $\mathbb{J}$ and $-\mathbb{J}$ are the only non-constant OCSes on this manifold (up to conformal transformations).

Coming back to our motivation, in [**20**] the authors proposed a new way to study the problem when Λ is a closed set in $\mathbb{R}^4$ of different type. The idea is to take the OCS $\mathbb{J}$, previously defined, and to push it forward via some function on the set we are interested in. To do this we need to be sure that the function f preserves the properties of $\mathbb{J}$. Well, if the function f is an injective Cullen regular function, then this is true. In addition, another interesting feature of Cullen regular functions is that it is possible to lift them in the space of twistors as specified in theorem 5.3 of [**20**] and later in this thesis. Thanks to these two facts the authors of [**20**] were able to construct and classify all the OCSes defined over $\mathbb{R}^4 \setminus \gamma$, where γ is a parabola[4].

To obtain the results in [**20**] the authors need to be sure that a slice regular function has several properties; among them they need to know that if it is injective then its real differential is everywhere non-singular. Moreover to show this and other results they make use of several rigidity properties of the class of slice regular functions, such as the Open Mapping theorem and the Maximum Modulus Principle. In this thesis we obtain and collect all the needed results to extend everything in our case (i.e.: when $\Omega_D \cap \mathbb{R} = \emptyset$), and, moreover, we start the study of twistor geometry via slice regular functions.

To be more precise, this work can be divided into four parts: the first (chapter 1) is about some basic definitions and results, while the other three, whose results are contained more or less in [**3**],[**4**] and [**5**] respectively, are briefly described in the following.

[4] In particular, they found that if one removes a parabola from $\mathbb{R}^4$, then has to remove as well a solid three dimensional paraboloid placed in a certain specific position w.r.t. γ.

In chapter 2 we show some rigidity results about slice regular functions. We will make use of *slice constant functions*: slice functions induced by locally constant stem functions. Of course, if D is connected and intersects $\mathbb{R}$, then a slice constant function $f = \mathcal{I}(F)$ defined on Ω_D is an actually constant function. An example of slice constant function is $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$, $f(\alpha + I\beta) = (1 - Ii)/2$. This function is constant on each $\mathbb{C}_J^+$ and is equal to one on $\mathbb{C}_i^+$ and to zero on $\mathbb{C}_{-i}^+$.

Let $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H} \in \mathcal{SR}(\Omega_D)$, $D_K^+ := \Omega_D \cap \mathbb{C}_K^+$, with $K \in \mathbb{S}$ and denote $f_K^+ := f \mid_{D_K^+}$. The next results, which were proven with the hypothesis $\Omega_D \cap \mathbb{R} \neq \emptyset$ respectively in **[9]**, **[27]** and the last two in **[22]**, are presented in this work, without this assumption, in the following formulations[5].

THEOREM 0.1. ***Identity Principle***. *Let Ω_D be a connected circular domain of $\mathbb{H}$. Given $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H} \in \mathcal{SR}(\Omega_D)$, with $V(f) = \{x \in \Omega_D \mid f(x) = 0\}$ be its zero locus. If there exists $K \neq J \in \mathbb{S}$ such that both $D_K^+ \cap V(f)$ and $D_J^+ \cap V(f)$ contain accumulation points, then $f \equiv 0$ on Ω_D.*

THEOREM 0.2. ***Maximum Modulus Principle***. *Let $f = \mathcal{I}(F) \in \mathcal{SR}(\Omega_D)$ with Ω_D connected circular domain. If there exist $J \neq K \in \mathbb{S}$ such that $||f_J^+||$ has relative maximum in $a \in D_J^+$ and $||f_K^+||$ has relative maximum in $b \in D_K^+$, then f is slice-constant on Ω_D.*

THEOREM 0.3. ***Minimum Modulus Principle***. *Let Ω_D be a connected circular domain and let $f : \Omega_D \to \mathbb{H}$ be a slice regular function. If $||f||$ has a local minimum point $p = x + yI \in D_I^+$ then either $f(p) = 0$ or exists a $J \in \mathbb{S}$ such that f_J^+ is constant.*

We will see then that a slice regular function $f : \Omega_D \to \mathbb{H}$, that is not slice-constant, can not admit a three dimensional submanifold $M_f \subset \Omega_D$ of constant values, but if it is constant on a curve γ then it must be constant on a surface that contains γ.

THEOREM 0.4. ***Open Mapping Theorem***. *Let $f : \Omega_D \to \mathbb{H}$ be a slice regular function non-slice constant, defined over a connected circular domain. Let S_f be the set of surfaces on which f is constant. Then*

$$f : \Omega_D \setminus S_f \to \mathbb{H}$$

is open.

We will see that, if f is not slice-constant, then the set S_f is closed with empty interior and that it must be removed from the domain of f to obtain the thesis of the Open Mapping Theorem. Indeed let us define f as:

$$f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}, \quad f(x) = x(1 - I_x i)/2, \quad x = \alpha + I_x\beta. \tag{3}$$

f is non-constant in every semislice except for $\mathbb{C}_{-i}^+$ in which is identically equal to zero. We have then that $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ is not open while $f : \mathbb{H} \setminus \mathbb{C}_{-i}^+ \to \mathbb{H}$ it is.

Since an important tool in the proofs of previous theorems is the representation formula, it is clear that they will fail if we do not assume "sliceness" (i.e. if we allow, in our

[5]For complex variables background we refer to classical book as **[1, 37, 42]**.

theory, examples such as the one in equations 1 and 2).

In chapter 3 we study some differential properties of slice regular functions. Precisely, we show that *the real differential of any injective slice regular function is invertible everywhere*. The result, which is a generalization of a theorem proved in [**20**], is obtained thanks to some new information regarding the first coefficients of a certain series expansion (called *spherical expansion*), and to a new general theorem which says that *the slice derivative $\partial f/\partial x$ of any injective slice function is everywhere different from zero*. A useful tool proven in this chapter is a new formula that relates slice and spherical derivatives of a slice regular function (see formula 18).

With the mentioned formula it is possible to explicit the real differential as follows. Let $f \in \mathcal{SR}(\Omega_D)$ and let $(df)_x$ denotes the real differential of f at $x = \alpha + I_x\beta \in \Omega_D \setminus \mathbb{R}$. If we identify $T_x\mathbb{H}$ with $\mathbb{H} = \mathbb{C}_{I_x} \oplus \mathbb{C}_{I_x}^{\perp}$, then for all $v_1 \in \mathbb{C}_{I_x}$ and $v_2 \in \mathbb{C}_{I_x}^{\perp}$,

$$(df)_x(v_1 + v_2) = v_1 \frac{\partial f}{\partial x}(x) + v_2 \partial_s f(x).$$

If $\alpha \in \Omega_D \cap \mathbb{R}$ then, the previous formula reduces to the following one

$$(df)_\alpha(v) = v \frac{\partial f}{\partial x}(\alpha) = v \partial_s f(\alpha).$$

Finally, in chapter 4 we apply all the previous results to achieve the differential geometry application mentioned above. First of all, given an injective slice regular function $f : \Omega_D \to \mathbb{H}$, we define the push forward over $f(\Omega_D \setminus \mathbb{R})$ of $\mathbb{J}$ via f as,

$$\mathbb{J}^f := (df)\mathbb{J}(df)^{-1}.$$

We, then, prove that $\mathbb{J}^f$ is an actual OCS and describe its action.

Now, the complex manifold $(\mathbb{H} \setminus \mathbb{R}, \mathbb{J})$ is biholomorphic to the open subset $\mathcal{Q}^+$ of the quadric

$$\mathcal{Q} = \{[X_0, X_1, X_2, X_3] \in \mathbb{CP}^3 \mid X_0X_3 = X_1X_2\}, \tag{4}$$

such that at least one of the following conditions is satisfied:

- $X_0 \neq 0$ and $X_2/X_0 \in \mathbb{C}^+$,
- $X_1 \neq 0$ and $X_3/X_1 \in \mathbb{C}^+$.

Here we are meaning $\mathbb{CP}^3$ as the twistor space of $\mathbb{S}^4 \simeq \mathbb{H} \cup \{\infty\} \simeq \mathbb{HP}^1$ (i.e.: the total space of a bundle parametrizing OCSes on $\mathbb{S}^4$), projecting in $\mathbb{HP}^1$ as $\pi : \mathbb{CP}^3 \to \mathbb{HP}^1$, $\pi[X_0, X_1, X_2, X_3] = [X_0 + X_1 j, X_2 + X_3 j]$. Starting from this correspondence between $\mathbb{H} \setminus \mathbb{R}$ and $\mathcal{Q}^+$, we prove that *if f is a slice function, then there exists a map from a subset $\mathcal{O}$ of $\mathcal{Q}^+$, $\tilde{f} : \mathcal{O} \subset \mathcal{Q}^+ \to \mathbb{CP}^3$ such that $\pi \circ \tilde{f} = f \circ \pi$* (see theorem 4.5). Moreover f is slice regular if and only if $\tilde{f}$ is holomorphic. This result is an extension of one proved in [**20**], where the domain Ω_D has nonempty intersection with the real line and the function f is regular.

From the last theorem, which is constructive, we are able to show that, up to projective transformations, the only non-singular surface over which the lift of a slice regular function can lies is $\mathcal{Q}$ itself. Then we show that, up to conformal transformation of $\mathbb{S}^4 \simeq \mathbb{H} \cup \{\infty\}$, all the non-singular quadrics in the Salamon-Viaclovsky classification (see theorem 1.11 in [**43**]), can be reached. Moreover, we find all the possible (singular) quadric and cubic surfaces (up to projective transformations), that can be expressed in this way.

Since the quadric in equation 4 is doubly ruled and biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, we pass then to study the geometry of lines through $\tilde{f}$. In fact, we show by construction that a sphere $\alpha + \mathbb{S}\beta \subset \mathbb{H}$ can be identified with the line

$$l_v := \{[1, u, \alpha + i\beta, (\alpha + i\beta)u] \,|\, u \in \mathbb{C} \cup \{\infty\}\} \subset \mathbb{CP}^3,$$

defined by fixing $v = \alpha + i\beta \in \mathbb{C}^+$. Furthermore it is possible to see that $\tilde{f}(v)$ is a line in $\mathbb{CP}^3$ too.

The line $\tilde{f}(l_v) \subset \mathbb{CP}^3$ is seen as a point in the Grassmannian $\mathbb{G}r_2(\mathbb{C}^4)$ or, equivalently, as a point in the Klein quadric in $\mathbb{P}(\bigwedge^2 \mathbb{C}^4) \simeq \mathbb{CP}^5$ via Plücker embedding. To study how these lines behave in the target surface of our twistor lift $\tilde{f}$, we recall the definition of *twistor transform* of f ([**20**], definition 5.6), i.e.: the map defined as

$$\mathcal{F} : D \to \mathbb{G}r(\mathbb{C}^4), \quad v \mapsto \tilde{f}(l_v).$$

For this map we extend a result proved in [**20**] which says that, with some technical hypotheses, *the twistor transform of a slice function f is a holomorphic curve if and only if the function f is regular*.

With this tool it is possible to characterize a certain class of linear holomorphic functions $\gamma : D \to \mathbb{G}r(\mathbb{C}^4)$, in terms of slice regular functions. As we will see, this result is interesting because shows the importance, in this context, of the set of slice regular functions that do not extend to the real line.

The last part of the work is devoted to the study of the function f in equation 3 as tool to generate OCSes over its image. As already said, this function is constant and equal to 0 if restricted to $\mathbb{C}^+_{-i}$ and equal to x if restricted to $\mathbb{C}^+_i$. Moreover, f restricted to $\mathbb{H} \setminus \mathbb{C}^+_{-i}$ is open and injective. For these reasons this function fit very well in our setting and then, we describe its image properly restricted, obtaining,

$$f(\mathbb{H} \setminus \mathbb{C}^+_{-i}) = \{x \in \mathbb{H} \mid x_1 > 0\}, \quad \text{where } x = x_0 + x_1 i + x_2 j + x_3 k.$$

Then we show that the twistor lift of f lies in the hypersurface $\mathcal{H} := \{X_3 = 0\} \subset \mathbb{CP}^3$: in this case the general theory (see Section 3 of [**43**]) says that $\mathcal{H}$ induces an OCS conformally equivalent to a constant one, defined over the image of f. This is actually true and, in fact, the conformal function that shows this equivalence is $g : \{q_1 > 0\} \to \{q_1 < 0\}$ defined by $g(q) = q^{-1}$. So, in particular, *the map $h = g \circ f : (\mathbb{H} \setminus \mathbb{C}^+_{-i}, \mathbb{J}) \to (\{q_1 < 0\}, \mathbb{J}_i)$ shows a biholomorphism between the two complex manifolds.*

This result closes the present research, but, of course, all these new notions and tools opened several questions which we hope to study in the future. Some of these questions are listed in the final appendix.

CHAPTER 1

Definitions and basic tools

1. Slice functions and regularity

In this chapter we state the main definitions and results in the slice regularity theory.

Let $x = x_0+ix_1+jx_2+kx_3$ be generic element of $\mathbb{H}$, the real skew algebra of quaternions, where

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k,\, jk = -kj = -1,\, ki = -ik = j.$$

We denote the usual conjugation with $x^c = x_0 - ix_1 - jx_2 - kx_3$. It is clear then that

- $(x^c)^c = x$;
- $(xy)^c = y^c x^c$;
- $x^c = x$, $\forall x \in \mathbb{R}$.

For every $x \in \mathbb{H}$, is defined its (squared) norm as $||x||^2 = xx^c$. Any $x \in \mathbb{H} \setminus \{0\}$ is then invertible and $x^{-1} = x^c/||x||^2$.

We denote by $\mathbb{S} \subset \mathbb{H}$ the sphere of imaginary units in $\mathbb{H}$:

$$\mathbb{S} := \{I \in \mathbb{H} \mid I^2 = -1\}.$$

Let now $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ be the real tensor product between $\mathbb{H}$ and the complex space $\mathbb{C}$. An element of $\mathbb{H}_{\mathbb{C}}$ is a sum $w = x + \sqrt{-1}y$, where $x, y \in \mathbb{H}$. The space $\mathbb{H}_{\mathbb{C}}$ is a complex alternative algebra[1] with a unity w.r.t. the product defined by the formula

$$(x + \sqrt{-1}y)(z + \sqrt{-1}w) := xz - yw + \sqrt{-1}(xw + yz).$$

The algebra $\mathbb{H}$ can be identified with the sub algebra $\{x + \sqrt{-1}y \mid y = 0\} \subset \mathbb{H}_{\mathbb{C}}$ and the unity of $\mathbb{H}_{\mathbb{C}}$ coincide with the one of $\mathbb{H}$. In $\mathbb{H}_{\mathbb{C}}$ are then defined two commuting conjugations:

- $w^c = (x + \sqrt{-1}y)^c = x^c + \sqrt{-1}y^c$;

[1]An alternative algebra A is an algebra in which the multiplication is *alternative*, that is, for any $x, y \in A$, $x(xy) = (xx)y$ and $(yx)x = y(xx)$.

- $\overline{w} = \overline{x + \sqrt{-1}y} = x - \sqrt{-1}y$.

As already said, the main object in this thesis are slice regular functions. To define them we will use the concept of stem functions introduced by Ghiloni and Perotti in [**30**]. The idea is to have a "generating function" which carries on all the information we need.

The definition is rather technical but it will turn out to be very useful. The general reference for most of this chapter is [**30**] and the next is definition 4 of the mentioned paper.

DEFINITION 1.1. A function $F : D \to \mathbb{H}_{\mathbb{C}}$ is called a *stem function* on D if it is complex intrinsec, i.e.: if the condition

$$F(\overline{z}) = \overline{F(z)} \tag{5}$$

holds for each $z \in D$ such that $\overline{z} \in D$. Moreover we call F continuous or differentiable if the two components of $F = F_1 + \sqrt{-1}F_2$ are respectively continuous or differentiable.

REMARK 1.1. There are no restrictions to assume that D is symmetric with respect to the real axis, i.e.:

$$D = conj(D) := \{z \in \mathbb{C} \,|\, \overline{z} \in D\}.$$

In fact, if this is not the case, F can be extended to $D \cup conj(D)$ by imposing equation (5). Moreover, a function $F : D \subset \mathbb{C} \to \mathbb{H}$ is a stem function if and only if the $\mathbb{H}$-valued components F_1, F_2 of $F = F_1 + \sqrt{-1}F_2$ form an even-odd pair with respect to the imaginary part of z, i.e.:

$$F_1(\overline{z}) = F_1(z),\ F_2(\overline{z}) = -F_2(z),\ \forall z \in D \,\text{s.t.}\ \bar{z} \in D.$$

REMARK 1.2. In remark 3 of [**30**] it is described, in a more general context the following construction: as a real vector space, $\mathbb{H}$ has dimension 4, so, let $\mathcal{B} = \{u_k\}_{k=1}^4$ be a basis for $\mathbb{H}$. The function F can be identified with a complex intrinsic curve in $\mathbb{C}^4$. Let $F(z) = F_1(z) + \sqrt{-1}F_2(z) = \sum_{k=1}^4 F_{\mathcal{B}}^k(z)u_k$, with $F_{\mathcal{B}}^k(z) \in \mathbb{C}$. Then

$$\tilde{F}_{\mathcal{B}} = (F_{\mathcal{B}}^1, F_{\mathcal{B}}^2, F_{\mathcal{B}}^3, F_{\mathcal{B}}^4) : D \to \mathbb{C}^4$$

satisfies $\tilde{F}_{\mathcal{B}}(\overline{z}) = \overline{\tilde{F}_{\mathcal{B}}(z)}$. Giving to $\mathbb{H}$ the unique manifold structure as a real vector space, we get that a stem function F is of class C^k or real-analytic if and only if the same property holds for $\tilde{F}_{\mathcal{B}}$. Moreover this notion of differentiability is independent of the choice of the basis of $\mathbb{H}$.

DEFINITION 1.2. Given any set $D \subset \mathbb{C}$ we define the *circularization* of D in $\mathbb{H}$ as the subset of $\mathbb{H}$ defined by:

$$\Omega_D := \{\alpha + J\beta \in \mathbb{H} \,|\, \alpha + i\beta \in D\,,\ J \in \mathbb{S}\}.$$

A set Ω_D of this type will be called *circular domain*. If $D \cap \mathbb{R} = \emptyset$, then $\Omega_D = D \times \mathbb{S}$ and it is called *product domain*.

REMARK 1.3. If Ω_D is such that $\Omega_D \cap \mathbb{R} \neq \emptyset$, then Ω_D is a so-called *slice domain* (see definition 1.14 of [**25**]).

From now on D will always be an open set of $\mathbb{C}$ and Ω_D will be its associated circular set. In the following we will use the notations

$$D_J := \Omega_D \cap \mathbb{C}_J, \quad D_J^+ := \Omega_D \cap \mathbb{C}_J^+,$$

where $\mathbb{C}_J := \{x = \alpha + J\beta \in \mathbb{H} \,|\, \alpha, \beta \in \mathbb{R}\}$ and $\mathbb{C}_J^+ := \{x = \alpha + J\beta \in \mathbb{H} \,|\, \alpha, \beta \in \mathbb{R}\,,\, \beta \geq 0\}$. If $D \cap \mathbb{R} = \emptyset$, then $D_J = D \times \{-J, J\}$ and $D_J^+ = D \times \{J\}$. The sets D_J and D_J^+ will be called respectively *slice* and *semislice*. We are now in position to remember the definition of slice function (see definition 5 of [**30**]).

DEFINITION 1.3. A function $f : \Omega_D \to \mathbb{H}$ is called a (left)[2] slice function if it is induced by a stem function $F = F_1 + \sqrt{-1}F_2$ on D, denoted by $f = \mathcal{I}(F)$, in the following way:

$$f(\alpha + J\beta) := F_1(\alpha + i\beta) + JF_2(\alpha + i\beta), \quad \forall\, x = \alpha + J\beta \in \Omega_D.$$

We will denote by $\mathcal{S}(\Omega_D)$ and by $\mathcal{S}^1(\Omega_D)$ the real vector spaces and right $\mathbb{H}$-module[3] of slice functions on Ω_D induced respectively by continuous and differentiable stem functions.

Since (F_1, F_2) is an even-odd pair w.r.t. β, then the slice function f is well defined, in fact $f(\alpha + (-J)(-\beta)) = F_1(\overline{z}) + (-J)F_2(\overline{z}) = F_1(z) + JF_2(z)$. Moreover, for the same reason, f is defined also on real points: in fact if $F = F_1 + \sqrt{-1}F_2$, then $F_2(\alpha) = 0$ for any $\alpha \in \mathbb{R}$.

So, as we pointed out in the introduction, a slice function is a quaternionic function of one quaternionic variable that is affine w.r.t. the imaginary unit. With this in mind it is not difficult to understand part of the behavior of such a function. Even if the notion of stem function seems useless, later this will turn out to be false. Many definitions, in fact, will result to be more "natural" given in the stem's language than in the quaternionic functions' one. Let us show now some examples.

EXAMPLE 1.1.
(1) Clearly the stem functions $z = Re(z) + \sqrt{-1}Im(z)$ and $\overline{z} = Re(z) - \sqrt{-1}Im(z)$ induces the slice functions x and x^c respectively.
(2) For any $a \in \mathbb{H}$,$F(z) := z^n a = Re(z^n)a + \sqrt{-1}(Im(z^n)a)$ induces the monomial $f(x) = x^n a \in \mathcal{S}(\mathbb{H})$.
(3) By linearity, we get all the standard polynomials $p(x) = \sum_{j=0}^n x^j a_j$ with right quaternionic coefficients. More generally, every convergent power series $\sum_j x^j a_j$, with (possibly infinite) convergence radius R, belongs to the space $\mathcal{S}(B_R)$, where B_R is the open ball of $\mathbb{H}$ centered in the origin with radius R.
(4) The two functions $G(z) := Re(z^n)a$ and $H(z) := \sqrt{-1}Im(z^n)a$ are complex intrinsic on $\mathbb{C}$. They induce respectively the slice functions $g(x) = Re(x^n)a$ and $h(x) = Im(x^n)a$.

An important property of slice functions is that, as usual affine functions, they can be recovered by knowing their values on two semislices. More precisely, we have the following theorem.

[2]Of course, there is an analogous definition for right slice functions when the element $J \in \mathbb{S}$ is placed on the right of $F_2(z)$.

[3]This means that for each real number λ, each $q \in \mathbb{H}$ and each couple of slice (differentiable) functions f and g, $(f\lambda + g)q \in \mathcal{S}^{(1)}$.

Theorem 1.1. *([**30**], proposition 6). Let $J, K \in \mathbb{S}$ with $J \neq K$. Then every $f \in \mathcal{S}(\Omega_D)$ is uniquely determined by its values on D_J^+ and D_K^+. More precisely we have the following formula*

$$f(x) = (I-K)(J-K)^{-1}f(\alpha+J\beta) - (I-J)(J-K)^{-1}f(\alpha+K\beta) \tag{6}$$

for all $I \in \mathbb{S}$, for all $x = \alpha + I\beta \in D_I$. In particular if $K = -J$, we get the following simpler formula

$$f(x) = \frac{1}{2}\left[f(\alpha+J\beta) + f(\alpha-J\beta) - IJ\left(f(\alpha+J\beta) - f(\alpha-J\beta)\right)\right].$$

This theorem was firstly proven in [**9**] for slice regular functions on slice domains, and was used to show an extension result. After that, Ghiloni and Perotti in [**30**] proved the same theorem for slice functions which are not, in general, regular. Moreover, the Representation formula 6 characterizes slice functions: if a function $f : \Omega_D \to \mathbb{H}$ can be reconstructed with formula 6 for any couple $J \neq K \in \mathbb{S}$, then f is a slice function.

If $I = J$ we have the trivial equality

$$f(x) = \frac{1}{2}\left[f(x) + f(x^c)\right] + \frac{1}{2}\left[f(x) - f(x^c)\right],$$

where clearly $\frac{1}{2}(f(x) + f(x^c)) = F_1(z)$ and $\frac{1}{2}(f(x) - f(x^c)) = JF_2(z)$. Having this in mind, we remember definition 6 of [**30**].

Definition 1.4. We define the *spherical derivative* of f in $x \in \Omega_D \setminus \mathbb{R}$ as

$$\partial_s f(x) := \frac{1}{2} Im(x)^{-1}(f(x) - f(x^c))$$

and the *spherical value* of f in $x \in \Omega_D$ as

$$v_s f(x) := \frac{1}{2}(f(x) + f(x^c)).$$

Remark 1.4. We have that $v_s f = \mathcal{I}(F_1(z))$ on Ω_D and $\partial_s f = \mathcal{I}(\frac{F_2(z)}{Im(z)})$ on $\Omega_D \setminus \mathbb{R}$. Given $x = \alpha + J\beta \in \Omega_D$, the spherical derivative is constant on every sphere $\mathbb{S}_x = \{y \in \mathbb{H} \,|\, y = \alpha + I\beta, I \in \mathbb{S}\}$. Moreover $\partial_s f = 0$ if and only if f is constant on $\mathbb{S}_x$, in other terms:

$$\partial_s(\partial_s(f)) = 0,$$

and, in this case, $f = v_s f$ on $\mathbb{S}_x$. If $\Omega_D \cap \mathbb{R} \neq \emptyset$, under some regularity hypothesis on F (e.g.: differentiability of F_2), $\partial_s f$ can be extended continuously as a slice function on Ω_D. The next theorem will precise this claim.

The following is a regularity result for slice functions depending on their stem functions.

Theorem 1.2. *([**30**], proposition 7). Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$.*

(1) If $F \in C^0(D)$ then $f, v_s f \in C^0(\Omega_D)$, $\partial_s f \in C^0(\Omega_D \setminus \mathbb{R})$;
(2) If $F \in C^{2s+1}(D)$, $s \in \mathbb{N}$, then $f, v_s f, \partial_s f \in C^s(\Omega_D)$;
(3) If $F \in C^\omega(D)$ then $f, v_s f, \partial_s f \in C^\omega(\Omega_D)$ where $C^\omega(D)$ and $C^\omega(\Omega_D)$ denote, respectively, the spaces of analytic functions on D and on Ω_D).

Let $D \subset \mathbb{C}$ be an open set. Given a stem function $F = F_1 + \sqrt{-1}F_2 : D \to \mathbb{H}_{\mathbb{C}}$ of class $\mathcal{C}^1$, the two functions

$$\frac{\partial F}{\partial z}, \frac{\partial F}{\partial \bar{z}} : D \to \mathbb{H}_{\mathbb{C}},$$

are stem functions. Explicitly:

$$\frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial F}{\partial \alpha} - \sqrt{-1}\frac{\partial F}{\partial \beta}\right) = \frac{1}{2}\left(\frac{\partial F_1}{\partial \alpha} + \frac{\partial F_2}{\partial \beta} - \sqrt{-1}\left(\frac{\partial F_1}{\partial \beta} - \frac{\partial F_2}{\partial \alpha}\right)\right),$$

and

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial \alpha} + \sqrt{-1}\frac{\partial F}{\partial \beta}\right) = \frac{1}{2}\left(\frac{\partial F_1}{\partial \alpha} - \frac{\partial F_2}{\partial \beta} + \sqrt{-1}\left(\frac{\partial F_1}{\partial \beta} + \frac{\partial F_2}{\partial \alpha}\right)\right).$$

The previous stem functions induce the continuous *slice derivatives*:

$$\frac{\partial f}{\partial x} = \mathcal{I}\left(\frac{\partial F}{\partial z}\right), \quad \frac{\partial f}{\partial x^c} = \mathcal{I}\left(\frac{\partial F}{\partial \bar{z}}\right).$$

REMARK 1.5. Let $y = \xi + J\eta \in \Omega_D \setminus \mathbb{R}$. Define $\omega := \xi + i\eta \in D$ and $\Phi_J : D \to \Omega_D$ by setting $\Phi_J(\alpha + i\beta) := \alpha + J\beta$. We recall definition 1.7 in **[20]** of $\partial_C f$ and definition 1.1 in **[20]** of $\overline{\partial}_C f$:

$$\partial_C f(y) = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - J\frac{\partial}{\partial \beta}\right)(f \circ \Phi_J)(\omega), \quad \overline{\partial}_C f(y) = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} + J\frac{\partial}{\partial \beta}\right)(f \circ \Phi_J)(\omega).$$

On $\Omega_D \setminus \mathbb{R}$, the derivatives $\partial f/\partial x$ and $\partial_C f$ of a slice regular function f coincide. In fact it holds:

$$\begin{aligned} 2\frac{\partial f}{\partial x}(y) &= \left(\frac{\partial F_1}{\partial \alpha}(\omega) + \frac{\partial F_2}{\partial \beta}(\omega)\right) + J\left(\frac{\partial F_2}{\partial \alpha}(\omega) - \frac{\partial F_1}{\partial \beta}(\omega)\right) \\ &= \left(\frac{\partial F_1}{\partial \alpha}(\omega) + J\frac{\partial F_2}{\partial \alpha}(\omega)\right) - J\left(\frac{\partial F_1}{\partial \beta}(\omega) + \frac{\partial F_2}{\partial \beta}(\omega)\right) \\ &= \left(\frac{\partial}{\partial \alpha} - J\frac{\partial}{\partial \beta}\right)(f \circ \Phi_J)(\omega) = 2\partial_C f(y). \end{aligned}$$

Similarly, $\partial f/\partial x^c = \overline{\partial}_C f$ on $\Omega_D \setminus \mathbb{R}$. Therefore the operators $\partial/\partial x$ and $\partial/\partial x^c$ extend the Cullen derivative ∂_C and $\overline{\partial}_C$ to each slice function on domains possibly intersecting the real axis.

While the spherical derivative control the behavior of a slice function f along the "spherical" directions determined by $\mathbb{S}$, the slice derivatives $\partial/\partial x$ and $\partial/\partial x^c$, give information about the behavior along the remaining directions (i.e.: along the (semi)slices).

Now, left multiplication by $\sqrt{-1}$ defines a complex structure on $\mathbb{H}_{\mathbb{C}}$ and, with respect to this structure, a C^1 stem function

$$F = F_1 + \sqrt{-1}F_2 : D \to \mathbb{H}_{\mathbb{C}}$$

is holomorphic if and only if satisfy the Cauchy-Riemann equations

$$\frac{\partial F_1}{\partial \alpha} = \frac{\partial F_2}{\partial \beta}, \frac{\partial F_2}{\partial \beta} = -\frac{\partial F_2}{\partial \alpha}, z = \alpha + i\beta \in D$$

or equivalently if

$$\frac{\partial F}{\partial \overline{z}} \equiv 0.$$

This condition is equivalent to require that, for any basis $\mathcal{B}$, the complex curve $\tilde{F}_{\mathcal{B}}$ defined in remark 1.2 is holomorphic.

We are now in position to define slice regular functions (see definition 8 in [**30**]).

Definition 1.5. A function $f \in \mathcal{S}^1(\Omega_D)$ is *(left) slice regular* if its stem function F is holomorphic. The set of slice regular functions will be denoted by

$$\mathcal{SR}(\Omega_D) := \{f \in \mathcal{S}^1(\Omega_D) \mid f = \mathcal{I}(F), F : D \to \mathbb{H}_{\mathbb{C}} \text{ holomorphic}\}.$$

Equivalently, a slice function $f \in \mathcal{S}^1(\Omega_D)$ is regular if the following equation holds:

$$\frac{\partial f}{\partial x^c}(\alpha + J\beta) = 0, \quad \forall \alpha + J\beta \in \Omega_D.$$

The set of slice regular functions is again closed under linear combinations with real coefficients and so it is a real vector space. Moreover it is also closed under right multiplication by a quaternion, i.e.: $\mathcal{SR}$ is a right $\mathbb{H}$-module.

Example 1.2. The polynomials $\sum_{j=0}^m x^j a_j$ and power series in the previous example 1.1 are non trivial slice regular functions.

Further examples of slice regular functions will be given conveniently in the following. For now, as stated by the following theorem, a slice regular function can be constructed by means of the Representation formula in theorem 1.1.

The next theorem gives, in fact, a characterization of slice regular functions, but to state it it will be useful to introduce the following notation. Given $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$, we denote the restrictions over a complex slice or a complex semi-slice, respectively, as

$$f_J := f|_{D_J} : D_J \to \mathbb{H}, \quad f_J^+ := f|_{D_J^+} : D_J^+ \to \mathbb{H}.$$

The proof of the following proposition can be obtained combining proposition 8 and remark 6 of [**30**].

Proposition 1.3. *([**30**]) Let $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$, then the following facts are equivalents:*

- *$f \in \mathcal{SR}(\Omega_D)$;*
- *the restriction f_J^+ is holomorphic for every $J \in \mathbb{S}$ with respect to the complex structures on D_J and $\mathbb{H}$ defined by left multiplication by J;*
- *two restrictions f_J^+, f_K^+ $(J \neq K)$ are holomorphic on D_J^+ and D_K^+ respectively (the possibility $K = -J$ is not excluded).*

Remark 1.6. We underline that, the third point says that, in order to get slice regularity of $f = \mathcal{I}(F)$, it is sufficient to assume that two restrictions f_J^+, f_K^+ $(J \neq K)$ are holomorphic on D_J^+ and D_K^+ respectively. The possibility $K = -J$ is not excluded.

The second point of the proposition shows that if the set D has nonempty intersection with the real line, then f is slice regular on Ω_D if and only if it is Cullen regular in the sense introduced by Gentili and Struppa in [**26, 27**] (see also definition 0.1 in the introduction). Moreover in theorem 2.11 of [**32**], the authors, showed, in the more general context of real alternative algebras, that requiring regularity (in the sense of Cullen) for a quaternionic function does not imply sliceness if the domain does not intersects the real axis. An example of quaternionic regular function which is not slice is the following, already given in formula 2 in the introduction: fix a $J \in \mathbb{S}$ and a real number $\lambda \notin \{-1, 0, 1\}$. Let $x = \alpha + I_x\beta$ be a non-real quaternion and define $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ as

$$f(x) = I_x + \lambda J I_x J.$$

This function, which is of class $\mathcal{C}^\infty$ and Cullen-regular, sends $\mathbb{H} \setminus \mathbb{R}$ into an ellipsoidal surface. The function has image equal to an ellipsoidal surface and so, even if it is constant and consequently regular in every semislice, it is not a slice function. This issue will condition many of the following results (especially the one regarding rigidity), meaning that, many theorems will be split into two parts: one that deals with the case in which the function is defined over real points and the other that explores the remaining case.

REMARK 1.7. If a function f is slice regular and of class $\mathcal{C}^2$ on its domain, then also its slice derivative $\frac{\partial f}{\partial x}$ is slice regular on the same domain.

REMARK 1.8. Since the spherical derivative $\partial_s f$ is $\mathbb{H}$-valued, then it is slice regular only when is locally constant. In fact, let $f : \Omega_D \to \mathbb{H}$ be a slice function induced by $F = F_1 + \sqrt{-1}F_2$. Then, imposing the Cauchy-Riemann equations to the stem function $G : D \to \mathbb{H}_\mathbb{C}$, $G(\alpha + i\beta) = F_2(\alpha + i\beta)/\beta$, means that

$$\begin{cases} \dfrac{\partial G_1}{\partial \alpha} = \dfrac{\partial G_2}{\partial \beta} \\ \dfrac{\partial G_1}{\partial \beta} = -\dfrac{\partial G_2}{\partial \alpha} \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dfrac{1}{\beta}\dfrac{\partial F_2}{\partial \alpha} = 0 \\ \dfrac{(\partial F_2/\partial \beta)\beta - F_2}{\beta^2} = 0, \end{cases}$$

but the first equation says that $F_2(\alpha + i\beta) = q(\beta)$, for some quaternionic function $q : D \to \mathbb{H}$, while the second says that $\frac{\partial q}{\partial \beta} = \frac{q(\beta)}{\beta}$, and so $q(\beta) = q_0\beta$, for some $q_0 \in \mathbb{H}$.

Another consolidated and well known result about slice regular functions is the *splitting lemma*. It says that any slice regular function, if properly restricted, admits a splitting into two actual complex holomorphic functions. A proof of this result can be found in [**9, 33**], the first with the additional hypothesis that the domain of definition intersects the real axis.

LEMMA 1.4. *Let $f \in \mathcal{SR}(\Omega_D)$. Then, for each $J \in \mathbb{S}$ and each $K \perp J$, $K \in \mathbb{S}$, there exist two holomorphic functions $g, h : D_J \to \mathbb{C}_J$ such that*

$$f_J = g + hK.$$

Observe that g and h are defined over the whole D_J. This means that, if D_J is disconnected and the disjoint union of D_1 and D_2, then, g and h could have unrelated different behavior on D_1 and D_2. A particular case is when $\Omega_D \cap \mathbb{R} = \emptyset$, where, *a priori* the function f and g can have different behaviors if restricted either to D_J^+ or D_J^-.

REMARK 1.9. As stated in Remark 2.5 of [**33**], an immediate consequence of the splitting lemma is that, given a sequence $\{f_n\}_{n\in\mathbb{N}}$ of slice regular functions on Ω_D, uniformly convergent on compact subsets of Ω_D, the limit of the sequence is slice regular on Ω_D.

2. Product of slice functions and their zero set

We now want to multiply slice regular functions. In general, the pointwise product of slice functions is not a slice function[4], so we need another notion of product. The following, introduced in [**9, 24**] for slice regular functions defined over domains that does intersect $\mathbb{R}$ and in definition 9 of [**30**] for slice functions (in the context of real alternative algebras), is the notion that we will use.

DEFINITION 1.6. Let $f = \mathcal{I}(F)$, $g = \mathcal{I}(G) \in \mathcal{S}(\Omega_D)$ the *(slice) product* of f and g is the slice function

$$f \cdot g := \mathcal{I}(FG) \in \mathcal{S}(\Omega_D).$$

Explicitely, if $F = F_1 + \sqrt{-1}F_2$ and $G = G_1 + \sqrt{-1}G_2$ are stem functions, then $FG = F_1G_1 - F_2G_2 + \sqrt{-1}(F_1G_2 + F_2G_1)$.

REMARK 1.10. Let $f(x) = \sum_j x^j a_j$ and $g(x) = \sum_k x^k b_k$ be polynomials or, more generally, converging power series with coefficients $a_j, b_k \in \mathbb{H}$. The usual product of polynomials, where x is considered to be a commuting variable, can be extended to power series in the following way: the *star product* $f * g$ of f and g is the convergent power series defined by setting

$$(f * g)(x) := \sum_n x^n \left(\sum_{j+k=n} a_j b_k \right).$$

In proposition 12 of [**30**] it was proved that the product of f and g, viewed as slice functions, coincide with the star product $f * g$, i.e.: $\mathcal{I}(FG) = \mathcal{I}(F) * \mathcal{I}(G)$. Indeed sometimes the slice product between f and g is denoted by $f * g$ (see [**27**] or [**22**]) and called *regular product*, to stress that this notion of product was born to preserve the regularity as the next proposition says.

PROPOSITION 1.5. *(*[**30**]*, proposition 11). If $f, g \in \mathcal{SR}(\Omega_D)$ then $f \cdot g \in \mathcal{SR}(\Omega_D)$*

In [**30**] it is also pointed out and proved that the regular product introduced in [**9, 24**] is generalized by this one if the domain Ω_D does not have real points. By the way, an idea to prove this theorem is simply to explicit the slice product in term of stem function and compute the Cauchy-Riemann equations.

[4]For instance, if $f(q) = qa$ and $g(q) = q$, with $a \in \mathbb{H} \setminus \mathbb{R}$, then $h(q) = f(q)g(q) = qaq$ is not a slice function.

REMARK 1.11. If f, g are slice functions then the spherical derivative of the product follows a *Leibnitz type formula*:

$$\partial_s(f \cdot g) = (\partial_s f)(v_s g) + (v_s f)(\partial_s g).$$

The slice product of two slice functions coincide with the punctual product if the first slice function is *real* (see definition 10 of [**30**]).

DEFINITION 1.7. The slice function $f = \mathcal{I}(F)$ is called *real* or *slice-preserving* or, again, *quaternionic intrinsec* if the $\mathbb{H}$-valued components F_1, F_2 are real valued.

The next proposition, stated in lemma 6.8 of [**29**], justifies the different names given in the previous definition.

PROPOSITION 1.6. *Let $f = \mathcal{I}(F)$ be a slice function. The following conditions are equivalent.*

- *f is real.*
- *For all $J \in \mathbb{S}$, $f(D_J) \subset \mathbb{C}_J$.*
- *For all x in the domain of f it holds $f(x) = (f(x^c))^c$.*

These functions are special since, in a certain sense, transpose the concept of complex function in our setting. In fact, if $h(z) = u(z) + iv(z)$ is a complex function defined over a certain domain $D \subset \mathbb{C}$, then the function $H : D \to \mathbb{H}_{\mathbb{C}}$ defined as $H(z) = u(z) + \sqrt{-1}v(z)$ is a stem function, and $\mathcal{I}(H)$ is a real slice function.

As stated in [**24**], if f is a slice regular function defined on $B(0, R)$, the ball of center zero and radius R for some $R > 0$, then it is real if and only if f can be expressed as a power series of the form

$$f(x) = \sum_{n \in \mathbb{N}} x^n a_n,$$

with a_n real numbers.

LEMMA 1.7. *Let $f = \mathcal{I}(F), g = \mathcal{I}(G) \in \mathcal{S}(\Omega_D)$, with f real, then the slice function $h : \Omega_D \to \mathbb{H}$, defined by $h := f \cdot g$ is such that*

$$h(x) = f(x)g(x).$$

Proof. The proof of this lemma can be found in a more general context in remark 7 of [**30**]. If $x = \alpha + J\beta$ belongs to $D_J = \Omega_D \cap \mathbb{C}_J$ and $z = \alpha + i\beta$, then

$$(f \cdot g)(x) = F_1(z)G_1(z) - F_2(z)G_2(z) + JF_1(z)G_2(z) + JF_2(z)G_1(z),$$

while

$$f(x)g(x) = F_1(z)G_1(z) + JF_2(z)JG_2(z) + F_1(z)JG_2(z) + JF_2(z)G_1(z).$$

If the components F_1, F_2 of the first stem function F are real-valued, then $(f \cdot g)(x) = f(x)g(x)$ for every $x \in \Omega_D$.

$\square$

In the next proposition we explicit the slice product as the pointwise product with the proper evaluations. This proposition was proved for regular functions defined on domains that intersect the real axis in [**9, 23, 24**].

PROPOSITION 1.8. *Let $f, g \in \mathcal{SR}(\Omega_D)$ then, for any $x \in \Omega_D \setminus V(f)$*

$$(f \cdot g)(x) = f(x)g(f(x)^{-1}xf(x)).$$

Proof. Let $x = \alpha + I\beta$, with $I \in \mathbb{S}$ and $z = \alpha + i\beta$. Since $f(x) = F_1(z) + IF_2(z)$ is invertible, we have

$$\begin{aligned}
(f \cdot g)(x) &= \mathcal{I}(FG)(x) \\
&= F_1(z)G_1(z) - F_2(z)G_2(z) + I(F_1(z)G_2(z) + F_2(z)G_1(z)) \\
&= (F_1(z) + IF_2(z))[G_1(z) + \\
&\quad +(F_1(z) + IF_2(z))^{-1}(IF_1(z)G_2(z) - F_2(z)G_2(z))] \\
&= (F_1(z) + IF_2(z))[G_1(z) + \\
&\quad (F_1(z) + IF_2(z))^{-1}I(F_1(z) + IF_2(z))G_2(z)]
\end{aligned}$$

but, since $F_1(z) + IF_2(z)$ is invertible, then $I' = (F_1(z) + IF_2(z))^{-1}I(F_1(z) + IF_2(z)) = f(x)^{-1}If(x) \in \mathbb{S}$. So if we call $x' = \alpha + I'\beta = f(x)^{-1}xf(x)$ we obtain the thesis

$$(f \cdot g)(x) = f(x)g(x').$$

□

Given any quaternionic function $f : \Omega \subset \mathbb{H} \to \mathbb{H}$ of one quaternionic variable we will denote its zero set as

$$V(f) := \{x \in \Omega \mid f(x) = 0\}$$

COROLLARY 1.9. *Let $f = \mathcal{I}(F), g = \mathcal{I}(G) \in \mathcal{S}(\Omega_D)$, with g real, then the slice function $h : \Omega_D \setminus V(g) \to \mathbb{H}$, defined by $h = \mathcal{I}(G^{-1}F)$ is such that*

$$h(x) = \frac{1}{g(x)} f(x),$$

and, moreover, h belongs to $\mathcal{SR}(\Omega_D \setminus V(g))$.

In view of defining a "slice inversion" also for non-real slice functions, we introduce the following objects. The following definitions appeared for the first time in [**9**], can be found also in [**22**] and [**24**]. Later they were generalized by Ghiloni and Perotti for slice functions in definition 11 of [**30**].

DEFINITION 1.8. Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$, then also $F^c(z) = F(z)^c := F_1(z)^c + \sqrt{-1}F_2(z)^c$ is a stem function. We set

- $f^c := \mathcal{I}(F^c) \in \mathcal{S}(\Omega_D)$, called *slice conjugate* of f;
- $CN(F) := F^cF$;
- $N(f) := f^c \cdot f = \mathcal{I}(CN(F))$ *symmetrization* or *normal function* of f.

The symmetrization of f is sometimes denoted by f^s.

REMARK 1.12. We have that $(FG)^c = G^cF^c$, and so $(f \cdot g)^c = g^c \cdot f^c$, i.e.:

$$N(f) = N(f)^c.$$

Moreover the next equalities holds true:

$$N(f \cdot g) = N(f)N(g) \text{ and } N(f^c) = N(f).$$

Let now spend a few words about the zero locus of slice functions. For more details see [**30**] and [**24**]. The next result is a reformulation of proposition 16 of [**30**].

PROPOSITION 1.10. *Let $f \in \mathcal{S}(\Omega_D)$. Then the restriction $f_{|\mathbb{S}_x}$ is injective or constant for all $x \in \Omega_D \setminus \mathbb{R}$. In particular, either $\mathbb{S}_x \subseteq V(f)$ or $\mathbb{S}_x \cap V(f)$ is at most a singleton.*

This proposition is very natural given the expression of slice functions as "affine functions w.r.t. the imaginary unit". The structure of $V(f)$ for slice functions is showed in the next theorem.

THEOREM 1.11. *([**30**], theorem 17). Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$. Let $x = \alpha + J\beta \in \Omega_D$, $z = \alpha + i\beta \in D$. One of the mutually exclusive statements holds:*

(1) $\mathbb{S}_x \cap V(f) = \emptyset$;
(2) $\mathbb{S}_x \subseteq V(f)$ (in this case x is called a real ($x \in \mathbb{R}$) or spherical ($x \notin \mathbb{R}$) zero of f);
(3) $\mathbb{S}_x \cap V(f)$ consists of a single, non-real point (in this case x is called an $\mathbb{S}$-isolated non-real zero of f).

These three possibilities correspond, respectively to the following properties of $F(z) \in \mathbb{H}_{\mathbb{C}}$:

(1) $CN(F)(z) = F(z)F(z)^c \neq 0$;
(2) $F(z) = 0$
(3) $F(z) \neq 0$ and $CN(F)(z) = 0$.

Simple examples of the cases previously described can be found in Example 3.6-7 of [**25**]. Anyway we report them here for completeness.

EXAMPLE 1.3.
- Fix $J \in \mathbb{S}$, then the slice function $f_1 : \mathbb{H} \to \mathbb{H}$ defined by $f_1(x) = (x - J) \cdot (x + J) = x^2 + 1$ vanishes at each imaginary unit $I \in \mathbb{S}$.
- Fix $J_1 \neq \pm J_2 \in \mathbb{S}$, then the slice function $f_2 : \mathbb{H} \to \mathbb{H}$ defined by $f_2(x) = (x - J_1) \cdot (x - J_2) = x^2 - x(J_1 + J_2) + J_1J_2$ vanishes at J_1 and in no other point in $\mathbb{S}$ (in particular $f_2(J_2) \neq 0$).

A simple corollary of the previous theorem is the following.

COROLLARY 1.12. *([**30**], corollary 19). The following statements hold:*

(1) A real slice function has no $\mathbb{S}$-isolated non-real zeros;
(2) For all $f \in \mathcal{S}(\Omega_D)$

$$V(N(f)) = \bigcup_{x \in V(f)} \mathbb{S}_x.$$

The idea of the first point in the previous theorem is that if $f = \mathcal{I}(F)$ is a real slice function then $0 = f(\alpha + I\beta) = F_1(\alpha + i\beta) + IF_2(\alpha + i\beta)$ if and only if $F_1(\alpha + i\beta) = 0$ and $F_2(\alpha + i\beta) = 0$.

The next two results regard the zero set of the slice product of two slice functions. They can be find in section 7.2 of [**30**].

Proposition 1.13. *([**30**], proposition 24). Let $f, g \in \mathcal{S}(\Omega_D)$. Then $V(f) \subset V(f \cdot g)$.*

In general, we can not conclude that $V(g) \subset V(f \cdot g)$ as the last example has shown.

Proposition 1.14. *([**30**], proposition 25). Let $x \in \mathbb{H}$. If $f, g \in \mathcal{S}(\Omega_D)$, then it holds:*

$$\bigcup_{x \in V(f \cdot g)} \mathbb{S}_x = \bigcup_{x \in V(f) \cup V(g)} \mathbb{S}_x$$

In the next theorems we add regularity property. For all of them we refer to section 7 of [**30**]. The first result is a "natural" generalization of a feature that holds in the complex case and suggest a possible statement of an "Identity Principle" that will be discussed in the next chapter.

Theorem 1.15. *([**30**], theorem 20). Let Ω_D be connected. If f is slice regular and $N(f)$ does not vanish identically, then*

$$\mathbb{C}_J \cap \bigcup_{x \in V(f)} \mathbb{S}_x$$

is closed and discrete in D_J for all $J \in \mathbb{S}$. If $\Omega_D \cap \mathbb{R} \neq \emptyset$, then $N(f) \equiv 0$ if and only if $f \equiv 0$.

Theorem 1.16. *Let $f \in \mathcal{SR}(\Omega_D)$. Let $x, y \in \mathbb{R}$, such that $x + \mathbb{S}y \subset \Omega_D$. The zeros of f^c on $x + \mathbb{S}y$ are in bijective correspondence with those of f. Moreover $N(f)$ vanishes exactly on the set $x + \mathbb{S}y$ on which f has a zero.*

The previous theorem was proven in [**24**] for power series and extended to all regular functions on domain with real point in [**9**] and [**23**]. In general, combining proposition 1.14 and corollary 1.12, we have the thesis since:

$$\bigcup_{x \in V(N(f))} \mathbb{S}_x = \bigcup_{x \in V(f) \cup V(f^c)} \mathbb{S}_x,$$

and,

$$\bigcup_{x \in V(f)} \mathbb{S}_x = \bigcup_{x \in V(N(f))} \mathbb{S}_x = \bigcup_{x \in V(f^c)} \mathbb{S}_x.$$

The next definition is needed for defining the multiplicity of a slice function at a point. Moreover it provide a set of polynomial functions that will give several information in other parts of the theory. References for this set of functions are section 7.2 of [**30**] and the whole paper [**33**], in which it plays a fundamental role.

DEFINITION 1.9. The *characteristic polynomial* of y is the slice regular function $\Delta_y(x) : \mathbb{H} \to \mathbb{H}$ defined by:

$$\Delta_y(x) := N(x-y) = (x-y)\cdot(x-y^c) = x^2 - x(y+y^c) + yy^c.$$

REMARK 1.13. The following facts about the characteristic polynomial are quite obvious. If the reader need more details we refer again to [**30**].

- Δ_y is a real slice function (this is obvious from the definition).
- Two characteristic polynomials Δ_y, Δ'_y coincide if and only if $\mathbb{S}_y = \mathbb{S}_{y'}$ (in fact, if $y = \alpha + I\beta$ and $y' = \gamma + K\delta$, then $\Delta_y = \Delta'_y$ if and only if $2\alpha = 2\gamma$ and $\alpha^2 + \beta^2 = \gamma^2 + \delta^2$).
- $V(\Delta_y) = \mathbb{S}_y$ (this is obvious from the previous results and examples).

The next theorem and corollary will turn out to be powerful instruments for many results. They practically say that if a slice regular function has a zero in a point x_0, then we can divide the function by $\Delta_{x_0}(x)$ or by $(x - x_0)$ (respectively if x_0 is a spherical zero or not).

THEOREM 1.17. **([30], *theorem 22*).** *Let $f \in \mathcal{SR}(\Omega_D)$ and $x_0 \in V(f)$. Then the following statements are true.*

- *If $x_0 \in \mathbb{R}$, then there exists $g \in \mathcal{SR}(\Omega_D)$ such that $f(x) = (x - x_0)g(x)$.*
- *If x_0 is not real, then there exists $h \in \mathcal{SR}(\Omega_D)$ and $a, b \in \mathbb{H}$ such that $f(x) = \Delta_{x_0}(x)h(x) + xa + b$, where,*
  - *$\mathbb{S}_{x_0} \subset V(f)$ if and only if $a = b = 0$;*
  - *$\mathbb{S}_{x_0} \cap V(f)$ is a singleton if and only if $a \neq 0$ (in this case $x_0 = -ba^{-1}$ and $b \neq 0$).*

REMARK 1.14. In the last point of the previous theorem, $a = \partial_s f(x_0)$.

COROLLARY 1.18. *If $f \in \mathcal{SR}(\Omega_D)$ and $x_0 \in V(f)$, then there exists $g \in \mathcal{SR}(\Omega_D)$ such that $f(x) = (x - x_0) \cdot g(x)$.*

REMARK 1.15. Observe that while in the previous theorem the products between $(x - x_0)$ and g and between Δ_{x_0} and h are pointwise, in this corollary it isn't. In this case the thesis is that f is equal to the slice product between $(x - x_0)$ and g.

And now, this following fundamental corollary will permit us to define the multiplicity of a zero for a slice regular function.

COROLLARY 1.19. *([**30**], corollary 23). If $f \in \mathcal{SR}(\Omega_D)$ and $x_0 \in V(f)$ then $\Delta_{x_0}(x)$ divides $N(f)$.*

Thanks to the last corollary, we are able to remember the following definition (see definition 14 in [**30**]).

DEFINITION 1.10. Let $f \in \mathcal{SR}(\Omega_D)$ such that $N(f)$ does not vanish identically. Given $n \in \mathbb{N}$ and $x_0 \in V(f)$, we say that x_0 is a zero of f of *total multiplicity* n, and we will denote it by $m_f(x_0)$, if $\Delta_{x_0}^n \mid N(f)$ and $\Delta_{x_0}^{n+1} \nmid N(f)$.

If $m_f(x_0) = 1$, then x_0 is called a *simple zero* of f.

The last notion, is equivalent to the one of *total multiplicity* stated in [**28, 25**] when the domain intersects $\mathbb{R}$. The adjective "total" was introduced to underline the fact that this integer takes into accounts both spherical and isolated orders of zero of a point.

As previously mentioned before the discussion around zero set structure, we will now introduce the notion of reciprocal in the framework of slice functions. This was firstly introduced in [**9, 24, 22, 47**] and then in [**3**] if the domain of definition has empty intersection with the real line.

DEFINITION 1.11. Let $f = \mathcal{I}(F) \in \mathcal{SR}(\Omega_D)$. We call the *slice reciprocal* of f the slice function

$$f^{-\cdot} : \Omega_D \setminus V(N(f)) \to \mathbb{H}$$

defined by

$$f^{-\cdot} = \mathcal{I}((F^c F)^{-1} F^c)$$

From the previous definition it follows that, if $x \in \Omega_D \setminus V(N(f))$, then

$$f^{-\cdot}(x) = (N(f)(x))^{-1} f^c(x).$$

The regularity of the reciprocal just defined follows thanks to corollary 1.9. The following proposition extends an already known one that can be found in [**22**], where the authors studied the case of slice functions defined on domains that intersects the real axis.

PROPOSITION 1.20. *Let $f \in \mathcal{SR}(\Omega_D)$ such that $V(f) = \emptyset$, then $f^{-\cdot} \in \mathcal{SR}(\Omega_D)$ and*

$$f \cdot f^{-\cdot} = f^{-\cdot} \cdot f = 1.$$

Proof. Since $V(f) = \emptyset$ then $V(N(f)) = \emptyset$. So $(N(f))^{-1}$ and $f^{-\cdot}$ are well defined and regular on the whole Ω_D. We may consider then their regular product with other regular functions $g : \Omega_D \to \mathbb{H}$. For all g, $(N(f)(x))^{-1} g(x) = ((N(f))^{-1} \cdot g)(x)$. Than we have

$$f^{-\cdot} \cdot f = (N(f))^{-1} \cdot f^c \cdot f = (N(f))^{-1} N(f) = 1$$

and

$$f \cdot f^{-\cdot} = f \cdot (N(f))^{-1} \cdot f^c = (N(f))^{-1} \cdot f \cdot f^c = (N(f))^{-1} N(f) = 1.$$

□

With the notion of slice reciprocal, we can state the following characterization, that, in a certain sense, results to us very natural. In our knowledge this theorem is new and stated for the first time in this thesis.

THEOREM 1.21. *Let Ω_D be an open domain of $\mathbb{H}$ and $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$ be a differentiable slice function. The following limit exists for all $x_0 = \alpha_0 + J_0 \beta_0 \in \Omega_D$*

$$\lim_{x \to x_0} (x - x_0)^{-\cdot} \cdot (f(x) - f(x_0)) = q, \tag{7}$$

if and only if f is slice regular and $\frac{\partial f}{\partial x}(x_0) = q$.

Proof. If f is slice regular, then, thanks to corollary 1.18, there exists a slice regular function $g : \Omega_D \to \mathbb{H}$, such that

$$f(x) - f(x_0) = (x - x_0) \cdot g(x).$$

But then the function $(x-x_0)^{-\cdot} \cdot (f(x)-f(x_0))$ is defined also in x_0 and $\lim_{x\to x_0}(x-x_0)^{-\cdot} \cdot (f(x)-f(x_0)) = \lim_{x\to x_0} g(x) = g(x_0)$. Since then $\frac{\partial}{\partial x}((x-x_0)\cdot g(x)) = g(x)+(x-x_0)\frac{\partial g}{\partial x}(x)$, then $g(x_0) = \frac{\partial f}{\partial x}(x_0)$.

Viceversa, if for each $x_0 = \alpha_0 + J_0\beta_0 \in \Omega_D \setminus \mathbb{R}$, the limit in 7 exists, then, for any differentiable curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{C}_{J_0}$, such that $\gamma(0) = x_0$ we must have the following result

$$\lim_{t\to 0} \left[(x - x_0)^{-\cdot} \cdot (f(x) - f(x_0))\right]_{|x=\gamma(t)} = q = \circledast.$$

Using proposition 1.8 and denoting $h(x) = (x - x_0)^{-\cdot}$, we have that

$$h(x) \cdot (f(x) - f(x_0)) = h(x)(f(h(x)^{-1}xh(x)) - f(x_0)).$$

If one choose γ to be defined as $\gamma(t) = \alpha_0 + J_0(t + \beta_0)$, then we have that $h(\gamma(t)) = (J_0\beta_0)^{-\cdot} = (J_0\beta_0)^{-1}$ and $(f(h(x)^{-1}xh(x)) - f(x_0)) = (f(\alpha_0 + J_0(t + \beta_0)) - f(x_0))$, and so,

$$\circledast = \lim_{t\to 0} (J_0t)^{-1}(f(\alpha_0 + J_0(t + \beta_0)) - f(\alpha_0 + J_0\beta_0)) = -J_0\frac{\partial f}{\partial \beta}(x_0) = q.$$

Analogously we obtain, for $\gamma(t) = (\alpha_0 + t) + J_0\beta_0$ that

$$\circledast = \lim_{t\to 0} t^{-1}(f((\alpha_0 + t) + J_0\beta_0) - f(\alpha_0 + J_0\beta_0)) = \frac{\partial f}{\partial \alpha}(x_0) = q,$$

and so

$$q = \frac{\partial f}{\partial \alpha}(x_0) = -J_0\frac{\partial f}{\partial \beta}(x_0)$$

and if we write the function f as $f(\alpha + J\beta) = F_1(\alpha + i\beta) + JF_2(\alpha + i\beta)$, then we obtain exactly the Cauchy-Riemann equations and the thesis. If $x_0 \in \Omega_D \cap \mathbb{R}$, we repeat the same argument considering a curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{C}_J$ for a fixed $J \in \mathbb{S}$, such that $\gamma(0) = 0$. Underlying that, if $x_0 \in \mathbb{R}$, then $(x - x_0)^{-\cdot} \cdot (f(x) - f(x_0)) = (x - x_0)^{-1}(f(x) - f(x_0))$. □

2.1. Power and spherical series. In this section we will outline some results about the possibility of expanding a slice (regular) function, in some sense, in power series. We are firstly interested in power series of the form $p(x) = \sum_{n\in\mathbb{N}} x^n a_n$, with $a_n \in \mathbb{H}$, or, more generally, of the type

$$p(x) = \sum_{n\in\mathbb{N}} (x - y)^{\cdot n} a_n, \tag{8}$$

where $y \in \mathbb{H}$ and $(x - y)^{\cdot n}$ denotes the n-th power of $x - y$ w.r.t. the slice product.

In **[21]** it was proved that the sum of a series of type 8, in the interior of its set of convergence, is a slice regular function. If y is a real number, then the slice power $(x - y)^{\cdot n}$ coincides with the usual power $(x - y)^n$ and the series converges on the interior of an Euclidean ball $B(y, R)$ centered in y with radius $R \geq 0$. Unfortunately in the same paper was proved that, if $y \notin \mathbb{R}$, the set of convergence can have empty interior. This set may actually be reduced to a disk centered in y and contained in the complex slice of the quaternionic space defined by $span_{\mathbb{R}} < 1, y >$. To be more specific, we introduce the following metric. Let $x, y = \xi + J\eta \in \mathbb{H}$ for some $\xi, \eta \in \mathbb{R}$ and $J \in \mathbb{S}$. Define

$$\sigma(x, y) := \begin{cases} ||x - y|| & \text{if } x \in \mathbb{C}_J \\ \sqrt{|Re(x) - Re(y)|^2 + (||Im(x)|| + ||Im(y)||)^2} & \text{if } x \notin \mathbb{C}_J. \end{cases}$$

The topology induced by σ is finer than the Euclidean one: a σ-ball of radius r centered in y has empty interior if $r \leq ||Im(y)||$ (for some picture see fig. 2.2 of [**25**]). A σ-ball of radius r centered in y will be denoted by $\Sigma(y, R) := \{x \in \mathbb{H} \,|\, \sigma(x, y) < R\}$. For the following definition we refer to [**21, 33**].

Definition 1.12. Given a function $f : \Omega \to \mathbb{H}$ defined on a non-empty open subset Ω in $\mathbb{H}$, we say that f is *σ analytic* or *power analytic*, if, for all $y \in \Omega$, there exists a non-empty σ-ball Σ centered at y and contained in Ω, and a series $\sum_{n\in\mathbb{N}}(x - y)^{\cdot n}a_n$ with coefficients in $\mathbb{H}$, which converges to $f(x)$ for each $x \in \Sigma \cap \Omega$.

We have the following characterizing result.

Theorem 1.22. *Let Ω_D be a connected circular set and $f : \Omega_D \to \mathbb{H}$ be any function. The following assertions hold.*

*(1) ([**21**], corollary 2). If $D \cap \mathbb{R} \neq \emptyset$, then f is a slice regular function if and only if f is a σ-analytic function.*

*(2) ([**33**], theorem 4.3). If $D \cap \mathbb{R} = \emptyset$, then f is a slice regular function if and only if f is a σ-analytic slice function.*

The next proposition (proved in [**21, 33**]), establishes an explicit form of the coefficients of the slice power expansion. It will be stated only for $y \in \mathbb{H} \setminus \mathbb{R}$, since, as already said, if y is real, then a power series w.r.t. the slice product coincides with a standard power series.

Proposition 1.23. *Let $y \in \mathbb{H} \setminus \mathbb{R}$ and let $P : \Sigma(y, R) \to \mathbb{H}$ be a function defined by a power series $P(x) = \sum_{n\in\mathbb{N}}(x - y)^{\cdot n}a_n$ centered in y with positive σ-radius of convergence. Then it holds, for each $n \in \mathbb{N}$,*

$$a_n = \frac{1}{n!}\partial_C^n P(y).$$

Another approach that avoids the difficulty of having possibly domains of convergence with empty interior was given in [**46**], where the powers $(x - y)^{\cdot n}$ were replaced by another family of slice regular polynomials of a quaternionic variable. The set of convergence of these series is always an open Euclidean set and every slice regular function has a series expansion of this type near every point of its domain of definition. To enter into details we start expliciting the family of polynomials taken in consideration. The new choice is given by the powers of the characteristic polynomial introduced in definition 1.9. More precisely, for each $m \in \mathbb{N}$ we define, the slice regular polynomial functions

$$\mathcal{S}_{y,2m}(x) := \Delta_y(x)^m, \quad \mathcal{S}_{y,2m+1}(x) := \Delta_y(x)^m(x - y). \tag{9}$$

Note that, since Δ_y is a real slice function, then $\Delta_y^{\cdot m} = \Delta_y^m$. As already said, differently from slice power series, series of type $\sum_{n\in\mathbb{N}} \mathcal{S}_{y,n}(x)s_n$ have convergence sets that are always open w.r.t. the Euclidean topology. More precisely, we introduce the following *Cassini* pseudometric[5]. If $x, y \in \mathbb{H}$, we set

$$u(x,y) := \sqrt{\| \Delta_y(x) \|}.$$

The function u turn out to be a pseudometric on $\mathbb{H}$, whose induced topology is strictly coarser than the Euclidean one. A u-ball of radius r centered in y will be denoted by $U(y,R) := \{x \in \mathbb{H} \,|\, u(x,y) < R\}$. In [**46, 33**] it is showed that the sets of convergence of series $\sum_{n\in\mathbb{N}} \mathcal{S}_{y,n}(x)s_n$ are u-ball centered at y and it is proved a corresponding Abel theorem (see fig. 8.1 in [**25**]). Moreover in [**33**], formulas for computing the coefficients are given. In this context, the following is the definition of analyticity. For the following definition we refer to [**46, 33**].

Definition 1.13. Given a function $f : \Omega \to \mathbb{H}$ defined on a non-empty open circular subset Ω in $\mathbb{H}$, we say that f is *u-analytic* or *spherical analytic*, if, for all $y \in \Omega$, there exists a non-empty u-ball U centered at y and contained in Ω, and a series $\sum_{n\in\mathbb{N}} \mathcal{S}_{y,n}(x)s_n$ with coefficients in $\mathbb{H}$, which converges to $f(x)$ for each $x \in U \cap \Omega$.

We have the following expected result.

Theorem 1.24. *Let Ω_D be a connected circular set and $f : \Omega_D \to \mathbb{H}$ be any function. The following assertions hold.*

*(1) ([**46**], corollary 4.3). If $D \cap \mathbb{R} \neq \emptyset$, thenf f is a slice regular function if and only if f is a spherical analytic function.*
*(2) ([**33**], theorem 5.8). If $D \cap \mathbb{R} = \emptyset$, then f is a slice regular function if and only if f is a spherical analytic slice function.*

Some considerations about the coefficients of the spherical expansion of a slice regular function will be given in chapter 3.

We observe now that, since quaternionic polynomials and converging power series are contained in the set of quaternionic holomorphic function (see [**18**]), then the following corollary holds true.

Definition 1.14. A sufficiently regular quaternionic function of one quaternionic variable $f : \Omega_D \to \mathbb{H}$ is said to be *quaternionic holomorphic* if satisfies the following equation

$$\overline{\mathcal{D}}\Delta f(x) = 0, \qquad (10)$$

[5]A *pseudometric space* (X,d) is a set X endowed with a non-negative function (called *pseudometric*), $d : X \times X \to \mathbb{R}^+$, such that

- For each $x \in X$, $d(x,x) = 0$;
- For each $x, y \in X$, $d(x,y) = d(y,x)$;
- For each $x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$.

Differently from a *metric space*, here we're not asking for $d(x,y)$ to be different from zero for any $x \neq y$.

where $x = \alpha + I\beta = x_0 + x_1 i + x_2 j + x_3 k$, Δ denotes the Laplacian in the four variables x_0, x_1, x_2, x_3, and $\overline{\mathcal{D}}$ denotes the standard Cauchy-Fueter operator:

$$\overline{\mathcal{D}} := \frac{1}{4}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right).$$

COROLLARY 1.25. *Any slice regular function $f : \Omega_D \to \mathbb{H}$ is quaternionic holomorphic. Moreover, since f satisfies equation 10, then it also satisfies the following equation:*

$$\Delta\Delta f = 0.$$

The last equality holds because, if we denote by $\mathcal{D} := \frac{1}{4}\left(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}\right)$, then $\mathcal{D}\overline{\mathcal{D}} = \Delta$. For more details about the previous corollary and about the theory of quaternionic holomorphic functions, we refer to [**18, 39, 40**].

3. Slice differential and slice differential forms

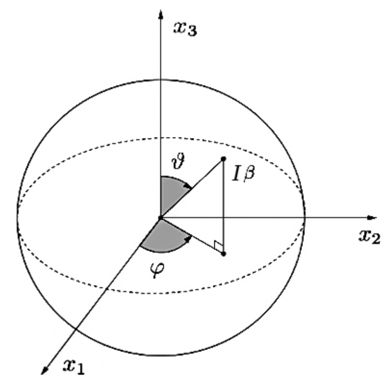
The content of this section is essentially an attempt of formalization of an important piece of the real differential of a slice regular function f: the *slice differential*. We will see that it carries on the slice information of the behavior of f, which, in some cases, can be enough to restore f. Let us start with some formal consideration.

Let $x \in \mathbb{H} \simeq \mathbb{R}^4$, $x = (x_0, x_1, x_2, x_3)$ with $(x_1, x_2, x_3) \neq (0,0,0)$ (i.e.: $x \in \mathbb{H} \setminus \mathbb{R}$). When we talk about slice functions we implicitly use the following change of coordinates:

$$(x_0, x_1, x_2, x_3) \mapsto (\alpha, \beta, I),$$

where $\alpha \in \mathbb{R}$, $\beta > 0$ and $I = I(\vartheta, \varphi) \in \mathbb{S}$ with the following equalities:

$$\begin{cases} \alpha &= x_0 \\ \beta &= \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \vartheta &= \arccos(\frac{x_3}{\beta}) \\ \varphi &= \arctan(\frac{x_2}{x_1}). \end{cases}$$



Spherical coordinates of $I\beta$

Let now $f : \Omega \subset \mathbb{R}^4 \to \mathbb{R}^4$ be any differentiable function. Then, its differential in these new coordinates, can be written in its domain, as follows

$$df = \left(\frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \beta} d\beta\right) + \frac{1}{\beta}\left(\frac{\partial f}{\partial \vartheta} d\vartheta + \frac{1}{\sin\vartheta}\frac{\partial f}{\partial \varphi} d\varphi\right), \tag{11}$$

where :

$$\begin{cases} d\alpha &= dx_0 \\ d\beta &= \sin\vartheta\cos\varphi dx_1 + \sin\vartheta\sin\varphi dx_2 + \cos\vartheta dx_3 \\ d\vartheta &= \cos\vartheta\cos\varphi dx_1 + \cos\vartheta\sin\varphi dx_2 - \sin\vartheta dx_3 \\ d\varphi &= -\sin\varphi dx_1 + \cos\varphi dx_2. \end{cases}$$

We would like, however, to consider also $\beta < 0$ (having in mind that a non-real quaternion x can be written both as $\alpha + I\beta$ and $\alpha + (-I)(-\beta)$). But in this case we have to take care that $d\beta(-\beta, I) = d\beta(\beta, -I) = -d\beta(\beta, I)$.

The aim of this section is to study the first part of the right hand side of equation 11, when f is a slice differentiable function.

We will start with the following general definition.

Definition 1.15. Let $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$. We define the *slice differential* $d_{sl}f$ of f as the following differential form:

$$\begin{array}{rccc} d_{sl}f: & (\Omega_D \setminus \mathbb{R}) & \to & \mathbb{H}^*, \\ & \alpha + I\beta & \mapsto & dF_1(\alpha + i\beta) + IdF_2(\alpha + i\beta). \end{array}$$

Remark 1.16. The one-form $\omega : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}^*$ defined as $\omega(\alpha + I\beta) = Id\beta$, represents the outer radial direction to the sphere $\mathbb{S}_x = \{\alpha + K\beta \,|\, K \in \mathbb{S}\}$. Then $\omega(\alpha + I(-\beta)) = \omega(\alpha + (-I)\beta) = -\omega(\alpha + I\beta)$. We can translate this observation in the language of slice forms. We've seen in example 1.1 that $h(x) = Im(x)$ is a slice function induced by $H(z) = \sqrt{-1}Im(z)$. Then we have $d_{sl}h(\alpha + I\beta) = Id\beta(\alpha + i\beta)$ and, thanks to the previous considerations $d_{sl}h(\alpha + (-I)(-\beta)) = -Id\beta(\alpha - i\beta) = Id\beta(\alpha + i\beta)$. Summarizing, we have that $d\beta(\bar{z}) = -d\beta(z)$. The same doesn't hold for $d\alpha$ which is a constant one form over $\mathbb{H}$ and for this reason in the next computations we will omit the variable (i.e.: $d\alpha = d\alpha(z) = d\alpha(\bar{z})$).

We can show now that the previous definition is well posed.

Proposition 1.26. *Definition 1.15 is well posed, i.e. if D is symmetric with respect to the real axis, then*

$$d_{sl}f(\alpha + I\beta) = d_{sl}f(\alpha + (-I)(-\beta)), \quad \forall \alpha + I\beta \in \Omega_D \setminus \mathbb{R}$$

Proof. Let $x = \alpha + J\beta \in \Omega_D \setminus \mathbb{R}$ and $z = \alpha + i\beta$, then,

$$\begin{aligned}
d_{sl}f(\alpha + (-I)(-\beta)) &= \\
&= \left(\frac{\partial F_1(\bar{z}) - IF_2(\bar{z})}{\partial\alpha}\right) d\alpha + \left(\frac{\partial F_1(\bar{z}) - IF_2(\bar{z})}{\partial\beta}\right) d\beta(\bar{z}) = \\
&= \frac{\partial F_1}{\partial\alpha}(\bar{z})d\alpha + \frac{\partial F_1}{\partial\beta}(\bar{z})(-1)d\beta(z) - I\left(\frac{\partial F_2}{\partial\alpha}(\bar{z})d\alpha + \frac{\partial F_2}{\partial\beta}(\bar{z})(-1)d\beta(z)\right) = \\
&= \frac{\partial F_1}{\partial\alpha}(z)d\alpha + \frac{\partial F_1}{\partial\beta}(z)d\beta(z) - I\left(-\frac{\partial F_2}{\partial\alpha}(z)d\alpha - \frac{\partial F_2}{\partial\beta}(z)d\beta(z)\right) = \\
&= \left(\frac{\partial F_1(z) + IF_2(z)}{\partial\alpha}\right) d\alpha + \left(\frac{\partial F_1(z) + IF_2(z)}{\partial\beta}\right) d\beta(z) = \\
&= d_{sl}f(\alpha + I\beta),
\end{aligned}$$

where the third equality holds thanks to the even-odd character of the couple (F_1, F_2).

□

To avoid ambiguity, in the following of this section we will consider always $\beta > 0$, so, to be more clear, the point $p = \alpha - J\beta$ will be intended as $p = \alpha + (-J)\beta$ and we will omit the argument of the one-form $d\beta$. We can represent, then, the slice differential as follows.

PROPOSITION 1.27. *Let $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$ with $D \subset \mathbb{C}^+$ (so that $\beta > 0$). Then, on $\Omega_D \setminus \mathbb{R}$, the following equality holds true.*

$$d_{sl}f = \frac{\partial f}{\partial \alpha}d\alpha + \frac{\partial f}{\partial \beta}d\beta.$$

Proof. The thesis follows from the following computations. Let $x = \alpha + I\beta \in \Omega_D$ and $z = \alpha + i\beta$, then

$$\begin{aligned} d_{sl}f(x) &= \left(\tfrac{\partial F_1}{\partial \alpha}(z)d\alpha + \tfrac{\partial F_1}{\partial \beta}(z)d\beta\right) + I\left(\tfrac{\partial F_2}{\partial \alpha}(z)d\alpha + \tfrac{\partial F_2}{\partial \beta}(z)d\beta\right) \\ &= \left(\tfrac{\partial F_1}{\partial \alpha}(z)d\alpha + I\tfrac{\partial F_2}{\partial \beta}(z)d\beta\right) + \left(\tfrac{\partial F_1}{\partial \beta}(z)d\beta + I\tfrac{\partial F_2}{\partial \beta}(z)d\beta\right) \\ &= \tfrac{\partial f}{\partial \alpha}(x)d\alpha + \tfrac{\partial f}{\partial \beta}(x)d\beta. \end{aligned}$$

□

It is clear from the definition that, if we choose the usual coordinate system, where $x = \alpha + I\beta$ with $\beta > 0$, then $d_{sl}x = d\alpha + Id\beta$ and $d_{sl}x^c = d\alpha - Id\beta$. We can now state the following theorem.

THEOREM 1.28. *Let $f \in \mathcal{S}^1(\Omega_D)$. Then the following equality holds:*

$$d_{sl}x\frac{\partial f}{\partial x}(x) + d_{sl}x^c\frac{\partial f}{\partial x^c}(x) = d_{sl}f(x), \quad \forall x \in \Omega_D \setminus \mathbb{R}.$$

Proof. The thesis is obtained after the following explicit computations:

$$\begin{aligned} d_{sl}x\frac{\partial f}{\partial x} + d_{sl}x^c\frac{\partial f}{\partial x^c} &= \frac{1}{2}\left[(d\alpha + Id\beta)\left(\frac{\partial F_1}{\partial \alpha} + \frac{\partial F_2}{\partial \beta} - I\left(\frac{\partial F_1}{\partial \beta} - \frac{\partial F_2}{\partial \alpha}\right)\right) + \right. \\ &\quad \left. + (d\alpha - Id\beta)\left(\frac{\partial F_1}{\partial \alpha} - \frac{\partial F_2}{\partial \beta} + I\left(\frac{\partial F_1}{\partial \beta} + \frac{\partial F_2}{\partial \alpha}\right)\right)\right] \\ &= \frac{1}{2}\left[d\alpha\frac{\partial F_1}{\partial \alpha} + d\alpha\frac{\partial F_2}{\partial \beta} - Id\alpha\frac{\partial F_1}{\partial \beta} + Id\alpha\frac{\partial F_2}{\partial \alpha}\right. \\ &\quad + Id\beta\frac{\partial F_1}{\partial \alpha} + Id\beta\frac{\partial F_2}{\partial \beta} + d\beta\frac{\partial F_1}{\partial \beta} - d\beta\frac{F_2}{\partial \alpha} + \\ &\quad + d\alpha\frac{\partial F_1}{\partial \alpha} - d\alpha\frac{\partial F_2}{\partial \beta} + Id\alpha\frac{\partial F_1}{\partial \beta} + Id\alpha\frac{\partial F_2}{\partial \alpha} + \\ &\quad \left. - Id\beta\frac{\partial F_1}{\partial \alpha} + Id\beta\frac{\partial F_2}{\partial \beta} + d\beta\frac{\partial F_1}{\partial \beta} + d\beta\frac{F_2}{\partial \alpha}\right] \\ &= d\alpha\frac{\partial F_1}{\partial \alpha} + Id\beta\frac{\partial F_2}{\partial \beta} + d\beta\frac{\partial F_1}{\partial \beta} + Id\alpha\frac{\partial F_2}{\partial \alpha} \\ &= \frac{\partial F_1}{\partial \alpha}d\alpha + \frac{\partial F_1}{\partial \beta}d\beta + I\left(\frac{\partial F_2}{\partial \alpha}d\alpha + \frac{\partial F_2}{\partial \beta}d\beta\right) \\ &= d_{sl}f. \end{aligned}$$

□

We have then the obvious corollary:

COROLLARY 1.29. *Let $f \in \mathcal{SR}(\Omega_D)$. Then the following equality holds:*

$$d_{sl} x \frac{\partial f}{\partial x}(x) = d_{sl} f(x), \quad \forall x \in \Omega_D \setminus \mathbb{R}.$$

At this point we want to embed the concept of slice differential in the theory of 1-forms. We begin with the following definition. This part of the thesis has not a proper application so far. Anyway it seems to us that this could be a useful tool for the analysis of slice functions in the future. To give a flavor of what can be done we will give a small non essential application, showing eventually a *Morera* type theorem.

DEFINITION 1.16. Let $D \subset (\mathbb{C}^+ \setminus \mathbb{R})$ (so $\beta > 0$), and let Ω_D be a circular domain in $\mathbb{H}$. A one-form $\omega : \Omega_D \setminus \mathbb{R} \to \mathbb{H}^*$ such that

$$\omega(\alpha + I\beta) = (\omega^{\alpha 1}(\alpha, \beta) d\alpha + \omega^{\beta 1}(\alpha, \beta) d\beta) + I(\omega^{\alpha 2}(\alpha, \beta) d\alpha + \omega^{\beta 2}(\alpha, \beta) d\beta),$$

where the coefficients ω^l_γ are continuous functions on Ω_D and will be called *slice differential one-form.*

From now on in this section, unless differently specified, we will suppose that $D \cap \mathbb{R} = \emptyset$ (and so $\Omega_D \cap \mathbb{R} = \emptyset$). For any $I \in \mathbb{S}$ and any slice form $\omega : \Omega_D \to \mathbb{H}^*$ we denote by ω_I the restriction

$$\omega_I := \omega|_{D_I^+} : D_I^+ \to \mathbb{H}^*.$$

We will use also the following notation: if ω is a slice form then

$$\begin{aligned}\omega(\alpha + I\beta) &= (\omega^{\alpha 1}(\alpha, \beta) + I\omega^{\alpha 2}(\alpha, \beta)) d\alpha + (\omega^{\beta 1}(\alpha, \beta) + I\omega^{\beta 2}(\alpha, \beta)) d\beta = \\ &= (\omega^{\alpha 1}(\alpha, \beta) d\alpha + \omega^{\beta 1}(\alpha, \beta) d\beta) + I(\omega^{\alpha 2}(\alpha, \beta) d\alpha + \omega^{\beta 2}(\alpha, \beta) d\beta) = \\ &= \omega^1(\alpha, \beta) + I\omega^2(\alpha, \beta),\end{aligned}$$

where, of course,

$$\omega^{\alpha 1} d\alpha + \omega^{\beta 1} d\beta = \omega^1, \quad \omega^{\alpha 2} d\alpha + \omega^{\beta 2} d\beta = \omega^2.$$

REMARK 1.17. If we want to define slice forms without imposing $\beta > 0$ we have to ask some conditions to the coefficients. These conditions can be expressed as $\omega^{\alpha 1}(\alpha, -\beta) = \omega^{\alpha 1}(\alpha, \beta)$, $\omega^{\beta 2}(\alpha, -\beta) = \omega^{\beta 2}(\alpha, \beta)$, $\omega^{\alpha 2}(\alpha, -\beta) = -\omega^{\alpha 2}(\alpha, \beta)$ and $\omega^{\beta 1}(\alpha, -\beta) = -\omega^{\beta 1}(\alpha, \beta)$.

As for slice functions also for slice forms we have a representation formula.

THEOREM 1.30. ***(Representation Formula)*** *Let $D \subset (\mathbb{C}^+ \setminus \mathbb{R})$. Let $J \neq K \in \mathbb{S}$. Every slice form $\omega : \Omega_D \to \mathbb{H}^*$ is uniquely determined by its values on the two distinct semislices D_J^+ and D_K^+. In particular the following formula holds:*

$$\omega(\alpha + I\beta) = (I - K)(J - K)^{-1}\omega(\alpha + J\beta) - (I - J)(J - K)^{-1}\omega(\alpha + K\beta), \tag{12}$$

for every $\alpha + I\beta \in \Omega_D$.

Before passing through the proof of the previous theorem, we observe that, if $K = -J$, the previous formula (12), can be written as:

$$\omega(\alpha + I\beta) = \frac{1}{2}\left[\omega(\alpha + J\beta) + \omega(\alpha - J\beta) - IJ\left(\omega(\alpha + J\beta) - \omega(\alpha - J\beta)\right)\right]. \tag{13}$$

Proof. To prove the theorem, we will show how to derive ω^1 and ω^2 from ω_J and ω_K . First of all we have,

$$\begin{aligned} \omega_J(\alpha + J\beta) - \omega_K(\alpha + K\beta) &= \omega^1(\alpha, \beta) + J\omega^2(\alpha, \beta) - \omega^1(\alpha, \beta) - K\omega^2(\alpha, \beta) \\ &= (J - K)\omega^2(\alpha, \beta), \end{aligned}$$

for all $\alpha + i\beta \in D$ with β greater than zero[6]. To determine ω^1 it is sufficient to consider the difference between ω_J and $J\omega^2$:

$$\omega_J - J\omega^2 = \omega^1(\alpha, \beta) + J\omega^2(\alpha, \beta) - J\omega^2.$$

Passing now to prove the formula, we already know that

$$\omega^2(\alpha, \beta) = (J - K)^{-1}(\omega(\alpha + J\beta) - \omega(\alpha + K\beta)), \quad \omega^1(\alpha, \beta) = \omega(\alpha + J\beta) - J\omega^2,$$

and so

$$\omega^1(\alpha, \beta) = \omega(\alpha + J\beta) - J(J - K)^{-1}(\omega(\alpha + J\beta) - \omega(\alpha + K\beta)).$$

Let's now $I \in \mathbb{S}$, then,

$$\begin{aligned} \omega(\alpha + I\beta) &= \omega^1(\alpha, \beta) + I\omega^2(\alpha, \beta) \\ &= \omega(\alpha + J\beta) - J(J - K)^{-1}(\omega(\alpha + J\beta) - \omega(\alpha + K\beta)) + \\ &\quad + I(J - K)^{-1}(\omega(\alpha + J\beta) - \omega(\alpha + K\beta)) \\ &= \omega(\alpha + J\beta) + (I - J)(J - K)^{-1}(\omega(\alpha + J\beta) - \omega(\alpha + K\beta)) \\ &= ((J - K) + (I - J))(J - K)^{-1}\omega(\alpha + J\beta) + (I - J)(J - K)^{-1}\omega(\alpha + K\beta) \\ &= (I - J)(J - K)^{-1}\omega(\alpha + J\beta) + (I - J)(J - K)^{-1}\omega(\alpha + K\beta). \end{aligned}$$

□

At this point a natural question arise: *are all the slice forms the slice differential of a slice function?* The answer, in general, is not. As a counterexample we can consider the slice form defined over $\mathbb{H} \setminus \mathbb{R}$ in the following way:

$$\omega : \alpha + I\beta \mapsto I\beta d\alpha + (-I\alpha)d\beta.$$

The previous slice form is not the slice differential of a slice function since, if this was the case, then $\omega = d_{sl}f$, with $f = \mathcal{I}(F_1 + \sqrt{-1}F_2)$ and $\frac{\partial F_2}{\partial \alpha} = \omega^{\alpha 2} = \beta$ while $\frac{\partial F_2}{\partial \beta} = \omega^{\beta 2} = -\alpha$. But,

$$\frac{\partial^2 F_2}{\partial \beta \partial \alpha} = \frac{\partial(\beta)}{\partial \beta} = 1 \neq -1 = \frac{\partial(-\alpha)}{\partial \alpha}\frac{\partial^2 F_2}{\partial \alpha \partial \beta}.$$

Definition 1.17. Let $D \subset (\mathbb{C}^+ \setminus \mathbb{R})$. A slice differential form $\omega : \Omega_D \to \mathbb{H}^*$ is called *slice-exact (s-exact)* in Ω_D if there exists a slice function $f : \Omega_D \to \mathbb{H}$ of class $\mathcal{C}^1$ such that $d_{sl}f(x) = \omega(x)$ on Ω_D.

[6] If we define slice forms without the request $\beta > 0$, at this point of the proof we have to add that, for $\beta < 0$, ω^2 is determined by oddness w.r.t. β.

The previous counterexample suggests a necessary condition for a slice form to be s-exact. The condition is the following.

DEFINITION 1.18. Let $D \subset (\mathbb{C}^+ \setminus \mathbb{R})$. A slice differential form with coefficients of class $\mathcal{C}^1$, $\omega : \Omega_D \to \mathbb{H}^*$ is called *slice-closed (s-closed)* in Ω_D if, written in coordinates

$$\omega(\alpha + I\beta) = \omega^\alpha(\alpha + I\beta)d\alpha + \omega^\beta(\alpha + J\beta)d\beta,$$

one has

$$\frac{\partial \omega^\alpha}{\partial \beta} = \frac{\partial \omega^\beta}{\partial \alpha}, \quad \forall \alpha + I\beta \in \Omega_D,$$

where $\omega^\alpha = \omega^{\alpha 1} + I\omega^{\alpha 2}$ and $\omega^\beta = \omega^{\beta 1} + I\omega^{\beta 2}$.

REMARK 1.18. If a slice form ω is such that $d\omega^1 = d\omega^2 = 0$, then ω is s-closed.

Thanks to theorem 1.30, we have the following proposition.

PROPOSITION 1.31. *Let $D \subset (\mathbb{C}^+ \setminus \mathbb{R})$. Let $\omega : \Omega_D \to \mathbb{H}^*$ be a slice differential form. If there exist two imaginary units $J \neq K \in \mathbb{S}$ such that the following equations holds:*

$$\begin{cases} \dfrac{\partial \omega_J^\alpha}{\partial \beta} = \dfrac{\partial \omega_J^\beta}{\partial \alpha}, & \forall \alpha + J\beta \in D_J^+, \\ \dfrac{\partial \omega_K^\alpha}{\partial \beta} = \dfrac{\partial \omega_K^\beta}{\partial \alpha}, & \forall \alpha + K\beta \in D_K^+, \end{cases} \tag{14}$$

then ω is s-closed.

Proof. The proof of the theorem follows applying formula 12 and deriving. □

The previous proposition simplifies the conditions to check if a form is closed. Moreover the result is sharp, meaning that asking only one condition in equation 14 it is not enough to obtain the thesis, as the following example will show.

EXAMPLE 1.4. Let $J \in \mathbb{S}$ be a fixed imaginary unit and let $\omega : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}^*$ be the following slice form:

$$\omega(\alpha + I\beta) = (J\beta + 1 + I(\beta + J))d\alpha + (1 - J\alpha + I(J - \alpha))d\beta.$$

This slice form is constructed with formula 13 where

$$\begin{aligned} \omega_J(\alpha, \beta) &= 2J\beta d\alpha - 2J\alpha d\beta \\ \omega_{-J}(\alpha, \beta) &= 2d\alpha + 2d\beta, \end{aligned}$$

and so

$$\frac{\partial \omega_{-J}^\alpha}{\partial \beta} = \frac{\partial \omega_{-J}^\beta}{\partial \alpha}, \quad \text{but,} \quad \frac{\partial \omega_J^\alpha}{\partial \beta} = J \neq -J = \frac{\partial \omega_J^\beta}{\partial \alpha}.$$

Now, if we compute the partial derivatives of ω_I, for a generic $I \in \mathbb{S}$, what we obtain is the following:

$$\frac{\partial \omega_I^\alpha}{\partial \beta} = J + I, \quad \frac{\partial \omega_I^\beta}{\partial \alpha} = -J - I,$$

which are equal if and only if $I = -J$.

The next is a natural result.

PROPOSITION 1.32. *Let $D \subset (\mathbb{C}^+ \setminus \mathbb{R})$. Let Ω_D be a circular open set in $\mathbb{H}$ and let $\omega : \Omega_D \to \mathbb{H}^*$ be a slice form with coefficients of class $\mathcal{C}^1$. If ω is s-exact then ω is s-closed.*

The next theorem is a characterization for s-exact slice form.

THEOREM 1.33. *Let $D \subset (\mathbb{C}^+ \setminus \mathbb{R})$ be an open and connected set and $\omega : \Omega_D \setminus \mathbb{R} \to \mathbb{H}^*$ a slice differential form. The following facts are equivalent:*

(1) ω is s-exact;
(2) for any $J \in \mathbb{S}$ and for any couple of path γ_1 and γ_2 contained in D_J, with same extremal points, the following equation hold,
$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega;$$
(3) for any $J \in \mathbb{S}$ and for any closed piecewise differentiable curve γ contained in D_J, the following equation hold,
$$\int_{\gamma} \omega = 0.$$
(4) there exist $J \neq K \in \mathbb{S}$ such that for any closed piecewise differentiable curve $\gamma : [0,1] \to (D_J \sqcup D_K)$, the following equation hold,
$$\int_{\gamma} \omega = 0.$$

REMARK 1.19. In the hypothesis of this theorem, since $D \subset (\mathbb{C}^+ \setminus \mathbb{R})$, then, automatically, $\beta > 0$ and $D_J = D_J^+$ for any J.

Proof. In the sequence (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), the only non-trivial implication to prove, and for which we give an argument, is (2) $\Rightarrow$ (1). Then we have that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and we have to prove (4) $\Rightarrow$ (1).

((2) $\Rightarrow$ (1)) Since for any $J \in \mathbb{S}$, D_J is connected, then any couple of points x and y in D_J can be connected with a curve. Let then x_0 be a fixed point in D_J and define $f : D_J \to \mathbb{H}$ as $f(x) := \int_\gamma \omega_J$, where γ is any curve from x_0 to x. The function f is well defined thanks to the hypothesis (2). It is clear then, using a little bit of standard analysis that $df = \omega_J$. The thesis is obtained repeating the same argument in another different semislice D_K, using the representation formula 12.

((4) $\Rightarrow$ (1)) Let $\gamma_L : [0,1] \to D_L$ be a closed piecewise differentiable curve in D_L for $L = J, K$, then,
$$0 = \int_{\gamma_L} \omega = \int_{\gamma_L} \omega_L,$$
where ω_L denotes the restriction of ω to D_L. But since γ_L lies on a complex plane then, for $L = J, K$, there exist $f_L : D_L \to \mathbb{H}$ such that $\omega_L = df_L$. Observe that since the domain

of f_L is D_L then it doesn't depend on L but only on the planar variables α and β. In particular we have

$$\omega_L = df_L = \frac{\partial f_L}{\partial \alpha} d\alpha + \frac{\partial f_L}{\partial \beta} d\beta.$$

Now, using representation formula 13, we have that for any $I \in \mathbb{S}$

$$\omega(\alpha + I\beta) = J(J-K)^{-1} df_K - K(J-K)^{-1} df_J + I(J-K)^{-1}(df_J - df_K)$$

but since $\beta > 0$, then the following expression

$$J(J-K)^{-1} f_K - K(J-K)^{-1} f_J + I(J-K)^{-1}(f_J - f_K)$$

defines a slice function $f : \Omega_D \to \mathbb{H}$ and, since all the imaginary units do not depend on α and β we obtain that $\omega = d_{sl} f$.

□

Using this theorem we can prove that not all the s-closed forms are s-exact. An example for such a case of this type is given by following slice form:

$$\begin{array}{rcl} \omega : \mathbb{H} \setminus \mathbb{R} & \to & \mathbb{H}^* \\ \alpha + I\beta & \mapsto & \dfrac{-(\beta-1)d\alpha + \alpha d\beta}{\alpha^2 + (\beta-1)^2}. \end{array}$$

This slice form is independent w.r.t $I \in \mathbb{S}$ and is s-closed (in particular $\omega = \omega^1 + I\omega^2$, with $\omega^2 \equiv 0$ and $d\omega^1 = 0$). We will show that ω is not s-exact by showing that point (3) of the previous theorem fails. Fix, in fact, $J \in \mathbb{S}$ and consider the differentiable circumference $\gamma : [0,1] \to \mathbb{C}_J^+$, defined as $\gamma(t) = (\frac{1}{2}\cos(2\pi t), \frac{1}{2}\sin(2\pi t) + 1)$. This is a circumference contained in $\mathbb{C}_J^+$, centered in $(0, J)$ with radius $1/2$. But then we have

$$\int_\gamma \omega = \int_\gamma \omega_J = 2\pi \neq 0,$$

and so ω is not s-exact.

At this point, we are going to state a *Poincaré type lemma*.

THEOREM 1.34. *Let D be a star-shaped open subset of $\mathbb{C}^+$ and let Ω_D its circularization. If a slice differential form $\omega : \Omega_D \setminus \mathbb{R} \to \mathbb{H}^*$ is s-closed then ω is s-exact.*

Proof. The proof follows a standard argument in calculus. We repeat the considerations here for completeness. Let $x_0 = \alpha_0 + J_0\beta_0 \in D_{J_0}^+$. We are going to show that the function $f_{J_0}^+ : D_{J_0}^+ \to \mathbb{H}$ defined by

$$\begin{aligned} f_{J_0}^+(\alpha + J_0\beta) := \int_0^1 [\omega_{J_0}^\alpha(\alpha_0 + t(\alpha-\alpha_0), \beta_0 + t(\beta-\beta_0))(\alpha-\alpha_0) + \\ + \omega_{J_0}^\beta(\alpha_0 + t(\alpha-\alpha_0), \beta_0 + t(\beta-\beta_0))(\beta-\beta_0)] dt, \end{aligned}$$

is a primitive of ω_{J_0}. To obtain the result it is sufficient to show that the partial derivatives of $f^+_{J_0}$ w.r.t. α and β are equal to $\omega^\alpha_{J_0}$ and $\omega^\beta_{J_0}$ respectively, i.e.:

$$\frac{\partial f^+_{J_0}}{\partial \alpha}(\alpha+J_0\beta)=\omega^\alpha_{J_0}(\alpha+J_0\beta),\quad \frac{\partial f^+_{J_0}}{\partial \beta}(\alpha+J_0\beta)=\omega^\beta_{J_0}(\alpha+J_0\beta),\quad \forall \alpha+J_0\beta\in D^+_{J_0}.$$

We have:

$$\begin{aligned}\frac{\partial f^+_{J_0}}{\partial \alpha}(\alpha+J_0\beta)=\int_0^1 [\frac{\partial \omega^\alpha_{J_0}}{\partial \alpha}&(\alpha_0+t(\alpha-\alpha_0),\beta_0+t(\beta-\beta_0))t(\alpha-\alpha_0)+\\ &+\omega^\alpha_{J_0}(\alpha_0+t(\alpha-\alpha_0),\beta_0+t(\beta-\beta_0))+\\ &+\frac{\partial \omega^\beta_{J_0}}{\partial \alpha}(\alpha_0+t(\alpha-\alpha_0),\beta_0+t(\beta-\beta_0))(\beta-\beta_0)]dt,\end{aligned}$$

and, since ω is s-closed ($\frac{\partial\omega^\alpha}{\partial\beta}=\frac{\partial\omega^\beta}{\partial\alpha}$), the last is equal to:

$$\begin{aligned}\int_0^1 t[\frac{\partial \omega^\alpha_{J_0}}{\partial \alpha}&(\alpha_0+t(\alpha-\alpha_0),\beta_0+t(\beta-\beta_0))(\alpha-\alpha_0)+\\ &+\frac{\partial \omega^\alpha_{J_0}}{\partial \beta}(\alpha_0+t(\alpha-\alpha_0),\beta_0+t(\beta-\beta_0))(\beta-\beta_0)]dt+\\ &+\int_0^1\omega^\alpha_{J_0}(\alpha_0+t(\alpha-\alpha_0),\beta_0+t(\beta-\beta_0))dt.\end{aligned}$$

Now, we observe that the first integral is equal to

$$\int_0^1 t[\frac{d\omega_{J_0}}{dt}(\alpha_0+t(\alpha-\alpha_0),\beta_0+t(\beta-\beta_0))]dt=\circledast,$$

which, integrated by parts, gives:

$$\circledast=[t\omega^\alpha_{J_0}(\alpha_0+t(\alpha-\alpha_0),\beta_0+t(\beta-\beta_0))]_{t=0}^{t=1}-\int_0^1\omega^\alpha_{J_0}(\alpha_0+t(\alpha-\alpha_0),\beta_0+t(\beta-\beta_0))dt,$$

and so

$$\frac{\partial f^+_{J_0}}{\partial \alpha}(\alpha+J_0\beta)=\omega^\alpha_{J_0}(\alpha+J_0\beta).$$

Obviously, the computations for $\frac{\partial f^+_{J_0}}{\partial\beta}$ are the same as before and so, repeating the same argument on another different semislice, say $D^+_{K_0}$, with $J_0\neq K_0\in\mathbb{S}$, and applying the representation formula, we obtain the thesis.

□

3.1. Morera's theorem. In this short subsection we will give an application of the theory of slice forms proving a Morera type theorem. As said before the notion of slice forms has not a proper application so far, meaning that there are not results that cannot be obtained without slice forms. In fact one can shows other "elementary" proof for the following Morera's theorem. Anyway we think that it can be useful to give a flavor of the basic techniques of slice forms. We start remembering the statements by Giacinto Morera regarding functions of a complex variable.

THEOREM 1.35. ***(Morera,1886)*** *If $f : D \to \mathbb{C}$ is defined and continuous in an open and connected set D, and if*

$$\int_\gamma f dz = 0,$$

for all closed and piecewise differentiable curves γ in D, then $f(z)$ is analytic in D.

In **[27]**, the authors states a Morera theorem in a first version for power series defined on a ball of center 0 and radius R. In **[25]** it is provided the following formulation.

THEOREM 1.36. *(**[25]**, propsition 6.2). Let Ω_D be a circular domain such that $\Omega_D \cap \mathbb{R} \neq \emptyset$ and let $f : \Omega_D \to \mathbb{H}$. If for each $I \in \mathbb{S}$, the restriction of f to D_I is continuous and satisfies*

$$\int_{\gamma_I} d_{sl} x f(x) = 0$$

for all rectifiable closed curve $\gamma_I : [0,1] \to D_I$, then f is regular in Ω_D.

We will prove an analogous result without the hypothesis on the domain and reducing the set of $I \in \mathbb{S}$ where to check the hypothesis only to two different imaginary units.

THEOREM 1.37. *Let $D \subset \mathbb{C}^+$ such that $\Omega_D \subset \mathbb{H}$ is a connected circular domain that does not intersect the real line in $\mathbb{H}$. Let $f : \Omega_D \to \mathbb{H}$ be a continuous slice function. If there are $I \neq J \in \mathbb{S}$ such that*

$$\int_\gamma d_{sl} x f(x) = 0, \tag{15}$$

for all closed and piecewise differentiable curves $\gamma : [0,1] \to (D_I^+ \sqcup D_J^+)$, then f is slice regular in Ω_D.

It must be noticed that the case $I = -J$ is not excluded.

We underline the fact that, if $f = \mathcal{I}(F_1 + \sqrt{-1}F_2)$ is a continuous slice function, then $d_{sl} x f(x)$ is a continuous slice form. Explicitly,

$$d_{sl} f = F_1 d\alpha - F_2 d\beta + I(F_2 d\alpha + F_1 d\beta).$$

Proof. Since $d_{sl} x f(x)$ is a slice form, then equation 15, with point (4) of theorem 1.33, says that it is also s-exact. And so there exist a slice differentiable function $g = \mathcal{I}(G) \in \mathcal{S}^1(\Omega_D)$ such that,

$$d_{sl} x f = d_{sl} g,$$

but this entails that,

$$\frac{\partial G_1}{\partial \alpha} d\alpha + \frac{\partial G_1}{\partial \beta} d\beta + I\left(\frac{\partial G_2}{\partial \alpha} d\alpha + \frac{\partial G_2}{\partial \beta} d\beta\right) = F_1 d\alpha - F_2 d\beta + I(F_2 d\alpha + F_1 d\beta), \quad \forall \alpha + I\beta \in \Omega_D$$

and so,

$$\begin{cases} \dfrac{\partial G_1}{\partial \alpha} = F_1 = \dfrac{\partial G_2}{\partial \beta} \\[2ex] \dfrac{\partial G_1}{\partial \beta} = -F_2 = -\dfrac{\partial G_2}{\partial \alpha}, \end{cases}$$

that means that g is regular and so also f is.

□

4. Slice affine functions

In the last part of this chapter we will introduce some notion about slice-constant and slice-affine functions. Slice-constant functions, firstly defined in [**3**], are the natural generalization, in the slice function theory setting, of constant holomorphic functions.

Definition 1.19. Let Ω_D be a connected circular domain and let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$. f is called slice constant if the stem function F is locally constant on D.

Proposition 1.38. *Let $f \in \mathcal{S}(\Omega_D)$ be a slice constant function, then f is slice regular.*

Proof. The proof is trivial because of the nature of the stem function that generate a slice constant function.

□

A simple characterization is given by the following theorem:

Theorem 1.39. *Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$. Then f is slice constant if and only if*

$$\frac{\partial f}{\partial x} = \mathcal{I}\left(\frac{\partial F}{\partial z}\right) \equiv 0.$$

Proof. Let F be locally constant, then in a connected component of D, $F = a + ib$, with $a, b \in \mathbb{H}$. This entails obviously that $\frac{\partial F}{\partial z} = 0$. Vice versa, let $f = \mathcal{I}(F) \in \mathcal{SR}(\Omega_D)$ such that $\frac{\partial F}{\partial z} \equiv 0$, then, recalling the intrinsic curve in remark 1.2, we have in a connected component of D, that

$$\frac{\partial F}{\partial z} = \sum_{k=1}^{4} \frac{\partial}{\partial z} F_{\mathcal{B}}^k(z) u_k = 0$$

and so $F_{\mathcal{B}}^k = c_k \in \mathbb{C}$, for every k, and so also $F = \sum_{k=1}^4 F_{\mathcal{B}}^k u_k = c' \in \mathbb{H}$.

□

Remark 1.20. The previous theorem tells that if we have a slice constant function $f \in \mathcal{SR}(\Omega_D)$ over a connected circular domain Ω_D, then, given $J \in \mathbb{S}$, if $x \in D_J^+ \setminus \mathbb{R}$

$$f(x) = a + Jb = a + \frac{Im(x)}{||Im(x)||} b, \quad a, b \in \mathbb{H}.$$

PROPOSITION 1.40. *Let Ω_D be a connected circular domain. Let $g : \Omega_D \to \mathbb{H}$ be a slice function. g is slice constant if and only if given any fixed $J \in \mathbb{S}$, g is a linear combination, with right quaternionic coefficients, of the two functions $g_\pm : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ defined by $g_\pm(\alpha + I\beta) = 1 \pm IJ$.*

Proof. Thanks to theorem 1.39, any linear combination of the two functions $g_\pm$ is slice constant since their slice derivative is everywhere zero. Vice versa, given a slice constant function $g = \mathcal{I}(G) : \Omega_D \to \mathbb{H}$, with $G := g_1 + \sqrt{-1}g_2$ its locally constant stem function, it holds $g(\alpha + I\beta) = g_1 + Ig_2$, but, thanks to the representation formula in 1.1, for any $J \in \mathbb{S}$, $g(\alpha + I\beta) = [(1 - IJ)(g_1 + Jg_2) + (1 + IJ)(g_1 - Jg_2)]/2$.

□

Now we will introduce the set of slice regular function that are affine slice by slice. This notion will be useful in the classification of rational curves in the Grassmannian in chapter 4.

DEFINITION 1.20. Let $f : \Omega_D \to \mathbb{H}$ be a slice regular function. f is called *slice affine* if its slice derivative is a slice constant function.

PROPOSITION 1.41. *Let $f : \Omega_D \to \mathbb{H}$ be a slice function. f is slice affine if and only if given any fixed $J \in \mathbb{S}$, g is a linear combination, with right quaternionic coefficient, of the four functions $f_\pm, g_\pm : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$, where $g_\pm$ are the same as before and $f_\pm(\alpha + I\beta) = (\alpha + I\beta)g_\pm(\alpha + I\beta)$.*

Proof. If f is a linear combination of $f_\pm$ and $g_\pm$ then it is obviously a slice affine function. Viceversa, since $\partial f/\partial x$ is a slice constant function, then, in the language of slice forms

$$d_{sl}f = d_{sl}x\frac{\partial f}{\partial x} = d_{sl}xg(x),$$

with $g = \mathcal{I}(g_1 + \sqrt{-1}g_2)$ a slice constant function. The previous equality, using the definition of slice form, is equivalent to the following one

$$\frac{\partial F_1}{\partial \alpha}d\alpha + \frac{\partial F_1}{\partial \beta}d\beta + I\left(\frac{\partial F_2}{\partial \alpha}d\alpha + \frac{\partial F_2}{\partial \beta}d\beta\right) = g_1 d\alpha - g_2 d\beta + I(g_2 d\alpha + g_1 d\beta),$$

that implies $F_1 = g_1\alpha - g_2\beta + q_1$ and $F_2 = g_2\alpha + g_1\beta + q_2$, for some couple q_1, q_2 of quaternions. But then, applying the representation formula in 1.1, and using the same argument as in the proof of the previous theorem we obtain,

$$\begin{aligned} f(\alpha + I\beta) &= = g_1\alpha - g_2\beta + I(g_2\alpha + g_1\beta) + q_1 + Iq_2 \\ &= \alpha(g_1 + Ig_2) + I\beta(g_1 + Ig_2) + q_1 + Iq_2 \\ &= (\alpha + I\beta)(g_1 + Ig_2) + q_1 + Iq_2 \\ &= (\alpha + I\beta)[(1 - IJ)(g_1 + Jg_2) + (1 + IJ)(g_1 - Jg_2)]/2+ \\ &\quad +[(1 - IJ)(q_1 + Jq_2) + (1 + IJ)(q_1 - Jq_2)]/2. \end{aligned}$$

□

REMARK 1.21. The set of slice constant functions contains the set of constant functions and the condition for a slice constant function $g = g_+q_+ + g_-q_-$ to be extended to $\mathbb{R}$ is that $q_+ = q_-$ (i.e.: g is a constant function). Analogously, a slice affine function $f = f_+q_{1+} + f_-q_{1-} + g_+q_{0+} + g_-q_{0-}$ extends to the real line if and only if $q_{1+} = q_{1-}$ and $q_{0+} = q_{0-}$ (i.e.: $f = xa + b$ is a $\mathbb{H}$-affine function). For slice constant function the assertion is trivial while for slice affine functions it requires a simple consideration regarding the limit of the function for β that approach 0 when β is lower or greater than zero. In formula, the previous condition is the following one:

$$\lim_{\substack{\beta\to 0\\ \alpha+I\beta\in\mathbb{C}_I^+}} f(\alpha+I\beta) = \lim_{\substack{\beta\to 0\\ \alpha+I\beta\in\mathbb{C}_I^-}} f(\alpha+I\beta).$$

REMARK 1.22. One can define, in general, the class of "slice polynomial" functions as the set of slice regular functions such that the n^{th} slice derivative vanishes for some n. This can be actually a useful notion in view of some researches regarding the number of counterimages of a slice regular function defined over a domain without real points. Anyway this theme is not explored in this thesis and so we will not spend any other words. We will remember this fact when in remark 3.6 we will motivate the construction of a new proof for theorem 3.11.

CHAPTER 2

Rigidity properties for slice regular functions

In this chapter we will explore some rigidity properties of slice regular functions. These are essentially generalizations of holomorphic functions properties in this quaternionic context. As in the complex setting, also in the quaternionic one it is very important to study rigidity properties and, in principle, one could ask if any result in complex analysis can be extended in this context. In particular, the content of this chapter will be used in many occasions later in this thesis. As already mentioned in section 3.1 of chapter 1 the theory of slice regular functions over domains that do not intersect the real line must contemplate examples of functions such that restricted to some complex plane $\mathbb{C}_J$ might behave in very different ways on $\mathbb{C}_J^+$ and $\mathbb{C}_{-J}^+$.

The results in this chapter were proved, with the additional hypothesis of nonempty intersection of the domain with the real axis, in [**9, 22, 23**] and then extended in [**3**].

1. Identity principle

In this section we will prove an analogous of the identity principle for slice regular functions. A suggestion that an analogous of the identity principle might be true came, among the other things, from theorem 1.15, where we said essentially that the set of zeros of a non-slice constant regular function restricted to a semislice is closed and discrete. In [**27**] the authors proved the statement for slice regular functions defined on open balls centered in the origin and later, in [**47**] it was extend to functions defined on generic circular domain with nonempty intersection with the real axis.

More precisely their statement, in our language, is the following.

THEOREM 2.1. *([**25**], theorem 1.12). Let $\Omega_D \subset \mathbb{H}$ be a connected domain such that $\Omega_D \cap \mathbb{R} \neq \emptyset$, and let $f : \Omega_D \to \mathbb{H}$ be a slice regular function. If there exists $I \in \mathbb{S}$ such that $D_I \cap V(f)$ has an accumulation point, then $f \equiv 0$ on Ω_D.*

Now, it is not possible to generalize this theorem as it is to the case in which the domain D does not intersect the real line. In fact, there is a counterexample.

EXAMPLE 2.1. Let $J \in \mathbb{H}$ be fixed. The slice regular function defined on $\mathbb{H} \setminus \mathbb{R}$ by

$$f(x) = 1 - IJ, \quad x = \alpha + I\beta \in \mathbb{C}_I^+$$

is induced by a locally constant stem function (and so is slice constant, see definition 1.19) and its zero set $V(f)$ is the half plane $\mathbb{C}_{-J}^+ \setminus \mathbb{R}$. The function can be obtained by the representation formula in theorem 1.1 by choosing the constant values 2 on $\mathbb{C}_J^+ \setminus \mathbb{R}$ and 0 on $\mathbb{C}_{-J}^+ \setminus \mathbb{R}$. Now, by definition, it is clear that this function does not satisfies the thesis of the theorem.

It is now clear that if we want to obtain an identity principle we must control the set of slice constant functions. The following theorem clarifies the situation.

THEOREM 2.2. ***(Identity Principle)*** *Let Ω_D be a connected open set of $\mathbb{H}$. Given $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H} \in \mathcal{SR}(\Omega_D)$, let $V(f) = \{x \in \Omega_D \mid f(x) = 0\}$ be its zero locus. If there exists $K \neq J \in \mathbb{S}$ such that both $D_K^+ \cap V(f)$ and $D_J^+ \cap V(f)$ contain accumulation points, then $f \equiv 0$ on Ω_D.*

Proof. Let $x_J = \alpha_1 + J\beta_1$, x_K be accumulation points of $V(f)$ respectively on $D_J^+ \cap V(f)$ and $D_K^+ \cap V(f)$. After having fixed a basis u_k for $\mathbb{H}$ and putting $z_0 = \alpha_1 + i\beta_1$ we have that (see remark 1.2),

$$\begin{aligned} 0 = f(x_J) &= F_1(\alpha_1 + i\beta_1) + JF_2(\alpha_1 + i\beta_1) \\ &= \sum_{k=1}^{4} F_{\mathcal{B}}^k(z_0) u_k, \end{aligned}$$

and so all the four components $F_{\mathcal{B}}^k$ vanishes at z_0, and since these are holomorphic, for the identity principle in the complex case, they are identically zero on D_J^+. Replacing J with K in the previous formula, we obtain that f is identically zero also on D_K^+. We now obtain the thesis thanks to the representation formula in theorem 1.1.

□

REMARK 2.1. An equivalent result is, of course, the following: if $f \in \mathcal{SR}(\Omega_D)$ is such that both $(f\mid_{D_K^+})^{-1}(q)$ and $(f\mid_{D_J^+})^{-1}(p)$ contain an accumulation point respectively in D_K^+ and in D_J^+, for some $J \neq K$ and $p, q \in \mathbb{H}$, then, using the representation formula 6, f is slice constant and equal to

$$f(\alpha + I\beta) = (I - K)(J - K)^{-1}p - (I - J)(J - K)^{-1}q$$

To prove the equivalence one apply the previous proof for $g = f - (I - K)(J - K)^{-1}p + (I - J)(J - K)^{-1}q$.

2. Maximum and minimum modulus principles

In this section we will generalize the maximum modulus principle stated in [**27**], to the case of regular functions defined over product domains. Before this we need a lemma. The proofs of the lemma and the theorem, follow the argument in [**27**], with the adjustments needed in our context.

LEMMA 2.3. *If $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$ is a slice regular function, and if $I \in \mathbb{S}$, then f_I^+ has the slice mean value property, i.e.: for any $x_0 = \alpha_0 + I\beta_0 \in \mathbb{C}_I^+$ and for each $r > 0$ such that the closed disc $B(x_0, r)$centered in x_0 with radius r is contained in D_I^+, the following equality holds*

$$f_I^+ = \frac{1}{2\pi}\int_0^{2\pi} f_I^+(x_0 + re^{I\vartheta})d\vartheta$$

Proof. If $x = \alpha + I\beta \in D_I^+$ then we know that $f(x) = F_1(z) + IF_2(z)$, with $z = \alpha + i\beta$. But then, for every point $x_0 = \alpha_0 + I\beta_0 \in D_I^+$, and all positive real number r such that $\overline{B(x_0, r)} \subset D_I^+$ we have,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} f(x_0 + re^{I\vartheta})d\vartheta &= \\
&= \frac{1}{2\pi}\int_0^{2\pi}(F_1(x_0' + re^{i\vartheta}) + IF_2(x_0' + re^{i\vartheta}))d\vartheta \\
&= F_1(x_0') + IF_2(x_0') = f(x_0)
\end{aligned}$$

where $x_0' = \gamma + i\delta$. The penultimate equality holds because, restricting to D_I^+ and passing through the complex curve in remark 1.2, $F_1(z) + \sqrt{-1}F_2(z) = \sum_{k=1}^4 F_{\mathcal{B}}^k(z)u_k$, with $F_{\mathcal{B}}^k(z) \in \mathbb{C}$, for each component we are in the hypothesis of the mean value property in the holomorphic case (more precisely we're considering here $f_I = \phi_I \circ F_{\mathcal{B}}^k$ for $F_{\mathcal{B}}^k$, where $\phi_I : \mathbb{C} \to \mathbb{C}_I$ is the isomorphism defined as $\phi_I(\alpha + i\beta) = \alpha + I\beta$), and so we're done.

□

THEOREM 2.4. ***(Maximum Modulus Principle)*** *Let $f = \mathcal{I}(F) \in \mathcal{SR}(\Omega_D)$ with Ω_D connected circular domain. If there exists $J \neq K \in \mathbb{S}$ such that $||f_J^+||$ has local maximum at $a \in D_J^+$ and $||f_K^+||$ has local maximum at $b \in D_K^+$, then f is slice-constant on Ω_D.*

Proof. If $f(a) = f(b) = 0$ the result is trivial. We will assume that at least one between $f(a)$ and $f(b)$ is different from zero. Let then $f(a) \neq 0$, using lemma 2.3 and following the proof in [**27**], we get that f is constant on D_J^+. Now, if $f(b) = 0$, the result is again trivial thanks to the representation formula in 1.1. If $f(b) \neq 0$, repeating the argument in [**27**], we get that f is constant also over D_K^+. The proof is concluded thanks to the representation formula in 1.1.

□

REMARK 2.2. It must be noticed that the hypothesis of double relative maximum on two different semislices of Ω_D is not removable. Indeed there is a counterexample. Let

$J \in \mathbb{S}$ be fixed. Then the function $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ defined by

$$f(x) = x(1 - IJ), \quad x = \alpha + I\beta,$$

is constant and equal to 0 on $\mathbb{C}^+_{-J} \setminus \mathbb{R}$, but is equal to $2x$ on $\mathbb{C}^+_J \setminus \mathbb{R}$.

We have the following trivial corollary

COROLLARY 2.5. *Let $f = \mathcal{I}(F) \in \mathcal{SR}(\Omega_D)$. If there exists $J \neq K \in \mathbb{S}$ such that $||f^+_J||$ has relative maximum in $a \in \Omega_D \cap \mathbb{C}^+_J$ and $||f^+_K||$ has relative maximum in $b \in \Omega_D \cap \mathbb{C}^+_K$ and $f(a) = f(b)$, then f is constant on Ω_D.*

Our next goal now is to obtain a minimum modulus principle that generalizes the one in [**22**] to the case of domains without real points. This will enable us to prove the open mapping theorem. For this reason we need some additional material regarding the symmetrization and the reciprocal of a slice regular function.

Thanks to proposition 1.8 we have the following lemma, the proof of which is identical to the one in [**22**] remembering that:

$$f^{-\cdot}(x) = N(f)(x)^{-1} f^c(x) = ((f^c \cdot f)(x))^{-1} f^c(x).$$

LEMMA 2.6. *([**22**], proposition 3.5). Let $f \in \mathcal{SR}(\Omega_D)$, then if we set $T_f(x) := f^c(x)^{-1} x f^c(x)$, we have*

$$f^{-\cdot}(x) = f(T_f(x))^{-1},$$

for all $x \in \Omega_D \setminus V(N(f))$.

PROPOSITION 2.7. *([**22**], proposition 3.6). Let $f \in \mathcal{SR}(\Omega_D)$, then T_f and T_{f^c} are mutual inverses w.r.t. composition. Moreover $T_f : \Omega_D \setminus V(f^c) \to \Omega_D \setminus V(f)$ is a diffeomorphism.*

We recall now the definition of the degenerate set of a function.

DEFINITION 2.1. Let $f \in \mathcal{S}(\Omega_D)$ and let $x = \alpha + I\beta \in \Omega_D$, $\beta > 0$ be such that $\mathbb{S}_x = \alpha + \mathbb{S}\beta \subset \Omega_D$. The 2-sphere $\mathbb{S}_x$ is said to be *degenerate* for f if the restriction $f|_{\mathbb{S}_x}$ is constant. The union D_f of all degenerate spheres for f is called *degenerate set* of f.

Observe that the degenerate set of a slice function is a circular domain. We will now state some properties of the degenerate set of a slice function. First of all, the degenerate set of a slice function can be described as the zero set of the spherical derivative as stated in the following proposition.

PROPOSITION 2.8. *Let f be a slice function over Ω_D, then we have the following equality:*

$$D_f = V(\partial_s f).$$

Moreover D_f is closed in $\Omega_D \setminus \mathbb{R}$.

Proof. The proof of the statement is trivial thanks to remark 1.4.

□

As usual, adding the regularity property implies several additional results as the following one.

PROPOSITION 2.9. *If $f \in \mathcal{SR}(\Omega_D)$ is non-constant, then the interior of D_f is empty.*

Proof. If ad absurdum there exists a point $p \in D_f$ and a circular connected neighborhood Ω_U of p such that $\Omega_U \subset D_f$ and f is non-constant, then, for all $\alpha + J\beta \in \Omega_U$

$$f(\alpha + \beta J) = F_1(\alpha + i\beta).$$

Since f is slice regular we have that

$$0 = \frac{\partial F_1}{\partial \overline{z}} = \frac{\partial F_1}{\partial \alpha} + \sqrt{-1}\frac{\partial F_1}{\partial \beta},$$

but then $\frac{\partial F_1}{\partial \alpha} = 0$ and $\frac{\partial F_1}{\partial \beta} = 0$ separately and so F_1 is equal to a constant in all $\Omega_U \subset D_f \subset \Omega_D$. Thanks to the identity principle we obtain that f is constant.

□

REMARK 2.3. As already stressed in the first chapter, if f is a slice function defined on Ω_D and $\mathbb{S}_x = \alpha + \mathbb{S}\beta \subset \Omega_D$ $(\beta > 0)$, is a non-degenerate sphere, then the restriction $f|_{\mathbb{S}_x}$ is a non-constant affine map of $\mathbb{S}_x$ onto a 2-sphere $b + \mathbb{S}c$ with $b, c \in \mathbb{H}$.

Thanks to this remark we have the following proposition:

PROPOSITION 2.10. *Let f be a slice regular function defined on Ω_D let $\alpha, \beta \in \mathbb{R}$, $\beta > 0$ be such that $\mathbb{S}_x = \alpha + \mathbb{S}\beta \subset \Omega_D$, $\mathbb{S}_x \nsubseteq D_f$. Then $||f|_{\mathbb{S}_x}||$ has a global minimum and a global maximum; moreover if the maximum and the minimum are different, then there are no other extremal point.*

We are now ready to state a formulation of the minimum modulus principle.

THEOREM 2.11. ***(Minimum Modulus Principle)*** *Let Ω_D be a connected circular domain and let $f : \Omega_D \to \mathbb{H}$ be a slice regular function. If $||f||$ has a local minimum point $p = x + Iy \in D_I^+$ then $f(p) = 0$ or exists a $I' \in \mathbb{S}$ such that $f_{I'}^+$ is constant.*

Proof. Suppose f does not have zeroes in $S = x + \mathbb{S}y$. The reciprocal $f^{-\cdot}$ is defined on $\Omega_D \setminus V(N(f))$ which includes S. Since $||f^{-\cdot}(q)|| = 1/||f(T_f(q))||$ for every q and T_f is a diffeomorphism, the fact that $||f||$ has a local minimum at $p = T_f(p')$ implies that $||f \circ T_f||$ has a local minimum at $p' = x + I'y$ and so, in particular, $||f \circ T_f(p')||$ is a local minimum if we restrict $||f \circ T_f||(p')$ to $D_{I'}^+$. As a consequence, $||f^{-\cdot}||$ has a maximum at p'. Now, by the argument of the maximum modulus principle $f^{-\cdot}$ is constant on $D_{I'}^+$. Hence, for almost any $\alpha + I'\beta \in \mathbb{C}_{I'}^+$ we have that $f^{-\cdot}(\alpha + I'\beta) = q \neq 0$

$$1 = (f^{-\cdot} \cdot f)(\alpha + I'\beta) = qf(\alpha + qI'q^{-1}\beta),$$

and so $f_{qI'q^{-1}}^+ = q^{-1}$.

Suppose now that, for every $J \in \mathbb{S}$, f_J^+ is non-constant, $||f||$ has a local minimum at $p = x + Iy$ and there exists a point $p' \in S$ such that $f(p') = 0$. But then $||f||$ has a local minimum also at p'. By the previous proposition, $||f||$ cannot have two distinct global minimum points on the same sphere S, unless $||f||$ is constant. As a consequence, either f is constant on S or $p = p'$. In both cases, $f(p) = f(p') = 0$.

□

It is clear that this theorem can be refined adding some hypothesis. For instance, if one ask for f to have two minimal points $p \neq q$ for its modulus that are sent by T_f on two different semislices, then one can conclude that f is slice constant. This case could happen for example when p, q belongs to the same sphere, because we know that T_f maps any 2-sphere to itself. Anyway this formulation of the minimum modulus principle is sufficient to prove the open mapping theorem.

3. Open mapping theorem

In the last part of this chapter we will expose an open mapping theorem. We remember that, for slice regular functions defined over circular domains intersecting the real axis, this was proved in [**22**].

We will expose now the introductory materials to state our extending version of this result.

What we need to know is something more about the zero set of a slice regular function. We have the following theorem.

THEOREM 2.12. *Let Ω_D be a connected circular domain such that $\Omega_D \cap \mathbb{R} = \emptyset$. Let $f \in \mathcal{SR}(\Omega_D)$ be a non-constant function. If $x_0 \in V(f)$ is not isolated in $V(f)$, then there exists a real surface $\mathcal{S} \subset \Omega_D$ such that $x_0 \in \mathcal{S} \subset V(f)$. Moreover, $V(f)$ does not contain any 3-manifold M.*

Proof. Let us start with the three dimensional case: writing Ω_D as the product $D \times \mathbb{S}$, we have that, if $V(f)$ contains a three-dimensional manifold M, then it can be split as $M_D \times M_{\mathbb{S}}$, with $M_D \subset D$ and $M_{\mathbb{S}} \subset \mathbb{S}$. Since M has dimension 3, then, it is not fully contained either in D or in $\mathbb{S}$ and it must contain either an open set of D times a curve in $\mathbb{S}$ or, conversely, an open set of $\mathbb{S}$ times a curve in D. But if M contains an open subset of $\Lambda \subset D$ times a curve in $\mathbb{S}$, then there are at least two imaginary units $I \neq J \in \mathbb{S}$ such that, denoting by Λ_K^+ the projection of Λ in D_K^+, $f(\Lambda_I^+) = f(\Lambda_J^+) \equiv 0$ and so, thanks to theorem 2.2, $f \equiv 0$. In the other case, if M contains an open set in $\mathbb{S}$ times a curve in D, then, thanks to formula 6, M contains the whole sphere and so f is equal to zero on a curve of degenerate spheres. Fixing then two different imaginary units $I \neq J \in \mathbb{S}$ we have that f_I^+ and f_J^+ are identically zero on a curve and so, again, $f \equiv 0$.

Let now $f = \mathcal{I}(F_1 + \sqrt{-1}F_2)$, $x = \alpha + I\beta \in \Omega_D$ and $z = \alpha + i\beta \in D$. If x is an accumulation point in $V(f) \cap \mathbb{S}_x$ then it is clear that the whole sphere $\mathbb{S}_x$ is contained in the zero locus of f. Analogously, if x is an accumulation point for $V(f) \cap D_I^+$, then $D_I^+ \subset V(f)$. Let us consider then the case in which x is a generic accumulation point that doesn't accumulate in any sphere or in any semislice. The point x belongs to $V(f)$ if and only if $F_1(z) + IF_2(z) = 0$. Since x doesn't accumulate in any sphere that intersects $V(f)$, then $F_2(z) \neq 0$. Then the zero locus of f is equal to

$$V(f) = V(\partial_s f) \sqcup \{x \in D \times \mathbb{S} \,|\, x = (z, -F_1(z)F_2(z)^{-1})\},$$

where $\Omega_D \simeq D^+ \times \mathbb{S}$ ($D^+ \subset \mathbb{C}^+$), is the isomorphism that send each $\alpha + I\beta \in \Omega_D$ to the couple $(\alpha + i\beta, I)$. Since x is an accumulation point in $V(f) \setminus V(\partial_s F)$, that means that, for any open disc centered in z and contained in D, there are infinite points w such that there exists an imaginary unit I_w for which $F_1(w) + I_w F_2(w) = 0$. Hence, for any $J \in \mathbb{S}$, the normal function $N(f)$ restricted to D_J^+ vanishes at infinite points that accumulates to $\alpha + J\beta$ and so, for the identity principle, $N(f) \equiv 0$. So, for any $z \in D$, there exists $I_z \in \mathbb{S}$, such that $(z, I_z) \in (D \times \mathbb{S}) \cap V(f)$. Now, the condition $N(f) \equiv 0$, translates in the following system

$$\begin{cases} g(F_1, F_1) - g(F_2, F_2) = 0 \\ g(F_1, F_2) = 0, \end{cases} \tag{16}$$

that means that, for any $z \in D$, $||-F_1(z)F_2(z)^{-1}|| = 1$ and $Re(F_1(z)F_2(z)^{-1}) = 0$ and so $F_1(z)F_2(z)^{-1} \in \mathbb{S}$. Finally, the set

$$\widetilde{V(f)} = \{x \in D \times \mathbb{S} \,|\, x = (z, -F_1(z)F_2(z)^{-1})\}$$

defines a surface in $D \times \mathbb{S}$ that contains the accumulation point x.

□

Given a non-constant slice regular function f, its zero locus contains isolated points, null-spheres and generic surfaces not contained in the degenerate set.

PROPOSITION 2.13. *Let Ω_D be a connected circular open domain. Let $f \in \mathcal{SR}(\Omega_D)$ be a slice regular function. If there exist $q \in \mathbb{H}$ such that $h = f - q$ admits two different surfaces S_1 and S_2 that are not degenerate spheres, in the zero locus (i.e.: $S_1, S_2 \subset V(h)$), then f is constant.*

Proof. Without loss of generality, we can suppose $q = 0$. Then, for any $z \in D$ there exist $I_1 \neq I_2 \in \mathbb{S}$ such that f vanishes both at (z, I_1) and (z, I_2) in $\Omega_D = D \times \mathbb{S}$. This will imply that the spherical derivative is everywhere equal to zero and so f is constant.

□

REMARK 2.4. The condition $N(f) \equiv 0$ defines a surface in Ω_D that can coincide with a semislice D_I^+, for some $I \in \mathbb{S}$, or not. We will see in the following chapter (see lemma 3.8), that the set of surfaces in which a slice regular function is constant is contained in a possibly bigger set that is closed and with empty interior.

Given a slice regular function $f : \Omega_D \to \mathbb{H}$ that is non-slice constant, the set of surfaces in Ω_D in which f is constant consists in the union of its degenerate set D_f and of S_f, the surface defined by $u : D \to D \times \mathbb{S}$, $u(z) = (z, -F_1(z)F_2(z)^{-1})$ extended, where $F(z) = 0$, with the unique element in $\mathbb{S}$ such that u is continuous, i.e.:

$$S_f = \bigcup_{\substack{c \in \mathbb{H} \\ N(f-c) \equiv 0}} \widetilde{V(f-c)}.$$

We can now state the open mapping theorem, where, again, if $\Omega_D \cap \mathbb{R} \neq \emptyset$, this result is proved in **[22]**.

THEOREM 2.14. ***(Open Mapping Theorem)*** *Let $f : \Omega_D \to \mathbb{H}$ be a slice regular function that is non-slice constant. Then*

$$f : \Omega_D \setminus (D_f \cup S_f) \to \mathbb{H}$$

is open.

The proof of this theorem follow the one in complex case.

Proof. Let U be an open set in $\Omega_D \setminus (D_f \cup S_f)$, the thesis is that $f(U)$ is open in $\mathbb{H}$. Let $p_0 \in f(U)$, then there exist $q_0 \in U$ such that $p_0 = f(q_0)$. Clearly, the function $f(q) - p_0$ vanishes in q_0. Now, theorem 1.11 tells that either q_0 is an isolated zero or is part of a sphere S where the function vanishes identically. Since by hypothesis we have removed all the surfaces on which f is constant from the domain of the function, the last option cannot hold and so q_0 is an isolated zero for f. We have then that there exists an open ball $B = B(q_0, r)$ such that $\overline{B} \subset U$ and $f(q) - p_0 \neq 0$ for all $q \in \partial B$, i.e.: there exists $\epsilon > 0$ such that $||f(q) - p_0|| \geq 3\epsilon$ for all $q \in \partial B$. We choose now an arbitrary p such that $||p - f(q_0)|| = ||p - p_0|| < \epsilon$ and we have the following inequality:

$$||f(q) - p|| \geq ||f(q) - p_0|| - ||p - p_0|| \geq 3\epsilon - \epsilon = 2\epsilon, \quad \forall q \in \partial B.$$

We have obtained that the minimum of $||f(q)-p||$ in $\overline{B}$ is strictly less then its minimum in ∂B, and so $||f(q)-p||$ must have a minimum in B. By theorem 2.11, either $f(q)-p$ vanishes at the point of minimum or there exists a semislice where the function is constant. Since, by hypothesis, f is non-constant in every semislice, then there exists a point $q \in B \subset U$ such that $f(q) = p$ and $p \in f(U)$ and the proof is concluded.

□

Of course, if we don't remove the set S_f from the domain Ω_D of the function f, then the previous theorem is no more true. Let us define f as in remark 2.2: let $J \in \mathbb{S}$ be a fixed imaginary unit and define

$$f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$$

$$f(x) = x(1 - IJ), \quad x = \alpha + I\beta$$

f is non-constant in every semislice except for $\mathbb{C}^+_{-J}$ in which is identically equal to zero. Moreover in every semislice except for $\mathbb{C}^+_{-J}$ it assumes non-purely real values. We will prove that $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ is not open while $f : \mathbb{H} \setminus \mathbb{C}^+_{-J} \to \mathbb{H}$ it is.

Given $B = B(y, r)$, with $y \in (\mathbb{H} \setminus \mathbb{R}) \cap \mathbb{C}^+_{-J}$, $r > 0$ such that $B \cap \mathbb{R} = \emptyset$, we will prove that $f(B)$ is not open. But this is trivial because $f(B) = \{0\} \sqcup D$, where $D \subset \mathbb{H} \setminus \mathbb{R}$ and the union is disjoint. This is true because, writing the function explicitly, it is easy to see that $f(\mathbb{H} \setminus \mathbb{C}^+_{-J}) \cap \mathbb{R} = \emptyset$. Let, in fact, $x = \alpha + I\beta \in \mathbb{H} \setminus \mathbb{C}^+_{-J}$,

$$f(\alpha + I\beta) = \alpha + I\beta - \alpha IJ + \beta J$$

and, since $I \neq \pm J$, then I, J and IJ describe independent vectors in $\mathbb{R}^4$; then, since $\beta > 0$, $Im(f(\alpha + I\beta)) \neq 0$. This entails that it is not possible to find a ball $B' \subset \mathbb{H}$ centered in zero such that $f(y) \in B' \subset f(B)$ because, otherwise, the intersection $B' \cap \mathbb{R}$ would be equal to an interval $(0 - \epsilon, 0 + \epsilon)$, for some $\epsilon > 0$ but $f(B) \cap \mathbb{R} = \{0\}$.

Let see now that the same function f restricted to $\mathbb{H} \setminus (\mathbb{R} \cup \mathbb{C}^+_{-J})$ is open. First of all, if $x = \alpha + I\beta$, then

$$f(x) = \alpha(1 + I \cdot J) + I\beta + J\beta + I \wedge J(-\sqrt{1 - (I \cdot J)^2}),$$

where $I \cdot J$ and $I \wedge J$ denote, respectively, the scalar and the vector products in $\mathbb{R}^3$. But then again,

$$f(x) = \alpha(1 + I \cdot J) + \left(\frac{\beta I + \beta J - \sqrt{1 - (I \cdot J)^2} I \wedge J}{2\beta^2 + \alpha^2(1 - (I \cdot J)^2)} \right) (2\beta^2 + \alpha^2(1 - (I \cdot J)^2)).$$

Now, the sets of the form

$$A = (\alpha - \epsilon, \alpha + \epsilon) + (\beta - \delta, \beta + \delta) B_I(R) \in \mathbb{H} \setminus (\mathbb{R} \cup \mathbb{C}^+_{-J}),$$

with $B_I(R) = B(I, R) \cap \mathbb{S}$, form a basis for the topology in $\mathbb{H} \setminus (\mathbb{R} \cup \mathbb{C}^+_{-J})$. So to prove that f is open we need to prove that $f(A)$ is open. Is clear that if we stay far from $\mathbb{C}^+_J$, then $\alpha(1 + I \cdot J)$ sends the set A in an open interval. It is also clear that $(2\beta^2 + \alpha^2(1 - (I \cdot J)^2))$ sends A into an open interval since $\beta - \delta > 0$. For the last part, since for $J \notin B_I(R)$, I, J and $I \wedge J$ are linear independent, then the function $\frac{\beta I + \beta J - \sqrt{1-(I \cdot J)^2} I \wedge J}{2\beta^2 + \alpha^2(1-(I \cdot J)^2)}$ sends A into an open set. If $I = J$ we have no problems, since the image of $\frac{\beta I + \beta J - \sqrt{1-(I \cdot J)^2} I \wedge J}{2\beta^2 + \alpha^2(1-(I \cdot J)^2)}$ contains a ball centered in J.

Remark 2.5. The last example shows also that, if the domain Ω_D of definition of a non-constant slice regular function f does not contains real points, then, in general, the set $f(\Omega_D)$ is not open in $\mathbb{H}$ (see theorem 7.4 of **[25]**).

CHAPTER 3

Real differential of a slice regular function

In this chapter we will describe some interesting differential properties of slice regular functions that will be useful in the next chapter.

We will start describing the real differential of a slice function. For this purpose, in addition to using what we already discussed in the previous pages, we will recall some results and constructions due to Caterina Stoppato (see [**46**]). Moreover, we will also use the concept of *spherical differential* that will be introduced right now.

Let $f \in \mathcal{S}^1(\Omega_D)$ be any differentiable slice function. We have seen that is possible to define its slice differential, considering, roughly speaking, the restriction of the real differential, outside of the real line, to each semislice. It is clear that this object does not exhaust the description of the real differential. What we are going to define is exactly the missing part.

Definition 3.1. Let $f \in \mathcal{S}^1(\Omega_D)$ be a differentiable slice function. We define its *spherical differential* as the following differentiable form

$$d_{sp}f : \Omega_D \setminus \mathbb{R} \to \mathbb{H}^*, \quad d_{sp}f(\alpha + I\beta) := df(\alpha + I\beta) - d_{sl}f(\alpha + I\beta),$$

where $df(\alpha + I\beta)$ denotes the real differential of f.

In the next pages we will give a more explicit description of the spherical differential of a slice function. Starting from equation 11, we have formally that,

$$d_{sp}f = df - d_{sl}f = \frac{1}{\beta}\left(\frac{\partial f}{\partial \vartheta}d\vartheta + \frac{1}{\sin\vartheta}\frac{\partial f}{\partial \varphi}d\varphi\right),$$

but since, for every $\alpha + I\beta \in \Omega_D \setminus \mathbb{R}$, a slice function f depends on $I = I(\vartheta, \varphi)$ in an affine way, then, if $x = \alpha + I\beta \in \Omega_D \setminus \mathbb{R}$ and $z = \alpha + i\beta \in D$

$$d_{sp}f(x) = \frac{1}{\beta}\left(\frac{\partial I}{\partial \theta}d\theta + \frac{1}{\sin\theta}\frac{\partial I}{\partial \varphi}d\varphi\right)F_2(z).$$

If $g : \mathbb{H} \to \mathbb{H}$ is the identity function, ($g(\alpha + I\beta) = \alpha + I\beta$), then

$$dg|_{\mathbb{H}\setminus\mathbb{R}} = d\alpha + Id\beta + dI\beta = d_{sl}x + \frac{1}{\beta}\left(\frac{\partial I}{\partial\theta}d\theta + \frac{1}{\sin\theta}\frac{\partial I}{\partial\varphi}d\varphi\right)\beta = d_{sl}x + d_{sp}x,$$

and so,

$$d_{sp}f = d_{sp}x\,\partial_s f$$

It seems then that, if $f \in \mathcal{SR}(\Omega_D)$, then its real differential satisfies the following equation:

$$df|_{\Omega_D\setminus\mathbb{R}} = d_{sl}x\frac{\partial f}{\partial x} + d_{sp}x\partial_s f,$$

where, the position of the elements of the cotangent space is on the *left*. As the reader could object, the previous are only formal considerations but, in the next pages everything will be proved in the case of slice regular functions (in particular see corollary 3.5). We remember firstly the notion of spherical analyticity and its consequences.

1. Coefficients of the spherical expansion

As already described in Section 2.1 of chapter 1, in [**33, 46**] the authors introduce, in slightly different contexts, a spherical series of the form

$$f(x) = \sum_{n\in\mathbb{N}} \mathcal{S}_{y,n}(x)s_n, \tag{17}$$

where $S_{y,n}$ were defined in equation 9.

Given a slice regular function $f : \Omega_D \to \mathbb{H}$, it is possible to construct its spherical coefficients $\{s_n\}$ (see [**46, 33**]), but the methods described in the cited papers allow a correct explanation and interpretation only for the first two coefficients which can be easily written as (see [**33**]),

$$\begin{aligned} s_1 &= \frac{1}{2}Im(y)^{-1}(f(y) - f(y^c)) = \partial_s f(y) \\ s_2 &= \frac{1}{2}Im(y)^{-2}(2Im(y)\frac{\partial f}{\partial x}(y) - f(y) + f(y^c)), \end{aligned}$$

and in particular

$$s_1 + 2Im(y)s_2 = \frac{\partial f}{\partial x}(y).$$

The following proposition, which has an independent interest, allows us to understand better the nature of s_2.

Proposition 3.1. *Let $f \in \mathcal{SR}(\Omega_D)$ be a slice regular function, then the following formula holds:*

$$\frac{\partial f}{\partial x}(x) = 2Im(x)\left(\frac{\partial}{\partial x}\partial_s f\right)(x) + \partial_s f(x), \quad \forall x = \alpha + J\beta \in \Omega_D. \tag{18}$$

Proof. Let $F = F_1 + \sqrt{-1}F_2$ the inducing stem function of f and let $x = \alpha + J\beta \in \Omega_D \setminus \mathbb{R}$ and $z = \alpha + i\beta$, then,

$$\frac{\partial f}{\partial x}(x) = \frac{1}{2}\left(\frac{\partial F_1}{\partial \alpha}(z) + J\frac{\partial F_2}{\partial \alpha}(z) - J\frac{\partial F_1}{\partial \beta}(z) + \frac{\partial F_2}{\partial \beta}(z)\right) = \circledast.$$

Using the slice regularity we have,

$$\circledast = \frac{\partial F_2}{\partial \beta}(z) + J\frac{\partial F_2}{\partial \alpha}(z) = 2J\left[\frac{1}{2}\left(\frac{\partial F_2}{\partial \alpha}(z) - J\frac{\partial F_2}{\partial \beta}(z)\right)\right](x).$$

Now $F_2(z) = \beta G(z)$, with $G = (F_2(z)/\beta)$ the stem function that induces the spherical derivative, then the last equation is equal to

$$\begin{aligned} & 2J\left[\frac{1}{2}\left(\beta\frac{\partial G}{\partial \alpha}(z) - J\beta\frac{\partial G}{\partial \beta}(z) - JG(z)\right)\right] = \\ = \quad & G(z) + 2J\beta\left(\frac{1}{2}\left(\frac{\partial G}{\partial \alpha}(z) - J\frac{\partial G}{\partial \beta}(z)\right)\right) \\ = \quad & \partial_s f(x) + 2Im(x)\left(\tfrac{\partial}{\partial x}\partial_s f\right)(x), \end{aligned}$$

where of course, in the last equality $\partial_s f$ and $\frac{\partial}{\partial x}\partial_s f$ are the slice functions induced by G and $\frac{1}{2}(\frac{\partial G}{\partial \alpha} - J\frac{\partial G}{\partial \beta})$ respectively.

At this point we have proven the theorem in the case in which the point x is not real. Now, if the function f is defined also on the real line, then, thanks to slice regularity we have, in particular, that f is of class $\mathcal{C}^\infty$. Therefore, recalling remark 1.4, we have that the spherical derivative and its slice derivative extends continuously to the real line and the proof of the theorem is concluded.

□

REMARK 3.1. Since the previous theorem holds for any $x_0 \in \Omega_D$, then , if $x_0 \in \mathbb{R}$, then we have that $\frac{\partial f}{\partial x}(x_0) = \partial_s f(x_0)$.

Thanks only to the previous formula we get the value of s_2.

COROLLARY 3.2. *Let $f \in \mathcal{SR}(\Omega_D)$ be a slice regular function with spherical expansion $f(x) = \sum_{n\in\mathbb{N}} \mathcal{S}_{y,n}(x)s_n$ centered in $x_0 \in \Omega_D$ then,*

$$s_2 = \frac{\partial}{\partial x}(\partial_s f)(x_0).$$

Of course, even if we will not use it, the last formula can be easily generalized to the case of n-th derivative, as the following corollary states.

COROLLARY 3.3. *Let $f \in \mathcal{SR}(\Omega_D)$ be a slice regular function, then we have the following formula:*

$$\left(\frac{\partial}{\partial x}\right)^{(n)} f(x) = ((A^{(n)}f))(x) + \partial_s(A)^{(n-1)}f(x), \quad \forall x = \alpha + J\beta \in \Omega_D,$$

for any $n \in \mathbb{N}$, where A denotes the operator $2Im(x)\frac{\partial}{\partial x}\partial_s$ and the apex $(A)^{(n)}$ denote the composition of the operator A with itself n times.

Proof. The proof is by induction. We yet know that $\frac{\partial f}{\partial x}(x) = 2J\beta\left(\frac{\partial}{\partial x}\partial_s f\right)(x)+\partial_s f(x)$, but then,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(x) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x)\\ &= 2J\beta\left(\frac{\partial}{\partial x}\partial_s\left(2J\beta\left(\frac{\partial}{\partial x}\partial_s f\right)(x)+\partial_s f(x)\right)\right)(x)+\\ &\quad +\partial_s\left(2J\beta\left(\frac{\partial}{\partial x}\partial_s f\right)(x)+\partial_s f(x)\right)(x), \quad \forall x \in \Omega_D,\end{aligned}$$

and since $\partial_s(\partial_s f) \equiv 0$ (see remark 1.4), we obtain the thesis for $n = 2$. With an analogous argument it is possible to complete the induction.

□

Since in this thesis we will not need other information about the spherical coefficients, we end this discussion here.

2. Rank of the real differential of a slice regular function

In [**20, 46**], the authors show the following theorem.

THEOREM 3.4. *([**46**], theorem 6.1). Let $f \in \mathcal{SR}(\Omega_D)$ and $x = \alpha + I\beta \in \Omega_D$. For all $v \in \mathbb{H}$, $||v|| = 1$, it holds*

$$\lim_{t\to 0}\frac{f(x+tv)-f(x)}{t} = vs_1 + (xv - vx^c)s_2,$$

where s_1 and s_2 are the first two coefficients of the spherical expansion of f.

The previous theorem has an important corollary (see equation 3.3 in [**20**]), that was anticipated in the introduction of this chapter.

COROLLARY 3.5. *Let $f \in \mathcal{SR}(\Omega_D)$ and let $(df)_x$ denote the real differential of f at $x = \alpha + I\beta \in \Omega_D \setminus \mathbb{R}$. If we identify $T_x\mathbb{H}$ with $\mathbb{H} = \mathbb{C}_I \oplus \mathbb{C}_I^\perp$, then for all $v_1 \in \mathbb{C}_J$ and $v_2 \in \mathbb{C}_J^\perp$,*

$$(df)_x(v_1+v_2) = v_1(s_1 + 2Im(x)s_2) + v_2 s_1 = v_1\frac{\partial f}{\partial x}(x) + v_2\partial_s f(x).$$

If $\alpha \in \Omega_D \cap \mathbb{R}$ then, the previous formula becomes the following one

$$(df)_\alpha(v) = v\frac{\partial f}{\partial x}(\alpha) = v\partial_s f(\alpha).$$

We will not give a proof of the previous theorem and corollary since the ones in [**46**] do not use the additional hypothesis of nonempty intersection between the domain and the real axis. The only feature needed for the proof is, in fact, the existence, for every slice regular function, of a spherical expansion, but this is true also if the domain of definition of f does not intersects the real line as was shown in **[33]** (see *theorem* 1.8). For the second part we only add that it can be seen as consequence of formula 18.

We now want to study the rank of a slice regular function. In **[20]** the authors prove that an injective slice regular function defined over a circular domain with real points, has invertible differential. The aim of the following pages is to extend this result to all injective slice regular functions. Let's start with a general result.

PROPOSITION 3.6. *(***[20]***, proposition 3.3). Let* $f \in \mathcal{SR}(\Omega_D)$ *and* $x_0 = \alpha + J\beta \in \Omega_D \setminus \mathbb{R}$.

- *If* $\partial_s f(x_0) = 0$ *then:*
  - df_{x_0} *has rank 2 if* $\frac{\partial f}{\partial x}(x_0) \neq 0$;
  - df_{x_0} *has rank 0 if* $\frac{\partial f}{\partial x}(x_0) = 0$.
- *If* $\partial_s f(x_0) \neq 0$ *then* df_{x_0} *is not invertible at* x_0 *if and only if* $1 + 2Im(x_0)s_2 s_1^{-1} = \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1}$ *belongs to* $\mathbb{C}_J^{\perp}$.

Let now $\alpha \in \Omega_D \cap \mathbb{R}$. df_{x_0} *is invertible at* α *if and only if its rank is not 0 at* $x_0 = \alpha + J\beta$. *This happens if and only if* $\partial_s f(x_0) = \frac{\partial f}{\partial x}(x_0) \neq 0$.

The proof of the previous statement can be found (with the appropriate change of notation), on **[20]** (proposition 3.3) or in **[25]** (proposition 8.18).

REMARK 3.2. As the previous theorem states, the rank of a slice regular function is always an even number[1].

DEFINITION 3.2. Let $f : \Omega \to \mathbb{H}$ any quaternionic function of quaternionic variable. We define the *singular set* of f as

$$N_f := \{x \in \Omega \,|\, df \text{ is not invertible at } x\}.$$

REMARK 3.3. If a slice regular function $f \in \mathcal{SR}(\Omega_D)$ is constant on a surface S, then $S \subset N_f$. This is obvious if S is in the degenerate set, but if S is not a degenerate sphere then this is true as well. If S is a semislice D_I^+ for some $I \in \mathbb{S}$, then the slice derivative of f on that semislice is everywhere zero and so $S \subset N_f$. Suppose now that S is not in the degenerate set, is not a semislice and $f|_S \equiv 0$, then $N(f) \equiv 0$ and the last translates in the system in equation 16. Deriving the first equation w.r.t. β and the second w.r.t. α we obtain, for each $z \in D$,

$$\begin{cases} g(\frac{\partial F_1}{\partial \beta}(z), F_1(z)) - g(\frac{\partial F_2}{\partial \beta}(z), F_2(z)) = 0 \\ g(\frac{\partial F_1}{\partial \alpha}(z), F_2(z)) + g(\frac{\partial F_2}{\partial \alpha}(z), F_1(z)) = 0. \end{cases}$$

If now $x_0 = \alpha_0 + I_0\beta_0 \in S$ and $z_0 = \alpha_0 + i\beta_0 \in D$, then $f(x_0) = 0$, and so, if S is not degenerate, $F_1(z_0) = -I_0 F_2(z_0)$. Evaluating the previous system in z_0 we obtain

$$\begin{cases} g(\frac{\partial F_1}{\partial \beta}(z_0), -I_0F_2(z_0)) - g(\frac{\partial F_2}{\partial \beta}(z_0), F_2(z_0)) = 0 \\ g(\frac{\partial F_1}{\partial \alpha}(z_0), F_2(z)) + g(\frac{\partial F_2}{\partial \alpha}(z_0), -I_0F_2(z_0)) = 0, \end{cases}$$

[1] This was also pointed out in **[41]**.

and, using regularity and the fact that for any $p, q, r \in \mathbb{H}$, $g(pq, r) = g(q, p^c r)$, we get,

$$\begin{cases} g(I_0(\frac{\partial F_1}{\partial \beta}(z_0) + I_0\frac{\partial F_2}{\partial \beta}(z_0)), F_2(z_0)) = \beta_0 ||\partial_s f(x_0)|| g(\frac{\partial f}{\partial x}(x_0)(\partial_s f)(x_0)^{-1}, 1) = 0 \\ g(I_0(\frac{\partial F_1}{\partial \alpha}(z_0) + I_0\frac{\partial F_2}{\partial \alpha}(z_0)), F_2(z_0)) = \beta_0 ||\partial_s f(x_0)|| g(\frac{\partial f}{\partial x}(x_0)(\partial_s f)(x_0)^{-1}, I_0) = 0, \end{cases}$$

and so for any $x_0 \in S$ we have that $x_0 \in N_f$.

The following theorem will characterize the set N_f of singular points of f. In particular, the next theorem generalizes a well known concept in real and complex analysis i.e.: the fact that if the differential of a function is singular in some point x_0, then, the function can be expanded in a neighborhood of x_0 as

$$f(x) = f(x_0) + o((x - x_0)^2).$$

THEOREM 3.7. *([**20**], proposition 3.6). Let $f \in \mathcal{SR}(\Omega_D)$ and let $x_0 = \alpha + \beta I \in \Omega_D$. Then $x_0 \in N_f$ if and only if there exists a point $\widetilde{x_0} \in \mathbb{S}_{x_0}$ and a function $g \in \mathcal{SR}(\Omega_D)$ such that the following equation hold:*

$$f(x) = f(x_0) + (x - x_0) \cdot (x - \widetilde{x_0}) \cdot g(x).$$

Equivalently, $x_0 \in N_f$ if and only if the function $f - f(x_0)$ has total multiplicity $n \geq 2$ in $\mathbb{S}_{x_0}$.

The proof of the last theorem is analogous to the one in proposition 3.6 of [**20**]. However, we will rewrite the proof in our setting with our notations. Before proving the last theorem we recall from [**20**] the following remark.

REMARK 3.4. For all $x_0 = \alpha + J\beta \in \mathbb{H} \setminus \mathbb{R}$, setting $\Psi(x) := (x - x_0)(x - x_0^c)^{-1}$ defines a stereographic projection of $\alpha + \mathbb{S}\beta$ onto the plane $\mathbb{C}_J^{\perp}$ from the point x_0^c. Indeed, if we choose $K \in \mathbb{S}$ with $K \perp J$ then for all $x = \alpha + \beta L$ with $L = tJ + uK + vJK \in \mathbb{S}$ we have $\Psi(x) = (L - J)(L + J)^{-1} = \frac{u+vJ}{1+t} JK$ and $\mathbb{C}_J \cdot K = (\mathbb{R} + \mathbb{R}J)JK = \mathbb{C}_J^{\perp}$.

We are now able to pass to the proof of the theorem.

Proof. If $x_0 \in \Omega_D \setminus \mathbb{R}$ then it belongs to D_f iff, f is constant on the sphere $\mathbb{S}_{x_0}$, i.e. there exists a slice regular function $g : \Omega_D \to \mathbb{H}$ such that

$$f(x) - f(x_0) = \Delta_{x_0}(x) g(x).$$

This happens if and only if the coefficient $s_1 = \partial_s f(x_0)$ in the spherical expansion vanishes.

Let now pass to the case $x_0 \in \Omega_D \setminus \mathbb{R}$, $x_0 \notin D_f$. Thanks to proposition 3.6, $x_0 \in N_f$ iff, $1 + 2Im(x_0)s_2 s_1^{-1} = p \in \mathbb{C}_J^{\perp}$. Thanks to remark 3.4, $p \in \mathbb{C}_J^{\perp}$ iff there exists $\widetilde{x_0} \in \mathbb{S}_{x_0} \setminus \{x_0^c\}$ such that $p = (\widetilde{x_0} - x_0)(\widetilde{x_0} - x_0^c)^{-1}$. The last formula is equivalent to

$$\begin{aligned} 2Im(x_0)s_2 s_1^{-1} &= (\widetilde{x_0} - x_0)(\widetilde{x_0} - x_0^c)^{-1} - (\widetilde{x_0} - x_0^c)(\widetilde{x_0} - x_0^c)^{-1} \\ &= (\widetilde{x_0} - x_0 - \widetilde{x_0} + x_0^c)(\widetilde{x_0} - x_0^c)^{-1} \\ &= -2Im(x_0)(\widetilde{x_0} - x_0^c)^{-1}, \end{aligned}$$

that is $s_1 = (x_0^c - \widetilde{x_0})s_2$. Writing then the first terms of the spherical expansion of f around x_0 we have:

$$\begin{aligned}
f(x) &= s_0 + (x - x_0)s_1 + \Delta_{x_0}(x)s_2 + \Delta_{x_0}(x)(x - x_0) \cdot h(x) \\
&= s_0 + (x - x_0)(x_0^c - \widetilde{x_0})s_2 + \Delta_{x_0}(x)s_2 + \Delta_{x_0}(x) \cdot (x - x_0)h(x) \\
&= s_0 + (x - x_0)(x_0^c - \widetilde{x_0})s_2 + \Delta_{x_0}(x)s_2 + (x - x_0) \cdot (x - x_0^c) \cdot (x - x_0) \cdot h(x) \\
&= s_0 + (x - x_0) \cdot [(x_0^c - \widetilde{x_0} + x - x_0^c)s_2 + \Delta_{\widetilde{x_0}}(x)h(x)] \\
&= s_0 + (x - x_0) \cdot (x - \widetilde{x_0}) \cdot [s_2 + (x - \widetilde{x_0^c}h(x))] \\
&= f(x_0) + (x - x_0) \cdot (x - \widetilde{x_0}) \cdot [s_2 + (x - \widetilde{x_0^c}h(x))],
\end{aligned}$$

for some slice regular function $h : \Omega_D \to \mathbb{H}$, where we used the following facts:

- $(x - x_0)(x_0^c - \widetilde{x_0}) = (x - x_0) \cdot (x_0^c - \widetilde{x_0})$ because the second factor is constant;
- $\Delta_{x_0}(x)(x - x_0) = \Delta_{x_0}(x) \cdot (x - x_0)$ because the first factor is a real slice function;
- $(x - x_0^c) \cdot (x - x_0) = \Delta_{x_0}(x)$;
- $\Delta_{x_0}(x) = \Delta_{\widetilde{x_0}}(x)$ because $\widetilde{x_0} \in \mathbb{S}_{x_0}$.

Finally, if $x_0 \in \Omega_D \cap \mathbb{R}$ then $s_1 = 0$ iff

$$f(x) = f(x_0) + (x - x_0)^2 \cdot l(x) = f(x_0) + (x - x_0) \cdot (x - x_0) \cdot l(x),$$

for some slice regular function $l : \Omega_D \to \mathbb{H}$.

□

For the main result we need, now, two lemmas, the first of which regards the topology of the singular set of a slice regular function, while the second regards the spherical behavior of an injective (possibly non regular) slice function.

LEMMA 3.8. *Let $f : \Omega_D \to \mathbb{H} \in \mathcal{SR}(\Omega_D)$ be non slice-constant. Then its singular set N_f has empty interior.*

Proof. Since $D_f = V(\partial_s f)$ then it is closed in Ω_D. So, since $D_f \subset N_f$, then the thesis is that $N_f \setminus D_f$ has empty interior.

Let $x_0 \in N_f \setminus D_f$ and ad absurdum let $R > 0$ be a real number such that the open Euclidean ball $B = B(x_0, R)$ centered in x_0 with radius R is fully contained in $N_f \setminus D_f$. For any $y \in B$ the spherical derivative $\partial_s f(x_0) \neq 0$ and, by the previous theorem 3.7, there exists a slice regular function $h_y : \Omega_D \to \mathbb{H}$ such that $N(f - f(y)) = \Delta_y(x)^2 h(x)$, where $N(f - f(y))$ is the normal function of $f - f(y)$. Computing the slice derivative of $N(f - f(y))$ and evaluating in $x = y$ we obtain

$$0 = \left[\frac{\partial N(f - f(y))}{\partial x}\right]_{x=y} = \left[\frac{\partial f}{\partial x} \cdot (f - f(y))^c\right]_{x=y}. \tag{19}$$

There are two cases 1) $\frac{\partial f}{\partial x}(y) = 0$ or 2) $\frac{\partial f}{\partial x}(y) \neq 0$. Case 2) implies, using formula 1.8, that

$$f\left(\frac{\partial f}{\partial x}(y)^{-1} y \frac{\partial f}{\partial x}(y)\right) = f(y).$$

Case 1) can be divided into two sub-cases: $i)$ $y = \alpha + I\beta$ is an isolated zero for the slice derivative in D_I^+ or $ii)$ $\frac{\partial f}{\partial x}_I^+ \equiv 0$. If $ii)$ holds true, then we change our point y considering another point $\omega \in B$ lying on another different semislice. Then, ω can only be an isolated zero on its semislice for the slice derivative of f (otherwise f would be slice constant). The only possibility is, therefore, case i). If we are in case $1), i)$ then we can find a positive real number r such that the two dimensional disc $\Delta = \Delta_I(x_0, r)$ is contained in $B \cap \mathbb{C}_I^+$ and, for any $x \in \Delta \setminus \{y\}$ we have $\frac{\partial f}{\partial x}(x) \neq 0$. For any $y' \in \Delta \setminus \{y\}$ we are in case 2) and, again, there are two sub cases: A) $\frac{\partial f}{\partial x}(y')^{-1}y'\frac{\partial f}{\partial x}(y') \neq y'$ or B) $\frac{\partial f}{\partial x}(y')^{-1}y'\frac{\partial f}{\partial x}(y') = y'$. If there is a point that satisfies case A), then f would be equal to some quaternion p both in y' and in $\frac{\partial f}{\partial x}(y')^{-1}y'\frac{\partial f}{\partial x}(y')$ and this would implies, using the representation theorem, that $f|_{\mathbb{S}_y} \equiv p$ that is $\mathbb{S}_y \in D_f$. So, the only possible case is, finally, B). But if condition B) holds true for any $y \in \Delta \setminus y'$, then

$$y\frac{\partial f}{\partial x}(y) = \frac{\partial f}{\partial x}(y)y,$$

and so, for any $y \in \Delta \setminus y'$, $\frac{\partial f}{\partial x}(y)$ belongs to $\mathbb{C}_I$ and so, thanks to theorem 2.2, this is true for any point in D_I^+. We claim that this is not possible. In fact, if $\alpha + I\beta = y \in B$, then

$$\frac{\partial f}{\partial x}(y)\partial_s f(y)^{-1} \in \mathbb{C}_I^{\perp}$$

and this is true if and only if

$$\frac{\partial f}{\partial x}(y)\partial_s f(y)^{-1}Im(y) = -Im(y)\frac{\partial f}{\partial x}(y)\partial_s f(y)^{-1}.$$

If, now, $\frac{\partial f}{\partial x}(y)$ belongs to $\mathbb{C}_I$ then it commutes with $Im(y)$ and so

$$\partial_s f(y)^{-1}Im(y) = -Im(y)\partial_s f(y)^{-1}$$

and $\partial_s f(y) \in \mathbb{C}_I^{\perp}$ for each $y \in D_I^+$. This implies that there exists an imaginary unit $J \in \mathbb{S}$ orthogonal to I and a function $g : D_I^+ \to \mathbb{R}$ such that, for any $y \in D_I^+$ it holds $\partial_s f(y) = \frac{1}{\beta}g(y)J$. Since the spherical derivative is independent from the imaginary unit I then it is g as well. Since $f = \mathcal{I}(F_1 + \sqrt{-1}F_2)$ is a slice regular function, then

$$\left(\frac{\partial f}{\partial x}\right)_I^+ = \frac{\partial F_2}{\partial \beta} - I\frac{\partial F_2}{\partial \alpha} = \left(\frac{\partial g}{\partial \beta} - I\frac{\partial g}{\partial \alpha}\right)J$$

and this is not possible since, as we said, the slice derivative belongs to $\mathbb{C}_I$.

□

Lemma 3.9. *Let $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$ be an injective slice function. Then for all $x = \alpha + J\beta \in \Omega_D \setminus \mathbb{R}$, $\partial_s f(x) \neq 0$.*

Proof. We know that $\partial_s f(x) = 0$ if and only if f is constant on the sphere $\mathbb{S}_x$ (see remark 1.4). But then if f is injective then $\partial_s f(x) \neq 0$ for all $x \in \Omega_D \setminus \mathbb{R}$.

□

Now we have that every injective slice regular function has real differential with rank at least equal to 2. The next step is to prove that for every injective slice regular function f the slice derivative $\frac{\partial f}{\partial x}$ is everywhere different from 0. To do that we need to introduce some tools from complex analysis. The main reference for the following is the book by Heins [**37**].

DEFINITION 3.3. Given a holomorphic function $f : D \subset \mathbb{C} \to \mathbb{C}$ we define the *multiplicity of f at a point $x \in D$* as the number:

$$n(x; f) := \inf\{k \in \mathbb{N} \setminus \{0\} \mid f^{(k)}(x) \neq 0\},$$

$f^{(k)}(x)$ denoting the k^{th} derivative of f w.r.t. z evaluated in x.

This notion of multiplicity, given for complex holomorphic functions, must not be confused with the one of total multiplicity given in chapter 1.

DEFINITION 3.4. Given a holomorphic function f defined over a region D we define the *valence* of f at $w \in \mathbb{C} \cup \{\infty\}$ as

$$v_f(w) := \begin{cases} +\infty, & \text{if the set } \{f(z) = w\} \text{ is infinite;} \\ \sum_{f(z)=w} n(z; f), & \text{otherwise.} \end{cases}$$

If f does not take the value w, then $v_f(w)$ is obviously equal to zero. It turns out that, for any $r > 0$, such that $\overline{D(x; r)} \subset D$, the valence at w of $f|_{D(x;r)}$ is constant on each component of $(\mathbb{C} \cup \{\infty\}) \setminus f(\partial D(x; r))$, where $D(x; r)$ denote the disc centered in x of radius r. Now we can pass to the quaternionic setting. We recall that any slice regular function admits a splitting into two complex holomorphic function as stated in lemma 1.4. So, let $f \in \mathcal{SR}(\Omega_D)$ and $J \perp K$ two elements of $\mathbb{S}$. Then there exists two holomorphic functions $f_1, f_2 : D_J \to \mathbb{C}_J$ such that

$$f_J = f_1 + f_2 K.$$

We can now state the following theorem.

THEOREM 3.10. *Let $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$ be an injective slice regular function. Then its slice derivative $\frac{\partial f}{\partial x}$ is always different from zero.*

Proof. What we want to prove is that, for any $x_0 = \alpha + J\beta \in \Omega_D$

$$\frac{\partial f}{\partial x}(x_0) \neq 0.$$

First of all, thanks to the identity principle 2.2 applied to the slice derivative of f, if $\partial f / \partial x$ is equal to zero in $y \in D_I^+ \subset \Omega_D$, for some $I \in \mathbb{S}$, then y is isolated in D_I^+. Since f is slice regular, for any K orthogonal to J in $\mathbb{S}$, there exist two holomorphic functions $f_1, f_2 : D_J^+ \to \mathbb{C}_J$ such that $f_J^+ = f_1 + f_2 K$. Thanks to lemma 2.1 of [**33**], we have then that

$$\frac{\partial f}{\partial x}(x_0) = \frac{\partial f_1}{\partial z} + \frac{\partial f_2}{\partial z} K,$$

and so the thesis becomes that at least one of the two derivatives $\frac{\partial f_1}{\partial z}$, $\frac{\partial f_2}{\partial z}$ is different from zero. Moreover, since f is injective, then also $f_J^+ = f|_{D_J^+}$ is injective. So, if one between f_1 and f_2 is constant, then the other one must be injective, and so we will have an injective holomorphic function and the thesis will follow trivially. Let's suppose then that both f_1 and f_2 are non-constant functions and fix the following notations:

$$n(x; f) := \inf\{k \in \mathbb{N} \setminus \{0\} \mid \frac{\partial^k f}{\partial x^k}(x) \neq 0\},$$

$$n_1(x; f) := \inf\{k \in \mathbb{N} \setminus \{0\} \mid f_1^{(k)}(x) \neq 0\},$$

$$n_2(x; f) := \inf\{k \in \mathbb{N} \setminus \{0\} \mid f_2^{(k)}(x) \neq 0\}.$$

Using again *lemma* 2.1 of [**33**], it is easy to see that, for every $x \in D_J^+$,

$$n(x; f) = \min(n_1(x; f), n_2(x; f)).$$

Moreover, since f is non-constant then the null set of its slice derivative restricted to the semislice D_J^+ is discrete. Let now $B_1 := B_1(x_0; r_1)$, $B_2 := B_2(x_0; r_2)$ be two balls such that their closure is contained in D_J^+ and f_i take the value $f_i(x_0)$ on $\overline{B_i}$ only at x_0 and such that $\frac{\partial f_i}{\partial z}(z) \neq 0$ for any $z \in B_i \setminus \{x_0\}$. Let now $B = B_1 \cap B_2$, then the valence $v_{f_i}(f_i(z))$ of $f_i|_B$ is constant and equal to $n_i(z; f)$ in the component of $(\mathbb{C}_J \cup \{\infty\}) \setminus f(\partial B)$ which contains $f_i(z)$. Since $n(x; f) = \min(n_1(x; f), n_2(x; f))$ and $n(x; f) = 1$ a.e. suppose that $\exists y \in B$ such that $1 = n(y; f) = n_1(y; f)$. Then n_1 is constant and equal to 1 in B and so we have the thesis.

□

REMARK 3.5. The proof of the previous statement works also to prove that a slice regular function $f : \Omega_D \to \mathbb{H}$ injective on a semislice $D_J^+ \subset \Omega_D$ has slice derivative nonzero over the same semislice D_J^+. We choose to formalize the theorem in the previous less general hypothesis only to simplify the reading.

THEOREM 3.11. *Let f be an injective slice regular function, then $N_f = \emptyset$.*

Proof. If, by contradiction, there exists $x_0 = \alpha + J\beta \in N_f \neq \emptyset$, then, thanks to theorem 3.7, the function $f - f(x_0)$ must have multiplicity n greater or equal to 2 in $\mathbb{S}_{x_0}$. This means that,

$$f(x) - f(x_0) = (x - x_0) \cdot g(x),$$

with $g \in \mathcal{SR}(\Omega_D)$ such that $g(x_1) = 0$ for some $x_1 \in \mathbb{S}_{x_0}$. Since f is injective, then $g(x_0) = \frac{\partial f}{\partial x}(x_0) \neq 0$ and $g(x_0^c) = \partial_s f(x_0) \neq 0$, and so $x_1 \neq x_0, x_0^c$. Now, whereas we know the values of g at x_0 and at x_0^c, we can apply the representation formula in theorem 1.1 to analyze the behavior over the sphere $\mathbb{S}_{x_0}$. The result is the following,

$$g(\alpha + I\beta) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(x_0) + \partial_s f(x_0) - IJ\left(\frac{\partial f}{\partial x}(x_0) - \partial_s f(x_0)\right)\right), \quad \forall I \in \mathbb{S}.$$

So, if there exists $I \in \mathbb{S}$ such that $g(\alpha + I\beta) = 0$, then,

$$\begin{aligned} \frac{\partial f}{\partial x}(x_0) + \partial_s f(x_0) &= IJ\left(\frac{\partial f}{\partial x}(x_0) - \partial_s f(x_0)\right) \\ \Leftrightarrow \quad \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} + 1 &= IJ\left(\frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} - 1\right) \\ \Leftrightarrow \quad \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} &= -(1-IJ)^{-1}(1+IJ), \end{aligned}$$

with $I \neq J, -J$, but then, since for $I \neq \pm J$ the product $-(1-IJ)^{-1}(1+IJ)$ has a non zero real part, $\frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1}$ does not belong to $\mathbb{C}_J^{\perp}$ and this is in contradiction with proposition 3.6.

□

EXAMPLE 3.1. Let $J \in \mathbb{S}$ be a fixed imaginary unit and $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ be the slice regular function defined at the end of the second chapter,

$$f(\alpha + I\beta) = (\alpha + I\beta)(1 - IJ).$$

We remember that this function is constructed, by means of the representation formula, to be equal to zero over the semislice $\mathbb{C}^+_{-J}$ and to be equal to $2x$ over the opposite semislice $\mathbb{C}^+_J$. What we want to show is that the restriction $f|_{\mathbb{H}\setminus(\mathbb{R}\cup\mathbb{C}^+_{-J})}$ is injective. This is trivial if we restrict the function to a semislice $\mathbb{C}^+_I$, so let $x_1 = \alpha_1 + \beta_1 I_1 \neq \alpha_2 + \beta_2 I_2 = x_2$, with $I_1 \neq I_2$, then

$$\begin{gathered} f(x_1) = f(x_2) \Leftrightarrow \\ \Leftrightarrow x_1(1 - I_1 J) = x_2(1 - I_2 J) \Leftrightarrow \\ \Leftrightarrow x_1 I_1(I_1 + J) = x_2 I_2(I_2 + J) \Leftrightarrow \\ \Leftrightarrow (x_2 I_2)^{-1}(x_1 I_1) = -\tfrac{1}{c}(I_2 + J)(I_1 + J), \end{gathered}$$

where $c = ||I_1 + J||^2 \neq 0$. Translating the variables x_1, x_2 into their components, we obtain that, the last equality is equivalent to the following one:

$$-\frac{1}{\alpha_2^2 + \beta_2^2}[-\beta_1\beta_2 + \alpha_1\beta_2 I_1 - \alpha_2\beta_1 I_2 + \alpha_1\alpha_2 I_1 I_2] = -\frac{1}{c}[I_2 I_1 + I_2 J + J I_1 - 1].$$

Now we can decompose the last equation into the system involving the real and imaginary parts as follows:

$$\begin{cases} \dfrac{c}{\alpha_2^2 + \beta_2^2}[\beta_1\beta_2 + \alpha_1\alpha_2 I_2 \cdot I_1] = 1 + I_1 \cdot I_2 + (I_1 + I_2) \cdot J \\ \\ \dfrac{c}{\alpha_2^2 + \beta_2^2}[\alpha_1\beta_2 I_1 - \alpha_2\beta_1 I_2 + \alpha_1\alpha_2 I_2 \wedge I_1] = I_2 \wedge I_1 + (I_2 - I_1) \wedge J \end{cases}$$

where $I \cdot J$ and $I \wedge J$ denote the scalar and the vector products[2] respectively in $\mathbb{R}^3$. We will work now on the second equation of the previous system. Firstly, multiplying scalarly

[2]Here we used the 'scalar-vector' notation.

the equation by $I_2 - I_1$, we obtain that

$$\alpha_1\beta_2 = -\alpha_2\beta_1.$$

Substituting $\alpha_1 = -\frac{\beta_1}{\beta_2}\alpha_2$ and multiplying scalarly by $I_1 + I_2$ it follows that

$$(I_2 \wedge I_1) \cdot J = \frac{c}{2}\frac{\alpha_1\beta_2}{\alpha_2^2 + \beta_2^2}.$$

Taking into account the previous results and multiplying scalarly by J and then by I_1 (or I_2), and supposing $\alpha_2 \neq 0$, we obtain the following two equalities:

$$(I_1 + I_2) \cdot J = -\frac{1}{2}\left[1 + c\frac{\alpha_2}{\beta_2}\frac{\alpha_2\beta_1}{\alpha_2^2 + \beta_2^2}\right], \quad I_1 \cdot I_2 = -\frac{1}{2}.$$

Putting all these ingredients in the first equation of the system one obtain that:

$$\frac{c\beta_1}{\alpha_2^2 + \beta_2^2}\left[\beta_2 + \frac{\alpha_2^2}{2\beta_2}\right] = -\frac{1}{2}\frac{c\beta_1}{\alpha_2^2 + \beta_2^2}\frac{\alpha_2^2}{\beta_2},$$

and this is possible if and only if $\beta_2^2 = -\alpha_2^2$, which is absurd. If now $\alpha_2 = 0$, following the first part of the same argument, we obtain $\alpha_1 = 0$ and so,

$$-\frac{c\beta_1}{\beta_2} = I_2I_1 + I_2J + JI_1 - 1. \tag{20}$$

But then, the imaginary part of $I_2I_1+I_2J+JI_1$, that is $I_2\wedge I_1+I_2\wedge J+J\wedge I_1$, must vanishes. This implies that $(I_2\wedge I_1)\cdot J = 0$ i.e.: $J = AI_1 + BI_2$, for some A and B real numbers both different from zero. In this case equation 20 becomes $A + B + 1 - \frac{c\beta_1}{\beta_2} = (1 + A + B)I_1I_2$ and so $I_1 \wedge I_2 = 0$. The last equalities (since $I_1 \neq I_2$), entails $I_1 = -I_2$ but this would imply $\frac{\beta_1}{\beta_2} = 0$ and this is not possible.

Since this function, with the proper restriction, is slice regular and injective then theorem 3.11 says that its real differential is always invertible. This fact could also be seen computing the slice and the spherical derivative. Indeed, since

$$\partial_s f(\alpha + I\beta) = \frac{\beta - \alpha J}{\beta},$$

is always different from zero, we need only to control that the product $\frac{\partial f}{\partial x}(\alpha+I\beta)(\partial_s f(\alpha+I\beta))^{-1}$ does not belong to $\mathbb{C}_I^{\perp}$. Now,

$$\frac{\partial f}{\partial x}(\alpha + I\beta)(\partial_s f(\alpha + I\beta))^{-1} = (1 - IJ)\left(\frac{\beta - \alpha J}{\beta}\right)^{-1} = \frac{\beta(1 - IJ)(\beta + \alpha J)}{\beta^2 + \alpha^2},$$

and so, whenever $I \neq -J$, the previous product has a nonzero real part and so does not belong to $\mathbb{C}_I^{\perp}$.

Remark 3.6. The reader could ask why we didn't follow the way of proving theorem 3.11 by Gentili, Salamon and Stoppato in **[20]**. The answer is that that proof does not work in the case in which the domain of the function does not have real points. This fact, rather than being a mere observation, gives space to interesting considerations that are not studied in this thesis. To be precise, the theorem that fails is the following:

Theorem 3.12. *([20], theorem 3.9) Let $f : \Omega_D \to \mathbb{H}$ be a non-constant regular function, and let $\Omega_D \cap \mathbb{R} \neq \emptyset$. For each $x_0 = \alpha + I\beta \in N_f$, there exists a $n > 1$, a neighborhood U of x_0 and a neighborhood T of $\mathbb{S}_{x_0}$ such that for all $x_1 \in U$, the sum of the total multiplicities of the zeros of $f - f(x_1)$ in T equals n.*

A counter example, if the domain does not have real points, is given by the function,

$$\begin{gathered} f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H} \\ \alpha + I\beta \mapsto (\alpha + I\beta)(1 - IJ), \end{gathered}$$

for a fixed $J \in \mathbb{S}$. As we have seen, this function is injective over $\mathbb{H} \setminus (\mathbb{R} \cup \mathbb{C}^+_{-J})$, and so, if we take $x_0 = -J \in N_f$, for any neighborhood U of $-J$ and any neighborhood T of $\mathbb{S}_{-J}$ the sum of total multiplicities of the zeros of $f - f(x_1)$, for any $x_1 \in U \setminus \mathbb{C}^+_{-J}$ is equal to 1. The previous function is constructed to be equal to 0 over $\mathbb{C}^+_{-J}$ and equal to $2x$ over $\mathbb{C}^+_J$, but other more complex examples can be build in this way, for example considering a function equal to some monomial x^m on a semislice and equal to another different monomial x^n on the opposite. This fact was already pointed out in remark 1.22 where was said that this feature will certainly be a starting point for future investigations.

CHAPTER 4

Applications to differential geometry

In this final chapter we will explore an interesting application of the theory of slice regular functions to some aspects of the complex geometry of $\mathbb{R}^4$. We will start with some basic definitions (for general references see book [**38**]).

Let (Ω^{2n}, g) be a $2n$-dimensional oriented Riemannian manifold. An *almost complex structure* over Ω is an endomorphism $J : T\Omega \to T\Omega$, defined over the tangent bundle such that $J^2 = -id$. An almost complex structure is said to be a *complex structure* if J is *integrable*, meaning, for instance, that the associate Nijenhuis tensor

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY],$$

vanishes everywhere for each couple of tangent vectors X and Y; it is said to be *orthogonal* if it preserves the Euclidean product, i.e. $g(JX,JY) = g(X,Y)$ for each couple of tangent vectors X and Y and preserves the orientation of Ω. Collecting everything, an *orthogonal complex structure (OCS)* is an almost complex structure which is integrable and orthogonal.

Of course, if J is an OCS w.r.t. some Riemannian metric g, then if we change g with a conformal equivalent metric h, J remains an OCS w.r.t. h as well. If Ω is a four dimensional open subset of $\mathbb{R}^4$, then the resulting theory is invariant by the group of conformal automorphisms of $\mathbb{H} \cup \{\infty\} \simeq \mathbb{S}^4$.

For an open subset Ω of $\mathbb{R}^4$ endowed with the standard Euclidean metric it is possible to construct standard OCSes, called *constant*, in the following way: think $\mathbb{R}^4$ as the space of real quaternions $\mathbb{H}$, then fix any element $q \in \mathbb{H}$ such that $q^2 = -1$. Identifying each tangent space $T_p\Omega$ with $\mathbb{H}$ himself, we define the complex structure everywhere by left multiplication by q, i.e. $J_p v = qv$. A standard example is the OCS defined everywhere as the left multiplication by i. Any OCS defined globally on $\mathbb{H}$ is known to be constant (see proposition 6.6 in [**49**]), moreover it was proven in [**43**] the following result.

THEOREM 4.1. *Let J be an OCS of class $\mathcal{C}^1$ on $\mathbb{R}^4 \setminus \Lambda$, where Λ is a closed set of zero 1-dimensional Hausdorff measure. Then either J is constant or J can be maximally extended to the complement of a point $\mathbb{R}^4 \setminus \{p\}$. In both cases, J is the push-forward of the standard OCS on $\mathbb{R}^4$ under a conformal transformation.*

In the same paper it was proven the following result which completely solve the situation in a very particular case.

THEOREM 4.2. *Let $\mathbb{J}$ be an OCS of class $\mathcal{C}^1$ on $\mathbb{R}^4 \setminus \Lambda$, where Λ is a round circle or a straight line, and assume that $\mathbb{J}$ is not conformally equivalent to a constant OCS. Then J is unique up to sign, and $\mathbb{R}^4 \setminus \Lambda$ is a maximal domain of definition for J.*

In this context it is possible to construct explicitly the OCS $\mathbb{J}$ as follows. As we already saw many times in this thesis, a point x in $\mathbb{H}$ can be written as $x = x_0 + x_1 i + x_2 j + x_3 k$ (with the usual multiplication rules of quaternions), or, if $x \in \mathbb{H} \setminus \mathbb{R}$, as $x = \alpha + I_x \beta$, where $\alpha = x_0$, $I_x = (x_1 i + x_2 j + x_3 k)/\sqrt{x_1^2 + x_2^2 + x_3^2}$ and $\beta = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then, for each $x = \alpha + I_x \beta \in \mathbb{H} \setminus \mathbb{R}$ we define $\mathbb{J}$ such as $\mathbb{J}_x v = I_x v$, for each $v \in T_x(\mathbb{H} \setminus \mathbb{R})$. Since the last is an OCS over $\mathbb{H} \setminus \mathbb{R}$, then $\mathbb{J}$ and $-\mathbb{J}$ are the only non-constant OCSes on this manifold (up to conformal transformations).

In **[20]** the authors proposed a new way to study the problem when Λ is a closed set in $\mathbb{R}^4$ of different type. The idea is to take the OCS previously defined $\mathbb{J}$ and to push it forward on the set we are interested in. To do this we need to be sure that the function f, considered to push forward, preserves the properties of $\mathbb{J}$. Well, if the function f is a slice regular function, then this is true.

What we are going to do now is to restore the theoretical work of **[20]** in our setting of slice regular functions on domains without real points. Then we will try to explore this construction from another point of view: we will show some results regarding the quality of the OCSes that can be reached by our construction. This will be done thanks to the twistorial interpretation of the theory, saying that any slice (regular) function $f : \Omega_D \to \mathbb{H}$ lifts to a (holomorphic) curve in the twistor space $\mathbb{CP}^3$ of $\mathbb{H} \cup \{\infty\} \simeq \mathbb{S}^4$. The complex projective space $\mathbb{CP}^3$ is in fact the twistor space of $(\mathbb{S}^4, g_{rnd})$ (i.e.: the total space of a bundle parametrizing orthogonal almost complex structures on $\mathbb{S}^4$), and we let $\pi : \mathbb{CP}^3 \to \mathbb{S}^4$ denotes the twistor projection with fiber $\mathbb{CP}^1$. We already know (see e.g.: **[43]**), that complex hypersurfaces in $\mathbb{CP}^3$ produce OCSes on subdomains of $\mathbb{S}^4$, wherever such a hypersurface is a single-valued graph. Conversely, any OCS J on a domain Ω generates a holomorphic hypersurface in $\mathbb{CP}^3$. Starting from that we will explore in more details this relation between these two theories.

At the end we will study a very particular case that fit very well in our theory and that will give a constructive result regarding the existence of a biholomorphism between a four dimensional open half space endowed with a constant OCS and a dense subset of $\mathbb{R}^4$ with a non-constant OCS.

1. Twistor lift

We can start the whole story thanks to theorems 2.14 and 3.11 that extend, as said, two results proved, respectively in [**22**] and in [**20**].

In particular, thanks to theorem 3.11 it is possible to define the push forward of the following OCS defined over $\mathbb{H}\setminus\mathbb{R}$.

DEFINITION 4.1. Let $p=\alpha+I_p\beta\in X=\mathbb{H}\setminus\mathbb{R}$. We define the following OCS over X:

$$\mathbb{J}_p v=\frac{Im(p)}{||Im(p)||}v=I_p v,$$

where v is a tangent vector of X in p, and we are identifying $T_pX\simeq\mathbb{H}$ and I_pv denotes the quaternionic multiplication between I_p and v.

Given an injective slice regular function $f:\Omega_D\to\mathbb{H}$ we define the pushforward of $\mathbb{J}$ via f on $f(\Omega_D\setminus\mathbb{R})$ as:

$$\mathbb{J}^f:=(df)\mathbb{J}(df)^{-1},$$

for any $v\in T_{f(p)}f(\Omega_D\setminus\mathbb{R})\simeq\mathbb{H}$.

The following theorem explains the action of the push-forward of $\mathbb{J}$ via a slice regular function.

THEOREM 4.3. *Let $f:\Omega_D\to\mathbb{H}$ be an injective slice regular function and $p=\alpha+I_p\beta\in\Omega_D$. Then*

$$\mathbb{J}^f_{f(p)}v=\frac{Im(p)}{||Im(p)||}v=I_pv.$$

Moreover $\mathbb{J}^f$ is an OCS on the image of f.

Proof. The theorem can be proved as in [**20**], but we will write again the proof using the representation in theorem 3.5 of the real differential of a slice regular function. The thesis follows thanks to the next computations. Let v be a tangent vector to $f(\Omega_D\setminus\mathbb{R})$ in $f(x)$

$$\mathbb{J}^f_{f(x)}v=(df)_x\mathbb{J}_x(df)^{-1}_{f(x)}v=\circledast.$$

Putting $(df)^{-1}_{f(x)}v=w$ and denoting by $w_\top$ and $w_\perp$, respectively, the tangential and orthogonal part of w w.r.t. $\mathbb{C}_{I_x}$, we obtain,

$$\begin{aligned}\circledast &= (df)_x\mathbb{J}_xw=(df)_xI_xw\\ &= I_xw_\top\frac{\partial f}{\partial x}(x)+I_xw_\perp\partial_sf(x)\\ &= I_x(df)_xw=I_xv.\end{aligned}$$

For the second part of the theorem we refer again in [**20**].

□

At this point one could ask if it is possible to construct a twistor theory also for slice regular functions that do not extend to the real line. To be more clear, if $\Omega_D\cap\mathbb{R}=\emptyset$, then *is it possible to construct its twistor lift as explained in theorem 5.3 of* [**20**]*?* Well, the answer is yes and it is explained in the next pages.

First of all we need to introduce coordinates for the sphere $\mathbb{S}$ of imaginary units. For this purpose we will follow the construction in section 4 of **[20]**. For any complex number u we define the following quaternion $Q_u := 1 + uj$. Let, now, ϕ be the following application:

$$\begin{array}{rccc} \phi: & \mathbb{C}\times\mathbb{C}^+ & \to & \mathbb{H} \\ & (u,v) & \mapsto & Q_u^{-1} v Q_u \end{array}$$

By direct computation it is clear that $\phi(u, \alpha + J\beta) \in \alpha + \mathbb{S}\beta$ and so, for any $J \in \mathbb{S}$, the number $\phi(u, J)$ belongs to $\mathbb{S}$ as well. Fix now J to be equal to i, then, for each $q \in \mathbb{H} \setminus \mathbb{R}$, there exists a unique couple $(u, v) \in \mathbb{C} \times \mathbb{C}^+$ such that $q = \phi(u, v)$. In particular, if $q = \alpha + I\beta$, $\beta > 0$, then,

$$\begin{cases} v &= \alpha + i\beta \\ u &= -i\frac{b+ic}{1+a}, \end{cases}$$

where $I = ai + bj + ck$. And, finally, we obtain the following representation,

$$\alpha + I\beta = \phi(u, v) = Q_u^{-1} v Q_u = \alpha + Q_u^{-1} i Q_u \beta.$$

At this point we can pass to reintroduce the twistorial interpretation of the quaternionic analysis. Let $\mathbb{HP}^1$ be the left quaternionic projective line, i.e. the set of equivalence classes $[q_1, q_2] = [pq_1, pq_2]$, for every $p \in \mathbb{H}^*$. We embed $\mathbb{H}$ in $\mathbb{HP}^1$, sending a quaternion q in the affine line $[1, q]$. This definition of quaternionic projective line enable us to define the twistor projection

$$\begin{array}{rccc} \pi: & \mathbb{CP}^3 & \to & \mathbb{HP}^1 \\ & [X_0, X_1, X_2, X_3] & \mapsto & [X_0 + X_1 j, X_2 + X_3 j]. \end{array}$$

This map is well defined, in fact, if we change the representative (X_0, X_1, X_2, X_3) in $\mathbb{C}^4\setminus\{0\}$, this will not affect the image in the quaternionic projective line. Moreover, the embedding of $\mathbb{H} \setminus \mathbb{R}$ in $\mathbb{HP}^1$ via the function $q \to [1, q]$, can be viewed, also, in the following way:

$$\begin{aligned} [1, q] &= [1, Q_u^{-1} v Q_u] = [Q_u, vQ_u] \\ &= [1 + uj, v + vuj] = \pi[1, u, v, uv], \end{aligned}$$

and so, we have obtained, as in **[20]**, the following proposition.

Proposition 4.4. *The complex manifold* $(\mathbb{H}\setminus\mathbb{R}, \mathbb{J})$ *is biholomorphic to the open subset* $\mathcal{Q}^+$ *of the quadric*

$$\mathcal{Q} = \{[X_0, X_1, X_2, X_3] \in \mathbb{CP}^3 \mid X_0X_3 = X_1X_2\}, \tag{21}$$

such that at least one of the following conditions is satisfied:

- $X_0 \neq 0$ *and* $X_2/X_0 \in \mathbb{C}^+$,
- $X_1 \neq 0$ *and* $X_3/X_1 \in \mathbb{C}^+$.

Now we have all the ingredients to state the following theorem which generalizes theorem 5.3 of **[20]**.

Theorem 4.5. *Let* D *be a domain of* $\mathbb{C}$ *and* $\Omega_D \subset \mathbb{H}$ *its circularization. Let* $f : \Omega_D \to \mathbb{H}$ *be a continuous slice function. Then* f *admits a twistor lift to* $\mathcal{O} = \pi^{-1}(\Omega_D \setminus \mathbb{R}) \cap \mathcal{Q}^+$, *i.e.: there exist a continuous function* $\tilde{f} : \mathcal{O} \to \mathbb{CP}^3$, *such that* $\pi \circ \tilde{f} = f \circ \pi$. *Moreover* f *is slice regular if and only if* $\tilde{f}$ *is a holomorphic map.*

As we said, this theorem was already proven in [**20**], when the domain D has nonempty intersection with the real line and the function f is regular. Our proof contemplate also the case in which f does not extends to the real line and it is not regular, so it is more general. To add this extension we will use the previous described formalism of stem functions to which we add this trivial lemma that is a consequence of lemma 6.11 of [**29**].

LEMMA 4.6. *Let $f = \mathcal{I}(F) : \Omega_D \to \mathbb{H}$ be a slice function induced by the stem function $F : D \to \mathbb{H}_{\mathbb{C}}$. Then, for each couple $I, J \in \mathbb{S}$ such that $I \perp J$, there exist two stem functions $F^{\top}, F^{\perp} : D \to \mathbb{C}_I \otimes_{\mathbb{R}} \mathbb{C}$, such that $f = f^{\top} + f^{\perp} J$ with $f^{\top} = \mathcal{I}(F^{\top})$, while $f^{\perp} = \mathcal{I}(F^{\perp})$.*

Now we pass to the proof of theorem 4.5.

Proof. Since f is a slice function, then it is induced by a stem function $F : D \to \mathbb{H}_{\mathbb{C}}$ such that, for $q = \alpha + I\beta \in \Omega_D$,

$$f(q) = f(\alpha + I\beta) = f(\alpha + Q_u^{-1} i Q_u \beta) = F_1(\alpha + i\beta) + Q_u^{-1} i Q_u F_2(\alpha + i\beta).$$

Thanks to the previous lemma f can be written also as $f = f^{\top} + f^{\perp} j$, with $f^{\top} = \mathcal{I}(F^{\top})$, $f^{\perp} = \mathcal{I}(F^{\perp})$, $F^{\top}, F^{\perp} : D \to \mathbb{C}_i \otimes_{\mathbb{R}} \mathbb{C}$. Now, each stem function splits into two components, $F^{\top} = F_1^{\top} + \sqrt{-1} F_2^{\top}$ and $F^{\perp} = F_1^{\perp} + \sqrt{-1} F_2^{\perp}$, and we define, for $i \in \mathbb{S}$, $F_i^{\top} = p_i \circ F^{\top}$ and $F_i^{\perp} = p_i \circ F^{\perp}$, where p_i is the map that sends $\sqrt{-1}$ to i (e.g.: if $w = x + \sqrt{-1} y \in \mathbb{H}_{\mathbb{C}}$, then $p_i(w) = x + iy$). To resume we have the following diagram

$$\begin{tikzcd} D \arrow[r, "F^{\top}, F^{\perp}"] \arrow[dr, "F_i^{\top}, F_i^{\top}"'] & \mathbb{C}_i \otimes_{\mathbb{R}} \mathbb{C} \arrow[d, "p_i"] \\ & \mathbb{C}_i \end{tikzcd}$$

Letting finally $q = \alpha + I\beta$ and $v = \alpha + i\beta$ and remembering that $Q_u = 1 + uj$, we can compute,

$$\begin{aligned} [1, f(q)] &= [1, f(Q_u^{-1}(\alpha + i\beta) Q_u)] \\ &= [1, f(\alpha + Q_u^{-1} i Q_u \beta)] \\ &= [1, F_1^{\top}(v) + Q_u^{-1} i Q_u F_2^{\top}(v) + F_1^{\perp}(v) j + Q_u^{-1} i Q_u F_2^{\perp}(v) j] \\ &= [Q_u, F_1^{\top} + uj F_1^{\top} + i F_2^{\top} + uij F_2^{\top} + F_1^{\perp} j + uj F_1^{\perp} j + i F_2^{\perp} j + uij F_2^{\perp} j] = \circledast, \end{aligned}$$

where in the last equality we have omitted the variable v. Now, for any $w \in \mathbb{C}_i$, we have that $jw = w^c j$ and $jwj = -w^c$ and so, identifying $\mathbb{C}_i$ with $\mathbb{C}$,

$$\begin{aligned} \circledast &= [Q_u, F_1^{\top} + u F_1^{\top c} j + i F_2^{\top} + ui F_2^{\top c} j + F_1^{\perp} j - u F_1^{\perp c} + i F_2^{\perp} j - ui F_2^{\perp c}] \\ &= [Q_u, F_1^{\top} + i F_2^{\top} + (F_1^{\perp} + i F_2^{\perp}) j + u((F_1^{\top c} + i F_2^{\top c}) j - (F_1^{\perp c} + i F_2^{\perp c}))] \end{aligned}$$

We finally obtain the coordinates of the lift:

$$\tilde{f}[1, u, v, uv] = [1, u, p_i \circ F^{\top}(v) - u(p_i \circ F^{\perp c}(v)), p_i \circ F^{\perp}(v) + u(p_i \circ F^{\top c}(v))]. \tag{22}$$

But now, remembering that $F^{\top}, F^{\perp}$ are holomorphic stem functions, then, we have that f is slice regular if and only if $\tilde{f}$ is a holomorphic map.

□

REMARK 4.1. Starting with a regular slice function f, one can repeat the computations in the following way

$$\begin{array}{rcl}[1, f(q)] & = & [1, f(Q_u^{-1}(\alpha + i\beta)Q_u)] \\ & = & [1, f(\alpha + Q_u^{-1}(i)Q_u\beta)] \\ & = & [1, F_1(\alpha + i\beta) + Q_u^{-1}(i)Q_u F_2(\alpha + i\beta)] \\ & = & [Q_u, Q_u F_1(\alpha + i\beta) + iQ_u F_2(\alpha + i\beta)] \\ & = & [1 + uj, (1 + uj)F_1(\alpha + i\beta) + i(1 + uj)F_2(\alpha + i\beta)] \\ & = & [1 + uj, f(\alpha + i\beta) + ujf(\alpha - i\beta)] \\ & = & [1 + uj, f(v) + ujf(\bar{v})] = \circledast.\end{array}$$

At this point, using the splitting in lemma 1.4, we can write $f_i(v) = g(v) + h(v)j$, where $g, h : D_i \to \mathbb{C}_i$ are holomorphic functions. Denoting by $\hat{g}(v) = \overline{g(\bar{v})}$, $\hat{h}(v) = \overline{h(\bar{v})}$, we obtain

$$ujf(\alpha - i\beta) = u(\hat{g}(v)j - \hat{h}(v)),$$

and then

$$\begin{array}{rcl}\circledast & = & [1 + uj, g(v) + h(v)j - u\hat{h}(v) + u\hat{g}(v)j] \\ & = & \pi[1, u, g(v) - u\hat{h}(v), h(v) + u\hat{g}(v)],\end{array} \tag{23}$$

and so the lift coincide with the one computed in [**20**]

REMARK 4.2. It will be useful to notice that the twistor lift of a slice regular function is always a rational map over its image.

Thanks only to the general shape of the lift given in equation 22, we are able to prove the following result. Given a slice regular function f we will say that its twistor lift $\tilde{f}$ lies on a certain variety $\mathcal{S}$ if the image of $\tilde{f}$ is contained in $\mathcal{S}$.

THEOREM 4.7. *Let $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ be a slice regular function. Then its twistor lift lies over the quadric in equation 21 if and only if f is a real slice function.*

Proof. Since the parametrization of the lift $\tilde{f}$ is given by equation 22, then the condition of lying on the quadric 21 is encoded by the following system of equations

$$\begin{cases} p_i \circ F^{\perp c} = 0 = p_i \circ F^{\perp} & (24) \\ p_i \circ F^{\top c} = p_i \circ F^{\top}, & (25) \end{cases}$$

and so the slice regular function f with lifting equal to $\tilde{f}$ can be constructed, thanks to equation 24, to be equal to,

$$f(\alpha + I\beta) = f^{\top}(\alpha + I\beta) = F_1^{\top}(\alpha + i\beta) + IF_2^{\top}(\alpha + i\beta).$$

But, thanks to equation 25 we have that

$$F_1^{\top}(\alpha + i\beta) + IF_2^{\top}(\alpha + i\beta) = F_1^{\top c}(\alpha + i\beta) + IF_2^{\top c}(\alpha + i\beta),$$

which implies that both $F_{1,2}^{\top}$ are real functions and so f is real.

The converse is trivial.

□

Of course it is interesting in the context of OCSes to classify projective hypersurfaces in $\mathbb{CP}^3$ under the action of the conformal group of $\mathbb{S}^4$ acting on $\mathbb{CP}^3$ as a subgroup of the holomorphic automorphism group $PGL(4,\mathbb{C})$ (this is explained in section 2.7 of [**43**]). But such a classification was already given for quadric surfaces in [**43**] as follows.

THEOREM 4.8. ***(Salamon-Viaclovsky)*** *Any non-singular quadric hypersurface in $\mathbb{CP}^3$ is equivalent under the action of the conformal group of $\mathbb{S}^4$ to the zero set of*

$$e^{\lambda+i\nu}X_0^2+e^{-\lambda+i\nu}X_1^2+e^{\mu-i\nu}X_2^2+e^{-\mu-i\nu}X_3^2, \tag{26}$$

or the zero set of

$$i(X_0^2+X_1^2)+k(X_1X_3-X_0X_2)+X_1X_2-X_0X_3, \tag{27}$$

where in the first case a couple of parameters (λ,μ,ν), (λ',μ',ν') define two quadrics in the same equivalence class if and only if (λ,μ,ν) and (λ',μ',ν') belong to the same orbit under the group Γ of transformation of $\mathbb{R}^3$ generated by the four maps

$$\begin{cases}(\lambda,\mu,\nu)\mapsto(\lambda,\mu,\nu+\frac{\pi}{2})\\ (\lambda,\mu,\nu)\mapsto(-\lambda,\mu,\nu)\\ (\lambda,\mu,\nu)\mapsto(\lambda,-\mu,\nu)\\ (\lambda,\mu,\nu)\mapsto(\mu,\lambda,-\nu),\end{cases}$$

while $k\in[0,1)$ is a complete invariant in the second case.

With this result the authors of [**43**] where able to describe the geometry of non-singular quadric surfaces under the twistor projection π. Defining the *discriminant locus* of a hypersurface $\mathcal{S}$ of degree d as the set of point D in $\mathbb{S}^4$ such that $\pi^{-1}(p)\cap\mathcal{S}$ has cardinality different from d then the following theorem describe the possible cases that can occur for $d=2$.

THEOREM 4.9. ***(Salamon-Viaclovsky)*** *For any non-degenerate quadric $\mathcal{Q}$ there are three possibilities.*

(1) $\mathcal{Q}$ is a real quadric with discriminant locus a circle in $\mathbb{S}^4$ and $\mathcal{Q}$ contains all the twistor lines over the circle.
(2) $\mathcal{Q}$ contains exactly one or exactly two twistor lines. In these cases the discriminant locus is a singular torus pinched at one or two points, respectively.
(3) $\mathcal{Q}$ does not contain any twistor lines. In this case the discriminant locus is a torus $\mathbb{T}^2\subset\mathbb{S}^4$ with a smooth unknotted embedding.

Moreover if $\mathcal{Q}$ is the zero locus of the polynomial in (26) with $0\le\lambda\le\mu$ and $0\le\nu<\pi/2$, then

(1) $\mathcal{Q}$ contains a family of twistor lines over a circle if and only if $\lambda=\mu=\nu=0$,
(2) $\mathcal{Q}$ contains exactly two twistor lines if and only if $\lambda=\mu\neq0$ and $\nu=\pi/2$,
(3) $\mathcal{Q}$ contains no twistor lines in the other cases.

Finally if $\mathcal{Q}$ is the zero locus of the polynomial in 27 with $k\in[0,1)$, then the corresponding quadric $\mathcal{Q}$ contains exactly one twistor line.

Now, the next result states that every non-singular quadric in the previous classification can be reached by the lift of a slice regular function. Before stating it and proving it we just remark that, given the Representation formula for slice functions, exhibiting a slice function is equivalent to exhibit its defining stem function or its splitting over a complex plane $\mathbb{C}_I$ for some $I \in \mathbb{S}$. In fact, in the next proofs we will construct g and h starting from equation 23.

THEOREM 4.10. *For any non-singular quadric in the classification of theorem 4.9 there is a equivalent one $\mathcal{Q}$ such that there exists a slice regular function f defined on a dense subset of $\mathbb{H} \setminus \mathbb{R}$, such that its twistor lift lies in $\mathcal{Q}$.*

Proof. For all the cases we will show the thesis exhibiting the splitting of f.

(1) If $\mathcal{Q}$ is given as in equation 26, then it translates in set of solutions of
$$e^{\lambda+i\nu} + e^{-\lambda+i\nu}u^2 + e^{\mu-i\nu}(g(v) - u\hat{h}(v))^2 + e^{-\mu-i\nu}(h(v) + u\hat{g}(v))^2 = 0.$$
Writing the previous equation as a polynomial in u and imposing the vanishing of the coefficients we obtain the following system
$$\begin{cases} e^{\lambda+i\nu} + e^{\mu-i\nu}g^2 + e^{-\mu-i\nu}h^2 = 0 \\ -e^{\mu}g\hat{h} + e^{-\mu}h\hat{g} = 0 \\ e^{-\lambda+i\nu} + e^{\mu-i\nu}\hat{h}^2 + e^{-\mu-i\nu}\hat{g}^2 = 0 \end{cases}$$
From the first and the last equations we obtain
$$h^2 = -e^{\mu+i\nu}(e^{\lambda+i\nu} + e^{\mu-i\nu}g^2), \quad \hat{h}^2 = -e^{-\mu+i\nu}(e^{-\lambda+i\nu} + e^{-\mu-i\nu}\hat{g}^2).$$
Take now the square of second equation e substitute the values of h^2 and $\hat{h}^2$:
$$e^{\mu}g^2(e^{-\lambda+i\nu} + e^{\mu-i\nu}\hat{g}^2) = e^{-\mu}\hat{g}^2(e^{\lambda+i\nu} + e^{\mu-i\nu}g^2),$$
that is
$$\hat{g} = \pm e^{\mu-\nu}g.$$
Taking now, for instance, $g(v) = v$, $\hat{g}(v) = e^{\mu-\nu}v$, $h = i(e^{\mu+i\nu}(e^{\lambda+i\nu} + e^{\mu-i\nu}g^2))^{1/2}$ and $\hat{h} = i(e^{-\mu+i\nu}(e^{-\lambda+i\nu} + e^{-\mu-i\nu}\hat{g}^2))^{1/2}$, we get the thesis in the first case.

(2) The last case is when $\mathcal{Q}$ is the zero locus of the polynomial in 27 with $k \in [0, 1)$. Imposing then the usual equations we obtain that $g, h : \mathbb{C}_i \setminus \mathbb{R} \to \mathbb{C}$ and $\hat{g}, \hat{h} : \mathbb{C}_i^- \setminus \mathbb{R} \to \mathbb{C}$ can be chosen as
$$g(v) = -\hat{g}(v) = v, \quad h(v) = 2i + v/2, \quad \hat{h}(v) = 2i - v/2.$$

It is now a matter of computation, using the Representation formula, to write the slice regular functions defined by the previous three cases.

□

REMARK 4.3. In the next section we will compute the points in $\mathbb{S}^4$ where the eventual twistor lines lie.

In the next theorem we will show that the result in theorem 4.7 exhausts the set of non-singular algebraic surfaces (up to projective transformations) of degree 2, that can be reached by the twistor lift of a slice regular function. Some suspects that a result of this kind must hold came from the fact that there aren't dominant[1] rational maps from $\mathcal{Q}$ to any smooth varieties of degree $d \geq 4$. In fact, any smooth quadric in $\mathbb{CP}^3$ is projectively isomorphic to $\mathcal{Q}$ (see, for instance, section 4 of [**34**]). Now, if $X \to Y$ is a dominant rational map between non-singular varieties in $\mathbb{CP}^3$, then $dimH^0(Y, K_Y) \leq dimH^0(X, K_X)$, where K_X and K_Y stands for the canonical bundle of the subscript variety (see chapter 2, Section 8 of [**35**]). But $dimH^0(\mathcal{S}, K_{\mathcal{S}})$ is greater or equal to 1 when the degree of $\mathcal{S}$ is greater or equal to 4 and it is 0 when $d = 2, 3$.

Anyway the specific statement and proof follow.

THEOREM 4.11. *Let $\mathcal{S}$ be a non-singular algebraic surface of degree $d \geq 2$ in $\mathbb{CP}^3$ and let $\tilde{f} : \mathcal{Q}^+ \to \mathcal{S}$ the twistor lift of a slice regular function and such that $\tilde{f}(\mathcal{Q}^+)$ is open in $\mathcal{S}$. Then $\mathcal{S}$ is projectively equivalent to $\mathcal{Q}$.*

Proof. Observe that for each fixed v_0 in $\mathbb{CP}^1$, the twistor lift $\tilde{f}$ of a generic slice regular function f, contains the whole line $l_{v_0} : \mathbb{CP}^1 \to \mathbb{CP}^3$ parametrized by $u \in \mathbb{CP}^1$. In formula

$$l_{v_0}[1, u] = [1, u, f^{\top}(v_0) - uf^{\perp c}(v_0), f^{\perp}(v_0) + uf^{\top c}(v_0)].$$

This is enough to prove the theorem since, from general facts about projective surfaces, we know that the number of lines over a non-singular surface of degree greater or equal to 3 in $\mathbb{CP}^3$ is always finite[2].

□

REMARK 4.4. The theory of lines or, in general, of rational curves over a surface is a very interesting and studied field. In particular we point out that several further information are stated about the nature of rational curves that can lie over a surface. Among the others we found theorem 1.1 in [**7**] and theorem 1 in [**50**], in which the authors state general formulas that implies that surfaces of degree greater or equal to 5 contain no lines. For the case in which the degree is equal to 3 we refer to [**17**] in which there is a summary of the whole story concerning the 27 lines over a cubic surface, while for degree equal to four we cite the classical paper [**44**] by Segre in which it is stated a upper bound on the number of lines over a quartic surface.

REMARK 4.5. The case studied in [**20**] gave rise to a quartic ruled surface and so it is coherent with our last result.

After the last result one can search for classes of singular varieties that can be reached by the twistor lift of a slice regular function. Of course, since the argument of the proof is general, one can exclude from this classification all the surfaces which are not ruled by lines. And so, we obtain the following theorems.

[1]Meaning a rational map with dense image.

[2] We would like to thanks Prof. Edoardo Ballico for the useful discussion about classical algebraic geometry.

THEOREM 4.12. *Up to projective transformation any quadric surface $\mathcal{Q} \subset \mathbb{CP}^3$ is such that there exists a slice regular function f such that its twistor lift $\tilde{f}$ lies on $\mathcal{Q}$.*

In the proof of this theorem, we will choose a particular union of two planes and a particular cone. Since the classification is projective this is enough to complete all the possible cases. If one is interested in singular quadric surfaces defined by different equations it may be possible to find no slice regular function whose lift realizes the chosen equation.

Proof. The smooth case is solved thanks to theorem 4.7 and by the fact that all non-singular quadric are projectively equivalent. Up to projective transformations there are only two classes of singular quadric surfaces: the unions of two planes and cones. We will show that there is a cone and a union of two planes that can be described with coordinates in accordance with equation 23.

(1) Let $\mathcal{P}$ be the union of two planes defined by the following equation
$$X_0^2 - X_2^2 = 0.$$
The slice regular function $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ defined by $f(\alpha + I\beta) = (\alpha + I\beta)(1 - Ii)\frac{j}{2}$ lifts as $\tilde{f}[1, u, v, uv] = [1, u, 1, v]$ and so lies in $\mathcal{P}$.

(2) Let $\mathcal{K}$ be the quadratic cone defined by the following equation
$$X_1^2 = X_2 X_3$$
Imposing then the usual equations we obtain that $g, h : \mathbb{C}_i \setminus \mathbb{R} \to \mathbb{C}_i$ can be chosen as
$$g(v) = \begin{cases} 0 & \text{if } v \in \mathbb{C}_i^+ \\ v & \text{if } v \in \mathbb{C}_i^-. \end{cases}, \quad h(v) = \begin{cases} 0 & \text{if } v \in \mathbb{C}_i^+ \\ -\frac{1}{v} & \text{if } v \in \mathbb{C}_i^-. \end{cases}$$

As before, it is now a matter of computation, using the Representation formula, to write the slice regular functions defined by the previous equation.

□

We will treat now the case of cubics surfaces. Firstly we will consider *non-normal* cubics and then cones. An algebraic variety X is said to be *normal* if it is normal at every point, meaning that the local ring at any point is an integrally closed domain. If X is a non-normal cubic surface, then its singular locus contains a 1 dimensional part (see [**16**], chapter 9.2).

THEOREM 4.13. *Let $\mathcal{C}$ be a non-normal cubic surface in $\mathbb{CP}^3$ that is not a cone. Then, up to projective isomorphisms, there exists a slice regular function f such that its twistor lift $\tilde{f}$ lies on $\mathcal{C}$.*

Proof. In theorem 9.2.1 of [**16**], the author says that, up to projective isomorphisms, the only non-normal cubic surfaces in $\mathbb{CP}^3$ that are not cones are the following two:

(1) $X_0 X_3^2 + X_1^2 X_2 = 0$,
(2) $X_0 X_1 X_3 + X_2 X_3^2 + X_1^3 = 0$.

Putting the coordinates of the lift in Remark 4.1 in the previous equations we obtain, respectively,

(1) $g(v) = -v^2$, $\hat{g}(v) = v$ and $h \equiv 0 \equiv \hat{h}$
(2) $g(v) = -1/v$, $\hat{g}(v) = v$, $\hat{h}(v) = 1/v^2$ and $h \equiv 0$

and so, if we put $x = \alpha + I\beta$ and $v = \alpha + i\beta$, the two slice regular functions are, respectively,

(1) $f_1 : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ defined by

$$f_1(x) = -x^2 \frac{(1 - Ii)}{2} + x\frac{(1 + Ii)}{2}, \tag{28}$$

(2) $f_2 : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ defined by

$$f_2(x) = -x^{-1} \frac{(1 - Ii)}{2} + x\frac{(1 + Ii)}{2} + x^{-2}\frac{(1 + Ii)}{2} j$$

□

The last case that we will threat is the case of cubic cones. The set of cubic cones can be defined by the equation

$$X_3^3 - (c + 1)X_3^2 X_1 + cX_3 X_1^2 - X_2^2 X_1 = 0, \tag{29}$$

where, if $c \in \mathbb{C} \setminus \{0, 1\}$, the surface is a cone over a non-singular plane cubic curve, while, in the case in which $c = 0, 1$ the surface is a cone over a nodal or cuspidal plane cubic curve respectively.

THEOREM 4.14. *Let $\mathcal{C}$ be cubic cone. Then there exist a slice regular function f defined on a dense subset of $\mathbb{H} \setminus \mathbb{R}$, such that, up to projective transformations, its twistor lift $\tilde{f}$ lies on $\mathcal{C}$.*

Proof. As in the previous theorems we will prove this result by exhibiting the splitting of the function f. If we impose equation 29 in the coordinates 23 we obtain that g and h must be identically zero while $\hat{g}$ and $\hat{h}$ must satisfy the following equation

$$\hat{g}^3 - (c + 1)\hat{g}^2 + c\hat{g} = \hat{h}^2.$$

Solving then in $\hat{h}$ or in $\hat{g}$, one finds the desired splitting of the slice regular function that give the thesis.

□

Since, up to projective transformations, the only cubic surfaces that contain infinite lines are cones and the non-normal ones, then, the projective classification is complete.

Of course, the functions seen in the previous proofs are not the only slice regular functions that solve the problem and give the thesis. One could ask for the "best" slice regular function such that its lift satisfies a certain algebraic equation, but this issue will not be treated in this paper and we propose it for some future work.

2. Rational curves on the Grassmannian

The aim of this section is to reconstruct the twistor transform defined in [**20**] for slice regular functions that are not defined on the real line. Moreover at the end we will characterize certain rational curves over the Grassmannian $\mathbb{G}r_2(\mathbb{C}^4)$.

The non-singular quadric in equation 21 is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the rulings are parametrized by u and v. A sphere $\alpha + \mathbb{S}\beta$ can be identified with the line,

$$l_{v_0} := \{[1, u, \alpha + i\beta, (\alpha + i\beta)u] \mid u \in \mathbb{C} \cup \{\infty\}\} \subset \mathbb{CP}^3,$$

defined by fixing $v_0 = \alpha + i\beta$. The line l_{v_0} can also be seen as a point in the Grassmannian $\mathbb{G}r_2(\mathbb{C}^4)$ or, equivalently, as a point in the Klein quadric in $\mathbb{P}(\bigwedge^2 \mathbb{C}^4) \simeq \mathbb{CP}^5$ via Plücker embedding.

Now, the left multiplication by j on $\mathbb{H}^2$ lifts in $\mathbb{C}^4$ as

$$[X_0, X_1, X_2, X_3] \xrightarrow{j\cdot} [-\overline{X_1}, \overline{X_0}, -\overline{X_3}, \overline{X_2}],$$

and the last induces a real structure σ over $\mathbb{CP}^5$ as follows,

$$\sigma : [\xi_1, \dots, \xi_6] \mapsto [\bar{\xi}_1, \bar{\xi}_5, -\bar{\xi}_4, -\bar{\xi}_3, \bar{\xi}_2, \bar{\xi}_6],$$

where $\{\xi_1, \dots, \xi_6\}$ represent the basis $\{e^{01}, e^{02}, e^{03}, e^{12}, e^{13}, e^{23}\}$ of $\bigwedge^2 \mathbb{C}^4$ and, of course, $e^{ij} := e^i \wedge e^j$. In the above coordinates we can explicit the equation of the Klein quadric as follows,

$$\xi_1\xi_6 - \xi_2\xi_5 + \xi_3\xi_4 = 0. \tag{30}$$

As explained in section 2 of [**45**], *a fixed point of σ corresponds to a j-invariant line in $\mathbb{CP}^3$, i.e. a (twistor) fibre of π.*

Example 4.1. Consider the coordinates founded in theorem 4.10 as functions defined on $\mathbb{CP}^1 \times \mathbb{CP}^1$. We want to find the twistor fiber mentioned in the previous result imposing equation $\sigma(\mathcal{F}(v)) = \mathcal{F}(v)$.

(1) If $\lambda = \mu \neq 0$ and $\nu = \pi/2$ we get, $\mathcal{F} : v \mapsto [1, c(1-v^2)^{1/2}, -v, v, \frac{1}{c}(1-v^2)^{1/2}, 1]$. Imposing $\sigma(\mathcal{F}(v)) = \mathcal{F}(v)$, we obtain $v = \pm 1$ (i.e. *two twistor lines* in correspondence of $x = \pm 1 \in \mathbb{R}$).

(2) If $\lambda = \mu = 0$ and $\nu \in (0, \pi/2)/2$ we get,

$$\mathcal{F} : v \mapsto [v^2 - \frac{e^{2i\nu} + v^2}{||e^{i\nu}||^2}, \frac{i}{||e^{i\nu}||}(e^{2i\nu} + v^2)^{1/2}, -v, v, \frac{i}{||e^{i\nu}||}(e^{2i\nu} + v^2)^{1/2}, 1].$$

Imposing $\sigma(\mathcal{F}(v)) = \mathcal{F}(v)$, we obtain no solution or *no twistor lines* (this because ω is a fixed non-real complex number).

(3) If $\mathcal{Q}$ is the zero set of the polynomial in equation 27, we get, $\mathcal{F} : v \mapsto [-(\frac{5}{4}v^2 + 4), 2i + \frac{v}{2}, -v, -v, 2i - \frac{v}{2}, 1]$. Imposing $\sigma(\mathcal{F}(v)) = \mathcal{F}(v)$, we obtain $v = -4i$ (i.e. *one twistor line* in correspondence of $x = -4i \in \mathbb{H}$).

At this point we can extend the definition given in [**20**] of twistor transform.

Definition 4.2. Let $D \subset \mathbb{C}^+$ be a domain and $f : \Omega_D \to \mathbb{H}$ be a slice function. We define its *twistor transform* of f as the following map:

$$\begin{array}{rccc} \mathcal{F}: & D & \to & \mathbb{G}r(\mathbb{C}^4) \\ & v & \mapsto & \tilde{f}(l_v). \end{array}$$

The following result extends theorem 5.7 of [**20**].

THEOREM 4.15. *Let D be a domain in $\mathbb{C}^+$. If $f : \Omega_D \to \mathbb{H}$ is a continuous slice function, then its twistor transform $\mathcal{F}$ defines a continuous curve over D. Moreover, every curve $\gamma : D \to \mathbb{G}r(\mathbb{C}^4)$, such that $\xi_6 \circ \gamma$ is never zero, is the twistor transform of a slice function $f : \Omega_D \to \mathbb{H}$. The function f is regular if and only if its twistor transform is a holomorphic curve.*

Proof. Given a slice function $f : \Omega_D \to \mathbb{H}$, its twistor lift is given, as in 22, by, $\tilde{f}[1, u, v, uv] = [1, u, p_i \circ F^\top(v) - u(p_i \circ F^{\perp c}(v)), p_i \circ F^\perp(v) + u(p_i \circ F^{\top c}(v))]$, where $f^\top$ and $f^\perp$ are the same as in formula 22. Fixing v, $\tilde{f}(l_v)$ is defined by the following linear equations:

$$\begin{cases} X_0(p_i \circ F^\top) - X_1(p_i \circ F^{\perp c}) - X_2 = 0 \\ X_0(p_i \circ F^\perp) + X_1(p_i \circ F^{\top c}) - X_3 = 0. \end{cases}$$

The coefficients of the last two equations determines the following generating vectors

$$e_1 = [p_i \circ F^\top, -p_i \circ F^{\perp c}, -1, 0], \quad e_2 = [p_i \circ F^\perp, p_i \circ F^{\top c}, 0, -1].$$

Using equation 30, then, the twistor transform can be made explicit as follows

$$\begin{aligned} \mathcal{F}(v) = [\xi_1, \ldots, \xi_6] = [&(p_i \circ F^\top)(v)(p_i \circ F^{\top c})(v) + (p_i \circ F^\perp)(v)(p_i \circ F^{\perp c})(v), \\ &(p_i \circ F^\perp)(v), -(p_i \circ F^\top)(v), (p_i \circ F^{\top c})(v), (p_i \circ F^{\perp c})(v), 1], \end{aligned}$$

where $\{\xi_i\} = \{e_1^h \wedge e_2^k\}_{0 \le h < k \le 3}$. But now that we have the explicit parametrization of $\mathcal{F}(v)$ it is clear that this is a holomorphic curve if and only if f is a slice regular function.
Vice versa, given a curve $\gamma : D \to \mathbb{G}r_2(\mathbb{C}^4)$ such that $\xi_6 \circ \gamma$ is never zero, we can assume $\xi_6 \circ \gamma = 1$ and recover the splittings of f as follows,

$$(p_i \circ F^\top) = -\xi_3 \circ \gamma, \quad (p_i \circ F^\perp) = \xi_2 \circ \gamma, \quad (p_i \circ F^{\top c}) = \xi_4 \circ \gamma, \quad (p_i \circ F^{\perp c}) = \xi_5 \circ \gamma.$$

Thanks to the Representation theorem we can now recover f and thanks to theorem 1.3 we obtain regularity.

□

From the proof, then, came out that the twistor transform $\mathcal{F}$ of a slice regular function f, can be represented in the following way,

$$\begin{aligned} \mathcal{F}(v) = [&(p_i \circ F^\top)(v)(p_i \circ F^{\top c})(v) + (p_i \circ F^\perp)(v)(p_i \circ F^{\perp c})(v), \\ &(p_i \circ F^\perp)(v), -(p_i \circ F^\top)(v), (p_i \circ F^{\top c})(v), (p_i \circ F^{\perp c})(v), 1]. \end{aligned}$$

REMARK 4.6. As for theorem 4.5, in the last theorem we could repeat the computations using the splitting lemma. The result would be the following,

$$\mathcal{F}(v) = [g(v)\hat{g}(v) + \hat{h}(v)h(v), h(v), -g(v), \hat{g}(v), \hat{h}(v), 1],$$

which coincide with the result in **[20]**.

We will now present some examples.

EXAMPLE 4.2.
- Let $f_1 : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ be the following slice regular function: $f(\alpha + I\beta) = 1 - Ii$. This function is equal to 2 over $\mathbb{C}_i$ and to 0 over $\mathbb{C}_{-i}$. Its twistor transform $\mathcal{F}_1 : \mathbb{C}^+ \to \mathbb{G}r(\mathbb{C}^4)$ is the constant function $v \mapsto [0, 0, -2, 0, 0, 1]$.
- Let $f_2 : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ be the following slice regular function: $f(\alpha + I\beta) = 1 + Ii$. This function is equal to 0 over $\mathbb{C}_i$ and to 2 over $\mathbb{C}_{-i}$. Its twistor transform $\mathcal{F}_2 : \mathbb{C}^+ \to \mathbb{G}r(\mathbb{C}^4)$ is the constant function $v \mapsto [0, 0, 0, 2, 0, 1]$.
- Let $f_3 : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ be the following slice regular function: $f(\alpha + I\beta) = (\alpha + I\beta)(1 - Ii)/2$. This function is equal to $(\alpha + I\beta)$ over $\mathbb{C}_i$ and to 0 over $\mathbb{C}_{-i}$. Its twistor transform $\mathcal{F}_3 : \mathbb{C}^+ \to \mathbb{G}r(\mathbb{C}^4)$ is the function $v \mapsto [0, 0, -v, 0, 0, 1]$.
- Let $f_4 : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ be the following slice regular function: $f(\alpha + I\beta) = (\alpha + I\beta)(1 + Ii)/2$. This function is equal to 0 over $\mathbb{C}_i$ and to $(\alpha + I\beta)$ over $\mathbb{C}_{-i}$. Its twistor transform $\mathcal{F}_4 : \mathbb{C}^+ \to \mathbb{G}r(\mathbb{C}^4)$ is the function $v \mapsto [0, 0, 0, v, 0, 1]$.

As said at the beginning of this section we want to characterize a certain set of linear holomorphic functions $\gamma : D \to \mathbb{G}r(\mathbb{C}^4)$ in terms of slice regular functions. We will restrict to the case in which $\xi_6 \circ \gamma$ is never zero, The theorem we are going to prove is the following.

THEOREM 4.16. *Let $\gamma : \mathbb{C}^+ \to \mathbb{G}r(\mathbb{C}^4)$ be a holomorphic curve such that $\xi_6 \circ \gamma$ is never zero. Then γ is affine if and only if there exist $A, B \in \mathbb{C}$, with $A/B \in \mathbb{C}^+ \cup \mathbb{R}$ such that γ is the twistor transform of a slice regular function f and $(A + xB) \cdot f$ is a slice affine function that satisfies*

$$h_i(Af_i - Bg_i, \bar{A}f_{-i} - \bar{B}g_{-i}) = 0, \tag{31}$$

where $f_{\pm i}$ are the values of the slice derivative of $(A+xB) \cdot f$ in $\mathbb{C}_{\pm i}$, $g_{\pm i}$ are the values of the slice constant function $(A+xB) \cdot f - x[(1-Ii)f_i + (1+Ii)f_{-i}]$ in $\mathbb{C}_{\pm i}$ and h_i denotes the hermitian product in $\mathbb{C}_i \oplus \mathbb{C}_i^{\perp} \simeq \mathbb{H}$.

Proof. A linear map $\gamma : \mathbb{C}^+ \to \mathbb{G}r(\mathbb{C}^4)$ is a map of the form,

$$\gamma(v) = [c_{11} + c_{12}v, c_{21} + c_{22}v, c_{31} + c_{32}v, c_{41} + c_{42}v, c_{51} + c_{52}v, c_{61} + c_{62}v],$$

intending the Grassmannian $\mathbb{G}r_2(\mathbb{C}^4)$ as the Klein quadric 30 in $\mathbb{CP}^5$. The condition $\xi_6 \circ \gamma \neq 0$ for all $v \in \mathbb{C}^+$ can be interpreted, of course, as $c_{61}/c_{62} \in \mathbb{C}^+ \cup \mathbb{R}$. Dividing everything by $c_{61} + c_{62}v$, we obtain

$$\gamma(v) = \left[\frac{c_{11} + c_{12}v}{c_{61} + c_{62}v}, \frac{c_{21} + c_{22}v}{c_{61} + c_{62}v}, \frac{c_{31} + c_{32}v}{c_{61} + c_{62}v}, \frac{c_{41} + c_{42}v}{c_{61} + c_{62}v}, \frac{c_{51} + c_{52}v}{c_{61} + c_{62}v}, 1\right],$$

and so, now $\xi_6 \circ \gamma = 1$. Substituting then the components of γ in equation 30, one obtain the following system of equations:

$$\begin{cases} c_{11}c_{61} - c_{21}c_{51} + c_{31}c_{41} = 0 \\ c_{11}c_{62} + c_{12}c_{61} - (c_{21}c_{52} + c_{22}c_{51}) + (c_{31}c_{42} + c_{32}c_{41}) = 0 \\ c_{12}c_{62} + c_{32}c_{42} + c_{22}c_{52} = 0 \end{cases}. \tag{32}$$

Moreover, since γ is a holomorphic function, then it will be the twistor transform of some slice regular function f such that

$$\begin{aligned} f_{\mathbb{C}_i^+}(\alpha+i\beta) &= -\frac{c_{31}+c_{32}(\alpha+i\beta)}{c_{61}+c_{62}(\alpha+i\beta)}+\frac{c_{21}+c_{22}(\alpha+i\beta)}{c_{61}+c_{62}(\alpha+i\beta)}j \\ f_{\mathbb{C}_{-i}^+}(\alpha-i\beta) &= \frac{\overline{c_{41}+c_{42}(\alpha+i\beta)}}{c_{51}+c_{52}(\alpha+i\beta)}+\frac{\overline{c_{21}+c_{22}(\alpha+i\beta)}}{c_{61}+c_{62}(\alpha+i\beta)}j. \end{aligned}$$

With the Representation Formula one obtain that, for each $\alpha+I\beta \in \mathbb{H}\setminus\mathbb{R}$,

$$\begin{aligned} 2f(\alpha+I\beta) &= [(1-Ii)f(\alpha+i\beta)+(1+Ii)f(\alpha-Ii)] \\ &= (c_{61}+(\alpha+I\beta)c_{62})^{-\cdot}\cdot[(\alpha+I\beta)(1-Ii)(-c_{32}+c_{22}j)+(1-Ii)(-c_{31}+c_{21}j)] \\ &\quad +(c_{61}+(\alpha+I\beta)c_{62})^{-\cdot}\cdot[(\alpha+I\beta)(1+Ii)(\bar{c}_{42}+\bar{c}_{52}j)+(1+Ii)(\bar{c}_{41}+\bar{c}_{51}j)], \end{aligned}$$

but then, $(c_{61}+(\alpha+I\beta)c_{62})\cdot f$ is a slice affine function. If now, one between c_{61} or c_{62} is equal to zero this correspond, respectively, to A or B equal to zero and so equation 31 holds true. If both c_{61} and c_{62} are non-zero, observe that, the first and the third equations in 32 can be written , respectively, as $h_i(g_i,g_{-i})=c_{11}A$ and $h_i(f_i,f_{-i})=c_{12}B$. Substituting these in the second equation of the system and since $(c_{21}c_{52}+c_{22}c_{51})-(c_{31}c_{42}+c_{32}c_{41})=h_i(g_i,f_{-i})+h_i(f_i,g_{-i})$, we get

$$h_i(g_i,g_{-i})\frac{B}{A}+h_i(f_i,f_{-i})\frac{A}{B}=h_i(g_i,f_{-i})+h_i(f_i,g_{-i}),$$

and so equation 31 holds true. The vice versa is trivial, following the proof in the opposite verse.

□

EXAMPLE 4.3. Simple examples of slice regular functions that satisfies the condition in equation 31, are all the functions of the following type:

$$\begin{array}{rrcl} f: & \mathbb{H}\setminus\mathbb{R} & \to & \mathbb{H} \\ & \alpha+I\beta & \mapsto & (Cx+D)^{-\cdot}\cdot(Ax+B)(1-Ii)/2, \end{array}$$

with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{R})$. In the next section we will explore one particular function in this set and then we will add some remarks to the whole family.

REMARK 4.7. The set of slice affine functions that satisfy 31 does not contain non constant slice functions that extend to the real line. In fact, as shown in remark 1.21, a slice affine function extends to $\mathbb{R}$ if the coefficients of first order are equal, i.e.: $f_+=f_-$, meaning that $h_i(f_i,f_{-i})\neq 0$.

3. Main Example

In this section we will study the following slice regular function

$$\begin{array}{rrcl} f: & \mathbb{H}\setminus\mathbb{R} & \to & \mathbb{H} \\ & \alpha+I\beta & \mapsto & (\alpha+I\beta)(1-Ii)/2 \end{array}$$

as tool to generate OCSes over its image. We will write also, for brevity, $f(x) = x(1-Ii)/2$, where $x = \alpha + I\beta \in \mathbb{H} \setminus \mathbb{R}$. As was shown in remark 2.2, this function is constant and equal to 0 if restricted to $\mathbb{C}^+_{-i}$ and equal to x if restricted to $\mathbb{C}^+_i$. In the same section was shown either theoretically and by explicit computations that the restriction to $\mathbb{H} \setminus \mathbb{C}^+_{-i}$ is an open function. In example 3.1 was proved that, if restricted to $\mathbb{H} \setminus \mathbb{C}^+_{-i}$, the function f is injective. For these reasons this function fit very well in the twistorial construction studied here. Moreover, this construction has a symbiotic aspect w.r.t. the function f. In fact, with the help of the twistor lift stated in theorem 4.5 it is possible to understand constructively the image of f. The next theorem precise this fact.

THEOREM 4.17. *If $q = q_0 + q_1 i + q_2 j + q_3 k$, then the function defined in equation 3 is such that $f(\mathbb{H} \setminus \mathbb{C}^+_{-i}) = \{q \in \mathbb{H} \mid q_1 > 0\}$. Moreover*

$$\bigcup_{I \in \mathbb{S}} f\,|_{\mathbb{C}^+_I}(\mathbb{R}) = \{q \in \mathbb{H} \mid q_1 = 0\},$$

where $f\,|_{\mathbb{C}^+_I}(\mathbb{R})$ means the unique extension to $\mathbb{R}$ of the function restricted to $\mathbb{C}^+_I$.

Proof. To prove the theorem we will use the twistor lift 22. In fact, thanks to theorem 4.5, it is possible to compute the image of a slice regular function by looking at the image of the projection to $\mathbb{H}$ of its twistor lift. Since, as already said, the function f is equal to the identity if restricted to $\mathbb{C}^+_i$ and to zero over the opposite semislice $\mathbb{C}^+_{-i}$, then its twistor lift is defined as follows:

$$\begin{array}{rccc} F: & \mathcal{Q}^+ \cap \pi^{-1}(\mathbb{H} \setminus \mathbb{C}^+_{-i}) & \to & \mathbb{CP}^3 \\ & [1, u, v, uv] & \mapsto & [1, u, v, 0], \end{array} \tag{33}$$

where, if $\alpha + I\beta \in \mathbb{H} \setminus \mathbb{C}^+_i$ and $I = ai + bj + ck$, then $u = -i\frac{b+ic}{a+1}$ and $v = \alpha + i\beta$, with $(a, b, c) \neq (-1, 0, 0)$ and $\beta > 0$. At the end what we want to compute is the image of the function $(1 + uj)^{-1}v$ and so these are the computations:

$$\begin{aligned} (1 + uj)^{-1}v &= \left(1 - \frac{b + ic}{a + 1}k\right)(\alpha + i\beta) \\ &= \frac{(a + 1)^2}{(a + 1)^2 + (b^2 + c^2)}\left(1 + \frac{bk - cj}{a + 1}\right)(\alpha + i\beta) \\ &= \frac{1}{2}[(a + 1)(\alpha + i\beta) + (\beta b - \alpha c)j + (\alpha b + \beta c)k]. \end{aligned}$$

So, the image of a quaternion $x = \alpha + (ai + bj + ck)\beta$ via f, with $ai + bc + ck \in \mathbb{S} \setminus \{-i\}$ and $\beta > 0$ is the quaternion

$$2f(x) = \alpha(a + 1) + \beta(a + 1)i + (\beta b - \alpha c)j + (\alpha b + \beta c)k.$$

Take now a generic quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$, this will be reached by f if and only if $q_1 > 0$. In fact the system

$$\begin{cases} \alpha(a+1) = q_0 \\ \beta(a+1) = q_1 \\ \beta b - \alpha c = q_2 \\ \alpha b + \beta c = q_3, \end{cases}$$

can be solved in the following way: the first two equations give $\alpha = q_0/(a+1)$ and $\beta = q_1/(a+1)$ and since $(a+1) \in (0,2]$, then $q_1 > 0$. If we put $B = b/(a+1)$ and $C = c/(a+1)$, the last two equations can be rewritten as

$$\begin{cases} q_1 B - q_0 C = q_2 \\ q_0 C + q_1 B = q_3, \end{cases}$$

which is a linear system such that the two equations are linearly independent, so the solutions is,

$$B = \frac{q_1 q_2 + q_0 q_3}{q_0^2 + q_1^2}, \quad C = \frac{q_1 q_3 - q_0 q_2}{q_0^2 + q_1^2}.$$

Now we remember that $a^2 + b^2 + c^2 = 1$ and so $B^2 + C^2 = \frac{1-a}{1+a}$ that entails $a = \frac{1-B^2-C^2}{1+B^2+C^2}$ which is always an admissible solution since it is always different from -1.

For the second part of the theorem, fix $I = ai + bj + ck \in \mathbb{S} \setminus \{-i\}$ and look for the following limit,

$$\lim_{\substack{\beta \to 0 \\ \alpha + I\beta \in \mathbb{C}_I^+}} f(\alpha + I\beta).$$

After restricting the function to $\mathbb{C}_I^+$ it is possible to extend it to $\mathbb{R}$ and also to look at the image via the twistor lift. Since f is continuous we obtain that, up to a factor 2, the previous limit is equal to

$$\alpha(a+1) - \alpha c j + \alpha b k = \alpha(a+1, 0, -c, b),$$

which is a straight line belonging to the set $\{q \in \mathbb{H} \mid q_1 = 0\}$ passing through the vector $(a+1, 0, -c, b)$. Taking the union, for (a,b,c) that runs over $\mathbb{S} \setminus \{-i\}$, it is clear that this will span the whole hyperplane $\{q_1 = 0\}$.

$\square$

The twistor lift of f lies in the hypersurface $\mathcal{H} := \{X_3 = 0\} \subset \mathbb{CP}^3$. In this case the general theory (see Section 3 of [**43**]) says that $\mathcal{H}$ induces an OCS conformally equivalent to a constant one, defined over the image of f. This is actually true and we will show that there is a specific conformal function from $\{q_1 > 0\} \subset \mathbb{H}$ to $\{q_1 < 0\}$ that sends $\mathbb{J}^f$ to i. The theorem is the following one.

THEOREM 4.18. *The complex metric manifold $(\{q_1 > 0\}, g_{Eucl}, \mathbb{J}^f)$ is conformally equivalent to $(\{q_1 < 0\}, g_{Eucl}, \mathbb{J}_i)$, where, with $\mathbb{J}_i$ we mean the left multiplication by i. The conformality is determined by the function $g : \{q_1 > 0\} \to \{q_1 < 0\}$ defined by $g(q) = q^{-1}$.*

Proof. The function g is of course a conformal map for the Euclidean metric. So, the only thing to prove is that the push-forward of $\mathbb{J}^f$ via g is exactly $\mathbb{J}_i$, meaning that, the following equality holds true

$$dg \circ \mathbb{J}^f = \mathbb{J}_i \circ dg.$$

But what is the actual shape of the two complex structures $\mathbb{J}^f$ and $\mathbb{J}_i$? The answer is easy and can be found analyzing the action against a generic tangent vector on a point. So we have that, if $v = (v_0, v_1, v_2, v_3)$ is a tangent vector over $p = f(\alpha + I\beta)$, then, $\mathbb{J}_i(p)v = (-v_1, v_0, -v_3, v_2)$, while $\mathbb{J}^f(p)v = (-av_1 - bv_2 - cv_3, av_0 - cv_2 + bv_3, bv_0 + cv_1 - av_3, cv_0 - bv_1 + av_2)$, where $ai + bj + ck = I$.

And so

$$\mathbb{J}_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbb{J}^f(p) = \begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -c & b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{pmatrix},$$

where $p = p_0 + p_1 i + p_2 j + p_3 k$ and, working on the computations in the proof of theorem 4.17,

$$a = \frac{p_0^2 + p_1^2 - p_2^2 - p_3^2}{\mid p \mid^2}, \quad b = 2\frac{p_0 p_3 + p_1 p_2}{\mid p \mid^2}, \quad c = 2\frac{p_1 p_3 - p_0 p_2}{\mid p \mid^2}.$$

Now, writing g as $g(q_0 + q_1 i + q_2 j + q_3 k) = (q_0, -q_1, -q_2, -q_3)/\mid q \mid^2$, one have that

$$dg(q) = \begin{pmatrix} \mid q \mid^2 - 2q_0^2 & -2q_0 q_1 & -2q_0 q_2 & -2q_0 q_3 \\ 2q_1 q_0 & -\mid q \mid^2 + 2q_1^2 & 2q_1 q_2 & 2q_1 q_3 \\ 2q_2 q_0 & 2q_2 q_1 & -\mid q \mid^2 + 2q_2^2 & 2q_2 q_3 \\ 2q_3 q_0 & 2q_3 q_1 & 2q_3 q_2 & -\mid q \mid^2 + q_3^2 \end{pmatrix} / \mid q \mid^4$$

and that,

$$\mathbb{J}_i \circ dg = \begin{pmatrix} -2q_1 q_0 & \mid q \mid^2 - 2q_1^2 & -2q_1 q_2 & -2q_1 q_3 \\ \mid q \mid^2 - 2q_0^2 & -2q_0 q_1 & -2q_0 q_2 & -2q_0 q_3 \\ -2q_3 q_0 & -2q_3 q_1 & -2q_3 q_2 & \mid q \mid^2 - 2q_3^2 \\ 2q_2 q_0 & 2q_2 q_1 & -\mid q \mid^2 + 2q_2^2 & 2q_2 q_3 \end{pmatrix} / \mid q \mid^4 .$$

It is now a matter of computation to show that $dg \circ \mathbb{J}^f = \mathbb{J}_i \circ dg$, but we will skip it. □

REMARK 4.8. The function $g(q) = q^{-1}$ in the previous theorem, was found using the following idea. The constant OCS $\mathbb{J}_i$ is described by the hyperplane $\{X_1 = 0\} \subset \mathbb{CP}^3$ (see Remark 2.3 of [**43**]) and so, starting from our lift $[1, u, v, 0]$ after changing the first two coordinates with the second two and dividing everything by $v (\neq 0)$, we obtain $[1, 0, v^{-1}, v^{-1}u]$ that projects to $[1, v^{-1}(1 + uj)]$, but now $v^{-1}(1 + uj) = ((1 + uj)^{-1}v)^{-1} = (f(q))^{-1}$.

REMARK 4.9. The last theorem and construction can be obtained using the following function as well: $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$, defined as

$$f(\alpha + I\beta) = (Cx + D)^{-\cdot} \cdot (Ax + B)\frac{(1 - Ii)}{2},$$

with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{R})$, $x = \alpha + I\beta$ and $z = \alpha + i\beta$. In fact, if we remove from the domain of this function the semislice $\mathbb{C}^+_{-i}$ over which is equal to zero, f is open and injective and its image is equal again to $\{q \in \mathbb{H} \mid q_1 > 0\}$. With easy computations one obtains that

$$f(\alpha + I\beta) = \begin{cases} \dfrac{(a+1)}{2\|Cz + D\|^2}[CA\|z\|^2 + DB + (BC + AD)\alpha] = q_0 \\[2ex] \dfrac{(a+1)}{2\|Cz + D\|^2}\beta = q_1 \\[2ex] \dfrac{(b\beta - c[CA\|z\|^2 + DB + (BC + AD)\alpha])}{2\|Cz + D\|^2} = q_2 \\[2ex] \dfrac{c\beta + b[CA\|z\|^2 + DB + (BC + AD)\alpha]}{2\|Cz + D\|^2} = q_3, \end{cases} \quad , \quad z = \alpha + i\beta,$$

and, with the same argument in the proof of theorem 4.17, we obtain that $q_1 > 0$ and, for any values of q_0, q_1, each q_2 and q_3 can be reached. Now, on the remaining first two components the function is exactly equal to

$$\frac{A(\alpha + i\beta) + B}{C(\alpha + i\beta) + D} = \frac{q_0 + iq_1}{(a+1)},$$

but, since A, B, C, D are taken such that the matrix they describe is in $SL(2, \mathbb{R})$, then since the function on the left describes an automorphism of the upper half complex space, it turns out that each q_0 and $q_1 > 0$ can be reached. The twistor lift of this function is

$$\begin{array}{cccc} \tilde{f}: & \mathcal{Q}^+ \cap \pi^{-1}(\mathbb{H} \setminus \mathbb{C}^+_{-i}) & \to & \mathbb{CP}^3 \\ & [1, u, v, uv] & \mapsto & [1, u, \frac{Av+B}{Cv+D}, 0] \end{array}$$

In the next remark we will show an idea that we haven't explored completely but might be a starting point for some future considerations.

REMARK 4.10. The twistor lift in equation 33, extends to a holomorphic mapping $\tilde{f} : \mathcal{Q} \to \mathbb{CP}^3$ by allowing v to take values in $\mathbb{C}$ rather than just in $\mathbb{C}^+$. However, even if

$\pi \circ \tilde{f} = f \circ \pi$ on $\mathcal{Q}^+ \cap \pi^{-1}(\mathbb{H} \setminus \mathbb{C}^+_{-i})$,

$$\begin{tikzcd} \mathcal{Q}^+ \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & \{X_3 = 0\} \arrow[d, "\pi"] \\ \mathbb{H} \setminus \mathbb{R} \arrow[r, "f"] & \{q_1 > 0\} \end{tikzcd}$$

this does not imply that the graph will commute once $\tilde{f}$ is extended. In fact we will have the following diagram,

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "1:1"] \arrow[d, "2:1"'] & \{X_3 = 0\} \arrow[d, "1:1"] \\ ? \arrow[r, dotted, dash, "{}^*1:2^*"] & ? \end{tikzcd}$$

where, the number upon the arrows are intended as *generically* and we don't know *a priori* what is in the two corners below and what is the meaning of the arrow that connects them. Also this arrow must represent something which behaves like $1:2$. This of course cannot be possible and suggest the possibility of approaching the issue using *multi-valued* functions. Anyway this example seems enough easy to be studied "by hands". So, first of all, we need to construct the "ghost function" that realize the second part of that $1:2$ cited before. So, when we extend $\tilde{f}$ to the whole $\mathcal{Q}$ we need the function that realizes the lifting $\tilde{f}[1, u, v, uv] = [1, u, v, 0]$, for $v \in \mathbb{C}^-$.

Of course theorem 4.5 is true if we substitute $\mathcal{Q}^-$ for $\mathcal{Q}^+$ and, in this case the lift of a slice regular function g is given by

$$\tilde{g}[1, u, v, uv] = [1, u, g^- - u\overline{h^+}, h^- + u\overline{g^+}],$$

where g^+, h^+, g^- and h^- are the same as in equation 22. So, the function that has lift $\tilde{g} : \mathcal{Q}^- \to \mathbb{CP}^3$, $\tilde{g}[1, u, v, uv] = [1, u, v, 0]$, is

$$\begin{array}{rccc} g: & \mathbb{H} \setminus \mathbb{R} & \to & \mathbb{H} \\ & \alpha + I\beta & \mapsto & (\alpha + I\beta)(1 + Ii)/2. \end{array}$$

This function can be thought as dual to f: it is equal to the identity over $\mathbb{C}^+_{-i}$ and equal to zero over $\mathbb{C}^+_i$; is injective and open over $\mathbb{H} \setminus \mathbb{C}^+_i$; its image, restricted on $\mathbb{H} \setminus \mathbb{C}^+_i$ is $\{q \in \mathbb{H} \mid q_1 < 0\}$ and

$$\bigcup_{I \in \mathbb{S}} g_{\mathbb{C}^+_I}(\mathbb{R}) = \{q \in \mathbb{H} \mid q_1 = 0\}.$$

Obviously $f(\mathbb{H} \setminus \mathbb{C}^+_{-i}) \cap g(\mathbb{H} \setminus \mathbb{C}^+_i) = \emptyset$.

In some sense the function g is the natural holomorphic extension slice by slice of the function f. The lift $\tilde{f}$ can be considered as the lift of the double-function

$$\begin{array}{rccc} h: & \mathbb{H} \setminus \mathbb{R} & \to & \mathbb{H} \\ & \alpha + I\beta & \mapsto & \{x(1 + Ii)/2, x(1 - Ii)/2\}, \end{array}$$

where the first "component" lifts in $\mathcal{Q}^+$ and the second in $\mathcal{Q}^-$. With this definition of h we can complete the previous commuting diagram as follows:

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & \{X_3 = 0\} \arrow[d, "\pi"] \\
\mathbb{H} \setminus \mathbb{R} \arrow[r, shift left] \arrow[r, shift right, "h"'] & \mathbb{H} \setminus \{q_1 = 0\}
\end{tikzcd}$$

Given all the properties of h we would like to apply the theory of section 1 to our example. In particular we would like to construct a non-constant OCS over $\mathbb{R}^4$ minus an hyperplane. Since the images of f and g does not intersects, then we can define the push forward of an OCS via h as the push forward via f into $\{q_1 > 0\}$ and via g into $\{q_1 < 0\}$:

$$\mathbb{J}^h_p := \begin{cases} \mathbb{J}^f_{f^{-1}(p)} & \text{if } p \in \{q_1 > 0\} \\ \mathbb{J}^g_{g^{-1}(p)} & \text{if } p \in \{q_1 < 0\}. \end{cases}$$

We have just constructed the following metric complex space

$$\big(\mathbb{H} \setminus \{q_1 = 0\}, \, \mathbb{J}^h, \, g_{Eucl}\big)$$

We end this discussion here and, as already said, we remand to some future work the possibility of specifying and developing the content of this remark.

Remark 4.11. The techniques used in this section can be useful to study more complicated examples such as the function in equation 28 which seems more complicated but carries on some interesting geometries.

APPENDIX A

Future works

Since we believe in the **fully independence of scientific research**, we think that the richness of the theory exposed in this thesis gives valid reasons to deepen our research. Nevertheless the final application to complex geometry justifies concretely the importance of the theory also to whom think of science differently from us. This is of course not the only application to other fields: another active basin, nowadays, is the quaternionic functional calculus developed from this function's theory.

Strongly motivated by these feelings we propose some further open problems and questions that we didn't have the time to explore in these few months of doctorate's studies. The following enumeration can be interpreted as a to do list for the future but the order is almost random.

(1) **Further rigidity properties.** In general, any further information on the nature of a slice regular function could be of some interest both by itself and in view of next points.
(2) **Bergman spaces.** In our knowledge there is still not a result regarding Bergman spaces of slice functions defined over domains without real points. In [**8**] are presented two kinds of Bergman spaces of slice regular functions: the first defined by a norm that takes in consideration the values of a function over the whole domain (that in this thesis is denoted by Ω_D), while the second defined on a norm that take in consideration the values of a function on a single complex plane (D_I for some $I \in \mathbb{S}$). Both interpretations could give several useful results in our context.
(3) **Coefficients of spherical expansion.** So far the only closed form of the coefficients of the spherical expansion in definition 1.13 is the one in theorem 3.7 of [**32**]. Anyway the construction given in this paper is quite hard to understand: for instance it is not clear if there is any relation between the coefficients of the

expansion of a slice regular function and those of its slice derivative. We think that further developments in this way could give interesting results.

(4) **Univalent functions.** As we saw in theorem 3.11 if a regular function is injective, hence its real differential is everywhere non-singular. Is there any result in the opposite direction? In the recent paper [**19**] the authors start some investigations about that issue obtaining some sufficient condition and, in particular, they prove an analogous of the Bieberbach-de Branges result in a special case (see theorem 3.11 in [**19**]). We wonder (and hope to study) if there is any condition in the general case as well and if a Bieberbach-de Branges type theorem can be proved.

(5) **Multivalued functions.** The last remark 4.10 shows that could be interesting to study, in some sense, multivalued functions. A first reason is explained in the remark itself while another reason is the possibility to extend in some way the Hurwitz theorem cited in remark 3.6 also to the case of a (multivalued/2-valued) slice regular function defined on a domain without real points.

(6) **Further on twistor lift.** A feature that we would like to explore is if the slice or the spherical derivative (or, in general the real differential), of a slice function can be interpreted in its twistor lift. Understanding a relation of this kind could be very useful in the description of the discriminant locus of the surface reached by the lift.

(7) **Discriminant loci.** As we said just before theorem 4.9, the discriminant locus of a surface $\mathcal{S}$ of degree d is the set of point $p \in \mathbb{S}^4$ such that $\pi^{-1}(p) \cap \mathcal{S}$ has cardinality different from d, where π is the usual twistor projection $\pi : \mathbb{CP}^3 \to \mathbb{HP}^1$ explored in chapter 4. To construct the discriminant locus of a surface $\mathcal{S}$ means to understand the domain of definition of the OCSes induced by $\mathcal{S}$. We would like to study the discriminant loci of all the surfaces constructed in chapter 4. In principle, in fact, these surfaces could generate interesting OCSes over some domain in $\mathbb{S}^4$.

(8) **Rational curves over the Grassmannian.** This point is very simple: we would like to remove from theorems 4.15 and 4.16 the technical hypothesis $\xi_6 \circ \gamma \neq 0$. This will involves the study of singularities of a slice function. A result of this kind will be useful for a complete description of the twistor lines over some classes of special surfaces in $\mathbb{CP}^3$.

(9) **Extensions to $*$-algebras.** We are quite sure that many results contained in Chapters 2 and 3 can be extended in the general setting of a real $*$-algebra. We would like to prove some of them using the general approach of [**30**].

(10) **Slice regular functions of several variables.** In [**31**] the authors introduce the definition of slice regular function of several variables in the general context of real $*$-algebras. With this definition they are able to prove a Hartogs type extension result. Adapting the definition in the quaternionic case we would like to study this set of functions. As in the one variable case any result in this direction could be of interest on its own and giving some interesting application.

(11) G_2 **geometry of submanifolds.** The less investigated (but maybe the most important) issue that we had the chance to think of is a particular application of the

theory of quaternionic functions (in the more generality) to the geometry of special submanifolds in a G_2-manifold. In section IV.2.A of [**36**], the authors explain that the graph of certain quaternionic functions is a area minimizing submanifold of $\mathbb{R}^7$ viewed as G_2-manifold . Since the study of slice regular function is now well under way we want to understand if it could be applied in this different context, maybe, constructing non-trivial kind of these submanifolds.

Bibliography

[1] Ahlfors, LV. Complex analysis. McGraw-Hill Book Co. New York. 1978.

[2] Alpay D, Colombo F, Sabadini I, Salomon G. The Fock space in the slice hyperholomorphic setting, to appear in Trends in Mathematics vol. ISAAC

[3] Altavilla A. Some properties for quaternionic slice-regular functions on domains without real points. to appear in Complex Variables and Elliptic Equations; doi: 10.1080/17476933.2014.889691.

[4] Altavilla A. On the real differential of a slice regular function. arXiv:1402.3993 [math.CV]

[5] Altavilla A. Some results in twistor geometry via slice regular function - *In preparation*

[6] Arcozzi N, Sarfatti G. Invariant metrics for the quaternionic Hardy space. arXiv:1312.5906v2 [math.CV]

[7] Clemens H. Curves on generic hypersurfaces. Ann. Sci. Ècole Norm. Sup. (4) 19 (1986), no. 4, 629–636.

[8] Colombo F, Gonzalez-Cervantes JO, Luna-Elizarraras ME, Sabadini I, Shapiro MV. On two approaches to the Bergman theory for slice regular functions, Advances in Hypercomplex Analysis; Springer INdAM Series 1, (2013), 39–54.

[9] Colombo F, Gentili G, Sabadini I, Struppa DC. Extension results for slice regular functions of a quaternionic variable. Adv. Math. 222 (2009), no. 5, 1793–1808.

[10] Colombo F, Sabadini I, Struppa DC. An extension theorem for slice monogenic functions and some of its consequences. Israel J. Math. 177 (2010), 369–389.

[11] Colombo F, Sabadini I, Struppa DC. Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions. Progress in Mathematics, 289. Birkhäuser/Springer Basel AG, Basel, 2011. vi+221 pp. ISBN: 978-3-0348-0109-6

[12] Colombo F, Sabadini I, Struppa DC. Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions. Progress in Mathematics, 289. Birkhäuser/Springer Basel AG, Basel, 2011.

[13] Cullen CG. An integral theorem for analytic intrinsic functions on quaternions. Duke Math J.. 1965; 32: pp. 139–148

[14] de Fabritiis C, Gentili G, Sarfatti G. Quaternionic Hardy spaces, Preprint, `www.math.unifi.it/users/sarfatti/Hardy.pdf`, (2013)

[15] De Leo S, Rotelli P P. Quaternionic analyticity. Appl. Math. Lett. 16 (2003), no. 7, 1077–1081.

[16] Dolgachev I. Classical algebraic geometry. A modern view. Cambridge University Press, Cambridge, 2012. xii+639 pp. ISBN: 978-1-107-01765-8

[17] Dolgachev I. Luigi Cremona and cubic surfaces. (Italian summary) Luigi Cremona (1830–1903) (Italian), 55–70, Incontr. Studio, 36, Istituto Lombardo di Scienze e Lettere, Milan, 2005.

[18] Fueter, R. Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen. Comment. Math. Helv. 7 (1934), no. 1, 307–330.

[19] Gal SG, Gonzalez-Cervantes JO, Sabadini I. On Some Geometric Properties of Slice Regular Functions of a Quaternion Variable. arXiv:1410.2623v1 [math.CV]

[20] Gentili G, Salamon S, Stoppato C. Twistor transforms of quaternionic functions and orthogonal complex structures. to appear in JEMS

[21] Gentili G, Stoppato C. Power series and analyticity over the quaternions. Math. Ann. 352 (2012), no. 1, 113–131.

[22] Gentili G, Stoppato C. The open mapping theorem for regular quaternionic functions. Ann. Sc. Norm. Super. Pisa Cl. Sci.. 2009; 5: pp. 805–815.

[23] Gentili G, Stoppato C. The zero sets of slice regular functions and the open mapping theorem. Hypercomplex analysis and applications, 95–107, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2011.

[24] Gentili G, Stoppato C. Zeros of regular functions and polynomials of a quaternionic variable. Michigan Math. J. 56 (2008), no. 3, 655–667.

[25] Gentili G, Stoppato C, Struppa DC. Regular Functions of a Quaternionic Variable. Springer London, Limited. Springer Monographs in Mathematics 2013.

[26] Gentili G, Struppa DC. A new approach to Cullen-regular functions of a quaternionic variable. C. R. Acad. Sci. Paris, Ser. I 342 (2006).

[27] Gentili G, Struppa DC. A new theory of regular functions of a quaternionic variable. Adv. Math.. 2007; 216: pp. 279–301.

[28] Gentili G, Struppa DC. On the multiplicity of zeroes of polynomials with quaternionic coefficients. Milan J. Math. 76 (2008), 15–25.

[29] Ghiloni R, Moretti V, Perotti A. Continuous slice functional calculus in quaternionic Hilbert spaces. Rev. Math. Phys. 25 (2013), no. 4, 1350006, 83 pp.

[30] Ghiloni R, Perotti A. Slice regular functions on real alternative algebras. Adv. Math.. 2011; 226: pp. 1662–1691.

[31] Ghiloni R, Perotti A. Slice regular functions of several Clifford variables. AIP Conference Proceedings, 1493, 734–738 (2012), DOI:http://dx.doi.org/10.1063/1.4765569

[32] Ghiloni R, Perotti A. Global differential equations for slice regular functions. Math. Nach., 287, No. 5-6, Pages 561–573, 2014, DOI: 10.1002/mana.201200318

[33] Ghiloni R, Perotti A. Power and spherical series over real alternative *-algebras. Indiana University Mathematics Journal, Vol. 63 (2014), No. 2, Pages 495–532.

[34] Griffiths P, Harris J. Principles of algebraic geometry. Pure and Applied Mathematics. Wiley-Interscience, New York, 1978. xii+813 pp.

[35] Hartshorne R. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

[36] Harvey R, Lawson HB. Calibrated geometries. Acta Math. 148 (1982), 47–157.

[37] Heins M. Complex function theory. Pure and Applied Mathematics, Vol. 28 Academic Press, New York-London 1968 xv+416 pp.

[38] Kobayashi S, Nomizu K. Foundations of differential geometry. Vol. II. Pure and Applied Mathematics. Wiley-Interscience, New York, 1996. xvi+468 pp.

[39] Laville G, Ramadanoff I. Elliptic Cliffordian functions. Complex Variables Theory Appl. 45 (2001), no. 4, 297–318.

[40] Pernas L. Holomorphie quaternionienne. Advances in Applied Clifford Algebras December 1998, Volume 8, Issue 2, pp 283–298

[41] Perotti A. Fueter Regularity and Slice Regularity: Meeting Points for Two Function Theories. Advances in Hypercomplex Analysis Springer INdAM Series Volume 1, 2013, pp 93-117

[42] Rudin W. Real and complex analysis. McGraw-Hill Book Co. New York.1987.

[43] Salamon S, Viaclovsky J. Orthogonal complex structure on domains in $\mathbb{R}^4$. Math. Ann. 343, 853–899 (2009)

[44] Segre B. The maximum number of lines lying on a quartic surface. Quart. J. Math., Oxford Ser. 14, (1943). 86–96.

[45] Shapiro G. On discrete differential geometry in twistor space. Journal of Geometry and Physics, Volume 68, June 2013, Pages 81–102

[46] Stoppato C. A new series expansion for slice regular functions. Adv. Math. 231 (2012), no. 3-4, 1401–1416.

[47] Stoppato C. Poles of regular quaternionic functions. Complex Var. Elliptic Equ. 54 (2009), no. 11, 1001–1018.

[48] Sudbery A. Quaternionic analysis. Math. Proc. Cambridge Philos. Soc. 85 (1979), no. 2, 199–224.

[49] Wood JC. Harmonic morphisms and Hermitian structures on Einstein 4-manifolds. Internat. J. Math. 3 (1992), no. 3, 415–439.

[50] Xu G. Subvarieties of general hypersurfaces in projective space. J. Differential Geom. 39 (1994), no. 1, 139–172.