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Constructivisation through Induction and Conservation

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Abstract

The topic of this thesis lies in the intersection between proof theory and algebraic logic. The main object of discussion, constructive reasoning, was introduced at the beginning of the 20th century by Brouwer, who followed Kant's explanation of human intuition of spacial forms and time points: these are constructed step by step in a finite process by certain rules, mimicking constructions with straightedge and compass and the construction of natural numbers, respectively.

The aim of the present thesis is to show how classical reasoning, which admits some forms of indirect reasoning, can be made more constructive. The central tool that we are using are induction principles, methods that capture infinite collections of objects by considering their process of generation instead of the whole class. We start by studying the interplay between certain structures that satisfy induction and the calculi for some non-classical logics. We then use inductive methods to prove a few conservation theorems, which contribute to answering the question of which parts of classical logic and mathematics can be made constructive.

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Notational conventions

Implication-like symbols	
$\supset$	Implication
$\Rightarrow$	Metalinguistic implication
$\rightarrow$	Sequent (also used for functions)
$\vdash$	Derivability
$\Vdash$	Forcing relation
$\triangleright$	Entailment relation

Equivalence, equality	
$\supset\subset$	Equivalence
$\Longleftrightarrow$	Metalinguistic equivalence
$\approx$	Equivalence (entailment relations)
$=$	Equality as an equivalence relation
$\equiv$	Definitional equality

Conjunctions, disjunctions	
$\wedge$	Conjunction
&	Metalinguistic conjunction
$\vee$	Disjunction
or	Metalinguistic disjunction

We use $\forall$ and $\exists$ for the universal and existential quantifier, respectively. Sometimes the same symbols will be used in metalinguistic way; the meaning will be clear from the context.

We often write metalinguistic implications $(A_1 \ \& \ A_2 \ \& \ \ldots) \Rightarrow B$ as rules:

$$\frac{A_1 \qquad A_2 \qquad \ldots}{B} \qquad \text{or} \qquad \frac{\{A_i : i \in I\}}{B}$$

where $I \subseteq \mathbb{N}$ is a set of indexes. We also write metalinguistic equivalences $A \Longleftrightarrow B$ as rules:

$$\frac{A}{\overline{B}}$$

Contents

1 Introduction

1.1 A brief introduction to constructive reasoning

Constructivism [12,19,20,49,114] is an approach to logic and mathematics in which it is necessary to "construct" an object to prove that it exists. This opposes to *classical* logic and mathematics, in which one can prove the existence of an object without "finding" that object explicitly, for instance by proving that its non-existence is contradictory.

However, for several centuries, mathematics and logic were considered two different disciplines and little to no attempt was made to apply one to the other. The first important result which uses non-constructive reasoning is probably Gauß's proof of the *fundamental theorem of algebra* from 1799 [76]:

> "Every non-constant single-variable polynomial with complex coefficients has at least one complex root."

In the 19th century there have been radical changes to the methodology of mathematics. In particular, mathematicians grew a preference for conceptual reasoning and abstract characterisations of mathematical concepts rather than computations. This led to an increasing confidence in dealing with "non-tangible" objects, such as the infinite.

Starting from the end of the 19th century, scepticism grew about this new way of reasoning, thus multiple schools of constructivism spread: *intuitionism* [12], which doesn't admit indirect reasoning, *predicativism* [50], which doesn't admit circular definitions, *finitism* [195], which doesn't admit actual infinity, *ultraconstructivism* [196], which doesn't admit anything that can't be computed in practice (e.g. too large natural numbers). For a more extended survey, we refer to [117]. Nowadays, most "mainstream" constructivism, such as Martin-Löf type theory **MLTT** [115, 116, 185] and Aczel's constructive Zermelo–Fraenkel set theory **CZF** [6, 7], is *intuitionism + predicativism*. Bishop-style constructive mathematics [18–20], which has its roots in intuitionism, has developed to such an extent that it is often considered a largely independent mathematical field [108, 117].

1.1.1 Intuitionism and the constructive content of classical mathematics

> "Constructive mathematics does not postulate a pre-existent universe, with objects lying around waiting to be collected and grouped into sets, like shells on a beach."
>
> E. Bishop [18]

According to *intuitionism*, a school of constructivism founded by Brouwer [28, 29, 114], mathematics is considered to be purely the result of the constructive and creative mental activity of humans rather than the discovery of fundamental principles claimed to exist in an objective reality.

Brouwer followed Kant's explanation of human intuition of spacial forms and time points [99]: these are constructed step by step in a finite number of steps by certain rules, mimicking constructions with straightedge and compass and the construction of natural numbers, respectively.

We can think of objects as data on a computer, and of functions as programs operating on data. An object is considered "valid" only when there is a program witnessing its construction. We call such programs *realisers*, and write $p\colon P$ for "p realises P". This follows Bishop's conjecture that, in the future, proofs would be compiled more or less directly into implementable code [18], and assigns to constructive reasoning a privileged role in the light of automated reasoning, see e.g. [158].

If the objects under consideration are logical statements, their realisers are *proofs*. The rules of the *Brouwer–Heyting–Kolmogorov Interpretation* (*BHK*) explain when a program realises a statement:

— Falsehood $\bot$ is never realised.

— Truth $\top$ is realised by a constant $*$.

— The conjunction $P \wedge Q$ is realised by a pair (p,q) such that $p\colon P$ and $q\colon Q$. In this case, we write $\pi_1(p,q)$ for p and $\pi_2(p,q)$ for q.

— The disjunction $P \vee Q$ is realised by a pair (k,p) such that either $k \equiv 0$ and $p\colon P$ or else $k \equiv 1$ and $q\colon Q$.

— The implication $P \supset Q$ is realised by a program which maps realisers of P to realisers of Q.

— The universal formula $\forall x \in A.P(x)$ is realised by a program which maps (a representation of) any $a \in A$ to a realiser of $P(a)$.

— The existential formula $\exists x \in A.P(x)$ is realised by a pair (p,q) such that p represents some $a \in A$ and $q\colon P(a)$.

We often write derivations as trees: by

$$\frac{A_1 \qquad A_2 \qquad \ldots}{B}$$

we mean that if it is the case that $A_1, A_2, \ldots$, then it is the case that B. We also informally use dots $\vdots$ to intend that there are some steps that are not made explicit. Two derivations can be composed:

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array} \quad \text{and} \quad \begin{array}{c} A \quad \Delta \\ \vdots \\ B \end{array} \quad \text{compose into} \quad \begin{array}{c} \Gamma \\ \vdots \\ A \quad \Delta \\ \vdots \\ B \end{array}$$

When a realiser is introduced as the argument of a function that we are constructing, we put it in square brackets. For instance:

$$\dfrac{\begin{array}{c}[p\colon P] \\ \vdots \\ f(p)\colon Q\end{array}}{p \mapsto f(p)\colon P \supset Q}$$

Notice that, whenever we have realisers $f\colon P \supset Q$ and $p\colon P$, then we get a realiser $f(p)\colon Q$. We can write this as a rule:

$$\dfrac{f\colon P \supset Q \qquad p\colon P}{f(p)\colon Q}$$

This principle is called *modus ponens*.

In intuitionistic logic, $\neg P$ is defined as $P \supset \bot$. Therefore, a realiser of $\neg P$ is a program which maps realisers of P to realisers of $\bot$:

$$\dfrac{\begin{array}{c}[p\colon P] \\ \vdots \\ f(p)\colon \bot\end{array}}{p \mapsto f(p)\colon \neg P}$$

This reasoning principle is of course called *proof of negation*.

Let's give an example:

Theorem 1.1.1 (Russell's Theorem [160]). *There is no set* $R \equiv \{x\colon x \notin x\}$.

Proof. Suppose that there is such R. This means that there is a realiser r of $\forall x(x \in R \supset\subset x \notin x)$. In particular, $r(R) \equiv R \in R \supset\subset R \notin R$. Let $A \equiv R \in R$. We show that $q \equiv (a \mapsto (\pi_1(r(R))(a))(a))$ is a realiser of $\neg A$:

$$\dfrac{\dfrac{\dfrac{\pi_1(r(R))\colon A \supset \neg A \quad [a\colon A]}{\pi_1(r(R))(a)\colon \neg A} \qquad [a\colon A]}{(\pi_1(r(R))(a))(a)\colon \bot}}{q \equiv a \mapsto (\pi_1(r(R))(a))(a)\colon \neg A}$$

Now:

$$\dfrac{q\colon \neg A \qquad \dfrac{\pi_2(r(R))\colon \neg A \supset A \quad q\colon \neg A}{(\pi_2(r(R)))(q)\colon A}}{q((\pi_2(r(R)))(q))\colon \bot}$$

Since we obtained a realiser of $\bot$, our assumption that R exists cannot hold. ■

Recall that a realiser of $P \vee Q$ is a pair (k, p) such that either $k \equiv 0$ and $p\colon P$ or else $k \equiv 1$ and $q\colon Q$. Therefore, in intuitionistic logic, if $P \vee Q$ is provable, then either P is provable or Q is provable. This is known as the *disjunction property*.

Classical logic can be defined as intuitionistic logic *plus* the principle of *excluded middle*, also known as *tertium non datur* (TND):

> "For any P, it holds that $P \vee \neg P$."

Take an *undecidable* statement P, that is statement such that neither P nor $\neg P$ can be proved.[1] If we assume TND, then the disjunction $P \vee \neg P$ holds, but this collides with the disjunction property. Therefore the two principles cannot hold simultaneously: in particular, we get that classical logic does not satisfy the disjunction property and intuitionistic logic does not satisfy TND.

This is not to say that intuitionists deny all instances of TND, or even that they assume its negation. In fact, the following holds:

[1] An easy way to produce an undecidable statement is to take any theory and drop some axiom; unless the axiom is redundant, it is undecidable in the resulting theory.

Theorem 1.1.2. *In intuitionistic logic, $\neg\neg(P \vee \neg P)$ is provable.*

Proof.

$$\dfrac{\dfrac{[a\colon \neg(P \vee \neg P)] \qquad \dfrac{\dfrac{\dfrac{[a\colon \neg(P \vee \neg P)] \quad \dfrac{[p\colon P]}{(0,p)\colon P \vee \neg P}}{a((0,p))\colon \bot}}{q_a \equiv p \mapsto a((0,p))\colon \neg P}}{(1,q_a)\colon P \vee \neg P}}{a((1,q_a))\colon \bot}}{a \mapsto a((1,q_a))\colon \neg\neg(P \vee \neg P)}$$

As we found a realiser of $\neg\neg(P \vee \neg P)$, it is provable. ■

Also intuitionists may use some instances of TND, but only those that they have proved. For instance, if we have a realiser $p\colon P$ or a realiser $q\colon \neg P$, then we can easily get a realiser of $P \vee \neg P$; and it can be shown from the axioms of Peano Arithmetic **PA** that any two natural numbers are equal or not equal.

One might think mistakenly that intuitionism adopts the existence of a "third" possibility, i.e. of some Q such that $\neg Q \wedge \neg\neg Q$. This statement is intuitionistically false because it states both $\neg Q$ and its negation $\neg\neg Q$, which is made impossible by the following:

Theorem 1.1.3 (Principium contradictionis)**.** *There is no proposition P such that $P \wedge \neg P$.*

Proof. If there were a proposition P such that both $p\colon P$ and $q\colon \neg P$, then $q(p)$ would be a realiser of $\bot$. ■

In classical logic, the reasoning principle of *proof by contradiction*, or *reductio ad absurdum* (RAA), can be used:

$$\dfrac{\begin{matrix}[q\colon \neg P] \\ \vdots \\ f(q)\colon \bot\end{matrix}}{q \mapsto f(q)\colon P}$$

In fact, suppose that by assuming the existence of a realiser $q\colon \neg P$ one can reach a realiser $f(q)\colon \bot$. By TND there is a realiser of $P \vee \neg P$, which is either of the form $(0, p)$ where $p\colon P$ or of the form $(1, q)$ where $q\colon \neg P$, but we must exclude the latter case if $\neg\neg P$ since otherwise by assumption we would get a realiser of $\bot$.

Proof of negation and RAA look and feel similar, but notice that in one case the conclusion has a negation removed and in the other added

$$\dfrac{\begin{array}{c}[p\colon P]\\ \vdots \\ f(p)\colon \bot\end{array}}{p \mapsto f(p)\colon \neg P} \qquad \dfrac{\begin{array}{c}[q\colon \neg P]\\ \vdots \\ f(q)\colon \bot\end{array}}{q \mapsto f(q)\colon P}$$

Unless we already believe in $\neg\neg P \supset P$, known as *double negation elimination* or *duplex negatio affirmat* (DNA), we cannot get one from the other by exchanging P and $\neg P$. The proof of a negation is *not* a proof by contradiction: they really are different reasoning principles.

We know that the proof of a negation is admissible in intuitionistic logic and mathematics. What about $\neg\neg P \supset P$? If it were the case, then by $\neg\neg(P \vee \neg P)$ we would get $P \vee \neg P$. Therefore, in intuitionistic logic, proofs by contradiction are not admitted in general.

We now present a couple of theorems, known in classic times, whose proofs are often said to be examples of proofs by contradiction but actually are proofs of negations.

Theorem 1.1.4. *Let p be a prime number, i.e. an integer that satisfies*[2]

$$p|a \cdot b \iff p|a \text{ or } p|a.$$

Then the number $\sqrt{p}$ is irrational.

> "From the assumption that the diagonal is commensurate, it follows that odd numbers are equal to evens."
>
> Aristotle [9]

[2]Here $a|b$ stands for "there is n such that $a \cdot n = b$". We often say that a is a *divisor* of b or that b is a multiple of a.

Proof. Suppose that $\sqrt{p} = \frac{a}{b}$ for some integers a, b. Assume without loss of generality that they have no common prime divisor. Then $p \cdot b^2 = a^2$. Since a^2 is a multiple of the prime number p, then also a is a multiple of p. Then we can write $a = p \cdot c$ for some integer c. Now $p \cdot b^2 = (2 \cdot c)^2 = 4 \cdot c^2$, thus $b^2 = 2 \cdot c^2$. Since b^2 is a multiple of the prime number p, then also b is a multiple of p. This contradicts the hypothesis that a and b have no common prime divisor. ■

Theorem 1.1.5. *There are infinitely many primes.*

> "Prime numbers are more than any assigned multitude prime numbers."
>
> Euclid [62]

Proof. Suppose there is just a finite number k of primes $p_1 = 2$, $p_2 = 3$, ..., p_k and consider $n = p_1 \cdot ... \cdot p_k + 1$. Choose a prime divisor p_i of n, i.e. such that $n = p_i \cdot n_1$ for some $n_1 \in \mathbb{N}$. By definition, we can also write $n = p_i \cdot n_2 + 1$ for some $n_2 \in \mathbb{N}$. Therefore $p_i \cdot n_1 = p_i \cdot n_2 + 1$, i.e. $p_i \cdot (n_1 - n_2) = 1$. In natural numbers, whenever $a \cdot b = 1$ we have $a = b = 1$, which contradicts the hypothesis that p_i is a prime number. ■

Many widely used principles of modern mathematics are incompatible with intuitionism since they imply TND over e.g. **CZF**. These include the axiom of choice [54] and and the generalised continuum hypothesis [27]. However, some weaker versions of these axioms are compatible with intuitionism, e.g. the *axiom of countable choice* [145]:

> "Any family of inhabited sets indexed by the natural numbers has a choice function."

It's worth pointing out that this form of choice suffices for many arguments that one encounters in analysis, so not all is lost. Furthermore, Richman [151] claims that the arguments for assuming countable choice are not compelling, and argues that it is possible to do mathematics even without it.

A turning point was 1967, when Bishop's book on Constructive Analysis [20] was published.

> "The thrust of Bishop's work was that both Hilbert and Brouwer had been wrong about an important point on which they had agreed. Namely both of them thought that if one took constructive mathematics seriously, it would be necessary to "give up" the most important parts of modern mathematics (such as, for example, measure theory or complex analysis). Bishop showed that this was simply false, and in addition that it is not necessary to introduce unusual assumptions that appear contradictory to the uninitiated."
>
> M. Beeson [13]

It is well worth pointing out that intuitionistic mathematics is a generalisation of classical mathematics, as was emphasised by Richman [149, 150], for a proof which avoids excluded middle and choice is still a classical proof. We thus have a shift of perspective in foundations: rather than developing constructive and classical mathematics separately, as in Brouwer's program, one studies which parts of classical mathematics can be directly translated into constructive terms.

It is in this frame that we see the importance of results about conservativity of classical logic over intuitionistic logic. The most well-known is surely Glivenko's Theorem [81, 82], which says that, in propositional logic, classical provability of a formula entails intuitionistic provability of the double negation of that formula.

1.1.2 A very short survey on proof systems

> "The main concern of proof theory is to study and analyze structures of proofs. A typical question in it is 'what kind of proofs will a given formula A have, if it is provable?', or 'is there any standard proof of A?'. In proof theory, we want to derive some logical prop-

> erties from the analysis of structures of proofs, by anticipating that these properties must be reflected in the structures of proofs. In most cases, the analysis will be based on combinatorial and *constructive* arguments. In this way, we can get sometimes much more information on the logical properties than with semantical methods, which will use set-theoretic notions like models, interpretations and validity."
>
> H. Ono [135]

Gentzen set as the task of his doctoral thesis [77,78] to develop a system of logic as close as possible to theorem proving in mathematics. He arguably succeeded, and came up with two different approaches: *natural deduction* and *sequent calculus*.

A system of *natural deduction* is specified by giving, for each logical connective and quantifier, introduction and elimination rules. The aim of natural deduction is to give a system of proof as close as possible to the "strategy of human reasoning". For instance, the introduction rule for $\wedge$ is given as

$$\frac{A \qquad B}{A \wedge B}\wedge\mathrm{I}$$

which can be read as "If we have a proof of A and a proof of B, then we have a proof of $A \wedge B$", which is exactly the clause for conjunction in the BHK Interpretation. Elimination rules, on the other hand, are obtained from introduction rules by applying the *inversion principle*:[3]

> "Whatever follows from the direct conditions/grounds for introducing a formula, must follow from that formula."

Again, we take conjunction as an example. For conjunction $A \wedge B$, the direct grounds are proofs of A and of B. Given that C follows

[3] Here we state the version by Negri and von Plato [132].

when both A and B are assumed, we find the elimination rule

$$\dfrac{A \wedge B \qquad \begin{matrix}[u\colon A] \quad [v\colon B] \\ \vdots \\ C\end{matrix}}{C}\ \wedge\mathrm{E}, u, v$$

which of course is tantamount to the more common "special elimination rules" obtained by setting $C \equiv A$ and $C \equiv B$:

$$\dfrac{A \wedge B}{A}\ \wedge\mathrm{E}_1 \qquad\qquad \dfrac{A \wedge B}{B}\ \wedge\mathrm{E}_2$$

When we do a proof in natural deduction, however, we lose trace of the compositions in the derivation. To be able to represent the composition of two derivations formally and to reason about its properties, we need a *sequent calculus*.

In a sequent calculus system, the open assumptions of each formula A in a derivation

$$\begin{matrix}\Gamma \\ \vdots \\ A\end{matrix}$$

are written out as a finite multiset[4] Γ in a sequent $\Gamma \to A$. The introduction rules of natural deduction become *right rules* of sequent calculus, and the elimination rules of natural deduction become *left rules* of sequent calculus. For instance, the rules for conjunction become:

$$\dfrac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \wedge B}\,\mathrm{R}\wedge \qquad\qquad \dfrac{A, B, \Gamma \to C}{A \wedge B, \Gamma \to C}\,\mathrm{L}\wedge$$

The calculus is then completed by the *initial sequent* (sometimes referred to as *axiom*, e.g. in [55])

$$P, \Gamma \to P,$$

[4]That is, a list with multiplicity but no order.

which affirms that a propositional variable P can prove itself. One proves *admissibility* of the *structural rules*, which assure (among other things) that derivations can be composed (rule of *Cut*) and that the multiset of assumptions can be enlarged (rule of *Weakening*).

Of course, there is no unique way to state the rules: there are several different calculi, of which the ones used in this thesis are collected in Appendix B. We just point out a few possible variants.

Rules of sequent calculus can have independent or shared contexts. For instance, rule R∧ stated as above has independent contexts Γ and Δ, but it can be stated as

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}\,\mathrm{R}\wedge$$

The two styles are equivalent in the presence of the structural rules.

A *multisuccedent* sequent has contexts both on the left and on the right, i.e. it is of the form $\Gamma \to \Delta$. The interpretation is that the conjunction of formulae in Γ implies the disjunction of formulae in Δ. This provides a natural representation of the division into cases often found in mathematical proofs. To illustrate, we give the shared-context multisuccedent version of L∧ and R∧:

$$\frac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}\,\mathrm{L}\wedge \qquad\qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}\,\mathrm{R}\wedge$$

We can view *sequent calculus* as the formal theory of the *derivability relation* →.

Another important class of calculi are those known as labelled sequent calculi. The basic idea of a *labelled sequent calculus* is the syntactical internalisation of Kripke semantics, which we are going to introduce.

A *Kripke model* [102] (X, R, val) is a set X of *possible worlds* together with an *accessibility relation* R, i.e. a binary relation between elements of X, and a valuation *val*, i.e. a function assigning one of the truth values 0 or 1 to an element x of X and an atomic formula P. The pair (X, R) is dubbed *Kripke frame*. The usual notation for

$val(x,P) \equiv 1$ is $x \Vdash P$. We read "xRy" as "y is *accessible* from x" and we read "$x \Vdash P$" as "formula P is *true* at world x" or just "x *forces* P".

Valuations are extended in a unique way to arbitrary formulae by means of inductive clauses, which depend on the logic. For instance, the usual inductive clause for conjunction and implication are

$x \Vdash A \wedge B$ if and only if $x \Vdash A$ and $x \Vdash B$,

$x \Vdash A \supset B$ if and only if $y \Vdash A \Rightarrow y \Vdash B$ for all y such that $x \leqslant y$.

We assume that $x \Vdash P$ is decidable for every $x \in X$ and each atomic formula P, which carries over to arbitrary formulae by the inductive clauses. Given a Kripke frame (X,R), we say that a formula A is *valid* in X if $val(x,A) = 1$ for every valuation val and for every $x \in X$.

An important feature of Kripke semantics is that a logic can be characterised by the mathematical properties of the accessibility relation R, such as *reflexivity* $(\forall x(xRx))$, *irreflexivity* $(\forall x(x\not R x))$, *transitivity* $(\forall x \forall y \forall z.(xRy \,\&\, yRz) \Rightarrow xRz)$. This will be further expanded in Chapter 2.

A labelled sequent calculus operates on labelled formulae $x\colon A$, to be read as "x forces A", and on relational formulae xRy. Inductive clauses are turned into formal rules. For instance, the inductive clause for conjunction stated above becomes

$$\frac{x\colon A, x\colon B, \Gamma \to \Delta}{x\colon A \wedge B, \Gamma \to \Delta}\,\mathrm{L}\wedge \qquad \frac{\Gamma \to \Delta, x\colon A \qquad \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \wedge B}\,\mathrm{R}\wedge$$

Also mathematical properties of the accessibility relation R can be turned into rules. For instance, reflexivity becomes

$$\frac{xRx, \Gamma \to \Delta}{\Gamma \to \Delta}\,\mathrm{Refl}_R$$

1.1.3 Avoiding circularity through induction

Consider *Richard's paradox* [148], here stated in the English version from [50]:

> "Let us consider all the real numbers which are definable in English by a finite number of words and let D be their collection. D is countable. We can then list all the elements of D, and mimic Cantor's diagonal proof of the non-denumerability of the real numbers to produce a new real number, r, which is different from each element of D. However, one can easily express in English a rendering of the "algorithm" that allows for the definition of r, so that r turns out to be a definable real number after all, and a contradiction arises."

Richard's paradox is engendered by a form of circularity: we define r by reference to the whole D, and therefore, so it is claimed, by reference to r itself.

Russell introduced the *vicious circle principle* to prevent the formation of collections such as D, and claimed that these are *ill-formed* [161]:

> "Whatever in any way concerns all or any or some of a class must not be itself one of the members of a class."

More examples of circularity:

(i) The liar paradox: *I'm lying.*

(ii) The Russell class $R \equiv \{x\colon x \notin x\}$, that is the class of all sets that do not contain themselves. See also Theorem 1.1.1.

(iii) The logicist definition of natural number:

$$N(n) \equiv \forall F(F(0) \,\&\, \forall x(F(x) \Rightarrow F(\mathrm{succ}\, x)) \Rightarrow F(n))$$

This urges us to distinguish between predicative and impredicative definitions [50]. A definition is *impredicative* if it defines an entity by reference to a class to which the entity itself belongs. A definition is *predicative* if it is not impredicative. Intuitively, predicative entities are those which are "built up from within".

One can avoid vicious circularity in the definition of $\mathbb{N}$ by defining it as the "smallest" class such that:

— 0 is a natural number,

— if n is a natural number, then $\operatorname{succ} n$ is a natural number.

Similarly, one can give the definition of *R-ideal*. Given a commutative ring R and $A \subseteq R$, the *R-ideal* $\langle A \rangle$ generated by A is defined as the "smallest" class such that:

— each element of A is in $\langle A \rangle$,

— if x and y are in $\langle A \rangle$, then $x + y$ is in $\langle A \rangle$,

— if x is in $\langle A \rangle$ and r is in R, then $x \cdot r$ is in $\langle A \rangle$.

These are prime examples of *inductive definitions*.

Formally, we should understand an *inductively defined class X* as being *freely generated* by a certain finite collection of *constructors*, each of which is a function $f : X^n \to X$, where $n \in \mathbb{N}$ can also be 0, in which case f is a *constant*. That is, the elements of X can be obtained by starting from nothing and applying the constructors repeatedly.

We briefly introduce entailment relations. More details will be presented in Section 5.2. Let S be a set. A *(single-conclusion) entailment relation* is a relation $\rhd \subseteq \operatorname{Fin}(S) \times S$ such that

$$\frac{}{a, U \rhd a}\,(\mathrm{R}) \qquad \frac{U \rhd a}{U, U' \rhd a}\,(\mathrm{M}) \qquad \frac{U \rhd a \qquad V, a \rhd b}{U, V \rhd b}\,(\mathrm{T})$$

for all finite $U, V \subseteq S$ and $a, b \in S$, where as usual $U, V \equiv U \cup V$ and $V, a \equiv V \cup \{a\}$. These rules are known as *reflexivity*, *monotonicity* and *transitivity*, respectively, and are collectively referred to as *structural rules*. Our focus thus is on *finite* subsets of S, for which we reserve the letters $U, V, W, \ldots$; we sometimes write $a_1, \ldots, a_n$ in place of $\{a_1, \ldots, a_n\}$ even if $n = 0$. Quite often an entailment relation is inductively generated from axioms by closing up with respect to the three rules above [157]. As we will see in Chapter 5, entailment relations were conceived as an abstract version of the derivability relation $\to$.

When we consider an inductively defined class X, we can define its *generating relation* $\rhd \subseteq \mathrm{Fin}(S) \times S$ as the entailment relation generated by all axioms of the form

$$a_1, \ldots, a_n \rhd f(a_1, \ldots, a_n)$$

for each constructor f. Intuitively, $U \rhd a$ means that a can be obtained by applying the constructors a finite numbers of times (possibly zero) to elements of U.

Consider a class S endowed with a relation $R \subseteq S \times S$. A predicate P on S is said to be *progressive* with respect to R if, whenever it holds for all predecessors (w.r.t. R) of a given element $b \in S$, then it also holds for b. As usual, P is said to be *universal* if it holds for all elements of S. The *(Noetherian) induction principle* for a relation R states that, whenever a predicate P is progressive, then it is universal. More explicitly, R satisfies the (Noetherian) induction principle if, for any given predicate P,

$$\forall b \in S(\forall a \in S.\, aRb \Rightarrow (P(a) \Rightarrow P(b))) \Rightarrow \forall b \in S\, P(b).$$

Similarly, the *(Noetherian) induction principle* for a relation $\rhd \subseteq \mathrm{Fin}(S) \times S$ states that, for any given predicate P,

$$\forall b \in S(\forall U \in \mathrm{Fin}(S).\, U \rhd b \Rightarrow (P[U] \Rightarrow P(b))) \Rightarrow \forall b \in S\, P(b), \quad (1.1)$$

where $P[U]$ is an abbreviation for $\forall u \in U\; P(u)$.

Notice that one can define an entailment relation $\blacktriangleright_P$ over S by postulating that $U \blacktriangleright_P b$ if and only if $P[U] \Rightarrow P(b)$; it is straightforward to check that $\blacktriangleright_P$ satisfies reflexivity, monotonicity and transitivity. Therefore, using this notation, we can rewrite (1.1) as

$$\rhd \subseteq \blacktriangleright_P \Rightarrow \forall b \in S\, P(b).$$

This can be useful in proof practice, since $\rhd \subseteq \blacktriangleright_P$ is tantamount to the condition that $\blacktriangleright_P$ satisfies all axioms and rules in the inductive definition of $\rhd$.

The *induction principle*, also known as *elimination rule* [185], for an inductively defined class X is then defined as the Noetherian induction principle for its generating relation $\rhd$.

This principle describes how to prove a proposition $P(x)$ about an arbitrary element $x \in X$. Such a proof is called *proof by induction*. Induction principles are a main tool for capturing the infinite by representing potentially incomplete processes of generation, and are usually more constructive than other classically equivalent principles. This well relates to Brouwer's conception of mathematics: mathematical construction can only be realised in a finite process, step by step like counting in arithmetic.

Here are a few formalised examples:

(i) Inductive definition of the *Boolean ring* 2:

$$\begin{aligned} 0&\colon 2 \\ 1&\colon 2 \end{aligned}$$

The only elements of 2 are 0 and 1. If constructors are all constant, then the class is said to be *extensionally defined*. The induction principle for 2 states:

$$(P(0) \;\&\; P(1)) \Rightarrow \forall x \in 2\; P(x).$$

(ii) Inductive definition of the class of *Natural numbers* $\mathbb{N}$:

$$\begin{aligned} 0&\colon \mathbb{N} \\ \mathrm{succ}&\colon \mathbb{N} \to \mathbb{N} \end{aligned}$$

Every element of $\mathbb{N}$ is either 0 or obtained by applying succ to some "previously constructed" element of $\mathbb{N}$. The induction principle for $\mathbb{N}$ states:

$$(P(0) \;\&\; \forall x \in \mathbb{N}(P(x) \Rightarrow P(\mathrm{succ}\, x))) \Rightarrow \forall x \in \mathbb{N}\; P(x).$$

(iii) Inductive definition of the *R-ideal $\langle A\rangle$ generated by $A \subseteq R$*, where R is a ring:

$$\begin{aligned} a&\colon \langle A\rangle && \text{for each } a \in A \\ +&\colon \langle A\rangle \times \langle A\rangle \to \langle A\rangle \\ \_\cdot r&\colon \langle A\rangle \to \langle A\rangle && \text{for each } r \in R \end{aligned}$$

Every element of $\langle A \rangle$ is either an element of A, or obtained by summing two "previously constructed" elements of $\langle A \rangle$, or obtained by multiplying a "previously constructed" element of $\langle A \rangle$ with an element of R. The induction principle for $\langle A \rangle$ states:

$$\begin{aligned}(\forall a \in A\, P(a) \;\&\; \forall a, b \in \langle A \rangle\, ((P(a) \;\&\; P(b)) \Rightarrow P(a+b)) \\ \&\; \forall a \in \langle A \rangle \forall r \in R\, (P(a) \Rightarrow P(a \cdot r))) \\ \Rightarrow \forall a \in \langle A \rangle\, P(a).\end{aligned}$$

1.2 Content and structure of the thesis, published material

Part I. Induction principles in labelled calculi for non-classical logics

The aim of this part is to investigate the properties of calculi that correspond to Kripke frame which satisfy induction principles. By doing so, we also get some insights on induction itself.

Chapter 2. Modal logic for induction We use modal logic to obtain syntactical, proof-theoretic versions of transfinite induction as axioms or rules within an appropriate labelled sequent calculus. While transfinite induction proper, also known as Noetherian induction, can be represented by a rule, the variant in which induction is done up to an arbitrary but fixed level happens to correspond to the Gödel–Löb axiom of provability logic. To verify the practicability of our approach in actual practice, we sketch a fairly universal pattern for proof transformation and test its use in several cases. Among other things, we give a direct and elementary syntactical proof of Segerberg's theorem that the Gödel–Löb axiom characterises precisely the (converse) well-founded and transitive Kripke frames.

This chapter is based on joint work with Sara Negri and Peter Schuster, and is a revised version of the following paper:

[67] Fellin, G., Negri, S. & Schuster, P. Modal logic for induction. *Advances In Modal Logic*. **13** pp. 209-227 (2020), Advances in Modal Logic 2020, Helsinki, Finland (on-line), August 24–28, 2020

Chapter 3. A terminating intuitionistic calculus In the labelled sequent calculus **G3I** for intuitionistic logic, we modify rule R⊃, by adding a variant of the principle of *a fortiori* in the left-hand side of premiss. In the resulting calculus $\mathbf{G3I_t}$, it is decidable whether any given sequent is derivable. In the negative case, the failed proof search gives a finite countermodel to the sequent on a reflexive, transitive and Noetherian Kripke frame.

This chapter is based on a joint work with Sara Negri and is still unpublished.

Part II. Conservation: Glivenko-style results

In the second part of the thesis, we apply methods based on induction in order to obtain a number of generalisations of Glivenko's theorem.

Chapter 4. Glivenko classes and constructive cut elimination in infinitary logic A constructivisation of the cut-elimination proof for sequent calculi for classical, intuitionistic and minimal infinitary logics with geometric rules—given in earlier work by Sara Negri [131]—is presented. This is achieved through a procedure where the non-constructive transfinite induction on the commutative sum of ordinals is replaced by two instances of Brouwer's Bar Induction. The proof of admissibility of the structural rules is made ordinal-free by introducing a new well-founded parameter called *proof embeddability*. Additionally, we extend to the infinitary case the proof of conservativity for the finitary Glivenko sequent classes given in earlier work by Negri [130].

This chapter is based on joint work with Sara Negri and Eugenio Orlandelli, and is a revised and extended version of the following paper:

[66] Fellin, G., Negri, S. & Orlandelli, E. Constructive cut-elimination in geometric logic. *27th International Conference On Types For Proofs And Programs (TYPES 2021)*. (2021)

That paper presented only the constructive cut elimination for classical and intuitionistic geometric logics based on Brouwer's Bar Induction. The main novelties of this chapter are that also minimal geometric logic is considered, that transfinite inductions on ordinals are replaced by well-founded induction with proof-embeddability, and that proofs of conservativity for the infinitary Glivenko classes are given.

Chapter 5. A general Glivenko–Gödel theorem for nuclei Glivenko's theorem says that, in propositional logic, classical provability of a formula entails intuitionistic provability of the double negation of that formula. We generalise Glivenko's theorem from double negation to an arbitrary nucleus, from provability in a calculus to an inductively generated abstract consequence relation, and from propositional logic to any set of objects whatsoever. The resulting conservation theorem comes with precise criteria for its validity, which allow us to instantly include Gödel's counterpart for first-order predicate logic of Glivenko's theorem. The open nucleus gives us a form of the deduction theorem for positive logic, and the closed nucleus prompts a variant of the reduction from intuitionistic to minimal logic going back to Johansson.

This chapter is based on joint work with Peter Schuster, and is a revised version of the following paper:

[68] Fellin, G. & Schuster, P. A General Glivenko-Gödel Theorem for Nuclei. *Proceedings Of The 37th Conference On The Mathematical Foundations Of Programming Semantics, MFPS 2021, Salzburg, Austria, August 29–September 3, 2021*. (2021)

The starting point for this chapter is previous joint work with Peter Schuster and Daniel Wessel, which was published in the paper

[70] Fellin, G., Schuster, P. & Wessel, D. The Jacobson Radical of a Propositional Theory. *The Bulletin Of Symbolic Logic*. pp. 1-20 (2021)

and in turn emerged from Fellin's MSc thesis [65] and Wessel's PhD thesis [191].

Chapter 6. Universal translation methods for nuclei Negative translations are well-known methods that turn classically valid formulae into intuitionistically valid ones. The most common are translations due to Kolmogorov, Gentzen, Kuroda and Krivine. As the name suggests, they rely on double negation, also known as the Glivenko nucleus. An attempt to generalise a variant of the Kuroda translation to arbitrary nuclei in logic was already done by van den Berg in [187]. The aim of this chapter is to further generalise negative translations from double negation to an arbitrary nucleus, from provability in a calculus to an inductively generated abstract consequence relation, and from propositional logic to any set of objects whatsoever.

This chapter is the continuation of the joint work with Peter Schuster presented in Chapter 5, and is still unpublished.

Contributions to publications

In publication [67], Fellin elaborated on co-author Schuster's idea to convert induction principles into modal formulae, and obtained the main results of Sections 2.3–2.4, which provide a general proof transformation pattern.

Publication [66] is mostly based on previous work by co-authors Negri and Orlandelli. In particular, it follows two earlier publications by Negri [130, 131]. The idea to define proof-embeddability (Section 4.4) is due to Fellin, who also took care of its applications in the proofs of structural rules (Section 4.5), cut-elimination (Lemmata 4.6.2–4.6.5) and Orevkov's theorems on infinitary Glivenko classes (Section 4.7).

Of publication [68], whereas the main idea goes back to Wessel and co-author Schuster, it was Fellin who came up with Example 5.3.6 and showed the importance of distinguishing between axioms and rules; thus introduced the concept of compatibility, which is crucial in the statement of the main results (Theorem 5.3.8 and Corollary 5.3.11) and developed the applications in logic (Section 5.5).

Part I

Induction principles in labelled calculi for non-classical logics

2 Modal logic for induction

2.1 Introduction

At least since Peano formalised what we all know as mathematical induction, induction as a proof principle has been the main tool for tidily unwrapping the potential infinite as generated by an a priori incomplete process. This is well reflected by the ubiquity of definitions and proofs by induction in today's ever more formal sciences.

Transfinite induction is a generalisation of mathematical induction from the natural numbers to less down-to-earth well-founded orders, such as the ordinal numbers. More precisely, if (and only if) any given order is well-founded, then *induction* holds: in the sense that a predicate holds everywhere in the given order provided that the predicate is progressive, i.e. propagates from all predecessors of a given element to the element itself.

As a rule of thumb, instances of induction are applicable more directly, and are better behaved proof-theoretically, than the corresponding instances of well-foundedness, which come as extremum principles or chain conditions (see, e.g., Proposition 2.4.2 below). Characteristic examples include Aczel's Set Induction [1–3,6,7] versus von Neumann and Zermelo's Axiom of Foundation or Regularity, and Raoult's Open Induction [15, 35, 143] as opposed to Zorn's Lemma.

Awareness of this phenomenon brought us to carry over to the inductive side some occurrences of well-foundedness in the modal logic of provability. Perhaps Segerberg's theorem [173], which stood right at the beginning of an impressive development [26], is the most prominent case: the Gödel–Löb axiom characterises exactly the (converse) well-founded and transitive Kripke frames.[1] The observation that those occurrences are rather about induction prompted the present investigation.

Inasmuch as instances of induction are about predicates or subsets, they typically go beyond the given logical level, and actually have a somewhat semantic flavour [41, 51]. By modal logic [21, 138, 142] we now obtain syntactical, proof-theoretic variants of induction: they are expressed as axioms or rules within an adequate labelled sequent calculus [125, 133]. While induction proper, for which we say Noetherian induction, can be mirrored by a rule (Lemma 2.3.3), the variant in which induction is done up to an arbitrary but fixed point of the given order, which we dub Gödel–Löb induction, happens to correspond (Lemma 2.3.1) to the homonymous axiom of provability logic [23, 24, 107, 178, 188].[2] In fact the usual way to define validity in a Kripke model for the modal operator $\Box$ lends itself naturally to capture universal validity up to a point.

To verify the practicability of our approach in proof practice, we give a fairly universal pattern for proof transformation, from rather algebraic inductive proofs to formal proofs with the required rules, and test this in several cases. Among other things, we prove with the corresponding modal rules that induction necessitates the order under consideration to be irreflexive (Lemma 2.4.1), and that every meet-closed inductive predicate on a poset propagates from the irreducible elements to any element whatsoever (Example 2.3.5) [153, 166, 167]. As a by product we gain the curiosity that Noetherian induction is tantamount to the corresponding chain

[1] See also, for example, Theorem 3.5 of [179], Example 3.9 of [21] and Teorema 7.2 of [138].

[2] This was also called axiom A3 [179], the Löb formula L [21] and axiom G or axiom W [84, 133].

condition plus irreflexivity (Proposition 2.4.2).[3] Last but not least we give a direct and elementary syntactical proof (Theorem 2.4.3) of Segerberg's aforementioned theorem that the Gödel–Löb axiom holds exactly in the (converse) well-founded and transitive Kripke frames. All this can also be useful in proof practice: while it might be cumbersome to prove directly that an induction principle holds for a given order, it is often easier to check properties such as irreflexivity and transitivity, or even chain conditions.

2.2 Basic modal logic K

Modal logic is obtained from propositional logic by adding the modal operator $\Box$ to the language of propositional logic. A *Kripke model* [102] (X, R, val) is a set X together with an *accessibility relation* R, i.e. a binary relation between elements of X, and a valuation val, i.e. a function assigning one of the truth values 0 or 1 to an element x of X and an atomic formula P. The usual notation for $val(x, P) \equiv 1$ is $x \Vdash P$.

We read "xRy" as "y is *accessible* from x" and we read "$x \Vdash P$" as "*x forces P*". Valuations are extended in a unique way to arbitrary formulae by means of inductive clauses:

$$x \nVdash \bot$$
$$x \Vdash A \supset B \text{ if and only if } x \Vdash A \Rightarrow x \Vdash B$$
$$x \Vdash A \wedge B \text{ if and only if } x \Vdash A \text{ and } x \Vdash B$$
$$x \Vdash A \vee B \text{ if and only if } x \Vdash A \text{ or } x \Vdash B$$
$$x \Vdash \Box A \text{ if and only if } \forall y (xRy \Rightarrow y \Vdash A)$$

We assume that $x \Vdash P$ is decidable for every $x \in X$ and each atomic formula P, which carries over to arbitrary formulae by the inductive clauses. With the intended applications in mind, in place of R we use the inverse accessibility relation $<$, i.e. we stipulate that $y < x$ if and only if xRy. The pair $(X, <)$ is then dubbed *Kripke frame*.

[3] Needless to say, this requires some countable choice.

We adopt the variant $\mathbf{G3K}_<$ (see Table 2.1) of the calculus **G3K** introduced in [125] for the *basic modal logic* **K** with the additional initial sequents

$$y < x, \Gamma \to \Delta, y < x \qquad (\sigma_<)$$

$$y = x, \Gamma \to \Delta, y = x \qquad (\sigma_=)$$

and the rules for equality (see Table 2.1). With $\neg A$ defined as $A \supset \bot$, the rules $L\neg, R\neg$ are special cases of $L \supset, R \supset$, and we do not give them explicitly.

The basic idea of the calculus is the syntactical internalisation of Kripke semantics: the calculus operates on labelled formulae $x\colon A$, to be read as "x forces A", and on relational formulae $y < x$. For each connective and for the modality $\Box$ the rules are obtained directly from the inductive forcing clauses for compound formulae.

As is common, we denote by $\mathbf{G3K}^*_<$ the extension of $\mathbf{G3K}_<$ with additional rules corresponding to frame properties $*$. The situation is as as laid out in Table 2.2, in which we use the common abbreviation $\forall y < x\, A$ for $\forall y(y < x \Rightarrow A)$.

Theorem 2.2.1. *The calculus* $\mathbf{G3K}_<$ *satisfies the following structural properties:*

(i) Sequents of the forms

$$x\colon A, \Gamma \to \Delta, x\colon A$$
$$x\colon A \supset B, x\colon A, \Gamma \to \Delta, x\colon B$$
$$\to x\colon \Box(A \supset B) \supset (\Box A \supset \Box B)$$

are derivable in $\mathbf{G3K}^*_<$ *for arbitrary modal formulae A and B.*

(ii) The rule of substitution

$$\frac{\Gamma \to \Delta}{\Gamma[y/x] \to \Delta[y/x]}\, Subs$$

is height-preserving admissible in $\mathbf{G3K}^*_<$.

Initial sequents

$x\colon P, \Gamma \to \Delta, x\colon P$

$x\colon \Box A, \Gamma \to \Delta, x\colon \Box A$

$y < x, \Gamma \to \Delta, y < x$

$x = y, \Gamma \to \Delta, x = y$

Propositional rules

$$\frac{x\colon A, x\colon B, \Gamma \to \Delta}{x\colon A \wedge B, \Gamma \to \Delta}\,\mathrm{L}\wedge \qquad \frac{\Gamma \to \Delta, x\colon A \qquad \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \wedge B}\,\mathrm{R}\wedge$$

$$\frac{x\colon A, \Gamma \to \Delta \qquad x\colon B, \Gamma \to \Delta}{x\colon A \vee B, \Gamma \to \Delta}\,\mathrm{L}\vee \qquad \frac{\Gamma \to \Delta, x\colon A, x\colon B}{\Gamma \to \Delta, x\colon A \vee B}\,\mathrm{R}\vee$$

$$\frac{\Gamma \to \Delta, x\colon A \qquad x\colon B, \Gamma \to \Delta}{x\colon A \supset B, \Gamma \to \Delta}\,\mathrm{L}\supset \qquad \frac{x\colon A, \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,\mathrm{R}\supset$$

$$\frac{}{x\colon \bot, \Gamma \to \Delta}\,\mathrm{L}\bot$$

Modal rules

$$\frac{y\colon A, x\colon \Box A, y < x, \Gamma \to \Delta}{x\colon \Box A, y < x, \Gamma \to \Delta}\,\mathrm{L}\Box \qquad \frac{y < x, \Gamma \to \Delta, y\colon A}{\Gamma \to \Delta, x\colon \Box A}\,\mathrm{R}\Box \quad (y \text{ fresh})$$

Rules for equality

$$\frac{x = x, \Gamma \to \Delta}{\Gamma \to \Delta}\,\mathrm{Ref}_{=} \qquad \frac{x = z, x = y, y = z, \Gamma \to \Delta}{x = y, y = z, \Gamma \to \Delta}\,\mathrm{Trans}_{=}$$

$$\frac{y < z, x = y, x < z, \Gamma \to \Delta}{x = y, x < z, \Gamma \to \Delta}\,\mathrm{Repl}_{<_1} \qquad \frac{x < y, z = y, x < z, \Gamma \to \Delta}{z = y, x < z, \Gamma \to \Delta}\,\mathrm{Repl}_{<_2}$$

$$\frac{y\colon P, x = y, x\colon P, \Gamma \to \Delta}{x = y, x\colon P, \Gamma \to \Delta}\,\mathrm{Repl}_{At}$$

Table 2.1: The sequent calculus $\mathbf{G3K}_{<}$.

Frame property	**Rule**
Reflexivity $\forall x(x < x)$	$\dfrac{x < x, \Gamma \to \Delta}{\Gamma \to \Delta}\text{Ref}$
Irreflexivity $\forall x(x \not< x)$	$\dfrac{}{x < x, \Gamma \to \Delta}\text{Irref}$
Transitivity $\forall x \forall y < x \forall z < y(z < x)$	$\dfrac{x < z, x < y, y < z, \Gamma \to \Delta}{x < y, y < z, \Gamma \to \Delta}\text{Trans}$

Table 2.2: Additional rules for $\mathbf{G3K}^*_<$ and the corresponding frame properties.

(iii) The rules of weakening

$$\frac{\Gamma \to \Delta}{x\colon A, \Gamma \to \Delta}\,LW \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, x\colon A}\,RW$$

are height-preserving admissible in $\mathbf{G3K}^*_<$.

(iv) The rules of contraction

$$\frac{x\colon A, x\colon A, \Gamma \to \Delta}{x\colon A, \Gamma \to \Delta}\,LC \qquad \frac{\Gamma \to \Delta, x\colon A, x\colon A}{\Gamma \to \Delta, x\colon A}\,RC$$

$$\frac{y < x, y < x, \Gamma \to \Delta}{y < x, \Gamma \to \Delta}\,LC_< \qquad \frac{\Gamma \to \Delta, y < x, y < x}{\Gamma \to \Delta, y < x}\,RC_<$$

are height-preserving admissible in $\mathbf{G3K}^*_<$.

(v) The rule of replacement

$$\frac{y\colon A, x = y, x\colon A, \Gamma \to \Delta}{x = y, x\colon A, \Gamma \to \Delta}\,Repl$$

is height-preserving admissible in $\mathbf{G3K}^*_<$.

(vi) The rule of necessitation

$$\frac{\rightarrow x\colon A}{\rightarrow x\colon \Box A}\,N$$

is admissible in $\mathbf{G3K}^*_<$.

(vii) All the rules of the system $\mathbf{G3K}^*_<$ *are height-preserving invertible.*

(viii) The cut rule

$$\frac{\Gamma \rightarrow \Delta, x\colon A \qquad x\colon A, \Gamma' \rightarrow \Delta'}{\Gamma, \Gamma' \rightarrow \Delta, \Delta'}\,Cut$$

is admissible in $\mathbf{G3K}^*_<$.

For a proof see Section 11.4 of [133].

Since we add the initial sequents $\sigma_<, \sigma_=$, we also need the following:

Lemma 2.2.2. *Rules*

$$\frac{\Gamma \rightarrow \Delta, y < x \qquad y < x, \Gamma' \rightarrow \Delta'}{\Gamma, \Gamma' \rightarrow \Delta, \Delta'}\,Cut_< \qquad \frac{\Gamma \rightarrow \Delta, y = x \qquad y = x, \Gamma' \rightarrow \Delta'}{\Gamma, \Gamma' \rightarrow \Delta, \Delta'}\,Cut_=$$

are admissible in $\mathbf{G3K}^*_<$.

Proof. The proof is induction as the proof of admissibility of Cut (see [133], Theorem 11.9), from which we exclude the cases in which the cut formula is principal as no rule has instances of $=, <$ as principal formulae. All the remaining cases are completely analogous to their counterparts in the proof of admissibility of Cut. ■

Two important results, to which we will collectively refer as *completeness*, carry over from [127]:

Theorem 2.2.3. *Let* $\Gamma \rightarrow \Delta$ *be a sequent in the language of* $\mathbf{G3K}^*_<$ *. Then either the sequent is derivable in* $\mathbf{G3K}^*_<$ *or it has a Kripke countermodel with properties* $*$.

Corollary 2.2.4. *If a sequent* $\Gamma \rightarrow \Delta$ *is valid in every Kripke model with the frame properties* $*$, *then it is derivable in the system* $\mathbf{G3K}^*_<$.

2.2.1 Connective-like rules for propositional variables

In some of the applications below, we will need to add a propositional variable P to the language of **K** that will have a "connective-like" behavior. For instance, suppose that we want a variable P to behave at x as $Q(x) \Rightarrow R(x)$. In order to avoid self-referential definitions, we ask Q and R not to contain P. We then add the following clause to the definition of *val*:

$$x \Vdash P \text{ if and only if } Q(x) \Rightarrow R(x)$$

Doing so, we further add to $\mathbf{G3K}^{*}_{<}$ a pair of rules that mirror the logical rules:

$$\frac{\Gamma \rightarrow \Delta, Q(x) \qquad R(x), \Gamma \rightarrow \Delta}{x\colon P, \Gamma \rightarrow \Delta}\,\mathrm{L}P \qquad \frac{Q(x), \Gamma \rightarrow \Delta, R(x)}{\Gamma \rightarrow \Delta, x\colon P}\,\mathrm{R}P$$

In order to get height-preserving invertibility, we require that the bottom sequent is not an initial sequent with $x\colon P$ principal; in other words, Δ does not contain $x\colon P$ in LP, and Γ does not contain $x\colon P$ in RP.

Since they have the same behaviour as the logical connectives, all proofs given or referred to in the last section can easily be generalised to extensions of $\mathbf{G3K}_{<}$ by rules of this kind.

We just need to point out that in the proof of admissibility of Cut, we need to do induction on a different notion of the *size* of the cut formula:

$$\begin{aligned}
size(x = y) &\equiv size(x < y) \equiv size(x\colon A) \equiv size(x\colon \bot) \equiv 0 && A \neq P \text{ atomic} \\
size(x\colon A \circ B) &\equiv \sup(size(x\colon A), size(x\colon B)) + 1 && \circ \in \{\supset, \wedge, \vee\} \\
size(x\colon \Box A) &\equiv size(x\colon A) + 1 \\
size(x\colon P) &\equiv \sup(size(Q(x)), size(R(x))) + 1
\end{aligned}$$

We then need to be careful when considering the case in which the cut formula is principal in both premisses. For instance when we transform

$$\dfrac{\dfrac{Q(x),\Gamma \rightarrow \Delta, R(x)}{\Gamma \rightarrow \Delta, x\colon P}\,\mathrm{R}P \qquad \dfrac{\Gamma' \rightarrow \Delta', Q(x) \quad R(x),\Gamma' \rightarrow \Delta'}{x\colon P, \Gamma' \rightarrow \Delta'}\,\mathrm{L}P}{\Gamma,\Gamma' \rightarrow \Delta,\Delta'}\,\mathrm{Cut}$$

into

$$\dfrac{\dfrac{\Gamma' \rightarrow \Delta', Q(x) \qquad \dfrac{Q(x),\Gamma \rightarrow \Delta, R(x) \quad R(x),\Gamma' \rightarrow \Delta'}{Q(x),\Gamma,\Gamma' \rightarrow \Delta,\Delta'}\,\mathrm{Cut}_{(<,=)}}{\Gamma,\Gamma',\Gamma' \rightarrow \Delta,\Delta',\Delta'}\,\mathrm{Cut}_{(<,=)}}{\Gamma,\Gamma' \rightarrow \Delta,\Delta'}\,\text{LC,RC (multiple times)}$$

we have to take into consideration that $Q(x), R(x)$ may be instances of $<, =$.

2.3 Induction principles

Induction principles are typically not expressible within a first-order language. We now present them as ordinary rules of labelled sequent calculus. To start with, we recall *Noetherian Induction* and define *Gödel–Löb Induction*:

$$\forall y(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y\, Ey \qquad \text{(N-Ind)}$$

$$\forall x(\forall y < x(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y < x\, Ey) \qquad \text{(GL-Ind)}$$

They prompt us to consider two rules and an axiom on top of $\mathbf{G3K}_<$:[4]

$$\dfrac{y\colon \Box A, \Gamma \rightarrow \Delta, y\colon A}{\Gamma \rightarrow \Delta, y\colon A}\,\mathrm{NI} \qquad \dfrac{y < x, y\colon \Box A, \Gamma \rightarrow \Delta, y\colon A}{\Gamma \rightarrow \Delta, x\colon \Box A}\,\mathrm{R}\Box\text{-GLI}$$

$$\Box(\Box A \supset A) \supset \Box A \qquad (W)$$

Both rules come with the variable condition that y does not appear in Γ, Δ.

[4]Rule *R□-GLI* is called *R□-L* in [133].

Lemma 2.3.1. *Let a Kripke frame $(X,<)$ be given. The following are equivalent:*

(a) Axiom W is valid in X for every formula A.

(b) Axiom W is valid in X for every propositional variable A.

(c) Gödel–Löb Induction holds in X, i.e.

$$\forall x(\forall y < x(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y < x\, Ey) \tag{GL-Ind}$$

for any given predicate $E(x)$ on X.

Proof. (a)⇒(b). Trivial.

(b)⇒(c). Given $E(x)$, pick a propositional variable A and take a valuation such that $x \Vdash A$ if and only if $E(x)$. Then by expanding the definitions we have the following:

$$\begin{aligned}
& x \Vdash \Box(\Box A \supset A) \supset \Box A \\
\Longrightarrow\ & x \Vdash \Box(\Box A \supset A) \Rightarrow x \Vdash \Box A \\
\Longrightarrow\ & \forall y < x\, y \Vdash \Box A \supset A \Rightarrow \forall y < x\, y \Vdash A \\
\Longrightarrow\ & \forall y < x\,(y \Vdash \Box A \Rightarrow y \Vdash A) \Rightarrow \forall y < x\, y \Vdash A \\
\Longrightarrow\ & \forall y < x\,(\forall z < y\, z \Vdash A \Rightarrow y \Vdash A) \Rightarrow \forall y < x\, y \Vdash A \\
\Longrightarrow\ & \forall y < x\,(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y < x\, Ey
\end{aligned}$$

(c)⇒(a). Given a formula A, define $E(x)$ as $x \Vdash A$. Then:

$$\begin{aligned}
& \forall y < x\,(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y < x\, Ey \\
\Longrightarrow\ & \forall y < x\,(\forall z < y\, z \Vdash A \Rightarrow y \Vdash A) \Rightarrow \forall y < x\, y \Vdash A \\
\Longrightarrow\ & \forall y < x\,(y \Vdash \Box A \Rightarrow y \Vdash A) \Rightarrow \forall y < x\, y \Vdash A \\
\Longrightarrow\ & \forall y < x\, y \Vdash \Box A \supset A \Rightarrow \forall y < x\, y \Vdash A \\
\Longrightarrow\ & x \Vdash \Box(\Box A \supset A) \Rightarrow x \Vdash \Box A \\
\Longrightarrow\ & x \Vdash \Box(\Box A \supset A) \supset \Box A
\end{aligned}$$

■

Lemma 2.3.2. *The following are equivalent over* $\mathbf{G3K}_<$ *without* $R\Box$ *(including the structural rules):*

(i) *Rule R□-GLI,*

(ii) *Rule R□ plus axiom W.*

Proof. Claim 1: R□-GLI⇒R□.

$$\cfrac{\cfrac{y < x, \Gamma \to \Delta, y : A}{y < x, y : \Box A, \Gamma \to \Delta, y : A}\,\text{LW}}{\Gamma \to \Delta, x : \Box A}\,\text{R}\Box\text{-GLI}$$

Claim 2: R□-GLI⇒*W*.

$$\cfrac{\cfrac{\cfrac{y < x, y : \Box A \supset A, y : \Box A, x : \Box(\Box A \supset A) \to y : A}{y < x, y : \Box A, x : \Box(\Box A \supset A) \to y : A}\,\text{L}\Box}{x : \Box(\Box A \supset A) \to x : \Box A}\,\text{R}\Box\text{-GLI}}{\to x : \Box(\Box A \supset A) \supset \Box A}\,\text{R}\supset$$

Claim 3: R□+*W*⇒R□-GLI.

$$\cfrac{\cfrac{\cfrac{y < x, y : \Box A, \Gamma \to \Delta, y : A}{y < x, \Gamma \to \Delta, y : \Box A \supset A}\,\text{R}\supset}{\Gamma \to \Delta, x : \Box(\Box A \supset A)}\,\text{R}\Box \qquad x : \Box(\Box A \supset A) \to x : \Box A}{\Gamma \to \Delta, x : \Box A}\,\text{Cut}$$

where $x : \Box(\Box A \supset A) \to x : \Box A$ is derivable from Axiom *W* by invertibility of *R*⊃. ■

Therefore the sequent calculus $\mathbf{G3KGL}_<$ obtained by replacing R□ by R□-GLI is an extension of $\mathbf{G3K}_<$. If we further add the mathematical rules Trans and Irref and remove the initial sequents $\sigma_<, \sigma_=$, we get the calculus **G3KGL** [125].

Lemma 2.3.3. *Let a Kripke frame* $(X, <)$ *be given. The following are equivalent:*

(a) *Rule*

$$\cfrac{y : \Box A, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, y : A}\,NI$$

where y does not occur in Γ, Δ, is sound in X. That is, whenever the top sequent is valid in X, then also the bottom sequent is valid in X.

(b) For every propositional variable A, in X we have

$$\forall y\,(y \Vdash \Box A \Rightarrow y \Vdash A) \Rightarrow \forall y\, y \Vdash A$$

for any given valuation $\Vdash$ on X.

(c) Noetherian Induction holds in X, i.e.

$$\forall y(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y\, Ey \tag{N-Ind}$$

for any given predicate $E(x)$ on X.

Proof. (a)⇒(b). Suppose that, for all y, $y \Vdash \Box A$ implies $y \Vdash A$. It follows that the sequent $y\colon \Box A \to y\colon A$ is valid, hence, by hypothesis, also $\to y\colon A$ is valid, which means that for all y, $y \Vdash A$.

(b)⇒(c). Given $E(x)$, pick a propositional variable A and take a valuation such that $x \Vdash A$ if and only if $E(x)$. Then:

$$\begin{aligned} &\forall y\,(y \Vdash \Box A \Rightarrow y \Vdash A) \Rightarrow \forall y\, y \Vdash A \\ \Longrightarrow &\forall y\,(\forall z < y\, z \Vdash A \Rightarrow y \Vdash A) \Rightarrow \forall y\, y \Vdash A \\ \Longrightarrow &\forall y\,(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y\, Ey \end{aligned}$$

(c)⇒(a). Given a formula A, define $E(x)$ as $x \Vdash A$. Suppose that the sequent $y\colon \Box A, \Gamma \to \Delta, y\colon A$, where y does not occur in Γ, Δ, is valid. By the variable condition, we can leave out the contexts and get $\forall y\,(\forall z < y\, z \Vdash A \Rightarrow y \Vdash A)$. By N-Ind on $E(x) \equiv x \Vdash A$, this yields $\forall y\, y \Vdash A$ By bringing the contexts back in, we conclude that $\Gamma \to \Delta, y\colon A$ is valid. ∎

The lemmata proved in this section allow us to transform rather algebraic proofs using induction into tree-like derivations in modal logic, following a certain pattern:

Proof transformation pattern Let X be a set endowed with a binary relation $<$. Suppose that we need to show either

(i) a statement of the form $\forall y\, E(y)$ by way of N-Ind, or

(ii) a statement of the form $\forall x \forall y < x\, E(y)$ by way of GL-Ind.

We consider $(X, <)$ as a Kripke frame, and build a Kripke model as follows. First, we consider a suitable subformula $U(x)$ of $E(x)$ such that it can be encoded in a sequent $Q(x) \to R(x)$, and fix a propositional variable P. We define a valuation such that $val\colon (x, P) \equiv 1$ if and only if $U(x)$. This is done by adding (variants of) the following rules to the calculus:

$$\frac{\Gamma \to \Delta, Q(x) \qquad R(x), \Gamma \to \Delta}{x\colon P, \Gamma \to \Delta}\,\mathrm{L}P \qquad \frac{Q(x), \Gamma \to \Delta, R(x)}{\Gamma \to \Delta, x\colon P}\,\mathrm{R}P$$

By means of P, we find a formula A such that $x \Vdash A$ if and only if $E(x)$. We then proceed as follows:

(i) For N-Ind: Derive the sequent $y\colon \Box A \to y\colon A$ by using $\mathbf{G3K}_<$ plus RP and LP, then apply rule NI:

$$\frac{\begin{array}{c}\vdots\\ y\colon \Box A \to y\colon A\end{array}}{\to y\colon A}\,\mathrm{NI}$$

(ii) For GL-Ind: Derive the sequent $y < x, y\colon \Box A \to y\colon A$ by using $\mathbf{G3K}_<$ plus RP and LP, then apply rule R$\Box$-GLI:

$$\frac{\begin{array}{c}\vdots\\ y < x, y\colon \Box A \to y\colon A\end{array}}{\Gamma \to \Delta, x\colon \Box A}\,\mathrm{R}\Box\text{-GLI}$$

We point out that this pattern is not fully general, as we do not yet have a universal method to find the subformula $U(x)$ needed to define the valuation.

2.3.1 Examples

Example 2.3.4. GL-Ind implies that $\forall y < x(y \neq x)$.[5]

Proof (algebraic). In order to apply GL-Ind, we need to show that $\forall y < x(\forall z < y(z \neq x) \Rightarrow y \neq x)$. Fix $y < x$ such that $\forall z < y(z \neq x)$. We need to show that $y \neq x$. Suppose $y = x$. Then $x < x$ and $\forall z < x(z \neq x)$, from which we derive $x \neq x$. Therefore $y \neq x$ and we proved our claim. ■

Proof (modal). Fix x. Pick P such that $y \Vdash P$ if and only if $y = x$. This corresponds to the rules

$$\frac{y = x, \Gamma \to \Delta}{y\colon P, \Gamma \to \Delta}\,\mathrm{L}P \qquad \frac{\Gamma \to \Delta, y = x}{\Gamma \to \Delta, y\colon P}\,\mathrm{R}P$$

Then our thesis is equivalent to say that $\to x\colon \Box\neg P$ is derivable in $\mathbf{G3K}_<$ plus R□-GLI, LP and RP:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{y = x, y < y, y\colon \Box\neg P \to y\colon \bot, y = x}{y = x, y < y, y\colon \Box\neg P \to y\colon \bot, y\colon P}\,\mathrm{R}P}{y\colon \neg P, y = x, y < y, y\colon \Box\neg P \to y\colon \bot}\,\mathrm{L}\neg}{y = x, y < y, y\colon \Box\neg P \to y\colon \bot}\,\mathrm{L}\Box}{y = x, y < x, y\colon \Box\neg P \to y\colon \bot}\,\mathrm{Repl}}{y < x, y\colon \Box\neg P, y\colon P \to y\colon \bot}\,\mathrm{L}P}{y < x, y\colon \Box\neg P \to y\colon \neg P}\,\mathrm{R}\supset}{\to x\colon \Box\neg P}\,\mathrm{R}\Box\text{-GLI}$$

■

Example 2.3.5. What follows is a somewhat more general formulation of the fact that by Noetherian induction every meet-closed predicate on a poset propagates from the irreducible elements to any element whatsoever [153, 166, 167].

[5] If we observe that $\forall y < x(y \neq x)$ is just a variant of irreflexivity $\forall x(x \not< x)$, then this result will be for free once we have proved Lemma 2.4.1 and Theorem 2.4.3.

Consider a ternary predicate $x = y \circ z$. We say that x is $\circ$-*reducible* (for short $R^\circ(x)$) if there are $y < x$ and $z < x$ such that $x = y \circ z$.

Let $E(x)$ be a predicate satisfying

$$\frac{x = y \circ z \quad E(y) \quad E(z)}{E(x)} \tag{$*$}$$

for every y, z. Then N-Ind implies $\forall x(R^\circ(x) \vee E(x)) \Rightarrow \forall x E(x)$.

Proof (algebraic). Assume that $\forall x(R^\circ(x) \vee E(x))$. In order to apply induction, we need to show that $\forall x(\forall y < x E(y) \Rightarrow E(x))$. Fix x such that $\forall y < x E(y)$. It now suffices to show $E(x)$. By assumption, we can distinguish two cases:

— Case $E(x)$: Trivial.

— Case $R^\circ(x)$: Take $y < x$ and $z < x$ such that $x = y \circ z$. By $\forall y < x E(y)$ we know that $E(y)$ and $E(z)$. From this we deduce $E(x)$ by $(*)$. ■

Proof (modal). Pick a propositional variable P such that $x \Vdash P$ if and only if $E(x)$. The hypothesis $(*)$ can be written as:

$$\frac{x\colon P, y\colon P, z\colon P, x = y \circ z, \Gamma \to \Delta}{y\colon P, z\colon P, x = y \circ z, \Gamma \to \Delta}\,(*)$$

The definition of being $\circ$-reducible can be used in the calculus via the rule

$$\frac{x = y \circ z, y < x, z < x, \Gamma \to \Delta}{R^\circ(x), \Gamma \to \Delta}\,\mathrm{L}R^\circ$$

where y, z are fresh, together with the appropriate $\mathrm{R}R^\circ$ rule. The thesis becomes that from the sequent $\to R^\circ(x), x\colon P$ we can derive

$\rightarrow x\colon P$ in $\mathbf{G3K}_<$ using NI, $(*)$, $\mathrm{L}R^\circ$ and $\mathrm{R}R^\circ$. In fact:

$$
\dfrac{\rightarrow R^\circ(x), x\colon P \qquad \dfrac{\dfrac{\dfrac{\dfrac{x = y \circ z, y < x, z < x, x\colon P, z\colon P, y\colon P, x\colon \Box P \rightarrow x\colon P}{x = y \circ z, y < x, z < x, z\colon P, y\colon P, x\colon \Box P \rightarrow x\colon P}\,(*)}{x = y \circ z, y < x, z < x, y\colon P, x\colon \Box P \rightarrow x\colon P}\,\mathrm{L}\Box}{x = y \circ z, y < x, z < x, x\colon \Box P \rightarrow x\colon P}\,\mathrm{L}\Box}{R^\circ(x), x\colon \Box P \rightarrow x\colon P}\,\mathrm{L}R^\circ}{\dfrac{x\colon \Box P \rightarrow x\colon P}{\rightarrow x\colon P}\,\mathrm{NI}}\,\mathrm{Cut}
$$

■

2.4 Consequences

In this section we apply the tools that we have just developed, in order to revisit certain common properties of the accessibility relation $<$. In particular, this will lead us to useful characterisations of the induction principles that can simplify the task of controlling that they hold in a given structure. We will further shed some more light on the role of transitivity in the calculus.

2.4.1 Irreflexivity & Noetherianity

The binary relation $<$ on X is said to be *irreflexive* if $\forall x(x \not< x)$, which corresponds to the following rule

$$
\dfrac{}{x < x, \Gamma \rightarrow \Delta}\,\mathrm{Irref}
$$

Lemma 2.4.1. *Noetherian Induction implies irreflexivity.*[6]

Proof. To show this claim, we use the syntactical proof pattern introduced in Section 2.3. Pick P such that $x \Vdash P$ if and only if $x < x$, i.e. such that

$$
\dfrac{x < x, \Gamma \rightarrow \Delta}{x\colon P, \Gamma \rightarrow \Delta}\,\mathrm{L}P \qquad\qquad \dfrac{\Gamma \rightarrow \Delta, x < x}{\Gamma \rightarrow \Delta, x\colon P}\,\mathrm{R}P
$$

[6] This lemma is a formal direct version of "every well-founded relation is irreflexive", to be compared with "Set Induction implies $\forall x(x \notin x)$" [1–3, 6, 7] as a direct version of "Foundation implies $\forall x(x \notin x)$" in axiomatic set theory.

Then we just need to show $\rightarrow x\colon \neg P$ in **G3K** plus NI, LP and RP:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x\colon \Box\neg P, x<x \rightarrow x<x}{x<x, x\colon \Box\neg P \rightarrow x\colon P}\,\mathrm{R}P}{x\colon \neg P, x<x, x\colon \Box\neg P \rightarrow}\,\mathrm{L}\neg}{x<x, x\colon \Box\neg P \rightarrow}\,\mathrm{L}\Box}{x\colon P, x\colon \Box\neg P \rightarrow}\,\mathrm{L}P}{x\colon \Box\neg P \rightarrow x\colon \neg P}\,\mathrm{R}\neg}{\rightarrow x\colon \neg P}\,\mathrm{NI}
$$

From this we also get admissibility of the rule version of irreflexivity:

$$
\dfrac{\rightarrow x\colon \neg P \qquad \dfrac{\dfrac{x<x, \Gamma \rightarrow \Delta, x<x}{x<x, \Gamma \rightarrow \Delta, x\colon \neg P}\,\mathrm{R}P}{x\colon \neg P, x<x, \Gamma \rightarrow \Delta}\,\mathrm{L}\neg}{x<x, \Gamma \rightarrow \Delta}\,\mathrm{Cut}
$$

■

As in mathematical practice one often talks about ascending chains, we now occasionally switch back to R. So let $y < x$ if and only if xRy: that is, $<$ and R are converse to each other. Notice that $<$ is irreflexive if and only if so is R.

An *infinite R-sequence* is a sequence $(x_i)_{i\in\mathbb{N}}$ of elements of X such that $x_i R x_{i+1}$ for all $i \in \mathbb{N}$. An infinite R-sequence $(x_i)_{i\in\mathbb{N}}$ is *convergent* if there is $i \in \mathbb{N}$ such that $x_j = x_i$ for all $j > i$. We say that R is *well-founded* if there is no infinite R-sequence; and that R is *Noetherian*—for short, R satisfies Noeth—if every infinite R-sequence converges.

While the first and second item of the next lemma are well-known to be equivalent, the occurrence of irreflexivity in the third item is due to the fact that a priori R and $<$ need not possess this feature of an order relation.

Proposition 2.4.2. *The following are equivalent:*

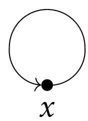
(a) $<$ satisfies Noetherian Induction.

(b) *R is well-founded.*

(c) *R is irreflexive and Noetherian.*

Proof. The equivalence of the first and the second item is folklore. See Lemma 2.4.1 for a formal proof that Noetherian Induction implies irreflexivity. If R is well-founded, i.e. there are no infinite R-sequences at all, then R is trivially Noetherian. As for the converse, if R is irreflexive, then no infinite R-sequence converges; whence if, in addition, R is Noetherian, then R is well-founded. ■

Notice in this context that if R is Noetherian, it is not always the case that $<$ satisfies N-Ind. In fact, the relation R with the following graph



does not satisfy N-Ind because it is not irreflexive, but R is Noetherian because the only infinite R-sequence, which is $xRxRxR...$, converges.

2.4.2 Transitivity & Induction

The binary relation $<$ on X is said to be *transitive* if $\forall x \forall y < x \forall z < y(z < x)$, which corresponds to the following rule

$$\frac{z < x, z < y, y < x, \Gamma \to \Delta}{z < y, y < x, \Gamma \to \Delta} \text{Trans}$$

In the light of Proposition 2.4.2, what we prove next in $\mathbf{G3K}_{<}$ is a formal version of Segerberg's theorem [173] that the Gödel–Löb axiom describes exactly the (converse) well-founded transitive Kripke frames.

Theorem 2.4.3. *The following are equivalent:*

(i) Gödel–Löb Induction,

(ii) Noetherian Induction + Transitivity.

Proof. Claim 1: GL-Ind⇒N-Ind. It suffices to show that NI is admissible in $\mathbf{G3KGL}_<$:

$$\dfrac{\dfrac{\dfrac{\dfrac{x\colon \Box A,\Gamma \to \Delta, x\colon A}{y\colon \Box A,\Gamma \to \Delta, y\colon A}\,\text{Subs}}{y<x, y\colon \Box A,\Gamma \to \Delta, y\colon A}\,\text{LW}}{\Gamma \to \Delta, x\colon \Box A}\,\text{R}\Box\text{-GLI} \qquad x\colon \Box A,\Gamma \to \Delta, x\colon A}{\dfrac{\Gamma,\Gamma \to \Delta,\Delta, x\colon A}{\Gamma \to \Delta, x\colon A}\,\text{LC,RC (multiple times)}}\,\text{Cut}$$

Claim 2: GL-Ind⇒Trans. To show this claim, we use the syntactical proof pattern introduced in Section 2.3. Fix x. Pick P such that $y \Vdash P$ if and only if $y < x$, i.e. such that

$$\dfrac{y<x,\Gamma \to \Delta}{y\colon P,\Gamma \to \Delta}\,\text{L}P \qquad\qquad \dfrac{\Gamma \to \Delta, y<x}{\Gamma \to \Delta, y\colon P}\,\text{R}P$$

It suffices to show that rule Trans is admissible in $\mathbf{G3KGL}_<$ plus LP and RP:

$$\dfrac{\to x\colon \Box(\Box P \wedge P) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z<x, z<y, y<x,\Gamma \to \Delta}{z<x, y\colon \Box P, y\colon P, x\colon \Box(\Box P \wedge P), z<y, y<x,\Gamma \to \Delta}\,\text{LW}}{z\colon P, y\colon \Box P, y\colon P, x\colon \Box(\Box P \wedge P), z<y, y<x,\Gamma \to \Delta}\,\text{L}P}{y\colon \Box P, y\colon P, x\colon \Box(\Box P \wedge P), z<y, y<x,\Gamma \to \Delta}\,\text{L}\Box}{y\colon \Box P \wedge P, x\colon \Box(\Box P \wedge P), z<y, y<x,\Gamma \to \Delta}\,\text{L}\wedge}{x\colon \Box(\Box P \wedge P), z<y, y<x,\Gamma \to \Delta}\,\text{L}\Box}{z<y, y<x,\Gamma \to \Delta}\,\text{Cut}$$

where $\to x\colon \Box(\Box P \land P)$ is derived as follows:[7]

$$\dfrac{\dfrac{y < x, y\colon \Box(\Box P \land P) \to y\colon \Box P \qquad \dfrac{y < x, y\colon \Box(\Box P \land P) \to y < x}{y < x, y\colon \Box(\Box P \land P) \to y\colon P}\,\mathrm{R}P}{y < x, y\colon \Box(\Box P \land P) \to y\colon \Box P \land P}\,\mathrm{R}\land}{\to x\colon \Box(\Box P \land P)}\,\mathrm{R}\Box\text{-GLI}$$

where $y < x, y\colon \Box(\Box P \land P) \to y\colon \Box P$ is derived as follows:

$$\dfrac{\dfrac{\dfrac{z\colon \Box P, z\colon P, z < y, z\colon \Box P, y < x, y\colon \Box(\Box P \land P) \to z\colon P}{z\colon \Box P \land P, z < y, z\colon \Box P, y < x, y\colon \Box(\Box P \land P) \to z\colon P}\,\mathrm{L}\land}{z < y, z\colon \Box P, y < x, y\colon \Box(\Box P \land P) \to z\colon P}\,\mathrm{L}\Box}{y < x, y\colon \Box(\Box P \land P) \to y\colon \Box P}\,\mathrm{R}\Box\text{-GLI}$$

Claim 3: N-Ind+Trans⇒GL-Ind. It suffices to show that Axiom W is derivable in $\mathbf{G3K}_{<}$ plus NI and Trans:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y\colon A, y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \to y\colon A \qquad \mathscr{D}_1}{y\colon \Box A \supset A, y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \to y\colon A}\,\mathrm{L}\supset}{y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \to y\colon A}\,\mathrm{L}\Box}{x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \to x\colon \Box A}\,\mathrm{R}\Box}{x\colon \Box(\Box(\Box A \supset A) \supset \Box A) \to x\colon \Box(\Box A \supset A) \supset \Box A}\,\mathrm{R}\supset$$

$$\dfrac{x\colon \Box(\Box(\Box A \supset A) \supset \Box A) \to x\colon \Box(\Box A \supset A) \supset \Box A}{\to x\colon \Box(\Box A \supset A) \supset \Box A}\,\mathrm{NI}$$

where $\mathscr{D}_1$ is the following derivation:

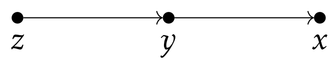
$$\dfrac{\dfrac{y\colon \Box A, y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \to y\colon A, y\colon \Box A \qquad \mathscr{D}_2}{y\colon \Box(\Box A \supset A) \supset \Box A, y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \to y\colon A, y\colon \Box A}\,\mathrm{L}\supset}{y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \to y\colon A, y\colon \Box A}\,\mathrm{L}\Box$$

[7] Notice that the sequent $\to x\colon \Box(\Box P \land P)$ corresponds to $\forall x \forall y < x(\forall z < y(z < x) \,\&\, y < x)$, which is a redundant version of transitivity as $y < x$ is repeated both in the premisses and in the conclusions. The reason why we need this version and not the "standard" one (as, for instance, in the case of Irref in Lemma 2.4.1), will become clear in the next subsection.

where $\mathscr{D}_2$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{z\colon \Box A \supset A, z < x, z < y, y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \rightarrow y\colon A, y\colon \Box A, z\colon \Box A \supset A}{z < x, z < y, y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \rightarrow y\colon A, y\colon \Box A, z\colon \Box A \supset A}\,\mathrm{L}\Box}{z < y, y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \rightarrow y\colon A, y\colon \Box A, z\colon \Box A \supset A}\,\mathrm{Trans}}{y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \rightarrow y\colon A, y\colon \Box A, y\colon \Box(\Box A \supset A)}\,\mathrm{L}\supset}{y < x, x\colon \Box(\Box(\Box A \supset A) \supset \Box A), x\colon \Box(\Box A \supset A) \rightarrow y\colon A, y\colon \Box A, y\colon \Box(\Box A \supset A)}\,\mathrm{R}\Box \qquad \blacksquare$$

Proposition 2.4.2 and Theorem 2.4.2 help to see that N-Ind$\not\Rightarrow$GL-Ind. In fact, the structure



satisfies both Noeth and Irref, but not Trans.

2.4.3 Transitivity & Cut

The rule Cut is known to be admissible in the calculus **G3GL** and thus, by equivalence, in **G3KGL** [133, Theorem 12.20]. As a consequence, Cut is also admissible in **G3KGL**$_<$ if we add Trans and Irref. Are these two rules really needed for Cut admissibility?

Lemma 2.4.4. *The following sequents are Cut-free derivable in* **G3KGL**$_<$*:*

(i) $x\colon \Box A \rightarrow x\colon \Box(A \wedge \Box A)$,[8]

(ii) $x\colon \Box(A \wedge \Box A) \rightarrow x\colon \Box\Box A$.

Proof. (i)

$$\dfrac{\dfrac{\dfrac{y\colon A, y < x, y\colon \Box(A \wedge \Box A), x\colon \Box A \rightarrow y\colon A}{y < x, y\colon \Box(A \wedge \Box A), x\colon \Box A \rightarrow y\colon A}\,\mathrm{L}\Box \qquad \mathscr{D}}{y < x, y\colon \Box(A \wedge \Box A), x\colon \Box A \rightarrow y\colon A \wedge \Box A}\,\mathrm{R}\wedge}{x\colon \Box A \rightarrow x\colon \Box(A \wedge \Box A)}\,\mathrm{R}\Box\text{-GLI}$$

[8] This is actually the redundant version of transitivity that we had in the proof of Theorem 2.4.3. Here, the definition of $y \Vdash A$ as $y < x$ is gained by the addition of the premiss $x\colon \Box A$.

where $\mathscr{D}$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{z\colon A, z\colon \Box A, z<y, z\colon\Box A, y<x, y\colon \Box(A\wedge\Box A), x\colon\Box A \to z\colon A}{z\colon A\wedge\Box A, z<y, z\colon\Box A, y<x, y\colon\Box(A\wedge\Box A), x\colon\Box A\to z\colon A}\,\mathrm{L}\wedge}{z<y, z\colon\Box A, y<x, y\colon\Box(A\wedge\Box A), x\colon\Box A\to z\colon A}\,\mathrm{L}\Box}{y<x, y\colon\Box(A\wedge\Box A), x\colon\Box A\to y\colon\Box A}\,\mathrm{R}\Box\text{-GLI}$$

(ii)

$$\dfrac{\dfrac{\dfrac{y\colon A, y\colon\Box A, y<x, y\colon\Box\Box A, x\colon\Box(A\wedge\Box A)\to y\colon\Box A}{y\colon A\wedge\Box A, y<x, y\colon\Box\Box A, x\colon\Box(A\wedge\Box A)\to y\colon\Box A}\,\mathrm{L}\wedge}{y<x, y\colon\Box\Box A, x\colon\Box(A\wedge\Box A)\to y\colon\Box A}\,\mathrm{L}\Box}{x\colon\Box(A\wedge\Box A)\to x\colon\Box\Box A}\,\mathrm{R}\Box\text{-GLI}$$

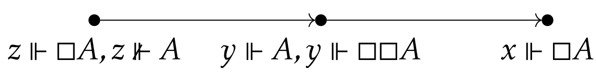
■

Theorem 2.4.5. *The Cut rule is not admissible in* **G3KGL**$_<$ *without Trans.*

Proof. If Cut were admissible, then by Lemma 2.4.4 the sequent $x\colon\Box A\to x\colon\Box\Box A$ would be Cut-free derivable. [9] Let's try to give a Cut-free proof:

$$\dfrac{\dfrac{\dfrac{y\colon A, z<y, z\colon\Box A, y<x, y\colon\Box\Box A, x\colon\Box A\to z\colon A}{y\colon A, z<y, z\colon\Box A, y<x, y\colon\Box\Box A, x\colon\Box A\to z\colon A}\,\mathrm{L}\Box}{y<x, y\colon\Box\Box A, x\colon\Box A\to y\colon\Box A}\,\mathrm{R}\Box\text{-GLI}}{x\colon\Box A\to x\colon\Box\Box A}\,\mathrm{R}\Box\text{-GLI}$$

Observe, however, that the upper-most sequent is not derivable in general. In fact, we have a countermodel:



[9]The sequent $x\colon\Box A\to x\colon\Box\Box A$ corresponds to transitivity the same way the sequent $x\colon\Box A\to x\colon\Box(A\wedge\Box A)$ corresponds to redundant transitivity from footnote 7. What we are showing is actually that the "standard" version of transitivity can be deduced from the redundant version by using Cut and that Cut is necessary in any proof of transitivity. This is why we needed the redundant version in the first place.

Notice that this is a non-transitive model. ■

As a consequence, we get that the assumption of Trans is necessary in the aforementioned proof of Cut-admissibility in $\mathbf{G3KGL}_{<}$. [10]

[10] At first glance, this may look a bit counterintuitive: a mathematical principle, transitivity, corresponds to a derivable sequent, but is also equivalent, modulo irreflexivity, to a structural rule. However, this is not really astonishing: Cut can be viewed as a form of transitivity, as it is a generalisation of the following:

$$\forall C \forall B (B \supset C \Rightarrow \forall A (A \supset B \Rightarrow A \supset C))$$

which is just transitivity of $\supset$ seen as a relation. This is also the reason for which the Cut in literature is sometimes called Trans, e.g. when dealing with Scott-style entailment relations (cf [171]; for recent work see, e.g., [30, 65, 70, 154, 155, 168, 192]).

3 A terminating intuitionistic calculus

3.1 Introduction

In his doctoral thesis [77, 78], Gentzen introduced sequent calculi for classical and intuitionistic logic. In particular, he solved the decision problem for intuitionistic propositional logic (**Int**) with a calculus that he called **LI**. However, Gentzen's original calculus lacked some desirable properties, such as the invertibility of rules which would eliminate the need for backtracking. Ever since then, many other approaches were proposed; we refer to [55] for an extended survey.

The labelled calculus **G3I** by Dyckhoff and Negri [56, 126, 133] reported in table 3.1 solves the problem of backtracking but loses the property of termination, see for instance the example of Peirce's Law in Subsection 3.3.3. In order to solve this problem, Negri [128, 129] showed how to add a loop-checking mechanism to ensure termination. However, it is desirable to avoid loop-checking since its effect on complexity isn't clear.

Corsi [47, 48] presented a calculus for **Int** which fulfils the termination property. The key to get termination is the addition of the following rule:

$$\frac{\Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\ \textit{a fortiori}$$

Initial sequent

$x \leqslant y, x\colon P, \Gamma \to \Delta, y\colon P$

Logical Rules

$$\frac{x\colon A, x\colon B, \Gamma \to \Delta}{x\colon A \wedge B, \Gamma \to \Delta}\,\mathrm{L}\wedge \qquad \frac{\Gamma \to \Delta, x\colon A \quad \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \wedge B}\,\mathrm{R}\wedge$$

$$\frac{x\colon A, \Gamma \to \Delta \quad x\colon B, \Gamma \to \Delta}{x\colon A \vee B, \Gamma \to \Delta}\,\mathrm{L}\vee \qquad \frac{\Gamma \to \Delta, x\colon A, x\colon B}{\Gamma \to \Delta, x\colon A \vee B}\,\mathrm{R}\vee$$

$$\frac{x \leqslant y, x\colon A \supset B, \Gamma \to \Delta, y\colon A \qquad x \leqslant y, x\colon A \supset B, y\colon B, \Gamma \to \Delta}{x \leqslant y, x\colon A \supset B, \Gamma \to \Delta}\,\mathrm{L}\supset$$

$$\frac{}{x\colon \bot, \Gamma \to \Delta}\,\mathrm{L}\bot \qquad \frac{x \leqslant y, y\colon A, \Gamma \to \Delta, y\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,\mathrm{R}\supset$$

Mathematical Rules

$$\frac{x \leqslant x, \Gamma \to \Delta}{\Gamma \to \Delta}\,\mathrm{Ref}_{\leqslant} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \to \Delta}{x \leqslant y, y \leqslant z, \Gamma \to \Delta}\,\mathrm{Trans}_{\leqslant}$$

Table 3.1: The sequent calculus **G3I**. Rule R⊃ has the condition that y is fresh.

This rule is logically equivalent to the formula $B \supset (A \supset B)$, which is the principle of *a fortiori*.

In the present chapter, we consider the labelled calculus **G3I** instead, and show that, a way to reach termination consists in modifying rule R⊃ as follows:

$$\frac{x \leqslant y, y\colon B \supset (A \supset B), y\colon A, \Gamma \to \Delta, y\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,\mathrm{R}\supset_t \quad (y \text{ fresh})$$

Although the idea comes from a similar terminating procedure [57] for the calculus **G3Grz** for the provability logic **Grz** , into which

Initial sequent As in **G3I**.

Logical Rules L$\wedge$, R$\wedge$, L$\vee$, R$\vee$, L$\supset$, L$\bot$ as in **G3I**.

$$\frac{x \leqslant y, y : B \supset (A \supset B), y : A, \Gamma \to \Delta, y : B}{\Gamma \to \Delta, x : A \supset B} \text{R}{\supset_t} \quad (y \text{ fresh})$$

Mathematical Rules As in **G3I**.

Table 3.2: The sequent calculus $\mathbf{G3I_t}$.

there **Int** is embeddable as detailed in Section 3.4, we notice that what we do is actually incorporating *a fortiori* into R$\supset$.

3.2 Structural properties

Consider sequent calculi **G3I** and $\mathbf{G3I_t}$ as presented in Tables 3.1 and 3.2, respectively.

Theorem 3.2.1. **G3I** *and* $\mathbf{G3I_t}$ *are equivalent in the sense that*

$$\mathbf{G3I} \vdash \Gamma \to \Delta \quad \textit{if and only if} \quad \mathbf{G3I_t} \vdash \Gamma \to \Delta$$

Proof. Suppose $\mathbf{G3I} \vdash \Gamma \to \Delta$. We transform it into a proof in $\mathbf{G3I_t}$ by using height-preserving weakening to add whenever needed the extra formula of the form $y : B \supset (A \supset B)$ in the premiss of R$\supset$. So $\mathbf{G3I_t} \vdash \Gamma \to \Delta$.

If $\mathbf{G3I_t} \vdash \Gamma \to \Delta$, consider the steps of R$\supset_t$ with a Cut with the extra (derivable) sequent $\to y : B \supset (A \supset B)$. We turn it into a premiss of R$\supset$. We conclude by admissibility of Cut in **G3I**. ■

Theorem 3.2.2. *All the structural properties hold for* $\mathbf{G3I_t}$. *In particular,*

(i) All sequents of the following form are derivable in $\mathbf{G3I_t}$*:*

a) $x \leqslant y, x{:}\, A, \Gamma \to \Delta, y{:}\, A$,

b) $x{:}\, A, \Gamma \to \Delta, x{:}\, A$.

(ii) *If* $\mathbf{G3I_t} \vdash \Gamma \to \Delta$, *then* $\mathbf{G3I_t} \vdash \Gamma(x/y) \to \Delta(x/y)$ *with the same derivation height.*

(iii) *The rules of weakening,*

$$\frac{\Gamma \to \Delta}{x{:}\, A, \Gamma \to \Delta}\,LW \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, x{:}\, A}\,RW \qquad \frac{\Gamma \to \Delta}{x \leqslant y, \Gamma \to \Delta}\,LW_{\leqslant}$$

are height-preserving admissible in $\mathbf{G3I_t}$.

(iv) *All rules of* $\mathbf{G3I_t}$ *are height-preserving invertible.*

(v) *The rules of contraction,*

$$\frac{x{:}\, A, x{:}\, A, \Gamma \to \Delta}{x{:}\, A, \Gamma \to \Delta}\,LC \qquad \frac{\Gamma \to \Delta, x{:}\, A, x{:}\, A}{\Gamma \to \Delta, x{:}\, A}\,RC$$

$$\frac{x \leqslant y, x \leqslant y, \Gamma \to \Delta}{x \leqslant y, \Gamma \to \Delta}\,LC_{\leqslant}$$

are height-preserving admissible in $\mathbf{G3I_t}$.

(vi) *The rule of cut,*

$$\frac{\Gamma \to \Delta, x{:}\, A \qquad x{:}\, A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}\,Cut$$

is admissible in $\mathbf{G3I_t}$.

Proof. The proofs of (i)–(v) are similar to those of [133, 12.25–12.29].

For the proof of (vi) we also follow [133, 12.30]. First, we observe that in all the cases of permutation of cuts that may give a clash with the variable conditions in the implication rules, and in the rules for $\leqslant$ in the case of geometric extensions, an appropriate substitution (see (ii)) prior to the permutation is used. The proof is thus by induction on the length of the cut formula, with a subinduction on the sum of the heights of the derivations of the premisses of cut. We consider in detail only the case of a cut with the cut formula principal in implication rules in both premisses. If the cut formula is $w\colon A \supset B$, the derivation is

$$\dfrac{\dfrac{x \leqslant y, y\colon B \supset (A \supset B), y\colon A, \Gamma \to \Delta, y\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,\text{R}{\supset}_t \quad \dfrac{x\colon A \supset B, x \leqslant z, \Gamma' \to \Delta', z\colon A \quad x\colon A \supset B, x \leqslant z, z\colon B, \Gamma' \to \Delta'}{x\colon A \supset B, x \leqslant z, x\colon B, \Gamma' \to \Delta'}\,\text{L}{\supset}}{x \leqslant z, \Gamma, \Gamma' \to \Delta, \Delta'}\,\text{Cut}$$

We apply the usual permutation and transform it into:

$$\dfrac{\dfrac{x \leqslant z^2, z\colon B \supset (A \supset B), \Gamma^2, \Gamma' \to \Delta^2, \Delta', z\colon B \qquad x \leqslant z, z\colon B, \Gamma, \Gamma' \to \Delta, \Delta'}{x \leqslant z^3, z\colon B \supset (A \supset B), \Gamma^3, \Gamma'^2 \to \Delta^3, \Delta'^2}\,\text{Cut}}{x \leqslant z, z\colon B \supset (A \supset B), \Gamma, \Gamma' \to \Delta, \Delta'}\,\text{LC,RC (multiple times)}$$

where the right sequent has been derived as follows

$$\dfrac{\dfrac{x \leqslant y, y\colon B \supset (A \supset B), y\colon A, \Gamma \to \Delta, y\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,\text{R}{\supset}_t \qquad x\colon A \supset B, x \leqslant z, z\colon B, \Gamma' \to \Delta'}{x \leqslant z, z\colon B, \Gamma, \Gamma' \to \Delta, \Delta'}\,\text{Cut}$$

and the left sequent has been derived as follows

$$\dfrac{x \leqslant z, \Gamma, \Gamma' \to \Delta, \Delta', z\colon A \qquad \dfrac{x \leqslant y, y\colon B \supset (A \supset B), y\colon A, \Gamma \to \Delta, y\colon B}{x \leqslant z, z\colon B \supset (A \supset B), z\colon A, \Gamma \to \Delta, z\colon B}\,\text{Subs}}{x \leqslant z^2, z\colon B \supset (A \supset B), \Gamma^2, \Gamma' \to \Delta^2, \Delta', z\colon B}\,\text{Cut}$$

with

$$\dfrac{\dfrac{x \leqslant y, y\colon B \supset (A \supset B), y\colon A, \Gamma \to \Delta, y\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,\text{R}{\supset}_t \qquad x\colon A \supset B, x \leqslant z, \Gamma' \to \Delta', z\colon A}{x \leqslant z, \Gamma, \Gamma' \to \Delta, \Delta', z\colon A}\,\text{Cut}$$

The two upper cuts, those on $x\colon A \supset B$ are of smaller derivation height, the other two on the smaller cut formulae $z\colon A$ and $z\colon B$. ■

Remark 3.2.3. As a consequence of admissibility of weakening, rule R⊃ of **G3I** is admissible in $\mathbf{G3I_t}$.

We now prove a few lemmata that will be useful later.

Lemma 3.2.4. *The rule*

$$\frac{x \leqslant y, \Gamma \to \Delta, x\colon A}{x \leqslant y, \Gamma \to \Delta, y\colon A}$$

is admissible in $\mathbf{G3I_t}$.

Proof. We prove it by induction on the height of the derivation of the premiss, with a subinduction on the length of A.[1]

$\underline{n = 0}$: The only nontrivial case is the one in which the premiss is an initial sequent and $x\colon A$ is principal. In this case, we can write the sequent as

$$x \leqslant y, w \leqslant x, w\colon A, \Gamma' \to \Delta, x\colon A,$$

where $\Gamma \equiv w \leqslant x, w\colon A, \Gamma'$. Observe that the sequent

$$w \leqslant y, x \leqslant y, w \leqslant x, w\colon A, \Gamma' \to \Delta, y\colon A$$

is initial. By transitivity, we get a derivation of

$$x \leqslant y, w \leqslant x, w\colon A, \Gamma' \to \Delta, y\colon A,$$

which is just $x \leqslant y, \Gamma \to \Delta, y\colon A$, as wanted.

$\underline{n > 0}$: The only nontrivial cases are those in which the last rule applied is a right rule and $x\colon A$ is principal. If the last rule applied is R$\wedge$ and $A \equiv B \wedge C$, then we have

$$\frac{x \leqslant y, \Gamma \to \Delta, x\colon B \quad x \leqslant y, \Gamma \to \Delta, x\colon C}{x \leqslant y, \Gamma \to \Delta, x\colon B \wedge C}\,\mathrm{R}\wedge$$

We can apply the induction hypothesis on the premisses and get

$$x \leqslant y, \Gamma \to \Delta, y\colon B,$$
$$x \leqslant y, \Gamma \to \Delta, y\colon C.$$

[1] While both Lemmata 3.2.4 and 3.2.5 could simply be seen as applications of Cut with the monotonicity sequent $x \leqslant y, x\colon A, \Gamma \to \Delta, y\colon A$ (which is derivable by Theorem 3.2.2(i)(a)), we prefer to do a direct proof: as we are going to notice later, from these we will obtain an alternative proof of cut-elimination.

We conclude by an application of R$\wedge$. If the last rule applied is R$\vee$ and $A \equiv B \vee C$, then we have

$$\frac{x \leqslant y, \Gamma \to \Delta, x\colon B, x\colon C}{x \leqslant y, \Gamma \to \Delta, x\colon B \vee C}\,\text{R}\vee$$

We can apply the induction hypothesis on the premiss and get

$$x \leqslant y, \Gamma \to \Delta, y\colon B, y\colon C.$$

We conclude by an application of R$\vee$. If the last rule applied is R$\supset_t$ and $A \equiv B \supset C$, then we have

$$\frac{x \leqslant z, x \leqslant y, z\colon C \supset (B \supset C), z\colon B, \Gamma \to \Delta, z\colon C}{x \leqslant y, \Gamma \to \Delta, x\colon B \supset C}\,\text{R}\supset_t$$

We can apply hp-weakening on the premiss and get

$$y \leqslant z, x \leqslant z, x \leqslant y, z\colon C \supset (B \supset C), z\colon B, \Gamma \to \Delta, z\colon C,$$

which, by an application of transitivity leads to

$$y \leqslant z, x \leqslant y, z\colon C \supset (B \supset C), z\colon B, \Gamma \to \Delta, z\colon C.$$

We conclude with an application of R$\supset_t$. ■

Lemma 3.2.5. *The rule*

$$\frac{x \leqslant y, x\colon A, y\colon A, \Gamma \to \Delta}{x \leqslant y, x\colon A, \Gamma \to \Delta}$$

is admissible in $\mathbf{G3I_t}$.

Proof. We prove it by induction on the height of the derivation of the premiss, with a subinduction on the length of A.[2]

$\underline{n = 0}$: The only nontrivial case is the one in which the premiss is an initial sequent and $y\colon A$ is principal. In this case, we can write the sequent as

$$x \leqslant y, y \leqslant z, x\colon A, y\colon A, \Gamma' \to \Delta', z\colon A,$$

[2]See footnote 1.

where $\Gamma \equiv y \leqslant z, \Gamma'$ and $\Delta \equiv \Delta', z\colon A$. Observe that the sequent

$$x \leqslant y, y \leqslant z, x \leqslant z, x\colon A, \Gamma' \to \Delta', z\colon A$$

is initial. By transitivity, we get a derivation of

$$x \leqslant y, y \leqslant z, x\colon A, \Gamma' \to \Delta', z\colon A,$$

which is just $x \leqslant y, x\colon A, \Gamma \to \Delta$, as wanted.

$\underline{n > 0}$: The only nontrivial cases are those in which the last rule applied is a left rule and $y\colon A$ is principal. If the last rule applied is L$\wedge$ and $A \equiv B \wedge C$, then we have

$$\frac{x \leqslant y, x\colon B \wedge C, y\colon B, y\colon C, \Gamma \to \Delta}{x \leqslant y, x\colon B \wedge C, y\colon B \wedge C, \Gamma \to \Delta}\,\mathrm{L}\wedge$$

Then, by hp-invertibility of L$\wedge$, we get

$$x \leqslant y, x\colon B, x\colon C, y\colon B, y\colon C, \Gamma \to \Delta,$$

to which the induction hypothesis can be applied:

$$x \leqslant y, x\colon B, x\colon C, \Gamma \to \Delta.$$

We conclude by an application of L$\wedge$. If the last rule applied is L$\vee$ and $A \equiv B \vee C$, then we have

$$\frac{x \leqslant y, x\colon B \vee C, y\colon B, \Gamma \to \Delta \qquad x \leqslant y, x\colon B \vee C, y\colon C, \Gamma \to \Delta}{x \leqslant y, x\colon B \vee C, y\colon B \vee C, \Gamma \to \Delta}\,\mathrm{L}\vee$$

Then, by hp-invertibility of L$\vee$, we get

$$x \leqslant y, x\colon B, y\colon B, \Gamma \to \Delta$$
$$x \leqslant y, x\colon C, y\colon C, \Gamma \to \Delta,$$

to which the induction hypothesis can be applied:

$$x \leqslant y, x\colon B, \Gamma \to \Delta$$
$$x \leqslant y, x\colon C, \Gamma \to \Delta.$$

We conclude by an application of L∨. If the last rule applied is L⊃ and $A \equiv B \supset C$, then we have

$$\dfrac{x \leqslant y, x\colon B \supset C, y\colon B \supset C, y \leqslant z, \Gamma' \to \Delta, z\colon B \quad x \leqslant y, x\colon B \supset C, y\colon B \supset C, z\colon C, y \leqslant z, \Gamma' \to \Delta}{x \leqslant y, x\colon B \supset C, y\colon B \supset C, y \leqslant z, \Gamma' \to \Delta}\text{L}\supset$$

where $\Gamma \equiv y \leqslant z, \Gamma'$. Then we can apply the induction hypothesis on the premisses:

$$x \leqslant y, x\colon B \supset C, y \leqslant z, \Gamma' \to \Delta, z\colon B$$
$$x \leqslant y, x\colon B \supset C, z\colon C, y \leqslant z, \Gamma' \to \Delta.$$

By hp-weakening, these lead to

$$x \leqslant z, x \leqslant y, x\colon B \supset C, y \leqslant z, \Gamma' \to \Delta, z\colon B$$
$$x \leqslant z, x \leqslant y, x\colon B \supset C, z\colon C, y \leqslant z, \Gamma' \to \Delta.$$

Now we can apply L⊃ in order to get

$$x \leqslant z, x \leqslant y, x\colon B \supset C, y \leqslant z, \Gamma' \to \Delta.$$

We conclude by an application of transitivity. ■

Lemma 3.2.6. *The rule*

$$\dfrac{x \leqslant y, x\colon B \supset (A \supset B), y\colon A, \Gamma \to \Delta, y\colon B, y\colon A \supset B}{x \leqslant y, x\colon B \supset (A \supset B), y\colon A, \Gamma \to \Delta, y\colon B}$$

is admissible.

Proof. Direction "⇑" is just an instance of weakening. By invertibility of $\text{R}\supset_t$ we get

$$\dfrac{\dfrac{x \leqslant y, x\colon B \supset (A \supset B), y\colon B \supset (A \supset B), y\colon A, y\colon A, \Gamma \to \Delta, y\colon B, y\colon B}{x \leqslant y, x\colon B \supset (A \supset B), y\colon B \supset (A \supset B), y\colon A, \Gamma \to \Delta, y\colon B}\text{RC}}{x \leqslant y, x\colon B \supset (A \supset B), y\colon A, \Gamma \to \Delta, y\colon B}\text{Lemma 3.2.5}$$

■

3.3 Soundness and completeness

3.3.1 Semantics

A *Kripke model* [102] (X, R, val) is a set X together with an *accessibility relation* R, i.e. a binary relation between elements of X, and a *valuation* val, i.e. a function assigning one of the truth values 0 or 1 to an element x of X and an atomic formula P. The usual notation is for $val(x, P) \equiv 1$ is $x \Vdash P$. In Kripke models for intuitionistic logic, the accessibility relation is a preorder, that is *reflexive*

$$\forall x(xRx)$$

and *transitive*

$$\forall x \forall y(yRx \Rightarrow \forall z(zRy \Rightarrow zRx)),$$

and therefore it is denoted by $\leqslant$. For convenience, we assume to have equality $=$ and a binary relation $<$ on X which is transitive and *irreflexive*, i.e.

$$\forall x(x \not< x),$$

and we define $\leqslant$ as its *reflexive closure*:

$$x \leqslant y \iff (x < y \text{ or } x = y).$$

As usual, we denote by $\geqslant$ the inverse relation of $\leqslant$. The inductive definition of truth of a proposition in **Int** in terms of Kripke semantics is:

$x \nVdash \bot$

$x \Vdash A \wedge B$ if and only if $x \Vdash A$ and $x \Vdash B$

$x \Vdash A \vee B$ if and only if $x \Vdash A$ or $x \Vdash B$

$x \Vdash A \supset B$ if and only if $y \Vdash A \Rightarrow y \Vdash B$ for all y such that $x \leqslant y$

Let $x \in X$. We say that $\leqslant$ *satisfies the semantic a fortiori property for* x if

$$\forall y \geqslant x(y \Vdash B \supset (A \supset B) \,\&\, y \Vdash A \Rightarrow y \Vdash B). \qquad (\mathrm{SAF}_x)$$

Let R be a relation on X. An *infinite R-sequence* is a sequence $(x_i)_{i\in\mathbb{N}}$ of elements of X such that $x_i R x_{i+1}$ for all $i \in \mathbb{N}$. An infinite R-sequence $(x_i)_{i\in\mathbb{N}}$ is *convergent* if there is $i \in \mathbb{N}$ such that $x_j = x_i$ for all $j > i$. We say that R is *Noetherian*—for short, R satisfies Noeth—if every infinite R-sequence converges.

Lemma 3.3.1. *Let $x \in X$. If $\leqslant$ is Noetherian and satisfies SAF_x, then*

$$\forall y > x(y \Vdash B \supset (A \supset B)).$$

Proof. Notice that the relation $<$ is transitive, irreflexive and Noetherian. Therefore it follows from Theorem 2.4.3 and Proposition 2.4.2 that its inverse $>$ satisfies the Gödel–Löb Induction, that is

$$\forall x(\forall y > x(\forall z > y\, Ez \Rightarrow Ey) \Rightarrow \forall y > x\, Ey) \tag{GL-Ind}$$

for any given predicate $E(x)$ on X. Therefore, if we let $E(x) \equiv x \Vdash B \supset (A \supset B)$, it suffices to show that

$$\forall y > x(\forall z > y(z \Vdash B \supset (A \supset B)) \Rightarrow y \Vdash B \supset (A \supset B)). \tag{3.1}$$

So let $y > x$ such that

$$\forall z > y(z \Vdash B \supset (A \supset B)). \tag{3.2}$$

We claim that $y \Vdash B \supset (A \supset B)$, i.e.

$$\forall z \geqslant y(z \Vdash B \Rightarrow z \Vdash A \supset B). \tag{3.3}$$

So let $z \geqslant y$ such that $z \Vdash B$. We have to prove $z \Vdash A \supset B$, i.e.

$$\forall w \geqslant z(w \Vdash A \Rightarrow w \Vdash B). \tag{3.4}$$

So let $w \geqslant z$ such that $w \Vdash A$. The claim is $w \Vdash B$.

— If $w = z$, then we already know that $z \Vdash B$.

— If $w > z$, then by transitivity $w > y$ and by (3.2) we get $w \Vdash B \supset (A \supset B)$. Since $w \Vdash A$ and by transitivity $w \geqslant x$, we can apply SAF_x and derive $w \Vdash B$.

Now unroll the proof to get claims (3.4), (3.3) and (3.1), and thus the main claim. ∎

Lemma 3.3.2. *Fix $x \in X$. If $\leqslant$ is Noetherian and satisfies SAF_x, then $x \Vdash B \supset (A \supset B)$.*

Proof. The claim is equivalent to

$$\forall y \geqslant x (y \Vdash B \Rightarrow y \Vdash A \supset B). \tag{3.5}$$

Fix $y \geqslant x$ such that $y \Vdash B$. We claim that $y \Vdash A \supset B$, i.e.

$$\forall z \geqslant y (z \Vdash A \Rightarrow z \Vdash B). \tag{3.6}$$

Fix $z \geqslant y$ such that $z \Vdash A$. We need to prove that $z \Vdash B$.

— If $z = y$, then we already know that $y \Vdash B$.

— If $z > y$, then by transitivity $z > x$ and by Lemma 3.3.1 we get $z \Vdash B \supset (A \supset B)$. Since $z \Vdash A$ and by transitivity $z \geqslant x$, we can apply SAF_x and derive $z \Vdash B$.

Now unroll the proof to get claims (3.6), and (3.5), and thus the main claim. ∎

Lemma 3.3.3 (Semantic Lemma)**.** *Fix $x \in X$. If $\leqslant$ is Noetherian, then the following are equivalent:*

(i) SAF_x.

(ii) $\forall y \geqslant x (y \Vdash A \Rightarrow y \Vdash B)$.

Proof. (ii)⇒(i): *A fortiori*.

(i)⇒(ii): Fix $y \geqslant x$ such that $y \Vdash A$. We claim that $y \Vdash B$.

— If $y = x$, then by Lemma 3.3.2 we get that $x \Vdash B \supset (A \supset B)$.

— If $y > x$, then by Lemma 3.3.1 we get that $y \Vdash B \supset (A \supset B)$.

In either case we have $y \Vdash B \supset (A \supset B)$ and $y \Vdash A$, thus we can apply SAF_x and get $y \Vdash B$. ∎

3.3.2 Proof search

Consider the proof search procedure as defined in [57]. We have the analogous of 5.3–6:

Theorem 3.3.4 (Soundness)**.** *If* $\mathbf{G3I_t} \vdash \Gamma \to \Delta$*, then* $\Gamma \to \Delta$ *is valid in every reflexive transitive and Noetherian frame.*

Proof. If $\mathbf{G3I_t} \vdash \Gamma \to \Delta$, then $\mathbf{G3I} \vdash \Gamma \to \Delta$ is valid in every reflexive transitive frame, a fortiori in every Noetherian one. ∎

Theorem 3.3.5. *Let* $\Gamma \to \Delta$ *be a sequent in the language of* $\mathbf{G3I_t}$*. Then it is decidable whether it is derivable in* $\mathbf{G3I_t}$*. If it is not derivable, the failed proof search gives a finite countermodel to the sequent on a reflexive, transitive and Noetherian frame.*

Proof. We adapt the proof of [57, Theorem 5.4], which in turn is an adaptation to labelled sequents of the method of reduction trees detailed for Gentzen's LK by Takeuti [182, Chapter 1, Paragraph 8].

For an arbitrary sequent $\Gamma \to \Delta$ in the language of $\mathbf{G3I_t}$ we apply, whenever possible, root-first the rules of $\mathbf{G3I_t}$, in a given order. The procedure will construct either a derivation in $\mathbf{G3I_t}$ or a countermodel.

1. Construction of the reduction tree: The reduction tree is defined inductively in stages as follows: Stage 0 has $\Gamma \to \Delta$ at the root of the tree. For each branch, stage $n > 0$ has two cases:

Case I: If the top-sequent is either an initial sequent or has some $x{:}\ A$, not necessarily atomic, on both left and right, or is a conclusion of L⊥, the construction of the branch ends.

Case II: Otherwise we continue the construction of the branch by writing, above its top-sequent, other sequents that are obtained by applying root-first the rules of $\mathbf{G3I_t}$ (except L⊥) whenever possible, in a given order and under suitable conditions.

There are 8 different stages: one for each logical rule, Ref and Trans. At stage $8+1$ we repeat stage 1, at stage $8+2$ we repeat stage 2, and so on until an initial sequent, or a conclusion of L⊥, or a *saturated branch* (defined below) is found.

The stages for the rules other than R$\supset_t$ are similar to those in [133, Theorem 11.28]. Note that all rules but L$\supset$ discard the principal formula; all such formulae however are available somewhere on the branch for when we need to discuss the counter model construction.

At the stage of construction relative to R$\supset_t$, we consider all labelled formulae of the form $x\colon A \supset (B \supset A)$ in the succedent. If the succedent of the top-sequent contains $y\colon B$, $y\colon (A \supset B)$ and the antecedent contains $x \leqslant y, x\colon B \supset (A \supset B), y\colon A$, then we need not further analyse $y\colon A \supset B$; this is justified by Lemma 3.2.6. More generally, if $x\colon B$ is in the succedent of any sequent on the branch, we do the same. For each of the remaining labelled boxed formulae $x_i\colon B_i \supset (A_i \supset B_i)$ for $i \in \{1,...,m\}$, we apply several times the rule $R \supset_t$, that is, we construct the step

$$\frac{x_1 \leqslant y_1,...,x_m \leqslant y_m, y_1\colon B_1 \supset (A_1 \supset B_1),...,y_m\colon B_m \supset (A_m \supset B_m), y_1\colon A_1,...,y_m\colon A_m, \Gamma \to \Delta, y_1\colon B_1,...,y_m\colon B_m}{\Gamma \to \Delta, x_1\colon A_1 \supset B_1,...,x_m\colon A_m \supset B_m}$$

Finally, we consider the cases of the frame rules Ref and Trans. As detailed in [56, 57], it is enough to instantiate Ref only on terms in the top-sequent.

Observe also that, because of height-preserving admissibility of contraction, once a rule has been considered, it need not be instantiated again on the same principal formulae (for L$\supset$ such principal formulae are pairs of the form $x \leqslant y, x\colon A \supset B$ and it need not be applied whenever its application produces a duplication of labelled formulae or relational atoms.

To show that the procedure terminates, it is enough to show that every branch in the reduction tree for a sequent $\Gamma \to \Delta$ is finite. Every branch contains one or more chains of labels $x_1 \leqslant y_1,...,x_m \leqslant y_m,...$; each label that was not already in the endsequent is introduced by a step of R$\supset_t$. By inspection of the rules of $\mathbf{G3I_t}$, it is clear that all the formulae that occur in the branch are subformulae of Γ, Δ or formulae of the form $A \supset (B \supset A)$ for some subformula $B \supset A$ of Γ, Δ. To ensure that all proper chains of labels in the reduction tree are finite, it is therefore enough to prove that rule R$\supset_t$ need not be applied twice to the same formula along a chain of labels.

Suppose that we have a chain $x_0 \leqslant x_1, \ldots, x_{n-1} \leqslant x_n$ in the antecedent and $x_0 \colon A \supset B, x_n \colon A \supset B$ in the succedent of a branch in the proof search and that R$\supset_t$ has been applied to $x_0 \colon A \supset B$. We need to show that there is no need to apply R$\supset_t$ to $x_n \colon A \supset B$. Suppose for simplicity that we have a chain of length 2, with $x_0 \equiv x, x_1 \equiv y, x_2 \equiv z$:

$$\dfrac{\begin{array}{c} x \leqslant y, y \leqslant z, y \colon B \supset (A \supset B), \Gamma' \to \Delta', z \colon A \supset B \\ \vdots \\ x \leqslant y, y \colon B \supset (A \supset B), y \colon A, \Gamma \to \Delta, y \colon B \end{array}}{\Gamma \to \Delta, x \colon A \supset B} \text{R}\supset_t$$

and assume that the top-sequent is closed under all the available rules (excluding R$\supset_t$) of the reduction procedure. By the closure properties for L$\supset$, the proof search continues with

$$x \leqslant y, y \leqslant z, y \colon B \supset (A \supset B), \Gamma' \to \Delta', z \colon B, z \colon A \supset B \tag{3.7}$$

and

$$x \leqslant y, y \leqslant z, y \colon B \supset (A \supset B), z \colon A \supset B, \Gamma' \to \Delta', z \colon A \supset B \tag{3.8}$$

Observe that, as observed above, by Lemma 3.2.6 we need not further analyse (3.8), and (3.7) follows from

$$x \leqslant y, y \colon B \supset (A \supset B), y \colon A, \Gamma \to \Delta, y \colon B$$

by admissibility of weakening (Theorem 3.2.2(iii)) and Lemma 3.2.4.

We can conclude that all the chains of labels in the tree are finite. To conclude that the branch is finite, it is enough to observe that it contains only a finite number of such chains (the number of chains is bounded by a function of the number of disjunctions or commas in the positive part of the endsequent; observe that this argument would break down in the labelled calculus for intuitionistic logic because here we rely on the fact that propositional rules have premisses in which the active formulae are strictly simpler than the principal formula). The general case, where the chain is longer than just $x \leqslant y, y \leqslant z$, is similar.

A branch which either ends in an initial sequent or in a sequent with the same labelled formula, even compound, in both the antecedent and succedent, or at the conclusion of L$\bot$, or has a top-sequent amenable to any of the reduction steps, is called *unsaturated*. Every other branch is said to be *saturated*.

2. Construction of the countermodel: If the reduction tree for $\Gamma \to \Delta$ is not a derivation, it has at least one saturated branch. Let $\Gamma^* \to \Delta^*$ be the union (respectively, of the antecedents and succedents) of all the sequents $\Gamma_i \to \Delta_i$ of the branch up to its top-sequent. We define a Kripke model that forces all the formulae in Γ^* and no formula in Δ^* and is therefore a countermodel to the sequent $\Gamma \to \Delta$.

Consider the frame X, the nodes of which are the labels that appear in the relational atoms in Γ^* and the order on which is given by these relational atoms. Clearly, the construction of the reduction tree imposes the frame properties on the countermodel: Ref and Trans hold because the branch is saturated. Morever, any label that appears in the sequent will appear in a relational atom (and thus in the frame X), because the rule Ref has been applied. Noetherianity clearly holds because all the strictly ascending chains in the countermodel are finite by construction.

The model is defined as follows. First, the interpretation $[[x]]$ of each label x is just x itself. As for the valuation, for each labelled atomic formula $x : P$ in Γ^* we stipulate that $x \Vdash P$. Since the top-sequent is not initial, for all labelled atomic formulae $y : Q$ in Δ^* we infer that $y \nVdash Q$. We then show by induction on $size(A)$ that $x \Vdash A$ if $x : A$ is in Γ^* and that $x \nVdash A$ if $x : A$ is in Δ^*. Therefore we have a countermodel to the endsequent $\Gamma \to \Delta$.

— If A is atomic, then the claim holds by the definition of the model.

— If $A \equiv \bot$, it cannot be in Γ^*, by definition of saturated branch: so $x \nVdash A$.

— If $A \equiv B \wedge C$ is in Γ^*, then by the saturation of the branch we also have $x\colon B$ and $x\colon C$ in Γ^*. By the induction hypothesis, $x \Vdash B$ and $x \Vdash C$, and therefore $x \Vdash B \wedge C$.

— If $A \equiv B \wedge C$ is in Δ^*, then by the saturation of the branch either $x\colon B$ or $x\colon C$ in Δ^*. By the induction hypothesis, $x \nVdash B$ or $x \nVdash C$, and therefore $x \nVdash B \wedge C$.

— If $A \equiv B \vee C$ is in Γ^*, then by the saturation of the branch either $x\colon B$ or $x\colon C$ in Γ^*. By the induction hypothesis, $x \Vdash B$ or $x \Vdash C$, and therefore $x \Vdash B \vee C$.

— If $A \equiv B \vee C$ is in Δ^*, then by the saturation of the branch we also have $x\colon B$ and $x\colon C$ in Δ^*. By the induction hypothesis, $x \nVdash B$ and $x \nVdash C$, and therefore $x \nVdash B \vee C$.

— If $A \equiv B \supset C$ is in Γ^*, then for any occurrence of $x \leqslant y$ in Γ^* we find, by saturation and by the construction of the reduction tree, either an occurrence of $y\colon B$ in Δ^* or an occurrence of $y\colon C$ in Γ^*. By the induction hypothesis, in the former case $y \nVdash B$, and in the latter $y \Vdash C$, so in both cases $x \Vdash B \supset C$.

— If $A \equiv B \supset C$ is in Δ^*, we consider the step where it is analysed. If $x\colon C$ is in the succedent of that step (or any succedent below it), then by the induction hypothesis $x \Vdash B$. Since $x \leqslant x$ is also in Γ^* by construction of the reduction tree, it follows that $x \Vdash B \supset C$. Otherwise there is $x \leqslant y$ in Γ^* and $y\colon C$ in Δ^*. By the induction hypothesis $y \nVdash C$, and therefore $x \nVdash A$. ■

Corollary 3.3.6. *If a sequent $\Gamma \to \Delta$ is valid in every reflexive, transitive and Noetherian frame, then it is derivable in* $\mathbf{G3I_t}$.

Corollary 3.3.7. *A formula A is provable in* **Int** *if and only if the sequent $\to x\colon A$ is derivable in* $\mathbf{G3I_t}$ *for some (or any) label x.*

We observe that completeness implies in particular closure of our sequent calculus with respect to Cut, so we have an indirect proof of admissibility of the Cut rule, which was proved directly in Theorem 3.2.2, see also footnote 1. The proof of Theorem 3.3.5 is also of interest because it establishes the finite model property for **Int** and gives a constructive decision procedure for it, i.e. an algorithm that, given a sequent, constructs either a derivation or a countermodel.

3.3.3 An example: Peirce's Law

Consider Peirce's Law:

$$((P \supset Q) \supset P) \supset P.$$

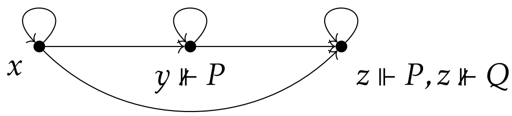
If we try to do a derivation of $\to x\colon ((P \supset Q) \supset P) \supset P$ in **G3I**, we get

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{y \leqslant w, z \leqslant w, y \leqslant z, y \leqslant y, x \leqslant y, w\colon P, z\colon P, y\colon (P \supset Q) \supset P \to y\colon P, z\colon Q, w\colon Q}\,\mathrm{R}{\supset}}{z \leqslant w, y \leqslant z, y \leqslant y, x \leqslant y, w\colon P, z\colon P, y\colon (P \supset Q) \supset P \to y\colon P, z\colon Q, w\colon Q}\,\mathrm{Trans}}{y \leqslant z, y \leqslant y, x \leqslant y, z\colon P, y\colon (P \supset Q) \supset P \to y\colon P, z\colon Q, z\colon P \supset Q}\,\mathrm{R}{\supset} \quad \vdots}{y \leqslant z, y \leqslant y, x \leqslant y, z\colon P, y\colon (P \supset Q) \supset P \to y\colon P, z\colon Q}\,\mathrm{L}{\supset}}{y \leqslant y, x \leqslant y, y\colon (P \supset Q) \supset P \to y\colon P, y\colon P \supset Q}\,\mathrm{R}{\supset} \quad \vdots}{y \leqslant y, x \leqslant y, y\colon (P \supset Q) \supset P \to y\colon P}\,\mathrm{L}{\supset}}{x \leqslant y, y\colon (P \supset Q) \supset P \to y\colon P}\,\mathrm{Ref}_{\leqslant}}{\to x\colon ((P \supset Q) \supset P) \supset P}\,\mathrm{R}{\supset}$$

We see that the left branch is generating a loop and therefore does not terminate. If we try to do a derivation of $\to x\colon ((P \supset Q) \supset P) \supset P$ in $\mathbf{G3I_t}$ instead, we get

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y \leqslant z, y \leqslant y, x \leqslant y, z\colon P \supset ((P \supset Q) \supset P), z\colon P, y\colon P \supset (((P \supset Q) \supset P) \supset P), y\colon (P \supset Q) \supset P \to y\colon P, z\colon Q, z\colon P \supset Q \quad \vdots}{y \leqslant z, y \leqslant y, x \leqslant y, z\colon P \supset ((P \supset Q) \supset P), z\colon P, y\colon P \supset (((P \supset Q) \supset P) \supset P), y\colon (P \supset Q) \supset P \to y\colon P, z\colon Q}\,\mathrm{L}{\supset}}{y \leqslant y, x \leqslant y, y\colon P \supset (((P \supset Q) \supset P) \supset P), y\colon (P \supset Q) \supset P \to y\colon P, y\colon P \supset Q}\,\mathrm{R}{\supset_t} \quad \vdots}{y \leqslant y, x \leqslant y, y\colon P \supset (((P \supset Q) \supset P) \supset P), y\colon (P \supset Q) \supset P \to y\colon P}\,\mathrm{L}{\supset}}{x \leqslant y, y\colon P \supset (((P \supset Q) \supset P) \supset P), y\colon (P \supset Q) \supset P \to y\colon P}\,\mathrm{Ref}_{\leqslant}}{\to x\colon ((P \supset Q) \supset P) \supset P}\,\mathrm{R}{\supset_t}$$

This time, the proof search algorithm defined in the proof of Theorem 3.3.5 tells us that the top-sequent of the left branch need not be further analysed, and it helps us in constructing a countermodel:



Let's check that actually $x \nVdash ((P \supset Q) \supset P) \supset P$, which is equivalent to the statement that

$$\forall x_1 \geqslant x(\forall x_2 \geqslant x_1(\forall x_3 \geqslant x_2(x_3 \Vdash P \Rightarrow x_3 \Vdash Q) \Rightarrow x_2 \Vdash P) \Rightarrow x_1 \Vdash P)$$

does not hold. We check that this does not hold for $x_1 \equiv y$. Since $y \nVdash P$, we just need to show that

$$\forall x_2 \geqslant y(\forall x_3 \geqslant x_2(x_3 \Vdash P \Rightarrow x_3 \Vdash Q) \Rightarrow x_2 \Vdash P).$$

We have two cases: if $x_2 \equiv y$, then our claim follows from $y \leqslant z$ and $z \Vdash P \nRightarrow z \Vdash Q$; if $x_2 \equiv z$, then our claim follows a fortiori from $z \Vdash P$.

3.4 Embedding into Grzegorczyk logic

We recall that modal logic is obtained by adding the modal operator $\Box$ to the language of propositional logic, and inductive clauses for valuations of Kripke frames are the following:

$x \nVdash \bot$

$x \Vdash A \supset B$ if and only if $x \Vdash A \Rightarrow x \Vdash B$

$x \Vdash A \wedge B$ if and only if $x \Vdash A$ and $x \Vdash B$

$x \Vdash A \vee B$ if and only if $x \Vdash A$ or $x \Vdash B$

$x \Vdash \Box A$ if and only if $\forall y(x \leqslant y \Rightarrow y \Vdash A)$

The provability logic **Grz** (Grzegorczyk logic) [10, 57, 86] is an extension of basic modal logic **K** with the additional schemata

$$\Box A \supset A \qquad \text{(Ax. T)}$$

$$\Box A \supset \Box\Box A \qquad \text{(Ax. 4)}$$

$$\Box(G(A) \supset A) \supset A \qquad \text{(Ax. Grz)}$$

Initial sequent

$x\colon P, \Gamma \to \Delta, x\colon P$

Propositional rules

$$\frac{x\colon A, x\colon B, \Gamma \to \Delta}{x\colon A \wedge B, \Gamma \to \Delta}\,\mathrm{L}\wedge \qquad \frac{\Gamma \to \Delta, x\colon A \qquad \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \wedge B}\,\mathrm{R}\wedge$$

$$\frac{x\colon A, \Gamma \to \Delta \qquad x\colon B, \Gamma \to \Delta}{x\colon A \vee B, \Gamma \to \Delta}\,\mathrm{L}\vee \qquad \frac{\Gamma \to \Delta, x\colon A, x\colon B}{\Gamma \to \Delta, x\colon A \vee B}\,\mathrm{R}\vee$$

$$\frac{\Gamma \to \Delta, x\colon A \qquad x\colon B, \Gamma \to \Delta}{x\colon A \supset B, \Gamma \to \Delta}\,\mathrm{L}\supset \qquad \frac{x\colon A, \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,\mathrm{R}\supset$$

$$\frac{}{x\colon \bot, \Gamma \to \Delta}\,\mathrm{L}\bot$$

Modal rules

$$\frac{x \leqslant y, y\colon A, x\colon \Box A, \Gamma \to \Delta}{x \leqslant y, x\colon \Box A, \Gamma \to \Delta}\,\mathrm{L}\Box \qquad \frac{x \leqslant y, y\colon G(A), \Gamma \to \Delta, y\colon A}{\Gamma \to \Delta, x\colon \Box A}\,\mathrm{R}\Box\mathrm{Z}$$

Mathematical rules

$$\frac{x \leqslant x, \Gamma \to \Delta}{\Gamma \to \Delta}\,\mathrm{Ref}_{\leqslant} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \to \Delta}{x \leqslant y, y \leqslant z, \Gamma \to \Delta}\,\mathrm{Trans}_{\leqslant}$$

Table 3.3: The sequent calculus **G3Grz**. Rule R$\Box$ has the condition that y is fresh.

where $G(A) \equiv \Box(A \supset \Box A)$. **Grz** is characterised by reflexive, transitive and Noetherian frames [57]. The sequent calculus **G3Grz** for **Grz** (see table B.6) satisfies all usual structural rules, including hp-invertibility of its rules [57].

As shown in [57], an indirect decision procedure for **Int** is obtained through faithfulness of the embedding of **Int** into **Grz** via the translation $\_^{\Box}$ inductively defined as

$$\begin{aligned} P^{\Box} &\equiv \Box P \\ \bot^{\Box} &\equiv \bot \\ (A \wedge B)^{\Box} &\equiv A^{\Box} \wedge B^{\Box} \\ (A \vee B)^{\Box} &\equiv A^{\Box} \vee B^{\Box} \\ (A \supset B)^{\Box} &\equiv \Box(A^{\Box} \supset B^{\Box}) \end{aligned}$$

Labels and relational atoms are left unchanged.

Remark 3.4.1. The translation of $R{\supset}_t$ is the following:

$$\frac{x \leqslant y, y\colon \Box(B^{\Box} \supset \Box(A^{\Box} \supset B^{\Box})), y\colon A^{\Box}, \Gamma^{\Box} \to \Delta^{\Box}, y\colon B^{\Box}}{\Gamma^{\Box} \to \Delta^{\Box}, x\colon \Box(A^{\Box} \supset B^{\Box})} \quad (y \text{ fresh})$$

If we set $A \equiv \top$, this is equivalent to

$$\frac{x \leqslant y, y\colon \Box(B^{\Box} \supset \Box B^{\Box}), \Gamma^{\Box} \to \Delta^{\Box}, y\colon B^{\Box}}{\Gamma^{\Box} \to \Delta^{\Box}, x\colon \Box B^{\Box}} \quad (y \text{ fresh})$$

which is an instance of $R\Box Z$, the rule that allows decidability in the calculus **G3Grz** for Grzegorczyk logic.

We now want to give a proof of faithfulness alternative to the one is given in [57] by using $\mathbf{G3I_t}$ in place of **G3I** . We first need a few lemmata:

Lemma 3.4.2. *If there is a derivation of*

$$x\colon A \supset B, \Gamma \to \Delta \tag{3.9}$$

in **G3Grz** *of height n, then there are derivations of*

$$\Gamma \to \Delta, x\colon A \tag{3.10}$$

$$x\colon B, \Gamma \to \Delta \tag{3.11}$$

in **G3Grz** *of height at most n. If, moreover, $x\colon A \supset B$ is used as the principal formula somewhere in the given derivation of (3.9), then the derivations of (3.10) and (3.11) have height at most $n-1$.*

Proof. We slightly modify the usual argument for hp-invertibility of L$\supset$ (see, e.g. [125, Proposition 4.11]). The proof proceeds by induction on n.

$\underline{n = 0}$: Trivial.

$\underline{n > 0}$: If $x\colon A \supset B$ is principal in the last rule applied in the derivation of (3.9), then the two branches are derivations of (3.10) and (3.11) of height at most $n-1$. If it is not principal and the last rule applied is *rule*, then we proceed as usual by applying the induction hypothesis to the previous step(s) followed by *rule*. ■

Lemma 3.4.3. *The rule*

$$\frac{x \leqslant y, x\colon A^{\Box}, y\colon A^{\Box}, \Gamma \to \Delta}{x \leqslant y, x\colon A^{\Box}, \Gamma \to \Delta}$$

with the condition that the top-sequent is saturated under transitivity, is hp-admissible in **G3Grz**.

Proof. We prove it by induction on the height of the derivation of the premiss, with a subinduction on the length of A.

$\underline{n = 0}$: Trivial.

$\underline{n > 0}$: The only nontrivial cases are those in which the last rule applied is a left rule and $y\colon A^{\Box}$ is principal. Cases L$\wedge$ and L$\vee$ are dealt with as in Lemma 3.2.5, and L$\Box$ as in [57, Lemma 3.14]. The assumption of saturation under transitivity makes the application of Trans in [57, Lemma 3.14] unnecessary, thus ensuring height preservation. ■

Lemma 3.4.4. *The rule*

$$\frac{x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), \Gamma \to \Delta}{x\colon A^{\Box}, x\colon \Box(B \supset C), \Gamma \to \Delta}$$

with the condition that the top-sequent is saturated under transitivity, is hp-admissible in **G3Grz**.

Proof. Suppose that there is a derivation of

$$x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), \Gamma \to \Delta \tag{3.12}$$

of height n. We prove by induction on n that there is a derivation of

$$x\colon A^{\Box}, x\colon \Box(B \supset C), \Gamma \to \Delta \tag{3.13}$$

of height n.

<u>$n = 0$</u>: All cases are trivial.

<u>$n > 0$</u>: The cases in which the principal formula is in Γ or Δ are trivial.

Suppose that the principal formula is $x\colon A^{\Box}$, and consider the case in which $A \equiv A_1 \wedge A_2$, which means that we have a derivation

$$\frac{x\colon A_1^{\Box}, x\colon A_2^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), \Gamma \to \Delta}{x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), \Gamma \to \Delta}\ \mathrm{L}\wedge$$

We can apply hp-weakening on the premiss and get

$$x\colon A^{\Box}, x\colon A_1^{\Box}, x\colon A_2^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), \Gamma \to \Delta,$$

to which we can apply the induction hypothesis:

$$x\colon A^{\Box}, x\colon A_1^{\Box}, x\colon A_2^{\Box}, x\colon \Box(B \supset C), \Gamma \to \Delta.$$

We conclude by L$\wedge$ and contraction.

Suppose that the principal formula is $x\colon A^{\Box}$, and consider the case in which $A \equiv A_1 \vee A_2$, which means that we have a derivation

$$\frac{x\colon A_1^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), \Gamma \to \Delta \qquad x\colon A_2^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), \Gamma \to \Delta}{x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), \Gamma \to \Delta}\ \mathrm{L}\vee$$

We can apply hp-weakening on the premisses and get

$$x\colon A^{\Box}, x\colon A_1^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), \Gamma \to \Delta$$
$$x\colon A^{\Box}, x\colon A_2^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), \Gamma \to \Delta$$

to which we can apply the induction hypothesis:

$$x\colon A^{\Box}, x\colon A_1^{\Box}, x\colon \Box(B \supset C), \Gamma \to \Delta$$
$$x\colon A^{\Box}, x\colon A_2^{\Box}, x\colon \Box(B \supset C), \Gamma \to \Delta$$

We conclude by L$\vee$ and contraction.
Suppose that the principal formula is $x\colon A^{\Box}$, and consider the case in which $A \equiv A_1 \supset A_2$ or $A \equiv P$, which means that we have a derivation

$$\dfrac{y\colon A, x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), x \leqslant y, \Gamma' \to \Delta}{x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), x \leqslant y, \Gamma' \to \Delta}\,\mathrm{L}\Box$$

where $\Gamma \equiv x \leqslant y, \Gamma'$. We can apply the induction hypothesis to the premiss:

$$y\colon A, x\colon A^{\Box}, x\colon \Box(B \supset C), x \leqslant y, \Gamma' \to \Delta$$

We conclude by L$\supset$.
Now suppose that the principal formula is $x\colon \Box((A^{\Box} \supset B) \supset C)$. This means that we have

$$\dfrac{y\colon (A^{\Box} \supset B) \supset C, x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), x \leqslant y, \Gamma' \to \Delta}{x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), x \leqslant y, \Gamma' \to \Delta}\,\mathrm{L}\Box$$

where $\Gamma \equiv x \leqslant y, \Gamma'$. We can assume that $y\colon (A^{\Box} \supset B) \supset C$ is used as the principal formula somewhere above this instance of L$\Box$: if not, then we could find a derivation of (3.12) without this instance of L$\Box$, this would have lesser height and therefore we could apply the induction hypothesis to it. By applying hp-weakening to the premiss, we obtain a derivation of

$$y\colon A^{\Box}, y\colon (A^{\Box} \supset B) \supset C, x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), x \leqslant y, \Gamma' \to \Delta$$

of height $n-1$ and such that $y\colon (A^{\Box} \supset B) \supset C$ is used as the principal formula somewhere above. Now by Lemma 3.4.2 on invertibility of L$\supset$ we get derivations of

$$y\colon C, y\colon A^{\Box}, x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), x \leqslant y, \Gamma' \to \Delta \tag{3.14}$$

$$y\colon A^{\Box}, x\colon A^{\Box}, x\colon \Box((A^{\Box} \supset B) \supset C), x \leqslant y, \Gamma' \to \Delta, y\colon A^{\Box} \supset B, \tag{3.15}$$

both of height $n-2$. Now we can apply the induction hypothesis to (3.14) and get a derivation of

$$y\colon C, y\colon A^{\Box}, x\colon A^{\Box}, x\colon \Box(B \supset C), x \leqslant y, \Gamma' \to \Delta \tag{3.16}$$

of height $n-2$. By applying hp-invertibility of R$\supset$ and hp-contraction to (3.15), we get a derivation of

$$y\colon A^{\square}, x\colon A^{\square}, x\colon \square((A^{\square} \supset B) \supset C), x \leqslant y, \Gamma' \to \Delta, y\colon B$$

of height $n-2$, to which we can apply the induction hypothesis and get a derivation of

$$y\colon A^{\square}, x\colon A^{\square}, x\colon \square(B \supset C), x \leqslant y, \Gamma' \to \Delta, y\colon B \tag{3.17}$$

of height $n-2$. Now we can apply L$\supset$ to (3.16) and (3.17) and get a derivation of

$$y\colon B \supset C, y\colon A^{\square}, x\colon A^{\square}, x\colon \square(B \supset C), x \leqslant y, \Gamma' \to \Delta$$

of height $n-1$, which by an application of L$\square$ gives a derivation of

$$y\colon A^{\square}, x\colon A^{\square}, x\colon \square(B \supset C), x \leqslant y, \Gamma' \to \Delta$$

of height n. We conclude by Lemma 3.4.3. ∎

Now we are able to prove faithfulness:

Theorem 3.4.5 (Faithfulness)**.** *Let* $\Gamma \to \Delta$ *be a sequent in the language of* $\mathbf{G3I_t}$ *. If*

$$\mathbf{G3Grz} \vdash \Gamma^{\square} \to \Delta^{\square},$$

then

$$\mathbf{G3I_t} \vdash \Gamma \to \Delta.$$

Proof. By induction on the height of the derivation of $\Gamma^{\square} \to \Delta^{\square}$. We assume that $\Gamma^{\square} \to \Delta^{\square}$ is saturated under transitivity: this can be done without loss of generality since it is equivalent to apply Trans in the proof search as soon as possible, and it is innocuous because the rule operates on labels already introduced.

$\underline{n=0}$: If it is an initial sequent or the conclusion of L$\perp$, then it can be translated smoothly into the corresponding initial sequent or rule in $\mathbf{G3I_t}$.

$\underline{n>0}$: First, notice that rules for $\supset$ cannot produce a sequent of this form. If it is the conclusion of a rule for $\perp, \wedge, \vee$, then it can be translated smoothly into the corresponding initial sequent or rule in $\mathbf{G3I_t}$. If it is derived by a modal rule, then the principal formula can be of the form $\Box P$ or of the form $\Box(A^\Box \supset B^\Box)$. We have four cases:

— If $\Box P$ is principal on the left, we have

$$\frac{x \leqslant y, y\colon P, x\colon \Box P, \Gamma'^\Box \to \Delta'^\Box}{x \leqslant y, x\colon \Box P, \Gamma'^\Box \to \Delta'^\Box}\,\mathrm{L}\Box$$

which is translated into the admissible $\mathbf{G3I_t}$ step

$$\frac{x \leqslant y, y\colon P, x\colon P, \Gamma' \to \Delta'}{x \leqslant y, x\colon P, \Gamma' \to \Delta'}$$

— If $\Box P$ is principal on the right, we have

$$\frac{x \leqslant y, y\colon G(P), \Gamma'^\Box \to \Delta'^\Box, y\colon P}{\Gamma'^\Box \to \Delta'^\Box, x\colon \Box P}\,\mathrm{R}\Box\mathrm{Z}$$

which, as seen in Remark 3.4.1, is the translation of a step of rule $\mathrm{R}\supset_t$ with $\top \supset P$ as the principal formula.

— If $\Box(A^\Box \supset B^\Box)$ is principal on the left, we have

$$\frac{x \leqslant y, y\colon A^\Box \supset B^\Box, x\colon \Box(A^\Box \supset B^\Box), \Gamma'^\Box \to \Delta'^\Box}{x \leqslant y, x\colon \Box(A^\Box \supset B^\Box), \Gamma'^\Box \to \Delta'^\Box}\,\mathrm{L}\Box$$

from which, by hp-invertibility of L⊃ in **G3Grz** we have

$$\mathbf{G3Grz} \vdash x \leqslant y, y\colon B^\Box, x\colon \Box(A^\Box \supset B^\Box), \Gamma'^\Box \to \Delta'^\Box$$
$$\mathbf{G3Grz} \vdash x \leqslant y, x\colon \Box(A^\Box \supset B^\Box), \Gamma'^\Box \to \Delta'^\Box, y\colon A^\Box$$

to which the induction hypothesis applies:

$$\mathbf{G3I_t} \vdash x \leqslant y, y\colon B, x\colon A \supset B, \Gamma' \to \Delta'$$
$$\mathbf{G3I_t} \vdash x \leqslant y, x\colon A \supset B, \Gamma' \to \Delta', y\colon A$$

We conclude by an application of L⊃.

— If $\Box(A^\Box \supset B^\Box)$ is principal on the right, we have

$$\frac{x \leqslant y, y\colon G(A^\Box \supset B^\Box), \Gamma'^\Box \to \Delta'^\Box, y\colon A^\Box \supset B^\Box}{\Gamma'^\Box \to \Delta'^\Box, x\colon \Box(A^\Box \supset B^\Box)}\,\mathrm{R}\Box\mathrm{Z}$$

from which, by hp-invertibility of R⊃ in **G3Grz** we have

$$\mathbf{G3Grz} \vdash x \leqslant y, y\colon G(A^\Box \supset B^\Box), y\colon A^\Box, \Gamma'^\Box \to \Delta'^\Box, y\colon B^\Box.$$

By Lemma 3.4.4, it follows that

$$\mathbf{G3Grz} \vdash x \leqslant y, y\colon \Box(B^\Box \supset \Box(A^\Box \supset B^\Box)), y\colon A^\Box, \Gamma'^\Box \to \Delta'^\Box, y\colon B^\Box,$$

to which the induction hypothesis applies:

$$\mathbf{G3I_t} \vdash x \leqslant y, y\colon B \supset (A \supset B), y\colon A, \Gamma' \to \Delta', y\colon B$$

We conclude by $\mathrm{R}\!\supset_t$. ■

Part II

Conservation: Glivenko-style results

4 Glivenko classes and constructive cut elimination in infinitary logic

4.1 Introduction

Notable parts of algebra and geometry can be formalised as *coherent theories* over first-order classical or intuitionistic logic. Their axioms are *coherent implications*—i.e., universal closures of implications $D_1 \supset D_2$, where both D_1 and D_2 are built up from atoms using conjunction, disjunction and existential quantification. Examples include all algebraic theories, such as group theory and ring theory, all essentially algebraic theories, such as category theory [72], the theory of fields, the theory of local rings, lattice theory [177], projective and affine geometry [133,177], the theory of separably closed local rings (aka "strictly Henselian local rings") [98, 133, 194].

Although wide, the class of coherent theories leaves out certain axioms used in algebra—such as the axioms of torsion abelian groups or of Archimedean ordered fields, or in the theory of connected graphs, as well as in the modelling of epistemic social notions such as common knowledge. All the latter examples can however be axiomatised by means of *geometric axioms*: a generalisation of coherent axioms that admits infinitary disjunctions.

Coherent and geometric implications give a Glivenko sequent class [137], as shown by Barr's Theorem:

Theorem 4.1.1 (Barr's Theorem [11]). *If $\mathscr{T}$ is a coherent (geometric) theory and A is a coherent (geometric) sentence provable from $\mathscr{T}$ with (infinitary) classical logic, then A is provable from $\mathscr{T}$ with (infinitary) intuitionistic logic.*

Barr's Theorem[1] has its origin, through appropriate completeness results, in the theory of sheaf models, with the following formulation:

Theorem 4.1.2 ([112], Ch.9, Thm.2). *For every Grothendieck topos $\mathscr{E}$ there exists a complete Boolean algebra* **B** *and a surjective geometric morphism* $Sh(\mathbf{B}) \longrightarrow \mathscr{E}$.

Barr's theorem provides an important conservativity result for classical and intuitionistic geometric theories. Orevkov [137] has established some well-known conservativity results of classical logic over intuitionistic and minimal first-order logics with equality. These results generalise the finitary Barr's Theorem by considering further classes of sequent for which conservativity holds. In particular, [137] isolates seven classes of single-succedent sequents—the so-called *Glivenko sequent classes*—defined in terms of the absence of positive or negative occurrence of particular logical symbols (in a first-order language with equality) where classical derivability implies intuitionistic or even minimal derivability. The same paper also shows that these classes are optimal:[2] any class of sequents for which classical derivability implies intuitionisitc derivability is contained in one of these seven classes. The interest of such conservativity results is twofold. First, since proofs in intuitionistic

[1] Barr's theorem is often alleged to achieve more in that it also allows to eliminate uses of the axiom of choice. That such formulations of Barr's theorem should be taken with caution is demonstrated in [146] where *internal* vs. *external* addition of the the axiom of choice is considered and it is shown that the latter preserves conservativity whereas the former does not.

[2]Barr's Theorem corresponds to Orevkov's first class.

logic obtain a computational meaning via the Curry-Howard correspondence, such results identify some classical theories having a computational content. Second, since it may be easier to prove theorems in classical than in intuitionistic or minimal logic and since there are more well-developed automated theorem provers for classical than for sub-classical logics, such results simplify the search for theorems in intuitionistic (and minimal) theories.

Orevkov's results on Glivenko sequent classes have not received much attention despite their usefulness in analysing the computational content of classical theories. One of the main reasons for this is the complexity of Orevkov's [137] proofs. In recent year simpler proofs of conservativity results for some Glivenko sequent classes has been given [93, 121, 169]. An extremely simple and purely logical proof of the first-order Barr's Theorem for coherent theories has been given in [124] by means of **G3**-style sequent calculi: it is shown how to express coherent implications by means of rules that preserve the admissibility of the structural rules of inference. As a consequence, Barr's theorem is proved by simply noticing that a proof in **G3C.G**—i.e. the calculus for classical logic extended with rules expressing geometric implications—is also a proof in the intuitionistic multisuccedent calculus **G3I.G**. A purely logical proof of Barr's Theorem for infinitary geometric theories has been given [131]. This work considers the **G3**-style calculi for classical and intuitionistic infinitary logic $\mathbf{G3[CI]}_\omega$.

This simple and purely logical proof of Barr's Theorem has been extended to geometric theories in [131]. This work considers the **G3**-style calculi for classical and intuitionistic infinitary logic $\mathbf{G3[CI]}_\omega$ (with finite sequents instead of countably infinite sequents) and their extension with rules expressing geometric implications $\mathbf{G3[CI]}_\omega\mathbf{.G}$. To illustrate, the geometric axiom of torsion abelian groups

$$\forall x. \bigvee_{n>0} nx = 0$$

is expressed by the infinitary rule:

$$\frac{\{nx = 0, \Gamma \rightarrow \Delta \mid n > 0\}}{\Gamma \rightarrow \Delta}$$

The main results in [131] are that in $\mathbf{G3[CI]}_\omega.\mathbf{G}$ all rules are height-preserving invertible, the structural rules of weakening and contraction are height-preserving admissible, and cut is admissible. Hence, Barr's Theorem for geometric theories is proved in [131] as it was done in [124] for coherent ones: a proof in $\mathbf{G3C}_\omega.\mathbf{G}$ is also a proof in the intuitionistic multisuccedent calculus $\mathbf{G3I}_\omega.\mathbf{G}$.

The aforementioned proof of first-order Barr's Theorem has further been extended to cover all other first-order Glivenko sequent classes in [130]. In this chapter we extend the purely logical proof of the infinitary Barr's Theorem given in [131] to cover all other infinitary Glivenko sequent classes: for each class we give a purely constructive proof of conservativity of classical infinitary logic and of a class of classical geometric theories over intuitionistic and minimal infinitary logics and geometric theories, respectively.

We observe that the cut-elimination procedure given in Sect. 4.1 of [131] is not constructive. This is an instance of a typical limitation of cut eliminations in infinitary logics [64, 111, 182] since these proofs use the 'natural' (or Hessenberg) commutative sum of ordinals $\alpha \# \beta$:

$$(\omega^{\alpha_m} + \cdots + \omega^{\alpha_0}) \# (\omega^{\beta_n} + \cdots + \omega^{\beta_0}) = (\omega^{\gamma_{m+n+1}} + \cdots + \omega^{\gamma_0})$$

where $\gamma_{m+n+1}, \ldots, \gamma_0$ is a decreasing permutation of $\alpha_m, \ldots, \alpha_0, \beta_n, \ldots, \beta_0$; see [184, 10.1.2B]. The resort to the natural sum is inescapable for proofs using the cut-height—i.e., the sum of the derivation-height of the premisses of cut—as inductive parameter: it ensures that we can apply the inductive hypothesis when permuting the cut upwards in the derivation of one of the premisses. Nevertheless, it makes the proof non-constructive since

> [its] definition utilises the Cantor normal form of ordinals to base ω. This normal form is not available in **CZF**

(or **IZF**) and thus a different approach is called for. [146, p. 369]

This makes the conservativity results about infinitary Glivenko classes less appealing from the perspective of constructivists: cut is necessary to prove completeness of geometric theories—since they are axiomatised via geometric rules—and a non-constructive proof of cut elimination implies that we are working in a classical meta-theory.

To overcome this drawback we constructivise[3] the proof of (height-preserving) admissibility of the structural rules for $\mathbf{G3[CIM]}_\omega.\mathbf{G}$ by giving procedures that avoid completely the need for ordinal numbers: transfinite inductions on (sums of) ordinals are replaced by inductions on well-founded trees and by Brouwer's principle of Bar Induction—see Theorem 4.6.7.[4] In particular, we capture the fact that a derivation $\mathscr{D}_1$ is "smaller" than $\mathscr{D}_2$ if each branch of $\mathscr{D}_1$ is "smaller" than a branch of $\mathscr{D}_2$ by introducing a new well-founded inductive parameter called *proof embeddability*. This allows us to compare derivations without explicitly giving them an height, and thus to replace the transfinite inductions on the height of derivations used in [66, 131] with well-founded inductions on this new parameter. This will allow us to give an ordinal-free proof of invertibility and of the admissibility of the structural rules of weakening and contraction.[5] Next, we build on these results to give a constructive and ordinal free proof of cut-elimination for geometric logics. In order to do so we replace the Dragalin-style proof adopted in [131] with a (modification of a) proof strategy introduced in [118] for fuzzy logics. This strategy replaces the induc-

[3]By "constructive" here we mean not relying on classical logical principles such as excluded middle or linearity of ordinals but we do not mean acceptable in all schools of constructive mathematics.

[4]See [146, §7] for a different proof, based on constructive ordinals, of cut elimination in infinitary logic. The proof in [8] does not use ordinals, but it is inherently classical in that it uses a one-sided calculus based on De Morgan's dualities.

[5]Even if all proofs in [66] make no use of non-constructive assumptions about ordinals, we prefer to avoid completely the assumption of total ordering.

tion on the natural sum of the heights of the derivations of the two premisses of Cut with two separate well-founded inductions with proof embeddability on the derivation of the right and left premiss, respectively. Finally, we use an instances of Brouwer's Bar Induction to prove that an uppermost instance of Cut is admissible and than another instance to prove that all instances of Cut are admissible. Bar Induction is needed to avoid considering a Cut of maximal rank as in [118]—since this would need the trichotomy of ordinals—and, hence, to obtain a constructive and ordinal-free proof of the admissibility of Cut in $\mathbf{G3[CIM]}_\omega.\mathbf{G}$.

The chapter is organised as follows. Sections 4.2 and 4.3 introduce sequent calculi for infinitary logics and for geometric theories, respectively. Next, Section 4.4 introduces the notion of proof-embeddability and Section 4.5 proves that all rules of $\mathbf{G3[CIM]}_\omega.\mathbf{G}$ are proof embeddable invertible and that the structural rules of weakening and contraction are proof embeddable admissible. Building on these results Section 4.6 presents an ordinal-free and constructive proof of the admissibility of Cut. Finally, Section 4.7 proves conservativity results of classical logic (theories) over intuitionistic and minimal logics (theories) for the infinitary Glivenko sequent classes.

4.2 Syntax and sequent calculi for infinitary logics

Let $\mathscr{S}$ be a signature containing, for every $n \in \mathbb{N}$, a countable (i.e., finite, possibly empty, or countably infinite) set $REL_n^{\mathscr{S}}$ of n-ary predicate letters $P_1^n, P_2^n, \ldots$, and a countable set CON of individual constants $c_1, c_2, \ldots$. Let VAR be a denumerable set of variables $x_1, x_2, \ldots$. The language contains the following logical symbols: $=, \top, \bot, \wedge, \vee, \supset, \forall, \exists$, as well as countable conjunction $\bigwedge_{n>0}$ and countable disjunction $\bigvee_{n>0}$.

The sets TER of the of *terms* is the union of VAR and CON. The set of *formulae* of the language $\mathscr{L}_\omega^{\mathscr{S}}$ is generated by:

$$A ::= P_i^n t_1, \ldots, t_n \mid t_1 = t_2 \mid \top \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \forall x A \mid \exists x A$$
$$\mid \bigwedge_{n>0} A_n \mid \bigvee_{n>0} A_n$$

where $t_i \in TER$, $P_i^n \in REL_n^{\mathscr{S}}$, and $x \in VAR$.
We use the following metavariables:

- x, y, z for variables and $\vec{x}, \vec{y}, \vec{z}$ for lists thereof;
- t, s, r for terms;
- P, Q, R for atomic formulae;
- A, B, C for formulae.

We use $A(\vec{x})$ to say that the variables having free occurrences in A are included in $\vec{x}$. We follow the standard conventions for parentheses. The formulae $\top$, $\neg A$ and $A \supset\subset B$ are defined as expected. When considering (infinitary) classical logic we can shrink the set of primitive logical symbols by means of the well-known De Morgan's dualities (including $\bigvee_{n>0} A_n \supset\subset \neg \bigwedge_{n>0} \neg A$), however also in the classical case we consider a language where all operators (excluding $\neg$ and $\supset\subset$) are taken as primitive. This is not just useful but even necessary since our purpose is to extract the constructive content of classical proofs and many of the interdefinabilities do not hold in intuitionistic logic.

The notions of *free* and *bound occurrences* of a variable in a formula are the usual ones. We posit that no formula may have infinitely many free variables. A *sentence* is a formula without free occurrences of variables. Given a formula A, we use $A[t/x]$ to denote the formula obtained by replacing each free occurrence of x in A with an occurrence of t, provided that t is free for x in A—i.e., no new occurrence of t is bound by a quantifier.

Sequents $\Gamma \to \Delta$ have a finite multiset of formulae on each side. The inference rules for $\bigvee_{n>0}$ are thus:

$$\frac{\{\Gamma, A_n \to \Delta \mid n > 0\}}{\Gamma, \bigvee_{n>0} A_n \to \Delta} \text{ L}\bigvee \qquad \frac{\Gamma \to \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \to \Delta, \bigvee_{n>0} A_n} \text{ R}\bigvee.$$

Observe that $L\bigvee$ has countably many premisses, one for each $n > 0$. The rules for $\bigwedge_{n>0}$ are dual to the above ones.

Derivations built using these rules are thus (in general) infinite trees, with countable branching but where (as may be proved by induction on the definition of derivation) each branch has finite length. The *leaves* of the trees are those where the two sides have an atomic formula in common, and also instances of rules $L\bot$, $R\top$. To make this precise, we give a formal definition of the notion of *derivation* $\mathscr{D}$ and its *end-sequent*.

Definition 4.2.1 (Derivations and their end-sequent)**.**

(i) Any sequent $\Gamma \to \Delta$, where some atomic formula occurs in both Γ and Δ, is a derivation with *end-sequent* $\Gamma \to \Delta$.

In minimal logic, any sequent $\bot, \Gamma \to \Delta, \bot$, is a derivation with *end-sequent* $\bot, \Gamma \to \Delta, \bot$.

(ii) Let $\beta \leqslant \omega$. If each $\mathscr{D}_n$, for $0 < n < \beta$, is a derivation with end-sequent $\Gamma_n \to \Delta_n$ and

$$\frac{\ldots \quad \Gamma_n \to \Delta_n \quad \ldots}{\Gamma \to \Delta}\ R$$

is an instance of a rule with β premisses, then

$$\frac{\ldots \quad \mathscr{D}_n \begin{cases} \quad \vdots \\ \Gamma_n \to \Delta_n \end{cases} \quad \ldots}{\Gamma \to \Delta}\ R$$

is a derivation with *end-sequent* $\Gamma \to \Delta$.[6] If **X** is a calculus, we use $\mathbf{X} \vdash \Gamma \to \Delta$ to say that $\Gamma \to \Delta$ is derivable in the calculus **X**.

[6]Derivations can thus be represented as (infinite) trees, where the nodes are the sequents in the derivation, and a nodes that corresponds to a premiss of a rule is an immediate successor of the node that corresponds to the conclusion of such rule. Therefore, a node that corresponds to the conclusion of a rule with β premisses has β immediate successors.

Derivations and formulae can be associated with ordinals, but we don't need this association here and actually depart from the ordinal approach for the reasons explained above. For the definition of ordinal height of a derivation and ordinal depth of a formula in infinitary logic we refer the reader to [131].

Definition 4.2.2 (Sequent calculi for infinitary logics with equality)**.**

(i) $\mathbf{G3C}_\omega$ is defined by the rules in Table 4.1;

(ii) $\mathbf{G3I}_\omega$ is defined as $\mathbf{G3C}_\omega$ with the exception of rules L$\supset$, R$\supset$, R$\forall$, and R$\bigwedge$ that are defined as in Table 4.2.

By $\mathbf{G3[CI]}_\omega$ we denote any one of the two calculi above. Observe that a multi-succedent intuitionistic calculus as the one we use is closer to a classical calculus than the usual calculus with the restriction that the succedent of sequents should consist of at most one formula (used, for example in [146]). As in the finitary case such a multi-succedent choice is particularly useful for proving Glivenko-style results [130].

As usual, we consider only derivations of *pure sequents*—i.e., sequents where no variable has both free and bound occurrences. We say that $\Gamma \to \Delta$ is $\mathbf{G3[CI]}_\omega$-*derivable*, and we write $\mathbf{G3[CI]}_\omega \vdash \Gamma \to \Delta$, if there is a $\mathbf{G3[CI]}_\omega$-derivation of $\Gamma \to \Delta$ or of an alphabetic variant of $\Gamma \to \Delta$. A rule is said to be *admissible* in $\mathbf{G3[CI]}_\omega$, if, whenever its premisses are $\mathbf{G3[CI]}_\omega$-derivable, also its conclusion is $\mathbf{G3[CI]}_\omega$-derivable. A rule is said to be *invertible* in $\mathbf{G3[CI]}_\omega$, if, whenever its conclusion is $\mathbf{G3[CI]}_\omega$-derivable, also its premisses are $\mathbf{G3[CI]}_\omega$-derivable. In each rule depicted in Tables 4.1, 4.2, and 4.3 the multisets Γ and Δ are called *contexts*, the formulae occurring in the conclusion are called *principal*, and the formulae occurring in the premiss(es) only are called *active*.

Initial sequent

$P, \Gamma \to \Delta, P$

Propositional rules

$$\frac{}{\bot, \Gamma \to \Delta}\ \mathrm{L}\bot \qquad \frac{}{\Gamma \to \Delta, \top}\ \mathrm{R}\top$$

$$\frac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}\ \mathrm{L}\wedge \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}\ \mathrm{R}\wedge$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\ \mathrm{L}\vee \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\ \mathrm{R}\vee$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\ \mathrm{L}\supset \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\ \mathrm{R}\supset$$

Rules for quantifiers

$$\frac{A[y/x], \forall x A, \Gamma \to \Delta}{\forall x A, \Gamma \to \Delta}\ \mathrm{L}\forall \qquad \frac{\Gamma \to \Delta, A[z/x]}{\Gamma \to \Delta, \forall x A}\ \mathrm{R}\forall \quad (y \text{ fresh})$$

$$\frac{A[z/x], \Gamma \to \Delta}{\exists x A, \Gamma \to \Delta}\ \mathrm{L}\exists \quad (y \text{ fresh}) \qquad \frac{\Gamma \to \Delta, A[y/x], \exists x A}{\Gamma \to \Delta, \exists x A}\ \mathrm{R}\exists$$

Infinitary rules

$$\frac{A_k, \bigwedge A_n, \Gamma \to \Delta}{\bigwedge A_n, \Gamma \to \Delta}\ \mathrm{L}\bigwedge \qquad \frac{\{\Gamma \to \Delta, A_i \mid i > 0\}}{\Gamma \to \Delta, \bigwedge A_n}\ \mathrm{R}\bigwedge$$

$$\frac{\{A_i, \Gamma \to \Delta \mid i > 0\}}{\bigvee A_n, \Gamma \to \Delta}\ \mathrm{L}\bigvee \qquad \frac{\Gamma \to \Delta, \bigvee A_n, A_k}{\Gamma \to \Delta, \bigvee A_n}\ \mathrm{R}\bigvee$$

Rules for equality

$$\frac{s = s, \Gamma \to \Delta}{\Gamma \to \Delta}\ \mathrm{Ref} \qquad \frac{P[t/x], s = t, P[s/x], \Gamma \to \Delta}{s = t, P[s/x], \Gamma \to \Delta}\ \mathrm{Repl}$$

Table 4.1: The calculus $\mathbf{G3C}_\omega$.

Initial sequent	As in $\mathbf{G3C}_\omega$.
Rules	As in $\mathbf{G3C}_\omega$, except for the following:

$$\frac{A \supset B, \Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\ \mathrm{L}\supset \qquad \frac{A, \Gamma \to B}{\Gamma \to \Delta, A \supset B}\ \mathrm{R}\supset$$

$$\frac{\Gamma \to A[z/x]}{\Gamma \to \Delta, \forall x A}\ \mathrm{R}\forall \qquad \frac{\{\Gamma \to A_i \mid i > 0\}}{\Gamma \to \Delta, \bigwedge A_n}\ \mathrm{R}\bigwedge$$

Table 4.2: The calculus $\mathbf{G3I}_\omega$.

Initial sequent	As in $\mathbf{G3I}_\omega$, plus
	$\bot, \Gamma \to \Delta, \bot$
Rules	As in $\mathbf{G3I}_\omega$, except for L$\bot$.

Table 4.3: The calculus $\mathbf{G3M}_\omega$.

$$\frac{\ldots \qquad Q_{n_1}(\vec{x}, \vec{y}_n), \ldots, Q_{n_m}(\vec{x}, \vec{y}_n), P_1(\vec{x}), \ldots, P_k(\vec{x}), \Gamma \to \Delta \qquad \ldots}{P_1(\vec{x}), \ldots, P_k(\vec{x}), \Gamma \to \Delta}\ \mathrm{L}_G$$

Table 4.4: Geometric rule L_G expressing the geometric sentence (G)

4.3 From geometric implications to geometric rules

By a *geometric implication* we mean the universal closure of an implicative formula whose antecedent and consequent are positive formulae (i.e., formulae constructed from atomic formulae and $\bot, \top$ using only $\wedge$, $\vee$, $\exists$, and $\bigvee_{n>0}$). More precisely

Definition 4.3.1 (Geometric implication)**.**

— A formula is *Horn* iff it is built from atoms and $\top$ using only $\wedge$;

— A formula is *geometric* iff it is built from atoms and $\top$, $\bot$ using only $\wedge$, $\vee$, $\exists$, and $\bigvee_{n>0}$;

— A sentence is a *geometric implication* iff it is of the form $\forall \vec{x}(A \supset B)$ where A and B are geometric formulae.

By a *coherent implication* we mean a geometric implication without occurrences of $\bigvee_{n>0}$.

As is well known, for geometric implications we have a normal form theorem.

Theorem 4.3.2 (Geometric normal form (GNF))**.** *Any geometric implication is equivalent to a possibly infinite conjunction of sentences of the form*

$$\forall \vec{x}(A \supset B)$$

where A is Horn and B is a possibly infinite disjunction of existentially quantified Horn formulae.

This normal form theorem is important because, as shown in [124] for coherent implications and in [131] for geometric ones, we can extract from a sentence G in GNF a *geometric rule* L_G (where the name L_G indicates that it is a *left rule*) that can be added to a sequent calculus without altering its structural properties. To be more precise, let us consider the following sentence G in GNF:

$$\forall\vec{x}(P_1(\vec{x}) \wedge \cdots \wedge P_k(\vec{x}) \supset \bigvee_{n>0} \exists\vec{y}(Q_{n_1}(\vec{x},\vec{y}) \wedge \cdots \wedge Q_{n_m}(\vec{x},\vec{y}))) \qquad (G)$$

Such a sentence G determines the (finitary or infinitary) *geometric rule* given in Table 4.4 with one premiss for each of the countably many disjuncts in $\bigvee_{n>0}(Q_{n_1}(\vec{x},\vec{y}) \wedge \cdots \wedge Q_{n_m}(\vec{x},\vec{y}))$. The variables in $\vec{y}_n$ are chosen to be *fresh*, i.e. are not in the conclusion; and without loss of generality they are all distinct. The list $\vec{y}_n$ of variables may vary as n varies, and maybe no finite list suffices for all the countably many cases. The variables $\vec{x}$ (finite in number) may be instantiated with arbitrary terms. Henceforth we shall normally omit mention of the variables.

We need also a further condition:

Definition 4.3.3 (Closure condition)**.** Given a calculus with geometric rules, if it has a rule with an instance with repetition of some principal formula such as:

$$\frac{\cdots \qquad Q_1,\ldots,Q_n,P_1,\ldots,P_{k-2},P,P,\Gamma \to \Delta \qquad \cdots}{P_1,\ldots,P_{k-2},P,P,\Gamma \to \Delta}\ \mathrm{L}^c_G$$

then also the *contracted instance*

$$\frac{\cdots \qquad Q_1,\ldots,Q_m,P_1,\ldots,P_{k-2},P,\Gamma \to \Delta \qquad \cdots}{P_1,\ldots,P_{k-2},P,\Gamma \to \Delta}\ \mathrm{L}^c_G$$

has to be included in the calculus.

As for the finitary case [124], also in the infinitary case the condition is unproblematic, since each atomic formula contains only a finite number of variables and therefore so are the instances; it follows that, for each geometric rule, the number of rules that have to be added is finite. Moreover, in many cases contracted instances need not be added since they are admissible in the calculus without them.

To illustrate, we consider the coherent rule Repl for equality given in Table 4.1:

$$\frac{P[t/x], s = t, P[s/x], \Gamma \to \Delta}{s = t, P[s/x], \Gamma \to \Delta} \text{ Repl}$$

This rule generates contracted instances only when its two principal formulae (as well as its active formula) are copies of the same reflexivity atom $t = t$. In this case, after having applied contraction, we can replace the instance of Repl with an instance of Ref (instead of Repl^c). That is we can transform:

$$\frac{t = t, t = t, t = t, \Gamma \to \Delta}{t = t, t = t, \Gamma \to \Delta} \text{ Repl} \qquad \text{into} \qquad \cfrac{\cfrac{t = t, t = t, t = t, \Gamma \to \Delta}{t = t, t = t, \Gamma \to \Delta} \text{ LC}}{t = t, \Gamma \to \Delta} \text{ Ref}$$

But this does not hold in general. For example, if $<$ is an Euclidean relation, we must add both of the following rules:

$$\frac{s < r, t < s, t < r, \Gamma \to \Delta}{t < s, t < r, \Gamma \to \Delta} \text{ Euc} \qquad \text{and} \qquad \frac{s < s, t < s, \Gamma \to \Delta}{t < s, \Gamma \to \Delta} \text{ Euc}^c$$

otherwise the valid sequent $t < s \to s < s$ would not be contraction-free derivable. In presence of Ref, no added rule is needed.

Theorem 4.3.4 ([131])**.** *If we add to the calculus* $\mathbf{G3[CI]}_\omega$ *a finite or infinite family of geometric rules* L_G*, then we can prove all of the geometric sentences* G *from which they were determined.*

In the following, we shall denote with $\mathbf{G3[CI]}_\omega.\mathbf{G}$ any extension of $\mathbf{G3[CI]}_\omega$ with a finite or infinite family of geometric rules L_G (together with all needed contracted instances thereof).

Before proceeding with the structural properties, we give some examples of geometric axioms and their corresponding rules.

Example 4.3.5 (Geometric axioms and rules)**.**

(i) The axiom of **torsion Abelian groups**, $\forall x. \bigvee_{n>1}(nx = 0)$, becomes the rule

$$\frac{\dots \quad nx = 0, \Gamma \to \Delta \quad \dots}{\Gamma \to \Delta} \text{ R}_{Tor}$$

(ii) The axiom of **Archimedean ordered fields**, $\forall x. \bigvee_{n \geqslant 1}(x < n)$, becomes the rule

$$\frac{\dots \quad x < n, \Gamma \to \Delta \quad \dots}{\Gamma \to \Delta} \; \mathrm{R}_{Arc}$$

(iii) The axiom of **connected graphs**,

$$\forall xy.x = y \vee \bigvee_{n \geqslant 1} \exists z_0 \dots \exists z_n (x = z_0 \wedge y = z_n \wedge z_0 R z_1 \wedge \cdots \wedge z_{n-1} R z_n)$$

becomes the rule

$$\frac{x = y, \Gamma \to \Delta \quad xRy, \Gamma \to \Delta \quad \dots \quad x = z_0, y = z_n, z_0 R z_1, \dots, z_{n-1} R z_n, \Gamma \to \Delta \quad \dots}{\Gamma \to \Delta} \; \mathrm{R}_{Conn}$$

4.4 Embeddable derivation

The proofs given in [66, 131] make use of transfinite inductions on the height of derivations, which are quite powerful tools. We claim, however, that they are in a certain sense too powerful: they are often non-constructive and, as it will be shown, can be avoided.

Usually, in order to compare two derivations, one assigns ordinal numbers, called *heights*, to them, then compares these parameters. As heights are inductively defined by means of the branches of the derivation, this becomes a comparison between branches. Our main observation is that, in order to compare two derivations, what we actually need is just the fact that $\mathscr{D}$ is "smaller" than $\mathscr{D}'$ if each branch of $\mathscr{D}$ is "smaller" than a branch of $\mathscr{D}'$, without explicitly "measuring" them.

We make this precise by inductively defining simultaneously the relations $\prec$ and $\preccurlyeq$ between derivations. We read $\mathscr{D} \preccurlyeq \mathscr{D}'$ as "$\mathscr{D}$ is embeddable in $\mathscr{D}'$" and $\mathscr{D} \prec \mathscr{D}'$ as "$\mathscr{D}$ is strictly embeddable in $\mathscr{D}'$".

In what follows, we say that a derivation $\mathscr{D}$ is

— *trivial* if it is an initial or empty sequent;

— *composite*, or *nontrivial*, if has the following form:

$$\mathscr{D}\left\{\frac{\{\mathscr{D}_i\}}{\Gamma \to \Delta}\right.$$

It is decidable whether $\mathscr{D}$ is trivial or composite, and the two properties are mutually exclusive.

Definition 4.4.1 (proof-embeddability)**.**

(i) If $\mathscr{D}$ and $\mathscr{D}'$ are trivial, then $\mathscr{D} \preccurlyeq \mathscr{D}'$.

(ii) If

$$\mathscr{D}\left\{\frac{\{\mathscr{D}_i\}}{\Gamma \to \Delta}\right. \qquad \text{and} \qquad \mathscr{D}'\left\{\frac{\{\mathscr{D}'_j\}}{\Gamma' \to \Delta'}\right.$$

and for each $\mathscr{D}_i$ there is $\mathscr{D}'_j$ such that $\mathscr{D}_i \preccurlyeq \mathscr{D}'_j$, then $\mathscr{D} \preccurlyeq \mathscr{D}'$.[7]

(iii) If $\mathscr{D} \preccurlyeq \mathscr{D}'$ and

$$\mathscr{D}''\left\{\frac{\dots \quad \mathscr{D}' \quad \dots}{\Gamma \to \Delta}\right.$$

then $\mathscr{D} \prec \mathscr{D}''$.

(iv) If $\mathscr{D} \prec \mathscr{D}'$ then $\mathscr{D} \preccurlyeq \mathscr{D}'$.

This is a compact but unusual way to do parallel inductive definitions. An equivalent, more standard way to do this is to first define $\preccurlyeq$ by taking clauses (i)–(iii), where in the latter $\prec$ is replaced by $\preccurlyeq$, and then to define $\prec$ by taking clause (iii) alone. In this way, clause (iv) becomes automatic.

Remark 4.4.2.

(i) By definition, $\mathscr{D} \prec \mathscr{D}'$ implies $\mathscr{D} \preccurlyeq \mathscr{D}'$.

[7]One may be mislead here by assuming that the correspondence between branches implies that the two derivations have the same structure. However, this is not the case as the correspondence is not required to be injective nor surjective.

(ii) Note that, in general, $\mathscr{D} \prec \mathscr{D}'$ is not the same as the conjunction of $\mathscr{D} \preccurlyeq \mathscr{D}'$ and $\mathscr{D} \neq \mathscr{D}'$; and similarly $\mathscr{D} \preccurlyeq \mathscr{D}'$ is not the same as the disjunction of $\mathscr{D} \prec \mathscr{D}$ and $\mathscr{D} = \mathscr{D}'$. However, it can be shown that $\mathscr{D} \prec \mathscr{D}'$ if and only if $\mathscr{D} \preccurlyeq \mathscr{D}'$ and $\mathscr{D}' \not\preccurlyeq \mathscr{D}$.

Lemma 4.4.3. *Let $\mathscr{D}$ be a trivial derivation.*

(i) $\mathscr{D}' \preccurlyeq \mathscr{D}$ if and only if $\mathscr{D}'$ is trivial, and there is no $\mathscr{D}''$ such that $\mathscr{D}'' \prec \mathscr{D}$.

(ii) $\mathscr{D} \preccurlyeq \mathscr{D}'$ for every $\mathscr{D}'$, and $\mathscr{D} \prec \mathscr{D}''$ for every nontrivial $\mathscr{D}''$.

Proof. Straightforward. ■

Lemma 4.4.4. *The relation $\preccurlyeq$ is a (non-strict) preorder, i.e. it is reflexive and transitive.*

Proof. Reflexivity: Take a derivation $\mathscr{D}$. We prove that

$$\mathscr{D} \preccurlyeq \mathscr{D} \tag{4.1}$$

by structural induction on $\mathscr{D}$. If $\mathscr{D}$ is trivial, then $\mathscr{D} \preccurlyeq \mathscr{D}$ by clause (i) of the definition. If

$$\mathscr{D}\left\{\frac{\{\mathscr{D}_i\}}{\Gamma \to \Delta}\right.$$

with each $\mathscr{D}_i$ satisfying (4.1), then $\mathscr{D} \preccurlyeq \mathscr{D}$ by clause (ii) of the definition.

Transitivity: Take a derivation $\mathscr{D}$. We prove that

$$\forall \mathscr{D}' \forall \mathscr{D}''.(\mathscr{D} \preccurlyeq \mathscr{D}' \,\&\, \mathscr{D}' \preccurlyeq \mathscr{D}'') \Rightarrow \mathscr{D} \preccurlyeq \mathscr{D}'' \tag{4.2}$$

by structural induction on $\mathscr{D}$. If $\mathscr{D}$ is trivial, see Lemma 4.4.3. Suppose that

$$\mathscr{D}\left\{\frac{\{\mathscr{D}_i\}}{\Gamma \to \Delta}\right.$$

with each $\mathscr{D}_i$ satisfying (4.2). Consider $\mathscr{D}', \mathscr{D}''$ such that $\mathscr{D} \preccurlyeq \mathscr{D}'$ and $\mathscr{D}' \preccurlyeq \mathscr{D}''$. By Lemma 4.4.3, since $\mathscr{D}$ is composite, then so must be $\mathscr{D}'$, and similarly since $\mathscr{D}'$ is composite, then so must be $\mathscr{D}''$:

$$\mathscr{D}'\left\{\frac{\{\mathscr{D}'_j\}}{\Gamma' \to \Delta'}\right. \qquad \text{and} \qquad \mathscr{D}''\left\{\frac{\{\mathscr{D}''_k\}}{\Gamma'' \to \Delta''}\right.$$

For every $\mathscr{D}_{i^*}$, we show that there is a (finite) chain

$$\mathscr{D}_{i^*} \preccurlyeq \ldots \preccurlyeq \mathscr{D}''_{k^*}$$

for some $\mathscr{D}''_{k^*}$. We do a proof by cases, depending on whether $\mathscr{D} \preccurlyeq \mathscr{D}'$ and $\mathscr{D}' \preccurlyeq \mathscr{D}''$ are witnessed by clause (ii) or (iii):

— Suppose that both $\mathscr{D} \preccurlyeq \mathscr{D}'$ and $\mathscr{D}' \preccurlyeq \mathscr{D}''$ are witnessed by clause (ii). In particular, there is $\mathscr{D}'_{j^*}$ such that $\mathscr{D}_{i^*} \preccurlyeq \mathscr{D}'_{j^*}$, for which in turn there is $\mathscr{D}''_{k^*}$ such that

$$\mathscr{D}_{i^*} \preccurlyeq \mathscr{D}'_{j^*} \preccurlyeq \mathscr{D}''_{k^*}.$$

— Suppose that $\mathscr{D} \preccurlyeq \mathscr{D}'$ is witnessed by clause (iii) and that $\mathscr{D}' \preccurlyeq \mathscr{D}''$ is witnessed by clause (ii). This means that there is $\mathscr{D}_{j^*}$ such that $\mathscr{D} \preccurlyeq \mathscr{D}'_{j^*}$ and for each $\mathscr{D}'_j$ there is $\mathscr{D}''_k$ such that $\mathscr{D}'_j \preccurlyeq \mathscr{D}''_k$. In particular, there is $\mathscr{D}''_{k^*}$ such that

$$\mathscr{D}_{i^*} \preccurlyeq \mathscr{D} \preccurlyeq \mathscr{D}'_{j^*} \preccurlyeq \mathscr{D}''_{k^*},$$

where $\mathscr{D}_{i^*} \preccurlyeq \mathscr{D}$ because of reflexivity and clause (iii).

— Suppose that $\mathscr{D}' \preccurlyeq \mathscr{D}''$ is witnessed by clause (iii). This means that there is $\mathscr{D}''_{k^*}$ such that $\mathscr{D}' \preccurlyeq \mathscr{D}''_{k^*}$. It follows that

$$\mathscr{D}_{i^*} \preccurlyeq \mathscr{D} \preccurlyeq \mathscr{D}' \preccurlyeq \mathscr{D}''_{k^*},$$

where $\mathscr{D}_{i^*} \preccurlyeq \mathscr{D}$ because of reflexivity and clause (iii).

We apply (4.2) to the chain, possibly multiple times, and get $\mathscr{D}_{i^*} \preccurlyeq \mathscr{D}''_{k^*}$. We can now apply clause (ii) to conclude that $\mathscr{D} \preccurlyeq \mathscr{D}''$. ■

Lemma 4.4.5.

(i) If $\mathscr{D} \preccurlyeq \mathscr{D}'$ *and* $\mathscr{D}' \prec \mathscr{D}''$*, then* $\mathscr{D} \prec \mathscr{D}''$*.*

(ii) If $\mathscr{D} \prec \mathscr{D}'$ *and* $\mathscr{D}' \prec \mathscr{D}''$*, then* $\mathscr{D} \prec \mathscr{D}''$*.*

(iii) If $\mathscr{D} \prec \mathscr{D}'$ *and* $\mathscr{D}' \preccurlyeq \mathscr{D}''$*, then* $\mathscr{D} \prec \mathscr{D}''$*.*

Proof.

(i) By definition of $\prec$, we have $\mathscr{D}^*$ such that $\mathscr{D}' \preccurlyeq \mathscr{D}^*$ and

$$\mathscr{D}''\left\{\frac{\cdots \quad \mathscr{D}^* \quad \cdots}{\Gamma \to \Delta}\right.$$

By transitivity of $\preccurlyeq$ (Lemma 4.4.4), we have that $\mathscr{D} \preccurlyeq \mathscr{D}^*$. We conclude that $\mathscr{D} \prec \mathscr{D}''$ by clause (iii) of the definition.

(ii) If $\mathscr{D} \prec \mathscr{D}'$, then in particular $\mathscr{D} \preccurlyeq \mathscr{D}'$, so the claim follows from (i).

(iii) We do a proof by cases, depending on whether $\mathscr{D}' \preccurlyeq \mathscr{D}''$ is witnessed by clause (i), (ii) or (iv), whereas clause (iii) does not apply:

— If it is witnessed by clause (i), i.e. $\mathscr{D}'$ and $\mathscr{D}''$ are trivial, then there is no such $\mathscr{D}$, and the claim is vacuously satisfied.

— Suppose that it is witnessed by clause (ii), i.e.

$$\mathscr{D}'\left\{\frac{\{\mathscr{D}'_j\}}{\Gamma' \to \Delta'}\right. \qquad \text{and} \qquad \mathscr{D}''\left\{\frac{\{\mathscr{D}''_k\}}{\Gamma'' \to \Delta''}\right.$$

and for each $\mathscr{D}'_j$ there is $\mathscr{D}''_k$ such that $\mathscr{D}'_j \preccurlyeq \mathscr{D}''_k$. By definition of $\mathscr{D} \prec \mathscr{D}'$, we have $\mathscr{D} \preccurlyeq \mathscr{D}'_{j^*}$ for some $\mathscr{D}'_{j^*}$, hence by transitivity $\mathscr{D} \preccurlyeq \mathscr{D}''_{k^*}$ for the corresponding $\mathscr{D}''_{k^*}$. It follows that $\mathscr{D} \prec \mathscr{D}''$.

— If it is witnessed by clause (iv), i.e. $\mathscr{D}' \prec \mathscr{D}''$, then $\mathscr{D} \prec \mathscr{D}''$ follows from (ii). ∎

We say that a property E of derivations is *progressive*, if

$$\forall \mathscr{D}.(\forall \mathscr{D}' \prec \mathscr{D}(E\mathscr{D}')) \Rightarrow E\mathscr{D}.$$

Theorem 4.4.6. *Strict proof embeddability $\prec$ satisfies Noetherian induction, i.e. it satisfies $\forall \mathscr{D}(E\mathscr{D})$ for every progressive property E.*

Proof. Consider a progressive property E. It is enough to show that

$$\forall \mathscr{D}' \prec \mathscr{D}(E\mathscr{D}') \tag{4.3}$$

for every derivation $\mathscr{D}$. We proceed by structural induction on $\mathscr{D}$. If $\mathscr{D}$ is trivial, then it has no predecessors (Lemma 4.4.3) and the claim holds. Suppose that

$$\mathscr{D}\left\{\frac{\{\mathscr{D}_i\}}{\Gamma \to \Delta}\right.$$

with each $\mathscr{D}_i$ satisfying (4.3). Consider $\mathscr{D}' \prec \mathscr{D}$. By definition, $\mathscr{D}' \preccurlyeq \mathscr{D}_{i^*}$ for some $\mathscr{D}_{i^*}$. We claim that $E\mathscr{D}''$ for each $\mathscr{D}'' \prec \mathscr{D}'$. In fact, given any such $\mathscr{D}''$, by Lemma 4.4.5 we have that $\mathscr{D}'' \prec \mathscr{D}_{i^*}$. The claim follows by the fact that $\mathscr{D}_{i^*}$ satisfies (4.3). Since E is progressive, we get $E\mathscr{D}'$. ∎

Corollary 4.4.7. *The relation $\prec$ is a strict partial order, i.e. it is irreflexive and transitive.*

Proof. Transitivity is Lemma 4.4.5(ii), while irreflexivity follows from Noetherian induction (see e.g. Lemma 2.4.1). ∎

Given a calculus **G**, by $\mathbf{G} \vdash^{\mathscr{D}} \Gamma \to \Delta$ we mean that there is a derivation $\mathscr{D}$ with end-sequent $\Gamma \to \Delta$ in calculus **G**.

We say that a rule

$$\frac{\Gamma \to \Delta}{\Gamma' \to \Delta'}$$

is *proof embeddable* admissible (for short pe-admissible) if for each derivation $\mathscr{D}$ of $\Gamma \to \Delta$ there is a derivation $\mathscr{D}'$ of $\Gamma' \to \Delta'$ such that $\mathscr{D}' \preccurlyeq \mathscr{D}$.

The notion of pe-admissibility is used in place of hp-admissibility for the calculi $\mathbf{G3[CI]}_\omega.\mathbf{G}$, and is studied in the following sections.

4.5 Structural rules

We present here the results concerning the admissibility of the structural rules, cut excluded, in the calculi $\mathbf{G3[CI]_{\omega}.G}$. All these results have been proved in Sect. 4 of [131] by simple transfinite induction on ordinals, either on the depth of a formula or on the height of a derivation, here replaced by proof-embeddability both in the statement of the results and in their proofs.

Lemma 4.5.1 (α-conversion)**.** *If* $\mathbf{G3[CI]_{\omega}.G} \vdash^{\mathscr{D}_1} \Gamma \to \Delta$ *then*

$$\mathbf{G3[CI]_{\omega}.G} \vdash^{\mathscr{D}_2} \Gamma' \to \Delta'$$

with $\mathscr{D}_2 \preccurlyeq \mathscr{D}_1$*, for* $\Gamma' \to \Delta'$ *a bound alphabetic variant of* $\Gamma \to \Delta$*.*

Proof. Similar to the proof of hp-α-conversion in [169]. ■

Lemma 4.5.2 (Substitution)**.** *If* $\mathbf{G3[CIM]_{\omega}.G} \vdash^{\mathscr{D}_1} \Gamma \to \Delta$ *then*

$$\mathbf{G3[CIM]_{\omega}.G} \vdash^{\mathscr{D}_2} \Gamma[t/x] \to \Delta[t/x]$$

(for t free for x in Γ, Δ*) with* $\mathscr{D}_2 \preccurlyeq \mathscr{D}_1$*.*

Proof. Similar to the proof of hp-substitution in [169]. ■

Theorem 4.5.3 (Weakening)**.** *The left and right rules of weakening:*

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}\ LW \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}\ RW$$

are pe-admissible in $\mathbf{G3[CIM]_{\omega}.G}$*.*

Proof. Similar to the proof of hp-weakening in [169]. ■

Lemma 4.5.4 (Invertibility)**.**

(i) Each rule of $\mathbf{G3C_{\omega}.G}$ *is pe-invertible.*

(ii) Each rule of $\mathbf{G3[IM]_{\omega}.G}$ *except* $R\supset$*,* $R\forall$*, and* $R\bigwedge$ *is pe-invertible.*

Proof. The proof for rules L$\forall$, R$\exists$, L$\bigwedge$ and R$\bigvee$ follows from Theorem 4.5.3. For the other rules we proceed by well-founded induction with proof-embeddability.

We consider the case of L$\bigvee$, i.e. a sequent $\bigvee_{n>0} A_n, \Gamma \to \Delta$. If it is an initial sequent, then each $A_n, \Gamma \to \Delta$ is also an initial sequent. If it is an instance of L$\bot$, then there's nothing to prove. Let us consider the last (proper) rule and distinguish the case in which $\bigvee_{n>0} A_n$ is a side formula and the case in which it is the principal formula. In the former case the last rule can have one, two or denumerably many premisses. The derivation $\mathscr{D}$ has the form

$$\mathscr{D}_m \left\{ \frac{\begin{array}{c} \vdots \\ \{\bigvee_{n>0} A_n, \Gamma_m \to \Delta_m \mid m \in I\} \end{array}}{\bigvee_{n>0} A_n, \Gamma \to \Delta} \, rule \right.$$

where I is either $\{1\}, \{1,2\}$ or $\mathbb{N}$. Clearly $\mathscr{D}_m \prec \mathscr{D}$ for each m. By inductive hypothesis, we have derivations $\mathscr{D}_{mn} \preccurlyeq \mathscr{D}_m$ of $A_n, \Gamma_m \to \Delta_m$. Then we get derivations

$$\mathscr{D}_{mn} \left\{ \frac{\begin{array}{c} \vdots \\ \{A_n, \Gamma_m \to \Delta_m \mid m \in I\} \end{array}}{A_n, \Gamma \to \Delta} \, rule \right.$$

which are embeddable in $\mathscr{D}$. If instead $\bigwedge_{n>0} A_n$ is principal, the derivation $\mathscr{D}$ has the form

$$\mathscr{D}_n \left\{ \frac{\begin{array}{c} \vdots \\ \{A_n, \Gamma \to \Delta \mid n > 0\} \end{array}}{\bigvee_{n>0} A_n, \Gamma \to \Delta} \, \mathrm{L}\bigvee \right.$$

and we just need to observe that $\mathscr{D}_n \preccurlyeq \mathscr{D}$.

The proof for other rules is similar. ∎

Theorem 4.5.5 (Contraction)**.** *The left and right rules of contraction:*

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta} \, LC \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A} \, RC$$

are pe-admissible in **G3**[**CIM**]$_\omega$.**G**.

Proof. By simultaneous Noetherian induction with $\prec$ on the left and right contraction rule. Consider the left rule. If it is an initial sequent, then the conclusion is also an initial sequent and is embeddable. If the contraction formula A is not principal in the last rule, we have the derivation $\mathscr{D}$

$$\mathscr{D}_m \left\{ \frac{\begin{array}{c}\vdots \\ \{A, A, \Gamma_m \to \Delta_m \mid m \in I\}\end{array}}{A, A, \Gamma \to \Delta} \, rule \right.$$

where I is either $\{1\}, \{1,2\}$ or $\mathbb{N}$. Clearly $\mathscr{D}_m \prec \mathscr{D}$ for each m. By induction hypothesis we have derivations $\mathscr{D}'_m \preccurlyeq \mathscr{D}_m$ of $A, \Gamma_m \to \Delta_m$. Then the derivation

$$\mathscr{D}'_m \left\{ \frac{\begin{array}{c}\vdots \\ \{A, \Gamma_m \to \Delta_m \mid m \in I\}\end{array}}{A, \Gamma \to \Delta} \, rule \right.$$

is as wanted.

We're left with the case in which the contraction formula is principal in the last rule. Consider the case of $\bigvee_{n>0} A_n$ in L$\bigvee$. We have the derivation $\mathscr{D}$

$$\mathscr{D}_n \left\{ \frac{\begin{array}{c}\vdots \\ \{\bigvee_{n>0} A_n, A_n, \Gamma \to \Delta \mid n > 0\}\end{array}}{\bigvee_{n>0} A_n, \bigvee_{n>0} A_n, \Gamma \to \Delta} \, \mathrm{L}\bigvee \right.$$

where clearly $\mathscr{D}_n \prec \mathscr{D}$. By pe-invertibility of L$\bigvee$ we obtain derivations $\mathscr{D}'_n \preccurlyeq \mathscr{D}_n$ of $A_n, A_n, \Gamma \to \Delta$, and thus $\mathscr{D}'_n \prec \mathscr{D}$ by clause (ii) of proof-embeddability, cf. Def. 4.4.1. By induction hypothesis, we now get derivations $\mathscr{D}''_n \preccurlyeq \mathscr{D}'_n$ of $A_n, \Gamma \to \Delta$. By transitivity,

$\mathscr{D}''_n \preccurlyeq \mathscr{D}_n$. In conclusion, we get the derivation $\mathscr{D}'$

$$\mathscr{D}''_n \left\{ \begin{array}{c} \vdots \\ \{A_n, \Gamma \to \Delta \mid n > 0\} \end{array} \right. \over \bigvee_{n>0} A_n, \Gamma \to \Delta \; \text{L}\bigvee$$

which is embeddable in $\mathscr{D}$. The proof for other invertible rules is similar.

Consider the case of $A \supset B$ principal in intuitionistic R$\supset$. We have the derivation $\mathscr{D}$

$$\mathscr{D}^- \left\{ \begin{array}{c} \vdots \\ A, \Gamma \to B \end{array} \right. \over \Gamma \to \Delta, A \supset B, A \supset B \; \text{R}\supset$$

where clearly $\mathscr{D}^- \prec \mathscr{D}$. We easily get the derivation $\mathscr{D}'$

$$\mathscr{D}^- \left\{ \begin{array}{c} \vdots \\ A, \Gamma \to B \end{array} \right. \over \Gamma \to \Delta, A \supset B \; \text{R}\supset$$

which is embeddable in $\mathscr{D}$. Again, the proof for other non-invertible rules is similar. ■

4.6 Constructive cut-elimination

We are now ready to prove that the following context-sharing rule of cut:

$$\frac{\Gamma \to \Delta, C \quad C, \Gamma \to \Delta}{\Gamma \to \Delta} \text{Cut}$$

is eliminable to the calculus $\mathbf{G3[CI]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ obtained by extending $\mathbf{G3[CI]}_\omega.\mathbf{G}$ with Cut. In order to give a proof of cut elimination that uses only constructively admissible proof-theoretical tools we must avoid the 'natural' (or Hessenberg) commutative sum of ordinals: we cannot use the cut-height as inductive parameter as is done in Gentzen- and Dragalin-style proofs. In order to avoid it, we make use of a proof strategy introduced in [118] for fuzzy logics that has been extensively used in the context of hypersequent calculi; see [31, 92, 105]. This proof strategy can be seen as a simplified and local version of the proof given by H.B. Curry in [52]. The proof is based on two main lemmata (Lemmata 4.6.4 and 4.6.5 below) that are proved by induction on the derivation of the right and of the left premiss of cut, respectively. Moreover, (almost) all non-principal instances of cut are taken care by separate lemmata (Lemma 4.6.2 and 4.6.3) which shows that Cut can be permuted upwards with respect to rule instances not having the cut formula among their principal formulae.

Observe that, differently from [31, 105, 118], we will not consider an arbitrary instance of Cut of maximal rank (i.e., such that its cut formula has maximal depth among the cut formulae occurring in the derivation), but we will always consider an uppermost instance of Cut, i.e. a cut the premisses of which are cut-free derivations. Otherwise, in Lemmata 4.6.4 and 4.6.5 as well as in Theorem 4.6.7, we would have to assume that ordinals are linearly/totally ordered; but in a constructive setting this assumption implies the law of excluded middle [6]. In Theorem 4.6.7 we will proceed, instead, by using two instances of Brouwer's principle of Bar Induction, the first one will be used to show that an uppermost instance of Cut is eliminable and the second to show that all instances of Cut are eliminable. Note that although it is a constructively admissible principle, Bar Induction increases the proof-theoretic strength of **CFZ**, cf. [146].

Definition 4.6.1 (Cut-substitutive rule)**.** A sequent rule *Rule* is *cut-substitutive* if each instance of cut with cut formula not principal in the last rule instance *Rule* of one of the premisses of cut can be permuted upwards w.r.t. *Rule* as in the following example:

$$\dfrac{\dfrac{A,\Gamma\to\Delta,B,C}{\Gamma\to\Delta,A\supset B,C}\ \mathrm{R}\supset \qquad C,\Gamma\to\Delta,A\supset B}{\Gamma\to\Delta,A\supset B}\ \mathrm{Cut} \quad\rightsquigarrow\quad \dfrac{\dfrac{A,\Gamma\to\Delta,B,C \qquad \dfrac{C,\Gamma\to\Delta,A\supset B}{A,C,\Gamma\to\Delta,B}\ \text{pe-inv}}{A,\Gamma\to\Delta,B}\ \mathrm{Cut}}{\Gamma\to\Delta,A\supset B}\ \mathrm{R}\supset$$

Lemma 4.6.2. *Each rule of* $\mathbf{G3C}_\omega.\mathbf{G}$ *is cut-substitutive.*

Proof. By inspecting the rules in Tables 4.1 it is immediate to realise that each of them is cut-substitutive because they are all pe-invertible (using Lemma 4.5.2 for rules L∃, R∀, and for geometric rules with a variable condition). ■

Lemma 4.6.3. *Each rule of* $\mathbf{G3[IM]}_\omega.\mathbf{G}$ *except R⊃, R∀ and R$\bigwedge$ is cut-substitutive.*

Proof. Same as for $\mathbf{G3C}_\omega$. ■

Lemma 4.6.4 (Right reduction)**.** *If we are in* $\mathbf{G3[CIM]}_\omega.\mathbf{G}$ *and all of the following hold:*

(i) $\mathscr{D}_1 \vdash \Gamma \to \Delta, A$

(ii) $\mathscr{D}_2 \vdash A, \Gamma \to \Delta$

(iii) A is principal in the last rule instance applied in $\mathscr{D}_1$

(iv) If $A \equiv \exists x B$ *or* $A \equiv \bigvee_{n>0} B_n$*, then A is not principal in the last rule instance applied in* $\mathscr{D}_2$

Then there is a $\mathbf{G3[CIM]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$*-derivation* $\mathscr{D}$ *concluding* $\Gamma \to \Delta$ *containing only cuts on proper subformulae of A.*

Proof. By Noetherian induction with proof-embeddability in the derivation $\mathscr{D}_2$ of $A, \Gamma \to \Delta$.

If $\mathscr{D}_2$ is a one node tree, since A cannot be principal in the initial sequent, then the conclusion of Cut is also initial.

Else, we have two cases depending on whether A is principal in the last rule instance applied in $\mathscr{D}_2$ or not.

In the latter case, if we are in $\mathbf{G3C}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$, the lemma holds thanks to Lemma 4.6.2. If we are in $\mathbf{G3I}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$ and the last step of $\mathscr{D}_2$ is not by one of R$\supset$, $R\forall$, and R$\bigwedge$ then it holds by Lemma 4.6.3. In the remaining three cases, we have two cases according to whether $\mathscr{D}_1$ ends with a step by an invertible rule or not. In the latter case, $\mathscr{D}_1$ ends with one of R$\supset$, $R\forall$, and R$\bigwedge$. We permute the cut upwards in the right premiss. To illustrate, we consider the case of R$\bigwedge$. We transform

$$\dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \to B[y/x]\end{array}\right.}{\Gamma \to \Delta', \bigwedge\limits_{n>0} A_n, \forall x B}\,\mathrm{R}\forall \qquad \dfrac{\mathscr{D}_{2i}\left\{\begin{array}{c}\vdots\\ \{\forall xB,\Gamma\to A_i \,|\, i>0\}\end{array}\right.}{\forall xB,\Gamma \to \Delta', \bigwedge\limits_{n>0}A_n}\,\mathrm{R}\bigwedge}{\Gamma\to\Delta',\bigwedge\limits_{n>0}A_n}\,\mathrm{Cut}$$

into

$$\dfrac{\dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \to B[y/x]\end{array}\right.}{\Gamma\to\forall xB}\,\mathrm{R}\forall \quad \mathscr{D}_{2i}\left\{\begin{array}{c}\vdots\\ \{\forall xB,\Gamma\to A_i \,|\, i>0\}\end{array}\right.}{\{\Gamma\to A_i\,|\,i>0\}}\,\text{i.h.}_i,\ i>0}{\Gamma\to\Delta',\bigwedge\limits_{n>0}A_n}\,\mathrm{R}\bigwedge$$

If, instead, $\mathscr{D}_1$ ends by an invertible rule, then we apply invertibility, thus transforming the derivation into one having only cuts on proper subformulae of A. For example, if $\mathscr{D}_1$ ends with a step by

R∧, we transform

$$\dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta',\bigwedge_{n>0}A_n,B\end{array}\right. \quad \mathscr{D}_{12}\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta',\bigwedge_{n>0}A_n,C\end{array}\right.}{\Gamma\to\Delta',\bigwedge_{n>0}A_n,B\wedge C}\,\mathrm{R}\wedge \quad \mathscr{D}_2\left\{\begin{array}{c}\vdots\\ B\wedge C,\Gamma\to\Delta',\bigwedge_{n>0}A_n\end{array}\right.}{\Gamma\to\Delta',\bigwedge_{n>0}A_n}\,\mathrm{Cut}$$

into

$$\dfrac{\mathscr{D}_{12}\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta',\bigwedge_{n>0}A_n,C\end{array}\right. \quad \dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta',\bigwedge_{n>0}A_n,B\end{array}\right.}{C,\Gamma\to\Delta',\bigwedge_{n>0}A_n,B}\,\mathrm{LW} \quad \dfrac{\mathscr{D}_{2}\left\{\begin{array}{c}\vdots\\ B\wedge C,\Gamma\to\Delta',\bigwedge_{n>0}A_n\end{array}\right.}{B,C,\Gamma\to\Delta',\bigwedge_{n>0}A_n}\,\text{pe-inv}}{C,\Gamma\to\Delta',\bigwedge_{n>0}A_n}\,\mathrm{Cut}}{\Gamma\to\Delta',\bigwedge_{n>0}A_n}\,\mathrm{Cut}$$

Next, we consider the case with A is principal in the last rule instance applied in $\mathscr{D}_2$. We have cases according to the shape of A.

If $A \equiv P$ for some atomic formula P, then the last rule instance in $\mathscr{D}_2$ is by a geometric rule (rules for equality included) L_G concluding $P_1,\dots,P,\dots,P_k,\Gamma'' \to \Delta',P$ and $\mathscr{D}_1$ is the one node tree $P,\Gamma' \to \Delta',P$. The conclusion of cut is the initial sequent $P,\Gamma' \to \Delta',P$ which is cut-free derivable.

The cases with $A \equiv \bot$, $A \equiv \top$ or $A \equiv B \circ C$, for ($\circ \in \{\wedge\vee,\supset\}$), are left to the reader.

If $A \equiv \forall x B$ we transform (if we are in $\mathbf{G3[IM]}_\omega.\mathbf{G}+\{\mathbf{Cut}\}$, Δ is not in the premiss of R∀)

$$\dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta,B[y/x]\end{array}\right.}{\Gamma\to\Delta,\forall xB}\,\mathrm{R}\forall \quad \dfrac{\mathscr{D}_{21}\left\{\begin{array}{c}\vdots\\ B[t/x],\forall xB,\Gamma\to\Delta\end{array}\right.}{\forall xB,\Gamma\to\Delta}\,\mathrm{L}\forall}{\Gamma\to\Delta}\,\mathrm{Cut}$$

into the following derivation having only cuts on proper subformulae of A (if we are in $\mathbf{G3[IM]}_{\omega}.\mathbf{G}+\{\mathbf{Cut}\}$ then Δ is introduced in $\mathscr{D}_{11}$ by pe-weakenings):

$$\dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta,B[y/x]\end{array}\right.}{\Gamma\to\Delta,B[t/x]}\,\text{Subs} \qquad \dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta,B[y/x]\end{array}\right.}{\Gamma\to\Delta,\forall xB}\,\text{R}\forall \quad \mathscr{D}_{21}\left\{\begin{array}{c}\vdots\\ B[t/x],\forall xB,\Gamma\to\Delta\end{array}\right.}{B[t/x],\Gamma\to\Delta}\,\text{i.h.}}{\Gamma\to\Delta}\,\text{Cut}$$

If $A\equiv\bigwedge_{n>0}B_n$ we transform (Δ not in the premisses of $\text{R}\bigwedge$ if we are in $\mathbf{G3[IM]}_{\omega}.\mathbf{G}+\{\mathbf{Cut}\}$)

$$\dfrac{\dfrac{\mathscr{D}_{1i}\left\{\begin{array}{c}\vdots\\ \{\Gamma\to\Delta,B_i\,|\,i>0\}\end{array}\right.}{\Gamma\to\Delta,\bigwedge_{n>0}B_n}\,\text{R}\bigwedge \qquad \dfrac{\mathscr{D}_{21}\left\{\begin{array}{c}\vdots\\ B_k,\bigwedge_{n>0}B_n,\Gamma\to\Delta\end{array}\right.}{\bigwedge_{n>0}B_n,\Gamma\to\Delta}\,\text{L}\bigwedge}{\Gamma\to\Delta}\,\text{Cut}$$

into the following derivation having only cuts on proper subformulae of A (if we are in $\mathbf{G3[IM]}_{\omega}.\mathbf{G}+\{\mathbf{Cut}\}$ then Δ is introduced in $\mathscr{D}_{1k}$ by pe-weakenings):

$$\dfrac{\mathscr{D}_{1k}\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta,B_k\end{array}\right. \qquad \dfrac{\dfrac{\dfrac{\mathscr{D}_{1i}\left\{\begin{array}{c}\vdots\\ \{\Gamma\to\Delta,B_i\,|\,i>0\}\end{array}\right.}{\Gamma\to\Delta,\bigwedge_{n>0}B_n}\,\text{R}\bigwedge}{B_k,\Gamma\to\Delta,\bigwedge_{n>0}B_n}\,\text{LW} \quad \mathscr{D}_{21}\left\{\begin{array}{c}\vdots\\ B_k,\bigwedge_{n>0}B_n,\Gamma\to\Delta\end{array}\right.}{B_k,\Gamma\to\Delta}\,\text{i.h.}}{\Gamma\to\Delta}\,\text{Cut}$$

■

Lemma 4.6.5 (Left reduction). *If we are in* $\mathbf{G3[CIM]}_{\omega}.\mathbf{G}$ *and all of the following hold:*

(i) $\mathscr{D}_1 \vdash \Gamma \to \Delta, A$

(ii) $\mathscr{D}_2 \vdash A, \Gamma \to \Delta$

Then there is a $\mathbf{G3[CIM]}_\omega.\mathbf{G}$*-derivation* $\mathscr{D}$ *concluding* $\Gamma \to \Delta$ *containing only cuts on proper subformulae of* A

Proof. By Noetherian induction with proof-embeddability in the derivation $\mathscr{D}_1$ of $\Gamma \to \Delta, A$.

If $\mathscr{D}_1$ is a one node tree, the lemma obviously holds.

Else, we have two cases depending on whether A is principal in the last rule instance applied in $\mathscr{D}_1$ or not.

In the latter case, the lemma holds thanks to Lemma 4.6.2 or 4.6.3 (if the last step of $\mathscr{D}_1$ is by an intuitionistic non-invertible rule we proceed as in the analogous case of Lemma 4.6.4). In the former case we have cases according to the shape of A.

If A is an atomic formula, or $\bot$, or $\top$ or $B \circ C$ ($\circ \in \{\wedge, \vee \supset\}$), or $\forall x B$, or $\bigwedge B_n$, the lemma holds thanks to Lemma 4.6.4.

If $A \equiv \exists x B$ we transform:

$$\dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \to \Delta, \exists x B, B[t/x]\end{array}\right.}{\Gamma \to \Delta, \exists x B}\,\mathrm{R}\exists \qquad \mathscr{D}_2\left\{\begin{array}{c}\vdots\\ \exists x B, \Gamma \to \Delta\end{array}\right.}{\Gamma \to \Delta}\,\mathrm{Cut}$$

into the following derivation having only cuts on proper subformulae of A:

$$\dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots\\ \Gamma \to \Delta, \exists x B, B[t/x]\end{array}\right. \qquad \mathscr{D}_2\left\{\begin{array}{c}\vdots\\ \exists x B, \Gamma \to \Delta\end{array}\right.}{\Gamma \to \Delta, B[t/x]}\,\text{i.h.} \qquad \dfrac{\mathscr{D}_2\left\{\begin{array}{c}\vdots\\ \exists x B, \Gamma \to \Delta\end{array}\right.}{B[t/x], \Gamma \to \Delta}\,\text{pe-inv}}{\Gamma \to \Delta}\,\mathrm{Cut}$$

If $A \equiv \bigvee B_n$ we transform:

$$
\dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots \\ \Gamma \to \Delta, \bigvee_{n>0} B_n, B_k\end{array}\right.}{\Gamma \to \Delta, \bigvee_{n>0} B_n}\,\mathrm{R}\bigvee \qquad \mathscr{D}_2\left\{\begin{array}{c}\vdots \\ \bigvee_{n>0} B_n, \Gamma \to \Delta\end{array}\right.}{\Gamma \to \Delta}\,\mathrm{Cut}
$$

into the following derivation:

$$
\dfrac{\dfrac{\mathscr{D}_{11}\left\{\begin{array}{c}\vdots \\ \Gamma \to \Delta, \bigvee_{n>0} B_n, B_k\end{array}\right. \quad \mathscr{D}_2\left\{\begin{array}{c}\vdots \\ \bigvee_{n>0} B_n, \Gamma \to \Delta\end{array}\right.}{\Gamma \to \Delta, B_k}\,\mathrm{i.h.} \qquad \dfrac{\mathscr{D}_2\left\{\begin{array}{c}\vdots \\ \bigvee_{n>0} B_n, \Gamma \to \Delta\end{array}\right.}{B_k, \Gamma \to \Delta}\,\mathrm{pe\text{-}inv}}{\Gamma \to \Delta}\,\mathrm{Cut}
$$

■

In order to prove Cut elimination in a constructive way we use Bar Induction as done in [174, p. 18] for ω-arithmetic. This strategy avoids the assumption of total ordering of ordinal numbers. Before proving the theorem we introduce Brouwer's principle of (decidable) Bar Induction.

Definition 4.6.6 (Bar Induction)**.** Let B and I be unary predicates (the so-called 'base predicate' and 'inductive predicate', respectively) of finite lists of natural numbers (to be denoted by $u, v, \dots$). If:

(i) B is decidable;

(ii) Every infinite sequence of natural numbers has a finite initial segment satisfying B;

(iii) $B(u)$ implies $I(u)$ for every finite list u;

(iv) If $I(u * n)$ holds for all $n \in \mathbb{N}$ then $I(u)$ holds;

Then I holds for the empty list of natural numbers.

Theorem 4.6.7 (Cut elimination)**.** *Cut is admissible in* $\mathbf{G3[CI]}_\omega$.

Proof. Throughout this proof, we use finite lists of natural numbers to index (partial) branches of trees, i.e. directed paths from the root to a node, possibly a leaf. Consider a tree such that each node has immediate successors either indexed by ω or else by some $k < \omega$, and such that each branch has finite length, then:

- The empty list $\{\}$ indexes the root of the tree.
- Given any infinite sequence of numbers, we have $B(u)$ for every finite initial segment u that represents a full branch $\mathscr{R}$ of the tree, i.e., a root-to-leaf path (simply because every leaf is either an atomic formula, or $\top$ or $\bot$) and by construction of the representation there are such u.
- Suppose that u indexes a partial branch $\mathscr{R}$ of the tree and that the last node a has immediate successor nodes indexed by $k < \omega$, and let a natural number n be given. Let $m = n \mod k$: that is, m is the remainder of n after division by k. Then $u * n$ indexes $\mathscr{R}$ extended with the m^{th} immediate successor node of a. For example, in the case of a 2-premiss rule, odd numbers index the left premiss, even numbers the right premiss.

Notice that the above gives a partial surjective map, with decidable domain, from sequences of natural numbers to branches in the given tree. Moreover, this ensures that every infinite sequence has an initial segment that indexes a branch of the tree.[8]

Let d be a derivation in the calculus $\mathbf{G3[CI]}_\omega.\mathbf{G} + \{\mathbf{Cut}\}$. The proof consists of two parts, each building on an appropriate Bar Induction.

[8] Since the number of nodes of the tree is at most countable, one may also define an encoding such that the correspondence is unique. This however would require more effort and we would lose the property that every infinite sequence has an initial segment that indexes a branch of the tree.

— **Part 1.** We use Bar Induction to show that an uppermost instance of Cut with cut-formula C occurring in d is admissible. We use the method defined above to index the branches of the formation tree of the formula C—where C is the root of the tree and atomic formulae or $\top$ or $\bot$ are its leaves. Let $B(u)$ hold if u indexes a branch whose last element is an atom or $\bot$ or $\top$; let $I(u)$ hold if and u indexes a partial branch whose last element is a formula D such that an uppermost cut on D in $\mathbf{G3[CI]}_{\omega}.\mathbf{G} + \{\mathbf{Cut}\}$ is eliminable.

The following hold:

(i) $B(u)$ is decidable by simply comparing the list with the formation tree;

(ii) By definition of the indexing, the n^{th} element of the sequence identifies the n^{th} node in a branch of the formation tree of a formula. After a finite number of steps from the root we find an atom or $\bot$ or $\top$ since all branches of the tree are finite and this identifies an initial segment of the infinite sequence that satisfies B.

(iii) $B(u)$ implies $I(u)$ since cuts on atomic formulae, $\top$, or $\bot$ are admissible;

(iv) $I(u * n)$ for all n implies $I(u)$: by Lemma 4.6.5 an uppermost cut on some formula E can be reduced to cuts on proper subformulae of E.

By Bar Induction we conclude that the uppermost cut with cut-formula C is eliminable from $\mathbf{G3[CI]}_{\omega}.\mathbf{G} + \{\mathbf{Cut}\}$.

— **Part 2.** We show that all cuts can be eliminated from $\mathscr{D}$. We consider a derivation $\mathscr{D}$ in $\mathbf{G3[CI]}_{\omega}.\mathbf{G} + \{\mathbf{Cut}\}$ and, as above, we use lists of natural numbers to index branches of $\mathscr{D}$. Let $B(u)$ hold if u indexes a branch ending in a leaf of $\mathscr{D}$; let $I(u)$ hold if u indexes a partial branch whose last element has a cut-free derivation (i.e., it is $\mathbf{G3[CI]}_{\omega}.\mathbf{G}$-derivable). All conditions of Bar Induction are satisfied by this choice of B and I:

(i) $B(u)$ is decidable;

(ii) Given any infinite sequence of numbers, we have $B(u)$ for every finite initial segment u that represents a full branch $\mathscr{R}$ of the tree, i.e., a root-to-leaf path; and by construction of the representation there are such u.

(iii) $B(u)$ implies $I(u)$ since the leaves of $\mathscr{D}$ trivially have a cut-free derivation;

(iv) $I(u * n)$ for all n implies $I(u)$: having shown in part 1 that uppermost instances of Cut are admissible, if all the premisses of a rule instance in $\mathscr{D}$ have a cut-free derivation, then also its conclusion has a cut-free derivation.

By Bar Induction we conclude that the conclusion of $\mathscr{D}$ has a cut-free derivation. ■

Corollary 4.6.8. *The rule of context-free cut:*

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Sigma}{\Pi, \Gamma \to \Delta, \Sigma}\ Cut_{cf}$$

is admissible in $\mathbf{G3[CI]}_\omega.\mathbf{G}$.

Proof. This is an immediate consequence of Theorem 4.6.7 since rules Cut and Cut_{cf} are equivalent when weakening and contraction are admissible. ■

4.7 Orevkov's theorems on infinitary Glivenko classes

We follow Orevkov's notation and denote by $\circ^+$ positive and by $\circ^-$ negative occurrences of the connective or quantifier $\circ$ in a sequent.

Theorem 4.7.1 (Glivenko Class 1)**.** *If neither* $\supset^+$, *nor* $\forall^+$, *nor* $\bigwedge^+$ *occurs in* $\Gamma \to \Delta$ *and* $\mathbf{G3C}_\omega.\mathbf{G} \vdash^{\mathscr{D}} \Gamma \to \Delta$, *then* $\mathbf{G3I}_\omega.\mathbf{G} \vdash^{\mathscr{D}'} \Gamma \to \Delta$ *with* $\mathscr{D}' \preccurlyeq \mathscr{D}$.
If, moreover, $\bot^-$ *does not occur in* $\Gamma \to \Delta$ *then* $\mathbf{G3M}_\omega.\mathbf{G} \vdash^{\mathscr{D}'} \Gamma \to \Delta$.

Proof. Any derivation in $\mathbf{G3C}_\omega.\mathbf{G}$ uses only rules that follow the (infinitary) geometric rule scheme and logical rules. Observe that geometric implications contain no $\supset$, nor $\forall$, nor $\bigwedge$ in the scope of $\vee$ nor of $\bigvee$, which means that no instance of the rules that violates the intuitionistic restrictions is used, so the derivation directly gives (through the addition, where needed, of the missing implications in steps of L$\supset$) a derivation in $\mathbf{G3I}_\omega.\mathbf{G}$ of the same conclusion. Moreover, if $\bot^-$ does not occur in $\Gamma \to \Delta$ the derivation is a $\mathbf{G3I}_\omega.\mathbf{G}$-derivation. ■

This is actually Barr's theorem.

Orevkov's theorem for most other Glivenko classes works only if we restrict ourselves to geometric rules with at most one premiss – i.e., rules expressing geometric implications without disjunction in the succedent. Hence we introduce the following piece of notation.

Definition 4.7.2. L_{GS} stands for a one premiss geometric rule and $\mathbf{G3[CIM]}_\omega.\mathbf{S}$ stands for any extension of $\mathbf{G3[CIM]}_\omega$ with a finite or infinite family of such rules L_{GS}.

Lemma 4.7.3. *If neither* $\supset^+$*, nor* $\vee^-$*, nor* $\bigvee^-$ *occurs in* $\Gamma \to \Delta$ *and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathscr{D}} \Gamma \to \Delta$*, then*

— *if* Δ *is inhabited, then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathscr{D}'} \Gamma \to A$ *for some* $A \in \Delta$*;*

— *if* Δ *is empty, then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathscr{D}'} \Gamma \to \Delta$*;*

with $\mathscr{D}' \preccurlyeq \mathscr{D}$*.*

The same holds with respect to $\mathbf{G3M}_\omega.\mathbf{S}$ *if we assume additionally that no instance of* $\bot^-$ *occurs in* $\Gamma \to \Delta$*.*

Proof. By induction.

If $\Gamma \to \Delta$ is an initial sequent with principal formula some atomic formula P, the lemma holds by taking $A \equiv P$. If $\mathscr{D}$ ends with an instance of L$\bot$, we have two cases: if $\Delta \neq \emptyset$ we take $A \equiv D$ for some $D \in \Delta$; else we have that $\mathbf{G3I}_\omega.\mathbf{S} \vdash \Gamma \to \Delta$.

If the last step of $\mathscr{D}$ is an instance of L$\wedge$, then $\mathscr{D}$ has the form:

$$\mathscr{D}_1\left\{\begin{array}{c}\vdots\\ \dfrac{B,C,\Gamma'\to\Delta}{B\wedge C,\Gamma'\to\Delta}\ \mathrm{L}\wedge\end{array}\right.$$

We apply the inductive hypothesis to $\mathscr{D}_1$ and we obtain a derivation $\mathscr{D}_1'\preccurlyeq\mathscr{D}_1$ in $\mathbf{G3I}_\omega.\mathbf{S}$ of either $B,C,\Gamma'\to A$ for some $A\in\Delta$ or $B,C,\Gamma'\to\Delta$, depending on whether Δ is inhabited or not. In both cases, we get a derivation $\mathscr{D}'\preccurlyeq\mathscr{D}$ via an application of L$\wedge$:

$$\mathscr{D}_1'\left\{\begin{array}{c}\vdots\\ \dfrac{B,C,\Gamma'\to A}{B\wedge C,\Gamma'\to A}\ \mathrm{L}\wedge\end{array}\right. \qquad\qquad \mathscr{D}_1'\left\{\begin{array}{c}\vdots\\ \dfrac{B,C,\Gamma'\to\Delta}{B\wedge C,\Gamma'\to\Delta}\ \mathrm{L}\wedge\end{array}\right.$$

If the last step of $\mathscr{D}$ is an instance of R$\wedge$, then $\mathscr{D}$ has the form:

$$\dfrac{\mathscr{D}_1\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta',B\end{array}\right. \qquad \mathscr{D}_2\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta',C\end{array}\right.}{\Gamma\to\Delta',B\wedge C}\ \mathrm{R}\wedge$$

By applying the inductive hypothesis to $\mathscr{D}_1$ and $\mathscr{D}_2$, we obtain $\mathbf{G3I}_\omega.\mathbf{S}\vdash^{\mathscr{D}_1'}\Gamma\to B'$ with $B'\in\Delta',B$ and $\mathbf{G3I}_\omega.\mathbf{S}\vdash^{\mathscr{D}_2'}\Gamma\to C'$ with $C'\in\Delta',C$, such that $\mathscr{D}_1'\preccurlyeq\mathscr{D}_1$ and $\mathscr{D}_2'\preccurlyeq\mathscr{D}_2$. If $B'\equiv B$ and $C'\equiv C$ we get the following derivation $\mathscr{D}'\preccurlyeq\mathscr{D}$ in $\mathbf{G3I}_\omega.\mathbf{S}$:

$$\dfrac{\mathscr{D}_1'\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta',B\end{array}\right. \qquad \mathscr{D}_2'\left\{\begin{array}{c}\vdots\\ \Gamma\to\Delta',C\end{array}\right.}{\Gamma\to\Delta',B\wedge C}\ \mathrm{R}\wedge$$

Else we set $A\equiv B'$ or, if $B\equiv B'$, $A\equiv C'$, and we are done.

If $\mathscr{D}$ ends with an instance of R$\vee$ with premiss $\Gamma\to\Delta',B,C$ and conclusion $\Gamma\to\Delta',B\vee C$, then we have $\mathbf{G3C}_\omega.\mathbf{S}\vdash^{\mathscr{D}_1}\Gamma\to\Delta',B,C$ with $\mathscr{D}_1\prec\mathscr{D}$. By induction hypothesis, we get $\mathbf{G3I}_\omega.\mathbf{S}\vdash^{\mathscr{D}_1'}\Gamma\to D$ with

$\mathscr{D}_1' \preccurlyeq \mathscr{D}_1$ and D is either A, B, or in Δ. If $D \in \Delta$, then we're already done. If D is A or B we conclude by applying pe-weakening and R$\vee$.

If $\mathscr{D}$ ends with the following instance of L$\supset$:

$$\dfrac{\mathscr{D}_1 \left\{ \begin{array}{c} \vdots \\ \Gamma' \to \Delta, B \end{array} \right. \qquad \mathscr{D}_2 \left\{ \begin{array}{c} \vdots \\ C, \Gamma' \to \Delta \end{array} \right.}{B \supset C, \Gamma' \to \Delta} \text{ L}\supset$$

and Δ inhabited, by inductive hypothesis we have $\mathbf{G3I}_{\omega}.\mathbf{S} \vdash^{\mathscr{D}_1'} \Gamma' \to B'$ with $B' \in \Delta, B$ and $\mathbf{G3I}_{\omega}.\mathbf{S} \vdash^{\mathscr{D}_2'} C, \Gamma' \to D$ with $D \in \Delta$, such that $\mathscr{D}_1' \preccurlyeq \mathscr{D}_1$ and $\mathscr{D}_2' \preccurlyeq \mathscr{D}_2$. When $B' \equiv B$ we use the left and right right rules of weakening to obtain a derivation $\mathscr{D}_1'' \preccurlyeq \mathscr{D}_1'$ of $B \supset C, \Gamma' \to D, B$ and we obtain $\mathscr{D}' \preccurlyeq \mathscr{D}$ as follows:

$$\dfrac{\mathscr{D}_1'' \left\{ \begin{array}{c} \vdots \\ B \supset C, \Gamma' \to D, B \end{array} \right. \qquad \mathscr{D}_2' \left\{ \begin{array}{c} \vdots \\ C, \Gamma' \to D \end{array} \right.}{B \supset C, \Gamma' \to D} \text{ L}\supset$$

When, instead, $B' \equiv E$ for some $E \not\equiv B$, we conclude $C \supset B, \Gamma \to E$ by applying an instance of left weakening to the first derivation.

Next we consider the case with $\Delta = \emptyset$. By induction we have derivations $\mathbf{G3I}_{\omega}.\mathbf{S} \vdash^{\mathscr{D}_1'} \Gamma' \to B$ and $\mathbf{G3I}_{\omega}.\mathbf{S} \vdash^{D_2'} C, \Gamma' \to \Delta$, such that $\mathscr{D}_1' \preccurlyeq \mathscr{D}_1$ and $\mathscr{D}_2' \preccurlyeq \mathscr{D}_2$. By weakening to obtain a derivation $\mathscr{D}_1'' \preccurlyeq \mathscr{D}_1'$ of $B \supset C, \Gamma' \to \Delta, B$ and we obtain $\mathscr{D}' \preccurlyeq \mathscr{D}$ as follows:

$$\dfrac{\mathscr{D}_1'' \left\{ \begin{array}{c} \vdots \\ B \supset C, \Gamma' \to \Delta, B \end{array} \right. \qquad \mathscr{D}_2' \left\{ \begin{array}{c} \vdots \\ C, \Gamma' \to \Delta \end{array} \right.}{B \supset C, \Gamma' \to \Delta} \text{ L}\supset$$

The cases with $\mathscr{D}$ ending by a rule for the quantifiers are straightforward and can thus be omitted.

If $\mathscr{D}$ ends with an instance of L$\bigwedge$, we have simply to apply the inductive hypothesis to its premiss and an instance of L$\bigwedge$ to get a derivation (in $\mathbf{G3I}_{\omega}.\mathbf{S}$) $\mathscr{D}' \preccurlyeq \mathscr{D}$ of either $\Gamma \to \Delta$ or $\Gamma \to A$ for $A \in \mathscr{D}$ (depending on whether Δ is empty or not).

If $\mathscr{D}$ ends with the following instance of R$\bigwedge$:

$$\mathscr{D}_i \left\{ \begin{array}{c} \vdots \\ \{\Gamma \to \Delta', B_i \,|\, i > 0\} \end{array} \right. \over \Gamma \to \Delta', \bigwedge_{n>0} B_n \quad \text{R}\bigwedge$$

we apply the inductive hypothesis to obtain, for each $i > 0$, $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathscr{D}_i'} \Gamma \to C_i$ with $C_i \in \Delta', B_i$, such that $\mathscr{D}_i' \preccurlyeq \mathscr{D}_i$. If for some $j > 0$ we have $C_j \in \Delta'$, then by taking $A \equiv C_j$ we observe that $\mathscr{D}_i'$ is as wanted. Else $\mathscr{D}_i'$ is an intuitionistic derivation of $\Gamma \to B_i$ for all $i > 0$ and we conclude by an intuitionistic instance of R$\bigwedge$:

$$\mathscr{D}_i' \left\{ \begin{array}{c} \vdots \\ \{\Gamma \to B_i \,|\, i > 0\} \end{array} \right. \over \Gamma \to \bigwedge_{n>0} B_n \quad \text{R}\bigwedge$$

If $\mathscr{D}$ ends with the following instance of R$\bigvee$:

$$\mathscr{D}_1 \left\{ \begin{array}{c} \vdots \\ \Gamma \to \Delta', \bigvee_{n>0} B_n, B_k \end{array} \right. \over \Gamma \to \Delta', \bigvee_{n>0} B_n \quad \text{R}\bigvee$$

by inductive hypothesis $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathscr{D}_1'} \Gamma \to D$ with $D \in \Delta, \bigvee B_n, B_k$ and $\mathscr{D}_1' \preccurlyeq \mathscr{D}_1$. If $D \in \Delta, \bigvee B_n$, we conclude by taking $A \equiv D$. Else $\mathscr{D}_1'$ is a derivation of $\Gamma \to B_k$ and, by right weakening we get a derivation $\mathscr{D}_1'' \preccurlyeq \mathscr{D}_1'$ of $\Gamma \to B_k, \bigvee B_n$. Finally, we get a derivation $\mathscr{D}' \preccurlyeq \mathscr{D}$ in

G3I$_\omega$.S:

$$\mathscr{D}_1''\left\{\begin{array}{c}\vdots\\ \Gamma \to \bigvee_{n>0} B_n, B_k\end{array}\right. \over \Gamma \to \bigvee_{n>0} B_n \; \mathrm{R}\bigvee$$

If $\mathscr{D}$ ends with a one-premiss geometric rule (rules for equality included) L_{G^S}, then we have simply to apply the inductive hypothesis to the premiss and then an instance of L_{G^S} to obtain the desired conclusion. Observe that we had to exclude geometric rules with more than one premiss since the inductive hypothesis would have given us sequents with a possibly different succedent (for the same reason we had to exclude $\vee^-$ and $\bigvee^-$). ■

Theorem 4.7.4 (Glivenko Class 2)**.** *If neither* $\supset^+$*, nor* $\vee^-$*, nor* $\bigvee^-$ *occurs in* $\Gamma \to A$ *and* **G3C$_\omega$.S** $\vdash^{\mathscr{D}} \Gamma \to A$*, then* **G3I$_\omega$.S** $\vdash^{\mathscr{D}'} \Gamma \to A$ *with* $\mathscr{D}' \preccurlyeq \mathscr{D}$*.*
If, moreover, no instance of $\bot^-$ *occurs in* $\Gamma \to \Delta$ *then* **G3M$_\omega$.S** $\vdash^{\mathscr{D}'} \Gamma \to A$*.*

Proof. An immediate corollary of Lemma 4.7.3. ■

We list here two other corollaries of Lemma 4.7.3, the latter being an infinitary version of the result proved in [169].

Corollary 4.7.5. *If neither* $\supset^+$*, nor* $\vee^-$*, nor* $\bigvee^-$ *occurs in* $\Gamma \to \Delta$ *and* **G3C$_\omega$.S** $\vdash^{\mathscr{D}} \Gamma \to \Delta$*, then* **G3I$_\omega$.S** $\vdash^{\mathscr{D}'} \Gamma \to \Delta$ *with* $\mathscr{D}' \preccurlyeq \mathscr{D}$*.*
If, moreover, no instance of $\bot^-$ *occurs in* $\Gamma \to \Delta$ *then* **G3M$_\omega$.S** $\vdash^{\mathscr{D}'} \Gamma \to \Delta$*.*

Corollary 4.7.6. *Assume that no instance of* $\supset$ *occurs in* A *and that no instance of* $\bot^+$*,* $\vee^+$*,* $\bigvee^+$*, and* $\supset^-$ *occurs in* Γ*. If* **G3C$_\omega$.S** $\vdash^{\mathscr{D}} \Gamma \to A$*, then* **G3M$_\omega$.S** $\vdash^{\mathscr{D}'} \Gamma \to A$ *with* $\mathscr{D}' \preccurlyeq \mathscr{D}$*.*

Lemma 4.7.7. *If neither* $\supset^+$*, nor* $\forall^-$ *occurs in* $\Gamma \rightarrow \Delta$ *and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash \Gamma \rightarrow \Delta$*, then there is a classical derivation of* $\Gamma \rightarrow \Delta$ *such that all instances of rules in* $G_1 = \{L_{G^S}, R\wedge, R\vee, R\forall, R\exists, R\bigwedge, R\bigvee\}$ *precede all instances of rules in* $G_2 = \{L\wedge, L\vee, L\supset, L\exists, L\bigwedge, L\bigvee\}$*.*

Proof. First notice that in general it is possible to permute rules in $G2$ below rules in $G1$ since, having excluded instances of rule R$\supset$, the principal formula of rules in $G2$ cannot be active in rules in $G1$. In particular, instances of geometric rules have atomic formulae as active and, having excluded instances of R$\supset$, all active formulae of logical rules in $G1$ occur in the antecedents while principal formulae of rules in $G2$ occur in the succedents. Moreover instances of rule L$\exists$ can be permuted down with respect to instances of R$\forall$ and of geometric rules with a variable condition since their *eigenvariables* are necessarily distinct. ■

Theorem 4.7.8 (Glivenko class 3)**.**

(i) If neither $\supset^+$*, nor* $\forall^-$ *occurs in* $\Gamma \rightarrow A$ *and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash \Gamma \rightarrow A$*, then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash \Gamma \rightarrow A$*.*
If, moreover, no instance of $\bot^-$ *occurs in* $\Gamma \rightarrow A$ *then* $\mathbf{G3M}_\omega.\mathbf{S} \vdash \Gamma \rightarrow A$*.*

(ii) If neither $\supset^+$*, nor* $\forall^-$ *occurs in* $\Gamma \rightarrow \Delta$ *and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash \Gamma \rightarrow \Delta$*, then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash \Gamma \rightarrow \Delta$*.*
If, moreover, no instance of $\bot^-$ *occurs in* $\Gamma \rightarrow \Delta$ *then* $\mathbf{G3M}_\omega.\mathbf{S} \vdash \Gamma \rightarrow \Delta$*.*

Proof. We begin with item 1. By Lemma 4.7.7 we can transform the classical derivation of $\Gamma \rightarrow \Delta$ into a classical derivation where all instances of rules in group $G1$ precede instances of rules in $G2$. Then the lemma holds for the upper $G1$-component of this derivation by Theorem 4.7.4 and it holds for the lower $G2$-component since all rules instances applied therein are instances of rules identical in classical, intuitionistic and minimal logics.

Item 2 can be proved analogously using Corollary 4.7.5 instead of Theorem 4.7.4. ■

Theorem 4.7.9 (Glivenko class 4). *If neither* $\supset^-$*, nor* $\vee^+$*, nor* $\exists^+$ *nor* $\bigvee^+$ *occurs in* $\Gamma \to A$ *and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathscr{D}} \Gamma \to A$*, then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathscr{D}'} \Gamma \to A$ *with* $\mathscr{D}' \preccurlyeq \mathscr{D}$.
If, moreover, no instance of $\bot^-$ *occurs in* $\Gamma \to A$ *then* $\mathbf{G3M}_\omega.\mathbf{S} \vdash^{\mathscr{D}'} \Gamma \to A$.

Proof. Rules L$\supset$, $\vee^+$, $\exists^+$, and $\bigvee^+$ are the only rules of $\mathbf{G3C}_\omega.\mathbf{G}$ having instances with a single-succedent conclusion and a multi-succedent premiss. This implies that all sequents in the classical derivation $\mathscr{D}$ of $\Gamma \to A$ are single-succedent ones, hence all rule instances occurring in $\mathscr{D}$ satisfy the intuitionistic (and minimal) restriction. ■

Next we move to Glivenko classes 5, 6, and 7 which, roughly, are versions of classes 1,2, and 3 where the restriction on occurrences of $\supset^+$ is relaxed by allowing occurrences of $\supset^+$ having $\bot$ as succedent.

Lemma 4.7.10. *If* $\Gamma \to \Delta$ *does not contain* $\bot^-$, $\vee^+$, $\supset^+$, *or* $\bigvee^+$ *and* Δ *is either empty or* $\bot^+$ *occurs in each one of its formulae, then* $\mathbf{G3C}_\omega.\mathbf{G} \nvdash \Gamma \to \Delta$.

Proof. Since $\bot^-$ cannot occur in $\Gamma \to \Delta$ and since all formulae in Δ must contain an occurrence of $\bot^+$, $\Gamma \to \Delta$ cannot be the conclusion of an instance of L$\bot$ nor an initial sequent (atomic formulae do not contain occurrences of $\bot^+$). Having excluded the applicability of rules R$\vee$, R$\supset$ and $\bigvee^+$, we know that at least one branch of a proof-search tree for $\Gamma \to \Delta$ is such that $\bot^+$ occurs in each formula occurring in its succedents, hence that branch cannot reach an inital sequent. The lemma follows by the invertibility of the rules of $\mathbf{G3C}_\omega.\mathbf{G}$. ■

Corollary 4.7.11. *If* $\Gamma \to A$ *does not contain* $\neg^-$, $\vee^+$, $\supset$, *or* $\bigvee^+$, Γ *does not contain* $\bot$, *and* $\Gamma \to A$ *contains an occurrence of* $\neg^+$*, then* $\mathbf{G3C}_\omega.\mathbf{G} \nvdash \Gamma \to A$.

Proof. $\Gamma \to A$ satisfies the conditions of Lemma 4.7.10 since, by the restiction on implications, $\bot^+$ occurs in A. ■

Corollary 4.7.12. *If $\Gamma \to A$ does not contain $\neg^-$, $\vee^+$, $\supset$, or $\bigvee^+$, but it contains an occurrence of $\neg^+$, if* $\mathbf{G3C}_\omega.\mathbf{G} \vdash^{\mathscr{D}} \Gamma \to A$*, then* $\mathbf{G3I}_\omega.\mathbf{G} \vdash^{\mathscr{D}'} \Gamma \to A$ *with* $\mathscr{D}' \preccurlyeq \mathscr{D}$*.*
If, moreover, no instance of $\bot^-$ occurs in $\Gamma \to A$, then $\mathbf{G3M}_\omega.\mathbf{S}^{\mathscr{D}'}\Gamma \to A$*.*

Proof. If $\mathscr{D}$ is an initial sequent, then there's nothing to prove. Suppose it's not. If $\bot$ occurs in Γ, then the corollary holds because $\Gamma \to A$ is a conclusion of L$\bot$, thus we obtain a one-step derivation $\mathscr{D}'$ which is embeddable in any nontrivial derivation. Else it follows from Corollary 4.7.11. If no instance of $\bot^-$ occurs in $\Gamma \to A$, then we're always in the latter case. ■

Theorem 4.7.13 (Glivenko class 5)**.** *If neither $\supset^-$, nor $\vee^+$, nor $\forall^+$ nor $\bigvee^+$ occurs in $\Gamma \to A$, and $\supset^+$ occurs in $\Gamma \to A$ only in negations, and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathscr{D}} \Gamma \to A$*, then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathscr{D}'} \Gamma \to A$ *with* $\mathscr{D}' \preccurlyeq \mathscr{D}$*.*

Proof. If $\bot^+$ does not occur in $\Gamma \to A$, then the sequent is in Glivenko class 1 (see Theorem 4.7.1), else the theorem follows by Corollary 4.7.12. ■

Theorem 4.7.14 (Glivenko class 6)**.** *If neither $\supset^-$, nor $\vee$, nor $\bigvee$ occurs in $\Gamma \to A$, and $\supset^+$ occurs in $\Gamma \to A$ only in negations, and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash^{\mathscr{D}} \Gamma \to A$*, then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash^{\mathscr{D}'} \Gamma \to A$ *with* $\mathscr{D}' \preccurlyeq \mathscr{D}$*.*

Proof. If $\bot^+$ does not occur in $\Gamma \to A$, then the sequent is in Glivenko class 2 (see Theorem 4.7.4), else the theorem follows by Corollary 4.7.12. ■

Theorem 4.7.15 (Glivenko class 7)**.** *If neither $\supset^-$, nor $\vee^+$, nor $\forall^-$ nor $\bigvee^+$ occurs in $\Gamma \to A$, and $\supset^+$ occurs in $\Gamma \to A$ only in negations, and* $\mathbf{G3C}_\omega.\mathbf{S} \vdash \Gamma \to A$*, then* $\mathbf{G3I}_\omega.\mathbf{S} \vdash \Gamma \to A$*.*

Proof. If $\bot^+$ does not occur in $\Gamma \to A$, then the sequent is in Glivenko class 3 (see Theorem 4.7.8), else the theorem follows by Corollary 4.7.12. ■

5 A general Glivenko–Gödel theorem for nuclei

5.1 Introduction

Double negation over intuitionistic logic is a typical instance of a nucleus [5, 88, 97, 123, 159, 175, 176, 187]. Glivenko's theorem says that, in propositional logic, classical provability of a formula entails intuitionistic provability of the double negation of that formula [82]. This stood right at the beginning of the success story of negative translations, which have been put into the context of nuclei [187] or monads [60]. As compared to recent literature on Glivenko's theorem [61, 69, 70, 75, 87, 95, 109, 124, 130, 136, 140],[1] the purpose of the present chapter is to generalise Glivenko's theorem from double negation to an arbitrary nucleus, from provability in a calculus to an abstract consequence relation, and from propositional logic to any set of objects whatsoever.

To this end we move to a nucleus j over a Hertz–Tarski consequence relation in the form of a (single-conclusion) entailment relation $\rhd$ à la Scott [30, 171]. Assuming that $\rhd$ is inductively generated by axioms and rules, we propose two natural extensions (Section 5.3.1): $\rhd_j$ generalises the provability of double negation, and $\rhd^j$ is inductively defined by adding the generalisation of double nega-

[1] This list of references is by no means meant exhaustive.

tion elimination to the inductive definition of $\rhd$. By their very definitions, $\rhd^j$ satisfies all axioms and rules of $\rhd$, and $\rhd_j$ satisfies all axioms of $\rhd$. But when does $\rhd_j$ also satisfy all rules of $\rhd$? Our main result, Theorem 5.3.8, says that $\rhd^j$ extends $\rhd_j$, and that the two relations coincide precisely when $\rhd_j$ is closed under the non-axiom rules that are used to inductively generate $\rhd$, which of course is the case whenever there are no such non-axiom rules (Corollary 5.3.9).

In logic this gives us a multi-purpose conservation criterion (Theorem 5.5.2), by which propositional and predicate logic can be handled in parallel. The prime instance of course is Glivenko's theorem (Application 5.5.3(i)) as a syntactical conservation theorem (see also [69, 70]):

$$\Gamma \rhd_c \varphi \iff \Gamma \rhd_i \neg\neg\varphi$$

where $\rhd_c$ and $\rhd_i$ denote classical and intuitionistic propositional logic. Simultaneously we re-obtain Gödel's theorem (Application 5.5.3(ii)) which states that

$$\Gamma \rhd_c^Q \varphi \iff \Gamma \rhd_*^Q \neg\neg\varphi$$

where $\rhd_c^Q$ denotes classical predicate logic, and $\rhd_*^Q$ is any extension (by additional axioms) of intuitionistic predicate logic that satisfies the double negation shift:

$$\forall x \neg\neg\varphi \rhd \neg\neg\forall x\varphi$$

While the double negation nucleus $j\varphi \equiv \neg\neg\varphi$ is an instance of the continuation monad, it is tantamount to the same case $j\varphi \equiv \neg\varphi \supset \varphi$ of the Peirce monad [60]. What does our main result mean for other nuclei in logic? The Dragalin–Friedman nucleus $j\varphi \equiv \varphi \vee \bot$, a case of the closed nucleus, yields a variant of the reduction from intuitionistic to minimal logic going back to Johansson (Application 5.5.4). Last but not least, the open nucleus $j\varphi \equiv A \supset \varphi$ prompts a form of the deduction theorem for positive logic (Application 5.5.5).

Preliminaries

We intend to proceed in a constructive and predicative way, keeping the concepts elementary and the proofs direct. If a formal system is desired, our work can be placed in a suitable fragment of Aczel's *Constructive Zermelo–Fraenkel Set Theory* (**CZF**) [1–3, 6, 7] based on intuitionistic first-order predicate logic.

By a *finite set* we understand a set that can be written as $\{a_1, \ldots, a_n\}$ for some $n \geqslant 0$. Given any set S, let $\mathrm{Pow}(S)$ (respectively, $\mathrm{Fin}(S)$) consist of the (finite) subsets of S. We refer to [155] for further provisos to carry over to the present work.[2]

5.2 Entailment relations

Entailment relations are at the heart of this chapter. We briefly recall the basic notions, closely following [154, 155].

Let S be a set and $\rhd \subseteq \mathrm{Pow}(S) \times S$. Once abstracted from the context of logical formulae, all but one of Tarski's axioms of *consequence* [183] [3] can be put as

$$\frac{U \ni a}{U \rhd a} \qquad \frac{\forall b \in U (V \rhd b) \qquad U \rhd a}{V \rhd a} \qquad \frac{U \rhd a}{\exists U_0 \in \mathrm{Fin}(U)(U_0 \rhd a)}$$

where $U, V \subseteq S$ and $a \in S$. These axioms also characterise a finitary covering or Stone covering in formal topology [162];[4] see further [33, 34, 122, 123, 163, 164]. The notion of consequence has presumably been described first by Hertz [89–91]; see also [16, 106].

Tarski has rather characterised the set of consequences of a set of propositions, which corresponds to the *algebraic closure operator* $U \mapsto U^{\rhd}$ on $\mathrm{Pow}(S)$ of a relation $\rhd$ as above where

$$U^{\rhd} \equiv \{a \in S : U \rhd a\}.$$

[2]For example, we deviate from the terminology prevalent in constructive mathematics and set theory [6, 7, 18–20, 110, 119]: to reserve the term 'finite' to sets which are in *bijection* with $\{1, \ldots, n\}$ for a necessarily unique $n \geq 0$. Those exactly are the sets which are finite in our sense and are *discrete* too, i.e. have decidable equality [119].

[3]Tarski has further required that S be countable.

[4]This is from where we have taken the symbol $\rhd$, used also [32,189] to denote a 'consecution' [147].

Rather than with Tarski's notion, we henceforth work with its (tantamount) restriction to finite subsets, i.e. a *(single-conclusion) entailment relation*. [5] This is a relation $\rhd \subseteq \mathrm{Fin}(S) \times S$ such that

$$\frac{U \ni a}{U \rhd a}\,(\mathrm{R}) \qquad \frac{V \rhd b \qquad V', b \rhd a}{V, V' \rhd a}\,(\mathrm{T}) \qquad \frac{U \rhd a}{U, U' \rhd a}\,(\mathrm{M})$$

for all finite $U, U', V, V' \subseteq S$ and $a, b \in S$, where as usual $U, V \equiv U \cup V$ and $V, b \equiv V \cup \{b\}$. Our focus thus is on *finite* subsets of S, for which we reserve the letters $U, V, W, \ldots$; we sometimes write $a_1, \ldots, a_n$ in place of $\{a_1, \ldots, a_n\}$ even if $n = 0$.

Remark 5.2.1. The rule (R) is equivalent, by (M), to the axiom $a \rhd a$.

Redefining

$$T^{\rhd} \equiv \{a \in S : \exists U \in \mathrm{Fin}(T)(U \rhd a)\}$$

for *arbitrary* subsets T of S gives back an algebraic closure operator on $\mathrm{Pow}(S)$. By writing $T \rhd a$ in place of $a \in T^{\rhd}$, the entailment relations thus correspond exactly to the relations satisfying Tarski's axioms above.

Given an entailment relation $\rhd$, by setting $a \leqslant b \equiv a \rhd b$ we get a preorder on S; whence the conjunction $a \approx b$ of $a \leqslant b$ and $b \leqslant a$ is an equivalence relation.

Quite often an entailment relation is inductively generated from axioms by closing up with respect to the three rules above [157]. Some leeway is required in the present chapter by allowing for generating rules other than (R), (M), and (T). If, however, these three rules are the only rules employed for inductively generating an entailment relation, we stress this by saying that this is *generated only*

[5] In the present chapter there is no need for abstract *multi-conclusion* consequence or entailment à la Scott [170–172], Lorenzen's contributions to which are currently under scrutiny [42, 44]. The relevance of multi-conclusion entailment to constructive algebra, point-free topology, etc. has been pointed out in [30], and has widely been used, e.g. in [36–40, 45, 46, 110, 134, 134, 152, 154–156, 165, 168, 192, 193].

by axioms. Given an inductively generated entailment relation $\rhd$ and a set of axioms and rules P, then we call $\rhd$ *plus* P the entailment relation inductively generated by all axioms and rules that either are used for generating $\rhd$ or belong to P.

A main feature of inductive generation is that if $\rhd$ is an entailment relation generated inductively by certain axioms and rules, then $\rhd \subseteq \rhd'$ for every entailment relation $\rhd'$ satisfying those axioms and rules. By an *extension* $\rhd'$ of an entailment relation $\rhd$ we mean in general an entailment relation $\rhd'$ such that $\rhd \subseteq \rhd'$. We say that an extension $\rhd'$ of $\rhd$ is *conservative* if also $\rhd \supseteq \rhd'$ and thus $\rhd = \rhd'$ altogether [69, 70, 154, 155].

5.3 Nuclei over entailment relations

Throughout this section, fix a set S endowed with an entailment relation $\rhd$. We say that a function $j\colon S \supset S$ is a *nucleus (over* $\rhd$*)* if for all $a, b \in S$ and $U \in \mathrm{Fin}(S)$ the following hold:

$$\frac{U, a \rhd jb}{U, ja \rhd jb}\,\mathrm{L}j \qquad\qquad \frac{U \rhd b}{U \rhd jb}\,\mathrm{R}j$$

Unlike Lj, by (R) and (T) the rule Rj can be expressed by an axiom, viz.

$$b \rhd jb \tag{5.1}$$

Remark 5.3.1. The above notion of a nucleus includes as a special case the notion of a nucleus on a locale [5, 97, 123, 159, 175, 176], which is well-known as a point-free way to put subspaces. In fact, if S is a locale with partial order $\leq$, then

$$U \rhd a \iff \bigwedge U \leq a$$

defines an entailment relation [46] such that any given map $j : S \supset S$ is a nucleus on $\rhd$ precisely when j is a nucleus on the locale S. The latter means that j satisfies

$$ja \wedge jb \leq j(a \wedge b) \tag{5.2}$$

on top of the conditions for j being a closure operator on S, which can be put as $a \leq ja$ and

$$a \leq jb \implies ja \leq jb. \tag{5.3}$$

In the presence of $a \leq ja$, which is nothing but (5.1), the conjunction of (5.2) and (5.3) is equivalent to

$$c \wedge a \leq jb \implies c \wedge ja \leq jb,$$

which in turn subsumes Lj. So the two notions of a nucleus coincide.

Example 5.3.2.

(i) Every entailment relation $\rhd$ has the trivial nucleus $j \equiv \mathrm{id}$.

(ii) Consider an algebraic structure **S** with a unary self-inverse function j (e.g. take a group as **S** and the inverse as j). The entailment relation $\rhd$ of **S**-substructures is inductively defined by

$$a_1, \ldots, a_n \rhd f(a_1, \ldots, a_n) \tag{5.4}$$

for every n-ary function f in the language of **S**, including j. We want to show that j is a nucleus on $\rhd$. Axiom (5.1) is just (5.4) for $f \equiv j$, therefore rule Rj holds. In particular, $j^2 = \mathrm{id}$ implies $j(a) \rhd a$, which, together with (T), gives rule Lj. In conclusion, j is a nucleus on $\rhd$.

(iii) Double negation $\neg\neg$ is a nucleus over intuitionistic logic $\rhd_i$ as an entailment relation (see Subsection 5.5.1 for further details and Subsections 5.5.2–5.5.3 for more nuclei in logic).

5.3.1 Entailment relations induced by a nucleus, and conservation

Consider a nucleus j over an entailment relation $\rhd$. We define

— the *weak j-extension* (or *Kleisli extension*) of $\rhd$ as the relation $\rhd_j \subseteq \mathrm{Fin}(S) \times S$ defined by

$$U \rhd_j a \iff U \rhd ja$$

— the *strong j-extension* as the entailment relation $\rhd^j \subseteq \mathrm{Fin}(S) \times S$ inductively generated by the axioms and rules of $\rhd$ plus the *stability axiom* for j:

$$ja \rhd^j a \tag{5.5}$$

In the terminology coined before, $\rhd^j$ is nothing but $\rhd$ plus the stability axiom for j.

Remark 5.3.3. By (R) in the form of $a \rhd a$ (Remark 5.2.1), stability holds for $\rhd_j$ too, that is, $ja \rhd_j a$.

Under appropriate circumstances Remark 5.3.3 will help to obtain $\rhd^j \subseteq \rhd_j$; see Theorem 5.3.8 and Corollary 5.3.9.

Lemma 5.3.4. *Let S be a set with an entailment relation $\rhd$ and let j be a nucleus on $\rhd$.*

(i) $\rhd^j$ is an entailment relation that extends $\rhd$.

(ii) $\rhd_j$ is an entailment relation that extends $\rhd$.

Proof. (i) holds by the very definition of $\rhd^j$. As for (ii): By (5.1) and Remark 5.2.1, rule (R) is bestowed from $\rhd$ to $\rhd_j$. Rule (M) is inherited from $\rhd$, and so is rule (T) in view of Lj:

$$\dfrac{U \rhd ja \qquad \dfrac{V, a \rhd jb}{V, ja \rhd jb}\,\mathrm{L}j}{U, V \rhd jb}\,(\mathrm{T})$$

Finally, also $\rhd \subseteq \rhd_j$ is a consequence of (5.1). ∎

Remark 5.3.5. The nucleus j on $\rhd$ is a nucleus also on $\rhd_j$ and $\rhd^j$. In fact, by Lemma 5.3.4 both extensions inherit axiom (5.1) from $\rhd$, and actually satisfy the following strengthening of Lj:

$$\frac{U, a \rhd b}{U, ja \rhd b}\,\mathrm{L}j^+ \;.$$

While Lj^+ for $\rhd_j$ is just Lj for $\rhd$, stability $ja \rhd a$ is tantamount to Lj^+ for any entailment relation $\rhd$ whatsoever.

To understand better whether and when $\rhd_j$ coincides with $\rhd^j$, we first consider a concrete example.

Example 5.3.6. Consider deduction in minimal logic $\rhd_m$ with the nucleus $j\varphi \equiv \varphi \vee \bot$ (see Subsection 5.5.2 below for details). Propositional minimal logic $\rhd_m$ is inductively generated by certain axioms plus the rule

$$\frac{\Gamma, \varphi \rhd_m \psi}{\Gamma \rhd_m \varphi \supset \psi}\,\mathrm{R}\!\supset$$

which cannot be expressed as an axiom. By its very definition, $\rhd_m^j$ too satisfies R⊃. Does also $\rhd_{mj}$ satisfy this rule? If this were the case, then by definition of $\rhd_{mj}$ we would have

$$\frac{\Gamma, \varphi \rhd_m \psi \vee \bot}{\Gamma \rhd_m (\varphi \supset \psi) \vee \bot}$$

As $\bot \rhd_m \psi \vee \bot$, we would obtain $\rhd_m (\bot \supset \psi) \vee \bot$. However, since minimal logic has the disjunction property and neither disjunct is provable in general, this cannot be the case. So $\rhd_j$ does not satisfy rule R⊃.

The moral of Example 5.3.6 is that $\rhd$ may already have non-axiom rules, such as R⊃, which carry over to $\rhd^j$ by its very definition, and thus need to hold in $\rhd_j$ too for the former to be conservative over the latter. To deal with this issue, we say that a rule r that holds for $\rhd$ is *compatible* with j if r also holds for $\rhd_j$.

Remark 5.3.7.

(i) Rules (R), (M), (T) are compatible with every nucleus j, by Lemma 5.3.4.

(ii) Every composition r of compatible rules is compatible. In fact, the derivation that gives r in $\rhd$ can be translated smoothly into $\rhd_j$, as all applied rules are compatible.

This is very useful: if we want to check compatibility for all rules of an entailment relation $\rhd$, it suffices to check compatibility for any set of rules that generate $\rhd$.

(iii) Every axiom $a_1, \ldots, a_n \rhd b$ can be viewed as a rule with no premiss, and as such is compatible with every nucleus j, simply by Rj. Moreover, rules

$$\frac{U, b \rhd c}{U, a_1, \ldots, a_n \rhd c} \qquad \frac{U \rhd a_1 \quad \ldots \quad U \rhd a_n}{U \rhd b}$$

which are known respectively as *left* and *right rule* [65, 157] [6] are provably equivalent to the axiom $a_1, \ldots, a_n \rhd b$ and therefore are compatible with j.

(iv) If an entailment relation $\rhd$ is generated only by axioms, then every rule that holds for $\rhd$ is compatible with any nucleus j over $\rhd$.

Theorem 5.3.8 (Conservation for nuclei)**.** *Let S be a set with an entailment relation $\rhd$ inductively generated by axioms and rules, and let j be a nucleus on $\rhd$. Then $\rhd^j$ extends $\rhd_j$, that is $\rhd_j \subseteq \rhd^j$. Moreover, the following are equivalent:*

(a) *$\rhd^j$ is conservative over $\rhd_j$, that is, $\rhd^j \subseteq \rhd_j$;*

(b) *All non-axiom rules that generate $\rhd$ are compatible with j.*

[6] A reader familiar with structural proof theory may be reminded of the notion of left and right rules in sequent calculus [132, 133]. Though they look similar, the two concepts are not to be confused.

Proof. First recall that, by its very definition, $\rhd^j$ is inductively generated by rules (R), (M), (T), stability (5.5), and all rules that generate $\rhd$. In particular, $\rhd \subseteq \rhd^j$.

Now suppose that $U \rhd_j b$, i.e. $U \rhd jb$. Since $\rhd \subseteq \rhd^j$, also $U \rhd^j jb$. Then apply

$$\frac{U \rhd^j jb \qquad jb \rhd^j b}{U \rhd^j b}\,(\mathrm{T})$$

to show $\rhd_j \subseteq \rhd^j$.

<u>(a)⇒(b)</u> (b) follows directly from (a) and the fact that $\rhd^j$ satisfies all rules that generate $\rhd$.

<u>(b)⇒(a)</u> Let us consider one by one the axioms and rules that generate $\rhd^j$:

— $\rhd_j$ satisfies (R), (M), (T), since $\rhd_j$ is an entailment relation by Lemma 5.3.4.

— $\rhd_j$ satisfies stability (5.5) by Remark 5.3.3.

— $\rhd_j$ satisfies all rules that generate $\rhd$ since they are either compatible with j by hypothesis or axioms and thus compatible with j by Remark 5.3.7.

As $\rhd^j$ is the smallest extension of $\rhd$ satisfying these axioms and rules, we get $\rhd^j \subseteq \rhd_j$. ∎

Corollary 5.3.9. *Let S be a set with an entailment relation $\rhd$ inductively generated only by axioms, and let j be a nucleus on $\rhd$. Then $\rhd^j = \rhd_j$, that is, $\rhd^j$ is a conservative extension of $\rhd_j$.*

Let j be a nucleus over an entailment relation $\rhd$ inductively generated by axioms and rules, and let $\rhd_*$ be an extension of $\rhd$. We say that $\rhd_*$ is an *intermediate j-extension* of $\rhd$ if $\rhd_*$ is $\rhd$ plus $*$ where $*$ is a collection of axioms that are valid in $\rhd^j$. In particular, $\rhd \subseteq \rhd_* \subseteq \rhd^j$.

Remark 5.3.10. Since $\rhd \subseteq \rhd_*$, we have $\rhd^j \subseteq \rhd_*^j$. On the other hand, as all axioms in $*$ already hold for $\rhd^j$, we also have $\rhd_*^j \subseteq \rhd^j$. Therefore $\rhd_*^j = \rhd^j$.

Corollary 5.3.11 (Conservation for intermediate j-extensions)**.** *Let S be a set with an entailment relation $\rhd$ inductively generated by axioms and rules, let j be a nucleus on $\rhd$, and let $\rhd_*$ be an intermediate j-extension of $\rhd$. Then $\rhd^j$ extends $\rhd_{*j}$, that is $\rhd_{*j} \subseteq \rhd^j$. Moreover, the following are equivalent:*

(a) *$\rhd^j$ is conservative over $\rhd_{*j}$, that is, $\rhd^j \subseteq \rhd_{*j}$;*

(b) *All non-axiom rules that generate $\rhd$ hold for $\rhd_{*j}$.*

Proof. Follows from Theorem 5.3.8 for $\rhd_*$ by noticing that $\rhd_*^j = \rhd^j$ (Remark 5.3.10) and that all additional rules of $\rhd_*$ are axioms and thus already compatible with j (Remark 5.3.7). ∎

The following characterisation will prove useful in several applications:

Lemma 5.3.12. *Let S be a set with an entailment relation $\rhd$, and let j be a nucleus on $\rhd$. Let r be a rule holding for $\rhd$. The following are equivalent:*

(a) *Rule r is compatible with j.*

(b) *For every instance*

$$\frac{U_1 \rhd b_1 \quad \dots \quad U_n \rhd b_n}{U \rhd b}$$

of rule r, there is $\beta \in S$ such that $\beta \rhd jb$ and

$$\frac{U_1 \rhd jb_1 \quad \dots \quad U_n \rhd jb_n}{U \rhd \beta} \tag{5.6}$$

Proof. (a)⇒(b) If we take $\beta \equiv jb$, then (b) immediately follows by reflexivity and compatibility.

(b)⇒(a) Recall that $b \rhd jb$, and that from $U \rhd \beta$ and $\beta \rhd jb$ follows $U \rhd jb$ by (T). ∎

5.4 Logic as entailment

Throughout this section, the overall assumption is that S is a set of propositional or (first-order) predicate formulae containing $\top, \bot$, and closed under the connectives $\vee, \wedge, \supset, \neg$ for propositional logic and also under the quantifiers $\forall, \exists$ for predicate logic. Following [17, 144], by *(propositional) positive logic* $\rhd_p$ we mean the positive fragment of propositional intuitionistic logic. More precisely, we define $\rhd_p$ as the least entailment relation $\rhd$ that satisfies the *deduction theorem*

$$\frac{\Gamma, \varphi \rhd \psi}{\Gamma \rhd \varphi \supset \psi} \text{R}\supset$$

and the following axioms:

$$\begin{array}{ccc} \varphi, \psi \rhd \varphi \wedge \psi & \varphi \wedge \psi \rhd \varphi & \varphi \wedge \psi \rhd \psi \\ \varphi \rhd \varphi \vee \psi & \psi \rhd \varphi \vee \psi & \varphi \vee \psi, \varphi \supset \delta, \psi \supset \delta \rhd \delta \\ \varphi, \varphi \supset \psi \rhd \psi & & \\ \rhd \top & & \end{array}$$

Of course, we understand this as an inductive definition. The above system for positive logic [144] is equivalent to the **G3**-style calculus in Table B.11 taken from [17]; they inductively generate the same entailment relation.

On top of $\rhd_p$ we consider the following additional axioms:

$$\varphi \supset \bot \approx \neg\varphi \tag{PC}$$

$$\bot \rhd \varphi \tag{EFQ}$$

$$\neg\neg\varphi \rhd \varphi \tag{RAA}$$

They are known as *principium contradictionis*, *ex falso quodlibet sequitur* and *reductio ad absurdum*. The two directions of PC can also be expressed via the rules

$$\frac{\Gamma \rhd \varphi \quad \Gamma, \bot \rhd \psi}{\Gamma, \neg\varphi \rhd \psi} \text{L}\neg \qquad\qquad \frac{\Gamma, \varphi \rhd \bot}{\Gamma \rhd \neg\varphi} \text{R}\neg$$

In the presence of EFQ, the rule L¬ can be simplified as

$$\frac{\Gamma \rhd \varphi}{\Gamma, \neg\varphi \rhd \psi}\,\mathrm{L}\neg$$

Axiom EFQ is sometimes considered as a rule without premises:

$$\frac{}{\Gamma, \bot \rhd \varphi}\,\mathrm{L}\bot$$

We define:

— *minimal logic* $\rhd_m$ as $\rhd_p$ plus PC,

— *intuitionistic logic* $\rhd_i$ as $\rhd_m$ plus EFQ,

— *classical logic* $\rhd_c$ as $\rhd_i$ plus RAA.

Let $\rhd_*$ be $\rhd_p$ plus additional axioms. In particular, $\rhd_*$ satisfies the deduction theorem R⊃. The (first-order) predicate version $\rhd_*^Q$ of $\rhd_*$, which we also refer to as $\rhd_*$ *plus quantifiers*, is then obtained by adding quantifiers ∀ and ∃ to the language and the following rules to the inductive definition of $\rhd_*$:

$$\frac{\varphi[t/x], \Gamma, \forall x\varphi \rhd \delta}{\Gamma, \forall x\varphi \rhd \delta}\,\mathrm{L}\forall \qquad \frac{\Gamma \rhd \varphi[y/x]}{\Gamma \rhd \forall x\varphi}\,\mathrm{R}\forall$$

$$\frac{\Gamma, \varphi[y/x] \rhd \delta}{\Gamma, \exists x\varphi \rhd \delta}\,\mathrm{L}\exists \qquad \frac{\Gamma \rhd \varphi[t/x]}{\Gamma \rhd \exists x\varphi}\,\mathrm{R}\exists$$

with the condition that y has to be fresh in L∃ and R∀. Rules L∀ and R∃ can be expressed as axioms:

$$\forall x\varphi \rhd \varphi[t/x]$$
$$\varphi[t/x] \rhd \exists x\varphi$$

The definition of a nucleus j given in [187] requires j to be compatible with substitution, that is,

$$j(\varphi[t/x]) \equiv (j\varphi)[t/x]$$

We prefer not to have this as a general assumption, but to make explicit whenever we need it.

5.5 Conservation for nuclei in logic

Among the usual logical rules, R$\supset$, R$\forall$ and L$\exists$ are the only ones that cannot be expressed as axioms. Rule L$\exists$ is compatible with j for every nucleus j as it does not affect the right-hand side of the sequent. Therefore, when checking compatibility of rules with j, if we do not add other rules that cannot be expressed as axioms, then the only rules we have to check are R$\supset$ and R$\forall$.

Lemma 5.5.1. *Let $\rhd_*$ be $\rhd_p$ plus additional axioms, and let j be a nucleus on $\rhd_*$. Consider $\rhd_*$ as $\rhd$.*

(i) R$\supset$ is compatible with j if and only if

$$\varphi \supset j\psi \rhd_* j(\varphi \supset \psi)$$

(ii) If j is compatible with substitution, then R$\forall$ is compatible with j if and only if

$$\forall x j\varphi \rhd_*^{Q} j\forall x\varphi$$

Proof. We prove (i), the proof of (ii) is analogous. As for "if", by Lemma 5.3.12, R$\supset$ is compatible with j if and only if for every instance

$$\mathit{sake}\ \frac{\Gamma, \varphi \rhd_* \psi}{\Gamma \rhd_* \varphi \supset \psi}$$

of R$\supset$ there is $\beta \in S$ such that $\beta \rhd_* j(\varphi \supset \psi)$ and

$$\frac{\Gamma, \varphi \rhd_* j\psi}{\Gamma \rhd_* \beta}$$

By R$\supset$, the latter condition is satisfied if we set $\beta \equiv \varphi \supset j\psi$, for which the former condition reads as

$$\varphi \supset j\psi \rhd_* j(\varphi \supset \psi).$$

As for "only if", compatibility directly entails the desired criterion. In fact, as an instance of *modus ponens* we have

$$\varphi \supset j\psi, \varphi \rhd_* j\psi,$$

which by the very definition of $\rhd_j$ is nothing but

$$\varphi \supset j\psi, \varphi \rhd_{*j} \psi.$$

By compatibility, the deduction theorem carries over from $\rhd_*$ to $\rhd_{*j}$. Hence we get

$$\varphi \supset j\psi \rhd_{*j} \varphi \supset \psi,$$

which again by the definition of $\rhd_j$ yields the desired criterion:

$$\varphi \supset j\psi \rhd_* j(\varphi \supset \psi). \qquad \blacksquare$$

This gives us the following version of Corollary 5.3.11:

Theorem 5.5.2 (Conservation for nuclei in logic). *Let $\rhd$ be $\rhd_p$ plus additional axioms, let j be a nucleus on $\rhd$, and let $\rhd_*$ be $\rhd$ plus additional axioms such that $\rhd_* \subseteq \rhd^j$.*

(i) The following are equivalent in propositional logic:

a) $\Gamma \rhd^j \varphi \iff \Gamma \rhd_* j\varphi$ for all Γ, φ

b) $\rhd_*$ satisfies the following axiom:

$$\varphi \supset j\psi \rhd_* j(\varphi \supset \psi)$$

(ii) Let $\rhd^Q$, $\rhd_*^Q$, $\rhd^{Qj}$ be $\rhd$, $\rhd_*$, $\rhd^j$ plus quantifiers. If j is compatible with substitution, then the following are equivalent in predicate logic:

a) $\Gamma \rhd^{Qj} \varphi \iff \Gamma \rhd_*^Q j\varphi$ for all Γ, φ

b) $\rhd_*^Q$ satisfies the following axioms:

$$\varphi \supset j\psi \rhd_*^Q j(\varphi \supset \psi)$$
$$\forall x j\varphi \rhd_*^Q j\forall x\varphi$$

5.5.1 The Glivenko nucleus

Take intuitionistic logic $\rhd_i$ as $\rhd$, and define

$$j\varphi \equiv \neg\neg\varphi .$$

This j is well-known to be a nucleus over $\rhd_i$ [159, 187], which we call the *Glivenko nucleus*. As stability (5.5) equals RAA, the strong extension $\rhd_i^j$ of intuitionistic logic $\rhd_i$ is nothing but classical logic $\rhd_c$.

Since $\varphi \supset \neg\neg\psi \rhd_i \neg\neg(\varphi \supset \psi)$ follows, e.g., from [186, Lemma 6.2.2], and the Glivenko nucleus is compatible with substitution, we get the following instance of Theorem 5.5.2 where $\rhd$ is $\rhd_i$:

Application 5.5.3.

*(i) (**Glivenko's Theorem**) $\Gamma \rhd_c \varphi \iff \Gamma \rhd_i \neg\neg\varphi$ for all Γ, φ in propositional logic.*

*(ii) (**Gödel's Theorem**) Let $\rhd_*$ be $\rhd_i$ plus additional axioms such that $\rhd_* \subseteq \rhd_c$, and let $\rhd_i^Q$, $\rhd_*^Q$ and $\rhd_c^Q$ be $\rhd_i$, $\rhd_*$ and $\rhd_c$ plus quantifiers. The following are equivalent in predicate logic:*

a) $\Gamma \rhd_c^Q \varphi \iff \Gamma \rhd_*^Q \neg\neg\varphi$ for all Γ, φ;

b) $\forall x \neg\neg\varphi \rhd_*^Q \neg\neg\forall x\varphi$ for all φ.

Condition (b) in Application 5.5.3 is called *Double Negation Shift* (DNS) and is known to define a proper intermediate logic $\rhd_{\mathrm{DNS}}^Q$, that is, $\rhd_i^Q \subsetneq \rhd_{\mathrm{DNS}}^Q \subsetneq \rhd_c^Q$ [61].

Now let $j\varphi \equiv \neg\varphi \supset \varphi$. This j is a nucleus [159, 187], which we call the *Peirce nucleus*, as it is a special case of the Peirce monad [60]. Over intuitionistic logic, it is easy to show that the Glivenko nucleus is equivalent to the Peirce nucleus, i.e., $\neg\neg\varphi \approx_i \neg\varphi \supset \varphi$ for every φ.

5.5.2 The Dragalin–Friedman nucleus

Take minimal logic $\rhd_m$ as $\rhd$, and define

$$j\varphi \equiv \varphi \vee \bot .$$

This j is a nucleus, in fact a *closed nucleus* [159,187]. We refer to this j as the *Dragalin–Friedman nucleus*. As stability (5.5) is equivalent to EFQ, the strong extension $\rhd_m^j$ of minimal logic $\rhd_m$ is nothing but intuitionistic logic $\rhd_i$.

Since the Dragalin–Friedman nucleus is compatible with substitution, we get the following instance of Theorem 5.5.2 where $\rhd$ is $\rhd_m$:

Application 5.5.4. *Let $\rhd_*$ be $\rhd_m$ plus additional axioms such that $\rhd_* \subseteq \rhd_i$.*

(i) The following are equivalent in propositional logic:

a) $\Gamma \rhd_i \varphi \iff \Gamma \rhd_* \varphi \vee \bot$ for all Γ, φ;

b) $\varphi \supset (\psi \vee \bot) \rhd_* (\varphi \supset \psi) \vee \bot$ for all φ, ψ.

(ii) Let $\rhd_m^Q$, $\rhd_*^Q$ and $\rhd_i^Q$ be $\rhd_m$, $\rhd_*$ and $\rhd_i$ plus quantifiers. The following are equivalent in predicate logic:

a) $\Gamma \rhd_i^Q \varphi \iff \Gamma \rhd_*^Q \varphi \vee \bot$ for all Γ, φ;

b) $\varphi \supset (\psi \vee \bot) \rhd_*^Q (\varphi \supset \psi) \vee \bot$ and $\forall x(\varphi \vee \bot) \rhd_*^Q (\forall x \varphi) \vee \bot$ for all φ, ψ.

5.5.3 The deduction nucleus

Let $\rhd$ be $\rhd_p$ or $\rhd_p^Q$ plus additional axioms. We fix a propositional formula A and set

$$j\varphi \equiv A \supset \varphi .$$

This j, which we call the *deduction nucleus*, is an instance of the *open nucleus* [159, 187]. As for this j stability (5.5) is equivalent to $\rhd A$, the strong extension $\rhd^j$ is the smallest extension of $\rhd$ in which A is derivable.

The deduction nucleus is compatible with substitution, and the following axioms are easy to show (see, e.g., [186, Lemma 6.2.1] for the case of intuitionistic logic):

$$\varphi \supset (A \supset \psi) \rhd A \supset (\varphi \supset \psi)$$
$$\forall x(A \supset \varphi) \rhd A \supset \forall x\varphi$$

Hence we get the following instance of Theorem 5.5.2 where $\rhd = \rhd_*$ is $\rhd_p$ or $\rhd_p^Q$ plus additional axioms:

Application 5.5.5. *Let $\rhd$ be $\rhd_p$ or $\rhd_p^Q$ plus additional axioms. Then*

$$\Gamma \rhd^j \varphi \iff \Gamma \rhd A \supset \varphi$$

that is, $A \supset \varphi$ is derivable from Γ if and only if φ is derivable from Γ when assuming that A is derivable.

As $\Gamma \rhd^j \varphi$ also means that φ is derivable from $\Gamma \cup \{A\}$, Application 5.5.5 is a variant of the *deduction theorem*:

$$\Gamma, A \rhd \varphi \iff \Gamma \rhd A \supset \varphi$$

6 Universal translation methods for nuclei

6.1 Introduction

Negative translations are well-known methods that turn classically derivable formulae into intuitionistically derivable ones. More precisely, a negative translation is a mapping k that operates on formulae in predicate logic such that

(i) if φ is classically derivable from Γ, then $k\varphi$ is intuitionistically derivable from $k\Gamma$, with $k\Gamma \equiv \{k\psi \colon \psi \in \Gamma\}$, and

(ii) φ and $k\varphi$ are classically equivalent.

These translations have proved to be useful also in computer science [85], set theory [4], arithmetic, and analysis [180]. The first such translation is allegedly due to Kolmogorov [100], who observed that defining

$$\begin{aligned} k\varphi &\equiv \neg\neg\varphi, && \text{for } \varphi \in \{\bot, \top\} \text{ or atomic,} \\ k(\varphi * \psi) &\equiv \neg\neg(k\varphi * k\psi), && \text{for } * \in \{\wedge, \vee, \supset\}, \\ k(Qx\,\varphi) &\equiv \neg\neg(Qx\,k\varphi), && \text{for } Q \in \{\exists, \forall\}, \end{aligned}$$

it follows that classical provability of φ is equivalent to intuitionistic provability of $k\varphi$. This translation is known as the *Kolmogorov negative translation*.

A somewhat simpler version was developed by Gentzen [79,80]. Again, if one defines

$$\begin{aligned} k\varphi &\equiv \neg\neg\varphi, && \text{for } \varphi \in \{\bot, \top\} \text{ or atomic,} \\ k(\varphi * \psi) &\equiv k\varphi * k\psi, && \text{for } * \in \{\wedge, \supset\}, \\ k(\varphi \vee \psi) &\equiv \neg\neg(k\varphi \vee k\psi), \\ k(\forall x\, \varphi) &\equiv \forall x\, k\varphi, \\ k(\exists x\, \varphi) &\equiv \neg\neg(\exists x\, k\varphi), \end{aligned}$$

then classical provability of φ is equivalent to intuitionistic provability of $k\varphi$. This translation is usually referred to as the *Gödel–Gentzen negative translation*, even though Gödel's original translation [83] was somewhat in between Kolmogorov's and Gentzen's, as he defined

$$k(\varphi \supset \psi) \equiv \neg(k\varphi \wedge \neg k\psi),$$

which is intuitionistically equivalent to $\neg\neg(k\varphi \supset k\psi)$.

Kuroda [104] proposed a different translation, which can be decomposed as $k \equiv \neg\neg J$, where J is defined as

$$\begin{aligned} J\varphi &\equiv \varphi, && \text{for } \varphi \in \{\bot, \top\} \text{ or atomic,} \\ J(\varphi * \psi) &\equiv J\varphi * J\psi, && \text{for } * \in \{\wedge, \vee\ \supset\}, \\ J(\exists x\, \varphi) &\equiv \exists x\, J\varphi, \\ J(\forall x\, \varphi) &\equiv \neg\neg(\forall x\, J\varphi). \end{aligned}$$

As before, classical provability of φ is equivalent to intuitionistic provability of $k\varphi \equiv \neg\neg J\varphi$. This translation is known as the *Kuroda negative translation*. Murthy [120] defined a variant of the Kuroda translation by setting

$$J(\varphi \supset \psi) \equiv J\varphi \supset \neg\neg J\psi.$$

This has been studied in the literature for having somewhat nicer properties than Kuroda's original version, and is called *minimal Kuroda negative translation* [71,187].

Streicher and Reus [181] introduced a translation inspired by Krivine's work [103]. By defining

$$\begin{aligned}
D\varphi &\equiv \neg\varphi, && \text{for } \varphi \in \{\bot, \top\} \text{ or atomic,}\\
D(\varphi \supset \psi) &\equiv \neg D\varphi \wedge D\psi,\\
D(\varphi \wedge \psi) &\equiv D\varphi \vee D\psi,\\
D(\varphi \vee \psi) &\equiv D\varphi \wedge D\psi,\\
D(\exists x\, \varphi) &\equiv \neg\exists x \neg D\varphi,\\
D(\forall x\, \varphi) &\equiv \exists x\, D\varphi,
\end{aligned}$$

classical provability of φ is equivalent to intuitionistic provability of $k\varphi \equiv \neg D\varphi$. This translation is known as the *Krivine negative translation*.

Finally, Ferreira and Oliva [71] showed that the aforementioned minimal Kuroda, Krivine and Gödel–Gentzen negative translations are what they call maximal simplifications of the Kolmogorov negative translation, and observed that there is a fourth one: by defining

$$\begin{aligned}
E\varphi &\equiv \neg\varphi, && \text{for } \varphi \in \{\bot, \top\} \text{ or atomic,}\\
E(\varphi \supset \psi) &\equiv \neg E\varphi \wedge E\psi,\\
E(\varphi \wedge \psi) &\equiv \neg E\varphi \supset E\psi,\\
E(\varphi \vee \psi) &\equiv E\varphi \wedge E\psi,\\
E(\exists x\, \varphi) &\equiv \forall x\, E\varphi,\\
E(\forall x\, \varphi) &\equiv \neg\forall x \neg E\varphi,
\end{aligned}$$

classical provability of φ is equivalent to intuitionistic provability of $k\varphi \equiv \neg E\varphi$. We do not consider this in what follows, as in our approach this cannot be treated in the same way as the others.

Double negation over intuitionistic logic is a typical instance of a nucleus [5, 88, 97, 123, 159, 175, 176, 187]. As also observed by van den Berg [187] and Escardó and Oliva [59, 60], a generalised version of the Gödel–Gentzen negative translation for arbitrary nuclei has already been known in logic [4], in locale theory [97] and in topos theory [98], and van den Berg himself gives a generalisation of the minimal Kuroda negative translation for arbitrary nuclei in logic. These generalisations are called *j-translations*. Our aim is to give a somewhat more general and wider insight on j-translations, by considering arbitrary sets endowed with abstract consequence relations.

6.2 Preliminaries

For the the sake of reader's convenience, we briefly recall a few key concepts from Chapter 5 and [68].

Let S be a set and $\rhd \subseteq \mathrm{Fin}(S) \times S$. A *(single-conclusion) entailment relation* as considered in [154, 155] is a relation $\rhd \subseteq \mathrm{Fin}(S) \times S$ such that

$$\frac{}{a, U \rhd a}\,(\mathrm{R}) \qquad \frac{V \rhd b \qquad V', b \rhd a}{V, V' \rhd a}\,(\mathrm{T}) \qquad \frac{U \rhd a}{U, U' \rhd a}\,(\mathrm{M})$$

for all finite $U, U', V, V' \subseteq S$ and $a, b \in S$, where as usual $U, V \equiv U \cup V$ and $V, b \equiv V \cup \{b\}$. Our focus thus is on *finite* subsets of S, for which we reserve the letters $U, V, W, \ldots$; we sometimes write $a_1, \ldots, a_n$ in place of $\{a_1, \ldots, a_n\}$ even if $n = 0$.

Remark 6.2.1. The rule (R) is equivalent, by (M), to the axiom $a \rhd a$.

Given an entailment relation $\rhd$, by setting $a \leqslant b \equiv a \rhd b$ we get a preorder on S; whence the conjunction $a \approx b$ of $a \leqslant b$ and $b \leqslant a$ is an equivalence relation.

Quite often an entailment relation is inductively generated from axioms by closing up with respect to the three rules above [157]. Some leeway is required in the present paper by allowing for generating rules other than (R), (M), and (T). Given an inductively generated entailment relation $\rhd$ and a set of axioms and rules P, then we call $\rhd$ *plus* P the entailment relation inductively generated by all axioms and rules that either are used for generating $\rhd$ or belong to P.

A main feature of inductive generation is that if $\rhd$ is an entailment relation generated inductively by certain axioms and rules, then $\rhd \subseteq \rhd'$ for every entailment relation $\rhd'$ satisfying those axioms and rules. By an *extension* $\rhd'$ of an entailment relation $\rhd$ we mean in general an entailment relation $\rhd'$ such that $\rhd \subseteq \rhd'$. We say that an extension $\rhd'$ of $\rhd$ is *conservative* if also $\rhd \supseteq \rhd'$ and thus $\rhd = \rhd'$ altogether [69, 70, 154, 155].

We say that a function $j \colon S \supset S$ is a *nucleus (over $\rhd$)* if for all $a, b \in S$ and $U \in \mathrm{Fin}(S)$ the following hold:

$$\frac{U, a \rhd jb}{U, ja \rhd jb}\,\mathrm{L}j \qquad\qquad \frac{U \rhd b}{U \rhd jb}\,\mathrm{R}j$$

Unlike Lj, by (R) and (T) the rule Rj can be expressed by an axiom, viz.

$$b \rhd jb\,.$$

Given a nucleus j over an entailment relation $\rhd$, we have the following extensions of $\rhd$:

— the *weak j-extension* (or *Kleisli extension*) of $\rhd$ as the relation $\rhd_j \subseteq \mathrm{Fin}(S) \times S$ defined by

$$U \rhd_j a \iff U \rhd ja$$

— the *strong j-extension* as the entailment relation $\rhd^j \subseteq \mathrm{Fin}(S) \times S$ inductively generated by the axioms and rules of $\rhd$ plus the *stability axiom* for j:

$$ja \rhd^j a$$

In the terminology coined before, $\rhd^j$ is nothing but $\rhd$ plus the stability axiom for j.

Since $\rhd$ may already have non-axiom rules, which carry over to $\rhd^j$ by its very definition, they need to hold in $\rhd_j$ too for the former to be conservative over the latter. To deal with this issue, we say that a rule r that holds for $\rhd$ is *compatible* with j if r also holds for $\rhd_j$.

The main result in Chapter 5 is the following

Theorem 6.2.2 (Conservation for nuclei). *Let S be a set with an entailment relation $\rhd$ inductively generated by axioms and rules, and let j be a nucleus on $\rhd$. Then $\rhd^j$ extends $\rhd_j$, that is $\rhd_j \subseteq \rhd^j$. Moreover, the following are equivalent:*

(a) *$\rhd^j$ is conservative over $\rhd_j$, that is, $\rhd^j \subseteq \rhd_j$;*

(b) *All non-axiom rules that generate $\rhd$ are compatible with j.*

6.3 General j-translations

Let a set S endowed with an entailment relation $\rhd$ inductively generated by rules, and let j be a nucleus on $\rhd$.

We say that a function $k\colon S \to S$ is a

(i) *(weak) j-translation* if

$$\frac{U \rhd^j b}{\overline{kU \rhd kb}} \tag{6.1}$$

where $kU \equiv \{ku\colon u \in U\}$;

(ii) *strong j-translation* if it satisfies the following two conditions:

$$\frac{U \rhd^j b}{kU \rhd kb} \tag{6.2}$$

$$a \approx^j ka. \tag{6.3}$$

where $\approx^j$ is the intersection of the two directions of $\rhd^j$.

Remark 6.3.1. Let $k\colon S \to S$ be a strong j-translation. Then k is a j-translation. In fact, direction "only if" of (6.1) is just (6.2) and, since $\rhd^j$ extends $\rhd$, we have that $kU \rhd kb$ implies $kU \rhd^j kb$, which is equivalent to $U \rhd^j b$ by means of $a \approx^j ka$.

Remark 6.3.2. Let $k\colon S \to S$ be a strong j-translation. Then it satisfies

$$\frac{a, U \rhd kb}{ka, U \rhd kb}\,\mathrm{L}k$$

In fact:

$$\dfrac{\dfrac{\dfrac{a, U \rhd kb}{a, U \rhd^j kb}\,\text{ext.} \qquad \dfrac{}{kb \rhd^j b}\,(6.3)}{a, U \rhd^j b}\,(\mathrm{T})}{\dfrac{ka, kU \rhd kb}{ka, U \rhd kb}\,(\mathrm{T}) \text{ with } u \rhd ku \text{ for each } u \in U}\,(6.2)$$

It follows that a strong j-translation $k\colon S \to S$ is a nucleus if and only if $a \rhd ka$.

Theorem 6.3.3. *Let S be a set with an entailment relation $\rhd$ inductively generated by axioms and rules, and let j be a nucleus on $\rhd$. Then the following are equivalent:*

(a) The nucleus j is a strong j-translation;

(b) The nucleus j is a weak j-translation;

(c) All non-axiom rules that generate $\rhd$ are compatible with j.

As one may see from the proof, this is a variant of Theorem 6.2.2 where, instead of conservation, we have the equivalent condition that the nucleus j is a j-translation.

Proof. First, notice that from Theorem 6.2.2, (c) is equivalent to

$$\frac{U \rhd^j b}{U \rhd jb}$$

which, by simple applications of Lj and Rj, is equivalent to (6.2) for $k \equiv j$.

(a)⇒(b) follows from Remark 6.3.1. If j is a weak j-translation, then in particular it satisfies (6.2), so (b)⇒(c). Since (6.3) holds for all nuclei, (c)⇒(a). ∎

6.3.1 The Kolmogorov condition

Let a function $k\colon S \to S$ and a rule

$$\frac{\{U_i \rhd b_i \colon i \leqslant n\}}{U \rhd b}\, r$$

that holds for $\rhd$. We say that k satisfies the *Kolmogorov condition* on r if the following holds:

$$\frac{\{kU_i \rhd kb_i \colon i \leqslant n\}}{kU \rhd kb}\, r_k$$

Remark 6.3.4. Let a function $k\colon S \to S$.

(i) k trivially satisfies the Kolmogorov condition on the three structural rules.

(ii) k satisfies the Kolmogorov condition on any composition of rules on each of which k satisfies the Kolmogorov condition.

(iii) If k satisfies the Kolmogorov condition on all non-structural rules that generate $\rhd$, then k satisfies the Kolmogorov condition on all admissible rules of $\rhd$.

Theorem 6.3.5. *Let S be a set with an entailment relation $\rhd$ inductively generated by axioms and rules, and let j be a nucleus on $\rhd$. A function $k\colon S \to S$ such that $a \approx^j ka$ for every $a \in S$ is a strong j-translation if and only if the following two conditions hold:*

(i) k satisfies the Kolmogorov condition on all non-structural rules in the inductive definition of $\rhd$,

(ii) $kja \rhd ka$ for every $a \in S$.

Proof. Suppose that (i) and (ii) hold. We need to prove that $U \rhd^j b$ is equivalent to $kU \rhd kb$. If $kU \rhd kb$, then $kU \rhd^j kb$ since $\rhd^j$ is an extension of $\rhd$. Then $U \rhd^j b$ follows by applications of transitivity with instances of $u \rhd^j ku$ for each $u \in U$ and with $kb \rhd^j b$. Suppose that $U \rhd^j b$. We show that $kU \rhd kb$ by induction on the derivation of $U \rhd^j b$. The cases involving structural rules are trivial. The case of the stability axiom $ja \rhd^j a$ is tantamount to $kja \rhd ka$, which holds by hypothesis. Consider the case of a non-structural rule r in the inductive definition of $\rhd$, i.e.

$$\frac{\{U_i \rhd^j b_i : i \leqslant n\}}{U \rhd^j b}\, r$$

Then

$$\frac{\left\{\dfrac{U_i \rhd^j b_i}{kU_i \rhd kb_i}\,\text{ind.hyp.} : i \leqslant n\right\}}{kU \rhd kb}\, r_k$$

where r_k can be applied because of the Kolmogorov condition.

Now suppose that k is a j-translation. Notice that (ii) follows from $ja \rhd^j a$ by the fact that k is a j-translation. To prove (i), consider a rule

$$\frac{\{U_i \rhd b_i : i \leqslant n\}}{U \rhd b}\, r$$

in the inductive definition of $\rhd$. We need to show that k satisfies the Kolmogorov condition on this rule, which—as k is a j-translation—means that the rule also holds for $\rhd^j$ in place of $\rhd$. The latter is the case by the very definition of $\rhd^j$. ■

6.3.2 The Kuroda condition

Let a function $J : S \to S$ and a rule

$$\frac{\{U_i \rhd b_i : i \leqslant n\}}{U \rhd b}\, r$$

that holds for $\rhd$. We say that J satisfies the *Kuroda condition* on r if the following holds[1]

$$\frac{\{JU_i \rhd jJb_i : i \leqslant n\}}{JU \rhd jJb}\, r_J$$

Remark 6.3.6. Consider a function $J \colon S \to S$.

(i) J satisfies the Kuroda condition on the three structural rules. The only nontrivial case is (T):

$$\dfrac{JU \rhd jJa \qquad \dfrac{Ja, JV \rhd jJb}{jJa, JV \rhd jJb}\,\mathrm{L}j}{JU, JV \rhd jJb}\,(\mathrm{T})$$

(ii) J satisfies the Kuroda condition on any composition of rules on each of which J satisfies the Kuroda condition.

(iii) If J satisfies the Kuroda condition on all non-structural rules that generate $\rhd$, then J satisfies the Kuroda condition on all admissible rules of $\rhd$.

(iv) If J satisfies the Kolmogorov condition on an axiom, then J also satisfies the Kuroda condition on this axiom by means of Rj.

Lemma 6.3.7. *Let S be a set with an entailment relation $\rhd$ inductively generated by axioms and rules, and let j be a nucleus on $\rhd$. Let $J \colon S \to S$ satisfy the Kuroda condition on all rules that generate $\rhd$, and suppose that $Jja \rhd_j Ja$ for every $a \in S$. Then*[2]

$$U \rhd^j b \Longrightarrow JU \rhd_j Jb.$$

[1] Compare this definition with Lemma 5.3.12: the Kuroda condition for r can be viewed as compatibility of r with j modulo J.

[2] This can be viewed as conservation of $\rhd^j$ over $\rhd_j$ modulo J.

Proof. Suppose that $U \rhd^j b$. We show that $JU \rhd_j Jb$ by induction on the derivation of $U \rhd^j b$. If $U \rhd^j b$ is an instance of stability, then it is of the form $jb \rhd^j b$, and the claim is tantamount to $Jjb \rhd_j Jb$, which holds by assumption.

Suppose that

$$\frac{\{U_i \rhd b_i : i \leqslant n\}}{U \rhd b}\, r$$

is a structural rule or a rule in the inductive definition of $\rhd$. Then

$$\frac{\left\{\frac{}{JU_i \rhd jJb_i}\,\text{ind.hyp.}: i \leqslant n\right\}}{JU \rhd jJb}\, r_J$$

■

Theorem 6.3.8. *Let S be a set with an entailment relation $\rhd$ inductively generated by axioms and rules, and let j be a nucleus on $\rhd$. Let J satisfy the Kuroda condition on all rules that generate $\rhd$, and suppose that $a \approx^j Ja$ and $Jja \rhd_j Ja$ for every $a \in S$. Then $k \equiv jJ$ is a strong j-translation.*

Proof. (6.3) follows from Lemma 6.3.7 by direct applications of rule Lj. As for (6.3):

$$\frac{a \rhd^j Ja \qquad Ja \rhd^j jJa}{a \rhd^j jJa}\,(\mathrm{T}) \qquad\qquad \frac{jJa \rhd^j Ja \qquad Ja \rhd^j a}{jJa \rhd^j a}\,(\mathrm{T})$$

■

6.4 Seminuclei

Let a set S endowed with an entailment relation $\rhd$ inductively generated by rules. A function $d\colon S \to S$ is a *seminucleus* on $\rhd$ if d^2 is a nucleus on $\rhd$.

Example 6.4.1.

(i) The motivating example is of course negation $\neg$ in logic. Double negation is in fact a well-known nucleus, to which we refer as the *Glivenko nucleus*.

(ii) Since identity is a nucleus, every function which is *self-inverse*, i.e. that satisfy $f^2 a = a$, is a seminucleus.

(iii) Pseudo-complements in lattice theory are an example of seminuclei.

Given $p \in S$, a *pseudo-complement seminucleus* (for short *PCSN*) relative to p is a function $\neg_p \colon S \to S$ such that:

$$\frac{U \rhd a \qquad p, U \rhd b}{\neg_p a, U \rhd b}\,\mathrm{L}\neg_p \qquad\qquad \frac{a, U \rhd p}{U \rhd \neg_p a}\,\mathrm{R}\neg_p$$

The intuition behind this is that $\neg_p a$ mimics the behaviour of the implication $a \supset p$. In literature, this is commonly known as *pseudo-complement* [53, 58, 144]. Notice that, if $b \equiv p$, then rule $\mathrm{L}\neg_p$ can be simplified as

$$\frac{U \rhd a}{\neg_p a, U \rhd p}\,\mathrm{L}\neg_p'$$

which actually turns to be equivalent to $\mathrm{L}\neg_p$ itself. Moreover, it can also be expressed by an axiom, viz.

$$a, \neg_p a \rhd p,$$

which is kind of *modus ponens* for conclusion p. By modifying the rules for PCSN as

$$\frac{U \rhd a}{-a, U \rhd b}\,\mathrm{L}- \qquad\qquad \frac{a, U \rhd -a}{U \rhd -a}\,\mathrm{R}-$$

one gets that $-a$ mimics the behaviour of $a \supset p$ for all p together, which is intuitionistic negation:

$$U \rhd -a \iff \forall b \in S(a, U \rhd b). \tag{6.4}$$

We call this $-$ a *negative seminucleus*. Similarly to $\mathrm{L}\neg_p$, rule $\mathrm{L}-$ can also be expressed by an axiom, viz.

$$a, -a \rhd b.$$

Remark 6.4.2.

(i) Let $p \in S$. If there is a PCSN relative to p, then it is unique modulo equivalence $\approx$. In fact, if both $\neg_p$ and $\sim_p$ are PCSN relative to p, then we can apply $\mathrm{R}{\sim_p}$ to the axiom $a, \neg_p a \rhd p$ and obtain $\neg_p a \rhd \sim_p a$.

(ii) Analogously to (i), one proves that if there is a negative seminucleus $-$, then it is unique modulo equivalence $\approx$.

(iii) If there is a *unit* $e \in S$ (also known as *convincing element* [153, 154]), i.e. an element such that

$$\forall a \in S (e \rhd a),$$

e.g. $e \equiv \bot$, then $-$ is just $\neg_e$ modulo equivalence $\approx$.

Proposition 6.4.3. *Let $\rhd$ be an entailment relation on S, and let d be either a PCSN relative to p or a negative seminucleus on $\rhd$. Then d satisfies the following rules:*

$$\dfrac{a, U \rhd b}{db, U \rhd da}\,Cp \qquad \dfrac{U, a \rhd d^2 b}{U, d^2 a \rhd d^2 b}\,Ld^2 \qquad \dfrac{U \rhd b}{U \rhd d^2 b}\,Rd^2$$

In particular, d is a seminucleus.

Proof. We only prove the case in which $d \equiv \neg_p$, the other is similar. Rules Cp (*contraposition*) and Rd^2 are just obtained by subsequent applications of Ld' and Rd. As for Ld^2:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{db \rhd db}\,(\mathrm{R})}{d^2 b, db \rhd p}\,\mathrm{L}d'}{db \rhd d^3 b}\,\mathrm{R}d \qquad \dfrac{a, U \rhd d^2 b}{a, d^3 b, U \rhd p}\,\mathrm{L}d'}{\dfrac{a, db, U \rhd p}{db, U \rhd da}\,\mathrm{R}d}\,(\mathrm{T})}{d^2 a, U \rhd d^2 b}\,\mathrm{Cp}$$

∎

Remark 6.4.4. Let $\rhd$ be an entailment relation on S, and let d be either a PCSN relative to p or a negative seminucleus on $\rhd$.

(i) We have a generalisation of $\mathrm{L}d^2$:

$$\frac{a, U \rhd db}{d^2 a, U \rhd db}\,\mathrm{L}d^2_d$$

We prove its admissibility in the case in which $d \equiv \neg_p$, the other is similar:

$$\dfrac{\dfrac{\dfrac{\dfrac{a, U \rhd db \qquad \dfrac{\dfrac{}{b \rhd b}\,(\mathrm{R})}{b, db \rhd p}\,\mathrm{L}d'}{a, b, U \rhd p}\,(\mathrm{T})}{b, U \rhd da}\,\mathrm{R}d}{b, d^2 a, U \rhd p}\,\mathrm{L}d'}{d^2 a, U \rhd db}\,\mathrm{R}d$$

(ii) We have $d^3 a \approx da$. In fact, $da \rhd d^3 a$ and $d^3 a \approx da$ follow from direct applications of $\mathrm{R}d^2$ and $\mathrm{L}d^2_d$, respectively.

(iii) $U \rhd da$ if and only if $U \rhd_{d^2} da$. In fact, "only if" follows from $\rhd \subseteq \rhd_{d^2}$, while "if" is a direct application of (T) of $U \rhd d^3 a$ with $d^3 a \rhd da$. While the former is equivalent to $U \rhd_{d^2} da$ by definition of $\rhd_{d^2}$, the latter holds by (ii).

Theorem 6.4.5. *Let S be a set with an entailment relation $\rhd$ inductively generated by (axioms and) rules, and let d be a either a PCSN od a negative seminucleus on $\rhd$. If all rules that generate $\rhd$ are compatible with d^2, then*

$$U \rhd^{d^2} da \iff U \rhd da$$

Proof. Direction "$\Rightarrow$" holds in general since $\rhd \subseteq \rhd^{d^2}$. By Theorem 6.2.2, if all rules that generate $\rhd$ are compatible with d^2, then $\rhd^{d^2} \subseteq \rhd_{d^2}$ which, together with Remark 6.4.4(iii), gives "$\Leftarrow$". ∎

Let S be a set with an entailment relation $\rhd$. The *closure* of $U \in \mathrm{Fin}(S)$ with respect to $\rhd$ is

$$\langle U \rangle_{\rhd} \equiv \{a \in S : \; U \rhd a\}.$$

Lemma 6.4.6. *Let S be a set with an entailment relation $\rhd$, and let d be a either PCSN or a negative seminucleus on $\rhd$. Let $U \in \mathrm{Fin}(S)$. Then:*

$$\langle U \rangle_{\rhd_{d^2}} = \left\{ a \in S : \; \forall b \in S \; \frac{a \rhd db}{U \rhd db} \right\}. \tag{6.5}$$

Proof. Take $a \in \langle U \rangle_{\rhd_{d^2}}$, which means that $U \rhd d^2 a$. If $a \rhd db$, then $d^2 a \rhd db$ by $\mathrm{L}d^2_d$. We conclude $U \rhd db$ by (T). Now take a such that $\forall b \in S (a \rhd db \Rightarrow U \rhd db)$. From $a \rhd d^2 a$ thus follows $U \rhd d^2 a$, i.e. $a \in \langle U \rangle_{\rhd_{d^2}}$. ■

The *Jacobson radical* [3] of $U \in \mathrm{Fin}(S)$ with respect to $\rhd$ is

$$\mathrm{Jac}_{\rhd}(U) \equiv \left\{ a \in S : \; \forall b \in S \; \frac{\forall c \in S(a, b \rhd c)}{\forall c \in S(U, b \rhd c)} \right\}.$$

Remark 6.4.7. If there is a unit $e \in S$, then by (T) we have

$$\mathrm{Jac}_{\rhd}(U) = \left\{ a \in S : \; \forall b \in S \; \frac{a, b \rhd e}{U, b \rhd e} \right\}.$$

Theorem 6.4.8. *Let S be a set with an entailment relation $\rhd$, and let d be a negative seminucleus on $\rhd$. Let $U \in \mathrm{Fin}(S)$. Then*

$$\langle U \rangle_{\rhd_{d^2}} = \mathrm{Jac}_{\rhd}(U)$$

Proof. Apply (6.4) to (6.5). ■

The latter is sort of a generalisation of [70, Proposition 3 and Corollary 1].

[3]The notion of Jacobson radical originates in commutative ring theory [110] and carries over to distributive lattices [22, 43, 96]. Recent work [65, 70, 191] relates it to logic and arbitrary entailment relations.

6.5 j-translations in logic

In order to make this chapter more self-contained, we briefly recall how logic can be obtained as an entailment relation. We closely follow Section 5.4 and [68]. However, in the present chapter we only consider extensions of minimal predicate logic in place of positive propositional logic.

Throughout this section, the overall assumption is that S is a set of formulae of (first-order) predicate logic containing $\top, \bot$, and closed under the connectives $\vee, \wedge, \supset$ and the quantifiers $\forall, \exists$.[4] We consider by *minimal (predicate) logic* $\rhd_m$, which is the fragment of intuitionistic predicate logic without the principle of *ex falso quodlibet sequitur*. More precisely, $\rhd_m$ is defined as the least entailment relation $\rhd$ that satisfies the following rules

$$\frac{\Gamma, \varphi \rhd \psi}{\Gamma \rhd \varphi \supset \psi}\,\mathrm{R}{\supset} \qquad \frac{\Gamma, \varphi[y/x] \rhd \delta}{\Gamma, \exists x \varphi \rhd \delta}\,\mathrm{L}\exists \qquad \frac{\Gamma \rhd \varphi[y/x]}{\Gamma \rhd \forall x \varphi}\,\mathrm{R}\forall$$

with the condition that y has to be fresh in L∃ and R∀, and the following axioms:

$$\begin{array}{ccc} \varphi, \psi \rhd \varphi \wedge \psi & \varphi \wedge \psi \rhd \varphi & \varphi \wedge \psi \rhd \psi \\ \varphi \rhd \varphi \vee \psi & \psi \rhd \varphi \vee \psi & \varphi \vee \psi, \varphi \supset \delta, \psi \supset \delta \rhd \delta \\ \varphi, \varphi \supset \psi \rhd \psi & \varphi[t/x] \rhd \exists x \varphi & \forall x \varphi \rhd \varphi[t/x] \\ \rhd \top & & \end{array}$$

Of course, we understand this as an inductive definition. In this setting, negation $\neg$ is not given as a primitive operator, but it is rather defined by

$$\neg \varphi \equiv \varphi \supset \bot.$$

The above system for minimal logic is equivalent to the **G3**-style calculus in Table B.11; the two systems inductively generate the

[4]It is worth noting that, while we explicitly talk about logic, anything in this section can easily be transferred into any setting with logic-like operators, such as lattice theory, locale theory [97], topos theory [98] and such.

same entailment relation. On top of $\rhd_m$ we consider the following additional axioms:

$$\bot \rhd \varphi \tag{EFQ}$$
$$\neg\neg\varphi \rhd \varphi \tag{RAA}$$

They are known as *ex falso quodlibet sequitur* and *reductio ad absurdum*. As usual, we define:

— *intuitionistic logic* $\rhd_i$ as $\rhd_m$ plus EFQ,

— *classical logic* $\rhd_c$ as $\rhd_i$ plus RAA.

Throughout this section, we also suppose that the nucleus j is compatible with substitution, that is,

$$j(\varphi[t/x]) \equiv (j\varphi)[t/x].$$

Since this section contains many lengthy proofs by induction on either formulae or derivations, in order to facilitate the reading we decided to move the details to Appendix A.

We start by noticing that Theorem 6.3.3 already gives us some simple examples of j-translations.

Application 6.5.1.

(i) Let $\rhd$ *be* $\rhd_i$ *plus the* double negation shift *[60]*

$$\forall x\neg\neg\varphi \rhd \neg\neg\forall x\varphi.$$

Then the Glivenko nucleus $j \equiv \neg\neg$ *is a strong* j*-translation.*

(ii) Let $\rhd$ *be* $\rhd_m$ *plus*

$$\varphi \supset (\psi \vee \bot) \rhd (\varphi \supset \psi) \vee \bot, \text{ and}$$
$$\forall x(\varphi \vee \bot) \rhd (\forall x\varphi) \vee \bot.$$

Then the Dragalin–Friedman nucleus $j\colon \varphi \mapsto \varphi \vee \bot$ *is a strong* j*-translation.*

(iii) *Let $\rhd$ be $\rhd_m$ plus additional axioms, and fix a formula A. Then the* deduction nucleus $j\colon \varphi \mapsto A \supset \varphi$ *is a strong j-translation.*

Proof. By Theorem 6.3.3, each claim is tantamount to the condition that all non-axiom rules in the inductive definition of $\rhd$, i.e. R$\supset$, R$\forall$ and L$\exists$, are compatible with the nucleus j under consideration. Rule L$\exists$ is compatible with j for every nucleus j as it does not affect the right-hand side of the sequent. Therefore, the only rules we have to check are R$\supset$ and R$\forall$. By Lemma 5.5.1, their compatibilities with j are equivalent to

$$\varphi \supset j\psi \rhd j(\varphi \supset \psi), \tag{6.6}$$
$$\forall x j\varphi \rhd j\forall x\varphi, \tag{6.7}$$

respectively. Now we can prove all three parts:

(i) (6.6) holds as observed in Subsection 5.5.1, while (6.7) holds by hypothesis.

(ii) Both (6.6) and (6.7) hold by hypothesis.

(iii) Both (6.6) and (6.7) hold, as observed in Subsection 5.5.3. ∎

6.5.1 Translations à la Kolmogorov and à la Gödel–Gentzen

Let $\rhd$ be an extension of $\rhd_m$. Given a nucleus j on $\rhd$, we inductively define

(i) the *Kolmogorov j-function k* as

(i)	$k\varphi \equiv j\varphi,$	for $\varphi \in \{\bot, \top\}$ or atomic,
(ii)	$k(\varphi * \psi) \equiv j(k\varphi * k\psi),$	for $* \in \{\wedge, \vee, \supset\}$,
(iii)	$k(Qx\varphi) \equiv j(Qx\,k\varphi),$	for $Q \in \{\exists, \forall\}$.

This is named after the Kolmogorov negative translation, which k is obtained for $j \equiv \neg\neg$, as seen in Application 6.5.4(i).

(ii) the *Gödel–Gentzen j-function k* as

$$
\begin{array}{rll}
\text{(i)} & k\varphi \equiv j\varphi, & \text{for } \varphi \in \{\bot, \top\} \text{ or atomic,} \\
\text{(ii)} & k(\varphi * \psi) \equiv k\varphi * k\psi, & \text{for } * \in \{\wedge, \supset\}, \\
\text{(iii)} & k(\varphi \vee \psi) \equiv j(k\varphi \vee k\psi), & \\
\text{(iv)} & k(\forall x\, \varphi) \equiv \forall x\, k\varphi, & \\
\text{(v)} & k(\exists x\, \varphi) \equiv j(\exists x\, k\varphi). &
\end{array}
$$

This is named after the Gödel–Gentzen negative translation, which is k obtained for $j \equiv \neg\neg$, as seen in Application 6.5.4(i).

Let k be either the Kolmogorov j-function or the Gödel–Gentzen j-function. Notice that, since the nucleus j is compatible with substitution, a straightforward proof by induction gives that also k is compatible with substitution, that is

$$k(\varphi[t/x]) \equiv (k\varphi)[t/x].$$

Remark 6.5.2. If k is the Kolmogorov j-function or the Gödel–Gentzen j-function, then

$$\frac{\alpha, \Gamma \rhd k\beta}{j\alpha, \Gamma \rhd k\beta}\, \mathrm{L}j_k$$

In fact, if k is the Kolmogorov j-function, $k\beta \equiv j\beta'$ for some $\beta' \in S$, therefore rule Lj can be applied whenever we have $k\beta$ on the right-hand side; if k is the Gödel–Gentzen j-function, then the claim is proved by induction, see Appendix A.1 for details.

Proposition 6.5.3. *Let $\rhd$ be an extension of either $\rhd_m$ or $\rhd_i$ with additional rules R, let j be a nucleus on $\rhd$ and let k be either the Kolmogorov j-function or the Gödel–Gentzen j-function. Suppose that k satisfies the Kolmogorov condition on all elements of R. Then k is a j-translation if and only if*

$$kj\alpha \rhd k\alpha$$

for all $\alpha \in S$. In such case, k is a strong j-translation.

Proof. First, $\alpha \approx^j k\alpha$ is proved by induction on α. While it is straightforward to prove that k satisfies the Kolmogorov condition for every rule in the inductive definition of $\rhd_i$, k satisfies the Kolmogorov condition for every rule in R by hypothesis. We conclude by Theorem 6.3.5. See Appendix A.1 for details. ∎

Application 6.5.4.

(i) *Let $\rhd \equiv \rhd_i$, let $j \equiv \neg\neg$ be the Glivenko nucleus and let k be either the Kolmogorov j-function or the Gödel–Gentzen j-function. Then k is a strong j-translation, known as the* Kolmogorov negative translation *[71, 100] in the Kolmogorov j-function case or as the* Gödel–Gentzen negative translation *[71, 79, 80] in the Gödel–Gentzen j-function case.*

(ii) *Let $\rhd \equiv \rhd_m$, let j be the Dragalin–Friedman nucleus*

$$j\colon \alpha \mapsto \alpha \vee \bot$$

and let k be either the Kolmogorov j-function or the Gödel–Gentzen j-function. Then k is a strong j-translation. In the Gödel–Gentzen j-function case, it is known as the Friedman's A-translation of the negative translation *[60, 74]*

(iii) *Let $\rhd$ be an extension of either $\rhd_m$ or $\rhd_i$ with additional rules R. Let $A \in S$ and let j be the deduction nucleus*

$$j\colon \alpha \mapsto A \supset \alpha.$$

Suppose that the Kolmogorov j-function k satisfies the Kolmogorov condition on all elements of R. Then k is a j-translation if and only if $\rhd kA$. In particular, if k is a j-translation, then it is a strong j translation and $\rhd^j A$.

Proof.

(i) By Proposition 6.5.3, the claim is tantamount to $k\neg\neg\alpha \rhd k\alpha$. Notice that $j\bot \equiv \neg\neg\bot \approx \bot$ and thus $k(\neg\varphi) \approx \neg k\varphi$. Then our claim is tantamount to $jk\alpha \rhd k\alpha$, which follows from $k\alpha \rhd k\alpha$ by applying $\mathrm{L}j_k$.

(ii) By Proposition 6.5.3, the claim is tantamount to $k(\alpha \vee \bot) \rhd k\alpha$. Notice that $j\bot \equiv \bot \vee \bot \approx \bot$ and thus $k(\alpha \vee \bot) \approx k\alpha \vee \bot$. Then our claim is tantamount to $j^2 k\alpha \rhd k\alpha$, which follows from $k\alpha \rhd k\alpha$ by applying $\mathrm{L}j_k$.

(iii) By Proposition 6.5.3, k is a j-translation if and only if $k(A \supset \alpha) \rhd kA$ for every α. Case Kolmogorov j-function: Notice that the claim can be written as $A \supset (kA \supset k\alpha) \rhd kA$ for every α, which is equivalent to $\rhd kA$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, kA \rhd kA}\,(\mathrm{R})}{A \rhd kA \supset kA}\,\mathrm{R}\supset}{\rhd A \supset (kA \supset kA)}\,\mathrm{R}\supset \qquad A \supset (kA \supset kA) \rhd kA}{\rhd kA}\,(\mathrm{T}) \qquad \dfrac{\dfrac{\rhd kA \qquad \dfrac{}{k\alpha \rhd k\alpha}\,(\mathrm{R})}{kA \supset k\alpha \rhd k\alpha}\,\mathrm{L}\supset}{j(kA \supset k\alpha) \rhd k\alpha}\,\mathrm{L}j_k$$

Case Gödel–Gentzen j-function: Notice that the claim can be written as $kA \supset k\alpha \rhd kA$ for every α, which is equivalent to $\rhd kA$:

$$\dfrac{\dfrac{\dfrac{}{kA \rhd kA}\,(\mathrm{R})}{\rhd kA \supset kA}\,\mathrm{R}\supset \qquad kA \supset kA \rhd kA}{\rhd kA}\,(\mathrm{T}) \qquad \dfrac{\rhd kA \qquad \dfrac{}{k\alpha \rhd k\alpha}\,(\mathrm{R})}{kA \supset k\alpha \rhd k\alpha}\,\mathrm{L}\supset$$

■

6.5.2 Translations à la Kuroda

Let $\rhd$ be an extension of $\rhd_m$. Given a nucleus j on $\rhd$, define $J\colon S \to S$ inductively as follows:

(i) $J\varphi \equiv \varphi$, for $\varphi \in \{\bot, \top\}$ or atomic,

(ii) $J(\varphi \supset \psi) \equiv J\varphi \supset jJ\psi$,

(iii) $J(\varphi * \psi) \equiv J\varphi * J\psi$, for $* \in \{\wedge, \vee\}$,

(iv) $J(\exists x\, \varphi) \equiv \exists x\, J\varphi$,

(v) $J(\forall x\, \varphi) \equiv \forall x\, jJ\varphi$.

We call this J the *Kuroda j-function*. This definition follows van den Berg [187], and is based on the minimal Kuroda negative translation, which is $k \equiv jJ$ for $j \equiv \neg\neg$, see Application 6.5.6 below.

Since the nucleus j is compatible with substitution, a straightforward proof by induction gives that also the Kuroda j-function J is compatible with substitution, that is

$$J(\varphi[t/x]) \equiv (J\varphi)[t/x].$$

Proposition 6.5.5. *Let $\rhd$ be an extension of either $\rhd_m$ or $\rhd_i$ with additional rules R, let j be a nucleus on $\rhd$ and let J be the Kuroda j-function. Suppose that J satisfies the Kuroda condition on all elements of R and that*

$$Jj\alpha \rhd jJ\alpha$$

for all $\alpha \in S$. Then $k \equiv jJ$ is a j-translation. In such case, k is a strong j-translation.

Proof. First, $\alpha \approx^j J\alpha$ is proved by induction on α. While it is straightforward to prove that J satisfies the Kuroda condition for every rule in the inductive definition of $\rhd_i$, J satisfies the Kuroda condition for every rule in R by hypothesis. We conclude by Theorem 6.3.8. See Appendix A.2 for details. ∎

Application 6.5.6.

(i) *Let $\rhd \equiv \rhd_i$ and let $j \equiv \neg\neg$ be the Glivenko nucleus. Let J be the Kuroda j-function. Then $k \equiv jJ$ is a strong j-translation, known as the* minimal Kuroda negative translation *[71,104,120,187].*

(ii) *Let $\rhd \equiv \rhd_m$, let j be the Dragalin–Friedman nucleus*

$$j\colon \alpha \mapsto \alpha \vee \bot.$$

Let J be the Kuroda j-function. Then $k \equiv jJ$ is a strong j-translation.

(iii) *Let $\rhd$ be an extension of either $\rhd_m$ or $\rhd_i$ with additional rules R. Let $A \in S$ and let j be the deduction nucleus*

$$j\colon \alpha \mapsto A \supset \alpha.$$

Suppose that the Kuroda j-function J satisfies the Kuroda condition on all elements of R. Then $k \equiv jJ$ is a j-translation if and only if $A \rhd JA$. In particular, if k is a j-translation, then it is a strong j translation.

Proof.

(i) By Proposition 6.5.5, the claim is tantamount to $J\neg\neg\alpha \rhd \neg\neg J\alpha$. Notice that $j\bot \equiv \neg\neg\bot \approx \bot$ and thus $J(\neg\varphi) \approx \neg J\varphi$.

(ii) By Proposition 6.5.5, the claim is tantamount to $J(\alpha \vee \bot) \rhd J\alpha \vee \bot$, which holds since $J(\alpha \vee \bot) \equiv J\alpha \vee \bot$ by definition of J.

(iii) By Proposition 6.5.5, k is a j-translation if and only if $J(A \supset \alpha) \rhd A \supset J\alpha$ for every α. Notice that the latter can be written as $JA \supset (A \supset J\alpha) \rhd A \supset J\alpha$, for every α, which is equivalent to $A \rhd JA$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{JA, A \rhd JA}\,(\mathrm{R})}{JA \rhd A \supset JA}\,\mathrm{R}{\supset}}{\rhd JA \supset (A \supset JA)}\,\mathrm{R}{\supset} \qquad JA \supset (A \supset JA) \rhd A \supset JA}{\rhd A \supset JA}\,(\mathrm{T}) \qquad \dfrac{}{A, A \supset JA \rhd JA}\,\mathrm{ax.}}{A \rhd JA}\,(\mathrm{T})$$

and:

$$\dfrac{\dfrac{\dfrac{}{A \supset J\alpha, A \rhd J\alpha}\,\mathrm{ax.} \qquad A \rhd JA}{JA \supset (A \supset J\alpha), A \rhd J\alpha}}{JA \supset (A \supset J\alpha) \rhd A \supset J\alpha}\,\mathrm{R}{\supset}$$

■

6.5.3 Translations à la Krivine

Let $\rhd$ be an extension of $\rhd_m$, fix a formula π and consider the PCSN $\neg_\pi$. As for j, we assume that $\neg_\pi$ is compatible with substitution, that is

$$\neg_\pi(\varphi[t/x]) \equiv (\neg_\pi\varphi)[t/x].$$

Define $D\colon S \to S$ inductively as follows:

$$\begin{array}{rll} \text{(i)} & D\varphi \equiv \neg_\pi \varphi, & \text{for } \varphi \in \{\bot, \top\} \text{ or atomic,} \\ \text{(ii)} & D(\varphi \supset \psi) \equiv \neg_\pi D\varphi \wedge D\psi, & \\ \text{(iii)} & D(\varphi \wedge \psi) \equiv D\varphi \vee D\psi, & \\ \text{(iv)} & D(\varphi \vee \psi) \equiv D\varphi \wedge D\psi, & \\ \text{(v)} & D(\exists x\, \varphi) \equiv \neg_\pi \exists x \neg_\pi D\varphi, & \\ \text{(vi)} & D(\forall x\, \varphi) \equiv \exists x\, D\varphi. & \end{array}$$

We call this D the *Krivine $\neg_\pi$-function*. This is named after the Krivine negative translation which is $k \equiv \neg_\pi D$ for $\pi \equiv \bot$, as seen in Application 6.5.8(i). Since $\neg_\pi$ is compatible with substitution, a straightforward proof by induction gives that also the Krivine $\neg_\pi$-function D is compatible with substitution, that is

$$D(\varphi[t/x]) \equiv (D\varphi)[t/x].$$

Proposition 6.5.7. *Let $\rhd$ be an extension of either $\rhd_m$ or $\rhd_i$ with additional rules R, let $\neg_\pi$ be a PCSN on $\rhd$ and let D be the Krivine $\neg_\pi$-function. Suppose that D satisfies the Kolmogorov condition on all elements of R. Then $k \equiv \neg_\pi D$ is a $\neg_\pi^2$-translation if and only if*

$$k\neg_\pi^2 \alpha \rhd k\alpha$$

for all $\alpha \in S$. In such case, k is a strong $\neg_\pi^2$-translation.

Proof. Analogous to Proposition 6.5.3. See Appendix A.3 for details. ∎

Application 6.5.8. *Let $\rhd \equiv \rhd_i$ and $\varphi \equiv \bot$, which means $\neg_\pi \equiv \neg$. Let D be the Krivine $\neg_\pi$-function. Then $k \equiv \neg_\pi D$ is a strong $\neg_\pi^2$-translation, known as the* Krivine negative translation *[71, 181].*

Proof. By Proposition 6.5.7, the claim is tantamount to

$$\neg D \neg\neg \alpha \rhd \neg D\alpha.$$

We have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D\bot, D\alpha \triangleright D\alpha}\,(\mathrm{R})}{\neg D\alpha, D\bot, D\alpha \triangleright \bot}\,\mathrm{L}\neg'}{\neg D\alpha \wedge D\bot, D\alpha \triangleright \bot}\,\mathrm{L}\wedge}{D\neg\alpha, D\alpha \triangleright \bot}\,\text{dfn.}}{D\alpha \triangleright \neg D\neg\alpha}\,\mathrm{R}\neg \qquad \dfrac{\dfrac{\dfrac{}{\bot, D\alpha \triangleright \bot}\,(\mathrm{R})}{D\alpha \triangleright \neg\bot}\,\mathrm{R}\neg}{D\alpha \triangleright D\bot}\,\text{dfn.}}{\dfrac{D\alpha \triangleright \neg D\neg\alpha \wedge D\bot}{\dfrac{D\alpha \triangleright D\neg\neg\alpha}{\neg D\neg\neg\alpha \triangleright \neg D\alpha}\,\mathrm{Cp}}\,\text{dfn.}}\,\mathrm{R}\wedge$$

■

Part III

Conclusions, Appendices, Bibliography

Conclusive remarks, future and related work

The calculus **G3K** used as a basis for the results of Chapter 2 is classical, but the applications studied up to now have a purely constructive proof in their algebraic counterpart. This makes us confident that we can replace **G3K** by an intuitionistic modal calculus, such as the one presented in [113].

Furthermore, those applications have not yet suggested a general method to find the subformula $U(x)$ required to define the valuation; whence we will next try to pin down such a general method.

Other principles related to induction are worth a closer look. Apart from the notions of Noetherianity discussed in [51, 139], also the principles of transitivity and irreflexivity deserve further investigation, especially in connection with *Cut*-elimination, as well as the variant GH of the Gödel–Löb axiom [25].

There is already some work in progress on relating this approach with induction on ordinals, as ordinals can be characterised by Gödel–Löb Induction and some additional properties of the relation $<$. For instance, ω is characterised by Gödel–Löb Induction, the property that each node has a successor (*seriality*) and the property that each node is either zero or a successor.

Since the logic **Grz** studied in Chapter 3 is characterised by reflexive, transitive and Noetherian frames, we also intend to use the approach of Chapter 2 to define a variant of induction principle, which we may dub *Grzegorczyk induction* corresponding to rule R□Z:

$$\forall x.\forall y < x(GE(y) \Longrightarrow E(y)) \Longrightarrow \forall y < x\, E(y),$$

where $GE(y)$ is an abbreviation for $\forall z < y(E(z) \Longrightarrow \forall w < z\, E(w))$. This can be considered a weak form of induction compatible with reflexivity, and may give a different perspective of the semantics of both **Grz** and **Int** and may give some insights on the properties of the accessibility relation.

We then plan to extend the approach of Chapter 3 to extensions of **Int**, such as intermediate logics [56, 129], modal intuitionistic logic [113] and possibly bi-intuitionistic logic [141].

In Chapter 4, we have proved that classical derivability entails intuitionistic or even minimal derivability for seven infinitary Glivenko sequent classes. This result naturally extends the results presented in [130] for the finitary Glivenko sequent classes and those in [131] for the Barr's theorem for infinitary geometric theories. Moreover, we have also shown how to constructivise the cut-elimination procedure for geometric logics given in [131]: by introducing the notion of proof embeddability and by making use of Brouwer's principle of Bar Induction we have given an ordinal-free proof of cut-elimination that works within **IZF** (but not within **CZF**). The present proof strategy should allow to constructivise the cut-elimination procedure for other infinitary calculi such as those in [64, 111, 182].

One question that remains open is whether the seven infinitary Glivenko sequent classes considered here are optimal for conservativity or not. Orevkov [137] proved that this holds for the finitary case by listing the other possible classes of sequent and founding a sequent that is classically but not intuitionistically derivable in each class. We leave this question for future research.

In propositional lax logic (PLL) [63] the modality ◯ is characterised by axioms and rules corresponding [63, p. 2, (2)] to the ones of a (logical) nucleus. Also the rules Lj and Rj of Chapter 5 are counterparts of the rules ◯L and ◯R of PLL [63, p. 5]. We expect to gain insight by relating our approach to PLL, its semantics and applications. To start with, in the vein of [63, Lemma 2.1] rule Rj is tantamount to the inverse of Lj.

We will further study nuclei about other forms of negation: weak negation over positive logic [17], co-negation over dual logics [14] and strong negation over extensions of intuitionistic logic [101, 190]. It will be a challenge to include also other proof translation methods. For instance, Friedman's A-translation [73] makes use of the closed nucleus to prove Markov's rule; and Ishihara and Nemoto [94] use the same translation but work with the open nucleus to prove the independence-of-premiss rule.

A Details of the inductive proofs in Section 6.5

For the sake of brevity, we often write "(T) w/ stab." as the label of a rule if we are applying (T) and omit a branch which only consists of a leaf that is an instance of stability. Similarly, we often write "(T) w/ i.h." as the label of a rule if we are applying (T) and omit a branch which only consists of a leaf that is an instance of the induction hypothesis.

A.1 Translations à la Kolmogorov and à la Gödel–Gentzen

Proof of Remark 6.5.2

Proof by induction that the Gödel–Gentzen j-function k satisfies rule Lj_k:

— If $\beta \in \{\bot, \top, \varphi \vee \psi, \exists x\, \varphi\}$ or atomic, then $k\beta \equiv j\beta'$ for some $\beta' \in S$, so rule Lj can be applied.

— Suppose that $\beta \equiv \varphi \wedge \psi$. We have $\alpha, \Gamma \rhd k\varphi \wedge k\psi$, which is equivalent to have both $\alpha, \Gamma \rhd k\varphi$ and $\alpha, \Gamma \rhd k\psi$. By induction hypothesis, we have $j\alpha, \Gamma \rhd k\varphi$ and $j\alpha, \Gamma \rhd k\psi$, which together yield to $j\alpha, \Gamma \rhd k\varphi \wedge k\psi$.

— Suppose that $\beta \equiv \varphi \supset \psi$. We have $\alpha, \Gamma \rhd k\varphi \supset k\psi$, which is equivalent to $\alpha, \Gamma, k\varphi \rhd k\psi$. By induction hypothesis, we have $j\alpha, \Gamma, k\varphi \rhd k\psi$, which yields to $j\alpha, \Gamma \rhd k\varphi \supset k\psi$.

— Suppose that $\beta \equiv \forall x\, \varphi$. We have $\alpha, \Gamma \rhd \forall x\, k\varphi$, which is equivalent to $\alpha, \Gamma \rhd k\varphi[t/x]$ for all terms t. By induction hypothesis, we have $j\alpha, \Gamma \rhd k\varphi[t/x]$ for all terms t, which yields to $j\alpha, \Gamma \rhd \forall x\, k\varphi$. ■

Proof of Proposition 6.5.3

Case k **Kolmogorov** j**-function** Proof of $\alpha \approx^j k\alpha$ by induction on α:

— If $\alpha \in \{\bot, \top\}$ or is atomic, then $\alpha \approx^j k\alpha$ directly follows from $k\alpha \equiv j\alpha$.

— Suppose $\alpha \equiv \varphi \wedge \psi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \rhd^j k\varphi}\,\text{i.h.}}{\varphi, \psi \rhd^j k\varphi}\,(\text{M}) \qquad \dfrac{\dfrac{}{\psi \rhd^j k\psi}\,\text{i.h.}}{\varphi, \psi \rhd^j k\psi}\,(\text{M})}{\varphi, \psi \rhd^j k\varphi \wedge k\psi}\,\text{R}\wedge}{\varphi \wedge \psi \rhd^j k\varphi \wedge k\psi}\,\text{L}\wedge}{\varphi \wedge \psi \rhd^j j(k\varphi \wedge k\psi)}\,\text{R}j$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{k\varphi \rhd^j \varphi}\,\text{i.h.}}{k\varphi, k\psi \rhd^j \varphi}\,(\text{M}) \qquad \dfrac{\dfrac{}{k\psi \rhd^j \psi}\,\text{i.h.}}{k\varphi, k\psi \rhd^j \psi}\,(\text{M})}{k\varphi, k\psi \rhd^j \varphi \wedge \psi}\,\text{R}\wedge}{k\varphi \wedge k\psi \rhd^j \varphi \wedge \psi}\,\text{L}\wedge}{j(k\varphi \wedge k\psi) \rhd^j \varphi \wedge \psi}\,(\text{T})\text{ w/ stab.}$$

In conclusion, $\varphi \wedge \psi \approx^j k(\varphi \wedge \psi)$.

— Suppose $\alpha \equiv \varphi \vee \psi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \rhd^j k\varphi}\,\text{i.h.}}{\varphi \rhd^j k\varphi \vee k\psi}\,\text{R}\vee \qquad \dfrac{\dfrac{}{\psi \rhd^j k\psi}\,\text{i.h.}}{\psi \rhd^j k\varphi \vee k\psi}\,\text{R}\vee}{\varphi \vee \psi \rhd^j k\varphi \vee k\psi}\,\text{L}\vee}{\varphi \vee \psi \rhd^j j(k\varphi \vee k\psi)}\,\text{R}j$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{}{k\varphi \rhd^j \varphi}\,\text{i.h.}}{k\varphi \rhd^j \varphi \vee \psi}\,\text{R}\vee \qquad \dfrac{\dfrac{}{k\psi \rhd^j \psi}\,\text{i.h.}}{k\psi \rhd^j \varphi \vee \psi}\,\text{R}\vee}{k\varphi \vee k\psi \rhd^j \varphi \vee \psi}\,\text{L}\vee}{j(k\varphi \vee k\psi) \rhd^j \varphi \vee \psi}\,\text{(T) w/ stab.}$$

In conclusion, $\varphi \vee \psi \approx^j k(\varphi \vee \psi)$.

— Suppose $\alpha \equiv \varphi \supset \psi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{k\varphi \rhd^j \varphi}\,\text{i.h.} \qquad \dfrac{\dfrac{}{\psi \rhd^j k\psi}\,\text{i.h.}}{\psi, k\varphi \rhd^j k\psi}\,\text{(M)}}{\varphi \supset \psi, k\varphi \rhd^j k\psi}\,\text{L}\supset}{\varphi \supset \psi \rhd^j k\varphi \supset k\psi}\,\text{R}\supset}{\varphi \supset \psi \rhd^j j(k\varphi \supset k\psi)}\,\text{R}j$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \rhd^j k\varphi}\,\text{i.h.} \qquad \dfrac{\dfrac{}{k\psi \rhd^j \psi}\,\text{i.h.}}{k\psi, \varphi \rhd^j \psi}\,\text{(M)}}{k\varphi \supset k\psi, \varphi \rhd^j \psi}\,\text{L}\supset}{k\varphi \supset k\psi \rhd^j \varphi \supset \psi}\,\text{R}\supset}{j(k\varphi \supset k\psi) \rhd^j \varphi \supset \psi}\,\text{(T) w/ stab.}$$

In conclusion, $\varphi \supset \psi \approx^j k(\varphi \supset \psi)$.

— Suppose $\alpha \equiv \forall x\,\varphi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \rhd^j k\varphi}\,\text{i.h.}}{\varphi[y/x] \rhd^j k\varphi[y/x]}\,\text{Subs.}}{\forall x\,\varphi \rhd^j k\varphi[y/x]}\,\text{L}\forall}{\forall x\,\varphi \rhd^j \forall x\,k\varphi}\,\text{R}\forall\ (y\text{ fresh})}{\forall x\,\varphi \rhd^j j\forall x\,k\varphi}\,\text{R}j$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{k\varphi \rhd^j \varphi}\,\text{i.h.}}{k\varphi[y/x] \rhd^j \varphi[y/x]}\,\text{Subs.}}{\forall x\,k\varphi \rhd^j \varphi[y/x]}\,\text{L}\forall}{\forall x\,k\varphi \rhd^j \forall x\,\varphi}\,\text{R}\forall\ (y\text{ fresh})}{j\forall x\,k\varphi \rhd^j \forall x\,\varphi}\,\text{(T) w/ stab.}$$

In conclusion, $\forall x\,\varphi \approx^j k\forall x\,\varphi$.

— Suppose $\alpha \equiv \exists x\,\varphi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \rhd^j k\varphi}\,\text{i.h.}}{\varphi[y/x] \rhd^j k\varphi[y/x]}\,\text{Subs.}}{\varphi[y/x] \rhd^j \exists x\,k\varphi}\,\text{R}\exists}{\exists x\,\varphi \rhd^j \exists x\,k\varphi}\,\text{L}\exists\ (y\text{ fresh})}{\exists x\,\varphi \rhd^j j\exists x\,k\varphi}\,\text{R}j$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{k\varphi \rhd^j \varphi}\text{i.h.}}{k\varphi[y/x] \rhd^j \varphi[y/x]}\text{Subs.}}{k\varphi[y/x] \rhd^j \exists x\,\varphi}\text{R}\exists}{\exists x\, k\varphi \rhd^j \exists x\,\varphi}\text{L}\exists\ (y \text{ fresh})}{j\exists x\, k\varphi \rhd^j \exists x\,\varphi}\text{(T) w/ stab.}$$

In conclusion, $\exists x\,\varphi \approx^j k\exists x\,\varphi$.

Proof that k satisfies the Kolmogorov condition for every rule in the inductive definition of $\rhd_i$:

— Consider R⊃. We have to show that

$$\dfrac{k\Gamma, k\varphi \rhd k\psi}{k\Gamma \rhd \underbrace{k(\varphi \supset \psi)}_{j(k\varphi \supset k\psi)}}\text{R}{\supset_k}$$

which is an application of R⊃ followed by Rj. Case R∀ is similar.

— Consider $\varphi, \varphi \supset \psi \rhd \psi$. We have to show that $k\varphi, k(\varphi \supset \psi) \rhd k\psi$, i.e. $k\varphi, j(k\varphi \supset k\psi) \rhd k\psi$, which follows from $k\varphi, k\varphi \supset k\psi \rhd k\psi$ by an application of Lj_k. Cases $\varphi \wedge \psi \rhd \varphi$; $\varphi \wedge \psi \rhd \psi$; $\bot \rhd \varphi$; $\forall x\,\varphi \rhd \varphi[t/x]$ are similar.

— Consider $\varphi, \psi \rhd \varphi \wedge \psi$. We have to show that $k\varphi, k\psi \rhd k(\varphi \wedge \psi)$, i.e. $k\varphi, k\psi \rhd j(k\varphi \wedge k\psi)$, which follows from $k\varphi, k\psi \rhd k\varphi \wedge k\psi$ by an application of Rj. Cases $\varphi \rhd \varphi \vee \psi$; $\psi \rhd \varphi \vee \psi$; $\rhd \top$; $\varphi[t/x] \rhd \exists x\,\varphi$ are similar.

— Consider $\varphi \vee \psi, \varphi \supset \delta, \psi \supset \delta \rhd \delta$. We have to show that $k(\varphi \vee \psi), k(\varphi \supset \delta), k(\psi \supset \delta) \rhd k\delta$, i.e. $j(k\varphi \vee k\psi), j(k\varphi \supset k\delta), j(k\psi \supset k\delta) \rhd k\delta$, which follows from $k\varphi \vee k\psi, k\varphi \supset k\delta, k\psi \supset k\delta \rhd k\delta$ by applying Lj_k three times.

— Consider L∃. We have to show that

$$\dfrac{k\Gamma, k\varphi[y/x] \rhd k\delta}{k\Gamma, \underbrace{k\exists x\,\varphi}_{j\exists x\,k\varphi} \rhd k\delta}\,\mathrm{L}\exists_k$$

with y fresh, which is an application of L∃ followed by Lj_k.

Case k Gödel–Gentzen j-function In the proof of $\alpha \approx^j k\alpha$ by induction on α, cases that involve ∨ or ∃ are dealt with as in the case of the Kolmogorov j-function, while the remaining cases are trivial. Any Gödel–Gentzen j-function k satisfies the Kolmogorov condition for every rule in the inductive definition of $\rhd \in \{\rhd_m, \rhd_i\}$: Again, cases that involve ∨ or ∃ are dealt with as in the case of the Kolmogorov j-function, while the remaining cases are trivial. ■

A.2 Translations à la Kuroda

Proof of Proposition 6.5.5

The proof of $\alpha \approx^j J\alpha$ proceeds as the proof of $\alpha \approx^j k\alpha$ in Proposition 6.5.3 with slight adjustments in the derivations: we replace k by J, and apply (T) either with stability or with (5.1) in order to introduce j on the leaves to which we otherwise would not have the induction hypothesis. We give the proof of $\alpha \approx^j jJ\alpha$ in the case $\alpha \equiv \varphi \supset \psi$ as an example. Here the induction hypothesis states that $\varphi \approx^j jJ\varphi$ and $\psi \approx^j jJ\psi$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{J\varphi \rhd jJ\varphi}\,(5.1) \qquad \dfrac{}{jJ\varphi \rhd^j \varphi}\,\text{i.h.}}{J\varphi \rhd^j \varphi}\,(\mathrm{T}) \qquad \dfrac{\dfrac{}{\psi \rhd^j jJ\psi}\,\text{i.h.}}{\psi, J\varphi \rhd^j jJ\psi}\,(\mathrm{M})}{\varphi \supset \psi, J\varphi \rhd^j jJ\psi}\,\mathrm{L}\supset}{\varphi \supset \psi \rhd^j J\varphi \supset jJ\psi}\,\mathrm{R}\supset}{\varphi \supset \psi \rhd^j j(J\varphi \supset jJ\psi)}\,\mathrm{R}j$$

and

$$\dfrac{\dfrac{\dfrac{}{\varphi \rhd^j jJ\varphi}\text{i.h.} \quad \dfrac{}{jJ\varphi \rhd J\varphi}\text{stab.}}{\varphi \rhd^j J\varphi}(\text{T}) \quad \dfrac{\dfrac{}{jJ\psi \rhd^j \psi}\text{i.h.}}{jJ\psi, \varphi \rhd^j \psi}(\text{M})}{\dfrac{J\varphi \supset jJ\psi, \varphi \rhd^j \psi}{\dfrac{J\varphi \supset jJ\psi \rhd^j \varphi \supset \psi}{j(J\varphi \supset jJ\psi) \rhd^j \varphi \supset \psi}(\text{T}) \text{ w/ stab.}}\text{R}\supset}\text{L}\supset$$

Proof that J satisfies the Kuroda condition for every rule in the inductive definition of $\rhd_i$:

— Consider R⊃. We need to show that

$$\dfrac{J\Gamma, J\varphi \rhd jJ\psi}{J\Gamma \rhd j\underbrace{J(\varphi \supset \psi)}_{(J\varphi \supset jJ\psi)}}\text{R}\supset_J$$

which is an application of R⊃ followed by Rj. Case R∀ is similar.

— Consider $\varphi, \varphi \supset \psi \rhd \psi$. We have to show that $J\varphi, J(\varphi \supset \psi) \rhd jJ\psi$, i.e. $J\varphi, J\varphi \supset jJ\psi \rhd jJ\psi$, which is an instance of an axiom. Cases $\varphi \vee \psi, \varphi \supset \delta, \psi \supset \delta \rhd \delta$; $\bot \rhd \varphi$; $\forall x \varphi \rhd \varphi[t/x]$ are similar.

— Consider $\varphi, \psi \rhd \varphi \wedge \psi$. We have to show that $J\varphi, J\psi \rhd jJ(\varphi \wedge \psi)$, i.e. $J\varphi, J\psi \rhd j(J\varphi \wedge J\psi)$, which follows from $J\varphi, J\psi \rhd J\varphi \wedge J\psi$ by an application of Rj. Cases $\varphi \wedge \psi \rhd \varphi$; $\varphi \wedge \psi \rhd \psi$; $\varphi \rhd \varphi \vee \psi$; $\psi \rhd \varphi \vee \psi$; $\rhd \top$; $\varphi[t/x] \rhd \exists x \varphi$ are similar.

— Consider L∃. We have to show that

$$\dfrac{J\Gamma, J\varphi[y/x] \rhd jJ\delta}{J\Gamma, \underbrace{J\exists x \varphi}_{\exists x J\varphi} \rhd jJ\delta}\text{L}\exists_J$$

with y fresh, which is an application of L∃. ■

A.3 Translations à la Krivine

Proof of Proposition 6.5.7

Proof of $\alpha \approx^{j} k\alpha$ by induction on α:

— If $\alpha \in \{\bot, \top\}$ or atomic, then $\alpha \approx^{\neg_\pi^2} \neg_\pi D\alpha$ directly follows from $\neg_\pi D\alpha \equiv \neg_\pi^2 \alpha$.

— Suppose $\alpha \equiv \varphi \wedge \psi$. By induction hypothesis we have $\varphi \approx^{\neg_\pi^2} \neg_\pi D\varphi$ and $\psi \approx^{\neg_\pi^2} \neg_\pi D\psi$. Then

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(R)}
\UnaryInfC{$D\varphi, \psi \rhd^{\neg_\pi^2} D\varphi$}
\RightLabel{$L\neg_\pi'$}
\UnaryInfC{$D\varphi, \neg_\pi D\varphi, \psi \rhd^{\neg_\pi^2} \pi$}
\RightLabel{(T) w/ i.h.}
\UnaryInfC{$D\varphi, \varphi, \psi \rhd^{\neg_\pi^2} \pi$}
\AxiomC{}
\RightLabel{(R)}
\UnaryInfC{$D\psi, \varphi \rhd^{\neg_\pi^2} D\psi$}
\RightLabel{$L\neg_\pi'$}
\UnaryInfC{$D\psi, \varphi, \neg_\pi D\psi \rhd^{\neg_\pi^2} \pi$}
\RightLabel{(T) w/ i.h.}
\UnaryInfC{$D\psi, \varphi, \psi \rhd^{\neg_\pi^2} \pi$}
\RightLabel{$L\vee$}
\BinaryInfC{$D\varphi \vee D\psi, \varphi, \psi \rhd^{\neg_\pi^2} \pi$}
\RightLabel{$R\neg_\pi$}
\UnaryInfC{$\varphi, \psi \rhd^{\neg_\pi^2} \neg_\pi(D\varphi \vee D\psi)$}
\RightLabel{$L\wedge$}
\UnaryInfC{$\varphi \wedge \psi \rhd^{\neg_\pi^2} \neg_\pi(D\varphi \vee D\psi)$}
\end{prooftree}
$$

and

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{i.h.}
\UnaryInfC{$\neg_\pi D\varphi \rhd_{\neg_\pi^2} \varphi$}
\RightLabel{Cp}
\UnaryInfC{$\neg_\pi \varphi \rhd^{\neg_\pi^2} \neg_\pi^2 D\varphi$}
\RightLabel{(T) w/ stab.}
\UnaryInfC{$\neg_\pi \varphi \rhd^{\neg_\pi^2} D\varphi$}
\RightLabel{$R\vee$}
\UnaryInfC{$\neg_\pi \varphi \rhd^{\neg_\pi^2} D\varphi \vee D\psi$}
\RightLabel{Cp}
\UnaryInfC{$\neg_\pi(D\varphi \vee D\psi) \rhd^{\neg_\pi^2} \neg_\pi^2 \varphi$}
\RightLabel{(T) w/ stab.}
\UnaryInfC{$\neg_\pi(D\varphi \vee D\psi) \rhd^{\neg_\pi^2} \varphi$}
\AxiomC{}
\RightLabel{i.h.}
\UnaryInfC{$\neg_\pi D\psi \rhd_{\neg_\pi^2} \psi$}
\RightLabel{Cp}
\UnaryInfC{$\neg_\pi \psi \rhd^{\neg_\pi^2} \neg_\pi^2 D\psi$}
\RightLabel{(T) w/ stab.}
\UnaryInfC{$\neg_\pi \psi \rhd^{\neg_\pi^2} D\psi$}
\RightLabel{$R\vee$}
\UnaryInfC{$\neg_\pi \psi \rhd^{\neg_\pi^2} D\varphi \vee D\psi$}
\RightLabel{Cp}
\UnaryInfC{$\neg_\pi(D\varphi \vee D\psi) \rhd^{\neg_\pi^2} \neg_\pi^2 \psi$}
\RightLabel{(T) w/ stab.}
\UnaryInfC{$\neg_\pi(D\varphi \vee D\psi) \rhd^{\neg_\pi^2} \psi$}
\RightLabel{$R\wedge$}
\BinaryInfC{$\neg_\pi(D\varphi \vee D\psi) \rhd^{\neg_\pi^2} \varphi \wedge \psi$}
\end{prooftree}
$$

In conclusion, $\varphi \wedge \psi \approx^{\neg_\pi^2} \neg_\pi D(\varphi \wedge \psi)$.

— Suppose $\alpha \equiv \varphi \vee \psi$. By induction hypothesis we have $\varphi \approx^{\neg_\pi^2} \neg_\pi D\varphi$ and $\psi \approx^{\neg_\pi^2} \neg_\pi D\psi$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D\varphi, D\psi, \rhd^{\neg_\pi^2} D\varphi}\,(\mathrm{R})}{D\varphi, D\psi, \neg_\pi D\varphi \rhd^{\neg_\pi^2} \pi}\,\mathrm{L}\neg'_\pi}{D\varphi, D\psi, \varphi \rhd^{\neg_\pi^2} \pi}\,(\mathrm{T})\text{ w/ i.h.} \qquad \dfrac{\dfrac{\dfrac{}{D\varphi, D\psi, \rhd^{\neg_\pi^2} D\psi}\,(\mathrm{R})}{D\varphi, D\psi, \neg_\pi D\psi \rhd^{\neg_\pi^2} \pi}\,\mathrm{L}\neg'_\pi}{D\varphi, D\psi, \psi \rhd^{\neg_\pi^2} \pi}\,(\mathrm{T})\text{ w/ i.h.}}{D\varphi, D\psi, \varphi \vee \psi \rhd^{\neg_\pi^2} \pi}\,\mathrm{L}\vee}{D\varphi \wedge D\psi, \varphi \vee \psi \rhd^{\neg_\pi^2} \pi}\,\mathrm{L}\wedge}{\varphi \vee \psi \rhd^{\neg_\pi^2} \neg_\pi(D\varphi \wedge D\psi)}\,\mathrm{R}\neg_\pi$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_\pi D\varphi \rhd^{\neg_\pi^2} \varphi}\,\text{i.h.}}{\neg_\pi D\varphi \rhd^{\neg_\pi^2} \varphi \vee \psi}\,\mathrm{R}\vee}{\neg_\pi(\varphi \vee \psi) \rhd^{\neg_\pi^2} \neg_\pi^2 D\varphi}\,\text{C.p.}}{\neg_\pi(\varphi \vee \psi) \rhd^{\neg_\pi^2} D\varphi}\,(\mathrm{T})\text{ w/ stab.} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg_\pi D\psi \rhd^{\neg_\pi^2} \psi}\,\text{i.h.}}{\neg_\pi D\psi \rhd^{\neg_\pi^2} \varphi \vee \psi}\,\mathrm{R}\vee}{\neg_\pi(\varphi \vee \psi) \rhd^{\neg_\pi^2} \neg_\pi^2 D\psi}\,\text{C.p.}}{\neg_\pi(\varphi \vee \psi) \rhd^{\neg_\pi^2} D\psi}\,(\mathrm{T})\text{ w/ stab.}}{\neg_\pi(\varphi \vee \psi) \rhd^{\neg_\pi^2} D\varphi \wedge D\psi}\,\mathrm{R}\wedge}{\neg_\pi(D\varphi \wedge D\psi) \rhd^{\neg_\pi^2} \neg_\pi^2(\varphi \vee \psi)}\,\text{Cp}}{\neg_\pi(D\varphi \wedge D\psi) \rhd^{\neg_\pi^2} \varphi \vee \psi}\,(\mathrm{T})\text{ w/ stab.}$$

In conclusion, $\varphi \vee \psi \approx^{\neg_\pi^2} \neg_\pi D(\varphi \vee \psi)$.

— Suppose $\alpha \equiv \varphi \supset \psi$. By induction hypothesis we have $\varphi \approx^{\neg_\pi^2} \neg_\pi D\varphi$ and $\psi \approx^{\neg_\pi^2} \neg_\pi D\psi$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_\pi D\varphi \rhd^{\neg_\pi^2} \varphi}\,\text{i.h.}}{\neg_\pi D\varphi, D\psi \rhd^{\neg_\pi^2} \varphi}\,(\mathrm{M}) \qquad \dfrac{\dfrac{\dfrac{}{\neg_\pi D\varphi, D\psi \rhd^{\neg_\pi^2} D\psi}\,(\mathrm{R})}{\neg_\pi D\varphi, D\psi, \neg_\pi D\psi \rhd^{\neg_\pi^2} \pi}\,\mathrm{L}\neg'_\pi}{\neg_\pi D\varphi, D\psi, \psi \rhd^{\neg_\pi^2} \pi}\,(\mathrm{T})\text{ w/ i.h.}}{\neg_\pi D\varphi, D\psi, \varphi \supset \psi \rhd^{\neg_\pi^2} \pi}\,\mathrm{L}\supset}{\neg_\pi D\varphi \wedge D\psi, \varphi \supset \psi \rhd^{\neg_\pi^2} \pi}\,\mathrm{L}\wedge}{\varphi \supset \psi \rhd^{\neg_\pi^2} \neg_\pi(\neg_\pi D\varphi \wedge D\psi)}\,\mathrm{R}\neg_\pi$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \rhd^{\neg^2_\pi} \neg_\pi D\varphi}\,\text{i.h.}}{\neg_\pi\psi, \varphi \rhd^{\neg^2_\pi} \neg_\pi D\varphi}\,\text{(M)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_\pi D\psi \rhd^{\neg^2_\pi} \psi}\,\text{i.h.}}{\neg_\pi D\psi, \varphi \rhd^{\neg^2_\pi} \psi}\,\text{(M)}}{\neg_\pi\psi, \varphi \rhd^{\neg^2_\pi} \neg^2_\pi D\psi}\,\text{Cp}}{\neg_\pi\psi, \varphi \rhd^{\neg^2_\pi} D\psi}\,\text{(T) w/ stab.}}{\neg_\pi\psi, \varphi \rhd^{\neg^2_\pi} \neg_\pi D\varphi \wedge D\psi}\,\text{R}\wedge}{\dfrac{\dfrac{\neg_\pi(\neg_\pi D\varphi \wedge D\psi), \varphi \rhd^{\neg^2_\pi} \neg^2_\pi \psi}{\neg_\pi(\neg_\pi D\varphi \wedge D\psi), \varphi \rhd^{\neg^2_\pi} \psi}\,\text{(T) w/ stab.}}{\neg_\pi(\neg_\pi D\varphi \wedge D\psi) \rhd^{\neg^2_\pi} \varphi \supset \psi}\,\text{R}\supset}\,\text{Cp}
$$

In conclusion, $\varphi \supset \psi \approx^{\neg^2_\pi} \neg_\pi D(\varphi \supset \psi)$.

— Suppose $\alpha \equiv \forall x \varphi$. By induction hypothesis we have $\varphi \approx^{\neg^2_\pi} \neg_\pi D\varphi$, which in view of substitution can be written as $\varphi[y/x] \approx^{\neg^2_\pi} \neg_\pi D\varphi[y/x]$. Then

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D\varphi[y/x], \forall x\,\varphi \rhd^{\neg^2_\pi} D\varphi[y/x]}\,\text{(R)}}{\neg_\pi D\varphi[y/x], D\varphi[y/x], \forall x\,\varphi \rhd^{\neg^2_\pi} \pi}\,\text{L}\neg'_\pi}{\varphi[y/x], D\varphi[y/x], \forall x\,\varphi \rhd^{\neg^2_\pi} \pi}\,\text{(T) w/ i.h.}}{D\varphi[y/x], \forall x\,\varphi \rhd^{\neg^2_\pi} \pi}\,\text{L}\forall}{\exists x\, D\varphi, \forall x\,\varphi \rhd^{\neg^2_\pi} \pi}\,\text{L}\exists}{\forall x\,\varphi \rhd^{\neg^2_\pi} \neg_\pi \exists x\, D\varphi}\,\text{R}\neg_\pi
$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_\pi D\varphi[y/x] \rhd^{\neg_\pi^2} \varphi[y/x]}\;\text{i.h.}}{\neg_\pi \varphi[y/x] \rhd^{\neg_\pi^2} \neg_\pi^2 D\varphi[y/x]}\;\text{Cp}}{\neg_\pi \varphi[y/x] \rhd^{\neg_\pi^2} D\varphi[y/x]}\;\text{(T) w/ stab.}}{\neg_\pi \varphi[y/x] \rhd^{\neg_\pi^2} \exists x\, D\varphi}\;\text{R}\exists}{\neg_\pi \exists x\, D\varphi \rhd^{\neg_\pi^2} \neg_\pi^2 \varphi[y/x]}\;\text{Cp}}{\neg_\pi \exists x\, D\varphi \rhd^{\neg_\pi^2} \varphi[y/x]}\;\text{(T) w/ stab.}}{\neg_\pi \exists x\, D\varphi \rhd^{\neg_\pi^2} \forall x\, \varphi}\;\text{R}\forall$$

In conclusion, $\forall x\, \varphi \approx^{\neg_\pi^2} \neg_\pi D \forall x\, \varphi$.

— Suppose $\alpha \equiv \exists x\, \varphi$. By induction hypothesis we have $\varphi \approx^{\neg_\pi^2} \neg_\pi D\varphi$, which in view of substitution can be written as $\varphi[y/x] \approx^{\neg_\pi^2} \neg_\pi D\varphi[y/x]$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi[y/x] \rhd^{\neg_\pi^2} \neg_\pi D\varphi[y/x]}\;\text{i.h.}}{\varphi[y/x] \rhd^{\neg_\pi^2} \exists x \neg_\pi D\varphi}\;\text{R}\exists}{\exists x\, \varphi \rhd^{\neg_\pi^2} \exists x \neg_\pi D\varphi}\;\text{L}\exists}{\exists x\, \varphi \rhd^{\neg_\pi^2} \neg_\pi^2 \exists x \neg_\pi D\varphi}\;\text{R}\neg_\pi^2$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_\pi D\varphi[y/x] \rhd^{\neg_\pi^2} \varphi[y/x]}\;\text{i.h.}}{\neg_\pi D\varphi[y/x] \rhd^{\neg_\pi^2} \exists x\, \varphi}\;\text{R}\exists}{\exists x \neg_\pi D\varphi \rhd^{\neg_\pi^2} \exists x\, \varphi}\;\text{L}\exists}{\neg_\pi^2 \exists x \neg_\pi D\varphi \rhd^{\neg_\pi^2} \exists x\, \varphi}\;\text{(T) w/ stab.}$$

In conclusion, $\exists x\, \varphi \approx^{\neg_\pi^2} \neg_\pi D \exists x\, \varphi$.

Proof that k satisfies the Kolmogorov condition for every rule in the inductive definition of $\rhd_i$:

— Consider R⊃. We need to show that

$$\dfrac{\neg_\pi D\Gamma, \neg_\pi D\varphi \rhd \neg_\pi D\psi}{\neg_\pi D\Gamma \rhd \neg_\pi \underbrace{D(\varphi \supset \psi)}_{\neg_\pi D\varphi \wedge D\psi}}\ \text{R}{\supset}_{\neg_\pi D}$$

In fact

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D\psi \rhd D\psi}\,(\text{R})}{D\psi \rhd \neg_\pi^2 D\psi}\,\text{R}\neg_\pi^2 \qquad \dfrac{\neg_\pi D\varphi, \neg_\pi D\Gamma \rhd \neg_\pi D\psi}{\neg_\pi D\varphi, \neg_\pi^2 D\psi, \neg_\pi D\Gamma \rhd \pi}\,\text{L}\neg_\pi'}{\neg_\pi D\varphi, D\psi, \neg_\pi D\Gamma \rhd \pi}\,(\text{T})}{\neg_\pi D\varphi \wedge D\psi, \neg_\pi D\Gamma \rhd \pi}\,\text{L}\wedge}{\neg_\pi D\Gamma \rhd \neg_\pi(\neg_\pi D\varphi \wedge D\psi)}\,\text{R}\neg_\pi$$

— Consider $\varphi, \varphi \supset \psi \rhd \psi$. We have to show that $\neg_\pi D\varphi, \neg_\pi D\psi \rhd \neg_\pi D(\varphi \supset \psi)$, i.e. $\neg_\pi D\varphi, \neg_\pi D\psi \rhd \neg_\pi(\neg_\pi D\varphi \wedge D\psi)$. We have

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg_\pi D\varphi, D\psi, \neg_\pi D\varphi \rhd D\psi}\,(\text{R})}{\neg_\pi D\varphi, D\psi, \neg_\pi D\varphi, \neg_\pi D\psi \rhd \pi}\,\text{L}\neg_\pi'}{\neg_\pi D\varphi \wedge D\psi, \neg_\pi D\varphi, \neg_\pi D\psi \rhd \pi}\,\text{L}\wedge}{\neg_\pi D\varphi, \neg_\pi D\psi \rhd \neg_\pi(\neg_\pi D\varphi \wedge D\psi)}\,\text{R}\neg_\pi$$

— Consider $\varphi, \psi \rhd \varphi \wedge \psi$. We have to show that $\neg_\pi D\varphi, \neg_\pi D\psi \rhd \neg_\pi D(\varphi \wedge \psi)$, i.e. $\neg_\pi D\varphi, \neg_\pi D\psi \rhd \neg_\pi(D\varphi \vee D\psi)$. We have

$$\dfrac{\dfrac{\dfrac{\dfrac{}{D\varphi, \neg_\pi D\psi \rhd D\varphi}\,(\text{R})}{D\varphi, \neg_\pi D\varphi, \neg_\pi D\psi \rhd \pi}\,\text{L}\neg_\pi' \qquad \dfrac{\dfrac{}{D\psi, \neg_\pi D\varphi \rhd D\psi}\,(\text{R})}{D\psi, \neg_\pi D\varphi, \neg_\pi D\psi \rhd \pi}\,\text{L}\neg_\pi'}{D\varphi \vee D\psi, \neg_\pi D\varphi, \neg_\pi D\psi \rhd \pi}\,\text{L}\vee}{\neg_\pi D\varphi, \neg_\pi D\psi \rhd \neg_\pi(D\varphi \vee D\psi)}\,\text{R}\neg_\pi$$

— Consider $\varphi \wedge \psi \rhd \varphi$. We have to show that $\neg_\pi D(\varphi \wedge \psi) \rhd \neg_\pi D\varphi$, i.e. $\neg_\pi(D\varphi \vee D\psi) \rhd \neg_\pi D\varphi$, which follows from $D\varphi \rhd D\varphi \vee D\psi$ by an application of Cp. Cases $\varphi \wedge \psi \rhd \psi$; $\varphi \rhd \varphi \vee \psi$; $\psi \rhd \varphi \vee \psi$; $\forall x\, \varphi \rhd \varphi[t/x]$; are similar.

— Consider $\varphi \vee \psi, \varphi \supset \delta, \psi \supset \delta \vartriangleright \delta$. We have to show that $\neg_\pi D(\varphi \vee \psi), \neg_\pi D(\varphi \supset \delta), \neg_\pi D(\psi \supset \delta) \vartriangleright \neg_\pi D\delta$, i.e. $\neg_\pi(D\varphi \wedge D\psi), \neg_\pi(\neg_\pi D\varphi \wedge D\delta), \neg_\pi(\neg_\pi D\psi \wedge D\delta) \vartriangleright \neg_\pi D\delta$. We have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D\varphi, D\psi \vartriangleright D\varphi \wedge D\psi}\,\text{Axiom}}{D\varphi, D\psi, D\delta \vartriangleright D\varphi \wedge D\psi}\,\text{(M)}}{D\psi, D\delta, \neg_\pi(D\varphi \wedge D\psi) \vartriangleright \neg_\pi D\varphi}\,\text{Cp} \quad \dfrac{}{D\delta, \ldots \vartriangleright D\delta}\,\text{(R)}}{D\psi, D\delta, \neg_\pi(D\varphi \wedge D\psi) \vartriangleright \neg_\pi D\varphi \wedge D\delta}\,\text{R}\wedge}{D\delta, \neg_\pi(D\varphi \wedge D\psi), \neg_\pi(\neg_\pi D\varphi \wedge D\delta) \vartriangleright \neg_\pi D\psi}\,\text{Cp} \quad \dfrac{}{D\delta, \ldots \vartriangleright D\delta}\,\text{(R)}}{D\delta, \neg_\pi(D\varphi \wedge D\psi), \neg_\pi(\neg_\pi D\varphi \wedge D\delta) \vartriangleright \neg_\pi D\psi \wedge D\delta}\,\text{R}\wedge}{\neg_\pi(D\varphi \wedge D\psi), \neg_\pi(\neg_\pi D\varphi \wedge D\delta), \neg_\pi(\neg_\pi D\psi \wedge D\delta) \vartriangleright \neg_\pi D\delta}\,\text{Cp}$$

— Consider $\vartriangleright \top$. We have to show that $\vartriangleright \neg_\pi D\top$, i.e. $\vartriangleright \neg_\pi^2 \top$, which follows from $\vartriangleright \top$ by $\text{R}\neg_\pi^2$. Case $\varphi[t/x] \vartriangleright \exists x\, \varphi$ is similar.

— Consider $\bot \vartriangleright \varphi$. We have to show that $\neg_\pi D\bot \vartriangleright \neg_\pi D\varphi$, i.e. $\neg_\pi^2 \bot \vartriangleright \neg_\pi D\varphi$, which follows from $\bot \vartriangleright \neg_\pi D\varphi$ by $\text{L}\neg^2_{\pi_{\neg_\pi}}$.

— Consider R$\forall$. We have to show that

$$\dfrac{\neg_\pi D\Gamma \vartriangleright \neg_\pi D\varphi[y/x]}{\neg_\pi D\Gamma \vartriangleright \neg_\pi \underbrace{D\forall x\, \varphi}_{\exists x\, D\varphi}}\,\text{R}\forall_{\neg_\pi D}$$

with y fresh. In fact

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D\varphi[y/x] \vartriangleright D\varphi[y/x]}\,\text{(R)}}{D\varphi[y/x] \vartriangleright \neg_\pi^2 D\varphi[y/x]}\,\text{R}\neg_\pi^2 \quad \dfrac{\neg_\pi D\Gamma \vartriangleright \neg_\pi D\varphi[y/x]}{\neg_\pi^2 D\varphi[y/x], \neg_\pi D\Gamma \vartriangleright \pi}\,\text{L}\neg_\pi'}{D\varphi[y/x], \neg_\pi D\Gamma \vartriangleright \pi}\,\text{(T)}}{\exists x\, D\varphi, \neg_\pi D\Gamma \vartriangleright \pi}\,\text{L}\exists}{\neg_\pi D\Gamma \vartriangleright \neg_\pi \exists x\, D\varphi}\,\text{R}\neg_\pi$$

— Consider L$\exists$. We have to show that

$$\dfrac{\neg_\pi D\Gamma, \neg_\pi D\varphi[y/x] \vartriangleright \neg_\pi D\delta}{\neg_\pi D\Gamma, \neg_\pi \underbrace{D\exists x\, \varphi}_{\neg_\pi \exists x \neg_\pi D\varphi} \vartriangleright \neg_\pi D\delta}\,\text{L}\exists_{\neg_\pi D}$$

with y fresh, which is an instance of L$\exists$ followed by L$\neg^2_{\pi_{\neg_\pi}}$. ■

B Tables

In this appendix we collect the various calculi that are used in the present thesis.

Initial sequents

$x\colon P, \Gamma \to \Delta, x\colon P$ $\qquad$ $x\colon \Box A, \Gamma \to \Delta, x\colon \Box A$

$y < x, \Gamma \to \Delta, y < x$ $\qquad$ $x = y, \Gamma \to \Delta, x = y$

Propositional rules

$$\frac{x\colon A, x\colon B, \Gamma \to \Delta}{x\colon A \wedge B, \Gamma \to \Delta}\,\mathrm{L}\wedge \qquad \frac{\Gamma \to \Delta, x\colon A \quad \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \wedge B}\,\mathrm{R}\wedge$$

$$\frac{x\colon A, \Gamma \to \Delta \quad x\colon B, \Gamma \to \Delta}{x\colon A \vee B, \Gamma \to \Delta}\,\mathrm{L}\vee \qquad \frac{\Gamma \to \Delta, x\colon A, x\colon B}{\Gamma \to \Delta, x\colon A \vee B}\,\mathrm{R}\vee$$

$$\frac{\Gamma \to \Delta, x\colon A \quad x\colon B, \Gamma \to \Delta}{x\colon A \supset B, \Gamma \to \Delta}\,\mathrm{L}\supset \qquad \frac{x\colon A, \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,\mathrm{R}\supset$$

$$\frac{}{x\colon \bot, \Gamma \to \Delta}\,\mathrm{L}\bot$$

Modal rules

$$\frac{y\colon A, x\colon \Box A, y < x, \Gamma \to \Delta}{x\colon \Box A, y < x, \Gamma \to \Delta}\,\mathrm{L}\Box \qquad \frac{y < x, \Gamma \to \Delta, y\colon A}{\Gamma \to \Delta, x\colon \Box A}\,\mathrm{R}\Box \quad (y \text{ fresh})$$

Rules for equality

$$\frac{x = x, \Gamma \to \Delta}{\Gamma \to \Delta}\,\mathrm{Ref}_{=} \qquad \frac{x = z, x = y, y = z, \Gamma \to \Delta}{x = y, y = z, \Gamma \to \Delta}\,\mathrm{Trans}_{=}$$

$$\frac{y < z, x = y, x < z, \Gamma \to \Delta}{x = y, x < z, \Gamma \to \Delta}\,\mathrm{Repl}_{<_1} \qquad \frac{x < y, z = y, x < z, \Gamma \to \Delta}{z = y, x < z, \Gamma \to \Delta}\,\mathrm{Repl}_{<_2}$$

$$\frac{y\colon P, x = y, x\colon P, \Gamma \to \Delta}{x = y, x\colon P, \Gamma \to \Delta}\,\mathrm{Repl}_{At}$$

Table B.1: The sequent calculus $\mathbf{G3K}_{<}$.

Frame property	Rule
Reflexivity $\forall x(x < x)$	$\dfrac{x < x, \Gamma \to \Delta}{\Gamma \to \Delta}$ Ref
Irreflexivity $\forall x(x \not< x)$	$\dfrac{}{x < x, \Gamma \to \Delta}$ Irref
Transitivity $\forall x \forall y < x \forall z < y(z < x)$	$\dfrac{x < z, x < y, y < z, \Gamma \to \Delta}{x < y, y < z, \Gamma \to \Delta}$ Trans
Noetherian induction $\forall y(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y\, Ey$	$\dfrac{y\colon \Box A, \Gamma \to \Delta, y\colon A}{\Gamma \to \Delta, y\colon A}$ NI
Gödel–Löb induction $\forall x(\forall y < x(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y < x\, Ey)$	$\dfrac{y < x, y\colon \Box A, \Gamma \to \Delta, y\colon A}{\Gamma \to \Delta, x\colon \Box A}$ R□-GLI

Table B.2: Additional rules for $\mathbf{G3K}^*_<$ and the corresponding frame properties. Rule NI has the condition that y is not in Γ, Δ, rule R□-GLI has the condition that y is fresh.

Initial sequents	As in $\mathbf{G3K}^*_<$.
Propositional Rules	As in $\mathbf{G3K}^*_<$.
Modal Rules	L□ as in $\mathbf{G3K}^*_<$, R□-GLI.
Rules for equality	As in $\mathbf{G3K}^*_<$.

Table B.3: The sequent calculus $\mathbf{G3KGL}_<$.

Initial sequent

$x \leqslant y, x\colon P, \Gamma \to \Delta, y\colon P$

Logical Rules

$$\frac{x\colon A, x\colon B, \Gamma \to \Delta}{x\colon A \wedge B, \Gamma \to \Delta}\,\mathrm{L}\wedge \qquad \frac{\Gamma \to \Delta, x\colon A \quad \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \wedge B}\,\mathrm{R}\wedge$$

$$\frac{x\colon A, \Gamma \to \Delta \quad x\colon B, \Gamma \to \Delta}{x\colon A \vee B, \Gamma \to \Delta}\,\mathrm{L}\vee \qquad \frac{\Gamma \to \Delta, x\colon A, x\colon B}{\Gamma \to \Delta, x\colon A \vee B}\,\mathrm{R}\vee$$

$$\frac{x \leqslant y, x\colon A \supset B, \Gamma \to \Delta, y\colon A \qquad x \leqslant y, x\colon A \supset B, y\colon B, \Gamma \to \Delta}{x \leqslant y, x\colon A \supset B, \Gamma \to \Delta}\,\mathrm{L}\supset$$

$$\frac{}{x\colon \bot, \Gamma \to \Delta}\,\mathrm{L}\bot \qquad \frac{x \leqslant y, y\colon A, \Gamma \to \Delta, y\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,\mathrm{R}\supset$$

Mathematical Rules

$$\frac{x \leqslant x, \Gamma \to \Delta}{\Gamma \to \Delta}\,\mathrm{Ref}_{\leqslant} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \to \Delta}{x \leqslant y, y \leqslant z, \Gamma \to \Delta}\,\mathrm{Trans}_{\leqslant}$$

Table B.4: The sequent calculus **G3I**. Rule R$\supset$ has the condition that y is fresh.

Initial sequent As in **G3I**.

Logical Rules L$\wedge$, R$\wedge$, L$\vee$, R$\vee$, L$\supset$, L$\bot$ as in **G3I**.

$$\frac{x \leqslant y, y : B \supset (A \supset B), y : A, \Gamma \to \Delta, y : B}{\Gamma \to \Delta, x : A \supset B} \text{R}{\supset_t} \quad (y \text{ fresh})$$

Mathematical Rules As in **G3I**.

Table B.5: The sequent calculus $\mathbf{G3I_t}$.

Initial sequent

$x\colon P, \Gamma \to \Delta, x\colon P$

Propositional rules

$$\frac{x\colon A, x\colon B, \Gamma \to \Delta}{x\colon A \wedge B, \Gamma \to \Delta}\,L\wedge \qquad \frac{\Gamma \to \Delta, x\colon A \qquad \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \wedge B}\,R\wedge$$

$$\frac{x\colon A, \Gamma \to \Delta \qquad x\colon B, \Gamma \to \Delta}{x\colon A \vee B, \Gamma \to \Delta}\,L\vee \qquad \frac{\Gamma \to \Delta, x\colon A, x\colon B}{\Gamma \to \Delta, x\colon A \vee B}\,R\vee$$

$$\frac{\Gamma \to \Delta, x\colon A \qquad x\colon B, \Gamma \to \Delta}{x\colon A \supset B, \Gamma \to \Delta}\,L\supset \qquad \frac{x\colon A, \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,R\supset$$

$$\frac{}{x\colon \bot, \Gamma \to \Delta}\,L\bot$$

Modal rules

$$\frac{x \leqslant y, y\colon A, x\colon \Box A, \Gamma \to \Delta}{x \leqslant y, x\colon \Box A, \Gamma \to \Delta}\,L\Box \qquad \frac{x \leqslant y, y\colon G(A), \Gamma \to \Delta, y\colon A}{\Gamma \to \Delta, x\colon \Box A}\,R\Box Z$$

Mathematical rules

$$\frac{x \leqslant x, \Gamma \to \Delta}{\Gamma \to \Delta}\,Ref_{\leqslant} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \to \Delta}{x \leqslant y, y \leqslant z, \Gamma \to \Delta}\,Trans_{\leqslant}$$

Table B.6: The sequent calculus **G3Grz**. Rule R$\Box$ has the condition that y is fresh.

Initial sequent

$P, \Gamma \to \Delta, P$

Propositional rules

$$\frac{}{\bot, \Gamma \to \Delta}\ \mathrm{L}\bot \qquad \frac{}{\Gamma \to \Delta, \top}\ \mathrm{R}\top$$

$$\frac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}\ \mathrm{L}\wedge \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}\ \mathrm{R}\wedge$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\ \mathrm{L}\vee \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\ \mathrm{R}\vee$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\ \mathrm{L}\supset \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\ \mathrm{R}\supset$$

Rules for quantifiers

$$\frac{A[y/x], \forall x A, \Gamma \to \Delta}{\forall x A, \Gamma \to \Delta}\ \mathrm{L}\forall \qquad \frac{\Gamma \to \Delta, A[z/x]}{\Gamma \to \Delta, \forall x A}\ \mathrm{R}\forall \quad (y \text{ fresh})$$

$$\frac{A[z/x], \Gamma \to \Delta}{\exists x A, \Gamma \to \Delta}\ \mathrm{L}\exists \quad (y \text{ fresh}) \qquad \frac{\Gamma \to \Delta, A[y/x], \exists x A}{\Gamma \to \Delta, \exists x A}\ \mathrm{R}\exists$$

Infinitary rules

$$\frac{A_k, \bigwedge A_n, \Gamma \to \Delta}{\bigwedge A_n, \Gamma \to \Delta}\ \mathrm{L}\bigwedge \qquad \frac{\{\Gamma \to \Delta, A_i \mid i > 0\}}{\Gamma \to \Delta, \bigwedge A_n}\ \mathrm{R}\bigwedge$$

$$\frac{\{A_i, \Gamma \to \Delta \mid i > 0\}}{\bigvee A_n, \Gamma \to \Delta}\ \mathrm{L}\bigvee \qquad \frac{\Gamma \to \Delta, \bigvee A_n, A_k}{\Gamma \to \Delta, \bigvee A_n}\ \mathrm{R}\bigvee$$

Rules for equality

$$\frac{s = s, \Gamma \to \Delta}{\Gamma \to \Delta}\ \mathrm{Ref} \qquad \frac{P[t/x], s = t, P[s/x], \Gamma \to \Delta}{s = t, P[s/x], \Gamma \to \Delta}\ \mathrm{Repl}$$

Table B.7: The calculus $\mathbf{G3C}_\omega$.

Initial sequent As in $\mathbf{G3C}_\omega$.

Rules As in $\mathbf{G3C}_\omega$, except for the following:

$$\frac{A \supset B, \Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\, \mathrm{L}\supset \qquad \frac{A, \Gamma \to B}{\Gamma \to \Delta, A \supset B}\, \mathrm{R}\supset$$

$$\frac{\Gamma \to A[z/x]}{\Gamma \to \Delta, \forall x A}\, \mathrm{R}\forall \qquad \frac{\{\Gamma \to A_i \mid i > 0\}}{\Gamma \to \Delta, \bigwedge A_n}\, \mathrm{R}\bigwedge$$

Table B.8: The calculus $\mathbf{G3I}_\omega$.

Initial sequent As in $\mathbf{G3I}_\omega$, plus

$\bot, \Gamma \to \Delta, \bot$

Rules As in $\mathbf{G3I}_\omega$, except for L$\bot$.

Table B.9: The calculus $\mathbf{G3M}_\omega$.

$$\frac{\dots \qquad Q_{n_1}(\vec{x}, \vec{y}_n), \dots, Q_{n_m}(\vec{x}, \vec{y}_n), P_1(\vec{x}), \dots, P_k(\vec{x}), \Gamma \to \Delta \qquad \dots}{P_1(\vec{x}), \dots, P_k(\vec{x}), \Gamma \to \Delta}\, \mathrm{L}_G$$

Table B.10: Geometric rule L_G expressing the geometric sentence (G)

Generating axiom/rule	**Sequent calculus-like rule**
$\varphi, \psi \rhd \varphi \wedge \psi$	$\dfrac{\Gamma \rhd \varphi \qquad \Gamma \rhd \psi}{\Gamma \rhd \varphi \wedge \psi}\,\mathrm{R}\wedge$
$\varphi \wedge \psi \rhd \varphi$	$\dfrac{\Gamma, \varphi, \psi \rhd \delta}{\Gamma, \varphi \wedge \psi \rhd \delta}\,\mathrm{L}\wedge$
$\varphi \wedge \psi \rhd \psi$	
$\varphi \rhd \varphi \vee \psi$	$\dfrac{\Gamma \rhd \varphi}{\Gamma \rhd \varphi \vee \psi}\,\mathrm{R}\vee_1$
$\psi \rhd \varphi \vee \psi$	$\dfrac{\Gamma \rhd \psi}{\Gamma \rhd \varphi \vee \psi}\,\mathrm{R}\vee_2$
$\varphi \vee \psi, \varphi \supset \delta, \psi \supset \delta \rhd \delta$	$\dfrac{\Gamma, \varphi \rhd \delta \qquad \Gamma, \psi \rhd \delta}{\Gamma, \varphi \vee \psi \rhd \delta}\,\mathrm{L}\vee$
$\dfrac{\Gamma, \varphi \rhd \psi}{\Gamma \rhd \varphi \supset \psi}\,\mathrm{R}\supset$	
$\varphi, \varphi \supset \psi \rhd \psi$	$\dfrac{\Gamma \rhd \varphi \qquad \Gamma, \psi \rhd \delta}{\Gamma, \varphi \supset \psi \rhd \delta}\,\mathrm{L}\supset$
$\rhd \top$	$\dfrac{}{\Gamma \rhd \top}\,\mathrm{R}\top$
$\varphi \supset \bot \rhd \neg\varphi$	$\dfrac{\Gamma, \varphi \rhd \bot}{\Gamma \rhd \neg\varphi}\,\mathrm{R}\neg$
$\neg\varphi \rhd \varphi \supset \bot$	$\dfrac{\Gamma \rhd \varphi \qquad \Gamma, \bot \rhd \psi}{\Gamma, \neg\varphi \rhd \psi}\,\mathrm{L}\neg$
$\bot \rhd \varphi$	$\dfrac{}{\Gamma, \bot \rhd \varphi}\,\mathrm{L}\bot$
$\neg\neg\varphi \rhd \varphi$	(*not given*)
$\dfrac{\Gamma \rhd \varphi[y/x]}{\Gamma \rhd \forall x\varphi}\,\mathrm{R}\forall$ (y fresh)	
$\forall x\varphi \rhd \varphi[t/x]$	$\dfrac{\varphi[t/x], \Gamma, \forall x\varphi \rhd \delta}{\Gamma, \forall x\varphi \rhd \delta}\,\mathrm{L}\forall$
$\varphi[t/x] \rhd \exists x\varphi$	$\dfrac{\Gamma \rhd \varphi[t/x]}{\Gamma \rhd \exists x\varphi}\,\mathrm{R}\exists$
$\dfrac{\Gamma, \varphi[y/x] \rhd \delta}{\Gamma, \exists x\varphi \rhd \delta}\,\mathrm{L}\exists$ (y fresh)	

Table B.11: Axioms and rules that generate $\rhd_p$ and its common extensions; with corresponding sequent calculus-like rules.

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