Coupling for some partial differential equations driven by white noise

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Abstract

We prove, using coupling arguments, exponential convergence to equilibrium for reaction–diffusion and Burgers equations driven by space-time white noise. We use a coupling by reflection.

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1 Introduction

We are concerned with a stochastic differential equation in a separable Hilbert space H, with inner product $(\cdot,\cdot)$ and norm $|\cdot|$,

$$dX = (AX + b(X))dt + dW(t), \quad X(0) = x \in H, \tag{1.1}$$

where $A\colon D(A) \subset H \to H$ is linear, $b\colon D(b) \subset H \to H$ is nonlinear and W is a cylindrical Wiener process defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in H. Concerning A we shall assume that

Hypothesis 1.1

(i) *$A\colon D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup e^{tA}.*

(ii) *For any $t > 0$ the linear operator Q_t, defined as*

$$Q_t x = \int_0^t e^{sA} e^{sA^*} x ds, \;\; x \in H, \; t \geq 0, \tag{1.2}$$

is of trace class.

We shall consider situations where (1.1) has a unique mild solution $X(\cdot, x)$, that is a mean square adapted process such that

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A} b(X(s,x))ds + z(t), \quad \mathbb{P}\text{–a.s.}, \tag{1.3}$$

where $z(t)$ is the stochastic convolution

$$z(t) = \int_0^t e^{(t-s)A} dW(s). \tag{1.4}$$

It is well known that, thanks to Hypothesis 1.1, for each $t > 0$, $z(t)$ is a Gaussian random variable with mean 0 and covariance operator Q_t.

We will also assume that the solution has continuous trajectories. More precisely, we assume

$$X(\cdot, x) \in L^2(\Omega; C([0,T]; H)), \text{ for any } x \in H. \tag{1.5}$$

In this paper we want to study the exponential convergence to equilibrium of the transition semigroup

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad x \in H, \ t \geq 0,$$

where $\varphi \colon H \to \mathbb{R}$. We wish to use coupling arguments. It is well known that exponential convergence to equilibrium implies the uniqueness of invariant measure.

It seems that the first paper using a coupling method to prove uniqueness of the invariant measure and mixing property for a stochastic partial differential equation is [12]. There, an equation with globally Lipschitz coefficients is considered, some of them are also assumed to be monotone.

Coupling argument have also been used recently to prove ergodicity and exponential convergence to equilibrium in the context of the Navier-Stokes equation driven by very degenerate noises (see [7], [8], [10], [11]). The method has also been studied in [6] for reaction–diffusion equations and in [13] for Ginzburg–Landau equations.

Our interest here is different since we are interested in space-time white noises as in [12] but without the strong restrictions on the coefficients. Ergodicity is well known in the cases considered here. It can be proved by the Doob theorem (see [5]). Indeed, since the noise is non degenerate, it is not diffiucult to prove that the transition semigroup is strong Feller and irreducible. However, this argument does not imply exponential convergence to equilibrium and we think that it is important to study this question.

In this paper we shall follow the construction of couplings introduced in [9] (see also [1]) to treat both reaction diffusion and Burgers equations driven by white noise and obtain exponential convergence to equilibrium.

Note that exponential convergence to equilibrium for reaction–diffusion equations is well known. Anyway, we have chosen to treat this example because we think that the method presented here provides a very simple proof. Moreover, we recover the spectral gap property obtained in [3] by a totally different - and simpler - argument.

In the case of the Burgers equation driven by space-time white noise, it seems that our result is new.

The coupling method based on Girsanov transform introduced in [7], [8], [10], [11] can easily be used if the noise is nuclear. It is also possible it could be extended to our case. However, the extension is not straightforward and the method used here seems to be simpler. Moreover, it is not clear that, in the case of the reaction-diffusion equation, it is possible to prove the spectral gap property with this method.

Next section is devoted to describing the construction of the coupling used here, we follow [1]. Note that the coupling is constructed as the solution of a stochastic differential equation with discontinuous coefficients. In [1], the existence of the coupling is straightforward. It is easy to see that the corresponding martingale problem has a solution. This argument is difficult in infinite dimension and we have preferred to prove directly the existence of a strong solution. Section 3 is devoted to application to reaction–diffusion equations and section 4 to the Burgers equation driven by white noise.

We finally remark that our method extends to other equations such as reaction-diffusion equations or the stochastic Navier-Stokes equation in space dimension two with non degenerate noise. We have chosen to restrict our presentation to these two examples for clarity of the presentation.

2 Construction of the coupling

We shall consider the following system of stochastic differential equations:

$$\begin{cases} dX_1 = (AX_1 + b(X_1))dt + \frac{1}{\sqrt{2}}\, dW_1 + \frac{1}{\sqrt{2}}\, \left(1 - 2\frac{(X_1-X_2)\otimes(X_1-X_2)}{|X_1-X_2|^2}\right) dW_2 \\ \\ dX_2 = (AX_2 + b(X_2))dt + \frac{1}{\sqrt{2}}\, \left(1 - 2\frac{(X_1-X_2)\otimes(X_1-X_2)}{|X_1-X_2|^2}\right) dW_1 + \frac{1}{\sqrt{2}}\, dW_2 \\ \\ X_1(0) = x_1, \quad X_2(0) = x_2, \end{cases} \tag{2.1}$$

where W_1, W_2 are independent cylindrical Wiener processes. This corresponds to a coupling with reflection, see [1]. Equation (2.1) is associated to the Kolmogorov operator in $H \times H$ defined by

$$\mathcal{K}\,\Phi(x_1,x_2)$$

$$= \frac{1}{2}\,\mathrm{Tr}\,\left[\begin{pmatrix} 1 & 1-2\dfrac{(x_1-x_2)\otimes(x_1-x_2)}{|x_1-x_2|^2} \\ 1-2\dfrac{(x_1-x_2)\otimes(x_1-x_2)}{|x_1-x_2|^2} & 1 \end{pmatrix} D^2\Phi(x_1,x_2)\right]$$

$$+\left(\begin{pmatrix} Ax_1+b(x_1) \\ Ax_2+b(x_2) \end{pmatrix}, D\Phi(x_1,x_2)\right),$$

or, equivalently,

$$\begin{aligned} \mathcal{K}\,\Phi &= \frac{1}{2}\,\mathrm{Tr}\,\left[\Phi_{x_1x_1} + \left(2-4\frac{(x_1-x_2)\otimes(x_1-x_2)}{|x_1-x_2|^2}\right)\Phi_{x_1x_2} + \Phi_{x_2x_2}\right] \\ &\quad + (Ax_1+b(x_1),\Phi_{x_1}) + (Ax_2+b(x_2),\Phi_{x_2}). \end{aligned} \tag{2.2}$$

The following formula will be useful in the sequel.

Lemma 2.1 *Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be a $\mathcal{C}^2$ function and let Φ be defined by $\Phi(x_1,x_2) = f(|x_1-x_2|),\ x_1,x_2 \in H,\ f \in C^2(\mathbb{R})$, then*

$$\begin{aligned} \mathcal{K}\,\Phi(x_1,x_2) &= 2\,f''(|x_1-x_2|) + \frac{f'(|x_1-x_2|)}{|x_1-x_2|} \\ &\quad \times (A(x_1-x_2)+b(x_1)-b(x_2), x_1-x_2). \end{aligned} \tag{2.3}$$

Proof: We have

$$\Phi_{x_1} = -\Phi_{x_2}, \quad \Phi_{x_1x_1} = \Phi_{x_2x_2} = -\Phi_{x_1x_2},$$

$$\Phi_{x_1}(x_1,x_2) = f'(|x-y|)\,\frac{x_1-x_2}{|x_1-x_2|},$$

and

$$\begin{aligned} \Phi_{x_1x_1}(x_1,x_2) &= f'(|x_1-x_2|)\,\frac{|x_1-x_2|^2-(x_1-x_2)\otimes(x_1-x_2)}{|x_1-x_2|^3} \\ &\quad + f''(|x_1-x_2|)\,\frac{(x_1-x_2)\otimes(x_1-x_2)}{|x_1-x_2|^2}, \end{aligned}$$

The result follows. □

We will use functions f such that, for a suitable positive constant κ, we have

$$\mathcal{K}\Phi(x_1,x_2) \le -\kappa, \quad \text{for all } x_1,x_2 \in H. \tag{2.4}$$

Thanks to Lemma 2.1, we have to solve the following basic inequality, in the unknown f (notice that f has to be nonnegative),

$$2\, f''(|x_1 - x_2|) + \frac{f'(|x_1 - x_2|)}{|x_1 - x_2|}\, (A(x_1 - x_2) + b(x_1) - b(x_2), x_1 - x_2) \leq -\kappa. \qquad (2.5)$$

We now study problem (2.1). In our applications, it will be easy to verify that, for any $\varepsilon > 0$, it has a unique mild solution $X(t, x_1, x_2) = (X_1(t, x_1, x_2), X_2(t, x_1, x_2))$ on the random interval $[0, \tau^{\varepsilon}_{x_1,x_2}]$ where $\tau^{\varepsilon}_{x_1,x_2} = \inf\{t \in [0,T] \,|\, |X_1(t) - X_2(t)| \leq \varepsilon\}$.

Clearly $\tau^{\varepsilon}_{x_1,x_2}$ is increasing as $\varepsilon \to 0$ so that we can define $\tau_{x_1,x_2} = \lim_{\varepsilon\to 0} \tau^{\varepsilon}_{x_1,x_2}$ and get a unique mild solution $X(t, x_1, x_2) = (X_1(t, x_1, x_2), X_2(t, x_1, x_2))$ on the random interval $[0, \tau_{x_1,x_2})$.

Lemma 2.2 *$X(t, x_1, x_2)$ has a limit in $L^2(\Omega)$ when $t \to \tau_{x_1,x_2}$. Moreover, if $\tau_{x_1,x_2} < T$, we have $X_1(\tau_{x_1,x_2}, x_1, x_2) = X_2(\tau_{x_1,x_2}, x_1, x_2)$.*

Proof: Let us define

$$X_1^{\varepsilon}(t) = \begin{cases} X_1(t, x_1, x_2), \ t \leq \tau^{\varepsilon}_{x_1,x_2}, \\ \\ X(t, \tau^{\varepsilon}_{x_1,x_2}, X_1(\tau^{\varepsilon}_{x_1,x_2}, x_1, x_2), \ t \geq \tau^{\varepsilon}_{x_1,x_2}. \end{cases}$$

We have denoted by $X(\cdot, s, x)$ the solution of (1.1) with the condition $X(s, s, x) = x$ at time s instead of 0. It is not difficult to check that X_1 and $X(\cdot, x_1)$ have the same law. Let us write for $\eta_1, \eta_2 > 0$:

$$\begin{aligned} &\mathbb{E}(|X_1(\tau_{x_1,x_2} - \eta_1, x_1, x_2) - X_1(\tau_{x_1,x_2} - \eta_2, x_1, x_2)|^2) \\ &= \lim_{\varepsilon\to 0} \mathbb{E}(|X_1(\tau^{\varepsilon}_{x_1,x_2} - \eta_1) - X_1(\tau^{\varepsilon}_{x_1,x_2} - \eta_2)|^2) \\ &= \lim_{\varepsilon\to 0} \mathbb{E}(|X_1^{\varepsilon}(\tau^{\varepsilon}_{x_1,x_2} - \eta_1) - X_1^{\varepsilon}(\tau^{\varepsilon}_{x_1,x_2} - \eta_2)|^2) \\ &\leq \lim_{\varepsilon\to 0} \mathbb{E}(\sup_{t\in[0,T]} |X_1^{\varepsilon}(t - \eta_1, x_1, x_2) - X_1^{\varepsilon}(t - \eta_2, x_1, x_2)|^2). \end{aligned}$$

Since, X_1 and $X(\cdot, x_1)$ have the same law, we can write

$$\mathbb{E}(\sup_{t\in[0,T]} |X_1^{\varepsilon}(t - \eta_1) - X_1^{\varepsilon}(t - \eta_2)|^2) = \mathbb{E}(\sup_{t\in[0,T]} |X(t - \eta_1, x_1) - X(t - \eta_2, x_1)|^2).$$

By (1.5), we know that this latter term goes to zero so that we prove that $X_1(t)$ has a limit. We treat $X_2(t)$ exactly in the same way.

Finally, if $\tau_{x_1,x_2} < T$, then $|X_1(\tau^{\varepsilon}_{x_1,x_2}, x_1, x_2) - X_2(\tau^{\varepsilon}_{x_1,x_2}, x_1, x_2)| = \varepsilon$ for any $\varepsilon > 0$. Letting $\varepsilon \to 0$ we deduce the last statement. $\square$

We also consider the following equation

$$
\begin{cases}
dX_1 = (AX_1 + b(X_1))dt + \frac{1}{\sqrt{2}}\, 1\!\!1_{[0,\tau_{x_1,x_2}]}(t)dW_1 \\
\\
+\frac{1}{\sqrt{2}}\, 1\!\!1_{[0,\tau_{x_1,x_2}]}(t)\left(1 - 2\frac{(X_1-X_2)\otimes(X_1-X_2)}{|X_1-X_2|^2}\right)dW_2 + \frac{1}{\sqrt{2}}(dW_1 + dW_2)(1 - 1\!\!1_{[0,\tau_{x_1,x_2}]}(t)) \\
\\
dX_2 = (AX_2 + b(X_2))dt + \frac{1}{\sqrt{2}}\, 1\!\!1_{[0,\tau_{x_1,x_2}]}(t)\left(1 - 2\frac{(X_1-X_2)\otimes(X_1-X_2)}{|X_1-X_2|^2}\right)dW_1 \\
\\
+\frac{1}{\sqrt{2}}\, 1\!\!1_{[0,\tau_{x_1,x_2}]}(t)dW_2 + \frac{1}{\sqrt{2}}(dW_1 + dW_2)(1 - 1\!\!1_{[0,\tau_{x_1,x_2}]}(t)), \\
\\
X_1(0) = x_1, \quad X_2(0) = x_2.
\end{cases}
\tag{2.6}
$$

It is clear that for $t \le \tau_{x_1,x_2}$ the solutions of (2.1) and (2.6) do coincide, whereas for $t \ge \tau_{x_1,x_2}$ (2.6) reduce to

$$
\begin{cases}
dX_1 = (AX_1 + b(X_1))dt + \frac{1}{\sqrt{2}}(dW_1(t) + dW_2(t)), \quad t \ge \tau_{x_1,x_2}, \\
\\
dX_2 = (AX_2 + b(X_2))dt + \frac{1}{\sqrt{2}}(dW_1(t) + dW_2(t)), \quad t \ge \tau_{x_1,x_2}.
\end{cases}
\tag{2.7}
$$

Using Lemma 2.2, we easily prove that (2.6) has a unique solution. Moreover, since $\frac{1}{\sqrt{2}}(W_1(t)+W_2(t))$ is a cylindrical Wiener process, it follows that X_1 and X_2 have the same law as $X(\cdot,x_1)$ and $X(\cdot,x_2)$. In other words, (X_1,X_2) is a coupling of the laws of $X(\cdot,x_1)$ and $X(\cdot,x_2)$.

We are interested in the first time τ_{x_1,x_2} when $X_1(t,x_1,x_2))$ and $X_2(t,x_1,x_2))$ meet. That is τ_{x_1,x_2} is the stopping time

$$
\tau_{x_1,x_2} = \inf\{t > 0 : \ X_1(t,x_1,x_2) = X_2(t,x_1,x_2)\}. \tag{2.8}
$$

Our goal is first to show that

$$
\mathbb{E}(\tau_{x_1,x_2}) < +\infty. \tag{2.9}
$$

To prove (2.9) we look, following [1], for a Lyapunov function f such that (2.5) holds. This is motivated by next Proposition.

Proposition 2.3 *Assume that there exists a $\mathcal{C}^2$ function such that* (2.5) *holds. Let $x_1, x_2 \in H$ with $x_1 \ne x_2$. Then we have*

$$
\mathbb{P}(\tau_{x_1,x_2} = +\infty) = 0. \tag{2.10}
$$

Moreover,

$$
\mathbb{E}(\tau_{x_1,x_2}) \le f(|x_1 - x_2|). \tag{2.11}
$$

Proof. We shall write for simplicity,

$$X_1(t) = X_1(t, x_1, x_2), \quad X_1(t) = X_1(t, x_1, x_2).$$

Also, we can assume without loss of generality that $\kappa = 1$. Then we introduce the following stopping times:

$$S_N = \inf \{t \geq 0 : |X_1(t) - X_2(t)| > N\}, \quad N \in \mathbb{N},$$

and

$$\tau_{n,N} = \tau_{x_1,x_2}^{1/n} \wedge S_N.$$

By the Itô formula[1], Lemma 2.1 and (2.5), we have

$$\begin{aligned} &f(|X_1(t \wedge \tau_{n,N}) - X_2(t \wedge \tau_{n,N})|) \leq f(|x_1 - x_2|) \\ &+ \int_0^{t\wedge\tau_{n,N}} \frac{f'(|X_1(s) - X_2(s)|)}{|X_1 - X_2|} \tfrac{1}{\sqrt{2}}(X_1 - X_2, d(W_1 - W_2)) \\ &- (t \wedge \tau_{n,N}). \end{aligned} \tag{2.12}$$

It follows that

$$\mathbb{E}\big(f(|X_1(t \wedge \tau_{n,N}) - X_2(t \wedge \tau_{n,N})|)\big) \leq f(|x - y|) - \mathbb{E}(t \wedge \tau_{n,N}),$$

and

$$\mathbb{E}(t \wedge \tau_{n,N}) \leq f(|x - y|).$$

Consequently as $t \to \infty$ we find

$$\mathbb{E}(\tau_{n,N}) \leq f(|x - y|)$$

which yields as $n \to \infty$

$$\mathbb{E}(\tau_{x_1,x_2} \wedge S_N) \leq f(|x - y|).$$

By (1.5), we easily prove that $S_N \to \infty$ as $N \to \infty$ so that we get

$$\mathbb{E}(\tau_{x_1,x_2}) \leq f(|x - y|).$$

□

[1] The application of the Itô formula can be justified rigorously thanks to a regularization argument. This can be done easily in the applications considered hereafter.

3 Dissipative systems with white noise

We consider the case when there exist $\lambda \geq 0, a > 0$ such that

$$(A(x_1 - x_2) + b(x_1) - b(x_2), x_1 - x_2) \leq \lambda |x_1 - x_2|^2 - a|x_1 - x_2|^4, \tag{3.1}$$

for all $x_1, x_2 \in H$. We also assume that Hypothesis 1.1 and (1.5) hold.

A typical equation satisfying such assumptions is the following stochastic reaction-diffusion equation on $[0,1]$

$$\begin{cases} dX = (\partial_{\xi\xi} X - \alpha X^3 + \beta X^2 + \gamma X + \delta)dt + dW, \ t > 0, \ \xi \in (0,1), \\ \\ X(t,0) = X(t,1) = 0, \ t > 0, \\ \\ X(0,\xi) = x(\xi), \xi \in (0,1), \end{cases}$$

where $\alpha > 0$. In this case, we take $A = D_\xi^2$ on the domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$, $b(x) = -\alpha x^3 + \beta x^2 + \gamma x + \delta$. We could also consider the more general example where b is a polynomial of degree $2p+1$ with negative leading coefficient. Note that this equation is gradient, the invariant measure is known explicitly. However, we shall not use this fact. We could treat as well perturbation of this equation which are not gradient but satisfy (3.1)

Following the above discussion, we look for a positive function f such that

$$2\, f''(r) + f'(r) \left(\lambda r - a\, r^3\right) = -1. \tag{3.2}$$

Setting $f' = g$, (3.2) becomes

$$g'(r) - \frac{1}{2}\, g(r)(ar^3 - \lambda r) = -\frac{1}{2}. \tag{3.3}$$

whose general solution is given by

$$g(r) = e^{\frac{1}{8}\,(ar^4 - 2\lambda r^2)} g(0) - \frac{1}{2} \int_0^r e^{\frac{1}{8}\,(ar^4 - as^4 - 2\lambda r^2 + 2\lambda s^2)} ds.$$

Finally, we have

$$\begin{aligned} f(r) &= f(0) + \int_0^r e^{\frac{1}{8}\,(as^4 - 2\lambda s^2)} ds\ f'(0) \\ &\quad -\frac{1}{2} \int_0^r e^{\frac{1}{8}\,(as^4 - 2\lambda s^2)} \left[\int_0^s e^{-\frac{1}{8}\,(a\sigma^4 - 2\lambda\sigma^2)} d\sigma \right] ds. \end{aligned} \tag{3.4}$$

Setting $f(0) = 0$ and $f'(0) = \frac{1}{2} \int_0^\infty e^{-\frac{1}{8}\,(a\sigma^4 - 2\lambda\sigma^2)} d\sigma$ we obtain

$$f(r) = \frac{1}{2} \int_0^r e^{\frac{1}{8}\,(as^4 - 2\lambda s^2)} \left[\int_s^{+\infty} e^{-\frac{1}{8}\,(a\sigma^4 - 2\lambda\sigma^2)} d\sigma \right] ds \tag{3.5}$$

and

$$f'(r) = \tfrac{1}{2}\, e^{\frac{1}{8}\,(ar^4-2\lambda r^2)} \int_r^{+\infty} e^{-\frac{1}{8}\,(a\sigma^4-2\lambda\sigma^2)} d\sigma. \tag{3.6}$$

We need some properties on f.

Lemma 3.1 *We have $(ar^3 - \lambda r)f'(r) < 1$ for any $r \geq 0$.*

Proof: The function $r \mapsto ar^3 - \lambda r$ is increasing and positive if $r > \delta := \sqrt{\frac{\lambda}{a}}$ so that in this case

$$(ar^3 - \lambda r)f'(r) < \tfrac{1}{2}\, e^{\frac{1}{8}\,(ar^4-2\lambda r^2)} \int_r^{+\infty} (a\sigma^3 - \lambda\sigma) e^{-\frac{1}{8}\,(a\sigma^4-2\lambda\sigma^2)} d\sigma = 1.$$

Since for $0 \leq r \leq \delta$ we have $(ar^3 - \lambda r)f'(r) \leq 0$, the conclusion follows. □

Corollary 3.2 *f' is a decreasing positive function.*

Proof: Since f satisfies (3.2) , we deduce from Lemma 3.1 that $f'' < 0$ so that f' decreases. Moreover, f' is positive by (3.6). □

Lemma 3.3 *There exists Λ depending only on a, λ such that for any $r > 0$*

i) $f'(r) \leq \Lambda$

ii) $f(r) \leq \Lambda$

Proof: (i) follows obviously from Corollary 3.2. Let us show (ii). Fix $r > r_0 := \sqrt{\frac{\lambda}{a}}$. Since f is increasing, we have

$$f(r) \leq f(r_0), \quad r \leq r_0.$$

If $r > r_0$ we have by Lemma 3.1,

$$\begin{aligned} f(r) &= f(r_0) + \int_{r_0}^{r} f'(s)\, ds \\ &\leq f(r_0) + \int_{r_0}^{\infty} \tfrac{ds}{as^3-\lambda s} = f(r_0) + \tfrac{1}{2\lambda} \ln(1 + \tfrac{\lambda}{ar_0^2-\lambda}). \end{aligned}$$

Therefore $f(\infty) < \infty$ and ii) follows provided $\Lambda \geq \max\{f(\infty), f(r_0)\}$. □

The following results strengthen Proposition 2.3.

Proposition 3.4 *Let $x_1, x_2 \in H$ with $x_1 \neq x_2$, then there exists $\Lambda > 0$ such that*

$$\mathbb{E}\left(e^{\frac{1}{2\Lambda^2}\,\tau_{x_1,x_2}}\right) \leq e^{\frac{1}{\Lambda}}$$

Proof: We use the same notations as in the proof of Proposition 2.3. By (2.12) we have

$$t \wedge \tau_{n,N} \leq f(|x_1 - x_2|) + \int_0^{t\wedge\tau_{n,N}} \frac{f'(|X_1 - X_2|)}{|X_1 - X_2|} \frac{1}{\sqrt{2}}(X_1 - X_2, d(W_1 - W_2)),$$

so that

$$\mathbb{E}\left(e^{\alpha(t\wedge\tau_{n,N})}\right) \leq e^{\alpha f(|x_1 - x_2|)}\mathbb{E}\left(e^{\alpha\int_0^{t\wedge\tau_{n,N}} \frac{f'(|X_1-X_2|)}{|X_1-X_2|}\frac{1}{\sqrt{2}}(X_1-X_2,d(W_1-W_2))}\right). \tag{3.7}$$

On the other hand, since

$$\mathbb{E}\left(e^{\alpha\int_0^{t\wedge\tau_{n,N}} \frac{f'(|X_1-X_2|)}{|X_1-X_2|}\frac{1}{\sqrt{2}}(X_1-X_2,d(W_1-W_2))}\right) \leq \left(\mathbb{E}\left(e^{2\alpha^2\int_0^{t\wedge\tau_{n,N}} |f'(|X_1-X_2|)|^2 ds}\right)\right)^{1/2},$$

we deduce from Lemma 3.3 that

$$\mathbb{E}\left(e^{\alpha\int_0^{t\wedge\tau_{n,N}} \frac{f'(|X_1-X_2|)}{|X_1-X_2|}\frac{1}{\sqrt{2}}(X_1-X_2,d(W_1-W_2))}\right) \leq \left(\mathbb{E}\left(e^{2\alpha^2\Lambda^2(t\wedge\tau_{n,N})}\right)\right)^{1/2}$$

Substituting in (3.7) yields

$$\mathbb{E}\left(e^{\alpha(t\wedge\tau_{n,N})}\right) \leq e^{\alpha f(|x_1-x_2|)}\left(\mathbb{E}\left(e^{2\alpha^2\Lambda^2(t\wedge\tau_{n,N})}\right)\right)^{1/2}$$

Choosing $\alpha = \frac{1}{2\Lambda^2}$, we deduce

$$\mathbb{E}\left(e^{\frac{1}{2\Lambda^2}(t\wedge\tau_{n,N})}\right) \leq e^{\frac{1}{2\Lambda^2}f(|x_1-x_2|)} \leq e^{1/\Lambda}$$

Letting $n \to \infty$, $N \to \infty$ and arguing as in the proof of Proposition 2.3 we find the conclusion.

□

Corollary 3.5 *We have*

$$|P_t\varphi(x_1) - P_t\varphi(x_2)| \leq 2\,\|\varphi\|_0\, e^{\frac{1}{\Lambda}}\ e^{-\frac{1}{2\Lambda^2}t}$$

Proof: Let $x_1, x_2 \in H$. Since (X_1, X_2) is a coupling of $(X(\cdot, x_1), X(\cdot, x_2))$, we have

$$|P_t\varphi(x_1) - P_t\varphi(x_2)| = |\mathbb{E}(\varphi(X_1(t)) - \varphi(X_2(t)))| \leq \|\varphi\|_0 \, \mathbb{P}(\tau_{x_1,x_2} \geq t).$$

Now the conclusion follows Proposition 3.4. □

We end this section by proving that the spectral gap property holds. We thus recover a known result (see for instance [3]) with a totally different method.

Proposition 3.6 *Let ν be an invariant measure then, for any $p > 1$, there exist c_p, α_p such that*

$$|P_t\varphi - \bar{\varphi}|_{L^p(H,\nu)} \leq c_p e^{-\alpha_p t} \, |\varphi|_{L^p(H,\nu)}$$

Proof: By Corollary 3.5, we have the result for $p = \infty$. Using that P_t is a contraction semigroup on $L^1(H, \nu)$ and an interpolation argument, we obtain the result. □

4 Burgers equation

We take here $H = L^2(0,1)$ and denote by $\|\cdot\|$ the norm of the Sobolev space $H_0^1(0,1)$. We consider the equation

$$\begin{cases} dX = (AX + b(X))dt + dW, \\ \\ X(0) = x, \end{cases} \tag{4.1}$$

where

$$Ax = D_\xi^2 x, \quad x \in D(A) = H^2(0,1) \cap H_0^1(0,1)$$

and

$$b(x) = D_\xi(x^2), \quad x \in H_0^1(0,1).$$

It well known that problem (4.1) has a unique solution for any $x \in L^2(0,1)$ which we denote by $X(t,x)$, see [4]. It defines a transition semigroup $(P_t)_{t \geq 0}$. It is also known that it has a unique invariant measure and is ergodic (see [5]). The following result states the exponential convergence to equilibrium.

Theorem 4.1 *There exist constants $C, \gamma > 0$ such that for any $x_1, x_2 \in L^2(0,1)$, $\varphi \in C_b(L^2(0,1))$,*

$$|P_t\varphi(x_1) - P_t\varphi(x_2)| \leq ce^{-\gamma t}\|\varphi\|_0(1 + |x_1|^4 + |x_2|^4)$$

To prove this result, we want to construct a coupling for equation (4.1). It does not seem possible to apply directly the method of section 2. We shall first consider a cut off equation

$$\begin{cases} dX = (AX + D_\xi F_R(X))dt + dW, \\ \\ X(0) = x, \end{cases} \tag{4.2}$$

where $F_R : L^4(0,1) \to L^2(0,1)$ is defined by

$$F_R(x) = \begin{cases} x^2, \quad \text{if } |x|_{L^4} \leq R, \\ \\ \frac{R^2 x^2}{|x|_{L^4}^2}, \quad \text{if } |x|_{L^4} \geq R. \end{cases}$$

We have

$$|F_R(x) - F_R(y)| \leq 2R\, |x-y|_{L^4}, \quad x, y \in L^4(0,1) \tag{4.3}$$

and

$$|F_R(x)| \leq R^2, \quad x \in L^4(0,1). \tag{4.4}$$

We have denoted by $|\cdot|_{L^4}$ the norm in $L^4(0,1)$. The norm in $L^2(0,1)$ is still denoted by $|\cdot|$. It is not difficult to check that Hypothesis 1.1 and (1.5) hold so that the results of section 2 can be applied. We then need a priori estimates on the solutions of (4.1) so that we can control when the coupling for the cut-off equation can be used. These are given in section 4.2. Then, we construct a coupling for the Burgers equation which enables us to prove the result in section 4.4.

4.1 Coupling for the cut–off equation

Here $R > 0$ is fixed. We denote by the same symbol c_R various constants depending only on R.

Lemma 4.2 *There exists $\alpha > 0, \beta > 0, c_R > 0$ such that*

$$\frac{2\alpha}{1-\beta} \in [1,2), \tag{4.5}$$

and

$$|F_R(x) - F_R(y)| \leq c_R |x-y|^\alpha \|x-y\|^\beta, \quad x, y \in H_0^1(0,1). \tag{4.6}$$

Proof. First notice that by (4.3) and (4.4) it follows that

$$|F_R(x) - F_R(y)| \leq (2R)^{2-\gamma}\, |x-y|_{L^4}^\gamma, \quad x, y \in L^4(0,1),$$

for any $\gamma \in [0,1]$. Moreover, by the Sobolev embedding theorem we have $H^{1/4}(0,1) \subset L^4(0,1)$ and using a well known interpolatory inequality we find that

$$|x-y|_{L^4} \leq c|x-y|_{H^{1/4}} \leq c|x-y|^{3/4}\, \|x-y\|^{1/4}, \quad x, y \in H_0^1(0,1). \tag{4.7}$$

Consequently

$$|F_R(x) - F_R(y)| \leq c(2R)^{2-\gamma}\ |x-y|^{3\gamma/4}\|x-y\|^{\gamma/4}$$

Now setting $\alpha = 3\gamma/4,\ \beta = \gamma/4$, the conclusion follows choosing $\gamma \in [\frac{4}{7}, 1]$. $\Box$

We now construct the coupling for equation (4.2). For any $x, y \in D(A)$ we have, taking into account Lemma 4.2,

$$\begin{aligned}(A(x-y) + D_\xi F_R(x) - D_\xi F_R(y), x-y) \ &\leq\ -\|x-y\|^2 + |F_R(x) - F_R(y)|\ \|x-y\| \\ &\leq\ -\|x-y\|^2 + c_R\ |x-y|^\alpha\ \|x-y\|^{1+\beta}.\end{aligned}$$

Using the elementary inequality

$$uv \leq \tfrac{1+\beta}{2}\ (\epsilon u)^{\frac{2}{1+\beta}} + \tfrac{1-\beta}{2}\ (\epsilon^{-1} v)^{\frac{2}{1-\beta}}, \quad u, v, \epsilon > 0,$$

and choosing suitably ϵ we find

$$\begin{aligned}(A(x-y) + D_\xi F_R(x) - D_\xi F_R(y), x-y) \ &\leq\ -\tfrac{1}{2}\|x-y\|^2 + c_R\ |x-y|^{\frac{2\alpha}{1-\beta}} \\ &\leq\ -\tfrac{1}{2}\|x-y\|^2 + \tfrac{\pi^2}{4}|x-y|^2 + c_R\ |x-y|,\end{aligned}$$

since $\frac{2\alpha}{1-\beta} \in [1, 2)$. By the Poincaré inequality we conclude that

$$(A(x-y) + D_\xi F_R(x) - D_\xi F_R(y), x-y) \leq -\tfrac{\pi^2}{4}|x-y|^2 + c_R\ |x-y|.$$

Consequently (2.5) (with $\kappa = 1$) reduces to

$$2f_R''(r) - f_R'(r)(\tfrac{\pi^2}{4} r - c_R) = -1,$$

$$f_R''(r) - f_R'(r)(2\,a\,r - c_R) = -\tfrac{1}{2}, \qquad a = \tfrac{\pi^2}{16}.$$

Then

$$f_R'(r) = e^{ar^2 - c_R r} f_R'(0) - \tfrac{1}{2}\int_0^r e^{a(r^2-s^2) - c_R(r-s)} ds$$

and

$$f_R(r) = f_R(0) + f_R'(0)\int_0^r e^{as^2 - c_R s} ds - \tfrac{1}{2}\int_0^r ds \int_0^s e^{a(s^2-u^2) - c_R(s-u)} du.$$

Setting

$$f_R(0) = 0, \quad f_R'(0) = \tfrac{1}{2}\ e^{ar^2 - c_R r}\int_0^{+\infty} e^{-au^2 + c_R u} du,$$

we obtain the solution

$$f_R(r) := \tfrac{1}{2}\int_0^r e^{as^2 - c_R s}\left(\int_s^\infty e^{-a\xi^2 + c_R \xi} d\xi\right) ds. \tag{4.8}$$

We denote by $X_R(t,x)$ the solution of the cut-off equation (4.2). The corresponding coupling constructed above is denoted by $\big(X_{1,R}(t;x_1,x_2), X_{2,R}(t;x_1,x_2)\big)$. Then, setting

$$\tau_R^{x_1,x_2} = \inf\{t>0 : X_{1,R}(t;x_1,x_2) = X_{2,R}(t;x_1,x_2)\}.$$

By Proposition 2.3, we have

$$\mathbb{E}(\tau_R^{x_1,x_2}) \le f_R(|x_1-x_2|) \tag{4.9}$$

Remark 4.3 *Using similar arguments as in section 3, we can derive bounds on f_R and f_R' and prove following result for the transition semigroup associated the the cut-off equation. For all $\varphi \in C_b(H)$ we have*

$$|P_t^R\varphi(x_1) - P_t^R\varphi(x_2)| \le 2c\,\|\varphi\|_0\,(1+|x_1-x_2|)^{1/\Lambda}\,e^{-\frac{1}{2\Lambda^2}t}, \quad x_1,x_2\in H.$$

4.2 A priori estimates

Next result is similar to Proposition 2.3 in [2].

Lemma 4.4 *Let $\alpha \ge 0$ and*

$$z^\alpha(t) = \int_{-\infty}^t e^{(A-\alpha)(t-s)}dW(s).$$

Then, for any $p\in\mathbb{N}$, $\varepsilon>0$, $\delta>0$, there exists a random variable $K(\varepsilon,\delta,p)$ such that

$$|z^\alpha(t)|_{L^p} \le K(\varepsilon,\delta,p)\,\alpha^{-\frac14+\varepsilon}(1+|t|^\delta)$$

Moreover, all the moments of $K(\varepsilon,\delta,p)$ are finite.

Proof: Proceeding as in the proof of Proposition 2.1 in [2], we have

$$z^\alpha(t) = \int_{-\infty}^t \Big[(t-\sigma)^{\beta-1} - \alpha\int_\sigma^t (\tau-\sigma)^{\beta-1}e^{-\alpha(t-\tau)}d\tau\Big]e^{A(t-\sigma)}Y(\sigma)d\sigma$$

where

$$Y(\sigma) = \frac{\sin\pi\beta}{\pi}\int_{-\infty}^\sigma (\sigma-s)^{-\beta}e^{A(\sigma-s)}dW(s)$$

and $\beta\in(0,1/4)$. It is proved in [2] that for $\gamma\in[0,1]$

$$\begin{aligned}&\Big|(t-\sigma)^{\beta-1} - \alpha\int_\sigma^t(\tau-\sigma)^{\beta-1}e^{-\alpha(t-\tau)}d\tau\Big|\\ &\qquad\qquad \le c(\beta,\gamma)\,\alpha^{-\gamma}(t-\sigma)^{\beta-1-\gamma} + (t-\sigma)^{\beta-1}e^{-\alpha(t-\sigma)}.\end{aligned}$$

We deduce, by Poincaré inequality,

$$|z^\alpha(t)|_{L^p} \le c\int_{-\infty}^t \Big(\alpha^{-\gamma}(t-\sigma)^{\beta-1-\gamma} + (t-\sigma)^{\beta-1}e^{-\alpha(t-\sigma)}\Big)e^{-\lambda_p(t-\sigma)}|Y(\sigma)|_{L^p}d\sigma,$$

and, for $r>1$, $m\in\mathbb{N}$, by Hölder inequality we obtain if $\beta > \frac{1}{2m}$ and $\beta > \gamma + \frac{1}{2m}$

$$\begin{aligned}|z^\alpha(t)|_{L^p} &\le c\Big[\alpha^{-\gamma}\Big(\int_{-\infty}^t (t-\sigma)^{\frac{2m}{2m-1}(\beta-1-\gamma)}(1+|\sigma|^r)^{\frac{1}{2m-1}}e^{-\lambda_p(t-\sigma)}d\sigma\Big)^{\frac{2m-1}{2m}}\\ &\qquad + \Big(\int_{-\infty}^t (t-\sigma)^{(\beta-1)\frac{2m}{2m-1}}e^{-\frac{2m\alpha}{2m-1}(t-\sigma)}(1+|\sigma|^r)^{\frac{1}{2m-1}}d\sigma\Big)^{\frac{2m-1}{2m}}\Big]\\ &\qquad \times\Big(\int_{-\infty}^t (1+|\sigma|^r)^{-1}|Y(\sigma)|_{L^p}^{2m}\,d\sigma\Big)^{\frac{1}{2m}}\\ &\le c\Big(\alpha^{-\gamma}+\alpha^{-\beta+\frac{1}{2m}}\Big)(1+|t|^{\frac{r}{2m}})\Big(\int_{-\infty}^t (1+|\sigma|^r)^{-1}|Y(\sigma)|_{L^p}^{2m}\,d\sigma\Big)^{\frac{1}{2m}}\end{aligned}$$

and the first statement follows if γ, β, m are chosen so that

$$\tfrac{1}{2m} < \min(\tfrac{\delta}{r},\tfrac{\varepsilon}{2}), \qquad \tfrac14 - \tfrac{\varepsilon}{2} < \gamma + \tfrac{\varepsilon}{2} < \tfrac14.$$

Indeed, proceeding as in [2], we easily prove that

$$K(\varepsilon,\delta,p) = \Big(\int_{-\infty}^{\infty}(1+|\sigma|^r)^{-1}|Y(\sigma)|_{L^p}^{2m}\,d\sigma\Big)^{\frac{1}{2m}}$$

has all moments finite. □

Proposition 4.5 *Let $x\in L^2(0,1)$ and let $X(t,x)$ be the solution of (4.2).*

i) For any $\delta>0$, there exists a constant $K_1(\delta)$ such that for any $x\in L^4(0,1)$, $t\ge 0$

$$\mathbb{E}\left(\sup_{s\in[0,t]}|X(s,x)|_{L^4}^4\right) \le 4|x|_{L^4}^4 + K_1(\delta)(1+t^\delta)$$

ii) There exists a constant $K_2\ge 0$ such that for any $x\in L^4(0,1)$ and $t\ge 0$

$$\mathbb{E}(|X(t,x)|_{L^4}^4) \le (e^{-\pi^2 t/16}|x|_{L^4} + K_2)^4.$$

iii) There exists a constant K_3 such that for any $x\in L^2(0,1)$ and $t\in[1,2]$

$$\mathbb{E}(|X(t,x)|_{L^4}^4) \le K_3(1+|x|_{L^2}^4).$$

Proof: In the proof we shall denote by c several different constant. Let us prove (i). Fix $t>0, x\in L^4$ and $\delta>0$ and set $Y(s)=X(s,x)-z^\alpha(s)$, $Y(s)$ satisfies

$$\begin{cases} \dfrac{dY(s)}{ds} = AY(s) + D_\xi(Y(s)+z^\alpha(s))^2 + \alpha z^\alpha(s), \\ \\ Y(0) = x - z^\alpha(0) \end{cases}$$

By similar computations as in [2, Proposition 2.2], we have that

$$\tfrac{1}{4}\,\frac{d}{ds}|Y(s)|_{L^4}^4+\tfrac{3}{2}\int_0^1 |Y(s)|^2\,|D_\xi Y(s)|^2\,d\xi \le c|z^\alpha(s)|_{L^4}^{\frac{8}{3}}\,|Y(s)|_{L^4}^4+c|z^\alpha(s)|_{L^4}^4+\alpha^4|z^\alpha(s)|_{L^4}^4,$$

and, by the Poincaré inequality,

$$\frac{d}{ds}|Y(s)|_{L^4}^4 + \tfrac{3\pi^2}{8}|Y(s)|_{L^4}^4 \le c|z^\alpha(s)|_{L^4}^{\frac{8}{3}}|Y(s)|_{L^4}^4 + (c+\alpha^4(s))|z^\alpha(s)|_{L^4}^4. \tag{4.10}$$

We now choose α so large that

$$c|z^\alpha(s)|_{L^4}^{\frac{8}{3}} \le \tfrac{\pi^2}{8}, \quad s\in[0,t]. \tag{4.11}$$

For this we use Lemma 4.4 with

$$\varepsilon = \tfrac{1}{8}, \quad p=4 \quad \text{and } \delta \text{ replaced by } \tfrac{\delta}{8}.$$

We see that there exists $c>0$ such that (4.11) holds provided $\alpha = c\left(K(\frac{1}{8},\frac{\delta}{8},4)(1+t^{\delta/8}\right)^8$. So, by (4.10) it follows that

$$\frac{d}{ds}|Y(s)|_{L^4}^4 + \tfrac{\pi^2}{4}|Y(s)|_{L^4}^4 \le c\left(1+K(\tfrac{1}{8},\tfrac{\delta}{8},4)^8(1+t)^\delta\right), \quad s\in[0,t].$$

Consequently, by the Gronwall lemma, we see that

$$\begin{aligned} |Y(s)|_{L^4}^4 &\le e^{-\frac{\pi^2}{4}s}|Y(0)|_{L^4}^4 + c\int_0^s e^{-\frac{\pi^2}{4}(s-\sigma)}(1+K(\tfrac{1}{8},\tfrac{\delta}{8},4)^8(1+\sigma)^\delta d\sigma \\ &\le e^{-\frac{\pi^2}{4}s}|Y(0)|_{L^4}^4 + c(1+K(\tfrac{1}{8},\tfrac{\delta}{8},4)^8(1+t^\delta)), \quad s\in[0,t], \end{aligned}$$

which yields by (4.11)

$$\begin{aligned} |X(s,x)|_{L^4}^4 &\le 4e^{-\frac{\pi^2}{4}s}|x|_{L^4}^4 + 4|z^\alpha(0)|_{L^4}^4 + 4|z^\alpha(s)|_{L^4}^4 + c\,(1+K(\tfrac{1}{8},\tfrac{\delta}{8},4)^8(1+t^\delta)) \\ &\le 4e^{-\frac{\pi^2}{4}s}|x|_{L^4}^4 + c(1+K(\tfrac{1}{8},\tfrac{\delta}{8},4)^8(1+t^\delta)), \quad s\in[0,t]. \end{aligned} \tag{4.12}$$

Now, i) follows since $K(\frac{1}{8},\frac{\delta}{8},4)$ has finite moments.

To prove ii) we denote by $X(t,-t_0;x)$ the solution at time t of the Burgers equation with initial data x at the time $-t_0$. Since $X(t_0,x)$ and $X(0,-t_0;x)$ have the same law, it suffices to prove

$$\mathbb{E}(|X(0,-t_0;x)|_{L^4}^4) \le (e^{-\pi^2 t_0/16}|x|_{L^4}+K_2)^4, \quad t_0\ge 0.$$

We set $Y(t)=X(t,-t_0:x)-z^\alpha(t)$. Proceeding as above we find

$$|Y(t)|_{L^4}^4 \le e^{-\frac{\pi^2}{4}(t-s)}|Y(s)|_{L^4}^4 + c(1+K(\tfrac18,\tfrac18,4)^8(1+|s|)), \text{ for } 0\ge t\ge s.$$

Since $(a+b)^{1/4}\le a^{1/4}+b^{1/4}$, we obtain

$$|X(t,-t_0;x)|_{L^4} \le e^{-\frac{\pi^2}{16}(t-s)}|X(s,-t_0;x)|_{L^4}+c(1+K(\tfrac18,\tfrac18,4)^8(1+|s|))^{1/4}, \text{ for } 0\ge t\ge s.$$

We choose $-s=-t+1=1,2,\dots,n_0$ with $n_0=[t_0]$ and then $-s=-t+1=t_0$, we obtain

$$\begin{aligned}|X(0,-t_0,x)|_{L^4} &\le e^{-n_0\pi^2(t-s)/16}|X(-n_0,-t_0,x)|_{L^4}\\ &+c\textstyle\sum_{l=0}^{n-1} e^{-l\pi^2/16}(1+K(\tfrac18,\tfrac18,4)^8(1+l))^{1/4}\\ &\le e^{-t_0\pi^2(t-s)/16}|x|_{L^4}+c\textstyle\sum_{l=0}^{n-1} e^{-l\pi^2/16}(1+K(\tfrac18,\tfrac18,4)^8(1+l))^{1/4}\\ &+e^{-n_0\pi^2/16}(1+K(\tfrac18,\tfrac18,4)^8(1+t_0))^{1/4}\\ &\le e^{-t_0\pi^2(t-s)/16}|x|_{L^4}+c(1+K(\tfrac18,\tfrac18,4)^8)^{1/4},\end{aligned}$$

so, ii) follows.

To prove iii), we use, as in i), $Y(s)=X(s,x)-z^\alpha(s)$ and have

$$\tfrac12\frac{d}{ds}|Y(s)|_{L^2}^2+|D_\xi Y(s)|_{L^2}^2 \le c|z^\alpha(s)|_{L^4}^{\frac83}|Y(s)|_{L^2}^2+c|z^\alpha(s)|_{L^4}^4+\alpha^2|z^\alpha(s)|_{L^2}^2$$

so that, choosing α conveniently,

$$\sup_{[0,2]}|Y(s)|_{L^2}^2+\int_0^2|D_\xi Y(s)|_{L^2}^2ds \le |x|_{L^2}^2+c(1+\alpha^2).$$

For instance, we can take $\alpha=cK(\frac18,1,4)^8$.
Moreover

$$\frac{d}{ds}\big(s|Y(s)|_{L^4}^4\big)+\tfrac{\pi^2}{4}s|Y(s)|_{L^4}^4 \le |Y(s)|_{L^4}^4+c(1+\alpha^4)$$

so that for $t\in[1,2]$

$$|Y(t)|_{L^4}^4 \le \int_0^2|Y(s)|_{L^4}^4ds+c(1+\alpha^4).$$

Using the inequality (4.7), we have

$$|Y(t)|_{L^4} \le c\, |Y(t)|_{L^2}^{\frac{3}{4}}\, |D_\xi Y(t)|_{L^2}^{\frac{1}{4}}$$

and we deduce

$$|Y(t)|_{L^4}^4 \le c \int_0^2 |Y(s)|_{L^2}^3 |D_\xi Y(s)|_{L^2} ds + c(1+\alpha^4)$$

$$\le c(|x|_{L^2}^4 + (1+\alpha^2)^2) + c(1+\alpha^4)$$

for $t \in [1,2]$ and iii) follows. □

Next lemma is similar to Lemma 2.6 in Kuksin-Shirikyan [8].

Lemma 4.6 *Let $\rho_0 > 0$ and $\rho_1 > 0$, there exist $\alpha(\rho_0, \rho_1) > 0$ and $T(\rho_0,\rho_1) > 0$ such that for $t \in [T(\rho_0,\rho_1), 2T(\rho_0,\rho_1)]$ and $|x_1|_{L^4} \le \rho_0$, $|x_2|_{L^4} \le \rho_0$,*

$$\mathbb{P}\big(|X(t,x_1)|_{L^4} \le \rho_1 \text{ and } |X(t,x_2)|_{L^4} \le \rho_1\big) \ge \alpha(\rho_0,\rho_1).$$

Proof: Let X^0 be the solution of the deterministic Burgers equation

$$\begin{cases} \dfrac{dX^0}{dt}(t,x) = AX^0(t,x) + b(X^0(t,x)) \\ \\ X^0(0,x) = x. \end{cases}$$

Since, as easily checked, $(b(X^0), (X^0)^3) = 0$, we obtain by standard computations

$$|X^0(t,x)|_{L^4}^4 \le e^{-\pi^2 t/4} |x|_{L^4}^4 \le e^{-\pi^2 t/4} \rho_0^4, \quad \text{for all } |x|_{L^4} \le \rho_0.$$

We choose

$$T(\rho_0,\rho_1) = \frac{16}{\pi^2}\, \ln\Big(\frac{2\rho_0}{\rho_1}\Big).$$

Then, for $t \ge T(\rho_0,\rho_1)$

$$|X^0(t,x)|_{L^4} \le \frac{\rho_1}{2} \quad \text{for all } |x|_{L^4} \le \rho_0.$$

let

$$z(t) = \int_0^t e^{(t-s)A} dW(s)$$

then for any $\eta > 0$

$$\mathbb{P}\Big(\sup_{[0,2T_0(\rho_0,\rho_1)]} |z(t)|_{L^4} \le \eta\Big) > 0$$

and since the solution of the stochastic equation is a continuous function of z we can find η such that

$$|X(t,x_i)|_{L^4} \le |X^0(t,x_i)| + \frac{\rho_1}{2},\ i = 1,2,$$

for $t \in [0, 2T(\rho_0,\rho_1)]$ provided $|z(t)|_{L^4} \le \eta$ on $[0, 2T(\rho_0,\rho_1)]$. The conclusion follows easily. □

4.3 Construction of the coupling

Let $x_1, x_2 \in L^4(0,1)$, we construct $\big(X_1(\cdot; x_1, x_2), X_2(\cdot; x_1, x_2)\big)$ a coupling of $X(\cdot; x_1)$ and $X(\cdot; x_2)$ as follows. Fix $\rho_0 > 0$, $\rho_1 > 0$, $R > \max\{\rho_0, \rho_1\}$, $T > T_0 := T(\rho_0, \rho_1)$ (defined in Lemma 4.6), all to be chosen later.

We recall that $\big(X_{1,R}(t; x_1, x_2), X_{2,R}(t; x_1, x_2)\big)$ represents the coupling of $X_R(\cdot; x_1)$ and $X_R(\cdot; x_2)$ constructed in section 4.1, where $X_R(\cdot; x_1)$ and $X_R(\cdot; x_2)$ are the solutions of the cut-off equation (4.2).

We shall need also the coupling of $X_R(\cdot, T_0; x_1)$ and $X_R(\cdot, T_0; x_2)$ when the initial time is any $T_0 > 0$ instead of 0. We denote it by

$$\big(X_{1,R}(t, T_0; x_1, x_2), X_{2,R}(t, T_0; x_1, x_2)\big)$$

and in this case we shall write

$$\tau_R^{x_1,x_2} = \inf\{t > 0 : X_{1,R}(t + T_0, T_0; x_1, x_2) = X_{2,R}(t + T_0, T_0; x_1, x_2)\}.$$

Notice that $\tau_R^{x_1,x_2}$ does not depend on T_0 thank to the Markov property because the Burgers equation does not depend explicitely on time.

First we shall construct the coupling on $[0, T]$, defining $(X_1(t; x_1, x_2), X_2(t; x_1, x_2))$ as follows. If

$$x_1 \neq x_2,\ |x_1|_{L^4} \le \rho_0,\ |x_2|_{L^4} \le \rho_0,\ |X(T_0; x_1)|_{L^4} \le \rho_1 \text{ and } |X(T_0; x_2)|_{L^4} \le \rho_1, \quad (4.13)$$

we set

$$X_1(t; x_1, x_2) = X(t, x_1), \quad X_2(t; x_1, x_2) = X(t, x_2), \quad \text{for } t \in [0, T_0]$$

and for $i = 1, 2$

$$X_i(t; x_1, x_2) = \begin{cases} X_{i,R}(t, T_0; X(T_0, x_1), X(T_0, x_2)) \text{ if } T_0 \le t \le \min\{\widetilde{\tau}_R, T\}, \\[2ex] X(t, \tilde{\tau}_R; X_{i,R}(\tilde{\tau}_R, T_0; X(T_0, x_1), X(T_0, x_2)) \text{ if } \min\{\widetilde{\tau}_R, T\} < t \le T, \end{cases}$$

where

$$\widetilde{\tau}_R = \inf\{t \ge T_0 : \max\{|X_{i,R}(t, T_0; X(T_0, x_1), X(T_0, x_2))|_{L^4},\ i = 1, 2\} > R\}.$$

If (4.13) does not hold, we simply set

$$X_1(t; x_1, x_2) = X(t, x_1), \quad X_2(t; x_1, x_2) = X(t, x_2), \quad \text{for } t \in [0, T]$$

So, we have constructed the coupling on $[0, T]$. The preceding construction can be obviously generalized considering a time interval $[t_0, t_0 + T]$ and random initial data (η_1, η_2), $\mathcal{F}_{t_0}$-measurable. In this case we denote the coupling by

$$(X_1(t, t_0, \eta_1, \eta_2), X_2(t, t_0, \eta_1, \eta_2)).$$

Now we define the coupling $(X_1(t;x_1,x_2),X_2(t;x_1,x_2))$ for all time, setting by recurrence

$$X_i(t,x_1,x_2) := X_i(t,kT,X_1(kT,x_1,x_2),X_2(kT,x_1,x_2)), \quad i=1,2,\ t\in[kT,(k+1)T].$$

Let us summarize the construction of the coupling on $[0,T]$. We first let the original processes $X(\cdot,x_1)$, $X(\cdot,x_2)$ evolve until they are both in the ball of radius ρ_0. Then, we let them evolve and if at time T_0 they both enter the ball of radius ρ_1 we use the coupling of the truncated equation as long as the norm do not exceed R (so that if $\widetilde{\tau}_R \geq T-T_0$ we have a coupling of the Burgers equation having good properties). Then, if the coupling is successful, i.e. if $X_1(T;x_1,x_2)=X_2(T;x_1,x_2)$, we use the original Burgers equation and the solutions remain equal. Otherwise, we try again in $[T,2T]$ and so on.

4.4 Exponential convergence to equilibrium for the Burgers equation

We shall choose now ρ_0, ρ_1, R and T (recall that $T_0=T(\rho_0,\rho_1)$ is determined by Lemma 4.6). We first assume that

$$|x_1|_{L^4}\leq\rho_0 \qquad \text{and} \qquad |x_2|_{L^4}\leq\rho_0$$

and set

$$A=\left\{\tau_R^{X(T_0,x_1),X(T_0,x_2)}\leq T-T_0\right\} \quad \text{(the coupling is successfull in } [T_0,T]),$$

$$B=\left\{\sup_{t\in[T_0,T],i=1,2}|X_{i,R}(t,T_0;X(T_0,x_1),X(T_0,x_2))|_{L^4}\leq R\right\},$$

$$C=\left\{\max_{i=1,2}|X(T_0,x_i)|_{L^4}\leq\rho_1\right\}.$$

Then, we have

$$\mathbb{P}\big(X_1(T;x_1,x_2)=X_2(T,x_1,x_2)\big)\geq\mathbb{P}(A\cap B\cap C).$$

We are now going to estimate $\mathbb{P}(A\cap B\cap C)$. Concerning C we note that by Lemma 4.6 it follows that

$$\mathbb{P}\left(\max_{i=1,2}|X(T_0,x_i)|_{L^4}\leq\rho_1\right)\geq\alpha(\rho_0,\rho_1).$$

Moreover,

$$\mathbb{P}(A\cap B\cap C)=$$

$$\int\limits_{|y_i|_{L^4}\leq\rho_1,i=1,2}\mathbb{P}(A\cap B|X(T_0,x_1)=y_1,X(T_0,x_2)=y_2)\mathbb{P}(X(T_0,x_1)\in dy_1,X(T_0,x_2)\in dy_2). \tag{4.14}$$

But

$$\mathbb{P}(A \cap B | X(T_0, x_1) = y_1, X(T_0, x_2) = y_2)$$

$$= \mathbb{P}(\tau_R^{y_1,y_2} \leq T - T_0 \text{ and } \sup_{[T_0,T]} |X_{i,R}(t, T_0; y_1, y_2)|_{L^4} \leq R \quad \text{for } i = 1, 2)$$
$$\geq 1 - \mathbb{P}(\tau_R^{y_1,y_2} > T - T_0) - \sum_{i=1,2} \mathbb{P}\left(\sup_{[T_0,T]} |X_{i,R}(t, T_0; y_1, y_2)| > R\right) \tag{4.15}$$

By the Chebyshev inequality and (4.9) it follows that

$$\mathbb{P}(\tau_R^{y_1,y_2} \geq T - T_0) \leq \frac{1}{T - T_0}\, \mathbb{E}(\tau_R^{y_1,y_2}) \leq \frac{1}{T - T_0}\, f^R(|y_1 - y_2|)$$

and, since $\mathcal{L}\big(X_{i,R}(\cdot, T_0; y_1, y_2)\big) = \mathcal{L}\big(X_R(\cdot, T_0; y_i)\big) = \mathcal{L}\big(X_R(\cdot; y_i)\big)$, we have, taking into account Proposition 4.5-(i), that

$$\mathbb{P}\left(\sup_{[T_0,T]} |X_{i,R}(t, T_0; y_1, y_2)|_{L^4} > R\right) = \mathbb{P}\left(\sup_{[0,T-T_0]} |X_R(t, y_i)|_{L^4} > R\right)$$

$$= \mathbb{P}\left(\sup_{[0,T-T_0]} |X(t, y_i)|_{L^4} > R\right) \leq \frac{4\rho_1^4 + K_1(\delta)(1 + (T - T_0)^\delta)}{R^4}.$$

Consequently, if $|y_i|_{L^4} \leq \rho_1$ for $i = 1, 2$,

$$1 - \mathbb{P}(\tau_R^{y_1,y_2} > T - T_0) - \sum_{i=1,2} \mathbb{P}\left(\sup_{[T_0,T]} |X_{i,R}(t, T_0; y_1, y_2)| > R\right)$$

$$\geq 1 - \frac{1}{T - T_0}\, f_R(|y_1 - y_2|) - 2\frac{4\rho_1^4 + K_1(\delta)(1 + (T - T_0)^\delta)}{R^4}$$

$$\geq 1 - \frac{1}{T - T_0}\, f_R(2\rho_1) - \frac{8\rho_1^4 + 2K_1(\delta)(1 + (T - T_0)^\delta)}{R^4}$$

We deduce by (4.14) and (4.15) that for $|x_i|_{L^4} \leq \rho_0$, $i = 1, 2$,

$$\mathbb{P}\big(X_1(T, x_1, x_2) = X_2(T, x_1, x_2)\big) \geq \big(1 - \tfrac{1}{T-T_0}\, f_R(2\rho_1)$$
$$- \tfrac{8\rho_1^4 + 2K_1(\delta)(1+(T-T_0)^\delta)}{R^4}\big)\alpha(\rho_0, \rho_1). \tag{4.16}$$

We choose now $T - T_0 = 1$, $\rho_1 \leq 1$ and R such that

$$\frac{8 + 4K_1(\delta)}{R^4} \leq \frac{1}{4}. \tag{4.17}$$

Then, we take ρ_1 such that

$$f_R(2\rho_1) \leq \frac{1}{4} \tag{4.18}$$

(this is possible since $f_R(0) = 0$ and f_R is continuous).
It follows

$$\mathbb{P}\big(X_1(T;x_1,x_2) = X_2(T;x_1,x_2)\big) \geq \frac{1}{2}\,\alpha(\rho_0,\rho_1), \quad \text{for all } |x_1|_{L^4} \leq \rho_0,\ |x_2|_{L^4} \leq \rho_0. \tag{4.19}$$

To treat the case of arbitrary x_1, x_2, have to choose ρ_0. We proceed as in Kuksin-Shirikyan [8] and introduce the following Kantorovich functional

$$F_k = \mathbb{E}\big((1+\nu(|X_1(kT;x_1,x_2)|^4_{L^4} + |X_2(kT;x_1,x_2)|^4_{L^4}))1\!\!1_{X_1(kT;x_1,x_2)\neq X_2(kT;x_1,x_2)}\big),$$

where $k \in \mathbb{N}\cup\{0\}$ and ν is to be chosen later.

Proposition 4.7 *There exist positive numbers ρ_0, ν, γ such that*

$$\mathbb{P}(X_1(kT;x_1,x_2) \neq X_2(kT;x_1,x_2)) \leq e^{-\gamma k}\,(1+\nu(|x_1|^4_{L^4}+|x_2|^4_{L^4})), \quad x_1,x_2 \in L^4(0,1). \tag{4.20}$$

Proof. We shall denote by the same symbol c several different constants. Let us estimate F_1 in terms of $F_0 = (1+\nu(|x_1|^4_{L^4} + |x_2|^4_{L^4}))1\!\!1_{x_1\neq x_2}$. If $x_1 = x_2$ then $X_1(T;x_1,x_2) = X_2(T;x_1,x_2)$ a.s. and so, $F_1 = 0$. Let now $x_1 \neq x_2$. If $|x_1|_{L^4} > \rho_0$ then $X_1(T;x_1,x_2) = X(T,x_1)$ and $X_2(T;x_1,x_2) = X(T,x_2)$. Consequently, taking into account Proposition 4.5–(ii),

$$\begin{aligned} F_1 &= \mathbb{E}\left[(1+\nu(|X(T,x_1)|^4_{L^4} + |X(T,x_2)|^4_{L^4}))1\!\!1_{X(T,x_1)\neq X(T,x_2)}\right] \\ &\leq \mathbb{E}\big[1+\nu(|X(T,x_1)|^4_{L^4} + |X(T,x_2)|^4_{L^4})\big] \\ &\leq 1+\nu\big((e^{-\pi^2T/16}|x_1|_{L^4}+K_2)^4 + (e^{-\pi^2T/16}|x_2|_{L^4}+K_2)^4\big). \end{aligned}$$

Since $T \geq T - T_0 = 1$, there exists c such that

$$(e^{-\pi^2T/16}a+b)^4 \leq e^{-\pi^2T/8}a^4 + c\,b^4, \qquad \text{for any } a,b \geq 0 \tag{4.21}$$

and we deduce that, if $x_1 \neq x_2$ and $|x_1|_{L^4} > \rho_0$ (or if $x_1 \neq x_2$ and $|x_2|_{L^4} > \rho_0$),

$$F_1 \leq 1+\nu\big(e^{-\pi^2T/8}(|x_1|^4_{L^4}+|x_2|^4_{L^4}) + cK_2^4\big) \tag{4.22}$$

Let us now consider the case when $x_1 \neq x_2$, $|x_1|_{L^4} \leq \rho_0$ and $|x_2|_{L^4} \leq \rho_0$. Taking into account (4.19) and Proposition 4.5, we have

$$
\begin{aligned}
F_1 &= \mathbb{E}\big[(1+\nu(|X_1(T;x_1,x_2)|_{L^4}^4 + |X_2(T;x_1,x_2)|_{L^4}^4))1\!\!1_{X_1(T;x_1,x_2)\neq X_2(T;x_1,x_2)}\big] \\
&\leq \mathbb{P}(X_1(T;x_1,x_2) \neq X_2(T;x_1,x_2)) + \nu\mathbb{E}[|X_1(T;x_1,x_2)|_{L^4}^4 + |X_2(T;x_1,x_2)|_{L^4}^4] \\
&\leq 1 - \frac{1}{2}\,\alpha(\rho_0,\rho_1) + \nu\big(e^{-\pi^2 T/8}(|x_1|_{L^4}^4 + |x_2|_{L^4}^4) + cK_2^4\big)
\end{aligned}
\tag{4.23}
$$

since $\mathcal{L}(X_i(T;x_1,x_2)) = \mathcal{L}(X(T;x_i))$ and thanks to Proposition 4.5–ii) and (4.21).

To conclude, we shall choose ρ_0 and ν such that

$$
\begin{aligned}
q_1(\lambda) &:= \frac{1+\nu(e^{-\pi^2 T/8}\lambda + cK_2^4)}{1+\nu\lambda} \leq e^{-\gamma}, \quad \text{for } \lambda > \rho_0^4 \\
q_2(\lambda) &:= \frac{1-\frac{1}{2}\,\alpha(\rho_0,\rho_1) + \nu(e^{-\pi^2 T/8}\lambda + cK_2^4)}{1+\nu\lambda} \leq e^{-\gamma}, \quad \text{for } \lambda \leq 2\rho_0^4.
\end{aligned}
\tag{4.24}
$$

Note first that q_1 is decreasing in λ and tends to $e^{-\pi^2 T/8}$ as $\lambda \to \infty$. We choose ρ_0 such that

$$
\rho_0^4 = \frac{2(1+cK_2^4)}{1-e^{-\pi^2 T/8}}, \quad T = 1 + T_0(\rho_0,\rho_1).
$$

(It is easy to see that this equation has a solution). With this choice of ρ_0, it is also easy to check that $q_1(\rho_0^4) < 1$.
Choosing now

$$
\nu = \frac{\alpha(\rho_0,\rho_1)}{4cK_2^4}
$$

it is easy to check that

$$
q_2(\lambda) \leq \max\{e^{-\pi^2 T/8}, 1 - \tfrac{1}{4}\,\alpha(\rho_0,\rho_1)\}.
$$

Hence, choosing γ such that

$$
e^{-\gamma} \geq \max\{1 - \tfrac{1}{4}\,\alpha(\rho_0,\rho_1), q_1(\rho_0)\},
$$

(4.24) is fulfilled.

Now we continue the estimate of F_1. For any $\lambda \geq \rho_0^4$ we have from the first inequality in (4.24) that

$$
1 + \nu(e^{-\pi^2 T/8}\lambda + cK_2^4) \leq e^{-\gamma}(1+\nu\lambda).
$$

Then from (4.22) we deduce that if $x_1 \neq x_2$ and $|x_1|_{L^4} > \rho_0$ or $|x_2|_{L^4} > \rho_0$, we have

$$
F_1 \leq e^{-\gamma}(1+\nu(|x_1|_{L^4}^4 + |x_2|_{L^4}^4)) = e^{-\gamma}F_0.
$$

Moreover

$$1-\frac{1}{2}\,\alpha(\rho_0,\rho_1)+\nu\big(e^{-\pi^2T/8}\lambda+cK_2^4\big)\le e^{-\gamma}(1+\nu\lambda)$$

for any $\lambda\le 2\rho_0^4$. Then, by (4.23), we obtain for $x_1\ne x_2$, $|x_1|_{L^4}\le\rho_0$ and $|x_2|_{L^4}\le\rho_0$

$$F_1\le e^{-\gamma}(1+\nu(|x_1|_{L^4}^4+|x_2|_{L^4}^4))=e^{-\gamma}F_0.$$

Therefore, we have in any case

$$F_1\le e^{-\gamma}\,F_0,\quad x_1,x_2\in L^4(0,1).$$

It is not difficult to check that $(X_1(kT;x_1,x_2),X_2(kT;x_1,x_2))_{k\in\mathbb{N}}$ is a Markov chain so that we obtain for any $k\in\mathbb{N}$

$$F_{k+1}\le e^{-\gamma}\,F_k,\quad x_1,x_2\in L^4(0,1).$$

and, so

$$F_k\le e^{-k\gamma}\,F_0,\quad x_1,x_2\in L^4(0,1).$$

In particular

$$\mathbb{P}(X_1(kT;x_1,x_2)\ne X_2(kT;x_1,x_2))\le e^{-k\gamma}\,(1+\nu(|x_1|_{L^4}^4+|x_2|_{L^4}^4)),\quad x_1,x_2\in L^4(0,1).$$

□

By Proposition 4.7 the exponential convergence to equilibrium follows for $x_1,x_2\in L^4(0,1)$. If $x_1,x_2\in L^2(0,1)$, we write

$$\mathbb{P}(X_1(kT;X(1,x_1),X(1,x_2))\ne X_2(kT;X(1,x_1),X(1,x_2))$$

$$\le e^{-(k-1)\gamma}\,\big(1+\nu\mathbb{E}(|X(1,x_1)|_{L^4}^4+|X(1,x_2)|_{L^4}^4)\big)$$

and use Proposition 4.5–(iii) to conclude the proof of Theorem 4.1. □

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