Research Article

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Integral representation of local functionals depending on vector fields

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Abstract: Given an open and bounded set $\Omega \subseteq \mathbb{R}^n$ and a family $\mathbf{X} = (X_1, \dots, X_m)$ of Lipschitz vector fields on Ω, with $m \leq n$, we characterize three classes of local functionals defined on first-order X-Sobolev spaces, which admit an integral representation in terms of X, i.e.

$$F(u, A) = \int_A f(x, u(x), Xu(x))\, dx,$$

with f being a Carathéodory integrand.

Keywords: Integral representation, vector fields, variational functionals

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1 Introduction

The representation of local functionals as integral functionals of the form

$$F(u) = \int_\Omega f(x, u(x), Du(x))\, dx$$

has a very long history and exhibits a natural application when dealing with relaxed functionals and related Γ-limits in a suitable topology. In the Euclidean setting, this problem is now very well understood, and we refer the interested reader to the papers [2, 6–9] for a complete overview of the subject.

Recently, in [16], Franchi, Serapioni and Serra Cassano started the study of variational functionals driven by a family of Lipschitz vector fields. By a family of Lipschitz vector fields we mean an m-tuple $\mathbf{X} = (X_1, \dots, X_m)$, with $m \leq n$, where each X_j is a first-order differential operator with Lipschitz coefficients $c_{j,i}$ defined on a bounded open set $\Omega \subseteq \mathbb{R}^n$, i.e.

$$X_j(x) = \sum_{i=1}^{n} c_{j,i}(x)\partial_i, \quad j = 1, \dots, m.$$

Moreover, according to [20], we assume that the family $\mathbf{X}$ satisfies the structure assumption (LIC), which roughly means that $X_1(x), \dots, X_m(x)$ are linearly independent for a.e. $x \in \Omega$ as vectors of $\mathbb{R}^n$ (cf. Definition 2.1). We stress that this point of view is pretty general and encompasses, among other things, the Euclidean setting and many interesting sub-Riemannian manifolds.

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Since [16], the possibility to extend the classical results of the calculus of variations to the setting of variational functionals driven by vector fields has been the object of study of many papers. For example, the homogenization theory has been intensively studied so far in the setting of special sub-Riemannian manifolds, i.e. Carnot groups (see, for instance, [3, 18, 22]). More recently, in [20, 21] Maione, Pinamonti and Serra Cassano started the investigation of the Γ-convergence of translation-invariant local functionals $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$, with $\mathcal{A}$ being the class of all open subsets of Ω. In [20, Theorem 3.12], they found conditions under which F can be represented as

$$F(u, A) = \int_A f(x, Xu(x))\,dx \tag{1.1}$$

for any $A \subseteq \Omega$ open and $u \in L^p(\Omega)$ such that $u|_A \in W^{1,p}_{X,\text{loc}}(A)$ (cf. Definition 2.2 and [15]), and for a suitable $f : \Omega \times \mathbb{R}^m \to [0, \infty)$. Finally, they applied this characterization to prove a Γ-compactness theorem for integral functionals of the form (1.1) when $1 < p < +\infty$. Similar results have been proved in [22], under stronger conditions on the family $\mathbf{X}$. To conclude, we also point out that functional (1.1) was studied in [16] as far as its relaxation and in connection with the so-called Meyers–Serrin theorem for $W^{1,p}_X(\Omega)$.

Inspired by the results proved in [7, 8], the aim of the present paper is to extend the results achieved in [20] when we drop the assumption of translation-invariance. We find some sufficient and necessary conditions under which a local functional

$$F : W^{1,p}_{X,\text{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty]$$

admits an integral representation of the form

$$F(u, A) = \int_A f(x, u(x), Xu(x))\,dx \quad \text{for all } u \in W^{1,p}_{X,\text{loc}}(\Omega) \text{ and all } A \in \mathcal{A}, \tag{1.2}$$

for a suitable Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty)$. We point out that in this new framework, due to the lack of translation-invariance, a dependence of the integrand with respect to the function is expected. Let us observe that if F is defined on $L^p_{\text{loc}}(\Omega) \times \mathcal{A}$ instead of $W^{1,p}_{X,\text{loc}}(\Omega) \times \mathcal{A}$, under reasonable improvements of some assumptions, it is easy to extend the integral representation to get

$$F(u, A) = \int_A f(x, u(x), Xu(x))\,dx \quad \text{for all } A \in \mathcal{A} \text{ and all } u \in L^p_{\text{loc}}(\Omega) \text{ such that } u|_A \in W^{1,p}_{X,\text{loc}}(A).$$

The main goal of this paper is to obtain a representation formula as in (1.2) for the following three different classes of functionals:
(i) Convex functionals (Theorem 3.3).
(ii) $W^{1,\infty}$ weakly*-seq. l.s.c. functionals (Theorem 4.3).
(iii) None of the above (Theorem 5.6).

Unlike in Sobolev spaces, in this context no analogue of approximation results by a reasonable notion of piecewise X-affine function holds in general (cf. [20, Section 2.3]). To overcome this difficulty, we rely on the method employed in [20], consisting of three steps:
(i) Apply one of the classical results for Sobolev spaces [7, 8] to the functional, obtaining an integral representation with respect to a “Euclidean” Lagrangian f_e of the form

$$F(u, A) = \int_A f_e(x, u(x), Du(x))\,dx \quad \text{for all } u \in W^{1,p}_{\text{loc}}(\Omega) \text{ and all } A \in \mathcal{A}.$$

(ii) Find sufficient conditions on f_e that guarantee the existence of a “non-Euclidean” Lagrangian f such that

$$\int_A f_e(x, u(x), Du(x))\,dx = \int_A f(x, u(x), Xu(x))\,dx \quad \text{for all } A \in \mathcal{A} \text{ and all } u \in C^\infty(A). \tag{1.3}$$

(iii) Extend the previous equality to the whole space $W^{1,p}_{X,\text{loc}}(\Omega)$.

The second step crucially exploits third-argument convexity of the Euclidean Lagrangian f_e. Indeed, convexity of $f_e(x,u,\cdot)$ is sufficient to guarantee (1.3) (cf. Proposition 3.2). This is shown in [20], and the same ideas can be adapted to cases (i) and (ii) of convex and weakly*-seq. l.s.c. functionals, for which the convexity of $f_e(x,u,\cdot)$ is granted. On the contrary, due to the weaker assumptions on the functional, case (iii) is more demanding and requires a further step. In Section 5, we show that the convexity of $f_e(x,u,\cdot)$ is not necessary for (1.3). Thus, in order to find a more suitable notion of convexity, we define the weaker concept of *X-convexity* (cf. Definition 5.3), which strongly depends on the chosen family of vector fields. We show that, under a classical growth assumption on the functional, this new condition is equivalent to (1.3) (cf. Proposition 5.4). Finally, by slightly modifying a zig-zag argument due to Buttazzo and Dal Maso [8, Lemma 2.11], we show that X-convexity is a consequence of a reasonable lower semicontinuity assumption (cf. Lemma 5.5). This procedure allows to generalize the final case as well. Finally, for each of the previous results we show that our hypotheses are also necessary, in order to give a complete characterization of the classes of functionals studied.

The structure of the paper is the following. In Section 2, we briefly recall some basic facts about vector fields and X-Sobolev spaces. In Section 3, we get an integral representation result for a class of convex functionals. In Section 4, we deal with weakly*-sequentially l.s.c functionals. In Section 5, we drop both previous requirements, obtaining as well an integral representation result.

2 Vector fields and *X*-Sobolev spaces

2.1 Notation

Unless otherwise specified, we let $1 \le p < +\infty$ and $m, n \in \mathbb{N} \setminus \{0\}$ with $m \le n$, we denote by Ω an open and bounded subset of $\mathbb{R}^n$ and by $\mathcal{A}$ the family of all open subsets of Ω. Given two open sets A and B, we write $A \Subset B$ whenever $\overline{A} \subseteq B$. We set $\mathcal{A}_0$ to be the subfamily of $\mathcal{A}$ of all open subsets A of Ω such that $A \Subset \Omega$. For any $u, v \in \mathbb{R}^n$, we denote by $\langle u, v\rangle$ the Euclidean scalar product, and by $|v|$ the induced norm. We denote by $\mathcal{L}^n$ the restriction to Ω of the n-th dimensional Lebesgue measure, and for any set $E \subseteq \Omega$ we write $|E| := \mathcal{L}^n(E)$. Given an integrable function $f : \Omega \to \overline{\mathbb{R}}$, we write

$$\int_\Omega f(x)\,dx := \int_\Omega f(x)\,d\mathcal{L}^n(x).$$

Given $x \in \mathbb{R}^n$ and $R > 0$, we let $B_R(x) := \{y \in \mathbb{R}^n : |x - y| < R\}$, and given an integrable function $f : B_R(x) \to \mathbb{R}$ we denote its integral average by

$$\fint_{B_R(x)} f\,dx := \frac{1}{|B_R(x)|}\int_{B_R(x)} f\,dx.$$

We usually omit the variable of integration when writing an integral: for instance, given two functions $f : \Omega \times \mathbb{R} \to \overline{\mathbb{R}}$ and $u : \Omega \to \mathbb{R}$ such that $x \mapsto f(x, u(x))$ is integrable over Ω, we write its integral as $\int_\Omega f(x,u)\,dx$ instead of $\int_\Omega f(x,u(x))\,dx$. Finally, for $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ we set

$$\varphi_{x,u,\xi}(y) := u + \langle \xi, y - x\rangle. \tag{2.1}$$

2.2 Basic definitions and properties

We will always identify a first order differential operator $X := \sum_{i=1}^n c_i \frac{\partial}{\partial x_i}$ with the map

$$X(x) := (c_1(x), \dots, c_n(x)) : \Omega \to \mathbb{R}^n.$$

Definition 2.1. Let $m \le n$. We say that $\mathbf{X} := (X_1, \dots, X_m)$ is a *family of Lipschitz vector fields* on Ω if for any $j = 1, \dots, m$ and for any $i = 1, \dots, n$ there exists a function $c_{j,i} \in \operatorname{Lip}(\Omega)$ such that

$$X_j(x) = (c_{j,1}(x), \dots, c_{j,n}(x)).$$

We will denote by $C(x)$ the $m \times n$ matrix defined by

$$C(x) := [c_{j,i}(x)]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$$

We say that **X** satisfies the *linear independence condition* (LIC) on Ω if the set

$$N_X := \{x \in \Omega : X_1(x), \dots, X_m(x) \text{ are linearly dependent}\}$$

is such that $|N_X| = 0$. In this case, we set $\Omega_X := \Omega \setminus N_X$.

Let us point out that (LIC) embraces many relevant families of vector fields studied in the literature. In particular, neither the *Hörmander condition* for **X**, that is, each vector field X_j is smooth and the rank of the Lie algebra generated by $X_1, \dots, X_m$ equals n at any point of Ω, nor the (weaker) assumption that the X-gradient induces a *Carnot–Carathéodory metric* in Ω is requested. An exhaustive account of these topics can be found in [4].

Definition 2.2. Let $m \le n$, $u \in L^1_{\mathrm{loc}}(\Omega)$ and $v \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, and let **X** be a family of Lipschitz vector fields. We say that v is the *X-gradient* of u if for any $\varphi \in C^\infty_c(\Omega, \mathbb{R}^m)$ it holds that

$$-\int_\Omega u \sum_{j=1}^m \sum_{i=1}^n \frac{\partial}{\partial x_i}(c_{j,i}\varphi_j)\, dx = \int_\Omega \varphi \cdot v\, dx.$$

Whenever it exists, the X-gradient is shown to be unique a.e. In this case, we set $Xu := v$.

If $p \in [1, +\infty]$, we define the vector spaces

$$W^{1,p}_X(\Omega) := \{u \in L^p(\Omega) : Xu \in L^p(\Omega)\}$$

and

$$W^{1,p}_{X,\mathrm{loc}}(\Omega) := \{u \in L^p_{\mathrm{loc}}(\Omega) : u|_{A'} \in W^{1,p}_X(A') \text{ for all } A' \in \mathcal{A}_0\}.$$

We refer to them as *X-Sobolev spaces*, and to their elements as *X-Sobolev functions*.

The next proposition can be found in [15].

Proposition 2.3. *Let $p \in [1, +\infty]$. Then the vector space $W^{1,p}_X(\Omega)$, endowed with the norm*

$$\|u\|_{W^{1,p}_X(\Omega)} := \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega,\mathbb{R}^m)},$$

is a Banach space. Moreover, if $1 < p < +\infty$, it is a reflexive Banach space.

The following proposition tells us that X-Sobolev spaces are actually a generalization of the classical Sobolev spaces, both because each Sobolev function is in particular an X-Sobolev function, whatever **X** we choose, and because, as expected, the choice of the "standard" family of vector fields

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$$

gives rise to the classical Sobolev spaces.

Proposition 2.4. *The following facts hold:*
(i) *If $n = m$ and $c_{j,i}(x) = \delta_{j,i}$ for every $i, j = 1, \dots, n$, then $W^{1,p}(\Omega) = W^{1,p}_X(\Omega)$.*
(ii) *$W^{1,p}(\Omega) \subseteq W^{1,p}_X(\Omega)$, the inclusion is continuous and*

$$Xu(x) = C(x)Du(x)$$

for every $u \in W^{1,p}(\Omega)$ and a.e. $x \in \Omega$.

Let us notice that, with Ω being bounded, we have that

$$W^{1,\infty}(\Omega) \subseteq W^{1,p}(\Omega) \subseteq W^{1,p}_X(\Omega)$$

for any family **X** of Lipschitz vector fields. The following proposition tells us that the weak convergence in $W^{1,p}_X$ is weaker than the weak*-convergence in $W^{1,\infty}$.

Proposition 2.5. *Let* $\mathbf{X}$ *be a family of Lipschitz vector fields. Then, for any sequence* $(u_h)_h \subseteq W^{1,\infty}(\Omega)$ *and any* $u \in W^{1,\infty}(\Omega)$*, it follows that*

$$u_h \rightharpoonup^* u \text{ in } W^{1,\infty}(\Omega) \quad \text{implies} \quad u_h \rightharpoonup u \text{ in } W^{1,p}_X(\Omega).$$

Proof. This follows easily from [5, Theorem 3.10]. □

2.3 Approximation by regular functions

When dealing with representation theorems for local functionals defined on classical Sobolev spaces, a typical strategy is to exploit classical differentiation theorems for measures to get an integral representation of the form

$$F(u, A) = \int_A f_e(x, u, Du)\, dx$$

for classes of "simple" functions, that is, for instance, linear or affine functions. Then one can combine some semicontinuity properties of the functional together with approximation results by means of piecewise affine functions (see, for instance, [13, Chapter X, Proposition 2.9]) in order to extend the integral representation to all Sobolev functions. In this context, one of the main difficulties is that an analogue of [13, Chapter X, Proposition 2.9] does not hold. We mean that, if we call a C^∞-function X-affine when Xu is constant, then there are choices of $\mathbf{X}$ for which not all X-Sobolev functions can be approximated in $W^{1,p}_X$ by piecewise X-affine functions [20, Section 2.3]. So, as shown in Section 3, we have to adopt a different strategy. Anyway, we present some useful Meyers–Serrin-type results that are still true even in this non-Euclidean framework and that allow us to approximate X-Sobolev functions with smooth functions. For the following fundamental theorem, we refer to [17, Theorem 1.2.3].

Theorem 2.6. *Let* Ω *be an open subset of* $\mathbb{R}^n$*. For any* $u \in W^{1,p}_X(\Omega)$*, there exists a sequence*

$$u_\epsilon \in W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$$

such that

$$u_\epsilon \to u \quad \text{in } W^{1,p}_X(\Omega) \text{ as } \epsilon \to 0.$$

Proposition 2.7. *Given* $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$ *and let* $A' \Subset \Omega$*. Then there exists a function* $v \in W^{1,p}_X(\Omega)$ *which coincides with* u *on* A'*.*

Proof. Let φ be a smooth cut-off function between A' and Ω. It is straightforward to verify that the function $v(x) := \varphi(x)u(x)$ satisfies the desired requirements. □

The previous proposition, together with Theorem 2.6, allows to prove the following result.

Proposition 2.8. *Consider a function* $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$ *and an open set* $A' \Subset \Omega$*. Then there exists a sequence* $(u_\epsilon)_\epsilon \subseteq W^{1,p}_X(\Omega)$ *such that*

$$u_\epsilon|_{A'} \in W^{1,p}_X(A') \cap C^\infty(A') \quad \text{and} \quad u_\epsilon|_{A'} \to u|_{A'} \text{ in } W^{1,p}_X(A').$$

Proof. Let us fix $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$ and $A' \in \mathcal{A}_0$. By Proposition 2.7, we can find a function $\tilde{u} \in W^{1,p}_X(\Omega)$ such that $u|_{A'} = \tilde{u}|_{A'}$, and by Theorem 2.6 there exists a sequence $(u_\epsilon)_\epsilon \subseteq W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$ converging to $\tilde{u}$ in $W^{1,p}_X(\Omega)$. It is easy to see that

$$(u_\epsilon|_{A'})_\epsilon \subseteq W^{1,p}_X(A') \cap C^\infty(A').$$

Moreover, since $u|_{A'} = \tilde{u}|_{A'}$, we conclude that $u_\epsilon|_{A'} \to u|_{A'}$ in $W^{1,p}_X(A')$. □

2.4 Failure of a Lusin-type theorem

When dealing with integral representation in classical Sobolev spaces, one might exploit the following Lusin-type result (cf. [10, Theorem 13]).

Proposition 2.9. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, let $1 \le p \le +\infty$ and let $u \in W^{1,p}(\Omega)$. Then, for any $\epsilon > 0$, there exist $A_\epsilon \in \mathcal{A}$ and $v \in C^1(\overline{\Omega})$ such that $|A_\epsilon| \le \epsilon$ and $u|_{\Omega \setminus A_\epsilon} = v|_{\Omega \setminus A_\epsilon}$.*

Under reasonable assumptions (cf. [8, Lemma 2.7]), this result allows to extend an integral representation result from $C^1(\overline{\Omega}) \times \mathcal{A}$ to $W^{1,p}(\Omega) \times \mathcal{A}$. The following counterexample shows that an analogue of Proposition 2.9 does not hold in a general X-Sobolev space.

Counterexample 2.10. In this example, we speak about *approximate differentiability* and *approximate partial derivatives* according to [14, Section 3.1.2]. Let us take $n = 2$, $m = 1$, $\Omega = (0,1) \times (0,1)$ and $\mathbf{X} = X_1 = \frac{\partial}{\partial x}$ (which satisfies (LIC)). Let us consider a function $w : (0,1) \to \mathbb{R}$ which is bounded and continuous but not approximately differentiable for a.e. $x \in (0,1)$ (see, for instance, [23, p. 297]), and define the function $u : \Omega \to \mathbb{R}$ by

$$u(x,y) := w(y).$$

We have that $u \in L^\infty(\Omega)$, and it is constant with respect to x. Thus, for any $\varphi \in C_c^\infty(\Omega)$, we have that

$$-\int_\Omega u \frac{\partial \varphi}{\partial x}\, dx = -\int_0^1 dy\, w(y) \int_0^1 dx\, \frac{\partial \varphi}{\partial x} = 0,$$

and so $Xu = 0$. Hence $u \in W_X^{1,\infty}(\Omega)$, and in particular we have that $u \in W_X^{1,p}(\Omega)$ for any $p \in [1,+\infty]$. If it was the case that u satisfies the desired property, then we would have, for a.e. (x,y) in Ω, that u is approximately differentiable at (x,y) (see [19, Theorem 1]). Thus, according to [23, Theorem 12.2] and to the fact that u is constant with respect to x, we would have that for any $x \in (0,1)$ and for a.e. $y \in (0,1)$ the function $z \mapsto u(x,z) = w(z)$ is approximately differentiable at y, but this last assertion is in contradiction with our choice of w.

2.5 Algebraic properties of X

Here we present some algebraic properties of the coefficient matrix $C : \Omega \to \mathbb{R}^{m\times n}$. The following results have been achieved in [20, Section 3.2].

Definition 2.11. Suppose $\mathbf{X}$ is a family of Lipschitz vector fields. For any $x \in \Omega$, we define the linear map $L_x : \mathbb{R}^n \to \mathbb{R}^m$ by

$$L_x(v) := C(x)v \quad \text{if } v \in \mathbb{R}^n,$$

and

$$N_x := \ker(L_x), \quad V_x := \{C(x)^T z : z \in \mathbb{R}^m\}.$$

From standard linear algebra, we know that $\mathbb{R}^n = N_x \oplus V_x$, and so, for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$, there is a unique choice of $\xi_{N_x} \in N_x$ and $\xi_{V_x} \in V_x$ such that

$$\xi = \xi_{N_x} + \xi_{V_x}.$$

Finally, we define $\Pi_x : \mathbb{R}^n \to V_x \subset \mathbb{R}^n$ as the projection $\Pi_x(\xi) := \xi_{V_x}$.

These definitions make sense for a generic family of Lipschitz vector fields, but the following two propositions list some very useful invertibility and continuity properties that are typical of those families of vector fields satisfying (LIC).

Proposition 2.12. *Let $\mathbf{X}$ be a family of Lipschitz vector fields satisfying* (LIC) *on Ω. Then the following facts hold:*

(i) $\dim V_x = m$ *for each $x \in \Omega_X$ and $L_x(V_x) = \mathbb{R}^m$. In particular, $L_x : V_x \to \mathbb{R}^m$ is an isomorphism.*

(ii) *Let*

$$B(x) := C(x)C^T(x) \quad x \in \Omega.$$

Then, for each $x \in \Omega_X$, $B(x)$ is a symmetric invertible matrix of order m. Moreover, the map

$$B^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m),$$

defined by

$$B^{-1}(x)(z) := B(x)^{-1}z \quad \text{if } z \in \mathbb{R}^m,$$

is continuous.

(iii) *For each $x \in \Omega_X$, the projection Π_x can be represented as*

$$\Pi_x(\xi) = \xi_{V_x} = C(x)^T B(x)^{-1} C(x)\xi \quad \text{for all } \xi \in \mathbb{R}^n.$$

Remark 2.13. It is easy to see that $N_X = \{x \in \Omega : \det B(x) = 0\}$. Hence, N_X is closed in Ω.

Proposition 2.14. *Let $\mathbf{X}$ be a family of Lipschitz vector fields satisfying* (LIC) *on Ω. Then the map $L_x : V_x \to \mathbb{R}^m$ is invertible and the map $L^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, defined by*

$$L^{-1}(x) := L_x^{-1} \quad \text{if } x \in \Omega_X,$$

belongs to $\mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$.

2.6 Local functionals

We conclude this section by giving some definitions about increasing set functions, for which we refer to [12, Chapter 14], and local functionals defined on $W^{1,p}_X$. From now on, we assume that $\mathbf{X}$ is a family of Lipschitz vector fields satisfying (LIC) on Ω.

Definition 2.15. We say that $\omega : \Omega \times [0, +\infty) \to [0, +\infty)$ is a *locally integrable modulus of continuity* if and only if

$$r \mapsto \omega(x, r) \text{ is increasing, continuous and } \omega(x, 0) = 0 \text{ for a.e. } x \in \Omega,$$

and

$$x \mapsto \omega(x, r) \in L^1_{\text{loc}}(\Omega) \quad \text{for all } r \geq 0.$$

Definition 2.16. Let us consider a functional $F : \mathcal{F} \times \mathcal{A} \to [0, +\infty]$, where $\mathcal{F}$ is a functional space such that $C^1(\overline{\Omega}) \subseteq \mathcal{F}$. We make the following definitions:

(i) F satisfies the *strong condition* (ω) if there exists a sequence $(\omega_k)_k$ of locally integrable moduli of continuity such that

$$|F(v, A') - F(u, A')| \leq \int_{A'} \omega_k(x, r)\, dx \tag{2.2}$$

for any $k \in \mathbb{N}$, $A' \in \mathcal{A}_0$, $r \in [0, \infty)$ and $u, v \in C^1(\overline{\Omega})$ such that

$$|u(x)|, |v(x)|, |Du(x)|, |Dv(x)| \leq k,$$
$$|u(x) - v(x)|, |Du(x) - Dv(x)| \leq r$$

for all $x \in A'$.

(ii) F satisfies the *weak condition* (ω) if there exists a sequence $(\omega_k)_k$ of locally integrable moduli of continuity such that

$$|F(u + s, A') - F(u, A')| \leq \int_{A'} \omega_k(x, |s|)\, dx$$

for any $k \in \mathbb{N}$, $A' \in \mathcal{A}_0$, $s \in \mathbb{R}$ and $u \in C^1(\overline{\Omega})$ such that

$$|u(x)|, |u(x) + s|, |s| \leq k \quad \text{for all } x \in A'.$$

Definition 2.17. Let $\alpha : \mathcal{A} \to [0, +\infty]$ be a function. We make the following definitions:

(i) α is *increasing* if it holds that $\alpha(A) \leq \alpha(B)$ for any $A, B \in \mathcal{A}$ such that $A \subseteq B$.

(ii) α is *inner regular* if it is increasing and $\alpha(A) = \sup\{\alpha(A') : A' \Subset A\}$ for any $A \in \mathcal{A}$.

(iii) α is *subadditive* if it is increasing and, for any $A, B, C \in \mathcal{A}$ with $A \subseteq B \cup C$,

$$\alpha(A) \leq \alpha(B) + \alpha(C).$$

(iv) α is *superadditive* if it is increasing and, for any $A, B, C \in \mathcal{A}$ with $A \cap B = \emptyset$ and $A \cup B \subseteq C$,

$$\alpha(C) \geq \alpha(A) + \alpha(B).$$

(v) α is a *measure* if it is increasing and the restriction to $\mathcal{A}$ of a non-negative Borel measure.

Definition 2.18. Let us consider a functional

$$F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty].$$

We make the following definitions:

(i) F is a *measure* if, for any $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$,

$$F(u, \cdot) : \mathcal{A} \to [0, +\infty]$$

is a measure.

(ii) F is *local* if, for any $A' \in \mathcal{A}_0$ and $u, v \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$,

$$u|_{A'} = v|_{A'} \quad \text{implies} \quad F(u, A') = F(v, A').$$

(iii) F is *convex* if, for any $A' \in \mathcal{A}_0$, the function $F(\cdot, A') : W^{1,p}_X(\Omega) \to [0, +\infty]$ is convex.

(iv) F is *p-bounded* if there exist $a \in L^1_{\mathrm{loc}}(\Omega)$ and $b, c > 0$ such that, for any $A' \in \mathcal{A}_0$ and for any $u \in W^{1,p}_X(\Omega)$, it holds that

$$F(u, A') \leq \int_{A'} a(x) + b|Xu|^p + c|u|^p \, dx.$$

(v) F is *lower semicontinuous* (resp. *weakly sequentially lower semicontinuous*) if, for any $A' \in \mathcal{A}_0$,

$$F(\cdot, A') : W^{1,p}_X(\Omega) \to [0, +\infty]$$

is sequentially l.s.c. with respect to the strong (resp. weak) topology of $W^{1,p}_X(\Omega)$.

(vi) F is *weakly*-sequentially lower semicontinuous* if, for any $A' \in \mathcal{A}_0$,

$$F(\cdot, A') : W^{1,\infty}(\Omega) \to [0, +\infty]$$

is sequentially l.s.c. with respect to the weak*-topology of $W^{1,\infty}(\Omega)$.

3 Integral representation of convex functionals

In this section, we completely characterize a class of convex local functionals defined on $W^{1,p}_X$. As announced, we exploit [7, Lemma 4.1] to get an integral representation of the form

$$F(u, A) = \int_A f_e(x, u, Du) \, dx \quad \text{for all } A \in \mathcal{A} \text{ and all } u \in W^{1,p}(\Omega).$$

Then the forthcoming Propositions 3.1 and 3.2 guarantee the existence of a non-Euclidean Lagrangian f such that

$$\int_A f(x, u, Xu) \, dx = \int_A f_e(x, u, Du) \, dx \quad \text{for all } A \in \mathcal{A} \text{ and all } u \in C^\infty(A).$$

Finally, we extend the integral representation to the whole $W^{1,p}_{X,\mathrm{loc}}(\Omega)$.

The following propositions, which are almost totally inspired by [20, Theorem 3.5] and [20, Lemma 3.13], allow us to pass from a Euclidean to a non-Euclidean integral representation.

Proposition 3.1. *Let $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ be a Carathéodory function. Define $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty]$ by*

$$f(x, u, \eta) := \begin{cases} f_e(x, u, L^{-1}(x)(\eta)) & \text{if } (x, u, \eta) \in \Omega_X \times \mathbb{R} \times \mathbb{R}^m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Then the following facts hold:

(i) *f is a Carathéodory function.*

(ii) *If $f_e(x, \cdot, \cdot)$ is convex for a.e. $x \in \Omega$, then $f(x, \cdot, \cdot)$ is convex for a.e. $x \in \Omega$.*

(iii) *If $f_e(x,u,\cdot)$ is convex for a.e. $x \in \Omega$ and for any $u \in \mathbb{R}$, then $f(x,u,\cdot)$ is convex for a.e. $x \in \Omega$ and for any $u \in \mathbb{R}$.*

(iv) *If we assume that*

$$f_e(x,u,\xi) = f_e(x,u,\Pi_x(\xi)) \quad \text{for a.e. } x \in \Omega \text{ and all } (u,\xi) \in \mathbb{R}\times\mathbb{R}^n, \tag{3.2}$$

then it follows that

$$\int_A f_e(x,u,Du)\,dx = \int_A f(x,u,Xu)\,dx \quad \text{for all } A \in \mathcal{A} \text{ and all } u \in C^\infty(A). \tag{3.3}$$

Proof. (i) First we want to show that, for any $(u,\eta) \in \mathbb{R}\times\mathbb{R}^m$, the function $x \mapsto f(x,u,\eta)$ is measurable. Let us fix then $(u,\eta) \in \mathbb{R}\times\mathbb{R}^m$, define the function $\Phi : \Omega_X \to \mathbb{R}\times\mathbb{R}^n$ by $\Phi(x) := (u, L^{-1}(x)(\eta))$ and extend it to be zero on $\Omega \setminus \Omega_X$. By Proposition 2.14, $\Phi|_{\Omega_X}$ is continuous, and so in particular Φ is measurable. Noticing that

$$f(x,u,\eta) = f_e(x,\Phi(x)) \quad \text{for all } x \in \Omega_X,$$

with f_e being a Carathéodory function, and recalling [11, Proposition 3.7], we conclude that $x \mapsto f(x,u,\eta)$ is measurable. Let us define now the function

$$\Psi : \Omega_X \times \mathbb{R}\times\mathbb{R}^m \to \Omega_X \times \mathbb{R}\times\mathbb{R}^n, \quad \Psi(x,u,\eta) := (x,u,L^{-1}(x)(\eta)).$$

Since on Ω_X we have that $f = f_e \circ \Psi$, for any fixed $x \in \Omega_X$ such that $f_e(x,\cdot,\cdot)$ is continuous, $f(x,\cdot,\cdot)$ is the composition of a continuous function and a linear function, and so it is continuous.

(ii) If $x \in \Omega_X$ is such that $f_e(x,\cdot,\cdot)$ is convex, then $f = f_e \circ \Psi$ is the composition of a convex function and a linear function, and so it is convex.

(iii) This follows as (ii).

(iv) Assume that (3.2) holds. Let us fix $A \in \mathcal{A}$ and $u \in C^\infty(A)$. From the regularity of u, we have that $Xu(x) = C(x)Du(x)$. By Proposition 2.12, we get

$$\begin{aligned} L_x(\Pi_x(Du)) &= L_x(C(x)^T B(x)^{-1} C(x)Du) \\ &= C(x)C(x)^T B(x)^{-1} C(x)Du \\ &= B(x)B(x)^{-1}C(x)Du \\ &= C(x)Du \\ &= L_x(Du) \end{aligned}$$

and

$$\begin{aligned} f(x,u,Xu) &= f(x,u,C(x)Du) \\ &= f(x,u,L_x(Du)) \\ &= f(x,u,L_x(\Pi_x(Du))) \\ &= f_e(x,u,L_x^{-1}(L_x(\Pi_x(Du)))) \\ &= f_e(x,u,\Pi_x(Du)) \\ &= f_e(x,u,Du). \end{aligned}$$

Now, (3.3) follows by integrating over A. □

In the following result, we provide some sufficient conditions to guarantee (3.2).

Proposition 3.2. *Let $f_e : \Omega\times\mathbb{R}\times\mathbb{R}^n \to [0,+\infty]$ be a Carathéodory function such that the following conditions hold:*

(i) *$f_e(x,u,\cdot)$ is convex for a.e $x \in \Omega$ and any $u \in \mathbb{R}$.*

(ii) *There exist $a \in L^1_{\mathrm{loc}}(\Omega)$ and $b, c > 0$ such that*

$$f_e(x,u,\xi) \le a(x) + b|C(x)\xi|^p + c|u|^p \tag{3.4}$$

for a.e. $x \in \Omega$ and any $(u,\xi) \in \mathbb{R}\times\mathbb{R}^n$.

Then f_e satisfies (3.2).

Proof. This follows with some trivial modifications as in [20, Lemma 3.13]. □

Let us now state and prove the main result of this section.

Theorem 3.3. *Let $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty]$ be such that the following conditions hold:*
(i) *F is a measure.*
(ii) *F is local.*
(iii) *F is convex.*
(iv) *F is p-bounded.*
Then there exists a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty)$ such that

$$(u, \xi) \mapsto f(x, u, \xi) \quad \text{is convex for a.e. } x \in \Omega, \tag{3.5}$$
$$f(x, u, \xi) \le a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^m, \tag{3.6}$$

and the following representation formula holds:

$$F(u, A) = \int_A f(x, u, Xu)\, dx \quad \text{for all } u \in W^{1,p}_{X,\mathrm{loc}}(\Omega) \text{ and all } A \in \mathcal{A}. \tag{3.7}$$

Moreover, if $f_1, f_2 : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty)$ are two Carathéodory functions satisfying (3.5)–(3.7)*, then there exists $\tilde{\Omega} \subseteq \Omega$ such that $|\tilde{\Omega}| = |\Omega|$ and*

$$f_1(x, u, \xi) = f_2(x, u, \xi) \quad \text{for all } x \in \tilde{\Omega} \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^m. \tag{3.8}$$

Proof. We prove this theorem in five steps.

First step. Let

$$C := \max\{\sup\{|c_{j,i}(x)| : x \in \Omega\} : i = 1, \dots, n,\ j = 1, \dots, m\}.$$

Then, from our assumptions on $\mathbf{X}$, it follows that $0 < C < +\infty$. Let $\tilde{b} := C^p b$. By using (iv) and recalling that for all $u \in W^{1,p}(\Omega)$ we have that $Xu(x) = C(x)Du(x)$, it follows that

$$F(u, A') \le \int_{A'} a(x) + c|u|^p + \tilde{b}|Du|^p\, dx \quad \text{for all } A' \in \mathcal{A}_0 \text{ and all } u \in W^{1,p}(\Omega).$$

Thus, we can apply [7, Lemma 4.1] to get a Carathéodory function $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ such that

$$F(u, A) = \int_A f_e(x, u, Du)\, dx \qquad \text{for all } A \in \mathcal{A} \text{ and all } u \in W^{1,p}_{\mathrm{loc}}(\Omega), \tag{3.9}$$
$$f_e(x, u, \xi) \le a(x) + \tilde{b}|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^n,$$
$$f_e(x, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^n \to [0, \infty] \quad \text{is convex for a.e. } x \in \Omega. \tag{3.10}$$

Second step. We want to prove that f_e satisfies (3.2). By Proposition 3.2 and (3.10), we only need to prove (3.4). Let us then take $\Omega' \subseteq \Omega$ such that $|\Omega'| = |\Omega|$ and

$$(u, \xi) \mapsto f_e(x, u, \xi) \quad \text{is convex and finite for all } x \in \Omega',$$

and fix $x \in \Omega'$, $u \in \mathbb{Q}$ and $\xi \in \mathbb{Q}^n$. By (3.9), for any $R > 0$ small enough to ensure that $B_R(x) \Subset \Omega$, we have that

$$F(\varphi_{x,u,\xi}, B_R(x)) = \int_{B_R(x)} f_e(y, u + \langle \xi, y - x\rangle, \xi)\, dy,$$

and from (iv) we have that

$$F(\varphi_{x,u,\xi}, B_R(x)) \le \int_{B_R(x)} a(y) + c|u + \langle \xi, y - x\rangle|^p + b|C(y)\xi|^p\, dy,$$

where $\varphi_{x,u,\xi}$ is as in (2.1). Combining these two facts and dividing by $|B_R(x)|$, we obtain that

$$\fint_{B_R(x)} f_e(y, u + \langle \xi, x - y\rangle, \xi)\, dy \le \fint_{B_R(x)} a(y) + c|u + \langle \xi, y - x\rangle|^p + b|C(y)\xi|^p\, dy. \tag{3.11}$$

Since the integrand on the right-hand side is in $L^1_{\mathrm{loc}}(\Omega)$, and (3.11) holds indeed for all $A' \in \mathcal{A}_0$, the one on the left-hand side is in $L^1_{\mathrm{loc}}(\Omega)$ as well. Therefore, thanks to the Lebesgue theorem, we can find $\Omega_{u,\xi} \subseteq \Omega'$ such that $|\Omega_{u,\xi}| = |\Omega|$ and

$$f_e(x, u, \xi) \le a(x) + c|u|^p + b|C(x)\xi|^p \quad \text{for all } x \in \Omega_{u,\xi}.$$

Setting

$$\tilde{\Omega} := \bigcap_{(u,\xi)\in\mathbb{Q}\times\mathbb{Q}^n} \Omega_{u,\xi},$$

it holds that $|\tilde{\Omega}| = |\Omega|$ and

$$f_e(x, u, \xi) \le a(x) + c|u|^p + b|C(x)\xi|^p \quad \text{for all } x \in \tilde{\Omega} \text{ and all } (u, \xi) \in \mathbb{Q} \times \mathbb{Q}^n.$$

Since the map $(u, \xi) \mapsto f_e(x, u, \xi)$ is continuous for any $x \in \tilde{\Omega}$ and $\mathbb{Q} \times \mathbb{Q}^n$ is dense in $\mathbb{R} \times \mathbb{R}^n$, inequality (3.4) holds and the conclusion follows.

Third step. Thanks to the previous step, we can apply Proposition 3.1 (iv). Hence, we get

$$\int_A f_e(x, u, Du)\, dx = \int_A f(x, u, Xu)\, dx \quad \text{for all } A \in \mathcal{A},\ u \in C^\infty(A), \tag{3.12}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty]$ is the function defined in (3.1). First of all, we can assume that f is finite up to modifying it on a set of measure zero. Moreover, thanks to (3.10) and Proposition 3.1 (ii), we have that f satisfies (3.5). Now we want to prove that f satisfies (3.6). Let us fix $x \in \Omega$, $u \in \mathbb{Q}$ and $\xi \in \mathbb{Q}^n$. By (iv), (3.9) and (3.12), we have that

$$\begin{aligned}\int_{B_R(x)} f(y, \varphi_{x,u,\xi}, X\varphi_{x,u,\xi})\, dy &\le \int_{B_R(x)} a(y) + c|\varphi_{x,u,\xi}|^p + b|X\varphi_{x,u,\xi}|^p\, dy \\ &= \int_{B_R(x)} a(y) + c|u + \langle \xi, y - x\rangle|^p + b|C(y)\xi|^p\, dy,\end{aligned}$$

and so, dividing by $|B_R(x)|$, we get that

$$\fint_{B_R(x)} f(y, u + \langle \xi, y - x\rangle, C(y)\xi)\, dy \le \fint_{B_R(x)} a(y) + c|u + \langle \xi, y - x\rangle|^p + b|C(y)\xi|^p\, dy.$$

Arguing as in the second step, we can conclude that

$$f(x, u, C(x)\xi) \le a(x) + b|C(x)\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^n.$$

Finally, by recalling that for $x \in \Omega_X$ the map $L_x : V_x \to \mathbb{R}^m$ is surjective, inequality (3.6) follows.

Fourth step. Here we want to prove that (3.7) holds. Let us fix $u \in W^{1,p}_X(\Omega)$ and $A' \in \mathcal{A}_0$, and consider the two functionals

$$F_{A'}, G_{A'} : (\{v|_{A'} : v \in W^{1,p}_X(\Omega)\}, \|\cdot\|_{W^{1,p}_X(A')}) \to [0, +\infty]$$

defined by

$$F_{A'}(v|_{A'}) := F(v, A') \quad \text{and} \quad G_{A'}(v|_{A'}) := \int_{A'} f(x, v, Xv)\, dx,$$

respectively. Thanks to (iii), (iv), (3.5) and (3.6), they are convex and bounded on bounded subsets of

$$\{v|_{A'} : v \in W^{1,p}_X(\Omega)\}.$$

Hence, they are continuous (cf. [13, Lemma 2.1]). Moreover, from Proposition 2.8 we can find a sequence $(u_\epsilon)_\epsilon \subseteq W^{1,p}_X(\Omega)$ such that

$$(u_\epsilon|_{A'})_\epsilon \subseteq W^{1,p}_X(A') \cap C^\infty(A') \quad \text{and} \quad u_\epsilon|_{A'} \to u|_{A'} \text{ in } W^{1,p}_X(A').$$

From (3.9) and (3.12), we get that

$$\begin{aligned} F(u, A') &= \lim_{\epsilon \to 0} F(u_\epsilon, A') \\ &= \lim_{\epsilon \to 0} \int_{A'} f_e(x, u_\epsilon, Du_\epsilon) \\ &= \lim_{\epsilon \to 0} \int_{A'} f(x, u_\epsilon, Xu_\epsilon) \\ &= \int_{A'} f(x, u, Xu)\, dx, \end{aligned}$$

and so we assert that

$$F(u, A') = \int_A f(x, u, Xu)\, dx \quad \text{for all } u \in W^{1,p}_X(\Omega) \text{ and all } A' \in \mathcal{A}_0. \tag{3.13}$$

Let us take now $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$, $A \in \mathcal{A}$ and $A' \Subset A$, and, thanks to Proposition 2.7, take a function $v \in W^{1,p}_X(\Omega)$ such that $u|_{A'} = v|_{A'}$. Thus, from hypothesis (ii) and from (3.13), we have that

$$F(u, A') = F(v, A') = \int_{A'} f(x, v, Xv)\, dx = \int_{A'} f(x, u, Xu)\, dx. \tag{3.14}$$

Since by hypothesis the function $B \mapsto F(u, B)$ is inner regular (cf. [12, Theorem 14.23]), and noticing that the function $B \mapsto \int_B f(x, u, Xu)\, dx$ is inner regular, thanks to (3.14) we have that

$$\begin{aligned} F(u, A) &= \sup\{F(u, A') : A' \Subset A\} \\ &= \sup\Big\{ \int_{A'} f(x, u, Xu)\, dx : A' \Subset A \Big\} \\ &= \int_A f(x, u, Xu)\, dx, \end{aligned}$$

and so we can conclude that (3.7) holds.

Fifth step. Let us show the uniqueness of the Lagrangian. Fix then $x \in \Omega$, $u \in \mathbb{Q}$ and $\xi \in \mathbb{Q}^n$: since (3.7) holds both for f_1 and f_2, for any $R > 0$ small enough, we have that

$$\fint_{B_R(x)} f_1(y, u + \langle \xi, y - x\rangle, C(y)\xi)\, dy = \fint_{B_R(x)} f_2(y, u + \langle \xi, y - x\rangle, C(y)\xi)\, dy.$$

Since both integrand functions satisfy (3.6), they are both in $L^1_{\mathrm{loc}}(\Omega)$. Again, thanks to the Lebesgue theorem, there exists $\Omega_{u,\xi} \subseteq \Omega$ such that $|\Omega_{u,\xi}| = |\Omega|$ and

$$f_1(x, u, C(x)\xi) = f_2(x, u, C(x)\xi) \quad \text{for all } x \in \Omega_{u,\xi}.$$

If we set

$$\tilde\Omega := \bigcap_{(u,\xi)\in\mathbb{Q}\times\mathbb{Q}^n} \Omega_{u,\xi} \cap \{x \in \Omega : (3.5) \text{ and } (3.6) \text{ hold for } f_1 \text{ and } f_2\} \cap \Omega_X,$$

clearly we have $|\tilde\Omega| = |\Omega|$, and it holds that

$$f_1(x, u, C(x)\xi) = f_2(x, u, C(x)\xi) \quad \text{for all } x \in \tilde\Omega \text{ and all } (u, \xi) \in \mathbb{Q} \times \mathbb{Q}^n.$$

Since $(u, \xi) \mapsto f_1(x, u, \xi)$ and $(u, \xi) \mapsto f_2(x, u, \xi)$ are continuous for any $x \in \tilde\Omega$, and by recalling again that L_x is surjective for any $x \in \Omega_X$, we infer (3.8). □

The following theorem tells us that all hypotheses of Theorem 3.3 are also necessary.

Theorem 3.4. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty)$ be a Carathéodory function such that*

$$(u, \xi) \mapsto f(x, u, \xi) \qquad \text{is convex for a.e. } x \in \Omega, \tag{3.15}$$

$$f(x, u, \xi) \leq a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^m, \tag{3.16}$$

for some $b, c > 0$ and $a \in L^1_{\mathrm{loc}}(\Omega)$. If we set the functional $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty]$ as

$$F(u, A) := \int_A f(x, u, Xu)\, dx \quad \text{for all } u \in W^{1,p}_{X,\mathrm{loc}}(\Omega) \text{ and all } A \in \mathcal{A},$$

then F satisfies hypotheses (i)–(iv) *of Theorem 3.3.*

Proof. Let us fix $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$. Our aim is to prove that $\alpha(A) := F(u, A)$ is a measure. Notice that, with $f \geq 0$, α is increasing, and of course $\alpha(\emptyset) = 0$. Then, according to [12, Theorem 14.23], it suffices to show that α is subadditive, superadditive and inner regular. The first two properties are trivial, so let us focus on the third one. Let us fix $A \in \mathcal{A}$ and define the sequence of sets $(A_h)_h$ by $A_h := \{x \in A : \mathrm{dist}(x, \partial A) > \frac{1}{h}\}$. We have that $(A_h)_h \subseteq \mathcal{A}_0$, $A_h \Subset A_{h+1} \Subset A$ and $\bigcup_{h\in\mathbb{N}_+} A_h = A$. Thus by the monotone convergence theorem, we conclude that

$$\int_A f(x, u, Xu)\, dx = \int_A \lim_{h\to+\infty} \chi_{A_h} f(x, u, Xu)\, dx = \lim_{h\to+\infty} \int_{A_h} f(x, u, Xu)\, dx,$$

and so α is a measure. Property (ii) is straightforward, noticing that the X-gradients of two a.e. equal functions coincide a.e. Finally, (iii) and (iv) follow from (3.15) and (3.16). □

4 Integral representation of weakly*-sequentially lower semicontinuous functionals

In this section, we characterize a class of local functionals defined on $W^{1,p}_X$ for which we require neither translation-invariance nor convexity, but which are weakly*-sequentially lower semicontinuous in $W^{1,\infty}$. It is well known (cf. [1]) that, for an integral functional of the form

$$F(u, A) := \int_A f_e(x, u, Du)\, dx,$$

the weak*-lower semicontinuity is equivalent to the convexity in the third entry of f_e. Therefore, we can adopt the same strategy employed in the previous section, exploiting [8, Theorem 1.10] to get a Euclidean integral representation of the form

$$F(u, A) = \int_A f_e(x, u, Du)\, dx \quad \text{for all } A \in \mathcal{A} \text{ and all } u \in W^{1,p}(\Omega).$$

Again, Propositions 3.1 and 3.2 guarantee the existence of a non-Euclidean Lagrangian f such that

$$\int_A f(x, u, Xu)\, dx = \int_A f_e(x, u, Du)\, dx \quad \text{for all } A \in \mathcal{A} \text{ and all } u \in C^\infty(A).$$

We now start by proving a useful continuity result in $W^{1,p}_X$, whose classical version is usually known as *Carathéodory continuity theorem.*

Theorem 4.1. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty]$ be a Carathéodory function such that there exist $a \in L^1_{\mathrm{loc}}(\Omega)$ and $b, c > 0$ such that*

$$f(x, u, \xi) \leq a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^m. \tag{4.1}$$

Then it holds that, for any $A' \in \mathcal{A}_0$, the functional

$$F : W^{1,p}_X(A') \to [0, +\infty)$$

defined by

$$F(u) := \int_{A'} f(x, u, Xu)\, dx$$

is continuous with respect to the strong topology of $W^{1,p}_X(A')$.

Proof. We prove this theorem in two steps.

First step. Let us prove that F is lower semicontinuous. Fix $u \in W^{1,p}_X(A')$ and take a sequence $(u_h)_h \subseteq W^{1,p}_X(A')$ converging to u and such that there exists

$$\lim_{h\to+\infty} F(u_h) < +\infty.$$

Up to a subsequence, we can assume that $(u_h(x))_h$ converges to $u(x)$ and $(Xu_h(x))_h$ converges to $Xu(x)$ for a.e. $x \in A'$. Since f is Carathéodory, it follows that

$$\lim_{h\to\infty} f(x, u_h(x), Xu_h(x)) = f(x, u(x), Xu(x)) \quad \text{for a.e. } x \in \Omega.$$

Thanks to Fatou's lemma, we conclude that

$$\begin{aligned} F(u) &= \int_{A'} f(x, u, Xu)\, dx \\ &= \int_{A'} \liminf_{h\to+\infty} f(x, u_h, Xu_h) \\ &\le \liminf_{h\to+\infty} \int_{A'} f(x, u_h, Xu_h) \\ &= \lim_{h\to+\infty} F(u_h). \end{aligned}$$

Second step. Here we want to prove that F is upper semicontinuous. Again, fix $u \in W^{1,p}_X(A')$ and take a sequence $(u_h)_h \subseteq W^{1,p}_X(A')$ converging to u and such that there exists

$$\lim_{h\to+\infty} F(u_h) > -\infty.$$

Up to a subsequence, we can assume that $(u_h(x))_h$ converges to $u(x)$ and $(Xu_h(x))_h$ converges to $Xu(x)$ for almost every $x \in A'$. Let us define the sequence of functions

$$g_h(x) := -f(x, u_h, Xu_h) + C(|Xu_h|^p + |u_h|^p),$$

where $C := \max\{b, c\} > 0$. Using (4.1), we get

$$g_h(x) \ge -a(x) \quad \text{for a.e. } x \in A',$$

Since the right-hand side belongs to $L^1(A')$, we can apply Fatou's lemma and get that

$$\begin{aligned} \int_{A'} -f(x, u, Xu)\, dx + \|u\|_{W^{1,p}_X(A')} &= \int_{A'} \liminf_{h\to+\infty} g_h(x, u, Xu)\, dx \\ &= \int_{A'} \liminf_{h\to+\infty} (-f(x, u_h, Xu_h) + C(|Xu_h|^p + |u_h|^p))\, dx \\ &\le \liminf_{h\to+\infty} \int_{A'} -f(x, u_h, Xu_h) + C(|Xu_h|^p + |u_h|^p))\, dx \\ &= \lim_{h\to+\infty} \int_{A'} -f(x, u_h, Xu_h) + C \lim_{h\to+\infty} \|u_h\|_{W^{1,p}_X(A')} \\ &= \lim_{h\to+\infty} \int_{A'} -f(x, u_h, Xu_h) + \|u\|_{W^{1,p}_X(A')}, \end{aligned}$$

as desired. □

In the following proposition, we prove that the notion of lower semicontinuity introduced in Definition 2.18 is actually equivalent to a more useful condition.

Proposition 4.2. *Let $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty]$ be such that the following conditions hold:*
(i) *F is a measure.*
(ii) *F is local.*
Then the following conditions are equivalent:
(a) *F is lower semicontinuous.*
(b) *For all $A' \Subset \mathcal{A}_0$,*

$$F_{A'} : (\{u|_{A'} : u \in W^{1,p}_X(\Omega)\}, \|\cdot\|_{W^{1,p}_X(A')}) \to [0, +\infty]$$

defined by $F_{A'}(u|_{A'}) := F(u, A')$ is lower semicontinuous.

Proof. "(b) $\Longrightarrow$ (a)": It is straightforward.
"(a) $\Longrightarrow$ (b)": Fix an open set $A' \in \mathcal{A}_0$ and take $(u_h)_h$, u in $W^{1,p}_X(\Omega)$ such that

$$\|u_h|_{A'} - u|_{A'}\|_{W^{1,p}(A')} \to 0.$$

Now, for any $k \in \mathbb{N}$, take an open set A_k such that $A_k \Subset A_{k+1} \Subset A'$ and $\bigcup_{k=0}^{+\infty} A_k = A'$, and a smooth cut-off function φ_k between A_k and A'. For any $h, k \in \mathbb{N}$, define the functions $v^k := \varphi_k u$ and $v^k_h := \varphi_k u_h$. We have, for any $h, k \in \mathbb{N}$, that the v^k_h, v^k belong to $W^{1,p}_X(\Omega)$, $v^k_h|_{A_k} = u_h|_{A_k}$, $v^k|_{A_k} = u|_{A_k}$, and moreover

$$\lim_{h\to\infty} \|v^k_h - v^k\|_{W^{1,p}_X(\Omega)} = 0 \quad \text{for any } k \in \mathbb{N}.$$

Using (i) and (ii), we get

$$\begin{aligned}
F(u, A') &= \lim_{k\to\infty} F(u, A_k) \\
&= \lim_{k\to\infty} F(v^k, A_k) \\
&\le \lim_{k\to\infty} \liminf_{h\to\infty} F(v^k_h, A_k) \\
&= \lim_{k\to\infty} \liminf_{h\to\infty} F(u_h, A_k) \\
&\le \lim_{k\to\infty} \liminf_{h\to\infty} F(u_h, A') \\
&= \liminf_{h\to\infty} F(u_h, A'),
\end{aligned}$$

as desired. □

We are ready to state the main result of this section.

Theorem 4.3. *Let $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty]$ be such that the following conditions hold:*
(i) *F is a measure.*
(ii) *F is local.*
(iii) *F satisfies the weak condition (ω).*
(iv) *F is p-bounded.*
(v) *F is weakly*-sequentially lower semicontinuous.*
(vi) *F is lower semicontinuous.*
Then there exists a unique Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty)$ such that

$$\xi \mapsto f(x, u, \xi) \quad \text{is convex for a.e. } x \in \Omega \text{ and all } u \in \mathbb{R}, \tag{4.2}$$
$$f(x, u, \xi) \le a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^m, \tag{4.3}$$

and the following representation formula holds:

$$F(u, A) = \int_A f(x, u, Xu)\,dx \quad \text{for all } u \in W^{1,p}_{X,\mathrm{loc}}(\Omega) \text{ and all } A \in \mathcal{A}. \tag{4.4}$$

Remark 4.4. If we substitute hypotheses (v) and (vi) with
(v') F is weakly sequentially lower semicontinuous,
then the conclusions of Theorem 4.3 still hold. Indeed, thanks to Proposition 2.5, the latter is stronger than both (v) and (vi), even if not equivalent in general.

Proof of Theorem 4.3. We prove this theorem in two steps.

First step. By arguing as in the first step of the proof of Theorem 3.3, the restriction of F to $W^{1,p}_{\mathrm{loc}}(\Omega) \times \mathcal{A}$ satisfies all hypotheses of [8, Theorem 1.10]. Thus there exist $\tilde{b} > 0$ and a Carathéodory function

$$f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$$

such that

$$F(u, A) = \int_A f_e(x, u, Du)\, dx \qquad \text{for all } A \in \mathcal{A} \text{ and all } u \in W^{1,p}_{\mathrm{loc}}(\Omega), \tag{4.5}$$

$$f_e(x, u, \xi) \le a(x) + \tilde{b}|\xi|^p + c|u|^p \qquad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^n, \tag{4.6}$$

$$f_e(x, u, \cdot) : \mathbb{R}^n \to [0, \infty] \qquad \text{is convex for a.e. } x \in \Omega \text{ and all } u \in \mathbb{R}. \tag{4.7}$$

Now, arguing as in the second step of the proof of Theorem 3.3, from (4.6) and (4.7) and recalling Propositions 3.1 and 3.2, we obtain that

$$\int_A f_e(x, u, Du)\, dx = \int_A f(x, u, Xu)\, dx \quad \text{for all } A \in \mathcal{A},\ u \in C^\infty(A), \tag{4.8}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty]$ is the Carathéodory function defined in (3.1). Up to modifying f on a set of measure zero, we can assume that it is finite. Moreover, by arguing as in the third step of the proof of Theorem 3.3, f satisfies (4.2) and (4.3).

Second step. Here we prove that (4.4) holds. Let us start by fixing $u \in W^{1,p}_X(\Omega)$ and $A' \in \mathcal{A}_0$. Thanks to Proposition 2.8, we can find a sequence $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ such that

$$(u_h|_{A'})_h \subseteq W^{1,p}_X(A') \cap C^\infty(A') \quad \text{and} \quad u_h|_{A'} \to u|_{A'} \text{ in } W^{1,p}_X(A').$$

From this, (vi), (4.5), (4.8), Theorem 4.1 and Proposition 4.2, it follows that

$$\begin{aligned} F(u, A') &\le \liminf_{h\to+\infty} F(u_h, A') \\ &= \liminf_{h\to+\infty} \int_{A'} f_e(x, u_h, Du_h)\, dx \\ &= \lim_{h\to+\infty} \int_{A'} f(x, u_h, Xu_h)\, dx \\ &= \int_{A'} f(x, u, Xu)\, dx, \end{aligned}$$

and hence we obtain that

$$F(u, A') \le \int_{A'} f(x, u, Xu)\, dx \quad \text{for all } A' \in \mathcal{A}_0 \text{ and all } u \in W^{1,p}_X(\Omega). \tag{4.9}$$

To prove the converse inequality, fix $u_0 \in W^{1,p}_X(\Omega)$ and set

$$H : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty], \quad H(u, A) := F(u + u_0, A).$$

It is straightforward to check that H satisfies all hypotheses of the theorem. Hence, there exist a Carathéodory function $h : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty)$, $a_H \in L^1_{\mathrm{loc}}(\Omega)$ and $b_H, c_H > 0$ such that

$$h(x, u, \xi) \le a_H(x) + b_H|\xi|^p + c_H|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^m.$$

Moreover, it holds that

$$H(u,A)=\int_A h(x,u,Xu)\,dx \quad \text{for all } A\in\mathcal{A} \text{ and all } u\in C^\infty(A), \tag{4.10}$$

and

$$H(u,A')\le\int_{A'} h(x,u,Xu)\,dx \quad \text{for all } A'\in\mathcal{A}_0 \text{ and all } u\in W^{1,p}_X(\Omega). \tag{4.11}$$

Fix then $A'\in\mathcal{A}_0$. Arguing as before, we can find a sequence $(u_h)_h\subseteq W^{1,p}_X(\Omega)$ such that

$$(u_h|_{A'})_h\subseteq W^{1,p}_X(A')\cap C^\infty(A') \quad \text{and} \quad u_h|_{A'}\to u_0|_{A'} \text{ in } W^{1,p}_X(A').$$

Thus, thanks to Theorem 4.1, we get

$$\begin{aligned}
\int_{A'} h(x,0,0) &= H(0,A') && \text{(by (4.10))}\\
&= F(u_0,A')\\
&\le \int_{A'} f(x,u_0,Xu_0)\,dx && \text{(by (4.9))}\\
&= \lim_{h\to+\infty}\int_{A'} f(x,u_h,Xu_h)\,dx\\
&= \lim_{h\to+\infty} F(u_h,A')\\
&= \lim_{h\to+\infty} H(u_h-u_0,A')\\
&\le \lim_{h\to+\infty}\int_{A'} h(x,u_h-u_0,Xu_h-Xu_0)\,dx && \text{(by (4.11))}\\
&= \int_{A'} h(x,0,0)\,dx,
\end{aligned}$$

and all inequalities are indeed equalities. Since u_0 is arbitrarily chosen, we conclude that

$$F(u,A')=\int_{A'} f(x,u,Xu)\,dx \quad \text{for all } u\in W^{1,p}_X(\Omega) \text{ and all } A'\in\mathcal{A}_0.$$

The rest of the proof follows as in the proof of Theorem 3.3. □

The following theorem shows that the hypotheses of Theorem 4.3 are also necessary.

Theorem 4.5. *Let $f:\Omega\times\mathbb{R}\times\mathbb{R}^m\to[0,+\infty)$ be a Carathéodory function such that*

$$\begin{aligned}
&\xi\mapsto f(x,u,\xi) && \text{is convex for a.e. } x\in\Omega \text{ and all } u\in\mathbb{R},\\
&f(x,u,\xi)\le a(x)+b|\xi|^p+c|u|^p && \text{for a.e. } x\in\Omega \text{ and all } (u,\xi)\in\mathbb{R}\times\mathbb{R}^m,
\end{aligned} \tag{4.12}$$

for $b,c>0$ and $a\in L^1_{\mathrm{loc}}(\Omega)$, and define the functional $F:W^{1,p}_{X,\mathrm{loc}}(\Omega)\times\mathcal{A}\to[0,+\infty]$ by

$$F(u,A):=\int_A f(x,u,Xu)\,dx \quad \text{for all } u\in W^{1,p}_{X,\mathrm{loc}}(\Omega) \text{ and all } A\in\mathcal{A}.$$

Then F satisfies hypotheses (i)–(vi) *of Theorem 4.3.*

Proof. Condition (i) follows as in the proof of Theorem 3.4, while (ii) is trivial. In order to prove (iii), let us show that F satisfies the strong property (ω). This suffices, according to [8]. Since f is Carathéodory, the set

$$\Omega':=\{x\in\Omega:(u,\xi)\mapsto f(x,u,\xi) \text{ is continuous}\}$$

satisfies $|\Omega'| = |\Omega|$. For any $k \in \mathbb{N}$ and $\epsilon > 0$, set $E^k_\epsilon \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ as

$$E^k_\epsilon := \{(u, v, \xi, \eta) : |u|, |v|, |\xi|, |\eta| \leq k,\ |u - v|, |\xi - \eta| \leq \epsilon\}$$

and the function

$$\omega_k(x, \epsilon) := \begin{cases} \sup\{|f(x, u, \xi) - f(x, v, \eta)| : (u, v, \xi, \eta) \in E^k_\epsilon\} & \text{if } x \in \Omega', \\ 0 & \text{otherwise.} \end{cases}$$

We show that, for any k, ω_k is a locally integrable modulus of continuity. Let us then fix $\epsilon \geq 0$. Since $(u, \xi) \mapsto f(x, u, \xi)$ is continuous for almost every $x \in \Omega$, the supremum in the definition of ω_k can be taken over a countable subset of E^k_ϵ. Since for any (u, v, ξ, η) the function $x \mapsto |f(x, u, \xi) - f(x, v, \eta)|$ is measurable, we have that $\omega_k(\cdot, \epsilon)$ is measurable. We are left to show that it belongs to $L^1_{\text{loc}}(\Omega)$. Observe that by (4.12) it follows that, for any $(u, v, \xi, \eta) \in E^k_\epsilon$,

$$\begin{aligned} |f(x, u, \xi) - f(x, v, \eta)| &\leq 2|a(x)| + b|\xi|^p + b|\eta|^p + c|u|^p + c|v|^p \\ &\leq 2|a(x)| + 4k(b + c). \end{aligned}$$

Since the right-hand side does not depend on $(u, v, \xi, \eta) \in E^k_\epsilon$, we conclude that

$$\omega_k(x, \epsilon) \leq 2|a(x)| + 4k(b + c).$$

Hence, $\omega_k(\cdot, \epsilon) \in L^1_{\text{loc}}(\Omega)$. Fix now $x \in \Omega'$. Since $E^k_\epsilon \subseteq E^k_\delta$ for any $\epsilon \leq \delta$, we have that $\omega_k(x, \cdot)$ is increasing, and $\omega_k(x, 0) = 0$. Finally, its continuity follows from the continuity of $f(\cdot, u, \xi)$. Then $(\omega_k)_k$ is a sequence of locally integrable moduli of continuity. Let us recall that, if we define

$$C := \max\{\sup\{|c_{j,i}(x)| : x \in \Omega\} : i = 1, \dots, n,\ j = 1, \dots, m\},$$

it holds that $0 < C < +\infty$. Let us define now, for any $k \in \mathbb{N}$, the function

$$\tilde{\omega}_k(x, \epsilon) := \omega_{(\lfloor C \rfloor + 1)k}(x, C\epsilon) \quad \text{for all } x \in \Omega \text{ and all } \epsilon \geq 0.$$

Of course, we have that $(\tilde{\omega}_k)_k$ is still a sequence of locally integrable moduli of continuity: we show that such a sequence satisfies (2.2). Take $A' \in \mathcal{A}_0$, $k \in \mathbb{N}$, $\epsilon \geq 0$ and $u, v \in C^1(\overline{\Omega})$ such that

$$|u(x)|, |v(x)|, |Du(x)|, |Dv(x)| \leq k, \quad |u(x) - v(x)|, |Du(x) - Dv(x)| \leq \epsilon \qquad \text{for all } x \in A'.$$

Then it follows that

$$\begin{aligned} |Xu(x)| &= |C(x)Du(x)| \leq C|Du(x)| \leq Ck \leq (\lfloor C \rfloor + 1)k, \\ |Xv(x)| &= |C(x)Dv(x)| \leq C|Dv(x)| \leq Ck \leq (\lfloor C \rfloor + 1)k, \\ |Xu(x) - Xv(x)| &= |C(x)(Du(x) - Dv(x))| \leq C|Du(x) - Dv(x)| \leq C\epsilon. \end{aligned}$$

Thus we conclude that

$$|F(u, A') - F(v, A')| \leq \int_{A'} |f(x, u, Xu) - f(x, v, Xv)|\, dx \leq \int_{A'} \tilde{\omega}_k(x, \epsilon)\, dx,$$

and so also (iii) is proved. Condition (iv) follows easily from (4.12), while (vi) is a direct consequence of Theorem 4.1. Let us now define $H : W^{1,\infty}(\Omega) \times \mathcal{A} \to [0, +\infty]$ as the restriction to $W^{1,\infty}(\Omega) \times \mathcal{A}$ of F. Since for every $u \in W^{1,\infty}(\Omega)$ it holds that $Xu(x) = C(x)Du(x)$, if we define $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ by

$$f_e(x, u, \xi) := f(x, u, C(x)\xi),$$

we can easily notice that f_e is a Carathéodory function, convex in the third argument and such that

$$H(u, A) = \int_A f_e(x, u, Du)\, dx.$$

By applying [1, Theorem 2.1], condition (v) holds for H, and hence for F. □

5 Integral representation of non-convex functionals

In this section, we want to exploit [8, Theorem 1.8] to characterize a class of local functionals for which again we neither require translation-invariance nor convexity, and for which we want to weaken the assumption of weak*-sequential lower semicontinuity in Theorem 4.3. Convexity was a crucial assumption in Proposition 3.2 to guarantee the validity of (3.2), which can be easily seen to fail if we drop it. Indeed, we have the following example.

Example 5.1. Let us take $\Omega = B_1(0) \subseteq \mathbb{R}^2$, $m = 1$ and

$$X_1 := x\frac{\partial}{\partial y}.$$

Then X_1 is a Lipschitz vector field satisfying (LIC) on Ω, with $N_X := \{(x,y) \in \Omega : x = 0\}$. Clearly, for all $(x,y) \in \Omega_X$ we have

$$C((x,y))^T \cdot B^{-1}((x,y)) \cdot C((x,y)) = \begin{bmatrix} 0 \\ x \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{x^2} \end{bmatrix} \cdot \begin{bmatrix} 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus by Proposition 2.12, it follows that

$$\Pi_{(x,y)}(\xi_1, \xi_2) = (0, \xi_2) \quad \text{for all } (\xi_1,\xi_2) \in \mathbb{R}^2 \text{ and all } (x,y) \in \Omega_X. \tag{5.1}$$

Let us define the map $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to [0, +\infty)$ by

$$f_e((x,y), u, (\xi_1,\xi_2)) := \begin{cases} 1 - \xi_1^2 - \xi_2^2 & \text{if } \xi_1^2 + \xi_2^2 \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f_e is a bounded Carathéodory function not convex in the third entry. Moreover, for any $(x,y) \in \Omega_X$ and $(\xi_1,\xi_2) \in \mathbb{R}^2$ with $\xi_1^2 + \xi_2^2 \le 1$, thanks to (5.1) it holds that

$$f_e((x,y), u, \Pi_{(x,y)}(\xi_1,\xi_2)) = 1 - \xi_2^2.$$

We conclude that (3.2) does not hold.

On the other hand, it is easy to see that there are cases when Proposition 3.2 still holds even if the Lagrangian is not convex in the third argument, as the following example shows.

Example 5.2. Let us take n, m, $\mathbf{X}$ and Ω as in the previous example, and define the function

$$f_e : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to [0, +\infty)$$

by

$$f_e((x,y), u, (\xi_1,\xi_2)) := \begin{cases} 1 - \xi_2^2 & \text{if } |\xi_2| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then f_e is again a bounded Carathéodory function which is not convex in the third entry. Anyway, we can easily see that f_e satisfies (3.2).

At this point, we may ask ourselves if there is a way to weaken the convexity of f_e in the third entry which is still able to guarantee the validity of (3.2). In the previous example, we see that even if f_e is not globally convex in the third entry, it is anyway convex along the direction indicated by N_x. This leads us to the following definition.

Definition 5.3. We say that a Carathéodory function $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ is *X-convex* if, for a.e. $x \in \Omega$ and for any $u \in \mathbb{R}$, $t \in (0,1)$ and $\xi_1, \xi_2 \in \mathbb{R}^n$ such that $\xi_2 - \xi_1 \in N_x$, it holds that

$$f_e(x, u, t\xi_1 + (1-t)\xi_2) \le t f_e(x, u, \xi_1) + (1-t) f_e(x, u, \xi_2).$$

The following proposition tells us that X-convexity is the proper requirement that we have to assume on the Euclidean Lagrangian.

Proposition 5.4. *Let $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ be a Carathéodory function such that there exist $a \in L^1_{\mathrm{loc}}(\Omega)$ and $b, c > 0$ such that*

$$f_e(x, u, \xi) \le a(x) + b|C(x)\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{5.2}$$

Then the following facts are equivalent:
(i) *f_e is X-convex.*
(ii) *For a.e. $x \in \Omega$ and for any $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$, the function $g : N_x \to [0, +\infty]$ defined by $g(\eta) := f_e(x, u, \xi + \eta)$ is constant.*
(iii) *It holds*
$$f_e(x, u, \xi) = f_e(x, u, \Pi_x(\xi)) \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^n.$$

Proof. “(ii) $\iff$ (iii)”: Fix $x \in \Omega$ such that (ii) holds. For any $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$, we have that

$$f_e(x, u, \xi) = f_e(x, u, \xi_{N_x} + \Pi_x(\xi)) = f_e(x, u, \Pi_x(\xi)).$$

Conversely, take $x \in \Omega$ such that (iii) holds. For any $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and $\eta \in N_x$, it holds that

$$f_e(x, u, \xi + \eta) = f_e(x, u, \Pi_x(\xi + \eta)) = f_e(x, u, \Pi_x(\xi)) = f_e(x, u, \xi).$$

“(i) $\iff$ (ii)”: The fact that (ii) implies (i) is trivial. Conversely, assume (i) and fix $x \in \Omega$ such that (i) holds and $a(x) < +\infty$. Thanks to (5.2), we have that, for any fixed $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $\eta \in N_x$,

$$\begin{aligned} g(\eta) &= f_e(x, u, \xi + \eta) \\ &\le a(x) + b|C(x)\xi + C(x)\eta|^p + c|u|^p \\ &= a(x) + b|C(x)\xi|^p + c|u|^p \\ &< +\infty. \end{aligned}$$

Since the right-hand side does not depend on η, g is bounded on N_x. Since by assumption it is also convex on N_x, g is constant. □

In order to guarantee the X-convexity of the Euclidean Lagrangian, we exploit the zig-zag argument employed in [8, Lemma 2.11].

Lemma 5.5. *Let $F : W^{1,p}_{\mathrm{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty]$ be such that the following conditions hold:*
(i) *For all $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, the map $A \mapsto F(u, A)$ is a measure.*
(ii) *For all $u, v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and all $A' \in \mathcal{A}_0$,*
$$u|_{A'} = v|_{A'} \quad \text{implies} \quad F(u, A') = F(v, A').$$
(iii) *F satisfies the weak condition (ω).*
(iv) *For any $A' \in \mathcal{A}_0$ and $(u_h)_h \subseteq W^{1,p}(\Omega)$, $u \in W^{1,p}(\Omega)$ such that*
$$\lim_{h\to\infty} \|u_h - u\|_{W^{1,p}_X(\Omega)} = 0,$$
it holds
$$F(u, A') \le \liminf_{h\to\infty} F(u_h, A').$$

If for any $x \in \Omega$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ we define

$$f_e(x, u, \xi) := \limsup_{R\to 0} \frac{F(\varphi_{x,u,\xi}, B_R(x))}{|B_R(x)|}, \tag{5.3}$$

then it holds that f_e is X-convex.

Proof. A slight modification of [8, Lemma 2.10] ensures the existence of a sequence $(\omega_k)_k$ of locally integrable moduli of continuity and a set $\Omega' \subseteq \Omega$ such that $|\Omega'| = |\Omega|$ and all points in Ω' are Lebesgue points of $x \mapsto \omega_k(x, r)$ for any $k \in \mathbb{N}$ and for any $r \ge 0$. Moreover,

$$|f_e(x, u, \xi) - f_e(x, v, \xi)| \le \omega_k(x, |u - v|) \tag{5.4}$$

for any $x \in \Omega'$, $k \in \mathbb{N}$, $u, v \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ such that

$$|\xi|, |u|, |v| \leq k.$$

Take $x \in \Omega'$, $z \in \mathbb{R}$, $t \in (0,1)$ and $\xi_1 \neq \xi_2$ in $\mathbb{R}^n$ such that $\xi_2 - \xi_1 \in N_x$, and set $\xi := t\xi_1 + (1-t)\xi_2$. We want to prove that

$$f_e(x,z,\xi) \leq t f_e(x,z,\xi_1) + (1-t) f_e(x,z,\xi_2).$$

Let us define

$$\xi_0 := \frac{\xi_2 - \xi_1}{|\xi_2 - \xi_1|}$$

and, for any $h \in \mathbb{N}$, $k \in \mathbb{Z}$ and $i = 1, 2$, set

$$\begin{aligned}
\Omega^1_{h,k} &:= \Big\{ y \in \Omega : \frac{k-1}{h} \leq (\xi_0, y) < \frac{k-1+t}{h} \Big\}, \\
\Omega^2_{h,k} &:= \Big\{ y \in \Omega : \frac{k-1+t}{h} \leq (\xi_0, y) < \frac{k}{h} \Big\}, \\
\Omega^i_h &:= \bigcup_{k \in \mathbb{Z}} \Omega^i_{h,k}, \\
u(y) &:= z + (\xi, y - x) \quad \text{for all } y \in \Omega, \\
v_h(y) &:= \begin{cases} (1-t)\dfrac{k-1}{h}|\xi_2 - \xi_1| + z + (\xi_1, y - x) & \text{if } y \in \Omega^1_{h,k}, \\ -t\dfrac{k}{h}|\xi_2 - \xi_1| + z + (\xi_2, y - x) & \text{if } y \in \Omega^2_{h,k}. \end{cases}
\end{aligned}$$

Arguing as in the proof of [7, Lemma 2.11], we have that $v_h \to u$ uniformly on Ω. Hence, in particular, $v_h \to u$ strongly in $L^p(\Omega)$. Moreover, since $\xi_2 - \xi_1$ belongs to N_x and ξ is a convex combination of ξ_1 and ξ_2, both $\xi - \xi_1$ and $\xi - \xi_2$ belong to N_x. Thus for $i = 1, 2$ and for any $y \in \Omega^i_{h,k}$, we have that

$$|Xu(y) - Xv_h(y)| = |C(x)\xi - C(x)\xi_i| = |C(x)(\xi - \xi_i)| = 0.$$

Therefore, v_h converges to u strongly in $W^{1,p}_X(\Omega)$. Take now $k \in \mathbb{N}_+$ such that, for any $y \in \Omega$ and for any $h \in \mathbb{N}_+$,

$$|\xi_1|, |\xi_2|, |u_1(y)|, |u_2(y)|, |v_h(y)| \leq k.$$

Then, thanks to (5.4) and by noticing that (see [8, Lemma 2.4])

$$F(u, A) = \int_A f_e(x, u, Du)\,dx \quad \text{for all } u \text{ affine on } \Omega \text{ and all } A \in \mathcal{A},$$

arguing as in [7, Lemma 2.11] and setting $B^i_{h,R}(x) := B_R(x) \cap \Omega^i_h$ for $i = 1, 2$ and for any $R > 0$ such that $B_R(x) \Subset \Omega$, we obtain that

$$F(v_h, B_R(x)) \leq \int_{B^1_{h,R}(x)} f_e(y, u_1, Du_1)\,dy + \int_{B^2_{h,R}(x)} f_e(y, u_2, Du_2)\,dy + \int_\Omega w_k\Big(y, aR + \frac{b}{h}\Big),$$

with $a := |\xi_2 - \xi_1|$ and $b := at(1-t)$. Since v_h converges to u strongly in $W^{1,p}_X(\Omega)$ and thanks to hypothesis (iv), it is easy to see that

$$F(u, B_R(x)) \leq tF(u_1, B_R(x)) + (1-t)F(u_2, B_R(x)) + \int_\Omega w_k(y, \epsilon),$$

where this inequality holds for any $\epsilon > 0$ and for any $R \in (0, \frac{\epsilon}{a}]$. Dividing both sides by $|B_R(x)|$, passing to the limsup and recalling that x is a Lebesgue point of $y \mapsto w_k(y, \epsilon)$, we have that

$$f_e(x,z,\xi) \leq t f_e(x,z,\xi_1) + (1-t) f_e(x,z,\xi_2) + w_k(x, \epsilon).$$

By letting ϵ go to zero, the lemma is proved. □

We are now ready to state and prove the main result of this section.

Theorem 5.6. *Let $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty]$ be such that the following conditions hold:*
(i) *F is a measure.*
(ii) *F is local.*
(iii) *F satisfies the strong condition* (ω).
(iv) *F is p-bounded.*
(v) *F is lower semicontinuous.*
Then there exists a unique Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty)$ such that

$$f(x, u, \xi) \le a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^m, \tag{5.5}$$

and the following representation formula holds:

$$F(u, A) = \int_A f(x, u, Xu)\, dx \quad \text{for all } u \in W^{1,p}_{X,\mathrm{loc}}(\Omega) \text{ and all } A \in \mathcal{A}. \tag{5.6}$$

Proof. Let us consider the restriction of F to $W^{1,p}_{\mathrm{loc}}(\Omega) \times \mathcal{A}$. By arguing as in the first step of the proof of Theorem 3.3, it is easy to see that it satisfies all hypotheses of [8, Theorem 1.8]. Thus, if f_e is defined as in (5.3), it is a Carathéodory function and moreover there exists $\tilde{b} > 0$ such that

$$F(u, A) = \int_A f_e(x, u, Du)\, dx \quad \text{for all } A \in \mathcal{A} \text{ and all } u \in W^{1,p}_{\mathrm{loc}}(\Omega),$$

and

$$f_e(x, u, \xi) \le a(x) + \tilde{b}|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^m.$$

Moreover, thanks to Lemma 5.5, f_e is X-convex. So, recalling Proposition 5.4 and Proposition 3.1 (iv), we get that

$$\int_A f_e(x, u, Du)\, dx = \int_A f(x, u, Xu)\, dx \quad \text{for all } A \in \mathcal{A},\ u \in C^\infty(A),$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty]$ is the function defined in (3.1). Such an f can be supposed to be finite up to modifying it on a set of measure zero. By arguing as in the third step of the proof of Theorem 3.3, inequality (5.5) holds, while (5.6) follows exactly as in the last step of the proof of Theorem 4.3. Finally, uniqueness follows as usual. □

Proceeding exactly as in Theorem 4.5, we have the following theorem.

Theorem 5.7. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty)$ be a Carathéodory function such that*

$$f(x, u, \xi) \le a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega \text{ and all } (u, \xi) \in \mathbb{R} \times \mathbb{R}^m,$$

for $b, c > 0$ and $a \in L^1_{\mathrm{loc}}(\Omega)$. If we define the functional $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty]$ by

$$F(u, A) := \int_A f(x, u, Xu)\, dx \quad \text{for all } u \in W^{1,p}_{X,\mathrm{loc}}(\Omega) \text{ and all } A \in \mathcal{A},$$

then F satisfies hypotheses (i)–(v) *of Theorem 5.6.*

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