# Asymptotic behaviour of a Neumann parabolic problem with hysteresis

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**Abstract**

A parabolic equation in two or three space variables with a Preisach hysteresis operator and with homogeneous Neumann boundary conditions is shown to admit a unique global regular solution. A detailed investigation of the Preisach memory dynamics shows that the system converges to an equilibrium in the state space of all admissible Preisach memory configurations as time tends to infinity.

Classification: 35K55; 47J40; 35B40

Key words: parabolic equation; hysteresis; asymptotic behavior of solutions; Preisach model

# Introduction

We present here a qualitative study on the long time asymptotic stabilization of the solution to the equation

$$\frac{\partial}{\partial t}(f(u) + \mathcal{W}[\lambda, u]) - \Delta u \;=\; 0 \tag{0.1}$$

with given initial conditions, where $\Delta$ is the Laplace operator on a bounded regular domain $\Omega$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ with homogeneous Neumann boundary conditions, $f$ is an increasing function on an interval $[-R, R]$, and $\mathcal{W}[\lambda, \cdot]$ is a Preisach operator with initial memory configuration $\lambda$.

The original motivation for this problem comes from soil hydrology. The measurements carried out in [11] and interpreted in [9] show that the relation between the water pressure $p$ and moisture contents $s$ in the soil exhibits a Preisach-like hysteresis behavior. We write down the relation between $p$ and $s$ in the form $s = \tilde{f}(p) + \tilde{\mathcal{W}}[\lambda, p]$ under suitable assumptions on the function $\tilde{f}$ and the Preisach operator $\tilde{\mathcal{W}}$. Having in mind this application, we allow both $\tilde{f}$ and $\tilde{\mathcal{W}}$ to be bounded in their domains of definition.

Equation (0.1) can be interpreted as the local balance of liquid mass in a porous soil, provided we assume the simplified isotropic Darcy law $q = -\kappa(p)\nabla p$ between the mass flux $q$ and the pressure gradient $\nabla p$ with a positive proportionality factor $\kappa(p)$. For

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a more detailed discussion on further modeling issues, see [2, 3, 4, 28]. Assuming that no mass exchange with the exterior takes place, the mass balance then reads

$$\frac{\partial s}{\partial t} + \operatorname{div} q \;=\; 0 \;\text{ in } \Omega, \quad \langle q, n\rangle = 0 \;\text{ on } \partial\Omega\,, \tag{0.2}$$

where $n$ is the unit outward normal vector. We introduce formally a new variable

$$u \;=\; K(p) := \int_0^p \kappa(p')\,dp'.$$

This enables us to rewrite the balance equation (0.2) in the form

$$\frac{\partial}{\partial t}((\tilde f \circ K^{-1})(u) + \tilde{\mathcal{W}}[\lambda, K^{-1}(u)]) - \Delta u \;=\; 0 \;\text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \;\text{ on } \partial\Omega\,. \tag{0.3}$$

By [17, Theorem 4.17], the mapping $u \mapsto \tilde{\mathcal{W}}[\lambda, K^{-1}(u)])$ is again a Preisach operator, hence we obtain (0.1) with $f = \tilde f \circ K^{-1}$, $\mathcal{W}[\lambda, \cdot] = \tilde{\mathcal{W}}[\lambda, K^{-1}(\cdot)])$.

Figure 1 shows a typical diagram of the $u \mapsto s$ dependence. Preisach operators are characterized by the properties of *return point memory*, that is, minor loops return to their starting point, and *congruency*, that is, minor loops spanned between the same input values $u_1 < u_2$ have the same shape, see [23].

We consider the initial condition for $u$ within the invertibility domain of the constitutive operator (represented by the interval $[u_*, u^*]$ on Figure 1). By the parabolic maximum principle, the whole process takes place between these bounds. The main result we prove, namely that the solution $u$ tends uniformly to a constant as time tends to infinity, is indeed in agreement with the case without hysteresis as well as with the 1D case treated in [8], and therefore should not be surprising. The way to prove it is, however, rather difficult. This is due to the fact that the state space of the process described by (0.1) consists not only of all admissible space distributions of $u$, but also of all admissible space dependent memory configurations of the Preisach operator $\mathcal{W}$. More specifically, besides the space variable $x \in \Omega$ and time $t > 0$, an additional memory variable $r > 0$ comes into play and makes the asymptotic analysis more complicated. Note that, as it is often the case in the qualitative theory of PDEs with hysteresis, it is more convenient to use here the equivalent one-parametric representation of the Preisach operator, which goes back to [16], instead of the two-parametric one known from engineering literature, cf. [23].

The paper is organized as follows. The hypotheses and main results are stated in Section 1. Section 2 is devoted to a survey of less known results on memory properties of the Preisach model. A time-discrete scheme for Eq. (0.1) is presented in Section 3, where we derive the crucial estimates, which are necessary for our arguments. The convergence of the solution trajectory towards an equilibrium is then established in Section 4.

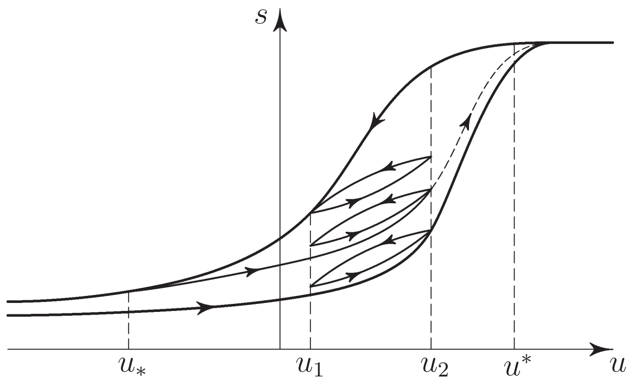


Figure 1: Return point memory and congruency of the Preisach diagram.

# 1 Statement of the problem

In a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $3$, with a Lipschitzian boundary and in the time interval $\mathbb{R}_+ = [0, \infty[$ we consider the evolution problem

$$\frac{\partial}{\partial t}(f(u) + w) - \Delta u \ = \ 0\,, \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given function, $\Delta$ is the Laplace operator, and

$$w(x,t) \ = \ \mathcal{W}[\lambda, u](x,t) \ = \ \int_0^\infty g(r, \mathfrak{p}_r[\lambda(x,\cdot), u(x,\cdot)](t))\, dr \tag{1.2}$$

is the output of a Preisach operator $\mathcal{W}$ with initial memory configuration $\lambda$ and generating function $g(r,v) = \int_0^v \varphi(r,z)\, dz$, where $\varphi$ is a given non-negative function. The notation in (1.2) and an elementary theory of the Preisach model are explained in detail below in Section 2.

Equation (1.1) is coupled with the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n} \ = \ 0 \quad \text{on } \partial\Omega \times \,]0, \infty[\ , \tag{1.3}$$

and with the initial condition

$$u(x,0) = u_0(x)\,, \quad w(x,0) = w_0(x) := \int_0^\infty g(r, P[\lambda(x,\cdot), u_0(x)](r))\, dr\,, \tag{1.4}$$

where $P$ is defined in (2.17).

**Hypothesis 1.1.** *We fix a constant $R > 0$, and make the following assumptions on the data.*

(i) *The function $f$ belongs to $W^{2,\infty}(-R,R)$, and there exist constants $f_1 > f_0 > 0$, $f_2 > 0$ such that $|f''(v)| \le f_2$ a. e., $f_0 \le f'(v) \le f_1$ for all $v \in [-R,R]$.*

(ii) *The function* $\varphi$ *in the definition of the Preisach operator restricted to* $Q(R) = ]0, R[ \times ]-R, R[$ *is measurable, and there exist a function* $\beta \in L^1(0, R)$ *and a constant* $\varphi_1 > 0$ *such that*

$$0 \ \leq \ \varphi(r,z) \ \leq \ \varphi_1 \,, \qquad \left|\frac{\partial \varphi}{\partial z}(r,z)\right| \ \leq \ \beta(r)$$

*for almost all arguments in* $Q(R)$ .

(iii) *The initial memory configuration* $\lambda$ *is a strongly measurable mapping from* $\Omega$ *to* $\Lambda_R$ *, where*

$$\Lambda_R \ := \ \{\lambda \in W^{1,\infty}(0,\infty)\,;\, |\lambda'(r)| \leq 1 \ a.\,e., \ \lambda(r) = 0 \ \ for \ r \geq R\} \tag{1.5}$$

*is endowed with the sup-norm.*

(iv) *The initial condition* $u_0$ *belongs to* $W^{2,2}(\Omega)$ *, and there exist constants* $-R < u_* < u^* < R$ *such that*

$$u_* \ \leq \ u_0(x) \ \leq \ u^* \quad \forall x \in \Omega\,.$$

For the meaning of Hypotheses 1.1 (ii) and (iii) we refer again to Section 2 below. In the 3D case, we will have to make a more restrictive assumption. To this end, we define the number

$$\gamma \ = \ f_2 + \int_0^R \beta(r)\, dr\,. \tag{1.6}$$

**Hypothesis 1.2.** *In the case* $d = 3$ *, assume in addition to Hypothesis* 1.1 *that*

$$\gamma^2 \int_\Omega \left(|\nabla u_0|^2 + |\Delta u_0|^2\right)\, dx \ < \ \frac{8 f_0^2}{81 \mu_4^4(\Omega)}\,,$$

*where* $\mu_4(\Omega)$ *is the constant in the Gagliardo-Nirenberg inequality* (3.39) *below corresponding to* $p = 4$ *.*

Note that if $f$ is linear and $\mathcal{W}$ is the so-called *Prandtl-Ishlinskii operator* corresponding to $\partial\varphi(r,z)/\partial z = 0$, then $\gamma = 0$ and Hypothesis 1.2 is satisfied automatically. Hypothesis 1.2 thus says that in 3D, either the initial datum must be close to a constant, or the combined hysteresis nonlinearity $f + \mathcal{W}$ must be somehow close to a Prandtl-Ishlinskii operator within the range $[u_*, u^*]$. Geometrically, the ascending/descending hysteresis branches have to be "almost convex/almost concave", respectively. A detailed discussion on convex hysteresis can be found in the monograph [18].

We now state the main result of the paper.

**Theorem 1.3.** *Let Hypotheses* 1.1–1.2 *hold. Then Problem* (1.1) – (1.4) *admits a unique continuous solution* $u$ *on* $\Omega \times \mathbb{R}_+$ *such that* $u_* \leq u(x,t) \leq u^*$ *for all* $(x,t) \in \Omega \times \mathbb{R}_+$ *,* $\partial_t u, \Delta u \in L^2(\Omega \times \mathbb{R}_+) \cap L^\infty(\mathbb{R}_+; L^2(\Omega))$ *,* $\partial_t \nabla u \in L^2(\Omega \times \mathbb{R}_+)$ *, and there exists a constant* $u_\infty \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty} \sup_{x\in\Omega} |u(x,t) - u_\infty| \ = \ 0\,. \tag{1.7}$$

*Moreover, also the instantaneous Preisach memory configurations converge to a limit configuration* $\lambda_\infty(x,r)$ *uniformly with respect to both variables* $x$ *and* $r$ *.*

**Remark 1.4.** Results on existence on bounded time intervals as well as on the asymptotic behavior for Dirichlet boundary conditions can already be found in early works by Visintin and his collaborators, see [14] and the survey in [27]. Uniqueness is based in a standard way on the Hilpert inequality, see [12]. The regularity, global bounds, and the long time stabilization results in the Neumann case, however, require a more sophisticated estimation technique, which we explain in detail in Sections 3–4. The difficulty of the asymptotic convergence argument is due to the fact that the state space contains not only the value of $u$ itself, but also all admissible hysteresis memory configurations, and the set of possible equilibria is thus very large.

For the reader's convenience, we first recall some basic concepts of the Preisach hysteresis model.

# 2 Hysteresis operators

The theory of hysteresis has a long history. One hundred years ago, Madelung in [22] proposed probably the first axiomatic approach to hysteresis by defining three experimental laws of what we call nowadays return point memory hysteresis (or "wiping-out property", cf. [23]). The model for ferromagnetic hysteresis proposed by Preisach in [24] is a prominent representative that possesses the return point memory property. Only recently, Brokate and Sprekels proved (see [7, Theorem 2.7.7]) that every return point memory hysteresis operator, which admits a specific initial memory configuration, has necessarily a Preisach-type memory structure. A basic mathematical theory of hysteresis operators has been developed by M. Krasnosel'skii and his collaborators. The results of this group are summarized in the monograph [15], which constitutes until now the main source of reference on hysteresis. Our presentation here is based on more recent results from [18, 19] that are needed here, in particular the alternative one-parametric formulation of the Preisach model based on variational inequalities.

## 2.1 The play operator in the space of regulated functions

The so-called *play operator* introduced in [15] is the main building block of the theory. For our purposes, it is convenient to work in the space $G_R(\mathbb{R}_+)$ of right-continuous regulated functions of time $t \in \mathbb{R}_+$, that is, functions $u : \mathbb{R}_+ \to \mathbb{R}$ that admit the left limit $u(t-)$ at each point $t > 0$, and the right limit $u(t+)$ exists and coincides with $u(t)$ for each $t \geq 0$. More information about regulated functions can be found e. g. in [1, 6, 13, 20, 26].

We endow the space $G_R(\mathbb{R}_+)$ with the system of seminorms

$$\|u\|_{[0,t]} \;=\; \sup\{|u(\tau)|\,;\, \tau \in [0,t]\} \qquad \text{for } u \in G_R(\mathbb{R}_+) \text{ and } t \in \mathbb{R}_+\,. \tag{2.1}$$

With the metric

$$d(u,v) \;=\; \sup_{T>0} \frac{\|u-v\|_{[0,T]}}{1+\|u-v\|_{[0,T]}} \qquad \text{for } u,v \in G_R(\mathbb{R}_+)\,, \tag{2.2}$$

the set $G_R(\mathbb{R}_+)$ becomes a Fréchet space. Similarly, $BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ will denote the space of right-continuous functions of bounded variation on each interval $[0,T]$ for any $T>0$, and $C(\mathbb{R}_+)$ is the space of continuous functions on $\mathbb{R}_+$. We have $BV_R^{\mathrm{loc}}(\mathbb{R}_+) \subset G_R(\mathbb{R}_+)$ and the embedding is dense, while $C(\mathbb{R}_+)$ is a closed subspace of $G_R(\mathbb{R}_+)$.

The uniform approximation problem for real-valued regulated functions by functions of bounded variation has actually an interesting solution. For each $u \in G_R(\mathbb{R}_+)$, a parameter $r>0$, and an initial condition $\xi_r^0 \in [u(0)-r, u(0)+r]$, there exists a unique $\xi_r \in BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ in the $r$-neighborhood of $u$ with minimal total variation, that is,

$$
\begin{aligned}
&|u(t)-\xi_r(t)| \le r \qquad \forall t \ge 0\,, && (2.3)\\
&\xi_r(0) \;=\; \xi_r^0\,, && (2.4)\\
&\operatorname*{Var}_{[0,t]} \xi_r \;=\; \min\left\{ \operatorname*{Var}_{[0,t]} \eta\,;\, \eta \in BV_R^{\mathrm{loc}}(\mathbb{R}_+),\ \eta(0)=\xi_r^0,\ \|u-\eta\|_{[0,t]} \le r \right\} \quad \forall t>0\,. && (2.5)
\end{aligned}
$$

This result goes back to A. Vladimirov and V. Chernorutskii for the case of continuous functions $u$; for a proof see [25]. An extension to $L^\infty(\mathbb{R}_+)$ has been done in [21]. The function $\xi_r$ can also be characterized as the unique solution of the variational inequality

$$
\begin{aligned}
|u(t)-\xi_r(t)| &\;\le\; r \qquad \forall t \ge 0\,, && (2.6)\\
\xi_r(0) &\;=\; \xi_r^0\,, && (2.7)\\
\int_0^t (u(\tau)-\xi_r(\tau)-y(\tau))\, d\xi_r(\tau) &\;\ge\; 0 && (2.8)\\
&\qquad \forall t \ge 0 \ \ \forall y \in G_R(\mathbb{R}_+)\,,\ \|y\|_{[0,t]} \le r\,,
\end{aligned}
$$

where the integration in (2.8) is understood in the Young or Kurzweil sense, see [20, 21]. If moreover $u$ is continuous, then $\xi_r$ is continuous, we can restrict ourselves to continuous test functions $y$, and (2.8) can be interpreted as the usual Stieltjes integral.

Let $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ denote the space of absolutely continuous functions on $\mathbb{R}_+$. It is an easy exercise to show that if $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, then the solution $\xi_r$ to (2.6)–(2.8) belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ and fulfils the variational inequality

$$
\dot\xi_r(t)\,(u(t)-\xi_r(t)-y) \;\ge\; 0 \ \text{ a.e.} \quad \forall y \in [-r,r]\,. \tag{2.9}
$$

Putting $y = u(t\pm h)-\xi_r(t\pm h)$ and letting $h \searrow 0$ we infer from (2.9) that

$$
\dot\xi_r(t)\,\dot u(t) \;=\; \dot\xi_r(t)^2 \ \text{ a.e.}. \tag{2.10}
$$

Let us consider the mapping $\hat{\mathfrak{p}}_r : \mathbb{R} \times G_R(\mathbb{R}_+) \to BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ which with each $\hat{\xi}_r^0 \in \mathbb{R}$ and $u \in G_R(\mathbb{R}_+)$ associates the solution $\xi_r$ of (2.6) – (2.8) with

$$
\xi_r^0 \;=\; \max\{u(0)-r, \min\{u(0)+r, \hat{\xi}_r^0\}\}\,. \tag{2.11}
$$

Then $\hat{\mathfrak{p}}_r$ is a hysteresis operator called the *play*. Alternative equivalent definitions of the play can be found in [7, 15, 27].

In order to model a more complex hysteresis memory behavior, it is convenient to consider the whole family $\{\xi_r\}_{r>0}$ corresponding to a fixed input $u \in G_R(\mathbb{R}_+)$. The parameter $r > 0$ then plays the role of *memory variable*. We introduce the *hysteresis memory state space*

$$\Lambda \;=\; \{\lambda : \mathbb{R}_+ \to \mathbb{R}\,;\; |\lambda(r) - \lambda(s)| \le |r - s| \;\; \forall r, s \in \mathbb{R}_+\,,\; \lim_{r\to+\infty} \lambda(r) = 0\}\,, \tag{2.12}$$

and choose the initial condition $\{\hat{\xi}_r^0\}_{r>0}$ in the form

$$\hat{\xi}_r^0 = \lambda(r) \qquad \text{for } \; r > 0\,, \tag{2.13}$$

where $\lambda \in \Lambda$ is given. We define the operators $\mathfrak{p}_r : \Lambda \times G_R(\mathbb{R}_+) \to BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ for $r > 0$ by the formula

$$\mathfrak{p}_r[\lambda, u] \;=\; \hat{\mathfrak{p}}_r[\lambda(r), u] \tag{2.14}$$

for $\lambda \in \Lambda$ and $u \in G_R(\mathbb{R}_+)$. Consistently with the definition we set $\mathfrak{p}_0[\lambda, u](t) = u(t)$ for all $t \ge 0$.

The following result was proved in [18, 21].

**Proposition 2.1.** *For every $\lambda \in \Lambda$, $u \in G_R(\mathbb{R}_+)$, and $t \ge 0$, the mapping $r \mapsto \lambda_t(r) = \mathfrak{p}_r[\lambda, u](t)$ belongs to $\Lambda$, and for all $\lambda_1, \lambda_2 \in \Lambda$, $u_1, u_2 \in G_R(\mathbb{R}_+)$ and $t \ge 0$ we have*

$$|\mathfrak{p}_r[\lambda_1, u_1](t) - \mathfrak{p}_r[\lambda_2, u_2](t)| \;\le\; \max\{|\lambda_1(r) - \lambda_2(r)|, \|u_1 - u_2\|_{[0,t]}\} \;\; \forall r \ge 0\,. \tag{2.15}$$

**Remark 2.2.** The play operator generates for every $t \ge 0$ a continuous *state mapping* $\Pi_t : \Lambda \times G_R(\mathbb{R}_+) \to \Lambda$ which with each $(\lambda, u) \in \Lambda \times G_R(\mathbb{R}_+)$ associates the state $\lambda_t \in \Lambda$ at time $t$, see Figure 2. Also in the sequel, we will interpret $\mathfrak{p}_r[\lambda, u](t)$ in both ways: as a function $\xi_r(t)$ of time parameterized by the memory variable $r$, or alternatively as function $\lambda_t(r)$ describing the memory configuration at time $t$.

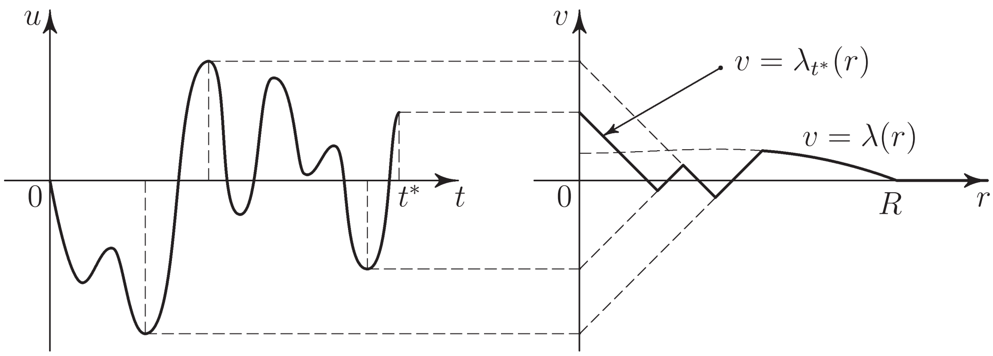


Figure 2: Evolution of the memory configuration.

In order to study further properties of the play, we first derive an explicit formula for $\mathfrak{p}_r[\lambda, u]$ if $u$ is a step function of the form

$$u(t) = \sum_{k=1}^{m} u_{k-1} \chi_{[t_{k-1}, t_k[}(t) \qquad \text{for} \quad t \ge 0 \tag{2.16}$$

with some given $u_i \in \mathbb{R}$, $i = 0, 1, \ldots, m-1$, where $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = +\infty$ is a given sequence and $\chi_A$ for $A \subset \mathbb{R}$ is the characteristic function of the set $A$, that is, $\chi_A(t) = 1$ for $t \in A$, $\chi_A(t) = 0$ otherwise. We define analogously to (2.11) for $\lambda \in \Lambda$ and $v \in \mathbb{R}$ the function $P[\lambda, v] : \mathbb{R}_+ \to \mathbb{R}$ by the formula

$$P[\lambda, v](r) \;=\; \max\{v - r, \min\{v + r, \lambda(r)\}\}\,. \tag{2.17}$$

In particular, $P$ is a mapping from $\Lambda \times \mathbb{R}$ to $\Lambda$. One can directly check as a one-dimensional counterpart of [20, Proposition 4.3] using the Young or Kurzweil integral calculus and the inequality

$$(P[\lambda, v](r) - \lambda(r))\;(v - P[\lambda, v](r) - z) \;\geq\; 0 \quad \forall |z| \leq r \tag{2.18}$$

that we have

$$\xi_r(t) = \sum_{k=1}^{m} \xi_{k-1}^{(r)} \chi_{[t_{k-1}, t_k[}(t) \qquad \text{for} \quad t \geq 0\,, \tag{2.19}$$

with

$$\xi_k^{(r)} = \lambda_k(r)\,, \quad \lambda_k = P[\lambda_{k-1}, u_k]\,, \quad \lambda_{-1} = \lambda\,. \tag{2.20}$$

for $k = 0, \ldots, m-1$.

As an example, consider the special case that the function $u$ in (2.16) is non-decreasing in some interval $[t_{k_0-1}, t_{k_1}]$, that is,

$$u_{k_0-1} \leq u_{k_0} \leq u_{k_0+1} \leq \cdots \leq u_{k_1}\,. \tag{2.21}$$

Then we have $\lambda_k(r) = \max\{u_k - r, \lambda_{k-1}(r)\}$ for $k = k_0, \ldots, k_1$. Let us fix some $k$ between $k_0$ and $k_1 - 1$ and set

$$r_k = \max\{r \geq 0\,;\, \lambda_{k-1}(r) + r = u_k\}\,, \quad \hat{r}_k = \max\{r \geq 0\,;\, \lambda_{k-1}(r) + r = u_{k+1}\}\,. \tag{2.22}$$

We have $u_{k+1} \geq u_k$, hence $\hat{r}_k \geq r_k$, and $\lambda_k(\hat{r}_k) = \lambda_{k-1}(\hat{r}_k)$. This yields $r_{k+1} \geq \hat{r}_k$. On the other hand, we have $\lambda_{k-1}(r) \leq \lambda_k(r)$ for all $r \geq 0$, hence $r_{k+1} \leq \hat{r}_k$. We conclude that

$$\lambda_{k+1} = P[\lambda_{k-1}, u_{k+1}]\,, \tag{2.23}$$

and by induction over $k$

$$\lambda_k = P[\lambda_{k_0-1}, u_k] \quad \forall k = k_0, \ldots, k_1\,. \tag{2.24}$$

The same result is obtained if $u$ is non-increasing in $[t_{k_0-1}, t_{k_1}]$.

Every function $u \in G_R(\mathbb{R}_+)$ can be approximated uniformly on every compact interval by step functions of the form (2.16). Proposition 2.1 enables us to extend the property (2.24) to the whole space $G_R(\mathbb{R}_+)$ and obtain for a function $u \in G_R(\mathbb{R}_+)$, which is monotone (non-decreasing or non-increasing) in an interval $[t_0, t_1]$ the representation formula

$$\mathfrak{p}_r[\lambda, u](t) \;=\; P[\lambda_{t_0}, u(t)](r) \;=\; \max\{u(t) - r, \min\{u(t) + r, \lambda_{t_0}(r)\}\} \tag{2.25}$$

for $t \in [t_0, t_1]$. Note that (2.25) has originally been used in [15] as alternative definition of the play on continuous piecewise monotone inputs, extended afterwards by density and continuity to the whole space of continuous functions.

More generally, the play possesses the *semigroup property* as a time-continuous version of (2.20), namely

$$\mathfrak{p}_r[\lambda, u](t+s) = \mathfrak{p}_r[\lambda_s, u(s+\cdot)](t) \tag{2.26}$$

for all $u \in G_R(\mathbb{R}_+)$, $\lambda \in \Lambda$ and $s, t \geq 0$.

For $K > 0$, let us consider subsets $\Lambda_K$ of the state space $\Lambda$ defined as

$$\Lambda_K \;=\; \{\lambda \in \Lambda\,;\, \lambda(r) = 0 \;\text{ for }\; r \geq K\} \tag{2.27}$$

in agreement with the notation in (1.5), and

$$\Lambda_\infty \;=\; \bigcup_{K>0} \Lambda_K\,. \tag{2.28}$$

The following property of the play is proved e. g. in [19, Lemma 3.1.2].

**Lemma 2.3.** *Let $u \in G_R(\mathbb{R}_+)$ and $t \geq 0$ be given. Set*

$$u_{\max}(t) = \sup_{\tau\in[0,t]} u(\tau)\,, \qquad u_{\min}(t) = \inf_{\tau\in[0,t]} u(\tau)\,. \tag{2.29}$$

*Then for all $\lambda \in \Lambda$ and $r > 0$ we have*

$$\mathfrak{p}_r[\lambda, u](\tau) \;\leq\; \max\{\lambda(r),\, u_{\max}(t) - r\} \quad \forall\, \tau \in [0,t]\,, \tag{2.30}$$

$$\mathfrak{p}_r[\lambda, u](\tau) \;\geq\; \min\{\lambda(r),\, u_{\min}(t) + r\} \quad \forall\, \tau \in [0,t]\,, \tag{2.31}$$

$$\mathfrak{p}_r[\lambda, u](t) \;=\; \lambda(r) \qquad \textit{for } r > \|m_\lambda(u(\cdot))\|_{[0,t]}\,, \tag{2.32}$$

*where for $v \in \mathbb{R}$ we put $m_\lambda(v) = \inf\{r \geq 0\,;\, |\lambda(r) - v| = r\}$. In particular, for $K > 0$, $\lambda \in \Lambda_K$ we have $\lambda_t \in \Lambda_{K_t}$ for all $t \geq 0$, where $K_t = \max\{K, \|u\|_{[0,t]}\}$.*

## 2.2 The Preisach operator

We describe here a construction of the Preisach model, which goes back to [16] and is equivalent to the classical model proposed in [24] and further investigated in [15, 23, 27]. The advantage of the present approach consists in a more straightforward derivation of analytical properties that are necessary for the a priori estimates in the next sections.

Given a non-negative locally bounded measurable function $\varphi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, we define a mapping $\mathcal{W}$, which with each initial memory distribution $\lambda \in \Lambda_\infty$ and input $u \in G_R(\mathbb{R}_+)$ associates the function

$$\mathcal{W}[\lambda, u](t) \;=\; \int_0^\infty g(r, \mathfrak{p}_r[\lambda, u](t))\, dr\,, \tag{2.33}$$

where we set

$$g(r,v) \;=\; \int_0^v \varphi(r,z)\,dz \qquad \text{for } v \in \mathbb{R}\,. \tag{2.34}$$

By (2.32), the definition is meaningful, since we integrate only over a finite interval on the right-hand side of (2.33).

The local Lipschitz continuity of $\mathcal{W}$ follows immediately from Proposition 2.1 and Lemma 2.3, and we state the result explicitly as follows.

**Proposition 2.4.** *Let* $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$ *and* $K > 0$ *be given, and let* $\mathcal{W}$ *be the operator* (2.33)*. Then for all* $u_1, u_2 \in G_R(\mathbb{R}_+)$, $\lambda_1, \lambda_2 \in \Lambda_K$, *and* $t \geq 0$ *we have*

$$\begin{aligned} &|\mathcal{W}[\lambda_1,u_1](t) - \mathcal{W}[\lambda_2,u_2](t)| \\ &\qquad \leq\; \max\{\|\lambda_1-\lambda_2\|_{[0,K]}\,, \|u_1-u_2\|_{[0,t]}\} \int_0^{R(t)} \operatorname*{sup\,ess}_{|z|\leq R(t)} \varphi(r,z)\,dr\ , \end{aligned} \tag{2.35}$$

*where we set* $R(t) = \max\{K, \|u_1\|_{[0,t]}\,, \|u_2\|_{[0,t]}\}$.

Let $u \in G_R(\mathbb{R}_+)$ and $0 \leq t_1 < t_2$ be arbitrarily chosen. Putting in (2.35) $\lambda_1 = \lambda_2 =: \lambda$ and $u_1 = u$, $u_2(t) = u(t)$ for $t \in [0, t_1[$ , $u_2(t) = u(t_1)$ for $t \in [t_1, t_2]$, $w = \mathcal{W}[\lambda, u]$, we obtain that

$$|w(t_2) - w(t_1)| \;\leq\; \|u - u(t_1)\|_{[t_1,t_2]} \int_0^{R(t_2)} \operatorname*{sup\,ess}_{|z|\leq R(t_2)} \varphi(r,z)\,dr\,. \tag{2.36}$$

In particular, if $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, then $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, and we have

$$\dot w(t) \;=\; \int_0^\infty \dot\xi_r(t)\,\varphi(r,\xi_r(t))\,dr\,, \tag{2.37}$$

almost everywhere, provided we denote as before $\xi_r(t) = \mathfrak{p}_r[\lambda,u](t)$. This and (2.10) yield

$$0 \;\leq\; \dot w(t)\,\dot u(t) \;\leq\; (\dot u(t))^2 \int_0^{R(t)} \operatorname*{sup\,ess}_{|z|\leq R(t)} \varphi(r,z)\,dr \qquad \text{a.e.} \tag{2.38}$$

This property of the operator $\mathcal{W}$ is called *piecewise monotonicity* or *local monotonicity*.

We may indeed consider both the input and the initial memory configuration $\lambda$ that additionally depend on a parameter $x$ (the space variable, say). If for instance $\lambda(x,\cdot)$ belongs to $\Lambda_\infty$ and $u(x,\cdot)$ belongs to $C(\mathbb{R}_+)$ for (almost) every $x$, then we define

$$\mathcal{W}[\lambda,u](x,t) \;:=\; \int_0^\infty g(r, \mathfrak{p}_r[\lambda(x,\cdot), u(x,\cdot)](t))\,dr\,. \tag{2.39}$$

# 3 Time discrete approximation

We fix a time step $\tau > 0$, the initial condition $u_0$ as in Hypothesis 1.1 (iv), and define in $\Omega$ for $k \in \mathbb{N}$ a recurrent system

$$\frac{1}{\tau}(f(u_k) + w_k - f(u_{k-1}) - w_{k-1}) - \Delta u_k \;=\; 0\,, \tag{3.1}$$

$$w_k(x) \;=\; \int_0^\infty g(r, \lambda_k(x,r))\, dr\,, \tag{3.2}$$

$$\lambda_0(x,r) \;=\; P[\lambda(x,\cdot), u_0(x)](r)\,, \quad \lambda_k(x,r) \;:=\; P[\lambda_{k-1}(x,\cdot), u_k(x)](r)\,, \tag{3.3}$$

$$\frac{\partial u_k}{\partial n} \;=\; 0 \quad \text{on} \;\; \partial\Omega\,. \tag{3.4}$$

Let us check that it admits a solution for every $k \in \mathbb{N}$. To this end, we define an auxiliary function

$$f^*(v) = \begin{cases} f(-R) + f'(-R)(v+R) & \text{for } \; v < -R\,, \\ f(v) & \text{for } \; v \in [-R, R]\,, \\ f(R) + f'(R)(v-R) & \text{for } \; v > R\,, \end{cases} \tag{3.5}$$

suitably extend the function $\varphi$ outside $Q(R)$ into $\varphi^*$ preserving the upper bound and such that $\varphi^*$ vanishes outside $Q(2R)$, and replace Eqs. (3.1)–(3.2) by

$$\frac{1}{\tau}(f^*(u_k) + w_k^* - f^*(u_{k-1}) - w_{k-1}^*) - \Delta u_k \;=\; 0\,, \tag{3.6}$$

$$w_k^*(x) \;=\; \int_0^\infty g^*(r, \lambda_k(x,r))\, dr\,, \tag{3.7}$$

where $g^*$ is the partial antiderivative to $\varphi^*$ with respect to the second variable as in (2.34).

The existence of a unique solution $u_k \in W^{2,2}(\Omega)$ to (3.3)–(3.7) in each step $k$ is easy. The problem is of the form

$$\hat{f}(x, u_k) - \Delta u_k \;=\; h_k \tag{3.8}$$

with $h_k \in L^2(\Omega)$ and $f_0/\tau \le \partial\hat{f}/\partial u_k \le (f_1 + 2\varphi_1 R)/\tau$, and we may use standard compactness and monotonicity arguments.

We now derive a series of crucial estimates.

**Estimate 1**.

For $k \in \mathbb{N}$ we denote

$$a_k(x) \;:=\; \frac{f^*(u_k(x)) + w_k^*(x) - f^*(u_{k-1}(x)) - w_{k-1}^*(x)}{u_k(x) - u_{k-1}(x)}$$

provided $u_k(x) - u_{k-1}(x) \ne 0$, otherwise we set $a_k(x) := f_0$. Eq. (3.6) has the form

$$\frac{a_k(x)}{\tau}(u_k(x) - u_{k-1}(x)) - \Delta u_k(x) \;=\; 0 \tag{3.9}$$

with $f_0 \le a_k(x) \le f_1 + 2\varphi_1 R$ a. e. We test (3.9) by $(u_k(x) - u^*)^+$ (the positive part of $u_k(x) - u^*$) and obtain

$$\int_\Omega \frac{a_k(x)}{\tau}\left((u_k(x) - u^*)^+\right)^2 dx + \int_\Omega \left|\nabla\left((u_k(x) - u^*)^+\right)\right|^2 dx$$
$$= \int_\Omega \frac{a_k(x)}{\tau}(u_{k-1}(x) - u^*)\,(u_k(x) - u^*)^+\, dx\,. \tag{3.10}$$

We have by Hypothesis 1.1 that $u_0(x) - u^* \le 0$, hence $(u_1(x) - u^*)^+ = 0$, that is, $u_1(x) - u^* \le 0$. By induction we obtain from (3.10) that $u_k(x) \le u^*$ a. e. for all $k \in \mathbb{N}$. Testing by $(u_* - u_k(x))^+$ we similarly conclude that

$$u_* \ \le \ u_k(x) \ \le \ u^* \qquad \text{a. e. } \forall k \in \mathbb{N}\,. \tag{3.11}$$

As a consequence, by virtue of Lemma 2.3, we see that the sequence $\{u_k\}$ is a solution to the original problem (3.1)–(3.4).

**Estimate 2.**

For $k = 0, 1, 2, \ldots$ and $x \in \Omega$ set $V_k(x) := f(u_k(x)) + w_k(x)$. Then (3.1) has for $k \in \mathbb{N}$ the form

$$\frac{1}{\tau}(V_k(x) - V_{k-1}(x)) - \Delta u_k(x) \ = \ 0\,. \tag{3.12}$$

Testing (3.12) by $u_k(x) - u_{k-1}(x)$, we obtain

$$\frac{1}{\tau}\int_\Omega (V_k(x) - V_{k-1}(x))(u_k(x) - u_{k-1}(x))\,dx + \int_\Omega \langle \nabla u_k(x), \nabla(u_k(x) - u_{k-1}(x))\rangle\,dx \ = \ 0\,, \tag{3.13}$$

with

$$\langle \nabla u_k, \nabla(u_k - u_{k-1})\rangle \ = \ \frac{1}{2}\left(|\nabla u_k|^2 - |\nabla u_{k-1}|^2 + |\nabla(u_k - u_{k-1})|^2\right)\,,$$

hence for every $n \in \mathbb{N}$ we have

$$\frac{1}{\tau}\sum_{k=1}^n \int_\Omega (V_k(x) - V_{k-1}(x))(u_k(x) - u_{k-1}(x))\,dx + \frac{1}{2}\int_\Omega |\nabla u_n(x)|^2\,dx \ \le \ \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2\,dx\,, \tag{3.14}$$

which yields, by monotonicity of $f, g$, and $P[\lambda, \cdot]$, that

$$\frac{f_0}{\tau}\sum_{k=1}^n \int_\Omega |u_k(x) - u_{k-1}(x)|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u_n(x)|^2\,dx \ \le \ \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2\,dx\,. \tag{3.15}$$

**Estimate 3.**

We denote $V_{-1}(x) = V_0(x) - \tau\,\Delta u_0(x)$, and test the identity

$$\frac{1}{\tau}(V_{k+1}(x) - 2V_k(x) + V_{k-1}(x)) - \Delta(u_{k+1}(x) - u_k(x)) \ = \ 0 \tag{3.16}$$

by $u_{k+1}(x) - u_k(x)$. The crucial point is now to estimate from below for $k \ge 1$ the term

$$D_k \ := \ (V_{k+1} - 2V_k + V_{k-1})(u_{k+1} - u_k) - \frac{1}{2}(V_{k+1} - V_k)(u_{k+1} - u_k) + \frac{1}{2}(V_k - V_{k-1})(u_k - u_{k-1})\,. \tag{3.17}$$

For $k \geq 0$ set

$$B_k(x) \;=\; \frac{V_{k+1}(x) - V_k(x)}{u_{k+1}(x) - u_k(x)} \tag{3.18}$$

with the convention $B_k(x) = f_0$ if $u_{k+1}(x) = u_k(x)$. Then

$$\begin{aligned} D_k(x) \;&=\; \frac{B_k(x)}{2}(u_{k+1}(x) - u_k(x))^2 + \frac{B_{k-1}(x)}{2}(u_k(x) - u_{k-1}(x))^2 \\ &\quad - B_{k-1}(x)(u_{k+1}(x) - u_k(x))(u_k(x) - u_{k-1}(x))\,. \end{aligned} \tag{3.19}$$

By monotonicity, we have $B_k(x) \geq f_0$ for all $k \geq 0$ and a. e. $x \in \Omega$, hence

$$D_k(x) \;\geq 0 \quad \text{whenever} \quad (u_{k+1}(x) - u_k(x))(u_k(x) - u_{k-1}(x)) \;\leq\; 0\,. \tag{3.20}$$

Assume now that for some $k$ and $x$ we have

$$u_{k-1}(x) \;<\; u_k(x) \;<\; u_{k+1}(x)\,. \tag{3.21}$$

To derive a lower bound for (3.19), we make use of the following elementary inequality.

**Lemma 3.1.** *For all positive numbers $B, b, p, q, \gamma$ we have the implication*

$$b - B \;\leq\; \gamma(p+q) \quad\Longrightarrow\quad \frac{1}{2}Bq^2 + \frac{1}{2}bp^2 - bpq + \frac{3}{2}\gamma pq^2 \;\geq\; 0\,. \tag{3.22}$$

*Proof.* The assertion is obvious if $B \geq b$. For $b > B$, we have

$$M \;:=\; \frac{1}{2}Bq^2 + \frac{1}{2}bp^2 - bpq + \frac{3}{2}\gamma pq^2 \;=\; \frac{1}{2}B(q-p)^2 + \frac{1}{2}(b-B)(p^2 - 2pq) + \frac{3}{2}\gamma pq^2\,,$$

hence $M \geq 0$ provided $p \geq 2q$. For $p < 2q$ we have

$$M \;\geq\; \frac{1}{2}\gamma(p+q)(p^2 - 2pq) + \frac{3}{2}\gamma pq^2 \;=\; \frac{1}{2}\gamma p\,(p^2 + q^2 - pq) \;\geq\; 0\,,$$

and the proof is complete. ■

We now resume the derivation of Estimate 3. Recall that the assumption (3.21) still holds. We are thus in the situation of (2.21), hence, by (2.24),

$$V_j(x) \;=\; \Phi(u_j(x)) \qquad \text{for } j = k-1, k, k+1\,, \tag{3.23}$$

where we set

$$\Phi(v) \;=\; f(v) + \int_0^\infty g(r, P[\lambda_{k-1}(x,\cdot), v](r))\,dr \qquad \text{for } v \in [u_*, u^*]\,. \tag{3.24}$$

We denote for simplicity $p(v,r) := P[\lambda_{k-1}(x,\cdot), v](r)$. Then

$$\Phi(v) \;=\; f(v) + \int_0^\infty \int_0^{p(v,r)} \varphi(r,z)\,dz\,dr \tag{3.25}$$

$$\Phi'(v) \;=\; f'(v) + \int_0^\infty \partial_v p(v,r)\,\varphi(r, p(v,r))\,dr\,. \tag{3.26}$$

Similarly as in (2.22) set $r_v := \max\{r \ge 0\,;\, \lambda_{k-1}(x,r) + r = v\}$. We have $\partial_v p(v,r) = 0$ for $r > r_v$, $p(v,r) = v - r$ for $0 < r < r_v$, hence

$$\Phi'(v) \;=\; f'(v) + \int_0^{r_v} \varphi(r, v-r)\, dr \qquad \text{for } v \in [u_*, u^*]\,. \tag{3.27}$$

For $u_* \le v_1 < v_2 \le u^*$ we thus have

$$\begin{aligned} \Phi'(v_2) - \Phi'(v_1) \;&=\; f'(v_2) - f'(v_1) + \int_0^{r_{v_1}} (\varphi(r, v_2 - r) - \varphi(r, v_1 - r))\, dr \\ &\quad + \int_{r_{v_1}}^{r_{v_2}} \varphi(r, v_2 - r)\, dr \\ &\ge\; -\left(f_2 + \int_0^R \beta(r)\, dr\right)(v_2 - v_1) \end{aligned} \tag{3.28}$$

according to Hypothesis 1.1.

Identities (3.23) and the Mean Value Theorem imply that there exist intermediate values $v_{k-1}(x) \in [u_{k-1}(x), u_k(x)]$ and $v_k(x) \in [u_k(x), u_{k+1}(x)]$ such that

$$B_k(x) \;=\; \Phi'(v_k(x))\,, \quad B_{k-1}(x) \;=\; \Phi'(v_{k-1}(x))\,. \tag{3.29}$$

Using $\gamma$ from (1.6), setting $p = u_k(x) - u_{k-1}(x)$, $q = u_{k+1}(x) - u_k(x)$, $B = B_k(x)$, $b = B_{k-1}(x)$, and combining Lemma 3.1 with (3.28)–(3.29), we obtain from (3.19) that

$$D_k(x) \;\ge\; -\frac{3}{2}\gamma\,(u_{k+1}(x) - u_k(x))^2 (u_k(x) - u_{k-1}(x))\,. \tag{3.30}$$

The case

$$u_{k-1}(x) \;>\; u_k(x) \;>\; u_{k+1}(x) \tag{3.31}$$

can be treated in a similar way. This time we have to introduce

$$r_v := \max\{r \ge 0; \lambda_{k-1}(x,r) - r = v\}$$

and this gives $v_2 > v_1 \;\Rightarrow\; r_{v_2} < r_{v_1}$. The counterparts of (3.27) and (3.28) then read

$$\Phi'(v) = f'(v) + \int_0^{r_v} \varphi(r, v+r)\, dr$$

and

$$\Phi'(v_2) - \Phi'(v_1) \le \left(f_2 + \int_0^R \beta(r)\, dr\right)(v_2 - v_1) \qquad \text{for } u_* \le v_1 < v_2 \le u^*\,.$$

The rest of the proof goes on with the obvious modifications, giving the following lower bound for $D_k(x)$:

$$D_k(x) \;\ge\; -\frac{3}{2}\gamma\,(u_k(x) - u_{k+1}(x))^2 (u_{k-1}(x) - u_k(x))\,. \tag{3.32}$$

From (3.16)–(3.20), (3.30) and (3.32) we thus obtain for $k \geq 1$ that

$$
\begin{aligned}
&\frac{1}{2\tau} \int_\Omega (V_{k+1}(x) - V_k(x))(u_{k+1}(x) - u_k(x))\, dx + \int_\Omega |\nabla (u_{k+1}(x) - u_k(x))|^2\, dx \\
&\leq \frac{1}{2\tau} \int_\Omega (V_k(x) - V_{k-1}(x))(u_k(x) - u_{k-1}(x))\, dx \\
&\quad + \frac{3\gamma}{2\tau} \int_\Omega (u_{k+1}(x) - u_k(x))^2 |u_k(x) - u_{k-1}(x)|\, dx\,. \qquad (3.33)
\end{aligned}
$$

The term corresponding to $k = 0$ will be treated directly. By (3.12) we have

$$
\begin{aligned}
&\frac{1}{\tau} \int_\Omega (V_1(x) - V_0(x))(u_1(x) - u_0(x))\, dx + \int_\Omega |\nabla (u_1(x) - u_0(x))|^2\, dx \\
&= \int_\Omega \Delta u_0(x)\, (u_1(x) - u_0(x))\, dx \\
&\leq \left( \frac{\tau}{f_0} \int_\Omega |\Delta u_0(x)|^2\, dx \right)^{1/2} \left( \frac{f_0}{\tau} \int_\Omega |u_1(x) - u_0(x)|^2\, dx \right)^{1/2} \\
&\leq \frac{\tau}{2f_0} \int_\Omega |\Delta u_0(x)|^2\, dx + \frac{1}{2\tau} \int_\Omega (V_1(x) - V_0(x))(u_1(x) - u_0(x))\, dx\,, \qquad (3.34)
\end{aligned}
$$

hence (we omit in the integrals the argument $(x)$ for simplicity)

$$
\frac{1}{2\tau} \int_\Omega (V_1 - V_0)(u_1 - u_0)\, dx + \int_\Omega |\nabla (u_1 - u_0)|^2\, dx \;\leq\; \frac{\tau}{2f_0} \int_\Omega |\Delta u_0|^2 dx\,. \qquad (3.35)
$$

For $k = 0, 1, 2, \ldots$ put

$$
W_k \;:=\; \frac{1}{2\tau^2} \int_\Omega (V_{k+1}(x) - V_k(x))(u_{k+1}(x) - u_k(x))\, dx\,,
$$

and note that

$$
\frac{f_0}{2\tau^2} \int_\Omega |u_{k+1}(x) - u_k(x)|^2\, dx \;\leq\; W_k \;\leq\; \frac{f_1 + \varphi_1 R}{2\tau^2} \int_\Omega |u_{k+1}(x) - u_k(x)|^2\, dx\,. \qquad (3.36)
$$

Using Hölder's inequality and (3.36), we may rewrite (3.33), (3.35) in the form (omitting again the argument $(x)$)

$$
W_0 + \frac{1}{\tau} \int_\Omega |\nabla (u_1 - u_0)|^2\, dx \;\;\leq\;\; \frac{1}{2f_0} \int_\Omega |\Delta u_0|^2 dx\,, \qquad (3.37)
$$

$$
W_k - W_{k-1} + \frac{1}{\tau} \int_\Omega |\nabla (u_{k+1} - u_k)|^2\, dx \;\;\leq\;\; \frac{3\gamma}{\tau\sqrt{2f_0}}\, W_{k-1}^{1/2} \left( \int_\Omega |u_{k+1} - u_k|^4\, dx \right)^{1/2} \qquad (3.38)
$$

for $k \geq 1$. We now use the Gagliardo-Nirenberg inequality

$$
\|v\|_{L^p(\Omega)} \;\leq\; \mu_p(\Omega) \left( \|v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}^{1-\eta} \|\nabla v\|_{L^2(\Omega)}^{\eta} \right)\,, \qquad (3.39)
$$

which holds for every bounded Lipschitzian domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ and every $v \in W^{1,2}(\Omega)$, provided $1/p > 1/2 - 1/d$ and $\eta = d(1/2 - 1/p)$, with a constant $\mu_p(\Omega) > 0$ independent of $v$ (see [5, 10]). We now distinguish the cases $d = 2$ and $d = 3$.

**Case d=2**.

Here, with the symbols $C_i$, $i = 1, \ldots, 7$, we denote positive constants independent of $k$, $x$ and $\tau$, but depending possibly on the data. Setting in (3.39) $v = u_{k+1} - u_k$, $p = 4$, and $\eta = 1/2$, we may remove the $L^4$-norm on the right-hand side of (3.38) and obtain

$$\begin{aligned} W_k - W_{k-1} + \frac{1}{\tau}\int_\Omega |\nabla(u_{k+1} - u_k)|^2\,dx \\ \leq \; C_1\, W_{k-1}^{1/2}\left(\tau W_k + W_k^{1/2}\left(\int_\Omega |\nabla(u_{k+1} - u_k)|^2\,dx\right)^{1/2}\right) \end{aligned} \tag{3.40}$$

by virtue of (3.36). From Young's inequality and (3.40) it follows that

$$\begin{aligned} W_k - W_{k-1} + \frac{1}{2\tau}\int_\Omega |\nabla(u_{k+1} - u_k)|^2\,dx \;&\leq\; C_2\,\tau\left(W_{k-1}^{1/2}\,W_k + W_{k-1}\,W_k\right) \\ &\leq\; C_3\,\tau\,(1 + W_{k-1})\,W_k\,. \end{aligned} \tag{3.41}$$

Using the inequality

$$\frac{W_k - W_{k-1}}{1 + W_{k-1}} \;=\; \frac{1 + W_k}{1 + W_{k-1}} - 1 \;\geq\; \log(1 + W_k) - \log(1 + W_{k-1}) \tag{3.42}$$

we obtain

$$\log(1+W_k) - \log(1+W_{k-1}) + \frac{1}{2\tau(1 + W_{k-1})}\int_\Omega |\nabla(u_{k+1} - u_k)|^2\,dx \;\leq\; C_3\,\tau W_k\,. \tag{3.43}$$

Invoking Estimate 2, more specifically the inequality (3.14), we conclude that

$$\log(1 + W_n) + \sum_{k=1}^{n}\frac{1}{2\tau(1 + W_{k-1})}\int_\Omega |\nabla(u_{k+1} - u_k)|^2\,dx \;\leq\; C_4 + \log(1 + W_0) \tag{3.44}$$

holds for every $n \in \mathbb{N}$ with a constant $C_4$ independent of $n$. By (3.37) we have that

$$W_0 \leq \frac{1}{2f_0}\int_\Omega |\triangle u_0|^2\,dx \leq C_5$$

hence $W_n \leq C_6$, where $C_5$ and $C_6$ are constants independent of $n$; so, by (3.36),

$$\frac{1}{\tau^2}\int_\Omega |u_n - u_{n-1}|^2\,dx + \frac{1}{\tau}\sum_{k=0}^{n}\int_\Omega |\nabla(u_{k+1} - u_k)|^2\,dx + \int_\Omega |\Delta u_n|^2 dx \;\leq\; C_7 \tag{3.45}$$

independently of $n$.

**Case d=3**.

We use again the interpolation inequality (3.39) with $v = u_{k+1} - u_k$, $p = 4$, and $\eta = 3/4$, to get the estimate

$$
\begin{aligned}
&\left(\int_\Omega |u_{k+1} - u_k|^4\, dx\right)^{1/2} \\
&\quad \le \; 2\mu_4^2(\Omega)\tau^2\left(\frac{2}{f_0}W_k + \left(\frac{2}{f_0}W_k\right)^{1/4}\left(\frac{1}{\tau^2}\int_\Omega |\nabla(u_{k+1} - u_k)|^2\, dx\right)^{3/4}\right) \\
&\quad \le \; \frac{\mu_4^2(\Omega)\tau^2}{f_0}(4 + \delta_*^3)\, W_k + \frac{3\mu_4^2(\Omega)}{2\delta_*}\int_\Omega |\nabla(u_{k+1} - u_k)|^2\, dx \qquad (3.46)
\end{aligned}
$$

with some constant $\delta_* > 0$ which has to chosen properly. More specifically, putting

$$
\delta_* \;=\; \delta\, W_{k-1}^{1/2}, \quad \delta \;=\; \frac{9\gamma\, \mu_4^2(\Omega)}{\sqrt{2f_0}}\,,
$$

we obtain from (3.38) that

$$
\begin{aligned}
W_k - W_{k-1} + \frac{1}{2\tau}\int_\Omega |\nabla(u_{k+1} - u_k)|^2\, dx \;&\le\; \frac{1}{3f_0}\,\delta_*(4 + \delta_*^3)\, \tau W_k \\
&\le\; \frac{1}{3f_0}\left(2 + \delta^2 W_{k-1}\right)^2 \tau W_k\,. \qquad (3.47)
\end{aligned}
$$

For $s > 0$ set

$$
h(s) \;=\; \frac{1}{(2 + \delta^2 s)^2}\,, \quad H(s) \;=\; \int_0^s h(\sigma)\, d\sigma \;=\; \frac{s}{2(2 + \delta^2 s)}\,.
$$

Since $H$ is increasing and concave, we have for all $k \in \mathbb{N}$ that

$$
(W_k - W_{k-1})\, h(W_{k-1}) \;\ge\; H(W_k) - H(W_{k-1})\,.
$$

Multiplying (3.47) by $h(W_{k-1})$, summing up over $k = 1, \dots, n$, and using (3.14) yields

$$
\begin{aligned}
H(W_n) - H(W_0) + \sum_{k=1}^n \frac{h(W_{k-1})}{2\tau}\int_\Omega |\nabla(u_{k+1} - u_k)|^2\, dx \;&\le\; \frac{1}{3f_0}\sum_{k=1}^n \tau W_k \qquad (3.48) \\
&\le\; \frac{1}{12 f_0}\int_\Omega |\nabla u_0|^2\, dx\,.
\end{aligned}
$$

Combining the above estimate with (3.37) we conclude that

$$
H(W_n) + \sum_{k=1}^n \frac{h(W_{k-1})}{2\tau}\int_\Omega |\nabla(u_{k+1} - u_k)|^2\, dx \;\le\; \frac{1}{8f_0}\int_\Omega \left(|\nabla u_0|^2 + |\Delta u_0|^2\right)\, dx\,. \qquad (3.49)
$$

We have by Hypothesis 1.2 that

$$
\lim_{s\to\infty} H(s) \;=\; \frac{1}{2\delta^2} \;=\; \frac{f_0}{81\gamma^2\mu_4^4(\Omega)} \;>\; \frac{1}{8f_0}\int_\Omega \left(|\nabla u_0|^2 + |\Delta u_0|^2\right)\, dx\,,
$$

hence $W_n$ is uniformly bounded above by a constant independent of $n$, which allows us to obtain the estimate (3.45) also for $d = 3$.

# 4 Proof of Theorem 1.3

This section is devoted to a detailed proof of Theorem 1.3. With the sequence $u_k(x)$ constructed in the previous section, we define for every choice of $\tau$ the functions

$$\left.\begin{array}{l} \hat{u}^{(\tau)}(x,t) = u_{k-1}(x) + \frac{t-(k-1)\tau}{\tau}(u_k(x) - u_{k-1}(x)) \\ \bar{u}^{(\tau)}(x,t) = u_k(x) \end{array}\right\} \tag{4.1}$$

for $x \in \Omega$ and $t \in [(k-1)\tau, k\tau[$ , $k = 1, 2, \ldots$ . From Estimates 1-3 it follows that

$$u_* \;\leq\; \hat{u}^{(\tau)}(x,t) \;\leq\; u^*\,, \tag{4.2}$$

$$\int_0^\infty \int_\Omega |\partial_t \hat{u}^{(\tau)}|^2(x,t)\,dx\,dt + \int_0^\infty \int_\Omega |\partial_t \nabla \hat{u}^{(\tau)}|^2(x,t)\,dx\,dt \;\leq\; C\,, \tag{4.3}$$

$$\int_\Omega |\partial_t \hat{u}^{(\tau)}|^2(x,t)\,dx + \int_\Omega |\Delta \bar{u}^{(\tau)}|^2(x,t)\,dx \;\leq\; C \tag{4.4}$$

with a constant $C$ independent of $\tau$ and $t$. In a standard way (e. g. as in [27, Chapter IX]), we pass to the limit as $\tau \searrow 0+$ (selecting a subsequence, if necessary) and using the continuity of the hysteresis operator $\mathcal{W}$, we obtain via compactness method a strong solution $u$ to (1.1)–(1.4) such that

$$u_* \;\leq\; u(x,t) \;\leq\; u^*\,, \tag{4.5}$$

$$\int_0^\infty \int_\Omega |\partial_t u|^2(x,t)\,dx\,dt + \int_0^\infty \int_\Omega |\partial_t \nabla u|^2(x,t)\,dx\,dt \;\leq\; C\,, \tag{4.6}$$

$$\int_\Omega |\partial_t u|^2(x,t)\,dx + \int_\Omega |\Delta u|^2(x,t)\,dx \;\leq\; C\,. \tag{4.7}$$

As already mentioned in Remark 1.4, this solution to (1.1)–(1.4) is unique by Hilpert's argument in [12], cf. also [27, Sect. IX.2].

Let us pass to the asymptotic behavior. We test (1.1) by $\partial_t u$ and obtain

$$\int_\Omega \partial_t u(x,t)\,\partial_t(f(u) + \mathcal{W}[\lambda,u])(x,t)dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u(x,t)|^2\,dx \;\leq\; 0 \tag{4.8}$$

for a. e. $t > 0$. Note that

$$\partial_t u\,(\partial_t f(u) + \partial_t \mathcal{W}[\lambda,u]) \geq \frac{1}{f_1 + \varphi_1 R}\,(\partial_t f(u) + \partial_t \mathcal{W}[\lambda,u])^2 \quad \text{a. e.} \tag{4.9}$$

We find an embedding constant $\mu > 0$ (by virtue of the Neumann boundary condition) and use (1.1) with (4.8)–(4.9) to obtain

$$\begin{aligned} \int_\Omega |\nabla u(x,t)|^2\,dx \;&\leq\; \mu \int_\Omega |\Delta u(x,t)|^2\,dx \\ &\leq\; \mu\,(f_1 + \varphi_1 R) \int_\Omega \partial_t u\,\partial_t(f(u) + \mathcal{W}[\lambda,u])(x,t)\,dx \\ &\leq\; -\frac{\mu\,(f_1 + \varphi_1 R)}{2}\frac{d}{dt}\int_\Omega |\nabla u(x,t)|^2\,dx \end{aligned} \tag{4.10}$$

hence

$$\int_\Omega |\nabla u(x,t)|^2\,dx \;\le\; \mathrm{e}^{-(2/(\mu\,(f_1+\varphi_1 R)))\,t}\int_\Omega |\nabla u_0(x)|^2\,dx\,. \tag{4.11}$$

From the Gagliardo-Nirenberg inequality (3.39) written in the form

$$\|\nabla u(\cdot,t)\|_{L^p(\Omega)} \;\le\; M_p\,\Big(\|\nabla u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u(\cdot,t)\|^{1-\eta}_{L^2(\Omega)}\|\Delta u(\cdot,t)\|^{\eta}_{L^2(\Omega)}\Big) \tag{4.12}$$

for a. e. $t>0$ with embedding constants $M_p$, and from the estimate (4.7) it follows that for every $2<p<6$ there exist constants $c_p, A_p > 0$ such that

$$\int_\Omega |\nabla u(x,t)|^p\,dx \;\le\; A_p\,\mathrm{e}^{-c_p\,t} \tag{4.13}$$

for a. e. $t>0$. Choosing $p=4$, say, we may use the embedding of $W^{1,p}(\Omega)$ into $C(\bar\Omega)$ and find $A>0$ and $c>0$ such that the deviation of $u$ from its integral mean $U(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx$ can be uniformly estimated by

$$|u(x,t) - U(t)| \;\le\; A\,\mathrm{e}^{-c\,t}\,. \tag{4.14}$$

The asymptotic convergence of $U(t)$ constitutes the most delicate part of the proof. Assume for contradiction that

$$\liminf_{t\to\infty} U(t) \;<\; \limsup_{t\to\infty} U(t)\,. \tag{4.15}$$

It follows from (4.5) that $u$ is globally bounded. We fix constants $\varepsilon>0$, $a<b\in\mathbb{R}$ such that

$$\varepsilon \;<\; \frac{f(b)-f(a)}{4\varphi_1 R}\,, \tag{4.16}$$

$$a-\varepsilon \;<\; \liminf_{t\to\infty} U(t) \;<a\;<\; b \;<\; \limsup_{t\to\infty} U(t) \;<\; b+\varepsilon\,. \tag{4.17}$$

We may find $\bar t>0$ such that

$$a-\varepsilon \;\le\; u(x,t) \;\le\; b+\varepsilon \quad \forall x\in\Omega \;\; \forall t\ge \bar t\,, \tag{4.18}$$

and a sequence $\bar t \le t_0 < t_1 < t_2 < \dots$, $\lim_{n\to\infty} t_n = \infty$, such that for all $x\in\Omega$ and $k=0,1,2,\dots$ we have

$$u(x,t_{2k}) \;\le\; a\,, \quad u(x,t_{2k+1}) \;\ge\; b\,. \tag{4.19}$$

We now fix some $x\in\Omega$, and for $r>0$, $t\ge 0$ set $\xi_r(x,t) = \mathfrak{p}_r[\lambda(x,\cdot),u(x,\cdot)](t)$, $\lambda_0(x,r) = \xi_r(x,t_0)$. Our aim is to estimate from below the difference between the values of $f(u)+\mathcal{W}[\lambda,u]$ at two consecutive times $t_{2k}, t_{2k+1}$, i. e. the quantity

$$f(u(x,t_{2k+1})) + \mathcal{W}[\lambda,u](x,t_{2k+1}) - f(u(x,t_{2k})) - \mathcal{W}[\lambda,u](x,t_{2k}).$$

Recall that by Lemma 2.3 and Hypothesis 1.1 (iii) we have $\xi_r(x,t) = 0$ for all $r \geq R$. Using (4.19), we certainly have

$$f(u(x,t_{2k+1})) + \mathcal{W}[\lambda,u](x,t_{2k+1}) - f(u(x,t_{2k})) - \mathcal{W}[\lambda,u](x,t_{2k})$$

$$\geq f(b) - f(a) + \int_0^R (g(r,\xi_r(x,t_{2k+1})) - g(r,\xi_r(x,t_{2k})))\, dr.$$

As the function $\varphi$ is non-negative, the function $g$ is nondecreasing in the second argument; for this reason we look for a lower bound for the quantity $\xi_r(x,t_{2k+1})$ and an upper bound for the term $\xi_r(x,t_{2k})$. We distinguish two cases.

If $r \leq \dfrac{b-a}{2}$, then we can use the following two inequalities which directly come from the definition of the play operator

$$\xi_r(x,t_{2k+1}) \geq\ b-r \qquad \xi_r(x,t_{2k}) \leq\ a+r \tag{4.20}$$

and deduce that

$$\int_0^{(b-a)/2} (g(r,\xi_r(x,t_{2k+1})) - g(r,\xi_r(x,t_{2k})))\, dr \geq \int_0^{(b-a)/2} (g(r,b-r) - g(r,a+r))\, dr \geq 0.$$

Let us assume now that $r > \dfrac{b-a}{2}$. The semigroup property (2.26) of the play enables us to consider $\lambda_0(x,\cdot)$ as new initial memory configuration for times $t > t_0$. By Lemma 2.3 and (4.20) we have for all $k$ and $r$ that

$$\xi_r(x,t_{2k+1})\ \leq\ \max\{\lambda_0(x,r), b-r+\varepsilon\} \leq\ \max\{a+r, b-r+\varepsilon\} \leq\ a+r+\varepsilon. \tag{4.21}$$

Applying once more Lemma 2.3 and putting $\lambda_1(x,r) = \xi_r(x,t_1)$, we obtain for $t \geq t_1$ that (the semigroup property (2.26) is used again)

$$\min\{\lambda_1(x,r), a+r-\varepsilon\}\ \leq\ \xi_r(x,t)\ \leq\ \max\{\lambda_1(x,r), b-r+\varepsilon\}\,, \tag{4.22}$$

which is equivalent to

$$\min\{0, a+r-\varepsilon-\lambda_1(x,r)\}\ \leq\ \xi_r(x,t)-\lambda_1(x,r)\ \leq\ \max\{0, b-r+\varepsilon-\lambda_1(x,r)\}\,. \tag{4.23}$$

We have $\lambda_1(x,r) \geq b-r$ for all $r$, hence

$$\xi_r(x,t)\ \leq\ \lambda_1(x,r) + \varepsilon\,. \tag{4.24}$$

On the other hand, (4.21) with $k=0$ gives

$$\xi_r(x,t_1) = \lambda_1(x,r) \leq\ a+r+\varepsilon \tag{4.25}$$

and so (4.23) entails

$$\xi_r(x,t)\ \geq\ \lambda_1(x,r) - 2\varepsilon\,. \tag{4.26}$$

Consequently, for $k = 1, 2, \dots$

$$
\begin{aligned}
& f(u(x,t_{2k+1})) + \mathcal{W}[\lambda,u](x,t_{2k+1}) - f(u(x,t_{2k})) - \mathcal{W}[\lambda,u](x,t_{2k}) \\
&\quad \geq \; f(b) - f(a) + \int_0^R \left(g(r,\xi_r(x,t_{2k+1})) - g(r,\xi_r(x,t_{2k}))\right) dr \\
&\quad \geq \; f(b) - f(a) + \int_{(b-a)/2}^R \left(g(r,\lambda_1(x,r) - 2\varepsilon) - g(r,\lambda_1(x,r) + \varepsilon)\right) dr \\
&\qquad + \int_0^{(b-a)/2} \left(g(r,b-r) - g(r,a+r)\right) dr \\
&\quad \geq \; f(b) - f(a) - 3\varphi_1 R\varepsilon \; \geq \; \frac{f(b)-f(a)}{4}\,. \qquad (4.27)
\end{aligned}
$$

The above inequality holds independently of $x$. Integrating (4.27) over $\Omega$ yields

$$
\begin{aligned}
& \int_\Omega \left(f(u(x,t_{2k+1})) + \mathcal{W}[\lambda,u](x,t_{2k+1}) - f(u(x,t_{2k})) - \mathcal{W}[\lambda,u](x,t_{2k})\right) dx \\
&\qquad\qquad \geq \; |\Omega|\, \frac{f(b)-f(a)}{4}\,. \qquad (4.28)
\end{aligned}
$$

On the other hand, integrating Eq. (1.1) we obtain

$$
\frac{d}{dt}\int_\Omega \left(f(u(x,t)) + \mathcal{W}[\lambda,u](x,t)\right) dx \; = \; 0 \qquad \text{a. e.}\;, \qquad (4.29)
$$

which is in contradiction with (4.28). Consequently, (4.15) does not hold, and we may put $u_\infty = \lim\limits_{t\to\infty} U(t)$. The convergence $u(x,t) \to u_\infty$ now follows from (4.14).

To check that the memory configurations $\lambda_t(x,r) = \mathfrak{p}_r[\lambda(x,\cdot),u(x,\cdot)](t)$ converge uniformly to some $\lambda_\infty(x,r)$, we use again Lemma 2.3 and the semigroup property (2.26). Indeed, for each $\varepsilon > 0$ we find $T > 0$ such that $|u(x,t) - u(x,T)| < \varepsilon$ for all $t \geq T$. Similarly as in (4.22), we obtain

$$
\min\{\lambda_T(x,r), u(x,T) + r - \varepsilon\} \; \leq \; \lambda_t(x,r) \; \leq \; \max\{\lambda_T(x,r), u(x,T) - r + \varepsilon\}\,. \qquad (4.30)
$$

From the elementary inequality $u(x,T) - r \leq \lambda_T(x,r) \leq u(x,T) + r$ it follows that $|\lambda_t(x,r) - \lambda_T(x,r)| \leq \varepsilon$, and the proof of Theorem 1.3 is complete.

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