# Von Zeipel's theorem for a magnetized circular flow around a compact object

**O. Zanotti · D. Pugliese**

**Abstract** We analyze a class of physical properties, forming the content of the so-called von Zeipel theorem, which characterizes stationary, axisymmetric, non-selfgravitating perfect fluids in circular motion in the gravitational field of a compact object. We consider the extension of the theorem to the magnetohydrodynamic regime, under the assumption of an infinitely conductive fluid, both in the Newtonian and in the relativistic framework. When the magnetic field is toroidal, the conditions required by the theorem are equivalent to integrability conditions, as it is the case for purely hydrodynamic flows. When the magnetic field is poloidal, the analysis for the relativistic regime is substantially different with respect to the Newtonian case and additional constraints, in the form of PDEs, must be imposed on the magnetic field in order to guarantee that the angular velocity $\Omega$ depends only on the specific angular momentum $\ell$. In order to deduce such physical constraints, it is crucial to adopt special coordinates, which are adapted to the $\Omega = \text{const}$ surfaces. The physical significance of these results is briefly discussed.

**Keywords** general relativity · magnetohydrodynamics · black holes · von Zeipel theorem

## 1 Introduction

In Newtonian hydrodynamics, the so-called *von Zeipel theorem* [45] specifies the physical conditions to which all stationary, axially symmetric, perfect flu-

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ids in circular motion must obey in order for the angular velocity $\Omega$ to depend only on the distance from the rotation axis. Abramowicz formulated the relativistic version[1] of the von Zeipel theorem [2,3], highlighting that the angular velocity $\Omega$ and the specific angular momentum $\ell$ have common iso-surfaces, with the topology of a cylinder, if and only if the rotating fluid is barotropic. This property was used soon after to build equilibrium solutions of geometrically thick disks (tori) around black holes [20,1,26]. In fact, the von Zeipel theorem for a purely hydrodynamic flow represents a set of integrability conditions for computing the equilibrium solution. Over the years, these analytic or semi-analytic solutions, sometimes referred to as "Polish doughnuts", turned out to be very useful to study various kinds of fluid instabilities and potentially detectable physical effects around compact objects, either black holes or neutron stars   [4,5,36,14,6,40,48,13,30,37,38,42].

Though rather simplified, geometrically thick discs are still attracting a lot of interest in high energy astrophysics. From one side, they are currently adopted as initial conditions in general relativistic hydrodynamic and magnetohydrodynamic numerical simulations of accretion flows [44,21,9,32,28], even in the presence of radiation fields [41,29,47]. Moreover, an additional indication supporting their astrophysical relevance is provided by the outcome of fully relativistic numerical simulations, which clearly show that high-density tori are indeed produced after the merger of neutron star binaries collapsing onto a black hole [39].

A substantial but necessary complication in the study of geometrically thick discs is represented by magnetic fields, as it is now generally accepted that in accretion flows magnetic fields are of fundamental importance to account for the outward transport of angular momentum induced by the magnetorotational instability. A few years ago, the Polish doughnut model was extended to flows with a toroidal magnetic field [25,7,46] and later adopted in various astrophysical applications [31,23]. However, in these works it is not discussed whether and how the von Zeipel theorem can be rephrased when a magnetic field is present. In this paper we fill this gap by studying the von Zeipel theorem for a magnetized circular flow around a compact object, by considering, separately, the case of a toroidal and of a poloidal magnetic field. While for the former case we find that von Zeipel theorem still provides an integrability condition, which is met, for instance, in the model by [25], for the latter case the magnetic field needs to be constrained by two additional partial differential equations. We emphasize that it is not our intention to compute an equilibrium model, which would involve the solution of the Grad-Shafranov equation, but rather to clarify the physical conditions required by the von Zeipel theorem for a magnetized flow. We also stress that in some circumstances, like for instance in the interior of neutron stars, the magnetic field has a twisted topology (see [17,16]), but in our analysis we have not

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[1]   The original version of the theorem by von Zeipel considers the role of a radiation field (see [18] for a modern account), while in this paper we have in mind the "von Zeipel's theorem in General Relativity", called in this way for the first time presumably by Thorne [2] and which has nothing to do with radiation effects.

considered this possibility, nor the issue of the various magnetic instabilities which may arise.

Since Newtonian physics can still provide useful comparisons, we first start from the Newtonian version of the von Zeipel theorem. Although our discussion is motivated by the astrophysical applications mentioned so far, in what follows we do not assume any particular form of the spacetime metric. Our only assumptions will be those of stationarity and axisymmetry for both gravity and matter, and of a perfect, infinitely conductive non-selfgravitating fluid in purely circular motion around a compact object.

We set the speed of light $c = 1$, the gravitational constant $G = 1$, and we adopt the Lorentz-Heaviside notation for the electromagnetic quantities, such that all $\sqrt{4\pi}$ factors disappear. Greek indices run from 0 to 3, Latin indices run from 1 to 3 and we use the Einstein summation convention of repeated indices.

## 2 Newtonian version

The Newtonian version of von Zeipel's theorem in the absence of magnetic fields states that the iso-density and iso-pressure surfaces within a rotating object coincide if and only if the angular velocity is a function of the distance from the rotation axis only [43]. If a magnetic field is present, a weaker version of this theorem can be proved. To this scope, we consider a stationary and axisymmetric system and a set of cylindrical coordinates $(r, \phi, z)$. The two relevant Euler equations can be written as

$$\partial_i \Phi + \frac{1}{\rho} \partial_i p - \delta_i^r \Omega^2 r = \frac{1}{\rho}((\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B})_i, \qquad i = r, z, \qquad (1)$$

where $\Phi$ is the gravitational potential, $\rho$ is the gas density, $p$ is the pressure, $\Omega$ is the angular velocity of the rotating fluid and $\mathbf{B}$ is the magnetic field. To simplify our treatment, in the following we separately consider the case of a purely toroidal and of a purely poloidal magnetic field, while keeping equation (1) as the reference equation.

### 2.1 Toroidal magnetic field

When the magnetic field is toroidal, the components of the poloidal vector on the right hand side of equation (1) are given by

$$((\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B})_i = -\partial_i(rB_\phi)\frac{B_\phi}{r}, \qquad i = r, z. \qquad (2)$$

If we take the partial derivative $\partial_j$ of (1) and then multiply it with $e_{kji}$, all the symmetric terms in $i$ and $j$ disappear, and we are left with

$$e_{kji}\frac{\partial_j \rho \partial_i p}{\rho^2} + e_{kji}\partial_j(\delta_i^r \Omega^2 r) = e_{kji}\partial_i(rB_\phi)\partial_j\left(\frac{B_\phi}{r\rho}\right). \qquad (3)$$

In the absence of a magnetic field, the right hand side of the above equation vanishes and we would have the statement of the theorem as reported in [43]. On the contrary, if the magnetic field is not zero, then we can only provide a sufficient condition for the theorem. Namely, we can say that if

1. the equation of state is barotropic, $p = p(\rho)$, and
2. (a) $B_\phi \sim r^{-1}$, or, (b) $B_\phi \sim r\rho$, or (c) $B_\phi/r\rho = f(rB_\phi)$,

then the angular velocity would only depend on the distance from the rotation axis.

Note, for instance, that [34] proposed an equilibrium model for a barotropic, magnetized and geometrically thick disc around a central object by assuming that the toroidal magnetic field scales like $B_\phi^2 \sim r^{2(\mu-1)}\rho^\mu$. This choice indeed satisfies von Zeipel's theorem and it corresponds to constraint 2(c) above, with $f(rB_\phi) = (rB_\phi)^{1-2/\mu}$. It should be noted that the fulfillment of von Zeipel's hypothesis not only guarantees that $\Omega = \Omega(r)$ but it also provides the functional dependencies among $B_\phi$, $\rho$ and $r$ that allows to write the Euler equation in a potential form. In fact, we essentially imposed the vanishing of the curl of the right hand side of equation (1), so we looked for those cases when $((\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B})/\rho$ can be written as the gradient of a scalar.

## 2.2 Poloidal magnetic field

In cylindrical coordinates, a poloidal magnetic field can be written as

$$\mathbf{B} = \frac{1}{r}\boldsymbol{\nabla}\Psi \times \mathbf{e}_{\boldsymbol{\phi}} = \left(-\frac{\partial_z\Psi}{r}, 0, \frac{\partial_r\Psi}{r}\right) , \qquad (4)$$

where $\mathbf{e}_{\boldsymbol{\phi}}$ is the unit vector along the $\phi$ direction and $\Psi$, that physically represents the magnetic flux through a $z = z_0$, $r = r_0$ circle, is the $\phi$-component of the electromagnetic vector potential.[2] While in the case of toroidal magnetic fields the Maxwell equations $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and $\boldsymbol{\nabla} \times \mathbf{E} = 0$ are automatically satisfied under the assumption of stationarity and axisymmetry, when the magnetic field is poloidal they need to be properly taken into account, and they lead to so called Ferraro's iso-rotation law, namely $\mathbf{B} \cdot \boldsymbol{\nabla}\Omega = 0$. Hence, a necessary condition to have $\Omega = \Omega(r)$ is that $B^r = 0$, or, equivalently, that $\Psi = \Psi(r)$. As a result,

$$(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} = (-B_z\partial_r B_z, 0, 0) , \qquad (5)$$

and by computing the curl of equation (1) we find

$$2\Omega r\boldsymbol{\nabla}\Omega \times \mathbf{e}_r = -\frac{\boldsymbol{\nabla}\rho}{\rho^2} \times \boldsymbol{\nabla}p - \frac{B_z}{\rho^2}\partial_z\rho\,\partial_r B_z\,\mathbf{e}_{\boldsymbol{\phi}} . \qquad (6)$$

From (6), a weaker version of von Zeipel's theorem for a poloidal magnetic field can be deduced. Namely, if

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[2] In non-relativistic studies, $\Psi$ is usually called the *magnetic stream function* or *magnetic flux function*.

1. the equation of state is barotropic, $p = p(\rho)$,
2. $B^r = 0$, and
3. (a) $B_z = \text{const}$, or, (b) $\rho = \rho(r)$,

then $\Omega = \Omega(r)$. We note, incidentally, that a Newtonian star with a dipolar magnetic field will not satisfy this version of von Zeipel's theorem, since $B^r \neq 0$.

## 3 General relativistic version

The general relativistic version of von Zeipel's theorem in the absence of magnetic fields is due to [2] and it states that in a stationary and axisymmetric system the surfaces of constant angular velocity $\Omega$ and the surfaces of constant specific angular momentum $\ell$ coincide if and only if the rotating fluid is barotropic, i.e. it has an equation of state $p = p(e)$, where $e$ is the total energy density. Now we consider how this theorem can be rephrased if magnetic fields are present, again by distinguishing between the two main magnetic field topologies. Let us first recall the form of the energy momentum tensor for an magnetized flow with infinite conductivity, namely [8]

$$T^{\alpha\beta} = (\omega + b^2)u^\alpha u^\beta + (p + b^2/2)g^{\alpha\beta} - b^\alpha b^\beta \,, \tag{7}$$

where $u^\alpha$ are the components of the four velocity of the fluid, $b^\alpha$ are the components of the four vector magnetic field, $b^2 = b_\alpha b^\alpha$, $g_{\alpha\beta}$ are the coefficients of the metric, while $\omega = e + p$ is the enthalpy density. If we adopt a system of coordinates $(t, x^1, x^2, \phi)$ where $t$ and $\phi$ are the coordinates associated to the temporal and axial Killing vectors, respectively, then the four velocity of the circular motion is $u^\alpha = u^t(1, 0, 0, \Omega)$, where $\Omega = u^\phi/u^t$ is the angular velocity. Finally, the specific angular momentum mentioned in the theorem is $\ell = -u_\phi/u_t$, which is related to $\Omega$ by

$$\Omega = -\frac{g_{t\phi} + g_{tt}\ell}{g_{\phi\phi} + g_{t\phi}\ell} \,. \tag{8}$$

The relativistic Euler equation can be written as [10]

$$(\omega + b^2)a_\alpha + \nabla_\alpha\left(p + \frac{b^2}{2}\right) + u_\alpha u^\pi \nabla_\pi(p + b^2) + u_\alpha b^2 \nabla_\pi u^\pi - \nabla_\pi(b_\alpha b^\pi) = 0 \,.$$

As a result of the symmetries, $u^\alpha \nabla_\alpha(p + b^2) = \nabla_\alpha u^\alpha = 0$. Moreover, the four acceleration for the circular motion is [2]

$$a_\alpha = -\nabla_\alpha \ln u^t + \frac{\ell}{1 - \Omega\ell}\nabla_\alpha\Omega \,. \tag{9}$$

Combining all this, the Euler equation (9) becomes

$$-\nabla_\alpha \ln u^t + u^t u_\phi \nabla_\alpha\Omega = -\frac{\nabla_\alpha(p + b^2/2)}{\omega + b^2} + \frac{\nabla_\pi(b^\pi b_\alpha)}{\omega + b^2} \,, \tag{10}$$

where $u^t u_\phi = \ell/(1 - \Omega\ell)$. In ideal magnetohydrodynamics, the fluid acceleration obeys a few additional contraction relations, namely [8]

$$a_\alpha b^\alpha = \nabla_\alpha b^\alpha = -\frac{b^\alpha \partial_\alpha p}{\omega} \, , \tag{11}$$

that will be used below. Finally, according to Ferraro's iso-rotation law valid for stationary and axisymmetric systems, the normals to the equipotentials of $\Omega$ and of $\Psi$ are parallel, hence $\Omega = \Omega(\Psi)$, (see [27,10,22]).

### 3.1 Toroidal magnetic field

When the magnetic field is purely toroidal, it has been shown by [25] that Eq. (10) can be rewritten as

$$-\nabla_\alpha \ln u^t + u^t u_\phi \nabla_\alpha \Omega + \frac{\nabla_\alpha p}{\omega} + \frac{\nabla_\alpha(\mathcal{L}b^2)}{2\mathcal{L}\omega} = 0 \, , \tag{12}$$

where $\mathcal{L} = g_{t\phi}^2 - g_{tt}g_{\phi\phi}$. If we now take the covariant derivative $\nabla_\beta$ of (12) and then we multiply it by the completely antisymmetric tensor $\epsilon^{\alpha\beta\gamma\delta}$, all the symmetric terms in $(\alpha, \beta)$ vanish, and we find

$$\epsilon^{\alpha\beta\gamma\delta}\nabla_\alpha \Omega \nabla_\beta(u^t u_\phi) = \epsilon^{\alpha\beta\gamma\delta}\frac{\nabla_\alpha p \nabla_\beta \omega}{\omega^2} - \epsilon^{\alpha\beta\gamma\delta}\nabla_\alpha(\mathcal{L}b^2)\nabla_\beta\left(\frac{1}{2\mathcal{L}\omega}\right) . \tag{13}$$

In the absence of the magnetic field, the last term on the right hand side of this equation would vanish, and we would have the statement of von Zeipel's theorem as provided in [2]. In the presence of a magnetic field, a weaker version of von Zeipel's theorem can be formulated by saying that $\Omega = const$ surfaces coincide with $\ell = const$ surfaces if

1. the equation of state is barotropic, i.e. $p = p(e)$, and
2. (a) $\mathcal{L}b^2 = const$, or (b) $\mathcal{L}\omega = const$, or (c) $\mathcal{L}b^2 = f(\mathcal{L}\omega)$.

Note that, for instance, in the equilibrium model proposed by [25] $b^2 \sim \mathcal{L}^{\eta-1}\omega^\eta$, with $\eta$ a real index, and this choice does satisfy von Zeipel theorem as it corresponds to constraint 2(c), with $f$ a power law, i.e. $\mathcal{L}b^2 \sim (\mathcal{L}\omega)^\eta$. Also note that, as expected, the Newtonian limits of constraints 2(a), 2(b) and 2(c) just found span the same physical conditions provided by their Newtonian analogs that we found in Sec. 2.1.

### 3.2 Poloidal magnetic field

In the relativistic framework, the treatment of the poloidal magnetic field is significantly more involved than that of toroidal magnetic fields, and in what follows we use several results proved by Bekenstein & Oron [10,11]. We first exploit the freedom in the choice of the coordinates system to adopt coordinates $x^1 = z$ and $x^2 = \chi$ such that $\chi$ is constant along $\Psi = const$

surfaces, i.e. $\Psi = \Psi(\chi)$, while $z$ is constant along the normals to those surfaces. In the following, and just for notational convenience, we will refer to $(z, \chi)$ as to the B-O coordinates. An example of such coordinates is reported in Appendix A for the Schwarzschild metric.

We now recall a property highlighted by [35], who showed that a spacetime containing a purely toroidal flow with a purely poloidal (or purely toroidal) magnetic field is circular, meaning that in the chosen B-O coordinates the metric can be written in "diagonal plus one" form as

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{zz}dz^2 + g_{\chi\chi}d\chi^2 \ . \tag{14}$$

Moreover, from the definition of magnetic field in terms of the electromagnetic tensor, namely $b^\alpha =^* F^{\alpha\beta}u_\beta$, where $^*F^{\alpha\beta}$ is the dual of $F^{\alpha\beta}$, using the fact that there is no meridional motion ($u^z = u^\chi = 0$) and that $F_{\alpha\beta}u^\beta = 0$ (because of infinite conductivity) one finds ([11])

$$b^t = b^\phi = 0 \, , \tag{15}$$

$$b^i = \frac{\epsilon^{ij}}{u^t \sqrt{-g}} \partial_j \Psi \qquad i, j = z, \chi \, , \tag{16}$$

where $\epsilon^{z\chi} = -\epsilon^{\chi z} = 1$, $\epsilon^{zz} = \epsilon^{\chi\chi} = 0$ and where $g$ is the determinant of the metric (14). Therefore, the magnetic field lines lie on the equipotentials of $\Psi$, namely

$$b^\alpha \partial_\alpha \Psi = 0 \, , \tag{17}$$

which, due to Ferraro's iso-rotation law, also implies

$$b^\alpha \partial_\alpha \Omega = 0 \, . \tag{18}$$

We therefore expect that the magnetic field lines lie on the surfaces of constant magnetic potential $\Psi$ (magnetic surfaces), which coincide with the surfaces of constant angular velocity $\Omega$. This property prevents the generation of a toroidal component of the magnetic field, even in the presence of differential rotation.[3] The choice of the coordinates $(z, \chi)$ as described above turns out to be rather useful, because the poloidal magnetic field components are

$$b^z = \frac{1}{u^t \sqrt{-g}} \partial_\chi \Psi \, , \tag{19}$$

$$b^\chi = 0 \, , \tag{20}$$

while, from Eq. (18), it also follows that

$$\partial_z \Omega = 0 \implies \Omega = \Omega(\chi) = \Omega(\Psi) \, . \tag{21}$$

We note that the coordinate $\chi$ resembles closely the cylindrical coordinate $r$ of the Newtonian case, for which we found $B^r = 0$ from Ferraro's iso-rotation

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[3] If the poloidal magnetic field lines lie on the $\Psi = $ const surfaces, and if these surfaces coincide with the $\Omega = $ const surfaces, then the magnetic field lines will not "feel" rotational effects.

law, in analogy to Eq. (20). By combining Eq. (11) and Eq. (20), the fluid acceleration along the $z$-direction can be written as

$$a_z = -\partial_z \ln u^t = \frac{\nabla_\alpha b^\alpha}{b^z} = -\frac{\partial_z p}{\omega} \,. \tag{22}$$

Focusing on the last equality of Eq. (22) and writing explicitly the four-divergence of a four-vector, we obtain

$$\partial_z \ln \left( \frac{b\sqrt{-g}}{\sqrt{g_{zz}}} \right) = -\frac{\partial_z p}{\omega} \,, \tag{23}$$

where $b = (b^z b_z)^{1/2}$. Equation (23) places a first condition on the derivative $\partial_z b$, and can be used to write $\partial_z b / b$ as function of the metric and of the field. The interesting aspect of this equation is that it only involves derivatives along the coordinate $z$. We emphasize that it has nothing to do with von Zeipel's theorem, and it is a consequence of the adopted B-O coordinates.

The Euler equation (10) can also be written in terms of the new coordinates $(z, \chi)$. Before doing that, we write the magnetic terms on the right hand side of Eq. (10) as

$$-\frac{1}{2}\nabla_\alpha b^2 + \nabla_\pi (b^\pi b_\alpha) = b^\pi (\partial_\pi b_\alpha - \partial_\alpha b_\pi) - b_\alpha b^z \partial_z \ln u^t \,, \tag{24}$$

where we have again used Eq. (11), with $a_z$ given by Eq. (22). Hence, we can now write (10) in the two components $\chi$ and $z$ to obtain

$$-\partial_\chi \ln u^t + u^t u_\phi \partial_\chi \Omega = -\frac{\partial_\chi p}{\omega + b^2} - \frac{b^z \partial_\chi b_z}{\omega + b^2} \tag{25}$$

$$-\partial_z \ln u^t = -\frac{\partial_z p}{\omega + b^2} - \frac{b^2 \partial_z \ln u^t}{\omega + b^2} \,. \tag{26}$$

After taking the z-derivative of the first equation and the $\chi$-derivative of the second one, we find

$$-\partial_z \partial_\chi \ln u^t + \partial_z(\mathcal{K}(\ell)\partial_\chi \Omega) = \partial_z \left( -\frac{\partial_\chi p}{\omega + b^2} \right) - \partial_z \left( \frac{b^z \partial_\chi b_z}{\omega + b^2} \right) \tag{27}$$

$$-\partial_\chi \partial_z \ln u^t = -\partial_\chi \left( \frac{\partial_z p}{\omega + b^2} \right) - \partial_\chi \left( \frac{b^2 \partial_z \ln u^t}{\omega + b^2} \right) \,, \tag{28}$$

where we have defined the term

$$\mathcal{K}(\ell, \Omega) \equiv u^t u_\phi = l/(1 - \Omega l) \,. \tag{29}$$

We now subtract Eq. (28) from Eq. (27) to find the following condition

$$\partial_z \left( K(\ell, \Omega) \right) \partial_\chi \Omega = \underbrace{\partial_z \left( \frac{\partial_\chi p}{\omega} \right) - \partial_z \left( \frac{\partial_\chi p}{\omega + b^2} \right) - \partial_z \left( \frac{b^z \partial_\chi b_z}{\omega + b^2} \right)}_{1^{st}-\text{term}}$$

$$+ \underbrace{\partial_\chi \left( \frac{\partial_z p}{\omega + b^2} \right) - \partial_z \left( \frac{\partial_\chi p}{\omega} \right) + \partial_\chi \left( \frac{b^2 \partial_z \ln u^t}{\omega + b^2} \right)}_{2^{nd}-\text{term}} \,, \tag{30}$$

where, for reasons that will be immediately transparent, we have added and subtracted the term $\partial_z \left( \frac{\partial_\chi p}{\omega} \right)$, and where we have used $\partial_z \Omega = 0$. A generalized von Zeipel theorem can therefore be formulated by saying that $\Omega = \text{const}$ surfaces coincide with $l = \text{const}$ surfaces if the right hand side of equation (30) vanishes. In fact, when this is the case, the left hand side must satisfy

$$\partial_z \left( K(\ell, \Omega) \right) \partial_\chi \Omega = 0 \implies \partial_z \left( K(\ell, \Omega) \right) = \partial_l \mathcal{K} \partial_z \ell + \partial_\Omega \mathcal{K} \partial_z \Omega = \partial_l \mathcal{K} \partial_z \ell = 0 \,, \tag{31}$$

from which it follows that $\partial_z l = 0$. Since $\partial_z \Omega = 0$, this means that the surfaces at $\ell = \text{const}$ are the same as those at $\Omega = \text{const}$. We therefore concentrate our attention on the right side of Eq. (30). Under the assumption of barotropic equation of state, namely that $p = p(e)$, and using Eq. (22) we obtain

$$\partial_z \partial_\chi \ln u^t = \partial_\chi \partial_z \ln u^t = \partial_\chi \left( \frac{\partial_z p}{\omega} \right) = \partial_z \left( \frac{\partial_\chi p}{\omega} \right) \,. \tag{32}$$

As a result, the $2^{nd} -$ term of Eq. (30) is identically zero, as dictated by Eq. (28). On the other hand, the $1^{st} -$ term of Eq. (30) can be analyzed to obtain an expression for the partial derivative of $b$ along $\chi$. In fact, by imposing the vanishing of the $1^{st} -$ term we find

$$\partial_z \left[ \frac{b^2}{\omega(\omega + b^2)} \partial_\chi p - \frac{b^z \partial_\chi b_z}{\omega + b^2} \right] = 0 \,. \tag{33}$$

Hence, the quantity in the square brackets must be a function of the coordinate $\chi$ only, and therefore of $\Psi$ only. Hence, (see Appendix B for the details)

$$\frac{b^2}{2(\omega + b^2)} \left[ 2 \frac{\partial_\chi p}{\omega} - \partial_\chi \ln(b^2 g_{zz}) \right] = \mathcal{P}(\Psi) \,. \tag{34}$$

The constraint expressed by Eq. (33) can be further manipulated, leading to

$$\partial_z \ln \left( \frac{\omega}{b^2} \right) \left[ \partial_\chi \ln(b^2 g_{zz}) - 2 \frac{\partial_\chi p}{\omega} \right] = 2 \left( \frac{\omega + b^2}{\omega} \right) \partial_\chi \partial_z \ln(b^2 \sqrt{-g}) \,. \tag{35}$$

We emphasize that Eq. (35) and Eq. (23) form a system of partial differential equations for the magnetic field $b$ and the rest mass density $\rho$. To recap, when the magnetic field is poloidal, $\Omega = \text{const}$ surfaces coincide with $\ell = \text{const}$ surfaces if

1. the equation of state is barotropic, i.e. $p = p(e)$, and
2. the magnetic field obeys the PDE (35), or, equivalently, if the left hand side of (34) is a function of $\Psi$ only, where $(z, \chi)$ are the B-O coordinates such that $b^\chi = 0$.

A special case for which Eq. (23) admits a relatively simple solutions is discussed in Appendix C. It is also worth commenting about the relevance of our result in the context of the solution of the Grad-Shafranov (GS) equation, which is a highly non-linear partial differential equation in the unknown flux

function $\Psi$. Notoriously, the GS equation is used to compute stationary and axisymmetric magnetohydrodynamics solutions admitting a number of integral of motions, which depend only on $\Psi$. In the relativistic regime it has been analyzed by several authors in various physical contexts, including astrophysical jets, relativistic stars and magnetospheres of compact objects (see [15], [33], [19], [24], [12], [22]). The GS equation, however, provides no information about the relation among $\Omega$ and $\ell$ and the solution that is obtained in general does not satisfy the conditions of von Zeipel theorem. Our results show that, in order for the GS equation to give a solution with the property that $\Omega = \Omega(\ell)$, there must by an additional function $\mathcal{P}(\Psi)$, defined by (34) in B-O coordinates, which depends only on $\Psi$.

## 4 Conclusions

We have studied the conditions under which the surfaces of constant angular velocity $\Omega$ coincide with those of constant specific angular momentum $\ell$ for a stationary and axisymmetric magnetized perfect fluid in circular motion around a compact object. In the case of a purely toroidal magnetic field, such conditions amount to integrability conditions, both in the Newtonian and in the relativistic regime. The relativistic treatment of the poloidal magnetic field is more involved, and it is convenient to adopt suitable coordinates $(z, \chi)$, such that $\chi$ is constant along $\Psi = \text{const}$ surfaces, while $z$ is constant along the normals to those surfaces. In this way it is possible to show that $\Omega = \text{const}$ surfaces coincide with $\ell = \text{const}$ surfaces if the equation of state is barotropic, and if a function $\mathcal{P}(\Psi)$ exists, given by Eq. (34), which depends only on $\Psi$. These results become relevant when the construction of an equilibrium solution with a poloidal magnetic field is considered, a task that we will consider in the future through the numerical solution of the Grad-Shafranov equation.

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## A Example of B-O coordinates in the Schwarzschild metric

As an example of how to choose the B-O coordinates $(z, \chi)$ in a specific context, let us briefly
consider the case of a magnetized fluid rotating with constant specific angular momentum
$\ell$ in a Schwarzschild metric. From the iso-rotation law we know that $\Omega = \Omega(\Psi)$, and since
we are looking for a coordinate $\chi = \chi(\Psi)$, we can simply take $\chi = \Omega$. However, in the
Schwarzschild metric $\Omega = -g_{tt}\ell/g_{\phi\phi}$ and, since $\ell = \text{const}$, we can choose

$$\chi = \frac{(1 - 2/r)}{r^2 \sin^2 \theta}.$$ (36)

On the other hand, the coordinate $z$ can be computed from the requirement that the orthogonality between $z$ and $\chi$ is preserved, i.e. $g_{z\chi} = 0$. Straightforward metric coefficients transformations yield

$$z = (r - 3)\cos\theta\,. \tag{37}$$

In terms of such B-O coordinates it is then possible to write the constraints expressed by Eq. (23) and by Eq. (35) to guarantee that $\Omega = \Omega(\ell)$.

## B Derivation of Eq. (35)

We first note that, in the coordinates $(\chi, z)$, $b^2 = b_\alpha b^\alpha = b_z b^z$, and

$$b^z = \frac{b}{\sqrt{g_{zz}}}\,, \quad b_z = b\sqrt{g_{zz}}\,, \tag{38}$$

$$b^z \partial_i b_z = \frac{\partial_i b^2}{2} - \frac{1}{2}\mathcal{M}_i b^2\,, \quad \mathcal{M}_i \equiv g_{zz}\partial_i g^{zz} = \partial_i \ln g^{zz} = -\partial_i \ln g_{zz}\,. \tag{39}$$

Using these definitions, it is possible to rewrite Eq. (33) as

$$\partial_z \left\{ \frac{b^2}{2(\omega + b^2)} \left[ 2\frac{\partial_\chi p}{\omega} - \partial_\chi \ln b^2 - \partial_\chi \ln g_{zz} \right] \right\} = 0. \tag{40}$$

If we expand the derivatives in Eq. (40), we obtain

$$2\partial_z \left( \frac{\partial_\chi p}{\omega} \right) - \partial_z \partial_\chi \ln(b^2 g_{zz}) + \partial_z \ln \left( \frac{b^2}{\omega + b^2} \right) \left[ 2\frac{\partial_\chi p}{\omega} - \partial_\chi \ln(b^2 g_{zz}) \right] = 0\,, \tag{41}$$

which, by using Eq. (23), can also be written as

$$-2\partial_\chi \partial_z \ln(b^2\sqrt{-g}) + \partial_z \ln \left( \frac{b^2}{\omega + b^2} \right) \left[ 2\frac{\partial_\chi p}{\omega} - \partial_\chi \ln(b^2 g_{zz}) \right] = 0\,, \tag{42}$$

or, equivalently

$$\partial_\chi \ln(b^2 g_{zz}) = 2\frac{\partial_\chi p}{\omega} - \frac{2\partial_\chi \partial_z \ln((b^2\sqrt{-g}))}{\partial_z \ln \left( \frac{b^2}{\omega + b^2} \right)}\,. \tag{43}$$

The term $\partial_z \ln \left( \frac{b^2}{\omega + b^2} \right)$ can now be replaced using the identity

$$\partial_z \ln(\omega + b^2) = \partial_z \ln b^2 + \frac{\omega}{\omega + b^2}\partial_z \ln \left( \frac{\omega}{b^2} \right)\,, \tag{44}$$

thus allowing to rewrite Eq. (43) as

$$\partial_z \ln \left( \frac{\omega}{b^2} \right) \left[ \partial_\chi \ln(b^2 g_{zz}) - 2\frac{\partial_\chi p}{\omega} \right] = 2 \left( \frac{\omega + b^2}{\omega} \right) \partial_\chi \partial_z \ln(b^2\sqrt{-g})\,, \tag{45}$$

which is Eq. (35) of the main text. An alternative expression in which a single term with a first order derivative of the magnetic field is also possible and it is given by

$$\partial_\chi \ln b = - \left( \frac{b^2 + \omega}{2\omega} \right) \frac{\partial_\chi \left( \frac{2\partial_z p}{\omega} - \partial_z \ln \frac{g_{zz}}{\sqrt{-g}} \right)}{\partial_z \ln \sqrt{\frac{-g\omega}{g_{zz}}} + \frac{\partial_z p}{\omega}} + \frac{\partial_\chi p}{\omega} - \partial_\chi \ln \sqrt{g_{zz}}\,. \tag{46}$$

## C Analysis of a special case: the polytropic equation of state

A particular case for which Eq. (23) allows a simple integration is for a polytropic equations of state of the type $p = k\omega^\Gamma$, with $\Gamma$ and $k$ constants. In such circumstances a solution of Eq. (23) is

$$b = f(\chi)e^{-\frac{k\Gamma\omega^{\Gamma-1}}{\Gamma-1}} \frac{\sqrt{g_{zz}}}{\sqrt{-g}}, \quad \Gamma \neq 1,$$ (47)

where $f(\chi)$ is an arbitrary function of $\chi$, to be fixed through Eq. (35), or, alternatively, through Eq. (46). Hence, Eq. (47) expresses the general form of the magnetic field, written in B-O coordinates, when a polytropic equation of state is adopted. The fulfillment of von Zeipel's hypothesis through the additional condition (46) provides a constrain on the function $f(\chi)$, which must satisfy

$$\frac{\partial_\chi f}{f} + f^2 f_1(\chi, z) + f_2(\chi, z) = 0,$$ (48)

where $f_1$ and $f_2$ are two functions of $(\chi, z)$ through $\omega$, the metric $\mathbf{g}$ and their derivatives. As $f$ is a function of $\chi$ only, after taking the $z$ derivative of Eq. (48), we find the relation $f(\chi)^2 = -\partial_z f_2/\partial_z f_1$, which could be in principle used to determine $\omega$, and ultimately the rest mass density $\rho$, in a given spacetime metric. Although the explicit form of $f_1$ and $f_2$ can be extracted from Eq. (46), the relevant point is that, at least formally, the general solution of Eq. (48) is

$$f(\chi) = \mp \frac{e^{\int_1^\chi -f_2(t,z)\,dt}}{\sqrt{c_1 + 2\int_1^\chi e^{2\int_1^s -f_2(t,z)\,dt} f_1(s,z)ds}},$$ (49)

where $c_1$ is a constant.