FEYNMAN PATH INTEGRALS FOR THE INVERSE QUARTIC OSCILLATOR

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Abstract. The Feynman path integral representation for the weak solution of the Schrödinger equation with an inverse quartic oscillator potential is given in terms of a well defined infinite dimensional oscillatory integral. An analytically continued Wiener integral representation for the solution is provided and an explicit description of the quantum dynamics associated to a not essentially self-adjoint Hamiltonian is given.

Key words: Feynman path integrals, Schrödinger equation, analytic continuation of Wiener integrals, quartic oscillator.

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1. Introduction

Since their first introduction [17], Feynman path integrals had represented an alternative and suggestive formulation of quantum theory. According to Feynman's proposal, the solution of the time dependent Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{1}$$

describing the dynamics of the state of a d-dimensional quantum particle moving in a conservative force field given by a potential V (m is the mass of the particle and $\hbar$ is the reduced Planck constant), should be represented by a "sum over all possible histories" of the system

$$\psi(t,x) = \text{“}const \int_{\{\gamma|\gamma(0)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))D\gamma\text{''}. \tag{2}$$

The heuristic expression (2) is intended as an integral over the the space of paths γ arriving at time t at the point x, S_t is the classical action functional of the system evaluated along the path γ:

$$S_t(\gamma) = \int_0^t \frac{m}{2}\dot\gamma(s)^2 ds - \int_0^t V(\gamma(s))ds,$$

and $D\gamma$ denotes a heuristic "flat" Lebesgue-type measure on the space of paths. Feynman's representation creates a connection between the classical Lagrangian description of the physical world and the quantum one, making very intuitive the study of the "semiclassical limit" of quantum mechanics, that is the study of the detailed behavior of the wave function ψ when the Planck constant is regarded as a small parameter converging to 0. Indeed, according to an heuristic application of the stationary phase method [15], the asymptotic behavior of the integral (2) should be determined by the paths which make stationary the phase functional S_t, that is, by Hamilton's least action principle, the classical orbits of the system.
Despite its fascinating features, the integral (2) is not defined in a mathematical rigorous way. Feynman himself was aware of the problem[1], nevertheless he extended the path integral approach to the description of the dynamics of more general quantum systems, including the quantum fields, and producing an heuristic calculus that, from a physical point of view, works even in cases other arguments fail.

The first rigorous mathematical realization of a Feynman-type formula is due to M. Kac [24, 25], who noted that by considering the heat equation with potential instead of the Schrödinger equation

$$\begin{cases} -\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{3}$$

(obtained formally by the substitution $t \mapsto it$) and by replacing the oscillatory term $e^{\frac{i}{\hbar}\int_0^t \dot\gamma(s)^2 ds}$ in (2) with the fast decreasing one $e^{-\frac{1}{\hbar}\int_0^t \dot\gamma(s)^2 ds}$ it is possible to realize rigorously the heuristic path integral formula

$$\psi(t,x) = \quad \text{"}const \int_{\{\gamma|\gamma(0)=x\}} e^{-\frac{1}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))D\gamma \quad \text{"}. \tag{4}$$

in terms of a well defined Gaussian integral with respect to the Wiener measure W:

$$\psi(t,x) = \int e^{-\int_0^t V(\sqrt{\hbar/m}\,\omega(s)+x)ds}\psi_0(\sqrt{\hbar/m}\,\omega(t)+x)dW(\omega). \tag{5}$$

Equation (5), the famous "Feynman-Kac formula" [36], is the starting point for the definition of Feynman path integral as the analytic continuation of Wiener integrals [11, 13, 32, 26, 23]. Indeed by introducing in equations (3) and (4) a suitable parameter λ, proportional for instance

[1] Actually Feynman writes "One must fell as Cavalieri must have felt in calculating the volume of a pyramid before the invention of the calculus "

to the the time t as in the case $\lambda = \lambda_1$,

$$-\lambda_1\hbar\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\hbar^2\Delta\psi + V(x)\psi$$
$$\psi(t,x) = \int e^{-\frac{1}{\lambda_1\hbar}\int_0^t V(\sqrt{\hbar/(m\lambda_1)}\omega(s)+x)ds}\psi_0(\sqrt{\hbar/(m\lambda_1)}\omega(t)+x)dW(\omega),$$

or to the Planck constant, as in the case $\lambda = \lambda_2$,

$$\lambda_2\frac{\partial}{\partial t}\psi = \frac{1}{2m}\lambda_2^2\Delta\psi + V(x)\psi$$
$$\psi(t,x) = \int e^{\frac{1}{\lambda_2}\int_0^t V(\sqrt{\lambda_2/m}\omega(s)+x)ds}\psi_0(\sqrt{\lambda_2/m}\omega(t)+x)dW(\omega),$$

or to the mass, as in the case $\lambda = \lambda_3$,

$$\frac{\partial}{\partial t}\psi = \frac{1}{2\lambda_3}\Delta\psi - iV(x)\psi,$$
$$\psi(t,x) = \int e^{-i\int_0^t V(\sqrt{1/\lambda_3}\omega(s)+x)ds}\psi_0(\sqrt{1/\lambda_3}\omega(t)+x)dW(\omega),$$

and by allowing λ to assume complex values, then one gets, at least heuristically, Schrödinger equation and its solution by substituting respectively $\lambda_1 = -i$, $\lambda_2 = i\hbar$, or $\lambda_3 = -im$. These procedures can be made completely rigorous under suitable analyticity and growing conditions on the potential V and initial datum ψ_0. Moreover, in some particular cases, the study of the asymptotic behavior of the "Wiener integral representation" of the solution of the Schrödinger equation when $\hbar \downarrow 0$ has been applied to the semiclassical limit of quantum mechanics [9, 10].

An alternative mathematical definition of Feynman path integrals can be obtained by means the "infinite dimensional oscillatory integrals" [6, 7, 14, 3], an infinite dimensional analogue of the classical oscillatory integrals on finite dimensional spaces [22]. This approach has also allowed the implementation of an infinite dimensional version of the stationary phase method, applied to the study of semiclassical limits [7, 34, 3, 2].

Other mathematical definitions of the Feynman integral has been proposed, for instance by means of white noise calculus [21], or in terms of finite dimensional approximations and Trotter-type formulae [18, 19], or by means of nonstandard analysis [5]. However all the existing approaches present a common problem: the potentials V which can be handled are of the type "quadratic plus bounded perturbation" (which is Fourier transform of measure). It is important to stress that the problem is not only technical, but quite fundamental. Indeed it has been proved [39] that in one dimension, if the potential is time independent and super-quadratic in the sense that $V(x) \geq C(1+|x|)^{2+\epsilon}$ at infinity, $C > 0$ and $\epsilon > 0$, then, as a function of (t, x, y), the fundamental solution $E(t, 0, x, y)$ of the time dependent Schrödinger equation is nowhere C^1. Recently in [8] the case of potentials which are polynomial of the form

$$V(x) = \frac{1}{2}x\Omega^2 x + \lambda|x|^4,$$

(where Ω^2 is a positive symmetric $d \times d$ matrix, $\lambda \in \mathbb{R}^+$) has been handles. In this case the quantum mechanical Hamiltonian $H : D(H) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ given on vector $\phi \in C_0^\infty(\mathbb{R}^d)$ by

$$H\phi(x) = -\frac{\Delta}{2m}\phi(x) + V(x)\phi(x) \tag{6}$$

is essentially self-adjoint and determines uniquely a quantum dynamics. The present paper is devoted to the study of quartic (double well) polynomial potentials unbounded from below, of the form

$$V(x) = -\lambda|x|^4 + \frac{1}{2}x\Omega^2 x, \tag{7}$$

with $\lambda \in \mathbb{R}^+$ and Ω^2 being a positive symmetric $d \times d$ matrix. In this case the quantum Hamiltonian $H = -\frac{\Delta}{2m} + V$ is not essentially self-adjoint as one can deduce by a limit point argument (see [35], theorem X.9) and the quantum evolution is not uniquely determined. Nelson [32] was the first one proposing Feynman path integrals as a tool defining the quantum dynamics in the case of not essentially self-adjoint Hamiltonians. In [32], by means of a generalized Trotter product formula and an analytic continuation technique, a strongly continuous contraction semigroup

$$U(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \qquad t \geq 0$$

is constructed and, given a $\psi \in L^2(\mathbb{R}^d)$, the vector $\psi(t) \equiv U(t)\psi$ satisfies the Schrödinger equation in a distributional way, i.e. for any $\phi \in L^2(\mathbb{R}^d)$ sufficiently regular, one has

$$i\hbar\frac{d}{dt}\langle\phi, \psi(t)\rangle = \langle H\phi, \psi(t)\rangle.$$

Even if the starting point of Nelson's derivation is a Wiener integral representation of the solution of an heat equation with imaginary potential, the evolution operators $U(t)$ are defined in an abstract way by means of a limiting procedure and, in general, a path integral representation for its matrix elements $\langle\phi, U(t)\psi\rangle$ cannot be defined (even for very regular vectors $\phi, \psi \in L^2(\mathbb{R}^2)$). A technical problem of Nelson's result, directly connected with the method of the proof (i.e. the application of the Fatou-Privaloff theorem) is a restriction to the allowed values of the mass parameter m, which cannot belong to a set N of Lebesgue measure 0.

In the present paper we show that for the particular (unbounded from below) potential of the form (7) both problems of Nelson's paper can

be overcome. Indeed by using the theory of infinite dimensional oscillatory integrals with polynomial phase function developed in [8] we can give mathematical meaning, for suitable vectors $\phi, \psi \in L^2(\mathbb{R}^d)$, to the Feynman path integral representation for the weak solution of the Schrödinger equation, i.e. the matrix elements $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi \rangle$. More precisely, we provide the definition of $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi \rangle$ in terms of an infinite dimensional oscillatory integral on a suitable Hilbert space of paths and, thanks to a Parseval-type equality, we prove that it can be computed in terms of a Wiener integral. This result provides a link between two different approaches to the mathematical definition of Feynman path integral (the analytic continuation approach and the infinite dimensional oscillatory integral approach). Moreover we prove that the Feynman integrals representing $\langle \phi, e^{-\frac{i}{\hbar}Ht}\psi \rangle$ coincide, for any value of the mass m, with the matrix elements of the abstract evolution operators $U(t)$ constructed by Nelson.
In section 2 we recall the main results on infinite dimensional oscillatory integrals, including the more recent developments. In section 3 the path integral representation for the weak solution of the Schrödinger equation with potential (7) is constructed. It is also proved that it coincides with the dynamics defined by Nelson's method. Section 4 is devoted to an alternative description of the evolution operator defined in section 3.

2. Infinite dimensional oscillatory integrals

In the present section we present the definition and the main results on infinite dimensional oscillatory integrals, for more details we refer to [6, 14, 3]. The leading idea is the extension of the main properties of classical oscillatory integrals on $\mathbb{R}^n$ [22] to the case the integration is performed on an infinite dimensional Hilbert space.
In the following we will denote by $(\mathcal{H}, \langle \, , \, \rangle)$ a (finite or infinite dimensional) real separable Hilbert space, whose elements are denoted by $x, y \in \mathcal{H}$ and the norm with $\| \, \|$. In the case where $\mathcal{H}$ is finite dimensional, $\mathcal{H} \equiv \mathbb{R}^n$, an oscillatory (Fresnel) integrals on $\mathcal{H}$, i.e. an expression of the form

$$\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\|x\|^2} f(x) dx \tag{8}$$

(where $f : \mathbb{R}^n \to \mathbb{C}$ is a measurable function, and $\hbar > 0$ is a real parameter) can be defined even if f is not summable (and the integral (8) is not well defined in Lebesgue sense) by means of a limiting procedure [22, 14].

Definition 1. *A function $f:\mathbb{R}^n\to\mathbb{C}$ is Fresnel integrable if and only if for each Schwartz test function $\phi\in\mathcal{S}(\mathbb{R}^n)$ such that $\phi(0)=1$ the limit*

$$\lim_{\epsilon\to 0}(2\pi i\hbar)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)\phi(\epsilon x)dx \tag{9}$$

exists and is independent of ϕ. In this case the limit is called the Fresnel integral of f and denoted by

$$\widetilde{\int_{\mathbb{R}^n}} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx \tag{10}$$

Remark 1. *It is important to stress that in the case f is not summable, for instance if $f(x)=1$ $\forall x\in\mathbb{R}^n$, the convergence of the Fresnel integral is given by the cancellations due to the oscillatory behavior of the integrand. This makes oscillatory integrals the suitable mathematical tool to represent the physical concept of coherent superposition, that is of interference. Oscillatory integrals, as well as their asymptotic expansions when the parameter $\hbar\downarrow 0$, find important applications in several branches of physics where wave phenomena are fundamental, such as for instance optics [37]. From this point of view, the extension of the definition of oscillatory integrals to the case the integration is performed on a (infinite dimensional) Hilbert space of paths and the application to quantum mechanics appear very natural*

The normalization constant $(2\pi i\hbar)^{-n/2}$ becomes fundamental in the generalization of definition 1 to the infinite dimensional case. Indeed in the case where the Hilbert space $\mathcal{H}$ is infinite dimensional, the oscillatory integral is defined as the limit of a sequence of finite dimensional approximations [14, 3].

Definition 2. *A function $f:\mathcal{H}\to\mathbb{C}$ is Fresnel integrable if and only if for any sequence P_n of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n\leq P_{n+1}$ and $P_n\to 1$ strongly as $n\to\infty$ (1 being the identity operator in $\mathcal{H}$), the finite dimensional approximations*

$$\widetilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2\hbar}\langle P_nx,P_nx\rangle} f(P_nx)d(P_nx),$$

are well defined (in the sense of definition 1) and the limit

$$\lim_{n\to\infty}\widetilde{\int_{P_n\mathcal{H}}} e^{\frac{i}{2\hbar}\langle P_nx,P_nx\rangle} f(P_nx)d(P_nx) \tag{11}$$

exists and is independent of the sequence $\{P_n\}$.
In this case the limit is called the Fresnel integral of f and is denoted

by

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx.$$

An "operational characterization" of the largest class of "Fresnel integrable functions" is still an open problem, even in finite dimension, but one can find some interesting subsets of it. In particular [14, 3, 6] by considering a function $f : \mathcal{H} \to \mathbb{C}$ that is the Fourier transform of a complex bounded variation measure μ_f on $\mathcal{H}$, i.e. $f(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu_f(y) \equiv \hat{\mu}_f(x)$, and a self adjoint trace-class operator $L : \mathcal{H} \to \mathcal{H}$, such that $(I - L)$ is invertible, one can see that the function $e^{-\frac{i}{2\hbar}\langle x,Lx\rangle} f(x)$ is Fresnel integrable and the corresponding Fresnel integral can be explicitly computed in terms of a well defined absolutely convergent integral with respect to a σ−additive measure, by means of the following Parseval-type equality:

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} e^{-\frac{i}{2\hbar}\langle x,Lx\rangle} f(x)dx = (\det(I-L))^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle \alpha,(I-L)^{-1}\alpha\rangle} \mu_f(d\alpha) \tag{12}$$

where $\det(I - L) = |\det(I - L)|e^{-\pi i \,\mathrm{Ind}\,(I-L)}$ is the Fredholm determinant of the operator $(I - L)$, $|\det(I - L)|$ its absolute value and $\mathrm{Ind}((I - L))$ is the number of negative eigenvalues of the operator $(I - L)$, counted with their multiplicity.

In [8] equation (12) has been generalized to the case of polynomial phase functions with quartic growth. In particular the results of [8] show that, under suitable assumption an infinite dimensional oscillatory integral on an Hilbert space $\mathcal{H}$ can be computed in terms of a Gaussian integral. Indeed let us consider the abstract Wiener space $(i, \mathcal{H}, B)$ built on $\mathcal{H}$ [20, 29] (see the appendix for the definition and the main results on abstract Wiener spaces). Let $V_4 : \mathcal{H} \to \mathbb{R}$ be a strictly positive, 4-th order homogeneous map, i.e. $V_4(\alpha x) = \alpha^4 V_4(x)$ for any $\alpha \in \mathbb{R}, x \in \mathcal{H}$, which is continuous in the $|\ |$-norm. As a consequence V_4 is continuous in the $\|\ \|$-norm, moreover it can be extended by continuity to a random variable $\bar{V}_4$ on B, with $\bar{V}_4|_{\mathcal{H}} = V_4$ and the stochastic extension $\tilde{V}_4$ of $V_4 : \mathcal{H} \to \mathbb{R}$ exists and coincides with $\bar{V}_4 : B \to \mathbb{R}$ μ−a.e. (see appendix for definition and properties of stochastic extensions). Let us consider a self-adjoint trace class operator $B : \mathcal{H} \to \mathcal{H}$. The quadratic form on $\mathcal{H} \times \mathcal{H}$:

$$x \in \mathcal{H} \mapsto \langle x, Bx\rangle$$

can be extended to a random variable on B, denoted again by $\langle\ \cdot\ , B\ \cdot\ \rangle$. Let us assume that the largest eigenvalue of B is strictly less than 1 (or, in other words, that $(I-B)$ is strictly positive). Then one can prove that the random variable $g(\cdot) := e^{\frac{1}{2\hbar}\langle\cdot,B\cdot\rangle}$ is μ-summable (see appendix). In this setting it is possible to extend Parseval type equality (12) to the case of infinite dimensional oscillatory integral with a polynomial (quartic plus quadratic) phase function of the following form

$$\widetilde{\int_H} e^{\frac{i}{2\hbar}\|x\|^2} e^{-\frac{i}{2\hbar}\langle x,Bx\rangle} e^{\frac{i}{\hbar}V_4(x)} f(x)dx \tag{13}$$

Theorem 1. *Let B be self-adjoint trace class, $(I-B)$ strictly positive, $f \equiv \hat{\mu}_f$, and let us suppose that the bounded variation measure μ_f satisfies the following assumption*

$$\int_H e^{\frac{\hbar}{4}\langle k,(I-B)^{-1}k\rangle} |\mu_f|(dk) < +\infty. \tag{14}$$

Then the infinite dimensional oscillatory integral (13) *exists and is given by:*

$$\int_H \mathbb{E}[e^{in(k)(\omega)e^{i\pi/4}} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} e^{-\frac{i}{\hbar}V_4(\omega)}]\mu_f(dk)$$

Moreover the function f on the real Hilbert space $\mathcal{H}$ can be extended to those vectors $y \in \mathcal{H}^c$ in the complex Hilbert space $\mathcal{H}^c$ of the form $y = zx$, $x \in \mathcal{H}$, $z \in \mathbb{C}$. It can be also uniquely extended to a random variable on E, denoted again by f, defined by

$$f^z(\omega) \equiv f(z\omega) \equiv \int_H e^{izn(k)(\omega)}\mu_f(dk), \quad \omega \in E, \tag{15}$$

and the integral (13) *is also equal to*

$$\mathbb{E}[e^{-\frac{i}{\hbar}V_4(\omega)} e^{\frac{1}{2\hbar}\langle\omega,B\omega\rangle} f(e^{i\pi/4}\omega)] \tag{16}$$

where the expectation is taken with respect to the Gaussian measure μ on E.

3. The weak solution of Schrödinger equation

Let us consider the Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(x) + V(x)\psi(t,x) \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{17}$$

with V given by (7), and the heuristic Feynman path integral representation for its solution:

$$\psi(t,x) = \int_{\gamma(t)=x} e^{\frac{im}{2\hbar}\int_0^t \dot\gamma^2(s)ds - \frac{i}{\hbar}\int_0^t \gamma(s)\Omega^2\gamma(s)ds + \frac{i\lambda}{\hbar}\int_0^t |\gamma(s)|^4 ds}\psi_0(\gamma(0))d\gamma.$$

Analogously, given a vector $\phi \in L^2(\mathbb{R}^d)$, the inner product $\langle\phi, \psi(t)\rangle$ should be given by:

$$\int_{\mathbb{R}^d} \bar{\phi}(x) \int_{\gamma(t)=x} e^{\frac{im}{2\hbar}\int_0^t \dot{\gamma}^2(s)ds - \frac{i}{\hbar}\int_0^t \gamma(s)\Omega^2\gamma(s)ds + \frac{i\lambda}{\hbar}\int_0^t |\gamma(s)|^4 ds} \psi_0(\gamma(0)) d\gamma dx. \tag{18}$$

Theorem 1 allows one to give a rigorous mathematical meaning to the heuristic expression (18) in terms of an infinite dimensional oscillatory integral on a suitable Hilbert space.

In the following we shall put for notation simplicity $m = 1$ but the whole discussion can be generalized to arbitrary values of the mass parameter. Let us consider the Cameron-Martin space H_t, that is the Hilbert space of absolutely continuous paths $\gamma : [0,t] \to \mathbb{R}^d$, with $\gamma(0) = 0$ and inner product $\langle\gamma_1, \gamma_2\rangle = \int_0^t \dot{\gamma}_1(s)\dot{\gamma}_2(s)ds$. The cylindrical Gaussian measure on H_t with covariance operator the identity extends to a σ-additive measure on the Wiener space $C_t = \{\omega \in C([0,t];\mathbb{R}^d) \mid \gamma(0) = 0\}$: the Wiener measure W. (i, H_t, C_t) is an abstract Wiener space.

Let us consider moreover the Hilbert space $\mathcal{H} = \mathbb{R}^d \times H_t$, and the Banach space $B = \mathbb{R}^d \times C_t$ endowed with the product measure $N(dx) \times W(d\omega)$, N being the Gaussian measure on $\mathbb{R}^d$ with covariance equal to the $d \times d$ identity matrix. $(i, \mathcal{H}, B)$ is an abstract Wiener space.

Let us consider two vectors $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$ and the symmetric operator $B : \mathcal{H} \to \mathcal{H}$ given by:

$$(x, \gamma) \longmapsto (y, \eta) = B(x, \gamma),$$

$$y = t\Omega^2 x + \Omega^2 \int_0^t \gamma(s)ds, \qquad \eta(s) = \Omega^2 x(ts - \frac{s^2}{2}) - \int_0^s \int_t^u \Omega^2\gamma(r)drdu. \tag{19}$$

Let us also introduce the homogeneous fourth order polynomial V_4 given by $V_4(x, \gamma) = \lambda \int_0^t |\gamma(s) + x|^4 ds$, and the function $f : \mathcal{H} \to \mathbb{C}$ given by

$$f(x, \gamma) = (2\pi i\hbar)^{d/2} e^{-\frac{i}{2\hbar}|x|^2} \bar{\phi}(x)\psi_0(\gamma(t) + x) \tag{20}$$

with this notation expression (18) can be realized as the following infinite dimensional oscillatory integral on $\mathcal{H}$:

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}(|x|^2+|\gamma|^2)} e^{-\frac{i}{2\hbar}\langle(x,\gamma), B(x,\gamma)\rangle} e^{\frac{i}{\hbar}V_4(x,\gamma)} f(x, \gamma) dx d\gamma \tag{21}$$

In the following we will denote by Ω_i, $i = 1, \ldots, d$, the eigenvalues of the matrix Ω.

Theorem 2. *Let us assume that for each $i = 1, \dots, d$ the following inequalities are satisfied*

$$\Omega_i t < \frac{\pi}{2}, \qquad 1 - \Omega_i \tan(\Omega_i t) > 0. \tag{22}$$

Let $\phi, \psi_0 \in L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$. Let μ_0 be the complex bounded variation measure on $\mathbb{R}^d$ such that $\hat{\mu}_0 = \psi_0$. Let μ_ϕ be the complex bounded variation measure on $\mathbb{R}^d$ such that $\hat{\mu}_\phi(x) = (2\pi i\hbar)^{d/2} e^{-\frac{i}{2\hbar}|x|^2}\bar{\phi}(x)$. Assume in addition that the measures μ_0, μ_ϕ satisfy the following assumption:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{\hbar}{4}(y+\cos(\Omega t)^{-1}(1-\cos(\Omega t))x)(1-\Omega\tan(\Omega t))^{-1}(y+\cos(\Omega t)^{-1}(1-\cos(\Omega t))x)}$$
$$e^{\frac{\hbar}{4}x\Omega^{-1}\tan(\Omega t)x}|\mu_0|(dx)|\mu_\phi|(dy) < \infty \tag{23}$$

Then the function $f : \mathcal{H} \to \mathbb{C}$, given by (20) is the Fourier transform of a bounded variation measure μ_f on $\mathcal{H}$ satisfying

$$\int_{\mathcal{H}} e^{\frac{\hbar}{4}\langle (y,\eta),(I-B)^{-1}(y,\eta)\rangle} |\mu_f|(dy d\eta) < \infty \tag{24}$$

(B being given by (19)) and the infinite dimensional oscillatory integral (21) is well defined and is given by:

$$\int_{\mathbb{R}^d \times H_t}\left(\int_{\mathbb{R}^d \times C_t} e^{ie^{i\pi/4}(x\cdot y+\sqrt{\hbar}n(\gamma)(\omega))} e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds}\right.$$
$$\left. e^{-i\frac{\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\omega(s)+x|^4ds} W(d\omega)\frac{e^{-\frac{|x|^2}{2\hbar}}}{(2\pi\hbar)^{d/2}}dx\right)\mu_f(dyd\gamma). \tag{25}$$

This is also equal to

$$(i)^{d/2}\int_{\mathbb{R}^d \times C_t} e^{\frac{1}{2\hbar}\int_0^t(\sqrt{\hbar}\omega(s)+x)\Omega^2(\sqrt{\hbar}\omega(s)+x)ds}$$
$$e^{-i\frac{\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\omega(s)+x|^4ds}\bar{\phi}(e^{i\pi/4}x)\psi_0(e^{i\pi/4}\sqrt{\hbar}\omega(t)+e^{i\pi/4}x)W(d\omega)dx. \tag{26}$$

Moreover the latter is equal to the inner product $\langle\phi, U(t)\psi_0\rangle$, with $U(t)$, $t \geq 0$, strongly continuous contraction semigroup and

$$i\hbar\frac{d}{dt}\langle\phi, U(t)\psi_0\rangle = \langle H\phi, U(t)\psi_0\rangle,$$

(H being given on the smooth vector $\phi \in C_0^\infty(\mathbb{R}^d)$ by (6)).

Proof: The proof of the first part of the theorem, i.e. equations (25) and (26), is a direct application of theorem 1. Assumptions (22), restricting the interval of values that the variable t can assume, implies that the operator $(I - B)$ is positive. Assumptions (23) on the measures

μ_0, μ_ϵ implies the inequality (24), i.e. the condition (14) for the application of theorem (1) (see [8] for more details). Let us now consider the second part of the theorem, i.e. the construction of the semigroup $U(t)$, $t \geq 0$, and the proof that its matrix elements $\langle \phi, U(t)\phi_0 \rangle$ are given by the infinite dimensional oscillatory integral (21).
Let us consider the heat equation with complex potential

$$\begin{cases} -\hbar\frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(t,x) + iV(x)\psi(t,x) \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{27}$$

with V given by (7). For any $\psi_0 \in L_2(\mathbb{R}^d)$, the integral

$$U_m(t)\psi_0(x) \equiv \int_{C_t} e^{i\frac{\lambda}{\hbar}\int_0^t|\sqrt{\hbar/m}\,\omega(s)+x|^4 ds} e^{-\frac{i}{2\hbar}\int_0^t(\sqrt{\hbar/m}\,\omega(s)+x)\Omega^2(\sqrt{\hbar/m}\,\omega(s)+x)ds}\psi_0(\sqrt{\hbar/m}\,\omega(t)+x)W(d\omega) \tag{28}$$

is convergent and defines a contraction operator $U_m(t)$, as

$$|U_m(t)\psi_0(x)| \leq \int_{C_t}|\psi_0(\sqrt{\hbar/m}\,\omega(t)+x)|W(d\omega) \leq (2\pi t\hbar/m)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{m}{2t\hbar}|x-y|^2}\psi_0(y)dy = K_m(t)|\psi_0|(x), \tag{29}$$

where $K_m(t)$ is the heat semigroup $K_m(t) = e^{\frac{t\hbar}{2m}\Delta}$. By writing the cylindrical approximations of the Wiener integral (28) one has

$$U_m(t)\psi_0(x) = \lim_{n\to\infty}(K_m(t/n)M_V(t/n))^n\psi_0(x) \tag{30}$$

where $M_V(t)$ is the group given by the multiplication operator $M_V(t) = e^{-itV}$. By mimicking Nelson's argument [32] one can see that the limit (30) can be taken in L^2, it defines a strongly continuous contraction semigroup $U_m(t)$ and for any $\psi_0 \in C_0^\infty(\mathbb{R}^d)$ the generator A_m is given by

$$A_m\psi_0 = \lim_{t\to 0}\frac{1}{t}(U_m(t)\psi_0 - \psi_0) = (\frac{\hbar}{2m}\Delta - \frac{i}{\hbar}V)\psi_0. \tag{31}$$

As Δ is a negative operator, for any $t \geq 0$ $K_m(t)$ is an holomorphic operator-valued function of m in the half plane $Re(m) > 0$. It follows that for any $\psi_0 \in L^2$ and for any $n \in \mathbb{N}$, the expression $U_n(m) := (K_m(t/n)M_V(t/n))^n\psi_0$ defines an L^2-valued function holomorphic in the half plane $Re(m) > 0$ and continuous on $Re(m) \geq 0$. Since the sequence of functions $\{U_n\}_{n\in\mathbb{N}}$ is uniformly bounded on $Re(m) \geq 0$ by $\|\psi_0\|$ and converges for $m > 0$, by Vitali's theorem it converges on the whole domain $Re(m) > 0$ and the limit

$$\lim_{n\to\infty}(K_m(t/n)M_V(t/n))^n\psi_0 \equiv U_m(t)\psi_0$$

defines a an holomorphic L^2-valued function $U_m(t)\psi_0$ on $Re(m)>0$. By analytic continuation, one can prove that $U_m(t)$ is a strongly continuous contraction semigroup whose generator in given on vectors $\psi_0\in C_0^\infty(\mathbb{R}^d)$ by equation (31).
The purely quantum mechanical - Schrödinger case is obtained for m in (27) purely imaginary, i.e. $m=-i$. Let us consider ϕ,ψ_0 satisfying the assumptions of the theorem. For $m>0$, the inner product $\langle\phi, U_m(t)\psi\rangle$ is given by

$$\int_{\mathbb{R}^d}\int_{C_t} e^{-\frac{i}{2\hbar}\int_0^t(\sqrt{\hbar/m}\,\omega(s)+x)\Omega^2(\sqrt{\hbar/m}\,\omega(s)+x)ds} e^{i\frac{\lambda}{\hbar}\int_0^t|\sqrt{\hbar/m}\,\omega(s)+x|^4ds}\psi_0(\sqrt{\hbar/m}\,\omega(t)+x)\bar{\phi}(x)W(d\omega)dx \quad (32)$$

By a change of variable $x\mapsto x/\sqrt{m}$ the latter becomes

$$\langle\phi, U_m(t)\psi_0\rangle = m^{-d/2}\int_{\mathbb{R}^d}\int_{C_t} e^{-\frac{i}{2\hbar m}\int_0^t(\sqrt{\hbar}\,\omega(s)+x)\Omega^2(\sqrt{\hbar}\,\omega(s)+x)ds} e^{i\frac{\lambda}{m^2\hbar}\int_0^t|\sqrt{\hbar}\,\omega(s)+x|^4ds}\psi_0(\sqrt{\hbar/m}\,\omega(t)+x/\sqrt{m})\bar{\phi}(x/\sqrt{m})W(d\omega)dx \quad (33)$$

By assumptions (22) and (23), the right hand side of (33) is an holomorphic function of m in the domain $\{Re(m)>0\}\cap\{Im(m)<0\}$ and continuous on the boundary. On the other hand, by previous considerations, the matrix element $\langle\phi, U_m(t)\psi_0\rangle$ is an holomorphic function of m in the domain $\{Re(m)>0\}$ and coincides with the functional integral (33) on the half line $m>0$. By uniqueness of analytic continuation, both sides of (33) coincides on the domain $\{Re(m)>0\}\cap\{Im(m)<0\}$. In particular there exists the limit $\lim_{m\to -i}\langle\phi, U_m(t)\psi_0\rangle$ and, by bounded convergence theorem, it is equal to (26). □

Remark 2. *The results of theorem 2 hold also in the case the potential V is of the form:*

$$V(x)=-\lambda|x|^4-\frac{1}{2}x\Omega^2x, \qquad x\in\mathbb{R}^d$$

with $\lambda\in\mathbb{R}^+$ and Ω^2 a positive symmetric $d\times d$ matrix. In this case conditions (22) can be dropped and condition (23) has to be replaced by:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{\hbar}{4}(y+\cosh(\Omega t)^{-1}(1-\cosh(\Omega t))x)(1+\Omega\tanh(\Omega t))^{-1}(y+\cosh(\Omega t)^{-1}(1-\cosh(\Omega t))x)} e^{-\frac{\hbar}{4}x\Omega^{-1}\tanh(\Omega t)x}|\mu_0|(dx)|\mu_\phi|(dy)<\infty \quad (34)$$

4. The evolution operator

The explicit Wiener integral representation (26) for the matrix elements $(\phi, U(t)\psi_0)$ allows one to describe more explicitly the evolution operator $U(t)_{t\in\mathbb{R}^+}$, i.e. the dynamics defined by the Feynman path integral.

Let us denote by D_{θ_1,θ_2} the sector of the complex plane given by:

$$D_{\theta_1,\theta_2} := \{z \in \mathbb{C}, z = \rho e^{i\alpha} \; : \; \rho > 0, \alpha \in (\theta_1, \theta_2)\}.$$

Let us denote by $\mathcal{S}_1$ the subset of $S(\mathbb{R}^d)$ made of the functions $\phi : \mathbb{R}^d \to \mathbb{C}$ such that

(1) the function $z \mapsto \phi(zx)$, $x \in \mathbb{R}^d$, $z \in \bar{D}_{0,\pi/4}$ is analytic on $D_{0,\pi/4}$ and continuous on $\bar{D}_{0,\pi/4}$.

(2) the function $x \mapsto \phi(e^{i\frac{\pi}{4}}x)$, $x \in \mathbb{R}^d$ is in L^2.

Analogously, we shall denote by $\mathcal{S}_2$ the subset of $S(\mathbb{R}^d)$ made of the functions $\phi : \mathbb{R}^d \to \mathbb{C}$ such that

(1) the function $z \mapsto \phi(zx)$, $x \in \mathbb{R}^d$, $z \in \bar{D}_{-\pi/4,0}$ is analytic on $D_{-\pi/4,0}$ and continuous on $\bar{D}_{-\pi/4,0}$.

(2) the function $x \mapsto \phi(e^{-i\frac{\pi}{4}}x)$, $x \in \mathbb{R}^d$ belongs to $L^2(\mathbb{R}^d)$.

As an example, the functions of the form

$$\phi(x) = P(x)e^{-\frac{x^2}{2}(1-i)} \tag{35}$$

(where P is a polynomial with complex coefficients) belong to $\mathcal{S}_1$, while the functions of the form

$$\phi(x) = P(x)e^{-\frac{x^2}{2}(1+i)} \tag{36}$$

belong to $\mathcal{S}_2$. As the Hermite functions, which can be obtained by applying to suitable functions of the form (35) (resp (36)) the unitary transformation $\phi(x) \mapsto e^{\frac{i}{2}x^2}\phi(x)$ (resp. $\phi(x) \mapsto e^{-\frac{i}{2}x^2}\phi(x)$), form a complete orthonormal system in $L^2(\mathbb{R}^d)$, it is simple to verify that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are dense in $L^2(\mathbb{R}^d)$. In the following we shall denote by $S_1 \subset \mathcal{S}_1$ resp. $S_2 \subset \mathcal{S}_2$ the dense subsets of L^2 made of vectors of the form (35) resp. (36).

Let us denote by $T : \mathcal{S}_1 \to \mathcal{S}_2$ the linear operator defined by

$$T\phi(x) = e^{i\frac{\pi}{8}d}\phi(e^{i\frac{\pi}{4}}x), \qquad \phi \in \mathcal{S}_1$$

and by $T^{-1} : \mathcal{S}_2 \to \mathcal{S}_1$ its inverse, defined by

$$T^{-1}\phi(x) = e^{-i\frac{\pi}{8}d}\phi(e^{-i\frac{\pi}{4}}x) \qquad \phi \in \mathcal{S}_2$$

One can easily verify that T and T^{-1} are both symmetric, this implies that, for $\phi \in S_2$, $\psi \in S_1$, one has $(T^{-1}\phi, T\psi) = (\phi, \psi)$. Moreover they are positive operators, so that they admit self-adjoint extensions [35].

By means of an analytic continuation argument, one can see the following:

Lemma 1. *Let $\psi \in S_1$ and $\phi \in S_2$, then*

$$\begin{aligned}\langle \phi, e^{-\frac{i}{\hbar}H_0 t}\psi\rangle &= \langle T^{-1}\phi, e^{-\frac{1}{\hbar}H_0 t}T\psi\rangle \\ &= e^{i\frac{\pi}{4}d}\int_{\mathbb{R}^d}\bar{\phi}(e^{i\frac{\pi}{4}}x)\int_{C_t}\psi(\sqrt{\hbar}e^{i\frac{\pi}{4}}\omega(t)+e^{i\frac{\pi}{4}}x)W(d\omega)dx \qquad (37)\end{aligned}$$

where H_0 is the free Hamiltonian given on $S(\mathbb{R}^d)$ by $H_0\psi(x) = -\frac{\hbar^2}{2}\Delta\psi(x)$.

Proof: Let us consider the function $f : \bar{D}_{-\pi/2,\pi/2} \to \mathbb{C}$ given by $f(z) = \langle \phi, e^{-\frac{z}{\hbar}H_0}\psi\rangle$. By the spectral properties of H_0, f is analytic on $D_{-\pi/2,\pi/2}$ and continuous on $\bar{D}_{-\pi/2,\pi/2}$. Let $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) = z^{d/2}\int_{\mathbb{R}^d}\bar{\phi}(\sqrt{z}x)\int_{C_t}\psi(\sqrt{\hbar}\sqrt{z}\omega(t)+\sqrt{z}x)W(d\omega)dx$$

by the analyticity properties of ϕ, ψ, one can easily verify that g is analytic on $D_{-\pi/2,\pi/2}$ and continuous on the closure $\bar{D}_{-\pi/2,\pi/2}$ of $D_{-\pi/2,\pi/2}$. Moreover, by Feynman-Kac formula, f and g coincide on $\mathbb{R}^+$. By the uniqueness of analytic continuation they coincide on the whole domain and by continuity $f(i) = g(i)$, i.e.

$$\langle \phi, e^{-\frac{i}{\hbar}H_0 t}\psi\rangle = e^{i\frac{\pi}{4}d}\int_{\mathbb{R}^d}\bar{\phi}(e^{i\frac{\pi}{4}}x)\int_{C_t}\psi(\sqrt{\hbar}e^{i\frac{\pi}{4}}\omega(t)+e^{i\frac{\pi}{4}}x)W(d\omega)dx$$

where the right hand side of the latter equality can be written as $\langle T^{-1}\phi, e^{-\frac{1}{\hbar}H_0 t}T\psi\rangle$. □

An analogous result can be obtained also when the free Hamiltonian H_0 is replaced by the quartic oscillator Hamiltonian (6), where the potential V is given by (7). In the following we shall assume $\Omega = 0$ but the same reasonings can be generalized to arbitrary values of the parameter $\Omega \in \mathbb{R}^+$, provided that the time t is sufficiently small (i.e. it satisfies assumptions (22)).

Under the assumptions of theorem 2, the integral (26), i.e. the transition amplitude $\langle \phi, U(t)\psi_0\rangle$, is well defined and, for $\phi \in S_2$, $\psi_0 \in S_1$, can be written as

$$\langle \phi, U(t)\psi_0\rangle = \langle T^{-1}\phi, V(t)T\psi_0\rangle, \qquad (38)$$

where $V(t)$ is the C_0-contraction semigroup defined by the Feynman-Kac type formula:

$$V(t)\psi(x) := \int_{C_t} e^{-i\frac{\lambda}{\hbar}\int_0^t|\sqrt{\hbar}\omega(s)+x|^4 ds}\psi(\sqrt{\hbar}\omega(t)+x)W(d\omega). \qquad (39)$$

The operator-theoretic results of semi groups of the form (39) have been investigated in [27] (see also [23], chapter 13.5). In particular the generator A of the semigroup $V(t) = e^{\frac{t}{\hbar}A}$ is given on smooth vectors $\psi \in S(\mathbb{R}^d)$ by

$$A\psi(x) = \frac{\hbar^2}{2}\Delta\psi(x) - Q(x)\psi(x), \quad Q(x) := +i\lambda|x|^4$$

with domain

$$D(A) = \{\psi \in H^1(\mathbb{R}^d) \,:\, -\frac{\hbar^2}{2}\Delta\psi + Q\psi \in L^2(\mathbb{R}^d)\}$$

By considering a vector $\psi \in \mathcal{S}_2$, one can easily verify that

$$THT^{-1}\psi = iA\psi,$$

so that formally, for $\psi \in \mathcal{S}_2, \psi \in \mathcal{S}_1$

$$\begin{aligned}\langle T^{-1}\phi, e^{\frac{t}{\hbar}A}T\psi\rangle &= \sum_n \frac{1}{n!}t^n\hbar^{-n}\langle T^{-1}\phi, A^nT\psi\rangle \\ = \sum_n \frac{(-i)^n}{n!}t^n\hbar^{-n}\langle T^{-1}\phi, TH^nT^{-1}T\psi\rangle &= \sum_n \frac{(-i)^n}{n!}t^n\hbar^{-n}\langle \phi, H^n\psi\rangle \end{aligned} \tag{40}$$

and analogously

$$\langle \phi, e^{-\frac{it}{\hbar}H}\psi\rangle = \sum_n \frac{(-i)^n}{n!}t^n\hbar^{-n}\langle \phi, H^n\psi\rangle \tag{41}$$

so that the time series of $\langle T^{-1}\phi, e^{\frac{t}{\hbar}A}T\psi\rangle$ and $\langle \phi, e^{-\frac{it}{\hbar}H}\psi\rangle$ coincide. For general vectors $\psi \in S_1$, which belongs to the domain of H^n for any $n \in \mathbb{N}$, each term of the series (40) and (41) is well defined, however the series are not convergent, but only asymptotic, and the equality (38) has to be proved by means of theorem 2. By the contraction property of the evolution operator $U(t)$ and equation (38), one can deduce that, for $\phi \in \mathcal{S}_2, \psi \in \mathcal{S}_1$, one has

$$|\langle T^{-1}\phi, e^{-\frac{t}{\hbar}A}T\psi\rangle| \leq \|\phi\|\|\psi\|,$$

so that for $\psi \in \mathcal{S}_1$, the vector $e^{-\frac{t}{\hbar}A}T\psi$ belongs to the domain of T^{-1*} and on the dense domain $\mathcal{S}_2$ the evolution operator can be explicitly written as

$$U(t) = T^{-1*}e^{\frac{t}{\hbar}A}T\psi, \qquad \psi \in \mathcal{S}_2.$$

Remark 3. *One can consider the class of potentials V such that the relation*

$$TH_V T^{-1}\psi = iA_V\psi, \qquad \psi \in \mathcal{S}_2$$

holds, with H being given on smooth vectors ψ by $H\psi(x) = -\frac{\hbar^2}{2}\Delta\psi(x) + V(x)\psi(x)$ and $A_1 = \frac{\hbar^2}{2}\Delta\psi(x) - Q(x)\psi(x)$, with Q a complex valued function with $Re(Q) \geq 0$, so that A_1 is formally dissipative ($-A_1$ is formally accretive). An $m-$accretive realization of $-A_1$ in $L^2(\mathbb{R}^d)$ generates a C_0-contraction semigroup $e^{\frac{t}{\hbar}A_V}$ defined, on suitable vectors $\psi \in L^2(\mathbb{R}^d)$ by a Feynman-Kac formula. It is easy to verify that the class of potential V with this property includes several higher order polynomial potential. This problem will be investigated in details in a subsequent paper [33].

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Appendix A. Abstract Wiener spaces

Let $(\mathcal{H}, \langle\ ,\ \rangle, \|\ \|)$ be a real real separable Hilbert space. Let ν be the finitely additive cylinder measure on $\mathcal{H}$, defined by its characteristic functional $\hat{\nu}(x) = e^{-\frac{1}{2}\|x\|^2}$. Let $|\ |$ be a "measurable" norm on $\mathcal{H}$, that is $|\ |$ is such that for every $\epsilon > 0$ there exist a finite-dimensional projection $P_\epsilon : \mathcal{H} \to \mathcal{H}$, such that for all $P \perp P_\epsilon$ one has

$$\nu(\{x \in \mathcal{H} |\ |P(x)| > \epsilon\}) < \epsilon,$$

where P and P_ϵ are called orthogonal ($P \perp P_\epsilon$) if their ranges are orthogonal in $(\mathcal{H}, \langle\ ,\ \rangle)$. One can easily verify that $|\ |$ is weaker than $\|\ \|$. Denoted by B the completion of $\mathcal{H}$ in the $|\ |$-norm and by i the continuous inclusion of $\mathcal{H}$ in B, one can prove that $\mu \equiv \nu \circ i^{-1}$ is a countably additive Gaussian measure on the Borel subsets of B. The triple $(i, \mathcal{H}, B)$ is called an *abstract Wiener space*. Given $y \in B^*$ one can easily verify that the restriction of y to $\mathcal{H}$ is continuous on $\mathcal{H}$, so that one can identify B^* as a subset of $\mathcal{H}$. Moreover B^* is dense in $\mathcal{H}$ and we have the dense continuous inclusions $B^* \subset \mathcal{H} \subset B$. Each element $y \in B^*$ can be regarded as a random variable $n(y)$ on (B, μ). A direct computation shows that $n(y)$ is normally distributed, with covariance $\|y\|^2$. More generally, given $y_1, y_2 \in B^*$, one has

$$\int_B n(y_1)n(y_2)d\mu = \langle y_1, y_2\rangle.$$

The latter result allows the extension to the map $n : \mathcal{H} \to L^2(E, \mu)$, because E^* is dense in $\mathcal{H}$. Given an orthogonal projection P in $\mathcal{H}$, with

$$P(x) = \sum_{i=1}^{n} \langle e_i, x\rangle e_i$$

for some orthonormal $e_1, \dots, e_n \in \mathcal{H}$, the stochastic extension $\tilde{P}$ of P on E is well defined by

$$\tilde{P}(\,\cdot\,) = \sum_{i=1}^{n} n(e_i)(\,\cdot\,)e_i.$$

Given a function $f : \mathcal{H} \to E_1$, where $(E_1, \|\ \|_{E_1})$ is another real separable Banach space, the stochastic extension $\tilde{f}$ of f to E exists if the functions $f \circ \tilde{P} : E \to E_1$ converge to $\tilde{f}$ in probability with respect to μ as P converges strongly to the identity in $\mathcal{H}$. If $g : E \to E_1$ is continuous and $f := g|_{\mathcal{H}}$, then one can prove [20] that the stochastic extension of f is well defined and it is equal to g μ−a.e. Moreover for any $h \in \mathcal{H}$ the sequence of random variables

$$\sum_{i=1}^{n} h_i n(e_i), \qquad h_i = \langle e_i, h\rangle$$

converges in $L^2(E, \mu)$, and by subsequences μ a.e., to the random variable $n(h)$.

Given a self-adjoint trace class operator $B : \mathcal{H} \to \mathcal{H}$, the quadratic form on $\mathcal{H} \times \mathcal{H}$:

$$x \in \mathcal{H} \mapsto \langle x, Bx\rangle$$

can be extended to a random variable on E, denoted again by $\langle\,\cdot\,, B\,\cdot\,\rangle$. Indeed for each increasing sequence of finite dimensional projectors P_n converging strongly to the identity, $P_n(x) = \sum_{i=1}^{n} e_i\langle e_i, x\rangle$ ($\{e_i\}$ being a CONS in $\mathcal{H}$), the sequence of random variables

$$\omega \in E \mapsto \sum_{i,j=1}^{n} \langle e_i, Be_j\rangle n(e_i)(\omega) n(e_j)(\omega)$$

is a Cauchy sequence in $L^1(E, \mu)$. By passing if necessary to a subsequence, it converges to $\langle\,\cdot\,, B\,\cdot\,\rangle$ μ−a.e.

Let us assume that the largest eigenvalue of B is strictly less than 1 (or, in other words, that $(I - B)$ is strictly positive). Then one can prove that the random variable $g(\,\cdot\,) := e^{\frac{1}{2}\langle\,\cdot\,, B\,\cdot\,\rangle}$ is μ-summable. Indeed by considering a CONS $\{e_i\}$ made of eigenvectors of the operator B, b_i being the corresponding eigenvalues, the sequence of random variables

$$g_n : E \to \mathbb{C}, \qquad \omega \mapsto g_n(\omega) = e^{\frac{1}{2}\sum_{i=1}^{n} b_i([n(e_i)(\omega)]^2},$$

converges to $g(\omega)$ μ-a.e..
On the other hand one has

$$\int_H g_n(\omega)d\mu(\omega) = \prod_{i=1}^n \int \frac{e^{-\frac{1}{2}(1-b_i)x_i^2}}{\sqrt{2\pi}}dx_i = (\prod_{i=1}^n (1-b_i))^{-1/2}$$

so that $\int g_n d\mu$ converges, as $n \to \infty$, to $(\det(I-B))^{-1/2}$, where $\det(I-B)$ denotes the Fredholm determinant of $(I-B)$, which is well defined as B is trace class. Moreover $0 \leq g_n \leq g_{n+1}$ for each n. It follows that, as $n \to \infty$, $\int g_n d\mu \to \int g d\mu = (\det(I-B))^{-1/2}$. By an analogous reasoning one can prove that for any $y \in H$, the sequence of random variables f_n:

$$\omega \mapsto f_n(\omega) = e^{\sum_{i=1}^n y_i n(e_i)(\omega)} e^{\frac{1}{2}\sum_{i=1}^n b_i([n(e_i)(\omega)]^2}$$

where $y_i = \langle y, e_i \rangle$, converges μ−a.e. as n goes to ∞ to the random variable $f(\cdot) = e^{n(y)(\cdot)}e^{\frac{1}{2}\langle \cdot, B\cdot \rangle}$ and that

$$\int f_n d\mu \to \int f d\mu = (\det(I-B))^{-1/2} e^{\frac{1}{2}\langle y, (I-B)^{-1}y \rangle}. \qquad (4.12)$$

(see [29, 26]).

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