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Linear systems with uncertain complex coefficients for AC sensitivity analysis

Daniele De Cecco, Franco Blanchini, Daniele Casagrande, Giulia Giordano, Erica Salvato

***Abstract*— This paper deals with the characterization of the set of exact solutions to uncertain complex linear systems, with a particular focus on those encountered in the frequency analysis of electrical networks. We assume that the real and imaginary parts of the uncertain parameters belong to predefined intervals, and we aim to characterize the set of all possible solutions. Our main result shows that sensitivity analysis with respect to variations of a single element can be performed exactly, as the sets of exact solutions for all variables are bounded by circular arcs. When several elements of the network are simultaneously subject to variations, the solution sets can be characterized by adopting appropriate circle arcs to approximate their boundaries, with considerable precision.**

Index terms - Algebraic linear systems, Sensitivity analysis, Rank-one matrices, AC-electrical networks, Circle arcs.

I. INTRODUCTION

In this paper, we face the problem of determining the set of all the possible complex solutions to linear algebraic systems in which some parameters are uncertain. This problem is particularly relevant to the frequency analysis of electrical networks, where impedances or admittances have uncertain yet bounded real and imaginary components within specified intervals. The components of the corresponding possible solutions lie in a region of the complex plane, which we characterize by leveraging the algebraic properties of the underlying model.

For *real* parameters, previous research has effectively addressed the aforementioned issue; see e.g. [9], [13], [15], [16]. Notably, an intriguing finding indicates that when the system matrix is the linear combination of rank-one matrices with uncertain coefficients, then the exact intervals for each real component can be determined [4], [7], [11], [12], [18]. A similar analytical approach has been presented in [1], [19] to determine the eigenvalues of uncertain mechanical systems.

Corresponding author: Franco Blanchini, Department of Mathematics and Computer Science, University of Udine, Italy (blanchini@uniud.it).

Daniele De Cecco and Daniele Casagrande are with the Polytechnic Department of Engineering and Architecture, University of Udine, Italy (dececco.daniele@spes.uniud.it, daniele.casagrande@uniud.it).

Giulia Giordano is with the Department of Industrial Engineering, University of Trento, Italy (giulia.giordano@unitn.it).

Erica Salvato is with the Department of Engineering and Architecture, University of Trieste, Italy (erica.salvato@units.it).

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As far as uncertain systems with complex parameters are concerned, important contributions can be found in [8], [10], [17], [22] and, more recently, in [14], [20], [21], [23], [24], [25]. However, it is worth emphasizing that these studies have a predominantly computational approach, with the primary objective of determining appropriate (often conservative) interval bounds for the real and imaginary components of the solutions.

In this paper we propose to characterize the boundary of the solution sets by the union of *circle arcs*. The inspiring reference is [8], where it is observed that, for a scalar equation (with just one unknown), the solution set forms a region in the complex plane bounded by circle arcs. To the best of our knowledge, the general n-dimensional case is still an open challenge. We employ here methodologies that draw from classical techniques such as [2], [3] and in particular from [5], [6], whose methods are specifically applied to the robust frequency analysis of uncertain transfer functions.

Our main findings can be summarized as follows.

- For a complex algebraic linear system featuring uncertain interval coefficients and subject to a rank-one condition, the set of admissible values for each unknown can be precisely characterized, and corresponds to the image of a complex-valued function, given by the ratio of two multi-affine functions of the uncertain parameters.
- The rank-one condition is demonstrated to align seamlessly with the sinusoidal steady-state sensitivity analysis of electrical networks (and vibrating systems).
- When the uncertainty affects a single impedance (or admittance), such as an unknown load in an electrical network, the exact admissible solution sets for all variables are shown to be delimited by *four circle arcs* that can be precisely computed.
- In challenging scenarios where all the components can be uncertain, we propose a method to achieve a remarkably accurate approximate representation by adopting suitable circle arcs.

Throughout the paper, we provide non-trivial examples of uncertain electrical AC-networks and conduct their sensitivity analysis.

II. SYSTEMS WITH UNCERTAIN COMPLEX PARAMETERS

We first recall a known result for real interval systems.

Theorem 2.1: [7], [11], [12] Given $A_0, A_1, \ldots, A_p \in \mathbb{R}^{n\times n}$, $b_0, b_1, \ldots, b_p \in \mathbb{R}^{n\times 1}$ and $\theta = (\theta_1, \ldots, \theta_p) \in [\theta_1^-, \theta_1^+] \times \cdots \times [\theta_p^-, \theta_p^+] \subset \mathbb{R}^p$, consider the equation $A(\theta)x = b(\theta)$ with $A(\theta) = A_0 + \sum_{i=1}^p A_i\theta_i$, and $b(\theta) = b_0 + \sum_{i=1}^p b_i\theta_i$. If $A(\theta)$ is non-singular for all values of θ and $\text{rank}[A_i|b_i] \leq 1$, for all $i = 1, \ldots, p$, then the set of the possible solutions is $[x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+]$ where, x_j^- and x_j^+ are the minimum and the maximum of all the 2^p solutions of x_j for θ *on the vertices*, namely, with $\theta \in \{\theta_1^-, \theta_1^+\} \times \cdots \times \{\theta_p^-, \theta_p^+\}$. □

The above neat "vertex" result does not extend to the complex case, to which the following analysis is dedicated.

Consider, then, the system of algebraic equations

$$\left[A_0 + \sum_{i=1}^{p} A_i \theta_i\right] z = b_0 + \sum_{i=1}^{p} b_i \theta_i \qquad (1)$$

where $z \in \mathbb{C}^n$ is the vector of unknowns, θ_i are real parameters, grouped in the vector $\theta \in \mathbb{R}^p$, bounded as

$$\Theta = \{\theta : \theta_i^- \leq \theta_i \leq \theta_i^+\}, \qquad (2)$$

while $A_i \in \mathbb{C}^{n\times n}$ and $b_i \in \mathbb{C}^{n\times 1}$, for all $i = 1, \ldots, p$.

Assumption 1: For all $i = 1, 2, \ldots, p$, $\text{rank}[A_i|b_i] \leq 1$. Moreover, $A_0 + \sum_{i=1}^{p} A_i\theta_i$ is non-singular for all θ in Θ.

Definition 1: A point $\theta \in \Theta$ is a *vertex* of Θ if $\theta_i \in \{\theta_i^-, \theta_i^+\}$ for all i. An *edge* of Θ is a set $\{\theta \in \Theta : \theta_k \in [\theta_k^-, \theta_k^+], \theta_i \in \{\theta_i^-, \theta_i^+\}, i \neq k\}$, where all but one the components of θ take one of their two extreme values.

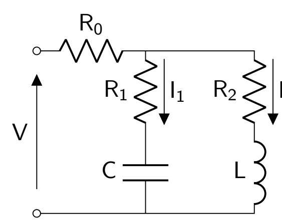

Fig. 1. A circuit representing a practical instance of the framework (1)-(2).

Example 2.1: Consider the simple circuit in Fig. 1. For a fixed frequency ω, the impedances are R_0, $R_1 + \jmath X_1$ and $R_2 + \jmath X_2$, where $X_1 = -1/(\omega C)$ and $X_2 = \omega L$. If we choose the currents I_1 and I_2 as unknowns, the corresponding system of linear equations is

$$\begin{bmatrix} R_0 + R_1 + \jmath X_1 & R_0 \\ -(R_1 + \jmath X_1) & R_2 + \jmath X_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}$$

Set $\theta_1 = R_0$, $\theta_2 = R_1$, $\theta_3 = X_1$, $\theta_4 = R_2$ and $\theta_5 = X_2$. Then, matrix A is

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\theta_1 + \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\theta_2 + \begin{bmatrix} \jmath & 0 \\ -\jmath & 0 \end{bmatrix}\theta_3 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\theta_4 + \begin{bmatrix} 0 & 0 \\ 0 & \jmath \end{bmatrix}\theta_5 ,$$

where $A_0 = 0$, while vector b is obtained as

$$b = \begin{bmatrix} V \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\theta_1 + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\theta_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\theta_3 + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\theta_4 + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\theta_5 .$$

Note that $\text{rank}[A_k|b_k] = 1$, for $k = 1, \ldots, 5$, which is a typical property of electric circuits (as well as vibrating systems). ■

Denote by z_i the i-th component of the unknown z, and define its solution set as:

$$\mathcal{Z}_i = \{z_i \in \mathbb{C} : z \text{ is a solution of (1) for some } \theta \in \Theta\} .$$

Our first result, whose proof is detailed in Sec. II-A, offers a characterization of the set $\mathcal{Z}_i$.

Theorem 2.2: Under Assumption 1, $\mathcal{Z}_i$ is the image of Θ through a complex valued function given by the ratio of two multi-affine polynomials. □

An interesting case is that of an electric circuit in which the real and imaginary parts of an impedance are unknown, as in the following result.

Theorem 2.3: Suppose that

$$A(\theta) = A_0 + A_1(\theta_R + \jmath\theta_I) , \quad b(\theta) = b_0 + b_1(\theta_R + \jmath\theta_I) , \ (3)$$

with $\theta = (\theta_R, \theta_I) \in \Theta = [\theta_R^-, \theta_R^+] \times [\theta_I^-, \theta_I^+]$. Under Assumption 1 the boundary of $\mathcal{Z}_i$ is the union of 4 circular arcs, which are the images of the edges of Θ. ■

To have a pictorial representation of the resulting set, the reader is referred to Figures 5 and 6.

A. Proofs of the results

Lemma 2.1: Let the matrices $M_1, \ldots, M_p \in \mathbb{C}^{n\times n}$ have rank at most one. The function $f : \mathbb{C}^p \to \mathbb{C}$ defined by

$$f(\theta_1, \ldots, \theta_p) = \det(M_0 + \textstyle\sum_{i=1}^{p} M_i\theta_i)$$

is multi-affine, i.e. it is affine with respect to each θ_i. □

Proof: Since $\text{rank}[M_k] \leq 1$ for $k = 1, \ldots, p$, there exist two column vectors v_k and w_k such that $M_k = v_k w_k^\top$. Also,

$$\det\left(\underbrace{M_0 + \textstyle\sum_{i\neq k} v_i w_i^\top \theta_i}_{\doteq \hat{M}} + \theta_k v_k w_k^\top\right) = \det\left(\hat{M} + \theta_k v_k w_k^\top\right)$$

$$= \det\left(\begin{bmatrix} \hat{M} + \theta_k v_k w_k^\top & 0 \\ w_k^\top & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} I & \theta_k v_k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{M} & -\theta_k v_k \\ w_k^\top & 1 \end{bmatrix}\right)$$

$$= 1 \times \left\{\det\left(\begin{bmatrix} \hat{M} & 0 \\ w_k^\top & 1 \end{bmatrix}\right) + \det\left(\begin{bmatrix} \hat{M} & -\theta_k v_k \\ w_k^\top & 0 \end{bmatrix}\right)\right\}$$

$$= \det\left(\hat{M}\right) + \theta_k \det\left(\begin{bmatrix} \hat{M} & -v_k \\ w_k^\top & 0 \end{bmatrix}\right) .$$

Hence the expression is affine in θ_k. ■

Lemma 2.2: Given $w_1, w_2, w_3, w_4 \in \mathbb{C}$ such that $w_3 + \theta w_4 \neq 0$ for all $\theta \in \mathbb{R}$, let $z : \mathbb{R} \to \mathbb{C}$ be the function defined by

$$z(\theta) = \frac{w_1 + \theta w_2}{w_3 + \theta w_4} . \qquad (4)$$

Then the image of $\mathbb{R}$ through z is a circle in the complex plane, possibly a degenerate one, namely a line. □

Proof: Let α_i and β_i ($i = 1, \ldots, 4$) denote the real and imaginary part of w_i and x and y denote the real and imaginary part of z. The condition on the denominator of $z(\theta)$ implies, in particular, that $w_3 \neq 0$. Assume, for the moment, that also $w_4 \neq 0$. Since $w_3 + \theta w_4 \neq 0$ for all (real) θ, the imaginary part of $-w_3/w_4$ must be different from zero, hence $\alpha_3\beta_4 - \alpha_4\beta_3 \neq 0$. Now, deriving θ from (4) yields

$$\theta = \frac{(\alpha_3 x - \beta_3 y - \alpha_1) + \jmath(\beta_3 x + \alpha_3 y - \beta_1)}{(\alpha_2 - \alpha_4 x + \beta_4 y) + \jmath(\beta_2 - \beta_4 x - \alpha_4 y)} . \qquad (5)$$

Since $\theta \in \mathbb{R}$, the two complex numbers at the numerator and denominator in (5) must be aligned, namely

$$\frac{\alpha_3 x - \beta_3 y - \alpha_1}{\alpha_2 - \alpha_4 x + \beta_4 y} = \frac{\beta_3 x + \alpha_3 y - \beta_1}{\beta_2 - \beta_4 x - \alpha_4 y} . \qquad (6)$$

Writing (6) as a quadratic equation in x and y yields

$$(\alpha_3\beta_4-\alpha_4\beta_3)(x^2+y^2)+(\alpha_2\beta_3+\alpha_4\beta_1-\alpha_3\beta_2-\alpha_1\beta_4)x+ \\ +(\alpha_2\alpha_3-\beta_1\beta_4+\beta_2\beta_3-\alpha_1\alpha_4)y+(\alpha_1\beta_2-\alpha_2\beta_1)=0\,. \quad (7)$$

Since $\alpha_3\beta_4 - \alpha_4\beta_3 \neq 0$, equation (7) represents a circle.

When instead $w_4 = 0$, we have that $\alpha_3\beta_4 - \alpha_4\beta_3 = 0$, so the equation represents a line. ■

We are now able to write the proof of Theorem 2.2.

Proof: (Theorem 2.2) It is well known that, by Cramer's rule, the k-th component of the solution of system (1) can be written as

$$z_k(\theta) = \frac{\det\left(C_0^{(k)} + \sum_{i=1}^p C_i^{(k)}\theta_i\right)}{\det\left(A_0 + \sum_{i=1}^p A_i\theta_i\right)} \quad (8)$$

where $C_i^{(k)}$ is obtained by replacing the k-th column of A_i with b_i, i.e.

$$C_i^{(k)} = [a_{i,1} \dots a_{i,k-1}\ b_i\ a_{i,k+1} \dots a_{i,n}]\,,$$

where $a_{i,j}$ is the j-th column of A_i. Since rank$[A_i|b_i] \leq 1$, $C_i^{(k)}$ has rank at most one as well (obviously, A_i also has rank at most one). Hence, Lemma 2.1 guarantees that both the numerator and the denominator of the fraction in (8) are multi-affine functions of θ. ■

Now we can also prove Theorem 2.3.

Proof: (Theorem 2.3) Again, by the well known Cramer's rule

$$z_i = \frac{\det\left(C_0 + C_1(\theta_R + \jmath\theta_I)\right)}{\det\left(A_0 + A_1(\theta_R + \jmath\theta_I)\right)}\,, \qquad i = 1, \dots, n\,, \quad (9)$$

where C_i is obtained by replacing the i-th column of A_i with b_i. Since C_1 and A_1 have rank one, by Lemma 2.1

$$z_i = \frac{c_0 + c_1(\theta_R + \jmath\theta_I)}{a_0 + a_1(\theta_R + \jmath\theta_I)} \quad (10)$$

for some complex c_0, c_1, a_0, a_1 not depending on θ_R or θ_I.

We want to show that the image $\mathcal{Z}_i$ of Θ through the function defined by (10) is delimited by four circle arcs (as in the pictorial representation in Figures 5 and 6). These four curves can be obtained by fixing one of the two parameters (either θ_I or θ_R) to one of its extreme admissible values and making the other one vary in its admissible interval. By Lemma 2.2, these curves are indeed circle arcs. The space enclosed by these arcs is clearly a subset of $\mathcal{Z}_i$. What we need to show is that these arcs are actually the boundary. Now the proof proceeds along the same lines as in [6], to which the reader is referred for details. ■

Our results can be extended to the case in which the variable of interest is not an individual unknown, but a linear combination of unknowns:

$$y = cz, \quad (11)$$

where c is a row vector. We can add this new equation to form the overall linear system

$$\begin{bmatrix} A(\theta) & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} b(\theta) \\ 0 \end{bmatrix}, \quad (12)$$

to which our theory can be applied without changes.

B. The case of multiple uncertain components

Theorem 2.3 applies when a single component has uncertain real and imaginary parts. When several components are uncertain, the situation is more involved, since θ is a hyper-rectangle in $\mathbb{R}^p$. Therefore, achieving a theoretical characterization of the solution set is hard. Yet, a heuristic procedure can be adopted to graphically represent this set:

- for each edge of Θ, draw the corresponding circle arc;
- let $\widetilde{\mathcal{Z}}$ be the minimal simply connected region containing all these arcs (roughly, the internal part).

Clearly, $\widetilde{\mathcal{Z}}$ represents a sub-region of the solution set. However, although examples in which the inclusion is strict may be constructed, the approximated subset $\widetilde{\mathcal{Z}}$ is an extremely accurate representation of the true solution set. Actually, any boundary point of the solution set is the image of a boundary point of Θ, not necessarily an edge.

Corollary 2.1: Let us generalise the expression of A in (3) as

$$A = A_0 + \sum_{h=1}^{p} A_h(\theta_R^h + \jmath\theta_I^h), \quad (13)$$

with rank$[A_h] = 1$. Each point of the boundary of $\mathcal{Z}$ is the image of some $\theta^* \in \Theta$ having the property that, for each h, the vector (θ_R^h, θ_I^h) is on the boundary of its admissible rectangle. Hence θ^* is on the boundary of Θ.

Proof: Simply apply Theorem 2.3 for each h. ■

As a consequence, if the uncertain coefficients are the real and imaginary parts of an impedance (or admittance), then any boundary point is such that at least one of the two parameters takes its value on the extrema of its admissible interval.

III. Steady-state sensitivity analysis of AC networks

Exploiting the fact that the rank-one condition is standard in electrical networks, we consider an electrical network in AC-regime, for which the unknowns are either currents multiplied by uncertain impedances or potentials multiplied by uncertain admittances: $V = (R + jX)I$, where $R \in [R^-, R^+]$ and $X \in [X^-, X^+]$ (or $I = (G + jH)V$, where $G \in [G^-, G^+]$ and $H \in [H^-, H^+]$), while current or voltage generators e are given. Then the linear system obtained by considering Kirchhoff's laws has an affine structure as in (1). Specifically, consider the generic column of matrix A, corresponding to an unknown current I_h. The non-zero terms in this column will be either ± 1, if the current I_h is considered in any node-balance equation, or the impedance $R_h + \jmath X_h$, if the potential $(R_h + \jmath X_h)I_h$ is considered in any loop potential equation.

The coefficients ± 1 will be part of A_0 (which has no rank assumptions). The rows with non-zero terms can be grouped to obtain

$$A = \begin{bmatrix} \dots & R_h + \jmath X_h & \dots \\ \dots & \vdots & \dots \\ \dots & R_h + \jmath X_h & \dots \\ \dots & \pm 1 & \dots \\ \dots & \vdots & \dots \\ \dots & 0 & \dots \end{bmatrix} = A_0 + M_1 R_h + \jmath M_2 X_h, \quad (14)$$

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$$\begin{bmatrix} Z_1+Z_2+Z_3 & -Z_2 & 0 & 0 & -Z_3 & 0 \\ -Z_2 & Z_2+Z_4+Z_{10} & -Z_{10} & 0 & 0 & -Z_4 \\ 0 & -Z_{10} & Z_{10}+Z_7+Z_8 & -Z_8 & -Z_7 & 0 \\ 0 & 0 & -Z_8 & Z_8+Z_9+Z_{11} & -Z_9 & 0 \\ -Z_3 & 0 & -Z_7 & -Z_9 & Z_3+Z_5+Z_6+Z_7+Z_9 & -Z_5 \\ 0 & -Z_4 & 0 & 0 & -Z_5 & Z_4+Z_5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} = \begin{bmatrix} e_1-e_2 \\ e_2 \\ -e_7 \\ 0 \\ e_7-e_5 \\ e_5 \end{bmatrix}$$

Fig. 2. Equation of the circuit from [10], [20], analysed in Section III-A.

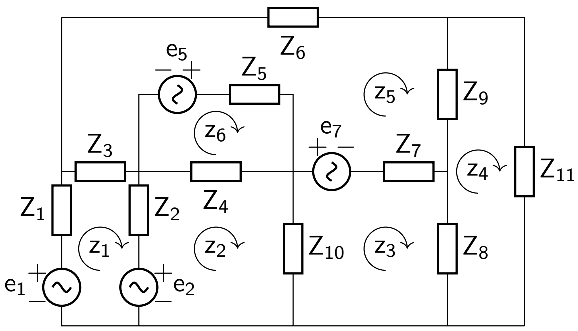

Fig. 3. Schematic of the circuit from [10], [20], analysed in Section III-A.

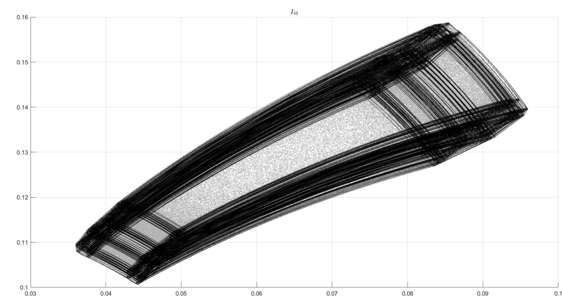

Fig. 4. The overall (approximate) estimation of the uncertainty set for the complex variable of the branch current through Z_{10}, $I_{10} = z_2 - z_3$: in black, the images of the edges (10^3 configurations per edge); in grey, 10^6 Monte Carlo solutions for random valid configurations.

where, again, A_0 does not depend on R_h and X_h. The rank-one condition is met, since R_h and X_h do not appear in any other column of A and, hence, M_1 and M_2 have a single non-zero column.

By duality, the same considerations can be repeated when an unknown voltage V_k appears in an equation. It will be either multiplied by ± 1, if V_k is considered in a loop potential equation, or multiplied by $G_h + \jmath H_h$ if the current $V_k(G_h + \jmath H_h)$ appears in a node-balance equation.

The rank-one condition holds even when choosing other unknown variables such as loop currents or node tensions, which are linear functions of the link currents or tensions.

Remark 3.1: Uncertain mutual inductances give rise to matrices that do not have rank one, and hence they cannot be considered. The simplest example is the system $V_1 = (R_1 + \jmath X_1)I_1 + X_{12}I_2$, $0 = (R_2 + \jmath X_2)i_1 + X_{12}I_1$.

Remark 3.2: Our results can be applied to elastic systems as well, given their analogy with linear electrical networks.

A. Sensitivity analysis of electrical networks

Consider the circuit in Fig. 3, reported from [10], [20]. Solving for the loop currents yields the complex-valued system of equations reported in Fig. 2. The nominal data given in [20], in terms of impedances: $Z_1 = Z_2 = Z_5 = Z_7 = 100 + \jmath 20$, $Z_3 = 100 + \jmath 30$, $Z_4 = 100 - \jmath 300$, $Z_6 = Z_8 = Z_9 = Z_{11} = 100$, $Z_{10} = 100 - \jmath 400$, and generators $e_1 = e_2 = 10$, $e_5 = e_7 = 100$. For a pictorial representation, we assume uncertainty over the complex impedances $Z_3, Z_4, Z_6, Z_8, Z_9, Z_{10}$ and Z_{11}, examining the sensitivity of all 6 unknowns $z_1, \ldots, z_6$ over these parameters. We assume a $\pm 20\%$ uncertainty for all parameters.

We sample 10^3 points from each edge of the parameter box Θ. For each component z_i, we plot its image solving the system for each of those parameter choices. These images include the (approximating) region of possible solutions. Fig. 4 shows the resulting overall sensitivity of the complex value of the current $I_{10} = z_2 - z_3$ through Z_{10}, which can be assessed by considering the extended system (11)–(12) with row vector $c = [\ 0 \ \ 1 \ -1 \ \ 0 \ \ 0\ ]$, while the matrix A and the column vector b are as in Figure 2.

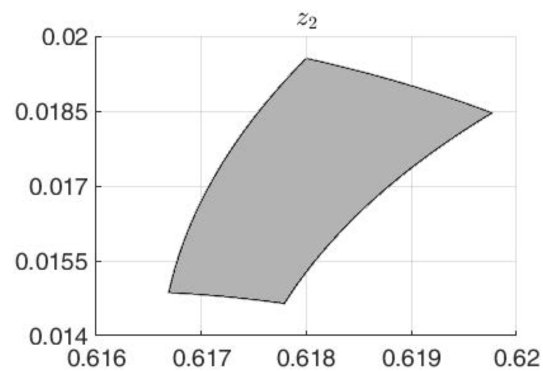

Fig. 5. Sensitivity of z_2 over Z_4 (exact).

In Fig. 5 and 6 only one complex parameter is considered at a time, therefore satisfying the hypothesis of Theorem 2.3 and thus ensuring that the boundary of the resulting edge set encloses all the possible solutions for the given parameter box.

For the sake of comparison, adopting specific optimization algorithms (Matlab routine *fmincon*) we have computed the minimum and maximum real parts of I_{10}, which are 0.6167 and 0.6198, respectively. The minimum and maximum imaginary parts of I_{10} are 0.0146 and 0.0196, respectively. These values are in perfect agreement with our theoretical results.

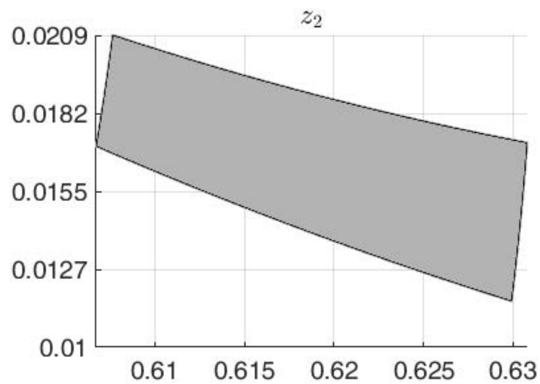

Fig. 6. Sensitivity of z_2 over Z_5 (exact).

TABLE I
RANDOM TEST RESULTS FOR $n = 6$.

p	Number of out-of-bounds points out of 10^6									
2	2	7	220	507	1019	3846	4167	4543	4825	8668
3	1	22	26	34	92	120	213	217	511	2538
4	0	0	0	0	0	0	3	5	7	14
5	0	0	0	0	0	0	0	0	0	0

B. Numerical random test to evaluate the accuracy of the representation

As a brief exploration of the multiple complex parameter case, we have performed the following test. We have randomly generated matrices associated with uncertain complex-valued systems. We have considered 10 random 6×6 systems for each value of $p \in \{2, 3, 4, 5\}$. For each system, we have generated random $A_0, \ldots, A_n$, b_0, and Θ and drawn the edge set. We have then computed 10^6 solutions of valid system configurations by means of a Monte Carlo approach. In particular, to mitigate the phenomenon of solutions naturally clustering around the one obtained by choosing the center value for each interval, we have sampled each interval with a U-shaped beta distribution with shape parameters $a = b = 0.3$. Table I shows the number of solutions that have been found to lie outside of the set characterized by all the circle arcs that are the image of the edges of the uncertainty set.

IV. CONCLUSION

This paper has addressed the sensitivity analysis of algebraic linear systems characterized by complex uncertain parameters. We have demonstrated that when the real and imaginary parts of a single parameter undergo interval variations, the admissible set for the components of the solution is bounded by circle arcs. Moreover, our findings offer an unexpectedly efficient heuristic for approximating solution sets in scenarios involving multiple parameter variations. Numerical random tests suggest that solutions lying outside the derived set are highly unlikely to be found.

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