EFFECTIVE METHODS TO COMPUTE THE TOPOLOGY OF REAL ALGEBRAIC SURFACES WITH ISOLATED SINGULARITIES

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Abstract. Given a real algebraic surface S in $\mathbb{RP}^3$, we propose a constructive procedure to determine the topology of S and to compute non-trivial topological invariants for the pair $(\mathbb{RP}^3, S)$ under the hypothesis that the real singularities of S are isolated. In particular, starting from an implicit equation of the surface, we compute the number of connected components of S, their Euler characteristics and the weighted 2-adjacency graph of the surface.

1. Introduction

Given a real algebraic surface S in $\mathbb{RP}^3$ by means of an implicit equation, the problem of recognizing the topology of the surface can be addressed at two different levels: either considering S only as an abstract topological space, or taking into account also its embedding in $\mathbb{RP}^3$ and looking at the topology of the pair $(\mathbb{RP}^3, S)$.

When the surface S is non-singular, the possible topological models for the connected components of S are given by the topological classification theorem for surfaces. Thus, if S is implicitely defined by an equation of even degree, all its connected components are orientable topological 2-manifolds and hence homeomorphic to a sphere with g handles (i.e. to a sphere or, if $g \geq 1$, to a torus with g holes); if the equation that defines S has an odd degree, then S contains exactly one non-orientable connected component homeomorphic to the connected sum of a projective plane and a sphere with g handles, while all the other components are orientable.

If we want to consider also how a surface is embedded in $\mathbb{RP}^3$, we say that two surfaces S, S' are ambient-homeomorphic if there exists a homeomorphism $\varphi : \mathbb{RP}^3 \to \mathbb{RP}^3$ such that $\varphi(S) = S'$; in this case we also say that the pairs $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are homeomorphic. At present there is no classification of the pairs $(\mathbb{RP}^3, S)$ up to homeomorphism even in the non-singular case and deciding whether two pairs $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are homeomorphic is a very hard problem, also for simple classes of surfaces such as tori (with one hole). Hence a useful contribution in this direction is to be able to compute topological invariants of the pair $(\mathbb{RP}^3, S)$.

A traditional approach to an algorithmical determination of the topology of a surface is via Cylindrical Algebraic Decomposition ([C] , see also [BPR]), which provides a cellular decomposition of the pair $(\mathbb{RP}^3, S)$. Different approaches have been proposed by other authors, starting from Gianni and Traverso ([GT]) who outlined a method based on the use of Morse theory; recently Mourrain and Técourt ([MT]) have proposed an algorithm that computes a simplicial complex isotopic to a given surface.

Developing the ideas in [GT], the papers [FGPT] and [FGL] give constructive answers to the problem of recognizing topologically a real algebraic non-singular surface proposing algorithms that compute the number of its connected components and the Euler characteristic of each of them, which determines them up to homeomorphism. In [FGLP] the non-singular surface is considered together with its embedding in $\mathbb{RP}^3$; the authors describe an algorithmical method to compute the

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"weighted adjacency graph" of the surface, which gives information both on the mutual disposition of the connected components and on their contractibility.

In this paper we address from a constructive point of view the same questions when the surface contains only isolated singularities. Our aim is to find a discrete set of data that are algorithmically computable, sufficient to determine the topology of the surface and that are non-trivial topological invariants for the pair $(\mathbb{RP}^3, S)$.

For our approach the basic topological information is that in a small 3-dimensional ball D, centered at an isolated singular point, $S \cap D$ is homeomorphic to the cone over the curve C obtained as the intersection of S with the boundary of the ball (see [M2]). Then, up to homeomorphism, the portion of S inside the ball can be seen as the space obtained taking the union of as many 2-dimensional disks as the connected components of C, choosing a point in each of these disks and collapsing these points to a single point. In this way we see the isolated singularity as the effect of two successive operations: first the glueing of a 2-cell (i.e. a subset homeomorphic to a closed 2-dimensional disk) along each connected component of C and then the collapsing of a set containing a point in each of the attached 2-cells.

Applying this procedure to all the singularities, we obtain a compact topological surface T without boundary such that S is homeomorphic to the topological quotient $T/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation that collapses suitable finite families of points of T. Thus our algorithm will determine topologically S by computing the Euler characteristics of the connected components of T and the families of points that, through a collapsing process, produce the singularities of S.

Furthermore, after defining the *weighted 2-adjacency graph* of S, we show that it is a topological invariant of the pair $(\mathbb{RP}^3, S)$ and we describe an algorithmical method to compute it.

This paper is a natural evolution of the articles [FGPT], [FGL] and [FGLP], which dealt with non-singular surfaces. Here we use the same basic ideas and techniques (use of a Morse projection, connecting paths, reduction to the affine case, etc.) but we insert them in a new procedure able to detect the presence of isolated singularities and to investigate their topological nature. Basically, while at a critical point most of the needed information is given by the index of that point, at an isolated singularity the necessary topological information will be obtained through the investigation of the curve where S intersects a small sphere centered at the singular point. Also to mantain the paper at a reasonable length, we have chosen to describe in detail only the topological results on which the algorithm bases its correctness and the organization of the main algorithm. As for the instrumental algorithmical techniques used as "black boxes", we only recall their essential features and refer the reader to the papers previously mentioned for a detailed presentation.

The main definitions, the necessary theoretical background and the list $D(S)$ of data invariant up to homeomorphism of the pair $(\mathbb{RP}^3, S)$ to be computed are contained in Section 2. In order to deal with isolated singularites we use a generalization of classical Morse theory to singular spaces, first introduced by Lazzeri ([L]), that we briefly recall in Section 3, where we prove also some related results necessary for the algorithm. In Section 4 we describe a constructive procedure to compute $D(S)$ when the surface is contained in an affine chart of $\mathbb{RP}^3$. This procedure can be applied if some preliminary tests have been positively passed, i.e. if the singularities of S are isolated and the working system of coordinates is a "good frame"; in Section 5 we present algorithms to perform both these tests and also some preliminary computations concerning the critical and singular points and some related data. When S is not affine, it is possible to construct a suitable compact algebraic surface $\widehat{S}$ in $\mathbb{R}^3$ and to recover $D(S)$ from $D(\widehat{S})$, which can be computed by means of the affine-case algorithm. This reduction procedure and the general-case algorithm are presented in Section 6 which contains also some examples.

2. Some remarks on the topology of surfaces with isolated singularities

Let S be the real projective algebraic surface in $\mathbb{RP}^3$ defined by the equation $F(x, y, z, t) = 0$, where F is a square-free homogeneous polynomial of degree d with real coefficients. A point $P \in S$ is called a singular point of the surface if it annihilates all the first partial derivatives of F; thus the set $Sing\, S$ of the singular points of S is an algebraic set.

Recall that a point $P \in S$ is called an *isolated point* of S if there exists an open neighborhood U of P in $\mathbb{RP}^3$ such that $S \cap U = \{P\}$; all isolated points of S are necessarily singular points. A singular point P of S is called an *isolated singular point* if it is isolated in $Sing\, S$, i.e. if there exists an open neighborhood U of P such that $(Sing\, S) \cap U = \{P\}$. We will consider only the case when each singular point of S is isolated in $Sing\, S$, so that $Sing\, S$ is a discrete set containing finitely many points. Note that we make no assumption on the singular locus $Sing\, S_{\mathbb{C}}$ of the complex projective surface $S_{\mathbb{C}}$ in $\mathbb{CP}^3$ defined by the equation $F = 0$; since F is square-free, $Sing\, S_{\mathbb{C}}$ cannot have dimension 2, but it can be a complex curve.

If $Q \in \mathbb{R}^3$ and $\epsilon \in \mathbb{R}, \epsilon > 0$, we will use the following notations to denote respectively the open ball, the closed ball and the sphere of radius ϵ centered at Q:
— $B(Q, \epsilon) = \{X \in \mathbb{R}^3 \mid d(X, Q) < \epsilon\}$
— $D(Q, \epsilon) = \{X \in \mathbb{R}^3 \mid d(X, Q) \leq \epsilon\}$
— $S(Q, \epsilon) = \{X \in \mathbb{R}^3 \mid d(X, Q) = \epsilon\}$,
where $d(\cdot, \cdot)$ denotes the Euclidean distance in $\mathbb{R}^3$. The previous notations make sense also for points Q in $\mathbb{RP}^3$ working in an affine chart $U \simeq \mathbb{R}^3$ containing Q.

If X is a topological space and Y a subspace, we will denote by $G(X, Y)$ the *adjacency graph of the pair* (X, Y), that is the graph whose vertices are the connected components of $X \setminus Y$ and where two distinct vertices Ω_1, Ω_2 are joined by an edge if and only if the topological closures of Ω_1 and Ω_2 are not disjoint.

If Z is a topological space, the space $Z \times [0, 1]/Z \times \{1\}$, obtained by collapsing to a point the subspace $Z \times \{1\}$, is called *the cone over* Z; we will denote it by $Cone(Z)$. Conventionally the cone over the empty set consists of a point. If $A \subseteq \mathbb{R}^3$ and P is a point in $\mathbb{R}^3$, by *cone over* A *with vertex* P we will mean the union of all segments joining P with any point $R \in A$ and we will denote it by $Cone(A, P)$. Similarly by convention $Cone(\emptyset, P) = \{P\}$.

The following result, proved by Milnor also in the complex case, gives the important information that locally at an isolated singularity S is topologically a cone:

Theorem 2.1. *(*[M2]*, Proposition 2.10) Let Q be an isolated singular point of S. Then there exists $r > 0$ such that for all positive $\epsilon \leq r$*

i) *$C(Q, \epsilon) := S \cap S(Q, \epsilon)$ is a non-singular curve (possibly empty)*
ii) *$S \cap D(Q, \epsilon)$ is homeomorphic to the cone over $C(Q, \epsilon)$.*

Any $r > 0$ such that $D(Q, r) \setminus \{Q\}$ contains no singular points of S and no critical points of the restriction to S of the function $X \to d(X, Q)^2$ satisfies the thesis of the previous theorem. For any $\epsilon \leq r$, we will call ϵ a *Milnor radius* at Q and $S(Q, \epsilon)$ a *Milnor sphere* at Q.

More precisely, using the Theorem of semialgebraic triviality for the function $X \to d(X, Q)$ and adapting suitably the proof of Theorem 9.3.6 in [BCR], we have:

Proposition 2.2. *Let $Q = (\alpha, \beta, \gamma)$ be an isolated singular point of S. Let r be a Milnor radius at Q both for the surface S and for the plane curve $S \cap \{z = \gamma\}$. Then for all $0 < \epsilon \leq r$ there exists a homeomorphism $\phi : D(Q, \epsilon) \to D(Q, \epsilon)$ such that:*

(1) *$\phi(S \cap D(Q, \epsilon)) = Cone(C(Q, \epsilon), Q)$;*
(2) *$\phi(D(Q, \epsilon) \cap \{z = \gamma\}) = D(Q, \epsilon) \cap \{z = \gamma\}$;*
(3) *$\phi(x) = x$ for all $x \in S(Q, \epsilon)$;*

(4) $\phi(S(Q,\epsilon')) = S(Q,\epsilon')$ *for all* $\epsilon' \leq \epsilon$.

As an immediate consequence, one also has

Corollary 2.3. *In the hypotheses of Proposition 2.2, we have*

$$\phi(S \cap D(Q,\epsilon) \cap \{z=\gamma\}) = Cone(C(Q,\epsilon) \cap \{z=\gamma\}, Q).$$

Our strategy to study S will be based on the possibility of modifying S inside Milnor spheres centered at the singular points, getting a topological surface $T \subset \mathbb{RP}^3$ (i.e. a 2-dimensional topological manifold) from which we can obtain again S, except its isolated points if any, by means of a suitable quotient. We will construct T as an application of the following

Construction 2.4. Let X be a non-empty topological 1-dimensional submanifold of a sphere $S(Q,r)$. Let L_1 be the adjacency graph of the pair $(S(Q,r), X)$; it is a tree having at least 2 vertices.

If L_1 has two vertices, then X is connected and we set $W(X) = Cone(X,Q)$.

Otherwise denote by $\mathcal{V}_1(L_1)$ the set of the vertices of L_1 of valency 1. Any $v \in \mathcal{V}_1(L_1)$ is a connected component of $S(Q,r) \setminus X$ homeomorphic to an open disk bounded by an oval $\omega(v)$ which appears in L_1 as the unique edge having v in its boundary. For $i \in \mathbb{N}$ denote by $\theta_i : S(Q,r) \to S(Q, \frac{1}{2^i}r)$ the function defined by $\theta_i(Y) = \frac{1}{2^i}(Y-Q)+Q$ and let

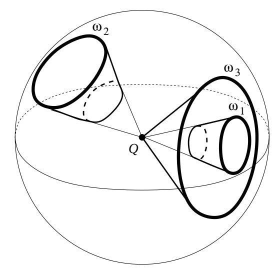

FIGURE 1. *The construction described in 2.4*

$$W_1(X) = \bigcup_{v \in \mathcal{V}_1(L_1)} \left(\left(Cone(\omega(v),Q) \setminus B(Q, \frac{1}{2}r) \right) \cup \theta_1(v) \right).$$

Then $W_1(X)$ is a union of disjoint 2-cells embedded in $D(Q,r)$ and having as boundary the curve $\bigcup_{v \in \mathcal{V}_1(L_1)} \omega(v) \subseteq X$.

Let L_2 be the graph obtained from L_1 removing the vertices in $\mathcal{V}_1(L_1)$; it is the adjacency graph of the curve $X_2 = X \setminus \bigcup_{v \in \mathcal{V}_1(L_1)} \omega(v)$ of $S(Q,r)$.

If L_2 has only one vertex (i.e. X_2 is empty), we set $W_2(X) = \emptyset$.

If L_2 has two vertices, we set $W_2(X) = Cone(X_2, Q)$.

If L_2 has at least three vertices, we consider the set $\mathcal{V}_1(L_2)$ of its vertices of valency 1 and set

$$W_2(X) = \bigcup_{v \in \mathcal{V}_1(L_2)} \left(\left(Cone(\omega(v),Q) \setminus B(Q, \frac{1}{2^2}r) \right) \cup \theta_2(v) \right).$$

We iterate the constructive procedure until we get $h \in \mathbb{N}$ such that L_h has at most one vertex; then we let

$$W(X) = W_1(X) \cup \ldots \cup W_{h-1}(X). \qquad \square$$

By construction we immediately get:

Lemma 2.5. *Let X be a non-empty topological 1-dimensional submanifold of a sphere $S(Q,r)$ and denote by $W(X)$ the subset of $D(Q,r)$ described in Construction 2.4. Then $W(X)$ is the union of finitely many disjoint 2-cells embedded in $D(Q,r)$ such that*

(1) *the boundary of $W(X)$ is the curve X,*
(2) $W(X) \cap S(Q,r) = X$,
(3) *the adjacency graph $G(D(Q,r), W(X))$ is isomorphic to the adjacency graph $G(S(Q,r), X)$.*

Assume at first, for simplicity, that Q is the only singular point of S. There are two possible situations:
– if Q is isolated in S, then S can be seen as the disjoint union of the point Q and the compact topological (not algebraic, in general) surface without boundary $S \setminus \{Q\}$,
– if Q is not isolated in S and $D(Q,\epsilon)$ is a Milnor ball at Q, then $S \setminus B(Q,\epsilon)$ is a topological surface having as its boundary the non-empty curve $C(Q,\epsilon) = S \cap S(Q,\epsilon)$. Denote by T the topological surface without boundary embedded in $\mathbb{RP}^3$ which is the union of $S \setminus B(Q,\epsilon)$ and the set $W(C(Q,\epsilon))$ obtained applying Construction 2.4 to the curve $C(Q,\epsilon)$. Thus T is obtained from S removing $S \cap B(Q,\epsilon)$ and attaching a 2-cell along each connected component of $C(Q,\epsilon)$. Choose a point in each of the attached 2-cells and denote by $Z(Q)$ the set of these points. Then S is homeomorphic to the topological quotient of T with respect to the equivalence relation that collapses $Z(Q)$ to a single point.

Coming back to the general case, henceforth we will denote by

- $Q_1, \ldots, Q_m$ the singularities of the surface that are not isolated points in S,
- $R_1, \ldots, R_s$ the isolated points in S.

Let ϵ be a small positive number such that $D(Q_i,\epsilon)$ is a Milnor ball at Q_i for $i = 1, \ldots, m$ and such that $D(Q_i,\epsilon) \cap D(Q_j,\epsilon) = \emptyset$ whenever $i \neq j$. In particular $R_h \notin \bigcup_{i=1}^m D(Q_i,\epsilon)$ for all $h = 1, \ldots, s$.

If we remove from S the points $R_1, \ldots, R_s$ and we apply Construction 2.4 to all the Milnor balls $D(Q_i,\epsilon)$ (i.e. we attach a 2-cell along each connected component of $S \cap S(Q_i,\epsilon)$ for all $i = 1, \ldots, m$), we get an embedded topological surface without boundary that we will denote by T henceforth. Again denote by $Z(Q_i)$ the set obtained choosing a point in each 2-cell attached to $C(Q_i,\epsilon)$. If $\mathcal{R}$ is the equivalence relation on T that collapses to a point each of the sets $Z(Q_1), \ldots, Z(Q_m)$, then S is homeomorphic to the disjoint union of $T/\mathcal{R}$ and of the set $\{R_1, \ldots, R_s\}$.

The topological type of the space obtained by collapsing finitely many points in a compact connected surface does not depend on the choice of these points, but only on their number. Therefore, if $T_1, \ldots, T_r$ are the connected components of T, the topological type of $T/\mathcal{R} = (T_1 \cup \ldots \cup T_r)/\mathcal{R}$ is completely determined by the topology of the T_i's and by the number n_{ij} of points in $Z(Q_i) \cap T_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, r$.

If a compact connected surface is orientable, by the topological classification theorem for surfaces it is homeomorphic to the connected sum of a sphere and g tori, i.e. it is homeomorphic to a sphere with g handles. The number g, called *genus*, is a topological invariant that determines a compact connected orientable surface up to homeomorphism. We can equivalently determine any orientable connected surface by computing its *Euler characteristic* χ, since it turns out that $\chi = 2 - 2g$ (which in particular is always an even integer).

Again by the topological classification theorem, any compact connected non-orientable surface is homeomorphic to the connected sum of either a projective plane or a Klein bottle and a compact connected orientable surface of genus g; the Euler characteristic is then respectively either $\chi = 1 - 2g$ or $\chi = -2g$, that is either odd or even. Note that a compact connected surface with an even Euler characteristic may be either orientable or non-orientable, so that in general the only knowledge of the characteristic is not sufficient to recognize topologically a compact connected surface, unless we know whether it is orientable or not. But it can be proved (see [V], 1.3.A) that the Euler characteristic of a non-orientable connected surface contained in $\mathbb{RP}^3$ is necessarily odd. Hence the knowledge of the Euler characteristics of the connected components of a compact surface embedded in $\mathbb{RP}^3$ completely determines the topological type of the surface.

Therefore, coming back to our situation, in order to determine the topological type of S, it will be sufficient to compute:

(1) the list $\chi(T) = [\chi_1, \ldots, \chi_r]$ of the Euler characteristics of the connected components $T_1, \ldots, T_r$ of T,
(2) m lists of non-negative integers, each having length r, say

$$l_1 = [n_{11}, \ldots, n_{1r}], \ldots, l_m = [n_{m1}, \ldots, n_{mr}]$$

where $n_{ij} = \#(Z(Q_i) \cap T_j)$,
(3) the number s of isolated points in S.

Example 2.6. The surface represented in Figure 2 has three isolated singularities Q_1, Q_2, R_1 and R_1 is an isolated point, so that $m = 2, s = 1$. The topological surface T constructed as explained above has three connected components T_1, T_2, T_3 and all of them are spheres. The singularity Q_1 can be obtained collapsing one point of T_1 and two points of T_2 to a single point; similarly Q_2 can be obtained collapsing three points chosen respectively in T_1, T_2 and T_3. Thus the topology of S is determined by the following data: $\chi(T) = [2, 2, 2],\ l_1 = [1, 2, 0],\ l_2 = [1, 1, 1],\ s = 1.$ □

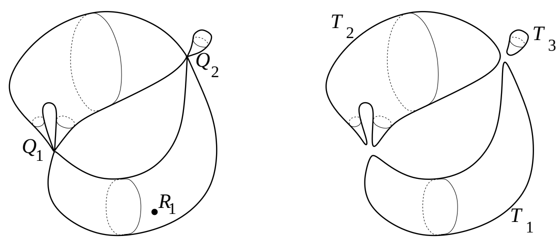

FIGURE 2. *A surface with three singular points (left-hand side) and the topological surface T associated to it (right-hand side).*

So far our topological investigation has not taken into account the way in which the surface is embedded in $\mathbb{RP}^3$ and in particular it gives no information about the mutual disposition of the connected components of S and the connected components (or *regions*) of $\mathbb{RP}^3 \setminus S$. In [FGLP] it was shown how additional information can be obtained in the case of a non-singular surface by computing the adjacency graph of the surface, where two distinct vertices (i.e. regions of $\mathbb{RP}^3 \setminus S$) are joined by an edge if and only if their topological closures are not disjoint. When S is non-singular, two adjacent regions of $\mathbb{RP}^3 \setminus S$ share in their boundaries a connected component of S. Hence the edges of the adjacency graph of S are in 1-1 correspondence with the connected components of the surface, with the only exception that, when S has an odd degree, the unique non-orientable component of S is not represented in the graph.

If S is singular, it may occur that the closures of two regions of the complement of S share only finitely many points; think for instance of the surface consisting of two spheres tangent at a common point. We will not consider two such regions "really adjacent" and will be interested in the *2-adjacency graph* $G(S)$ whose vertices are the regions of $\mathbb{RP}^3 \setminus S$, but in which two distinct vertices are joined by an edge if and only if the closures of the two regions of $\mathbb{RP}^3 \setminus S$ meet in a 2-dimensional subset of S. Observe that

(1) the 2-adjacency graph $G(S)$ just defined coincides with the ordinary adjacency graph when S is non-singular,
(2) the graph $G(S)$ is a topological invariant of the pair $(\mathbb{RP}^3, S)$,
(3) the isolated points of S, if any, are not represented in $G(S)$.

Unlike the non-singular case, there is not a bijective correspondence between the set of the edges of $G(S)$ and the set of the connected components of the surface even if S has an even degree: for instance if S consists of two cones with the same vertex, S is connected but $G(S)$ has two edges. Actually the next proposition shows that we recover similar properties if we consider the connected components of $S \setminus Sing\, S$; the proof of this result, that we insert for completeness, may be omitted with no influence on the comprehension of the rest of the paper.

Proposition 2.7. *Let $S_1, \ldots, S_n$ be the connected components of $S \setminus Sing\, S$. Then*

(1) *If the degree of S is odd, then there exists a unique i such that, if we set $\Gamma = S_i$, the set $\mathbb{RP}^3 \setminus \overline{\Gamma}$ is connected, while for any $j \neq i$ the set $\overline{S_j}$ disconnects $\mathbb{RP}^3$ into two connected regions.*

(2) *If the degree of S is even, for all j $\overline{S_j}$ disconnects $\mathbb{RP}^3$ into two connected regions.*

Proof. (1) The homology class $[S]$ in $H_2(\mathbb{RP}^3, \mathbb{Z}/2\mathbb{Z})$ is given by $[S] = \sum_{i=1}^{n} [\overline{S_i}]$. If S has an odd degree, $[S]$ is non-zero and hence there exists a component S_i of $S \setminus Sing\, S$ such that $[\overline{S_i}] \neq 0$.

Moreover for any $j \neq i$ necessarily $[\overline{S_j}] = 0$ because otherwise $[\overline{S_i}] \cdot [\overline{S_j}] \in H_1(\mathbb{RP}^3, \mathbb{Z}/2\mathbb{Z})$ would be non-trivial, which is impossible since $\overline{S_i} \cap \overline{S_j} \subset Sing\, S$ consists of isolated points.

Let $\Gamma = S_i$; we claim that $\mathbb{RP}^3 \setminus \overline{\Gamma}$ is arcwise connected (and hence connected). Namely for each $P, Q \in \mathbb{RP}^3 \setminus \overline{\Gamma}$ let $\alpha : [0,1] \to \mathbb{RP}^3$ be a continuous path in $\mathbb{RP}^3$ joining P and Q. If α does not intersect $\overline{\Gamma}$, the claim is proved; otherwise we can assume that α meets $\overline{\Gamma}$ in non-singular points of S (i.e. lying in Γ) and transversally. Since $\overline{\Gamma}$ is homologically non-trivial, then Γ is non-orientable; thus, if $\alpha(t_0) \in \Gamma$, there exists a loop γ in Γ passing through $\alpha(t_0)$ and orientation-reversing for Γ. If $n(t)$ is a normal vector to Γ along γ , since $\mathbb{RP}^3$ is orientable, then $n(1) = -n(0)$. It is therefore possible to join $\alpha(t_0 - \epsilon)$ with $\alpha(t_0 + \epsilon)$ without intersecting Γ following the normal $n(t)$. Repeating this construction for each point where α meets Γ, eventually we get a path lying in $\mathbb{RP}^3 \setminus \overline{\Gamma}$ and joining P and Q.

We have only to prove that, for each $S_j \neq \Gamma$, $\mathbb{RP}^3 \setminus \overline{S_j}$ is not connected. Otherwise, choosing a segment that meets transversally S_j in its medium point, and connecting the extremal points of this segment by means of a continuous path disjoint from $\overline{S_j}$, we would find a closed curve δ that meets $\overline{S_j}$ only in one point. Then if we consider the homology classes $[\delta] \in H_1(\mathbb{RP}^3, \mathbb{Z}/2\mathbb{Z})$ and $[\overline{S_j}] \in H_2(\mathbb{RP}^3, \mathbb{Z}/2\mathbb{Z})$, we would have $[\delta] \cdot [\overline{S_j}] = 1$, in contradiction with the fact that $[\overline{S_j}] = 0$.

(2) When S is even-degree, we get the thesis arguing in a similar way. □

As a consequence of the previous result, there is a 1-1 correspondence between the closures of the connected components of $S \setminus Sing\, S$ and the edges of $G(S)$, except for odd-degree surfaces when the closure of the component Γ given by Proposition 2.7 is not represented in $G(S)$.

Moreover, as a consequence of Lemma 2.5, it is not hard to see that

Proposition 2.8. *The 2-adjacency graph $G(S)$ is isomorphic to the adjacency graph $G(T)$.*

Apparently, not all information about the mutual position of couples of distinct regions of $\mathbb{RP}^3 \setminus S$ can be derived from the 2-adjacency graph $G(S)$. For instance, if S consists of two spheres tangent at a point and each external to the other, the graph $G(S)$ has 3 vertices and the two vertices corresponding to the interior parts of the two spheres are not joined by an edge, just as if the two spheres were disjoint. As a matter of fact, using the lists $l_1, \ldots, l_m$ relative to the singularities which are not isolated points in S, we can realize whether two regions not joined by an edge in $G(S) \simeq G(T)$, and therefore not adjacent with respect to the surface T, meet at one or more points of their boundaries after the collapsing process that yields S starting from T.

Recall that a subset $A \subset \mathbb{RP}^3$ is called *contractible* if any loop in A is contractible (i.e. homotopically trivial) as a loop in $\mathbb{RP}^3$, *non-contractible* otherwise. Using this notion we can endow the vertices of $G(T)$ with weights by means of the function $w_T : \{\text{vertices of } G(T)\} \to \{c, nc\}$ that marks each vertex of $G(T)$ (i.e. each region of $\mathbb{RP}^3 \setminus T$) as *contractible* or *non-contractible*.

Classical general results (see for instance [V]) ensure that the component Γ, if present, is always non-contractible, while the other components of T (that is the edges of $G(T)$) can be either contractible or non-contractible. Since it turns out that an edge of $G(T)$ is non-contractible if and only if its two vertices are both non-contractible, the knowledge of the function w_T is sufficient to know which components of T are contractible and which components are not. Let us recall that a contractible connected component W of T disconnects $\mathbb{RP}^3$ in two connected regions, only one of which is contractible and called the interior part of W. For such components it is possible to define a natural partial order relation: if W_i and W_j are two contractible connected components of T, we say that $W_i < W_j$ if W_i is contained in the interior part of W_j. In [FGLP] it is shown how the graph $G(T)$ can be endowed with a set of roots $r(T)$ from which it is possible to reconstruct the previous partial order.

The triple $(G(T), w_T, r(T))$ will be called the *weighted adjacency graph* of T.

We can fix the same system of weights $\{c, nc\}$ also on the vertices of the 2-adjacency graph $G(S)$ by means of $w_S : \{\text{vertices of } G(S)\} \to \{c, nc\}$. We will denote by $G_{nc}(S)$ the subgraph of $G(S)$ formed by the non-contractible vertices and by the edges having both vertices marked nc; instead we will denote by $\overline{G_c(S)}$ the subgraph formed by all the contractible vertices, all the edges where at least one vertex is contractible and all the vertices of these edges. By means of arguments similar to those used in the proof of Proposition 2.7, one can see that, if S has an odd degree, all the regions of $\mathbb{RP}^3 \setminus S$ are contractible and hence $G_{nc}(S) = \emptyset$. Instead, when the degree of S is even, the closure $\overline{S_j}$ of each connected component of $S \setminus Sing\, S$ disconnects $\mathbb{RP}^3$ into two connected regions and at least one of them is non-contractible (possibly both, as in the case of a one-sheeted hyperboloid); in particular $G_{nc}(S)$ is not empty.

In the singular case, it is no longer true that the weights on the two vertices of an edge are sufficient to determine the contractibility of the edge. For instance, if S is a cone, $G(S)$ has two vertices, one marked c, the other nc and still the only edge of $G(S)$ (i.e. the cone itself) is non-contractible. However the knowledge of the weights on the vertices of $G(S)$ is sufficient to define a partial order relation in the set of the closures of the connected components of $S \setminus Sing\, S$ as precised in the following:

Definition 2.9. *Let S_i and S_j be distinct connected components of $S \setminus Sing\, S$. We say that $\overline{S_i}$ is inside $\overline{S_j}$ if the following two conditions hold:*

1. *$\overline{S_j}$ disconnects $\mathbb{RP}^3$ into two regions and one of these is contractible,*
2. *S_i is contained in the contractible component of $\mathbb{RP}^3 \setminus \overline{S_j}$.*

All the information necessary to know the previous partial order among the sets $\overline{S_i}$ can be obtained by choosing some roots in the 2-adjacency graph $G(S)$. The way in which this can be done depends on the degree of S.

If S has an even degree, $G_{nc}(S)$ is connected and each connected component of $\overline{G_c(S)}$ is a tree that contains exactly one vertex weighted nc: we choose these vertices as a *set of roots* of $G(S)$. In this way the order induced on each connected component of $\overline{G_c(S)}$ by the only root contained in it coincides with the partial order described in Definition 2.9.

If S has an odd degree, $G_{nc}(S)$ is empty; however we are able to choose a root in $G(S)$ also in this case: we will call *root* of $G(S)$ the unique region of $\mathbb{RP}^3 \setminus S$ which is adherent to the connected component Γ given by Proposition 2.7.

Note that, while for even-degree surfaces the information about which vertices are the roots of $G(S)$ is obtained from the weights, for odd-degree surfaces this notion is independent of the weights on $G(S)$ because in this case each vertex is marked c. However this piece of information is quite important since sometimes it is the only one that allows to realize that two pairs $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are not homeomorphic: if, for instance, S consists of a projective plane and two topological spheres, only the knowledge of the root allows to recognize whether the two spheres are mutually external or one of them encircles the other one.

If $r(S)$ denotes the set of roots of S defined as above, the triple $(G(S), w_S, r(S))$ will be called the *weighted 2-adjacency graph* of S. With the previous definitions it is easy to see that

Proposition 2.10. *(i) The weighted 2-adjacency graph of S is an invariant of the pair $(\mathbb{RP}^3, S)$ up to homeomorphism.*

(ii) The weighted 2-adjacency graph of S is isomorphic to the weighted adjacency graph of T.

The isolated points of S do not appear at all in $G(S)$. While their number is sufficient for the topological characterization of S, in order to take into account the embedding of S in $\mathbb{RP}^3$ we need to know in which regions of the complement they lie. For that it will be sufficient to compute a list $q = [q_1, \dots, q_s]$ where q_i is the region of $\mathbb{RP}^3 \setminus T$ containing the i-th isolated point R_i.

We will collect all the mentioned data concerning the surface in a single list of data

$$D(S) = [\chi(T), G(T), w_T, r(T), l_1, \dots, l_m, q].$$

By the previous considerations we have that

(1) $D(S)$ is an invariant up to homeomorphism of the pair $(\mathbb{RP}^3, S)$,
(2) $D(S)$ completely determines the topological type of S,
(3) though not sufficient to determine the pair $(\mathbb{RP}^3, S)$, the set $D(S)$ gives useful information on the surface up to ambient-homeomorphism. For instance the surfaces S and S' represented in Figure 3 are homeomorphic but the pairs $(\mathbb{RP}^3, S)$ and $(\mathbb{RP}^3, S')$ are not homeomorphic since the weighted adjacency graphs of T and T' are not isomorphic.

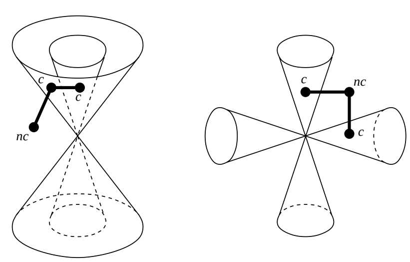

FIGURE 3. *Two surfaces that are homeomorphic, but not ambient-homeomorphic.*

The next sections will be devoted to show that all the data in $D(S)$ can be computed starting from an equation of S, even if T is not algebraic.

3. Passing through a critical or a singular point

In this section we start to describe our general strategy to compute the topology of a real algebraic surface S having only isolated singularities and we present the mathematical theoretical results on which the algorithm bases its correctness. In the present section we assume that S is contained in an affine chart of $\mathbb{RP}^3$ (as we will see, this is the basic case to be dealt with), so that we can see it as a compact affine surface in $\mathbb{R}^3$ defined by a polynomial equation $f(x, y, z) = 0$.

When S is non-singular it is possible to set up an algorithmical procedure to study S topologically using results from classical Morse theory. We refer to [FGPT], [FGL] and [FGLP] for a detailed description of such an algorithm; we recall here only its essential features as an help for the reader and for a later generalization.

Denote by $p : S \to \mathbb{R}$ the projection defined by $p(x, y, z) = z$. A point $P \in S$ is a *critical point* for p if it is non-singular but it annihilates the first partial derivatives f_x and f_y; in this case $p(P)$ is called a *critical value*. Recall (see [M2] Corollary 2.8) that p can have at most finitely many critical values.

A critical point P for p is called *non-degenerate* if it does not annihilate the determinant of the Hessian matrix $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$; by *index* of a non-degenerate critical point P one means the number of negative eigenvalues of the Hessian matrix of p at P with respect to local coordinates; it can be computed from f_z and the Hessian matrix of f. If S is non-singular the projection p is a *Morse function* if any critical point for p is non-degenerate.

For each $a \in \mathbb{R}$ we will denote by C_a the level curve $p^{-1}(a) = S \cap \{z = a\}$ and by S_a the level surface $p^{-1}([-N, a]) = S \cap \{z \leq a\}$ having C_a as its boundary.

In the non-singular case the basic idea to compute the topology of S is to find $N > 0$ such that $S \subset \mathbb{R}^2 \times [-N, N]$, to subdivide $\mathbb{R}^2 \times [-N, N]$ as the union of finitely many adjacent strips $\mathbb{R}^2 \times [a, b]$ each containing only one critical point and to compute iteratively the topology of S_b from the one of S_a by computing the Euler characteristics of all the connected components of the level surface. Recall that χ is a homotopic invariant, i.e. if X and Y are homotopically equivalent, then $\chi(X) = \chi(Y)$; hence a crucial tool to get information on the topology of S_b is given by the following result from classical Morse theory:

Theorem 3.1. *(cfr.* [M1]*, Theorem 3.1 and Theorem 3.2) Assume that* $[a, b]$ *is an interval such that* a *and* b *are regular values for the projection* p*.*

(1) *If the strip* $\mathbb{R}^2 \times (a, b)$ *contains no critical point for* p*, then* S_a *is homeomorphic to* S_b *(and furthermore* S_a *is a deformation retract of* S_b*).*
(2) *If* $\mathbb{R}^2 \times (a, b)$ *contains only one point* Q *which is a non-degenerate critical point for* p *of index* $k = 0, 1, 2$*, then* S_b *is homotopically equivalent to the space obtained from* S_a *by attaching a* k*-cell.*

This theorem, using only the index information, allows to compute how the global Euler characteristic of the level surface changes when passing through a (non-degenerate) critical point of index k, since it increases by $(-1)^k$. However it is not sufficient to reconstruct properly the topological type of S_b, i.e. its connected components and their Euler characteristics. To do that, it is also necessary to know how a k-cell is attached to the boundary of S_a. For instance in the case of a 1-cell it is important to detect whether it is attached to one or two components of S_a. This can be done by means of a more accurate analysis of the surface in the strip $\mathbb{R}^2 \times [a, b]$ based on the topological study of the level curves C_a and C_b and on a careful lifting procedure, for instance to recognize whether an oval of C_a and an oval of C_b bound the same connected component of $S \cap (\mathbb{R}^2 \times [a, b])$ or not.

This is why in the iterative step the main algorithm makes use of two special-purpose procedures already described in previous papers in the literature. The first one is used to study the *shape of a level curve* in correspondence of a regular value for p; in this case the level curve is an affine non-singular compact algebraic curve, so that all its connected components are ovals. Recall that an oval is called empty if it contains no other oval in its interior part, and a list $[\omega_1, \ldots, \omega_t]$ of ovals of a curve is called a *nest* of depth t if ω_1 is empty, ω_i is contained in the interior part of ω_{i+1} for all $i = 1, \ldots t-1$ (and any other oval containing ω_i contains also ω_{i+1}) and ω_t is not contained in the interior part of any oval of the curve.

In correspondence of a regular value, say for instance b, the pair $(\{z = b\}, C_b)$ is determined up to homeomorphism by the list of its nests or equivalently by the adjacency graph of the pair $(\{z = b\}, C_b)$, that we simply denote by $G(C_b)$. The first "black box" algorithmically computes $G(C_b)$ starting from the equation $f(x, y, b) = 0$ of C_b; several authors have proposed algorithms to do that (see for instance [GT], [CGT], [AMcC], [R], [CGVR]) and one can use whichever of them.

It is also possible (see for instance [FGPT]), by means of standard techniques, to enrich the curve-algorithm with special functions; namely, for a non-singular curve C,

– the function $findRegion$, given a point $P \in \mathbb{R}^2 \setminus C$, returns the connected component (or region) $findRegion(P)$ of $\mathbb{R}^2 \setminus C$ containing P,
– the function $findOvals$, given a point $P \in \mathbb{R}^2$, returns the list of the ovals of C containing P ordered by inclusion starting from the innermost oval,
– the function $findPoint$, given an oval ω of C, returns a point lying inside ω, more precisely a point R such that ω is the first oval of the sequence $findOvals(R)$.

The second special-purpose procedure in the iterative step is needed to lift and relate information from the level $\{z = a\}$ to the level $\{z = b\}$. This will be done by means of *connecting paths*: if $P \in \{a \leq z \leq b\} \setminus S$, we will denote by $pathUp(P, b)$ (resp. $pathDown(P, a)$) the final point $\alpha(1)$ of a continuous path $\alpha : [0, 1] \to \{a \leq z \leq b\}$ not intersecting S and such that $\alpha(0) = P$ and $\alpha(1) \in \{z = b\}$ (resp. $\alpha(1) \in \{z = a\}$).

In [FGLP] one can find a proof of the following

Proposition 3.2. *Assume that a and b are regular values for p and that $\mathbb{R}^2 \times (a, b)$ contains at most a point Q which is a non-degenerate critical point for p. Let Ω be a region of $(\mathbb{R}^2 \times [a, b]) \setminus S$ and let P be any point of Ω. Then*

(1) *if Q is critical of index 0 or of index 1, it is possible to construct an upward connecting path from P to $pathUp(P, b)$ contained in $\overline{\Omega}$*
(2) *if Q is critical of index 2 or of index 1, it is possible to construct a downward connecting path from P to $pathDown(P, a)$ contained in $\overline{\Omega}$*

The previous results, in particular Theorem 3.1, are important but not sufficient to handle the situation we are dealing with in this paper, i.e. when S contains isolated real singular points. However it is possible to get information about how the level surface changes when passing through an isolated singularity by using a generalization of Morse theory for singular spaces, first outlined by Lazzeri ([L]) and then furtherly developed by [P] and [GMcP]. First of all it is necessary to generalize the notion of Morse function:

Definition 3.3. (1) *A singular point $Q = (\alpha, \beta, \gamma) \in S$ is called* non-degenerate *with respect to the projection p if there exists no sequence $\{Q_n\}$ of non-singular points of the surface converging to Q such that the horizontal plane $\{z = \gamma\}$ passing through Q is the limit (in the Grassmannian of the 2-planes of $\mathbb{R}^3$) of the tangent planes $T_{Q_n}(S)$ to S at Q_n.*
(2) *The projection p is a* Morse function *on S if all the (smooth) critical points and all the real singular points of S are non-degenerate.*

From the theory presented in [L] one can obtain the following result that generalizes Theorem 3.1 to the singular case:

Theorem 3.4. *([L]) Let $Q = (\alpha, \beta, \gamma)$ be an isolated singular point for the surface S, which is non-degenerate w.r.t. the projection $p(x, y, z) = z$. Assume that $a < \gamma < b$ and that the strip $\mathbb{R}^2 \times [a, b]$ contains neither critical points for p nor singular points of S except Q. Then there exists $r > 0$ and a continuous function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

(1) *for $0 < \epsilon \leq r$ and for $0 < \eta \leq \eta(\epsilon)$ the plane set $S \cap \{z = \gamma - \eta\} \cap D(Q, \epsilon)$ is a smooth compact curve (possibly with boundary) whose diffeomorphism class does not depend on η, so that we can denote it simply by $M(Q)$,*
(2) *S_γ is a deformation retract of S_b,*
(3) *if K is a deformation retract of $M(Q)$, then S_b is homotopically equivalent to $S_{\gamma-\eta} \cup Cone(K, Q)$,*
(4) *if $\varphi : S_{\gamma-\eta} \to S_a$ is a homeomorphism, then S_b is homotopically equivalent to the space obtained attaching to S_a the set $Cone(K)$ along $K' = \varphi(K)$.*

Thus, if $M(Q)$ is simply an oval, the cone over it is topologically a disk so that passing through Q we have the attachment of a 2-cell to the boundary of S_a. Since an arc of curve is homotopically equivalent to a point, if $M(Q)$ consists of l arcs then we have the attachment of the cone over l points. In general if $M(Q)$ consists of h ovals and l arcs, S_b is obtained attaching to the boundary of S_a the cone over h ovals and l points.

Note that the previous description of the homotopy type of S_b with respect to S_a holds also when Q is a non-degenerate critical point of index k, and in fact the only one contained in the strip $\mathbb{R}^2 \times (a, b)$. Namely, if Q has index 2 then $M(Q)$ is an oval and the cone over it is a 2-cell; if Q has index 1 then $M(Q)$ is the union of 2 arcs, so that we can take as K a couple of points and the cone over them is topologically a 1-cell; if Q has index 0, then $M(Q)$ is empty and the cone over it is a point, i.e. a 0-cell. In other words Theorem 3.4 generalizes Theorem 3.1 replacing the index of the critical point with the plane curve $M(Q)$ as a source of information sufficient to describe the level surface S_b up to homotopy equivalence when passing through the point Q.

For computational reasons we will furtherly modify our source of topological information, proving that we can get a topological model of $M(Q)$ inspecting the intersection of S with a sphere $S(Q, \epsilon)$ centered at a singular point $Q = (\alpha, \beta, \gamma)$ provided that ϵ is a Milnor radius both for S and for the curve $S \cap \{z = \gamma\}$ at Q. This topological equivalence will be computationally relevant because it will allow us to perform our computations on the fixed sphere $S(Q, \epsilon)$ without the problem of computing an $\eta(\epsilon)$ sufficiently small so that the curve $S \cap \{z = \gamma - \eta\} \cap D(Q, \epsilon)$ is guaranteed to be topologically (and diffeomorphically) stable for all positive $\eta < \eta(\epsilon)$.

For the rest of the section we will assume that

- ϵ is a Milnor radius both for S and for the curve $S \cap \{z = \gamma\}$ at $Q = (\alpha, \beta, \gamma)$,
- $D(Q, \epsilon)$ is contained in the interior part of a strip $\mathbb{R}^2 \times [a, b]$ which contains no critical points for the projection p and no singular points of S except Q.

We will see that the asymptotic behaviour, for $\eta \to 0$, of the horizontal plane section $S \cap \{z = \gamma - \eta\}$ near the singular point Q is strictly related to the position of the connected components of the curve $C(Q, \epsilon) = S \cap S(Q, \epsilon)$ with respect to the circle $S(Q, \epsilon) \cap \{z = \gamma\}$. It will therefore be useful to introduce the following terminology:

Definition 3.5. *If $Q = (\alpha, \beta, \gamma)$, a subset X of $S(Q, \epsilon)$ is called*

1. *of type $(+)$ if $X \subseteq \{z > \gamma\}$*
2. *of type $(-)$ if $X \subseteq \{z < \gamma\}$*
3. *of type $(+-)$ if $X \cap \{z = \gamma\} \neq \emptyset$.*

Though $C(Q, \epsilon)$ is not a plane curve, we will call its connected components ovals. First of all let us prove that the type of an oval of $C(Q, \epsilon)$ does not change if we reduce the sphere radius:

Proposition 3.6. *Let ω be an oval of $C(Q, \epsilon)$. Denote by Y the connected component of $(S \cap D(Q, \epsilon)) \setminus \{Q\}$ containing ω. For $\epsilon' \leq \epsilon$, let $\omega(\epsilon') = Y \cap S(Q, \epsilon')$. Then, for all $\epsilon' \leq \epsilon$ we have:*

1. *If ω is an oval of type $(+)$ (resp. $(-)$), then $\omega(\epsilon')$ is an oval of type $(+)$ (resp. $(-)$).*
2. *If ω is an oval of type $(+-)$, then $\omega(\epsilon')$ is an oval of type $(+-)$ and $\omega(\epsilon') \cap \{z = \gamma\}$ contains as many points as $\omega \cap \{z = \gamma\}$.*

Proof. Let $\overline{Y} = Y \cup \{Q\}$. By Proposition 2.2 and Corollary 2.3, there exists a homeomorphism $\phi : D(Q, \epsilon) \to D(Q, \epsilon)$ such that $\phi(S \cap D(Q, \epsilon) \cap \{z = \gamma\}) = Cone(C(Q, \epsilon) \cap \{z = \gamma\}, Q))$ and $\phi(\overline{Y}) = Cone(\omega, Q)$. In particular $\overline{Y}$ is topologically a disk. Then $\overline{Y} \cap \{z = \gamma\}$ is homeomorphic to the cone $Cone(\omega \cap \{z = \gamma\}, Q)$.
If ω is of type $(+)$ (resp. $(-)$), then $\omega \cap \{z = \gamma\} = \emptyset$ and hence $Cone(\omega \cap \{z = \gamma\}, Q) = \{Q\}$. Then $\overline{Y} \cap \{z = \gamma\} = \{Q\}$ and $Y \cap \{z = \gamma\} = \emptyset$. It follows that $Y \subseteq \{z > \gamma\}$ (resp. $\{z < \gamma\}$) so that $\omega(\epsilon')$ is of type $(+)$ (resp. $(-)$.

In the case ω is of type $(+-)$ the thesis follows in the same way, observing that if $\omega \cap \{z = \gamma\}$ consists of k points, then $\overline{Y} \cap \{z = \gamma\}$ is homeomorphic to the cone over k points and so $\omega(\epsilon') \cap \{z = \gamma\}$ is homeomorphic to $Cone(\omega \cap \{z = \gamma\}, Q) \cap S(Q, \epsilon')$ hence it consists of exactly k points. □

Later on, it will be important also the following result on the positions of the ovals of $C(Q, \epsilon)$:

Lemma 3.7. *Let ω be an oval of $C(Q, \epsilon)$. Then ω is transversal to $\{z = \gamma\}$.*

Proof. By contradiction suppose that ω is tangent to $\{z = \gamma\}$ at a point P. Since ϵ is a Milnor radius for $S \cap \{z = \gamma\}$, then the curve $S \cap \{z = \gamma\}$ is transversal to the sphere $S(Q, \epsilon)$; hence the tangent line to that curve at P is transversal to $S(Q, \epsilon)$. Since the tangent line to ω at P is tangent to the sphere $S(Q, \epsilon)$, the tangent plane to S at P is generated by two transversal horizontal lines. Hence P is a critical point for the projection p, which is a contradiction. □

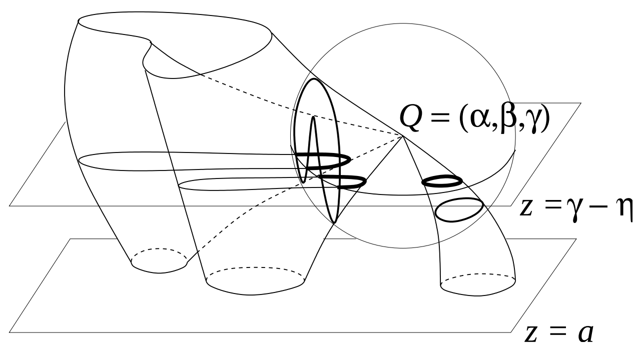

FIGURE 4. *Sections of a surface with a Milnor sphere centered at a singular point Q and with a horizontal plane $\{z = \gamma - \eta\}$ passing below the singularity.*

We are now ready to compare the local behaviour near a singular point of horizontal sections of S with the intersection of S with a Milnor sphere:

Proposition 3.8. *Let ω and Y be as in Proposition 3.6. Then there exists $\eta_0 > 0$ such that for all positive $\eta \leq \eta_0$ we have:*

(1) *if ω is an oval of type $(-)$, then $Y \cap \{z = \gamma - \eta\}$ consists of exactly one oval;*
(2) *if ω is an oval of type $(+-)$, and more precisely $\omega \cap \{z \leq \gamma\}$ is the union of k arcs, then $Y \cap \{z = \gamma - \eta\}$ is the union of k arcs too.*

Proof. (1) Let $\eta_0 > 0$ be such that $\omega \subset \{z < \gamma - \eta_0\}$. By connectivity, for any $\eta \leq \eta_0$ the set $Y \cap \{z = \gamma - \eta\}$ is not empty. Since $\omega \cap \{z = \gamma - \eta\} = \emptyset$ and the plane $\{z = \gamma - \eta\}$ is transversal to Y, then $Y \cap \{z = \gamma - \eta\}$ is a non-singular curve without boundary, i.e. consisting only of ovals. We have to prove that it consists of just one oval.

Suppose by contradiction that there exists $\eta' \leq \eta_0$ such that $Y \cap \{z = \gamma - \eta'\}$ contains at least two ovals. Then, since there are no critical points in the strip $\{\gamma - \eta_0 \leq z < \gamma\}$, the same is true for all sections $Y \cap \{z = \gamma - \eta\}$ with $\eta \leq \eta_0$; morover, for all $\eta_1 < \eta_2 \leq \eta_0$ the set $Y \cap \{\gamma - \eta_2 \leq z \leq \gamma - \eta_1\}$ is a disconnected cylinder. This means that $Y \cap \{\gamma - \eta_2 \leq z < \gamma\}$ is disconnected: otherwise if τ were a path in $Y \cap \{\gamma - \eta_2 \leq z < \gamma\}$ joining two points in two different components of $Y \cap \{z = \gamma - \eta_2\}$, it would exist a positive $\eta < \eta_2$ such that $\tau \subseteq Y \cap \{\gamma - \eta_2 \leq z \leq \gamma - \eta\}$ contradicting the fact that this set is a disconnected cylinder.

If $\overline{Y} = Y \cup \{Q\}$, the sets $U_\eta = \overline{Y} \cap \{\gamma - \eta \leq z \leq \gamma\}$, with $0 < \eta < \eta_0$, are a fundamental system of neighborhoods of Q in $\overline{Y}$, i.e. for any $\rho > 0$ there exists η such that $U_\eta \subseteq B(Q, \rho)$. Otherwise there exist ρ and a sequence of points $P_n \in \overline{Y} \cap \{\gamma - \frac{1}{n} \leq z \leq \gamma\}$ such that $P_n \notin B(Q, \rho)$. Since $\overline{Y}$

is compact, we can assume that $\{P_n\}$ converges to a point $P \in \overline{Y} \cap \{z = \gamma\} = \{Q\}$, contradicting the fact that $P_n \notin B(Q, \rho)$.

The sets U_η are connected, because any point $P \in U_\eta$ can be joined to Q by a path in U_η. Namely, if $\rho < \eta$, let η' be such that $U_{\eta'} \subseteq B(Q, \rho) \cap \overline{Y} \subseteq U_\eta$. Then, since $\overline{Y} \cap \{\gamma - \eta \leq z \leq \gamma - \eta'\}$ is a cylinder, P can be joined to a point in $U_{\eta'}$ and hence also to Q because $B(Q, \rho) \cap \overline{Y}$ is a cone.

We have so found a fundamental system of neighborhoods of Q in $\overline{Y}$ that are disconnected removing the point Q, in contradiction with the fact that $\overline{Y}$ is homeomorphic to a disk.

Observe in particular that U_η is topologically a disk.

(2) Since by Lemma 3.7 ω is transversal to $\{z = \gamma\}$, there exists $\eta_0 > 0$ such that $\omega \cap \{z \leq \gamma - \eta\}$ is the union of k arcs for all $\eta \leq \eta_0$. Furthermore, for $\eta \leq \eta_0$ by a transversality argument $Y \cap \{z = \gamma - \eta\}$ is a smooth curve with a boundary equal to $\omega \cap \{z = \gamma - \eta\}$ and hence it contains exactly k arcs .

We have to prove that this curve cannot contain any oval. Suppose that there exists η' such that $Y \cap \{z = \gamma - \eta'\}$ contains an oval σ. Since $\overline{Y}$ is homeomorphic to a disk, then σ disconnects $\overline{Y}$. Let Y' the connected component of $\overline{Y} \setminus \sigma$ not containing ω. It is clear that $Y' \cap S(Q, \epsilon) = \emptyset$.

Let $m = \min_{\overline{Y}'} z$ and $M = \max_{\overline{Y}'} z$. Since the function $p(x, y, z) = z$ cannot be constant on $\overline{Y}'$, then $M \neq m$, and since z is constant on the boundary of $\overline{Y}'$ then at least one of them must be achieved in an interior point R of $\overline{Y}'$.

If $Q \notin Y'$, such a point R should be a smooth point of S and thus a critical point for the function p on S, while no such point exists in the strip we are working in.

Suppose instead that $Q \in Y'$. If the maximum M were achieved in a point different from Q, we would have a contradiction as before. So we can suppose that $Y' \setminus \{Q\} \subseteq \{z < \gamma\}$. The set Y' is open in $\overline{Y}$, hence for ϵ' small enough $D(Q, \epsilon') \cap \overline{Y} \subseteq Y'$. Then $D(Q, \epsilon') \cap \overline{Y} = D(Q, \epsilon') \cap Y'$ and thus $S(Q, \epsilon') \cap \overline{Y} = S(Q, \epsilon') \cap Y'$. Since $Y' \setminus \{Q\} \subseteq \{z < \gamma\}$, we have $S(Q, \epsilon') \cap \overline{Y} \subseteq S(Q, \epsilon') \cap \{z < \gamma\}$ which means that $\omega(\epsilon') = Y \cap S(Q, \epsilon')$ should be an oval of type $(-)$, contradicting Proposition 3.6. □

As a consequence of Theorem 3.4 and Proposition 3.8, we can describe the change of the topology of S_b with respect to S_a in terms of the ovals of $C(Q, \epsilon)$: passing through Q we have the attachment of the cone over h ovals and l points, where h is the number of ovals of $C(Q, \epsilon)$ of type $(-)$ and l is the number of distinct arcs obtained intersecting all the ovals of $C(Q, \epsilon)$ of type $(+-)$ with the negative halfsphere $S(Q, \epsilon) \cap \{z \leq \gamma\}$.

Recall however that our true goal is not reconstructing globally the topology of S_b from that of S_a, but the topology of T_b from that of T_a and not globally but component by component. There are at least two remarkable differences: firstly $S \cap D(Q, \epsilon)$ is connected while $T \cap D(Q, \epsilon)$ may be non-connected; secondly, while S_γ is homotopically equivalent to S_b, in general T_γ is not homotopically equivalent to T_b because of the contributions due to the ovals of type $(+)$.

In order to detect the independent contributions to the topology of the various connected components of T_b when passing through Q, a more careful analysis of the situation is needed. Henceforth we will use the following

Notation 3.9. *If ω is an oval of $C(Q, \epsilon)$, we will denote by $T(\omega)$ the connected component of $(T \cap \{a \leq z \leq b\}) \setminus B(Q, \epsilon)$ containing ω in its boundary.*

Again a fundamental tool will be the possibility of constructing connecting paths starting from points on the Milnor sphere, which is easily achieved as a corollary of Proposition 3.2:

Corollary 3.10. *Assume that a and b are regular values for p and that $\mathbb{R}^2 \times (a, b)$ contains no critical point for p and a unique singular point $Q = (\alpha, \beta, \gamma)$. Let $D(Q, \epsilon)$ be a Milnor ball contained in $\mathbb{R}^2 \times (a, b)$ and let Ω be a region of $(\mathbb{R}^2 \times [a, b]) \setminus (S \cup D(Q, \epsilon))$. Then*

(1) *if* $P \in S(Q,\epsilon) \cap \{z > \gamma\} \cap \overline{\Omega}$, *then it is possible to construct an upward connecting path from* P *to* $pathUp(P,b)$ *contained in* $\overline{\Omega}$
(2) *if* $P \in S(Q,\epsilon) \cap \{z < \gamma\} \cap \overline{\Omega}$, *then it is possible to construct a downward connecting path from* P *to* $pathDown(P,a)$ *contained in* $\overline{\Omega}$.

Similarly to the non-singular case, by computing finitely many connecting paths starting from points on the Milnor sphere, it will be possible to detect which ovals of $C_a \cup C_b$ are contained in $T(\omega)$ and thus to lift correctly all the data.

Let us conclude the section with some results that give information about $T(\omega) \cap (C_a \cup C_b)$ according to the type of ω.

Proposition 3.11. *Let* ω *be an oval of* $C(Q,\epsilon)$. *If* ω *is of type* $(-)$ *(resp.*$(+)$*), then* $T(\omega)\cap\{z=a\}$ *(resp.* $T(\omega) \cap \{z = b\}$*) consists of a single oval* ω', *the boundary of* $T(\omega)$ *is* $\omega \cup \omega'$ *and* $T(\omega)$ *is homeomorphic to a cylinder.*

Proof. Assume for instance that ω is of type $(-)$ and denote by Y the connected component of $(S \cap D(Q,\epsilon)) \setminus \{Q\}$ containing ω. Let η_0 be such that $\omega \subset \{z < \gamma - \eta_0\}$; by Proposition 3.8 $Y \cap \{z = \gamma - \eta_0\}$ is an oval, say λ. The set $S \cap \{a \le z \le \gamma - \eta_0\}$ is a cylinder; let Λ be its connected component containing λ. Hence $\Lambda \cap \{z = a\}$ consists of a single oval ω'.

The set $\overline{Y} = Y \cup \{Q\}$ is topologically a disk; in the proof of Proposition 3.8 we saw that the sets $U_\eta = \overline{Y} \cap \{\gamma - \eta \le z \le \gamma\}$, with $0 < \eta \le \eta_0$, form a fundamental system of connected neighborhoods of Q in $\overline{Y}$ and that U_η is topologically a disk. Then $\overline{Y \setminus U_{\eta_0}}$ is a connected cylinder containing λ and ω and contained in $S \cap \{a \le z \le \gamma - \eta_0\}$; in particular $\overline{Y \setminus U_{\eta_0}} \subseteq \Lambda$ and hence $\omega \subset \Lambda$.

Furthermore, since $\Lambda \cup \overline{Y} = \Lambda \cup U_{\eta_0}$ and U_{η_0} is a disk with boundary λ, then $\Lambda \cup \overline{Y}$ is a disk with boundary ω', the set $\Lambda \setminus \overline{Y} = (\Lambda \cup \overline{Y}) \setminus \overline{Y}$ is a connected cylinder with boundary $\omega \cup \omega'$ and $\overline{\Lambda \setminus \overline{Y}}$ is a closed connected cylinder with boundary $\omega \cup \omega'$.

In order to get the thesis, it suffices to prove that $\overline{\Lambda \setminus \overline{Y}} = T(\omega)$.

First of all we claim that $\Lambda \cap D(Q,\epsilon)$ is connected. Namely otherwise, since $\Lambda \cap \{z = \gamma - \eta_0\} = \lambda$, there exists a connected component W of $\Lambda \cap D(Q,\epsilon)$ not containing λ, i.e. $W \cap \{z = \gamma - \eta_0\} = \emptyset$. Then the boundary of W is contained in $S(Q,\epsilon)$ and there exists a point $P \in W \cap B(Q,\epsilon)$ critical for the distance function $d(Q,X)^2$, contradicting the fact that $B(Q,\epsilon)$ is a Milnor ball for S.

Therefore $\Lambda \cap D(Q,\epsilon)$ is a connected set containing ω, hence $\Lambda \cap D(Q,\epsilon) \subseteq \overline{Y}$, which implies that $\overline{\Lambda \setminus \overline{Y}} \cap B(Q,\epsilon) = \emptyset$. Thus $\overline{\Lambda \setminus \overline{Y}}$ is a connected surface contained in $S \cap \{a \le z \le b\} \setminus B(Q,\epsilon)$ (and hence in $T \cap \{a \le z \le b\} \setminus B(Q,\epsilon)$) with boundary $\omega \cup \omega'$. Recall that if $N \subseteq M$ are two surfaces such that N is connected and the boundary of N is contained in the boundary of M, then N is a connected component of M. Then $\overline{\Lambda \setminus \overline{Y}}$ is the connected component of $T \cap \{a \le z \le b\} \setminus B(Q,\epsilon)$ containing ω, i.e. $\overline{\Lambda \setminus \overline{Y}} = T(\omega)$, which concludes the proof. □

This latter Proposition, together with the previous results, is sufficient to determine the contribution to the topology of T_b given by the ovals of type $(-)$ or of type $(+)$:

- if ω is of type $(+)$, attaching a 2-cell along ω to $S \setminus B(Q,\epsilon)$, there appears in T_b a new connected component homeomorphic to a disk and with boundary $\omega' \subset C_b$; it is therefore topologically equivalent to passing through a critical point of index 0;
- if ω is of type $(-)$, the attachment of a 2-cell along ω is topologically equivalent to the attachment of a 2-cell along an oval ω' of C_a (and hence equivalent to passing through a critical point of index 2); thus it contributes increasing by 1 the Euler characteristic of the connected component of T_b containing ω. Note that such a component may intersect $S(Q,\epsilon)$ also in other ovals of type $(-)$ or $(+-)$, thus its topology passing from level a to level b will be simultaneously influenced by the attachment of 2-cells along all these ovals.

As for the ovals of type $(+-)$, again using Theorem 3.4 and Proposition 3.8 we get some information about the effect of the attachment of a 2-cell along each of them. Namely we already know that, if ω is an oval of type $(+-)$ that intersects k arcs on the negative halfsphere, the attachment of a 2-cell along ω is homotopically equivalent to the attachment of the cone over k points to the boundary of $T_a(\omega)$, where $T_a(\omega)$ denotes the union of the connected components of T_a containing in their boundaries the ovals of $T(\omega) \cap C_a$. Hence all these components of T_a glue together into a component of T_b containing ω and the Euler characteristic of this latter component is increased by $1-k$ with respect to the sum of the Euler characteristics of the connected components of $T_a(\omega)$.

More precisely, if W is a connected component of T_b such that $W \cap S(Q,\epsilon)$ consists of ovals of type $(-)$ and/or $(+-)$, then $\chi(W)$ is equal to the sum of the Euler characteristics of the connected components of $W \cap \{z \leq a\}$ increased by 1 for each oval of type $(-)$ and increased by $1-k_i$ for each oval ω_i of type $(+-)$ that contains k_i negative arcs.

Of course now the situation is more complicated because the boundary of $T(\omega)$, apart from ω, can contain several ovals of C_a and several ovals of C_b that we need to detect by means of connecting paths. This can be done applying the following result to the connected components in which a region of type $(+-)$ of $S(Q,\epsilon) \setminus S$ is split by $\{z=\gamma\}$:

Proposition 3.12. *Let A be a region of $S(Q,\epsilon) \setminus (S \cup \{z=\gamma\})$. Then:*

1. *if A is contained in $\{z > \gamma\}$ and Ω is a region of $\{\gamma < z \leq b\} \setminus (S \cup D(Q,\epsilon))$ containing A in its boundary, then $\Omega \cap \{z=b\}$ is non-empty and connected,*
2. *if A is contained in $\{z < \gamma\}$ and Ω is a region of $\{a \leq z < \gamma\} \setminus (S \cup D(Q,\epsilon))$ containing A in its boundary, then $\Omega \cap \{z=a\}$ is non-empty and connected.*

Proof. Consider for instance case (1). By Corollary 3.10 we have that $\Omega \cap \{z=b\}$ is non-empty. If $P_1, P_2 \in \Omega \cap \{z=b\}$, there exist two paths joining respectively P_1 and P_2 to a point $R \in A$ and contained in Ω. Hence we have a path σ joining P_1 with P_2 and contained in $\{\gamma < z \leq b\} \setminus S$. Then there exists $\gamma' > \gamma$ such that σ is contained in $\{\gamma' \leq z \leq b\} \setminus S$. Since the interval $[\gamma', b]$ does not contain critical values, also the connected components of $\{\gamma' \leq z \leq b\} \setminus S$ are cylinders, hence σ can be deformed to a path in $\{z=b\} \setminus C_b$ joining P_1 and P_2. □

4. The compact affine case

In this section we describe a constructive procedure to compute the list of data $D(S) = [\chi(T), G(T), w_T, r(T), l_1, \dots, l_m, q]$ when the real algebraic surface S defined in $\mathbb{RP}^3$ by the homogeneous equation $F(x,y,z,t)=0$ and having only isolated singularities does not intersect in real points the plane "at infinity" $\{t=0\} \subset \mathbb{RP}^3$. In this case S is contained in the affine chart $\{[x,y,z,t] \in \mathbb{RP}^3 \mid t \neq 0\} \simeq \mathbb{R}^3$ and can be studied working in affine coordinates; namely $f(x,y,z) = F(x,y,z,1) = 0$ is an affine equation for S.

The projection $p : S \to \mathbb{R}$, $p(x,y,z) = z$ can have at most finitely many critical values and under our hypotheses S can have at most finitely many (real) singular points. Up to a generic linear change of coordinates we can assume (see [P]) that our system of coordinates (x,y,z) is a *good frame*, that is

- i) the projection p is a Morse function (i.e. any critical point for p and any singular point of S is non-degenerate)
- ii) if P_1 and P_2 are either singular or critical and $P_1 \neq P_2$, then $p(P_1) \neq p(P_2)$.

In this section we assume to have already checked that S has only isolated singularities and that the given system of coordinates (x,y,z) is a good frame; also we assume to have already computed the critical points and their indexes, the singularities and a Milnor radius at each of them (more precisely, for each singular point Q, an r which is a Milnor radius at Q both for the surface and for the plane curve obtained intersecting S with the horizontal plane through Q). In Section 5 we will see how these preliminary tests and computations can be performed.

Let $[-N, N]$ be an interval containing all the critical values of p and all the images through p of the singular points of S (that, for simplicity, we will call *singular values*). We can subdivide it as $[-N, N] = [-N = a_0, a_1] \cup [a_1, a_2] \cup \ldots \cup [a_u, a_{u+1} = N]$ so that each a_i is neither critical nor singular and each interval (a_i, a_{i+1}) contains only one critical value or one singular value.

Since the singular points are only finitely many, we can assume that $\epsilon \in \mathbb{Q}$ is positive and so small that

– $D(Q_i, \epsilon)$ is a Milnor ball at the singular point Q_i, for any $i = 1, \ldots, m$, both for S and for the curve $S \cap \{z = \gamma_i\}$ where $\{z = \gamma_i\}$ is the horizontal plane passing through Q_i,
– $D(R_j, \epsilon)$ is a Milnor ball for S at the isolated point R_j for any $j = 1, \ldots, s$,
– $D(Q_i, \epsilon) \cap \{z = a_h\} = \emptyset \quad \forall i = 1, \ldots, m$ and $\forall h = 0, \ldots, u + 1$
– $D(R_j, \epsilon) \cap \{z = a_h\} = \emptyset \quad \forall j = 1, \ldots, s$ and $\forall h = 0, \ldots, u + 1$.

Thus each Milnor ball of radius ϵ centered at a singular point is contained in a single open strip $\mathbb{R}^2 \times (a_h, a_{h+1})$ and does not intersect any level plane $\{z = a_h\}$.

As usual, denote by T the embedded topological surface without boundary obtained from S removing the points $R_1, \ldots, R_s$ and applying Construction 2.4 to all the Milnor disks $D(Q_i, \epsilon)$; then, for each $h = 0, \ldots, u + 1$, we have that

(1) $C_{a_h} = S \cap \{z = a_h\} = T \cap \{z = a_h\}$
(2) $T_{a_h} = T \cap \{z \leq a_h\}$ is a topological surface with boundary C_{a_h} obtained from S_{a_h} removing the points R_i that lie in $\{z < a_h\}$ and applying Construction 2.4 to the Milnor disks $D(Q_i, \epsilon)$ contained in $\{z < a_h\}$.

Thus each S_{a_h} is homeomorphic to the disjoint union of the isolated points R_j lying in $\{z < a_h\}$ and the quotient space $T_{a_h}/\mathcal{R}_h$, where $\mathcal{R}_h$ is the restriction to T_{a_h} of the equivalence relation $\mathcal{R}$ introduced in Section 2. Hence at each level we can get topological information on S_{a_h} studying the topological level surface T_{a_h}.

Our constructive procedure to compute the list of data $D(S)$ will be based on the following

Theorem 4.1. *(Iterative Step) Let $[a, b]$ be an interval such that a and b are regular values and $\mathbb{R}^2 \times (a, b)$ contains only one point Q which is either critical for p or singular for S. Then it is possible to compute*

$$Output(S_b) = \{G(C_b), \chi(T_b), G(T_b), M_b, l_1(T_b), \ldots, l_m(T_b), q(T_b)\}$$

starting from $Output(S_a)$, where

i) *$C_b = S \cap \{z = b\} = T \cap \{z = b\}$ and $G(C_b)$ is the adjacency graph of the pair $(\{z = b\}, C_b)$,*
ii) *$T_b = T \cap \{z \leq b\}$ and $\chi(T_b)$ is the list of the Euler characteristics of the connected components of T_b*
iii) *$G(T_b)$ is the adjacency graph of the pair $(\{z \leq b\}, T_b)$,*
iv) *$M_b : G(C_b) \to G(T_b)$ is the graph morphism that associates to each vertex v of $G(C_b)$ the vertex of $G(T_b)$ representing the region of $\{z \leq b\} \setminus T_b$ having in its boundary the region of $\{z = b\} \setminus C_b$ represented by v*
v) *$l_i(T_b) = [n_{i1}, \ldots, n_{ir}]$ where n_{ij} is the number of points of $Z(Q_i)$ lying in the j-th component of T_b (note that r depends on b)*
vi) *$q(T_b)$ is a list of length s where the i-th element is the region of $\{z \leq b\} \setminus T_b$ containing the i-th isolated point R_i if $R_i \in \{z \leq b\}$, it is 0 otherwise.*

Before proving this theorem, let us show that it easily allows to achieve our main goal:

Corollary 4.2. *There is an algorithmical procedure to compute $D(S)$.*

Proof. At at the initial step both C_{-N} and T_{-N} are empty, so that $G(C_{-N})$ and $G(T_{-N})$ consist of a single vertex. The lists $l_1,\dots,l_m,q$ are initialized as the zero lists; during the iterative procedure only the lists $l_i(T_b)$ concerning the singular points Q_i lying in $\{z<b\}$ are non-zero.

Since $T_N = T$, the data $\chi(T), G(T), l_1,\dots,l_m,q$ contained in the list $D(S)$ will be obtained after applying iteratively Theorem 4.1 to the intervals $[-N,a_1],\dots,[a_u,N]$. At the end of the iterative procedure, the only data that we still need to compute to get $D(S)$ are the function w_T and the roots $r(T)$. In the affine case this is straightforward: since T is contained in the affine chart $\{t\neq 0\}$ of $\mathbb{RP}^3$, all its components are contractible and all the regions of $\mathbb{RP}^3\setminus T$ are contractible except the only one external to all the components of T. The algorithm easily recognizes this external region as the only vertex in $G(T_{-N})$; we choose it as the only root of $G(T)$, mark it as non-contractible and mark as contractible all other vertices in $G(T)$. □

Example 4.3. Consider again the surface of Example 2.6 represented in the left-hand side of Figure 2. Focusing for instance our attention on the reconstruction of $\chi(T), l_1, l_2, q$, we want to see how these data (already announced in Example 2.6) are obtained at the end of the iterative procedure in the strips represented in Figure 5.

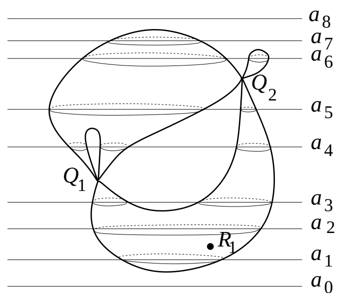

FIGURE 5. *Level planes and strips for the iterative reconstruction process.*

Output(S_{a_1}): $S_{a_1}=T_{a_1}$ is a disk, hence $\chi(T_{a_1})=[1],\ l_1=[0],\ l_2=[0],\ q=[0]$.
Output(S_{a_2}): we pass through a point which is isolated in S, hence $\chi(T_{a_2})=[1],\ l_1=[0],\ l_2=[0],\ q=[1]$ (where we label 1 the region of $\{z\le a_2\}\setminus T_{a_2}$ containing the isolated point).
Output(S_{a_3}): passing through a critical point of index 1 influences only the Euler characteristic and we get $\chi(T_{a_3})=[0],\ l_1=[0],\ l_2=[0],\ q=[1]$.
Output(S_{a_4}): the strip $\{a_3\le z\le a_4\}$ contains the singular point Q_1; T_{a_4} is the disjoint union of three disks and, apart from the isolated point R_1, S_{a_4} is homeomophic to $T_{a_4}/\mathcal{R}$ where $\mathcal{R}$ collapses to one point a set of three points lying respectively in the three connected components of T_{a_4}. Hence $\chi(T_{a_4})=[1,1,1],\ l_1=[1,1,1],\ l_2=[0,0,0],\ q=[1]$.
Output(S_{a_5}): two connected components of T_{a_4} glue together, so that T_{a_5} is the union of two disks and consequently the length of the lists l_1 and l_2 becomes 2. We get $\chi(T_{a_5})=[1,1],\ l_1=[1,2],\ l_2=[0,0],\ q=[1]$.
Output(S_{a_6}): we pass through the singular point Q_2; T_{a_6} is the union of a sphere and two disks; Q_2 is obtained collapsing one point in that sphere with two points chosen respectively in the two disks so that $\chi(T_{a_6})=[2,1,1],\ l_1=[1,2,0],\ l_2=[1,1,1],\ q=[1]$.
Output(S_{a_7}): only the Euler characteristics are modified passing through the critical point of index 2 contained in this strip and we get $\chi(T_{a_7})=[2,1,2],\ l_1=[1,2,0],\ l_2=[1,1,1],\ q=[1]$.
Output(S_{a_8}): passing through the last critical point of index 2 we get the final expected data

$$\chi(T_{a_8})=\chi(T)=[2,2,2],\quad l_1=[1,2,0],\quad l_2=[1,1,1],\quad q=[1]. \quad \square$$

The remaining part of the section will be devoted to the *Proof of Theorem 4.1*, i.e. to see how it is possible to compute *Output*(S_b) from *Output*(S_a) in the Iterative Step.

A crucial remark that we want preliminarily to emphasize is that all the needed computations, which have an algebraic nature, can be performed even if we are reconstructing the topology of T which is not an algebraic surface but a 2-dimensional topological manifold. In particular T is not defined by a polynomial equation. Nevertheless, as we will see, the only information about T that we need to compute concerns the behaviour of T outside the union of the Milnor balls centered at the singular points, where T coincides with S; this will enable us to study T outside those balls starting from the equation defining S.

Let us consider separately two cases according to the nature of the "special point" Q contained in the strip $\mathbb{R}^2 \times (a, b)$.

Case 1: Q is a critical point for p.

When the strip contains a unique critical point Q, using only the index of Q and computing finitely many connecting paths, it is possible (see [FGPT], [FGL] and [FGLP]) to detect the correspondence among the regions of $\{z = a\} \setminus C_a$ and those of $\{z = b\} \setminus C_b$, and hence to reconstruct $\chi(T_b), G(T_b)$ and M_b. As for the lists $l_i(T_b)$ and $q(T_b)$, we observe that
– if Q has index 0, a new component appears in T_b so that, for each i, we have that $length(l_i(T_b)) = length(l_i(T_a)) + 1$. More precisely, $l_i(T_b)$ is obtained from $l_i(T_a)$ inserting a zero in the new additional position, while $q(T_b) = q(T_a)$;
– if Q has index 1 and passing through it we have the glueing of two distinct connected components of T_a, say Σ_1 and Σ_2, then, for each i, $length(l_i(T_b)) = length(l_i(T_a)) - 1$. Each list $l_i(T_b)$ is obtained from $l_i(T_a)$ removing the position corresponding to one of the glued components, say Σ_2, and adding the integer number contained in the cancelled position to the integer appearing in $l_i(T_a)$ in the position corresponding to the component Σ_1. The glueing of two components goes along with the glueing of two regions of the complement, so it may occur that isolated points lying in different regions of $\{z \leq a\} \setminus T_a$ lie in the same region of $\{z \leq b\} \setminus T_b$: we modify $q(T_b)$ starting from $q(T_a)$ accordingly;
– if Q has index 2 or index 1 but there is no glueing of distinct components of T_a, then $l_i(T_b) = l_i(T_a)$ for all i and $q(T_b) = q(T_a)$.

Case 2: Q is a singular point for S.

As we know, the singular point Q may be either an isolated point for the surface or not. We can check which is the case by means of the curve $C(Q, \epsilon) = S \cap S(Q, \epsilon)$ where the surface meets the fixed Milnor sphere centered at $Q = (\alpha, \beta, \gamma)$, because Q is an isolated point for S if and only if $C(Q, \epsilon)$ is empty.

The algebraic curve $C(Q, \epsilon)$ is not plane, but it can be studied by investigating a plane curve homeomorphic to it. Namely, assuming that the "north pole" $N_\epsilon = (\alpha, \beta, \gamma + \epsilon)$ of $S(Q, \epsilon)$ does not lie in $C(Q, \epsilon)$ and denoting by $\psi : S(Q, \epsilon) \setminus \{N_\epsilon\} \to \mathbb{R}^2$ a stereographic projection from N_ϵ, the curve $\widetilde{C} = \psi(C(Q, \epsilon))$ is algebraic, compact, non-singular, homeomorphic to $C(Q, \epsilon)$ and the pairs $(S(Q, \epsilon) \setminus \{N_\epsilon\}, C(Q, \epsilon))$ and $(\mathbb{R}^2, \widetilde{C})$ are homeomorphic. By Lemma 3.7, $C(Q, \epsilon)$ is transversal to $S(Q, \epsilon) \cap \{z = \gamma\}$.

If $\widetilde{C}$ (and hence $C(Q, \epsilon)$) is empty, then Q is an isolated point for the surface, and in this case the reconstruction of $Output(S_b)$ is easy, because it is sufficient to lift the data from the level a to the level b by means of finitely many connecting paths. In order to update $q(T_b)$, we only need to detect the region of $\{z \leq b\} \setminus T_b$ containing Q. To do that, we compute the point $Q' = pathUp(Q, b)$: if $findRegion(Q') = \Sigma$ and $M_b(\Sigma) = R_\Sigma$, then Q lies in the region R_Σ of $\{z \leq b\} \setminus T_b$.

The case when Q is singular but not an isolated point in S is far less trivial. In this situation $C(Q, \epsilon)$ is non-empty and we know that $S \cap D(Q, \epsilon)$ is topologically a cone over $C(Q, \epsilon)$. In order to compute the topology of T_b, and hence of S_b, we need to determine, for each oval ω of $C(Q, \epsilon)$, the connected component of $T_b \setminus B(Q, \epsilon)$ in whose boundary it lies, or equivalently the connected component of T_b containing ω. Such a component can be found first detecting the connected component $T(\omega)$ of $(T \cap \{a \leq z \leq b\}) \setminus B(Q,\epsilon)$ containing ω in its boundary and then recovering, by means of the function M_a, the union $T_a(\omega)$ of the connected components of T_a containing in their boundaries the ovals of $T(\omega) \cap C_a$. By the results of the previous section, these data are sufficient to compute the effect on the Euler characteristic of the attachment of a 2-cell along ω and thus to update $\chi(T_b)$.

We need also to determine, for each ω, the ovals of $T(\omega) \cap C_b$ in order to update correctly $G(T_b)$ and M_b. By Proposition 3.11 the determination of $T(\omega) \cap C_a$ and $T(\omega) \cap C_b$ is fairly easy to do, by means of connecting paths, for the ovals of type $(+)$ or $(-)$, but it is more complicated for ovals of type $(+-)$. Just to point out why the ovals of type $(+-)$ need to be dealt with differently, observe that if ω is an oval of $C(Q,\epsilon)$ of type $(+)$ (resp. of type $(-)$), ω disconnects the positive (resp. negative) halfsphere containing it into two parts: the one containing the circle $S(Q,\epsilon) \cap \{z = \gamma\}$ will be called the exterior part of ω, while the other one will be called the interior part of ω. Using this terminology it is possible to arrange the ovals of $C(Q,\epsilon)$ of type $(+)$ and those of type $(-)$ in *nests*, extending the usual definition for plane curves recalled above.

For ovals of type $(+-)$ it is not possible to define an interior part; this will lead us to investigate such an oval as the common boundary of two regions of $S(Q,\epsilon) \setminus C(Q,\epsilon)$ both of type $(+-)$.

Since in general the Milnor sphere $S(Q,\epsilon)$ can contain several ovals of the three different types that simultaneously alter the topology of T_b when passing through Q, for the sake of clearness we think it helpful first to describe how the algorithm updates $Output(S_b)$ in the relatively simple situation described in the following example.

Example 4.4. Consider the surface represented in Figure 6.

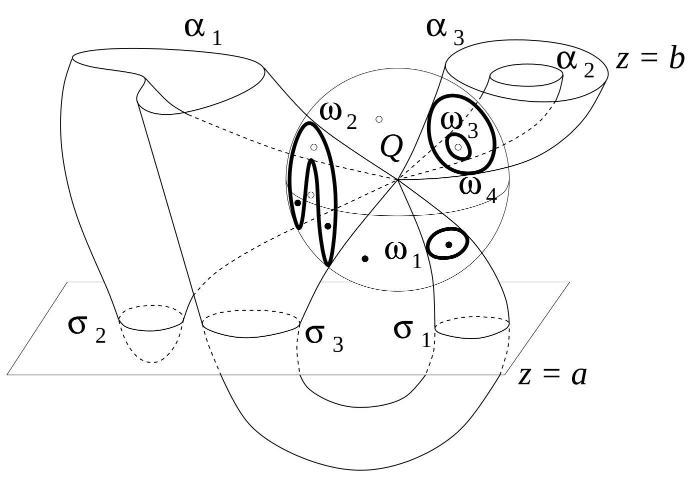

FIGURE 6. *Passing through a singular point Q which is not isolated in the surface.*

Here $C(Q,\epsilon)$ consists of one oval ω_1 of type $(-)$, one oval ω_2 of type $(+-)$ and a nest $[\omega_3, \omega_4]$ of type $(+)$ of depth 2.

The strategy to compute $\chi(T_b)$ and $G(T_b)$ is that of initializing them as $\chi(T_a)$ and $G(T_a)$, and then updating them recursively in consequence of the effects caused by the attachment of 2-cells along the ovals of $C(Q,\epsilon)$. Thus initially $\chi(T_b) = [1,0]$ and $G(T_b)$ is a tree with 3 vertices.

- First we consider the oval ω_1 of type $(-)$. By means of a downward connecting path starting from a point inside ω_1 and reaching level a, we see that the component $T(\omega_1)$ containing ω_1 intersects $\{z = a\}$ in the oval σ_1 and we have the attachment of a 2-cell along σ_1. Accordingly we temporarily update $\chi(T_b) = [1,1]$. No change occurs in $G(T_b)$.

- Then we consider the oval ω_2 of type $(+-)$. Since $\omega_2 \cap \{z < \gamma\}$ consists of two arcs, we know that the attachment of a 2-cell along ω_2 is equivalent to attaching to T_a the cone over a couple of points. We can determine $T(\omega_2) \cap C_a$ and also $T(\omega_2) \cap C_b$ as follows.
  (1) Denote by A and B the two regions of $S(Q,\epsilon) \setminus C(Q,\epsilon)$ having ω_2 as a common boundary; for instance let B be the one in the figure containing the north and the south pole of $S(Q,\epsilon)$.
  (2) A is split by $\{z = \gamma\}$ into three regions, thus let $A_1^+ = A \cap \{z > \gamma\}$ and let A_1^-, A_2^- be the two regions of $A \cap \{z < \gamma\}$. Choose three points $\xi_1^+ \in A_1^+$, $\xi_1^- \in A_1^-$, $\xi_2^- \in A_2^-$.
  (3) Similarly let B_1^+, B_2^+ be the two regions of $B \cap \{z > \gamma\}$ and $B_1^- = B \cap \{z < \gamma\}$; choose three points $\zeta_1^+ \in B_1^+$, $\zeta_2^+ \in B_2^+$, $\zeta_1^- \in B_1^-$.
  (4) For each of the chosen points lying in $\{z < \gamma\}$ compute the final points on $\{z = a\}$ of a downward connecting path starting from it and, by means of the function $findRegion$, determine the region of $\{z = a\} \setminus C_a$ containing it (this region is uniquely determined by Proposition 3.12). Considering the first negative arc on ω_2 as the common boundary of A_1^- and B_1^-, we see that the oval corresponding to it in $T(\omega_2) \cap C_a$ is the common boundary of the regions $findRegion(pathDown(\xi_1^-, a))$ and $findRegion(pathDown(\zeta_1^-, a))$, i.e. σ_2. Similarly we find that the other negative arc on ω_2 "corresponds" to σ_3.
  These computations give also further information: since we know that both ξ_1^- and ξ_2^- lie in the interior part of the same region A, we realize that the attachment of a 2-cell along ω_2 causes the glueing of two regions of $\{z \leq a\} \setminus T_a$ and the glueing of the distinct components of T_a, say T_a^1, T_a^2, bounded by σ_2 and σ_3 into a single connected component of T_b. Since the glueing is caused by the attachment of the cone over two points, respectively from σ_2 and σ_3, the new component has a Euler characteristic equal to the sum of the Euler characteristics of the glued components (as updated after the previous step) decreased by 1, i.e. $1+1-1 = 1$. Thus we temporarily update $\chi(T_b) = [1]$.
  The two previous connecting paths starting from points inside A can be seen as a way to associate to the vertex A of the adjacency graph $G(S(Q,\epsilon), C(Q,\epsilon))$ some vertices (two vertices in this case) of $G(C_a)$ and, via M_a, of $G(T_a)$. If we collaps the two vertices of $G(T_a)$ into a single vertex, this multivariate correspondence becomes univariate. Taking the quotient of the graph $G(T_a)$ according to the previous rule is the way to express in the adjacency graph the geometric phenomenon of the glueing of two components of T_a when passing through Q. Thus we temporarily update $G(T_b)$ as the mentioned quotient of $G(T_a)$.
  (5) We still need to detect $T(\omega_2) \cap C_b$ in order to update M_b in consequence of the attachment of a 2-cell along ω_2. We proceed in a similar way: we compute upward connecting paths up to level b starting from the points chosen in the regions A_1^+, B_1^+, B_2^+, we recognize the regions of $\{z = b\} \setminus C_b$ containing their final points and realize that $T(\omega_2) \cap C_b = \alpha_1$. If R_1 is the interior part of α_1, we set $M_b(R_1)$ to be the new region of $\{z \leq b\} \setminus T_b$ originated by the glueing of the two regions $M_a(findRegion(pathDown(\xi_1^-, a)))$ and $M_a(findRegion(pathDown(\xi_2^-, a)))$.
- Finally we consider the nest $[\omega_3, \omega_4]$ of type $(+)$. We choose a point θ inside ω_3, we compute $E = pathUp(\theta, b)$ and $findOvals(E) = [\alpha_2, \alpha_3]$. Thus we realize that in T_b there appear two new connected components, each topologically a disk, bounded respectively by α_2 and by α_3, hence for the last time we update $\chi(T_b)$ appending to the list two new positions filled with 1, thus getting $\chi(T_b) = [1, 1, 1]$.

In $G(T_b)$ the two new components appear as a path of 2 edges and 3 vertices to be attached to a vertex v in the temporarily updated $G(T_b)$. It is easy to detect such a v: the outermost oval ω_4 of the nest lies in the closure of the region B of $S(Q,\epsilon) \setminus C(Q,\epsilon)$ of type $(+-)$ that we have already dealt with before; so we attach the new path to the point of $G(T_b)$ given by $M_b(findRegion(pathUp(\zeta_2^+, b)))$ (which is equal to $M_b(findRegion(pathUp(\zeta_1^+, b)))$ in consequence of the previous step). Accordingly we complete the reconstruction of M_b.

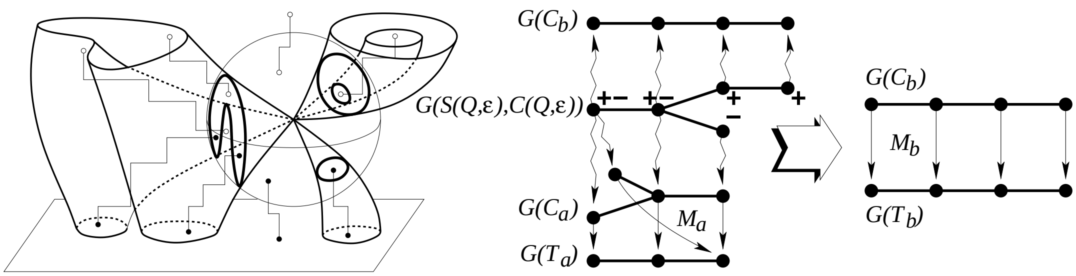

FIGURE 7. *Reconstruction of $G(T_b)$ and M_b passing through an isolated singular point as explained in Example 4.4; the polygonal lines in the left-hand part of the figure represent the connecting paths starting from the sample points on the Milnor sphere centered at the singularity; the induced correspondence from the vertices of $G(S(Q,\epsilon), C(Q,\epsilon))$ to the vertices of $G(C_a)$ and $G(C_b)$ is represented by means of wavy lines.*

The situation considered in Example 4.4 is of course a simplified case, but it contains all the phenomena that have to be taken into account; the general algorithm follows the same steps as in the example and is only made more complex because of the presence of multiple ovals and nests of the three kinds.

Hence we only sketch how the algorithm reconstructs $Output(S_b)$ in the iterative step, since we believe this is both sufficient for a whole understanding and even more efficace than a detailed description.

Iterative step: passing through a singularity Q not isolated in S.

Step 1. Preliminarily
– we compute the list of the nests of type $(+)$ and of type $(-)$ of $C(Q,\epsilon)$,
– for each nest $[\omega_1, \dots, \omega_n]$ of type $(-)$, we compute a point ξ^- lying in the interior part of ω_1,
– for each nest $[\omega_1, \dots, \omega_n]$ of type $(+)$, we compute a point ξ^+ lying in the interior part of ω_1,
– we detect the regions of $S(Q,\epsilon) \setminus C(Q,\epsilon)$ of type $(+-)$ and for each such region A we compute a set of points $\{\xi_1^+, \dots, \xi_p^+, \xi_1^-, \dots, \xi_q^-\}$ with $\xi_i^+ \in A_i^+$ for all $i = 1, \dots, p$ and $\xi_i^- \in A_i^-$ for all $i = 1, \dots, q$, where $A_1^+, \dots, A_p^+$ are the connected components of $A \cap \{z > \gamma\}$, $A_1^-, \dots, A_q^-$ are the connected components of $A \cap \{z < \gamma\}$.

In Section 5 we will see how, via stereographic projection, it is possible to perform these computations.

We initialize $\chi(T_b) = \chi(T_a)$ and $G(T_b) = G(T_a)$ and then we will update them recursively in consequence of the effects caused by the attachment of 2-cells along the ovals of $C(Q,\epsilon)$.

Step 2. We consider the nests of type $(-)$. The previous considerations guarantee that, if $[\omega_1, \dots, \omega_n]$ is a nest of type $(-)$ of $C(Q,\epsilon)$, when using Construction 2.4 in $D(Q,\epsilon)$ to construct T_b in the Milnor ball, we have the attachment of a 2-cell along each of the corresponding n ovals of C_a. To determine these ovals, it is sufficient to take a point ξ^- contained in the interior part of ω_1 and to compute $E^- = pathDown(\xi^-, a)$: the desired n ovals of C_a are the first n ovals of the list $findOvals(E^-)$. We can therefore modify the Euler characteristic of the connected components of

T_a having those n ovals in their boundaries (that we can detect by means of M_a) exactly as when we pass through a critical point of index 2. Accordingly we update $G(T_b)$ and M_b: in particular the presence of ovals of type $(-)$ causes no change in $G(T_b)$.

Step 3. We consider all the ovals of type $(+-)$.

(1) For each ω of this type we compute the number k of its "negative arcs" (again see Section 5).
(2) For each region A of $S(Q,\epsilon) \setminus C(Q,\epsilon)$ of type $(+-)$ we consider the points $\xi_1^+,\dots,\xi_p^+$, $\xi_1^-,\dots,\xi_q^-$ chosen in Step 1.
(3) For each "negative point" ξ_i^- in one of the regions $(+-)$ we compute a downward connecting path from ξ_i^- to level a and we compute the region $findRegion(pathDown(\xi_i^-,a))$.
(4) This correspondence between the regions of type $(+-)$ of $S(Q,\epsilon) \setminus C(Q,\epsilon)$ gives a multi-valued function from the corresponding set of vertices in the graph $G(S(Q,\epsilon), C(Q,\epsilon))$ to $G(C_a)$ and also, via M_a, to $G(T_a)$.
(5) Let $\sim$ be the equivalence relation on $G(T_a)$ that collapses the vertices that correspond to a unique vertex of $G(S(Q,\epsilon), C(Q,\epsilon))$ via the correspondence of (4). This relation $\sim$ also collapses edges of $G(T_a)$, i.e. connected components of T_a, that glue together passing through Q. Using this piece of information, we can (temporarily) update $\chi(T_b)$: if a new component W was originated by the glueing of some components $T_a^1,\dots,T_a^w$ of T_a by the attachment of a 2-cell to each of the ovals $\omega_1,\dots,\omega_t$ of type $(+-)$ having respectively $k_1,\dots,k_t$ negative arcs, then $\chi(W) = \chi(T_a^1) + \ldots + \chi(T_a^w) + (1-k_1) + \ldots + (1-k_t)$. Furthermore we update $G(T_b)$ as $G(T_a)/\sim$. Note that the correspondence of (4) becomes a function Φ from a subset of $G(S(Q,\epsilon), C(Q,\epsilon))$ to $G(T_a)/\sim = G(T_b)$.
(6) For each "positive point" ξ_i^+ in one of the regions $(+-)$ we compute an upward connecting path from ξ_i^+ up to level b and we compute the region $findRegion(pathUp(\xi_i^+,b))$. Let $G_{+-}(C_b)$ denote the subgraph of $G(C_b)$ obtained by taking the vertices (i.e. the regions of $\{z=b\} \setminus C_b$) reached by means of these upward connecting paths. Observe that, via these paths, we get a function Φ' from $G_{+-}(C_b)$ to $G(S(Q,\epsilon), C(Q,\epsilon))$ and also, via Φ, to $G(T_b)$. We can therefore update M_b on $G_{+-}(C_b)$: for each vertex α of $G_{+-}(C_b)$ we define $M_b(\alpha) = \Phi(\Phi'(\alpha))$.

Step 4. We consider all the nests of type $(+)$. We already know that each oval of type $(+)$ originates a new connected component in T_b and that each component is topologically a disk, which is sufficient to update $\chi(T_b)$. Hence each oval $(+)$ contributes to $G(T_b)$ with one more vertex and one more edge, but it is necessary to decide how the new edges have to be attached to the graph $G(T_b)$ as temporarily updated after the previous steps.

If there is only one nest $[\omega_1,\dots,\omega_n]$ of type $(+)$, we take a point ξ^+ inside ω_1 and compute $E^+ = pathUp(\xi^+,b)$. Then the first n ovals of $findOvals(E^+)$ respectively lie in the boundaries of the new components of T_b. These new components have to be attached to a vertex v of $G(T_b)$ as a path formed by n edges and $n+1$ vertices. In order to determine v, recall that the outermost oval ω_n of the nest lies in the closure of a unique region A of type $(+-)$ that we have already considered in the previous step of this reconstructive process. Then the n-length path corresponding to the nest $[\omega_1,\dots,\omega_n]$ has to be attached to $G(T_b)$ in the vertex corresponding to A. Accordingly we define M_b on these new vertices.

If there are more nests of type $(+)$, we must be more careful in the reconstruction process because different nests can share not only the region of type $(+-)$ containing in its closure the outermost ovals of those nests, but they can share also some ovals. However it is possible to adapt the procedure described here above to update $G(T_b)$ and M_b. For instance one considers recursively the nests of type $(+)$: if $[\omega_1,\dots,\omega_n]$ is such a nest and j is the least integer such that ω_j has already been met in another nest of type $(+)$ previously examined, then we attach a path with $j-1$ edges to the vertex of $G(T_b)$ corresponding to the region comprised between ω_{j-1} and ω_j. In this way we reproduce in $G(T_b)$ the structure of the subgraph of $G(S(Q,\epsilon), C(Q,\epsilon))$ relative to the ovals and regions of type $(+)$.

Step 5. We have so completed the computation of $\chi(T_b)$, $G(T_b)$ and of the action of M_b on the vertices and edges of $G(C_b)$ reached by means of the connecting paths starting from points in $S(Q,\epsilon)$ computed so far. At this point it is sufficient to complete the reconstruction of M_b via connecting paths starting from the regions of $\{z=a\} \setminus C_a$ not reached in the previous steps of this procedure.

Step 6. We update $q(T_b)$ taking into account whether some of the isolated points below level b lie in one of the regions that were glued when passing through Q.

Step 7. The only remaining task is the computation of the lists $l_1(T_b), \dots, l_m(T_b)$.

Since the length of each of them coincides with the number of connected components of T_b, if c distinct components of T_a glued together passing through Q giving origin to l distinct components of T_b and d is the number of ovals of $C(Q,\epsilon)$ of type $(+)$, then each list $l_i(T_b)$ has a length equal to $length(l_i(T_a)) - (c-l) + d$. Moreover, if Q is the j-th element of the list $[Q_1, \dots, Q_m]$, then we reconstruct the lists $l_i(T_b)$ as follows:

1) for all $i \neq j$, $l_i(T_b)$ is obtained from $l_i(T_a)$ inserting 0 in the final d positions and inserting in the position corresponding to a component originated by a glueing process the sum of the integers formerly contained in the positions in $l_i(T_a)$ corresponding to the components that glued into that one,

2) for each component W of T_b we compute the number of ovals of $W \cap S(Q,\epsilon)$ and we insert this integer number in the corresponding position in $l_j(T_b)$.

5. Preliminary tests and computations

In the previous section we have seen how it is possible to compute the set of data $D(S)$ assuming that S is an affine surface in $\mathbb{R}^3$ having at most isolated singularities, that the working system of coordinates is a good frame and assuming to have already computed the singular points, the critical points and their indexes and the other needed data concerning the behaviour of the surface locally at the singular points. In this section we describe how these preliminary tests and computations can be performed.

1. *Computation of the singular points.*

Using the notation of the previous section, assume that S is given as an affine surface in $\mathbb{R}^3$ by means of the defining equation $f(x,y,z)=0$ with f a square-free polynomial with real coefficients.

Denote by $J=(f, f_x, f_y, f_z)$ the ideal generated by f and its first partial derivatives, and by $V(J)$ the set of the complex zeros of J. We need to decide whether the set $V_{\mathbb{R}}(J)$ of the real zeros of J contains only finitely many points and, in this case, to compute all of them. This can be done by suitably modifying the procedure described in [FGPT], where the problem was deciding about the emptiness of $V_{\mathbb{R}}(J)$.

Since f is square-free, J cannot have dimension 2; if it has dimension 0 we can compute the finitely many points in $V(J)$ by means of any of the methods available in the literature and then select the real ones.

If J has dimension 1, $V(J)$ is a complex curve in $\mathbb{C}^3$. Up to a generic linear change of coordinates we can assume that the projection $\pi : \mathbb{C}^3 \to \mathbb{C}^2$, $\pi(x,y,z)=(y,z)$ is a "good projection" for $V(J)$, i.e. the restriction $\pi|_{V(J)} : V(J) \to \pi(V(J))$ is finite and $1-1$ except for at most a finite number of points. In [FGP] one can find a method, based on the use of the reduced lex Gröbner basis for J, to test whether π is a good projection. The Zariski closure of $\pi(V(J))$ is defined by the ideal $I = J \cap \mathbb{C}[y,z]$ which is the product of a principal ideal (g) generated by the *gcd* of a set of generators of I and a zero-dimensional ideal I_0 obtained from I by dividing out g from the generators of I.

The real points in $V(J)$ project to real points in $V(I)$, that is $\pi(V_{\mathbb{R}}(J)) \subseteq V_{\mathbb{R}}(I) = V_{\mathbb{R}}(I_0) \cup V_{\mathbb{R}}(g)$, but the inclusion can be strict because some complex points in $V(J)$ can project to real points in $V_{\mathbb{R}}(I)$. Since the projection π is good, this can happen only for finitely many points. As a consequence, $V_{\mathbb{R}}(g)$ cannot contain any 1-dimensional connected component intersecting the line at infinity, because S has no point at infinity and therefore $V_{\mathbb{R}}(J) \cap \{t = 0\} = \emptyset$. Moreover, the surface S has at most isolated real singularities if and only if $V_{\mathbb{R}}(g)$ contains no 1-dimensional compact components.

Note that, without changing $V(g)$, we can assume that g is square-free, so that $V(g)$ has at most finitely many singular points; moreover, since $V_{\mathbb{R}}(g)$ cannot contain any real line, dividing g by its univariate factors in z (necessarily without real roots) we can also assume that $V(g)$ does not contain any 1-dimensional irreducible component of critical points with respect to the projection to the z-axis $\sigma : \mathbb{C}^2 \to \mathbb{C},\ \sigma(y, z) = z$. The zero-set defined by the ideal $K = (g, g_y) \subseteq \mathbb{C}[y, z]$ contains the points of $V(g)$ which are either singular for $V(g)$ or critical with respect to the projection σ. Thus we can assume that the dimension of K is at most 0, so that we can easily compute $V_{\mathbb{R}}(K)$.

If $V_{\mathbb{R}}(g)$ contains some 1-dimensional compact component, then it necessarily contains some singular points or some critical points with respect to σ. Then, in order to check that $V_{\mathbb{R}}(g)$ contains no 1-dimensional components, it is sufficient to check that $V_{\mathbb{R}}(g) = V_{\mathbb{R}}(K)$.

If $\omega_1, \ldots, \omega_h$ are the points in $\sigma(V_{\mathbb{R}}(K))$ and we choose $\eta_0, \ldots, \eta_h \in \mathbb{Q}$ such that $\eta_0 < \omega_1 < \eta_1 < \omega_2 < \ldots < \eta_{h-1} < \omega_h < \eta_h$, then $V_{\mathbb{R}}(g) = V_{\mathbb{R}}(K)$ if and only if $V_{\mathbb{R}}(g) \cap \{z = \eta_i\} = \emptyset$, i.e. if and only if, for all i, the equation $g(y, \eta_i) = 0$ has no real roots, which is easy to check. If that is true, we are sure that S has only finitely many real singular points that lie in the fibers over the points in $V_{\mathbb{R}}(I_0 \cdot K)$; in order to compute them it is sufficient to compute the points in $V_{\mathbb{R}}(I_0 \cdot K, J)$, where the ideal $(I_0 \cdot K, J) \subseteq \mathbb{C}[x, y, z]$ is at most zero-dimensional since π has finite fibers.

2. *Good frame test and computation of the critical points.*

We want now to test that all (real) critical points for p and all singular points of S are non-degenerate and, if so, to compute the real critical points.

The zero-set defined in $\mathbb{C}^3$ by the ideal $K = (f, f_x, f_y)$ contains all the singular points of $S_{\mathbb{C}} = \{f = 0\} \subset \mathbb{C}^3$ and all the critical points of the projection $p : S_{\mathbb{C}} \subset \mathbb{C}^3 \to \mathbb{C},\ p(x, y, z) = z$.

Let u denote the product of all the univariate factors of f in the variable z. Then u cannot have any real root z_0, because otherwise the plane $\{z = z_0\}$ would be contained in S which has no points at infinity. Thus, dividing f by u (which does not modify the real zero-set), we can assume that f is not divisible by any univariate polynomial in z.

As a consequence, the ideal K cannot have dimension 2: otherwise, if $h = 0$ is the equation of a 2-dimensional irreducible component of $V(K)$, since h divides f, f_x and f_y, then h would be a univariate polynomial in z dividing f which cannot exist after our previous reduction.

A critical point $P \in S$ is degenerate for p if it annihilates the function $D(x, y, z) = \det H$, where $\det H$ denotes the determinant of the matrix $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$. Thus all real critical points for p are non-degenerate if and only if $V_{\mathbb{R}}(K, D) \subseteq V_{\mathbb{R}}(J)$. In order to test whether this condition holds, it is helpful to use the following result (for a proof see [FGPT], Proposition 6.5)

Lemma 5.1. *Any point P lying in a 1-dimensional component of $V(K)$ is necessarily either singular for $S_{\mathbb{C}}$ or degenerate.*

Hence, if we remove from $V(K)$ the points lying in $V(J) \cup V(D)$, we remove from $V(K)$ all the 1-dimensional components; in other words the ideal L defining the set $V(K) \setminus (V(J) \cup V(D)) = V(K) \setminus V(f_z \cdot D)$ is zero-dimensional and $V_{\mathbb{R}}(L)$ contains exactly the real non-degenerate critical points. Recall that L can be easily computed by saturating K with respect to $(f_z \cdot D)$.

If we denote by $\#A$ the number of elements of a finite set A, we get

Proposition 5.2. *All real critical points for p are non-degenerate if and only if $V_{\mathbb{R}}(K)$ is a finite set and $\#V_{\mathbb{R}}(K) = \#V_{\mathbb{R}}(L) + \#V_{\mathbb{R}}(J)$.*

Proof. We can split $V(K)$ as

$$V(K) = V(L) \cup V(J) \cup V(K, D)$$

where $V(L) \cap V(J) = \emptyset$ by construction. Since both $V(L)$ and $V_{\mathbb{R}}(J)$ are finite sets, from the previous splitting we get that $V_{\mathbb{R}}(K, D)$ is finite if and only if $V_{\mathbb{R}}(K)$ is finite.

If $V_{\mathbb{R}}(K)$ is not finite, then $V_{\mathbb{R}}(K, D)$ cannot be contained in $V_{\mathbb{R}}(J)$. If $V_{\mathbb{R}}(K)$ is finite, then $V_{\mathbb{R}}(K, D) \subseteq V_{\mathbb{R}}(J)$ if and only if $\#V_{\mathbb{R}}(K) = \#V_{\mathbb{R}}(L) + \#V_{\mathbb{R}}(J)$. □

Since we can check whether $V_{\mathbb{R}}(K)$ is a finite set by means of the procedure used to investigate $V_{\mathbb{R}}(J)$ in the previous Step 1, Proposition 5.2 gives a method to test whether all real critical points for p are non-degenerate. If this is true, we can compute the real critical points of p by computing $V_{\mathbb{R}}(L)$. Additional remarks concerning the possibility of computing only the real critical values and their indexes via eigenvalue computations, avoiding the whole computation of the real critical points, can be found in Section 5 of [FGLP].

Note that (by [M2] Corollary 2.8) the set of the critical values of $p : S_{\mathbb{C}} \subset \mathbb{C}^3 \to \mathbb{C}$ is a finite set and it is the image through p of the points in $V(K) \setminus V(J)$. If G is the ideal defining the Zariski closure $\overline{V(K) \setminus V(J)}^Z$, then the finite set $p(V(G)) \subset \mathbb{C}$ of the critical values of p can be computed as the set of the (real and complex) roots of $q(z)$, where $q(z)$ is a generator of the elimination ideal $G \cap \mathbb{Q}[z]$. In particular we can check that $q(p(Q)) \neq 0$ for each real singular point $Q \in S$, i.e. no real singular point lies at the same z-height of a (real or complex) critical point.

In order to complete our test that the system of coordinates (x, y, z) is a good frame, we only need to check that each real singular point is non-degenerate.

Denote $\nabla f(P) = (f_x(P), f_y(P), f_z(P))$. Recall that a singular point Q is degenerate for p if either $(0, 0, 1)$ or $(0, 0, -1)$ lies in the closure of the set of the unitary normal directions $n(P) = \frac{\nabla f(P)}{\|\nabla f(P)\|}$ to S in the non-singular points P lying in a small neighborhood of Q. Observe that, if P is a critical point, then $n(P) = (0, 0, \pm 1)$, hence $(0, 0, \pm 1)$ always belongs to the set of the normal directions to S. In order to distinguish whether $(0, 0, \pm 1)$ comes as a normal direction only from the critical points or also from some real singular point, we consider the affine variety $V = V(M) \subset \mathbb{C}^8$ defined by the ideal $M \subseteq \mathbb{C}[x, y, z, a, b, u, v, w]$

$$M = (f, \|\nabla f\|^2 a - 1, \|\nabla f\|^2 u - f_x^2, \|\nabla f\|^2 v - f_y^2, \|\nabla f\|^2 w - f_z^2, q(z)b - 1).$$

By definition if $(x, y, z, a, b, u, v, w) \in V(M)$ then $P = (x, y, z)$ is neither a singular point nor a critical point for p.

If $\pi : \mathbb{C}^8 \to \mathbb{C}^4$ denotes the projection $\pi(x, y, z, a, b, u, v, w) = (b, u, v, w)$, then $\overline{\pi(V)}^Z = \overline{\pi(V)}$ is defined by the elimination ideal $M \cap \mathbb{C}[b, u, v, w]$ that can be computed via lex Gröbner bases. Moreover $(b, 0, 0, 1) \notin \pi(V)$ for all $b \in \mathbb{C}$; otherwise there exist $P = (x, y, z)$ and $a \in \mathbb{C}$ such that $(P, a, b, 0, 0, 1) \in V$ so that P is non-singular and critical, in contradiction with the fact that $q(z) \neq 0$.

Proposition 5.3. *If $P \in S$ is a real degenerate singular point, then there exists $b \in \mathbb{R}$ such that $(b, 0, 0, 1) \in \overline{\pi(V)}$.*

Proof. Let $P_n = (x_n, y_n, z_n)$ be a sequence of real non-singular points on S converging to P and such that $n(P_n)$ converges to $(0, 0, \pm 1)$. As remarked above, $q(p(P)) \neq 0$, hence we can assume that $q(z_n) \neq 0$. Then

$$Y_n = (P_n, \frac{1}{\|\nabla f(P_n)\|^2}, \frac{1}{q(z_n)}, \frac{f_x(P_n)^2}{\|\nabla f(P_n)\|^2}, \frac{f_y(P_n)^2}{\|\nabla f(P_n)\|^2}, \frac{f_z(P_n)^2}{\|\nabla f(P_n)\|^2})$$

lies in V and the sequence $\{\pi(Y_n)\}$ converges to $(b,0,0,1)$, where $b=\frac{1}{q(p(P)}$ is the limit of $\frac{1}{q(z_n)}$. □

Corollary 5.4. *Let $g(b)$ be a generator of the ideal $M\cap\mathbb{Q}[b,u,v,w]$ specialized for $u=0,v=0,w=1$. If $g(b)$ has no real root, then on S there exist no real degenerate singular points for p.*

Let us conclude by showing that the latter condition is generically fulfilled:

Proposition 5.5. *Up to a generic change of coordinates $(b,0,0,1)\notin\overline{\pi(V)}$ for all $b\in\mathbb{R}$ (and hence S contains no real degenerate singular points for p).*

Proof. By [P], up to a generic change of coordinates in $\mathbb{C}^3$ we can assume that there exist no degenerate singular points for p in $S_{\mathbb{C}}$.

If $F(x,y,z,t)$ is the homogeneous polynomial obtained by homogenizing f with respect to the variable t, denote by $S^h_{\mathbb{C}}$ the projective surface in $\mathbb{CP}^3$ defined as $F(x,y,z,t)=0$. Since F is square-free, $Sing\,S^h_{\mathbb{C}}$ has at most dimension 1, hence we can assume that $S^h_{\mathbb{C}}\cap\{t=0\}$ is a curve Γ such that $Sing\,\Gamma$ consists of finitely many points and $Sing\,\Gamma=Sing\,S^h_{\mathbb{C}}\cap\{t=0\}$.

If $(\mathbb{CP}^3)^*$ denotes the dual space of $\mathbb{CP}^3$, consider the set Z of all points $(P,\tau)\in\Gamma\times(\mathbb{CP}^3)^*$ such that τ is the limit of tangent planes to $S^h_{\mathbb{C}}$ in points $P_n\in S^h_{\mathbb{C}}\setminus Sing\,S^h_{\mathbb{C}}$, with P_n converging to P and denote by $\Phi:Z\to(\mathbb{CP}^3)^*$ the projection on the second factor. If P is non-singular for $S^h_{\mathbb{C}}$, then $(P,\tau)\in Z$ if and only if τ is the tangent plane to $S^h_{\mathbb{C}}$ in P. If P is singular, the limits of tangent planes at points converging to P form an algebraic set $N(P)$ of dimension ≤ 1 (cfr. [GMcP]). Thus, if $Z'=Z\cap\big((\Gamma\setminus Sing\,\Gamma)\times(\mathbb{CP}^3)^*\big)$, the set

$$\overline{\Phi(Z)}^Z=\overline{\Phi(Z')}^Z\cup(\bigcup_{P\in Sing\,\Gamma}N(P))$$

is algebraic of dimension 1. Hence, up to a generic change of coordinates, we can assume that $\overline{\Phi(Z)}^Z$ does not intersect the line in $(\mathbb{CP}^3)^*$ representing the pencil of planes containing the line $\{z=t=0\}$ (i.e. the pencil of affine planes of equation $z=c$ with $c\in\mathbb{C}$).

Then we claim that $(b,0,0,1)\notin\overline{\pi(V)}$ for all $b\in\mathbb{R}$.

We already know that $(b,0,0,1)\notin\pi(V)$. If, by contradiction, $(b,0,0,1)\in\overline{\pi(V)}$, there exists a sequence of points $Y_n=(x_n,y_n,z_n,a_n,b_n,u_n,v_n,w_n)\in\mathbb{C}^8$ such that $P_n=(x_n,y_n,z_n)\in S_{\mathbb{C}}$, (u_n,v_n,w_n) converges to $(0,0,1)$ and b_n converges to b.

It cannot exist a sequence n_k such that P_{n_k} converges to a point $P\in\mathbb{C}^3$ and a_{n_k} converges to $a'\in\mathbb{C}$, because otherwise Y_{n_k} converges to $(P,a',b,0,0,1)\in V$ and hence $(b,0,0,1)\in\pi(V)$ which cannot happen.

Also, it cannot exist a sequence n_k such that P_{n_k} converges to a point $P\in\mathbb{C}^3$ and $|a_{n_k}|$ tends to infinity: in such a case $\|\nabla f(P_{n_k})\|$ would tend to 0, i.e. P would be a singular degenerate point.

The only possibility is that $\|P_n\|$ tends to infinity, so that there exists n_k such that P_{n_k} converges to a point $\overline{P}\in S^h_{\mathbb{C}}\cap\{t=0\}$ and $n(P_{n_k})$ tends to $(0,0,\pm1)$. But then the limit of the tangent planes to $S^h_{\mathbb{C}}$ in the points P_{n_k} would belong to the pencil of planes through the line $\{z=t=0\}$, which is a contradiction. □

3. <u>*Computation of the radius of a Milnor disk at a singular point.*</u>

Let $Q=(\alpha,\beta,\gamma)$ be a (real) singular point of S; by Theorem 2.1 there exists a real $r_0>0$ such that, for all positive $\epsilon\leq r_0$, $S\cap D(Q,\epsilon)$ is homeomorphic to the cone over $S\cap S(Q,\epsilon)$. In order to compute such a radius it suffices to compute an r_0 such that the disk $D(Q,r_0)$ contains neither singular points of S nor critical points for the function $\rho:S\to\mathbb{R}$, $\rho(x,y,z)=(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$ except Q itself.

The points that are either singular for S or critical for ρ are the points $P=(x,y,z)\in S$ such that the rank of the matrix $M=\begin{pmatrix} f_x & f_y & f_z\\ x-\alpha & y-\beta & z-\gamma\end{pmatrix}$ is lower or equal to 1. If we denote by

M_1, M_2, M_3 the determinants of the three square submatrices of order 2 of M, then the mentioned points are the real solutions of the system of four equations $f = 0, M_1 = 0, M_2 = 0, M_3 = 0$.

Let $d(x, y, z, r) = r - (x-\alpha)^2 - (y-\beta)^2 - (z-\gamma)^2$ and consider the ideal $G = (f, M_1, M_2, M_3, d) \subset \mathbb{C}[x, y, z, r]$. Then $V(G) = \Sigma \cup \Gamma \subset \mathbb{C}^4$, where

$$\Sigma = \{(P, \rho(P)) \mid P \in Sing\, S_{\mathbb{C}}\} \quad \text{and}$$

$$\Gamma = \{(P, \rho(P)) \mid f(P) = 0, P \notin Sing\, S_{\mathbb{C}} \text{ and } P \text{ is a critical point for } \rho\}.$$

If $\sigma : \mathbb{C}^4 \to \mathbb{C}$ is the projection defined by $\sigma(x, y, z, r) = r$, then $0 \in \sigma(V(G))$ since $(Q, 0) \in V(G)$; we look for a real positive r_0 such that $\{r \in \mathbb{R} \mid 0 < r < r_0\}$ does not contain any point in $\sigma(V_{\mathbb{R}}(G))$.

At first we can compute $\delta \in \mathbb{R}^+$ such that $\{z \in \mathbb{C} \mid 0 < |z| < \delta\} \cap \sigma(\Gamma) = \emptyset$. Namely, by Milnor's result on the critical values of polynomial maps already recalled, the set $\sigma(\Gamma)$ is finite in $\mathbb{C}$; in particular it coincides with its Zariski closure $\overline{\sigma(\Gamma)}^Z$. Then $\sigma(\Gamma) \subseteq \sigma(\overline{\Gamma}^Z) \subseteq \overline{\sigma(\Gamma)}^Z = \sigma(\Gamma)$ and hence $\sigma(\Gamma) = \sigma(\overline{\Gamma}^Z)$.

Since $\overline{\Gamma}^Z = \overline{V(G) \setminus \Sigma}^Z$ and $\Sigma = V(J, d)$, the algebraic set $\overline{\Gamma}^Z$ is the zero-set of the ideal I_Γ obtained by saturating G with respect to the ideal (J, d). Hence $\sigma(\Gamma) = \sigma(\overline{\Gamma}^Z)$ is the zero-set of the elimination ideal $I_\Gamma \cap \mathbb{C}[r]$ that we can easily compute. It is then sufficient to take $\delta = \min\{|\theta| \ : \ \theta \in V(I_\Gamma \cap \mathbb{C}[r]), \theta \neq 0\}$.

Thus the interval $(0, \delta) \subset \mathbb{R}$ does not contain any point in $\sigma(\Gamma_{\mathbb{R}})$, but it can still possibly contain some point in $\sigma(\Sigma_{\mathbb{R}})$. We can easily avoid this: since $\Sigma_{\mathbb{R}}$ is finite (by our hypothesis that the real singular locus $Sing\, S$ is finite), it is sufficient to compute $\eta = \min\{\|Q - P_i\| \mid P_i \in Sing\, S,\ P_i \neq Q\}$ and to take $r_0 = \min(\delta, \eta)$.

By means of the same procedure (in two variables) we can compute a real positive r_1 which is a Milnor radius at Q for the plane curve $S \cap \{z = \gamma\}$. Then $r = \min\{r_0, r_1\}$ is a Milnor radius at Q both for S and for $S \cap \{z = \gamma\}$ as needed for our algorithm.

4. *Computation of the "sample points" on a Milnor sphere.*

The reconstruction of $Output(S_b)$ when passing through a singular point Q used the ability to decide whether an oval on the Milnor sphere $S(Q, \epsilon)$ and a region of $S(Q, \epsilon) \setminus C(Q, \epsilon)$ is of type $(+)$, $(-)$ or $(+-)$ and the ability to choose on the sphere some "sample points" to be used as starting points for connecting paths (see Step 1 in the Iterative Step to reconstruct $Output(S_b)$ described in Section 4). We want now to see how all this can be done effectively.

Using the notation introduced in Section 4, if $\psi : S(Q, \epsilon) \setminus \{N_\epsilon\} \to \mathbb{R}^2$ is the stereographic projection that transforms the circle $S(Q, \epsilon) \cap \{z = \gamma\}$ onto the unit sphere $S^1 \subset \mathbb{R}^2$, the ovals of $C(Q, \epsilon)$ of type $(-)$ (resp. of type $(+)$) are mapped onto ovals of $\widetilde{C} = \psi(C(Q, \epsilon))$ internal to S^1 (resp. external to S^1), while the images of the ovals of type $(+-)$ intersect transversally S^1. A similar correspondence holds for the regions of $S(Q, \epsilon) \setminus C(Q, \epsilon)$. Accordingly we call of type $(+)$, $(-)$ or $(+-)$ the images through ψ of ovals and regions.

This allows us to solve our problem on $C(Q, \epsilon)$ working with the plane curve $\widetilde{C}$.

If $[\omega_1, \ldots, \omega_m]$ is a nest of $C(Q, \epsilon)$ of type $(+)$ or $(-)$, for all $i \in \{1, \ldots, m\}$ denote by $R(\omega_i)$ the region of $S(Q, \epsilon) \setminus C(Q, \epsilon)$ having in its boundary both ω_{i-1} and ω_i, and denote by $R(\omega_{m+1})$ the region "external to ω_m", more precisely such that $\overline{R(\omega_{m+1})} \cap \overline{R(\omega_m)} = \omega_m$. An analogous notation will be used also for the nests of the plane curve $\widetilde{C}$.

Observe that if $n = [\omega_1, \ldots, \omega_m]$ is a nest of type $(-)$ of $C(Q, \epsilon)$, then $\sigma_1 = \psi(\omega_1), \ldots, \sigma_m = \psi(\omega_m)$ are the first m ovals of a nest $\widetilde{n} = [\sigma_1, \ldots, \sigma_m, \ldots, \sigma_t]$ of $\widetilde{C}$ (having maybe a length greater than m) and m is the least integer in $\{1, \ldots, t\}$ such that $R(\sigma_{m+1})$ is of type $(+-)$.

A similar fact holds for a nest $n = [\omega_1, \ldots, \omega_m]$ of type $(+)$ of $C(Q, \epsilon)$ if and only if the north pole N_ϵ is not contained in the interior part of ω_i for any $i = 1, \ldots, m$. Otherwise let j be the least integer such that N_ϵ lies in the interior part of ω_j. Then $\psi(\omega_1), \ldots, \psi(\omega_{j-1})$ are the first ovals of a nest $\widetilde{n}$ of $\widetilde{C}$ while $\psi(\omega_j), \dots, \psi(\omega_m)$ are ovals of $\widetilde{C}$ containing 0 in their interior parts and so appearing among the ovals of $findOvals(0)$.

These observations explain why we can compute the needed data through the following steps:

(1) by means of the curve-algorithm compute the nests of $\widetilde{C}$,
(2) compute the points in $\widetilde{C} \cap S^1$ using for instance a rational parametrization of S^1,
(3) choose a point in each of the arcs of $S^1 \setminus \widetilde{C}$ and collect them into a set $\mathcal{M} = \{M_1, \dots, M_h\}$,
(4) for each $M_i \in \mathcal{M}$ compute the region $findRegion(M_i)$ of $\mathbb{R}^2 \setminus \widetilde{C}$ containing it (note that distinct points M_i, M_j may belong to the same region). The regions so found are precisely the regions of type $(+-)$. Since an oval is of type $(+-)$ if and only if the two adjacent regions are both $(+-)$, looking at the adjacency graph $G(\mathbb{R}^2, \widetilde{C})$ we determine also the ovals of type $(+-)$,
(5) if $M_{i_1}, \dots, M_{i_p}$ are the points in $\mathcal{M}$ lying in a region $\psi(A)$ of type $(+-)$, moving a little each point M_i outside and inside S^1, following for instance a radius, via ψ we compute a set of points $\{\xi^+_{i_1}, \dots, \xi^+_{i_p}\}$ lying in $A \cap \{z > \gamma\}$ and a set of points $\{\xi^-_{i_1}, \dots, \xi^-_{i_p}\}$ lying in $A \cap \{z < \gamma\}$. Note that in this way we can find more than one point in a connected component of $A \cap \{z > \gamma\}$ or $A \cap \{z < \gamma\}$. With respect to their use as described in Section 4, this is not a big problem, since the only nasty consequence is that one may have to compute more connecting paths than necessary and thus get repeatedly the same piece of information,
(6) let ω be an oval of type $(+-)$ and identify it as the common boundary of two regions Ω_1, Ω_2 of $\mathbb{R}^2 \setminus \widetilde{C}$. We can assume that the points $M_1, \dots, M_h$ are indexed following a cyclic order on S^1. Let e be the number of pairs (M_i, M_{i+1}) for $i = 1, \dots, h$ (setting $M_{h+1} = M_1$) such that $\{findRegion(M_i), findRegion(M_{i+1})\} = \{\Omega_1, \Omega_2\}$. Then ω contains $k = \frac{e}{2}$ arcs inside S^1 and k arcs outside S^1 (this datum was needed in Step 3 of the Iterative Step of Section 4),
(7) for each nest $\widetilde{n} = [\sigma_1, \dots, \sigma_t]$ of $\widetilde{C}$, by means of $findPoint$ choose a point $P(\widetilde{n})$ in the center of the nest (i.e. inside the innermost oval of $\widetilde{n}$).

Having already detected in (4) all the regions and all the ovals of type $(+-)$, we can recognize the nests of type $(+)$ or $(-)$ as the maximal paths of $G(\mathbb{R}^2, \widetilde{C}) = G(S(Q,\epsilon), C(Q,\epsilon))$ having one endpoint as their unique vertex of type $(+-)$: if the other endpoint is a region containing a point $P(\widetilde{n})$ such that $\|P(\widetilde{n})\| > 1$ (resp. $\|P(\widetilde{n})\| < 1$), then the corresponding nest is of type $(+)$ (resp. $(-)$).

More precisely, using the output in terms of nests given by the curve-algorithm applied to $\widetilde{C}$, consider first all the nests $\widetilde{n}$ such that $\|P(\widetilde{n})\| < 1$. In this case let j be the least integer in $\{1, \dots, t\}$ such that $R(\sigma_{j+1})$ is of type $(+-)$ (such an integer exists because of the previous considerations). Then $[\psi^{-1}(\sigma_1), \dots, \psi^{-1}(\sigma_j)]$ is a nest of $C(Q,\epsilon)$ of type $(-)$ and we choose $\psi^{-1}(P(\widetilde{n}))$ as ξ^-.

Consider now each nest $\widetilde{n} = [\sigma_1, \dots, \sigma_t]$ such that $\|P(\widetilde{n})\| > 1$. Take (if it exists) the least integer j in $\{1, \dots, t\}$ such that $R(\sigma_{j+1})$ is of type $(+-)$: then $[\psi^{-1}(\sigma_1), \dots, \psi^{-1}(\sigma_j)]$ is a nest of $C(Q,\epsilon)$ of type $(+)$ and we choose $\psi^{-1}(P(\widetilde{n}))$ as ξ^+.

If all regions $R(\sigma_1), \dots, R(\sigma_{t+1})$ are not of type $(+-)$, consider the list $findOvals(0) = [\eta_1, \dots, \eta_q]$ and let h be the least integer in $\{1, \dots, q\}$ such that η_h is neither of type $(-)$ nor of type $(+-)$ (all these ovals have already been detected so far). Then let j be the least integer in $\{1, \dots, t\}$ for which there exists $r \in \{h, \dots, q\}$ such that $R(\sigma_{j+1}) = R(\eta_{r+1})$ (these indexes can be detected inspecting the adjacency graph $G(\mathbb{R}^2, \widetilde{C})$). In particular η_r is an oval of type $(+)$. Then $[\psi^{-1}(\sigma_1), \dots, \psi^{-1}(\sigma_j), \psi^{-1}(\eta_r), \dots, \psi^{-1}(\eta_h)]$ is a nest of type $(+)$ and we choose $\psi^{-1}(P(\widetilde{n}))$ as ξ^+.

If $\psi^{-1}(\eta_q)$ has not been inserted in any of the nests of type (+), there is one more nest of type (+) on $S(Q,\epsilon)$ not detected yet: it is $[\psi^{-1}(\eta_q),\dots,\psi^{-1}(\eta_h)]$ and for it we choose the point N_ϵ as ξ^+.

6. The general case and examples

The Affine-Case-Algorithm described in Section 4 can be used to compute the set of data $D(S)$ only when S is contained in an affine chart of $\mathbb{RP}^3$. In this section we show that in the general case we can achieve the same goal constructing an affine real algebraic surface $\widehat{S}\subset\mathbb{R}^3$, computing $D(\widehat{S})$ by means of the Affine-Case-Algorithm and then recovering $D(S)$ from $D(\widehat{S})$.

The strategy is the same already used in [FGL] and [FGLP] respectively to compute the topological type and the weighted adjacency graph in the case of a non-singular algebraic surface. Here we see that the previous construction can be helpful even when S has isolated singularities and can be used to compute also the lists $l_1,\dots,l_m$ and q containing the needed information about the singularities of S. In the same spirit of the previous sections we only briefly recall the essential features of the construction of $\widehat{S}$ and some of its properties that can be found in detail in the two papers mentioned above; here we focus our attention on the new aspects due to the presence of the singularities and on the way to "descend" the data in $D(\widehat{S})$ concerning the singular points to recover the lists $l_1,\dots,l_m,q$.

Denote by $\pi: S^3\to\mathbb{RP}^3$ the map that associates to any point (x,y,z,t) of the 3-sphere S^3 the point of homogeneous coordinates $[x,y,z,t]$ in $\mathbb{RP}^3$; then each fiber contains two antipodal points on the sphere and S^3 turns out to be a 2-sheeted covering space of $\mathbb{RP}^3$. If S is defined by the homogeneous equation $F(x,y,z,t)=0$ and we lift S through π, the surface

$$\widetilde{S}=\pi^{-1}(S)=\{(x,y,z,t)\in\mathbb{R}^4 \mid F(x,y,z,t)=0\}\cap S^3$$

is invariant with respect to the antipodal map $ap: S^3\to S^3$ defined by $ap(v)=-v$. If $(0,0,0,1)\notin\widetilde{S}$ (what we can assume up to an affine translation of S) and if $\varphi: S^3\setminus\{(0,0,0,1)\}\to\mathbb{R}^3$ denotes the stereographic projection given by $\varphi(x,y,z,t)=(\frac{x}{1-t},\frac{y}{1-t},\frac{z}{1-t})$, then the image $\widehat{S}=\varphi(\widetilde{S})$ is a compact algebraic surface in $\mathbb{R}^3$, homeomorphic to $\widetilde{S}$ and defined implicitely by the polynomial equation $F(2X,\|X\|^2-1)=0$ where $X=(x,y,z)$. Furthermore, if $inv=\varphi\circ ap\circ\varphi^{-1}:\mathbb{R}^3\setminus\{0\}\to\mathbb{R}^3\setminus\{0\}$ denotes the involution $inv(X)=-\frac{X}{\|X\|^2}$ corresponding to ap via the stereographic projection, then $\widehat{S}$ is invariant with respect to inv.

In [FGL] and [FGLP] it was shown that, when S is non-singular, the ability to recognize the action of inv on the set of the connected components of $\widehat{S}$ and on the set of the regions of $\mathbb{R}^3\setminus\widehat{S}$, together with the topology of $\widehat{S}$ and the adjacency graph $G(\widehat{S})$, is sufficient to compute $\chi(S)$ and the weighted adjacency graph of S.

In the case we are examining, when S has at most isolated singularities, in order to compute $D(S)$ we make use of the topological surface T obtained from S applying the modifications of Construction 2.4 inside the Milnor disks at the singularities of S that are not isolated points. Also the topological surface $\varphi(\pi^{-1}(T))$ is invariant with respect to inv; more precisely inv induces an involution on the set $\mathcal{F}$ of the connected components of $\varphi(\pi^{-1}(T))$ and on the set $\mathcal{R}$ of the regions of $\mathbb{R}^3\setminus\varphi(\pi^{-1}(T))$.

Hence we can split $\mathcal{F}$ as the union $\mathcal{F}_1\cup\mathcal{F}_2$, where

$$\mathcal{F}_1=\{\widehat{Y}\in\mathcal{F}\mid inv(\widehat{Y})=\widehat{Y}\}\quad\text{and}\quad\mathcal{F}_2=\mathcal{F}\setminus\mathcal{F}_1$$

and split $\mathcal{R}$ as the union $\mathcal{R}_1\cup\mathcal{R}_2$, where

$$\mathcal{R}_1=\{\widehat{\Sigma}\in\mathcal{R}\mid inv(\widehat{\Sigma})=\widehat{\Sigma}\}\quad\text{and}\quad\mathcal{R}_2=\mathcal{R}\setminus\mathcal{R}_1.$$

Our *descending procedure* to derive the needed data relative to T (and hence to S) from the data on $\varphi(\pi^{-1}(T))$ is based on the following characterization:

Proposition 6.1. *Let Y be a connected component of T and Σ a region of $\mathbb{RP}^3 \setminus T$. Then*

(1) *$\varphi(\pi^{-1}(Y))$ is either a connected component of $\varphi(\pi^{-1}(T))$ (so that it belongs to $\mathcal{F}_1$) or it is the union of two distinct connected components of $\varphi(\pi^{-1}(T))$ transformed each into the other by inv,*
(2) *Y is non-contractible if and only if $\varphi(\pi^{-1}(Y)) \in \mathcal{F}_1$,*
(3) *Σ is non-contractible if and only if $\varphi(\pi^{-1}(\Sigma)) \in \mathcal{R}_1$.*

The previous results were proved in the mentioned papers [FGL] and [FGLP] for the components and regions of a non-singular algebraic surface; since the proof uses only the fact that $(S^3, \pi, \mathbb{RP}^3)$ is a double covering, it holds also for singular surfaces and even for topological 2-manifolds contained in $\mathbb{RP}^3$.

Since S has only isolated singularities, also the singularities of $\widehat{S}$ are isolated, so that we can compute $D(\widehat{S})$ by means of the Affine-Case-Algorithm; we obtain these data from the study of the topological 2-manifold $\widehat{T}$ associated to $\widehat{S}$ after applying the modification of Construction 2.4 inside the Milnor disks at the singularities of $\widehat{S}$. Note that $\widehat{T}$ is homeomorphic to $\varphi(\pi^{-1}(T))$ and also the pairs $(\mathbb{R}^3, \widehat{T})$ and $(\mathbb{R}^3, \varphi(\pi^{-1}(T)))$ are homeomorphic.

The fact that the pairs $(\mathbb{R}^3, \widehat{T})$ and $(\mathbb{R}^3, \varphi(\pi^{-1}(T)))$ are homeomorphic is very important because it allows us to recover $\chi(T), G(T), w_T$ and $r(T)$ by means of a "descending procedure" based on the properties of $\varphi(\pi^{-1}(T))$ with respect to *inv* but using the data relative to the surface $\widehat{T}$ computed by the Affine-Case-Algorithm. In particular the procedure to split the set $\mathcal{F}$ of the connected components of $\varphi(\pi^{-1}(T))$ and the set $\mathcal{R}$ of the regions of its complement can be performed working with the connected components of $\widehat{T}$. The procedure, described in detail in the previous papers, is based on the investigation of the plane level curve $\widehat{T} \cap \{z = 0\}$. Since we can assume that 0 is neither a critical value for the projection p nor a singular value for $\widehat{S}$, we can choose 0 as one of the levels a_i to be studied in the iterative procedure applied to $\widehat{S}$. In particular $\widehat{T} \cap \{z = 0\}$ coincides with $\widehat{S} \cap \{z = 0\}$ and it is a non-singular algebraic curve, invariant w.r.t. *inv*.

The only data of $D(S)$ that we still need to compute are the lists $l_1, \ldots, l_m, q$ that we want to derive from the analogous lists computed for $\widehat{S}$ by means of $\widehat{T}$. The reconstruction of $q(T)$ is straightforward: if R is an isolated point for S contained in a region Σ of $\mathbb{RP}^3 \setminus T$, then $\varphi(\pi^{-1}(R))$ consists of two points that lie in the same region of $\mathbb{R}^3 \setminus \widehat{T}$ if $\varphi(\pi^{-1}(\Sigma)) \in \mathcal{R}_1$ (i.e. it is connected) or that lie in different components if Σ splits into two regions of $\mathcal{R}_2$. Since we know the sets $\mathcal{R}_1$ and $\mathcal{R}_2$, accordingly we recover the list $q(T)$ of length s from the list $q(\widehat{T})$ of length $2s$.

Also the number of singularities that are not isolated points doubles when passing from S to $\widehat{S}$, hence for $\widehat{S}$ we get $2m$ lists $\widehat{l}_1, \ldots, \widehat{l}_{2m}$. Up to reordering we can assume that $\widehat{l}_1, \ldots, \widehat{l}_m$ are the lists relative to the points $\widehat{Q}_1, \ldots, \widehat{Q}_m$ and $\widehat{l}_{m+1}, \ldots, \widehat{l}_{2m}$ are relative to the points $inv(\widehat{Q}_1), \ldots, inv(\widehat{Q}_m)$.

Let us see how, for instance, from the lists $\widehat{l}_1$ and $\widehat{l}_{m+1}$ we recover the list l_1.

If $\mathcal{F}_1$ contains h components, say $\widehat{Y}_1, \ldots, \widehat{Y}_h$, and $\mathcal{F}_2$ contains $2k$ components $\widehat{Y}_{h+1}, \ldots, \widehat{Y}_{h+k}$, $inv(\widehat{Y}_{h+1}), \ldots, inv(\widehat{Y}_{h+k})$, then the length of both $\widehat{l}_1$ and $\widehat{l}_{m+1}$ is $h + 2k$, while the length of l_1 will be $h + k$. For simplicity assume that $\widehat{l}_1$ contains in the first h positions the data relative to the components in $\mathcal{F}_1$, in the successive k positions the data relative to $\widehat{Y}_{h+1}, \ldots, \widehat{Y}_{h+k}$ and in the last k positions the data relative to $inv(\widehat{Y}_{h+1}), \ldots, inv(\widehat{Y}_{h+k})$. In particular (if we denote by $l(i)$ the i-th element in a list l) we have that $\widehat{l}_1(j) = \widehat{l}_{m+1}(j)$ for each $j = 1, \ldots, h$ and, because of the pairing induced in $\mathcal{F}_2$ by *inv*, that $\widehat{l}_1(h + j) = \widehat{l}_{m+1}(h + k + j)$ and $\widehat{l}_1(h + k + j) = \widehat{l}_{m+1}(h + j)$ for all $j = 1, \ldots, k$.

Then it is easy to see that the list l_1 has to be filled according to the following rule

$l_1(j) = \widehat{l}_1(j) = (\widehat{l}_{m+1}(j)) \quad \forall j = 1, \dots, h \qquad$ and
$l_1(h+j) = \widehat{l}_1(h+j) + \widehat{l}_1(h+k+j) \quad \forall j = 1, \dots, k.$

Before exemplifying the descending procedure on some simple non-affine surfaces, let us conclude the paper with a brief schematic summary of how our algorithm and its two main functions work.

AFFINE-CASE-ALG
Input: $f(x, y, z) = 0$ with f a square-free polynomial in $\mathbb{Q}[x, y, z]$
Output: $D(S) = [\chi(T), G(T), w_T, r(T), l_1, \dots, l_m, q]$ if S has at most isolated singularities, error otherwise.

- Compute the singular locus: if S has non-isolated singularities, error
- Compute the real critical points, check that (x, y, z) is a good frame (otherwise perform a linear change of coordinates and start again) and compute the indexes of the critical points
- Split $[-N, N] = [-N = a_0, a_1] \cup [a_1, a_2] \cup \dots \cup [a_u, a_{u+1} = N]$
- Initialize $Output(S_{-N})$
- for $i = 1, \dots, u$ repeat $\quad Output(S_{a_i}) = lift(Output(S_{a_{i-1}}))$
  (the lifting and reconstruction process is explained in Section 4)
- Compute $r(T)$ choosing as root of $G(T_N)$ the only vertex in $G(T_{-N})$
- Compute w_T: mark the root as non-contractible vertex and mark all other vertices as contractible
- Assemble the list $D(S)$ extracting $\chi(T), G(T), l_1, \dots, l_m, q$ from $Output(S_N)$ and completing with $r(T)$ and w_T.

GENERAL-CASE-ALG
Input: $F(x, y, z, t) = 0$ with F a homogeneous square-free polynomial in $\mathbb{Q}[x, y, z, t]$
Output: $D(S) = [\chi(T), G(T), w_T, r(T), l_1, \dots, l_m, q]$ if S has at most isolated singularities, error otherwise.

- If the curve $C_\infty = \{F(x, y, z, 0) = 0\}$ is empty, then
  $D(S) =$ AFFINE-CASE-ALG$(F(x, y, z, 1))$
- else
  $f(x, y, z) = F(2X, \|X\|^2 - 1)$ with $X = (x, y, z)$
  $D(\widehat{S}) =$ AFFINE-CASE-ALG$(f(x, y, z))$
  $D(S) =$ descend $D(\widehat{S})$ (the descending process is explained in Section 6)

Example 6.2. Consider the surface represented in the left-hand side of Figure 8 consisting of a cone and two isolated points.

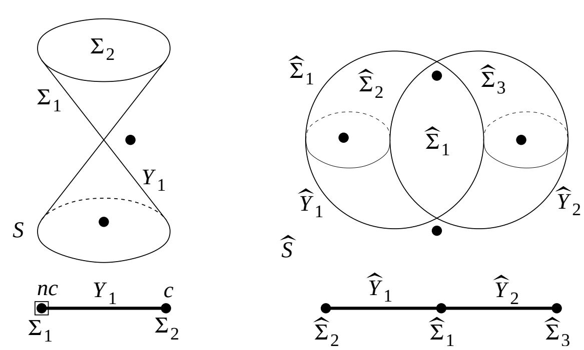

FIGURE 8. *An even degree non-affine surface.*

The right-hand side represents the doubled surface $\widehat{S}$ and the adjacency graph $G(\widehat{T})$, where $\widehat{T}$ is the union of two spheres. Note that the symbols used in the figure, according to the notation used in this section, indicate the components and regions of T and $\widehat{T}$, even if these surfaces are not represented in the figure.

Using the method described above we get the following data concerning $\widehat{T}$:
Regions: $inv(\widehat{\Sigma}_1) = \widehat{\Sigma}_1$ and $inv(\widehat{\Sigma}_2) = \widehat{\Sigma}_3$; hence, labelling by means of the index i each region $\widehat{\Sigma}_i$, we have $\mathcal{R}_1 = \{1\}$ and $\mathcal{R}_2 = \{2, 3\}$
Components: $inv(\widehat{Y}_1) = \widehat{Y}_2$; hence, again labelling by means of the index i each connected component $\widehat{Y}_i$, we get $\mathcal{F}_1 = \emptyset$ and $\mathcal{F}_2 = \{1, 2\}$. Morever $\chi(\widehat{T}) = [2, 2]$
Singularities: $\widehat{l}_1 = [1, 1]$, $\widehat{l}_2 = [1, 1]$, $q(\widehat{T}) = [1, 1, 2, 3]$.

The descending procedure yields $\chi(T) = [2]$, $l_1 = [2]$, $q(T) = [1, 2]$, so we recognize that T is a sphere and that S is the union of two isolated points and the space obtained collapsings two points in the sphere T. The weighted 2-adjacency graph of S is represented in Figure 8 below S.

Example 6.3. The surface S represented in the left-hand side of Figure 9 contains a cone and a plane passing through the vertex of the cone, thus there is only one singular point which is not isolated in S.

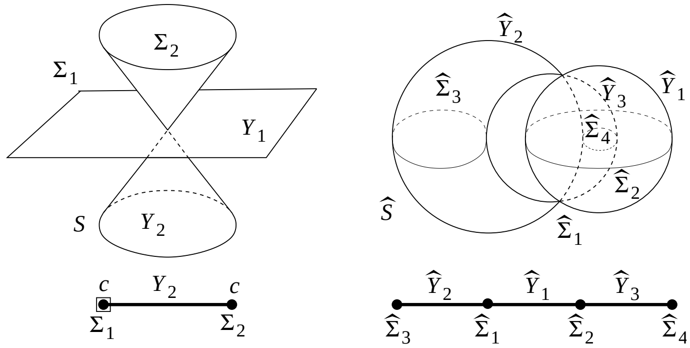

FIGURE 9. *An odd degree non-affine surface.*

Proceeding as in the previous example, using the notations appearing in the figure and the same way of labelling, we compute:
Regions: $inv(3) = 4$ and $inv(1) = 2$; hence $\mathcal{R}_1 = \emptyset$ and $\mathcal{R}_2 = \{1, 2, 3, 4\}$
Components: $inv(1) = 1, inv(2) = 3$; hence $\mathcal{F}_1 = \{1\}$ and $\mathcal{F}_2 = \{2, 3\}$. Moreover $\chi(\widehat{T}) = [2, 2, 2]$
Singularities: $\widehat{l}_1 = [1, 1, 1]$, $\widehat{l}_2 = [1, 1, 1]$, $q(\widehat{T}) = [\,]$.

By means of the descending procedure we get $\chi(T) = [1, 2]$, $l_1 = [1, 2]$, $q(T) = [\,]$, i.e. T is the disjoint union of a projective plane and a sphere and S is obtained from it collapsing a point in the plane with two points in the sphere.

Example 6.4. Also the surface in Figure 10 contains a projective plane and only one singular point.

The usual procedure yields:
Regions: $inv(3) = 4$ and $inv(1) = 2$; hence $\mathcal{R}_1 = \emptyset$ and $\mathcal{R}_2 = \{1, 2, 3, 4\}$
Components: $inv(1) = 1, inv(2) = 3$; hence $\mathcal{F}_1 = \{1\}$ and $\mathcal{F}_2 = \{2, 3\}$. Moreover $\chi(\widehat{T}) = [2, 2, 2]$
Singularities: $\widehat{l}_1 = [1, 2, 0]$, $\widehat{l}_2 = [1, 0, 2]$, $q(\widehat{T}) = [\,]$.

By means of the descending procedure we get $\chi(T) = [1, 2]$, $l_1 = [1, 2]$, $q(T) = [\,]$, i.e. T is the disjoint union of a projective plane and a sphere and S is obtained from it collapsing a point in the plane with two points in the sphere.

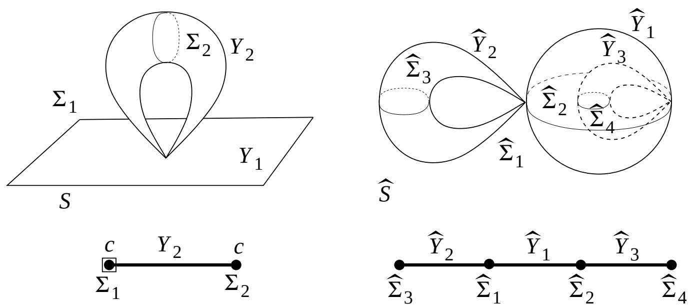

FIGURE 10. *Another odd degree non-affine surface.*

Observe that the output $D(S)$ obtained coincides with the one of the surface S of Example 9, in spite of the fact that the two surfaces cannot be mapped each into the other by means of a homeomorphism of $\mathbb{RP}^3$. However also in this case there is an invariant that distinguishes the two surfaces: the lists $\widehat{l}_1, \widehat{l}_2$. This shows that the method of "doubling" S into $\widehat{S}$ is not only a useful technical device to compute $D(S)$ but also provides new additional invariants by homeomorphism.

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