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Calculus of Variations

Indecomposable sets of finite perimeter in doubling metric measure spaces

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Abstract

We study a measure-theoretic notion of connectedness for sets of finite perimeter in the setting of doubling metric measure spaces supporting a weak (1, 1)-Poincaré inequality. The two main results we obtain are a decomposition theorem into indecomposable sets and a characterisation of extreme points in the space of BV functions. In both cases, the proof we propose requires an additional assumption on the space, which is called isotropicity and concerns the Hausdorff-type representation of the perimeter measure.

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Contents

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Introduction

The classical Euclidean theory of functions of bounded variation and sets of finite perimeter—whose cornerstones are represented, for instance, by [6,15,17,22,29,36] – has been successfully generalised in different directions, to several classes of metric structures. Amongst the many important contributions in this regard, we just single out the pioneering works [9–11,16,25,28]. Although the basic theory of BV functions can be developed on abstract metric measure spaces (see, e.g., [5]), it is in the framework of doubling spaces supporting a weak (1, 1)-Poincaré inequality (in the sense of Heinonen–Koskela [30]) that quite a few fine properties are satisfied (see [1,2,38]).

The aim of the present paper is to study the notion of *indecomposable set* of finite perimeter on doubling spaces supporting a weak (1, 1)-Poincaré inequality (that we call *PI spaces* for brevity). By indecomposable set we mean a set of finite perimeter E that cannot be written as disjoint union of two non-negligible sets F, G satisfying $\mathsf{P}(E) = \mathsf{P}(F) + \mathsf{P}(G)$. This concept constitutes the measure-theoretic counterpart to the topological notion of 'connected set' and, as such, many statements concerning connectedness have a correspondence in the context of indecomposable sets.

In the Euclidean framework, the main properties of indecomposable sets have been systematically investigated by Ambrosio et al. in [4]. The results of this paper are mostly inspired by (and actually extend) the contents of [4]. In the remaining part of the Introduction, we will briefly describe our two main results: the *decomposition theorem for sets of finite perimeter* and the *characterisation of extreme points in the space of BV functions*. In both cases, the natural setting to work in is that of PI spaces satisfying an additional condition—called *isotropicity*—which we are going to describe in the following paragraph.

Let $(\mathrm{X}, d, \mathfrak{m})$ be a PI space and $E \subset \mathrm{X}$ a set of finite perimeter; we refer to Sect. 1 for the precise definition of perimeter and the terminology used in the following. The perimeter measure $\mathsf{P}(E, \cdot)$ associated to E can be written as $\theta_E\, \mathcal{H} \llcorner_{\partial^e E}$, where $\mathcal{H}$ stands for the *codimension-one Hausdorff measure* on X, while $\partial^e E$ is the *essential boundary* of E (i.e., the set of points where neither the density of E nor that of its complement vanishes) and $\theta_E : \partial^e E \to [0, +\infty)$ is a suitable density function; cf. Theorem 1.23. The integral representation formula was initially proven in [1] only for Ahlfors-regular spaces, and this additional assumption has been subsequently removed in [2]. It is worth to point out that the weight function θ_E might (and, in some cases, does) depend on the set E itself; see, for instance, Example 1.27. In this paper, we mainly focus our attention on those PI spaces where θ_E is independent of E, which are said to be *isotropic* (the terminology comes from [7]). As we will discuss in Example 1.31, the class of isotropic PI spaces includes weighted Euclidean spaces, Carnot groups of step 2 and non-collapsed RCD spaces. Another key feature of the theory of sets of finite perimeter in PI spaces is given by the *relative isoperimetric inequality* (see Theorem 1.17 below), which has been obtained by M. Miranda in the paper [38].

Our main result (namely, Theorem 2.14) states that on isotropic PI spaces any set of finite perimeter E can be written as (finite or countable) disjoint union of indecomposable sets. Moreover, these components—called *essential connected components* of E—are uniquely determined and maximal with respect to inclusion, meaning that any indecomposable subset of E must be contained (up to null sets) in one of them. We propose two different proofs of this decomposition result, in Sects. 2 and 4, respectively. The former is a variational argument that was originally carried out in [4], while the latter is adapted from [33] and based on Lyapunov's convexity theorem. However, both approaches strongly rely upon three fundamental ingredients: representation formula for the perimeter measure, relative isoperimetric inequality, and isotropicity. We do not know whether the last one is in fact needed for the decomposition to hold (see also Example 2.16).

Furthermore, in Sect. 3 we study the extreme points in the space $\mathrm{BV}(\mathrm{X})$ of functions of bounded variation defined over X; we are again assuming $(\mathrm{X}, d, \mathfrak{m})$ to be an isotropic PI space. More precisely: call $\mathcal{K}(\mathrm{X}; K)$ the family of all those functions $f \in \mathrm{BV}(\mathrm{X})$ supported in K, whose total variation satisfies $|Df|(\mathrm{X}) \leq 1$ (where $K \subset \mathrm{X}$ is a fixed compact set). Then we can completely characterise (under a few additional assumptions) the extreme points of $\mathcal{K}(\mathrm{X}; K)$ as a convex, compact subset of $L^1(\mathfrak{m})$; see Theorem 3.8. It turns out that these extreme points coincide (up to a sign) with the normalised characteristic functions of *simple sets* (cf. Definition 3.1). In the Euclidean case, the very same result was proven by W. H. Fleming in [23,24] (see also [13]). Part of Sect. 3 is dedicated to some equivalent definitions of simple set: in the general framework of isotropic PI spaces, a plethora of phenomena concerning simple sets may occur, differently from what happens in $\mathbb{R}^n$ (see [4]). For more details, we refer to the discussion at the beginning of Sect. 3.1.

1 Preliminaries

For our purposes, by *metric measure space* we mean a triple $(\mathrm{X}, d, \mathfrak{m})$, where (X, d) is a complete and separable metric space, while $\mathfrak{m} \neq 0$ is a non-negative, locally finite Borel measure on X. For any open set $\Omega \subset \mathrm{X}$ we denote by $\mathrm{LIP}_{\mathrm{loc}}(\Omega)$ the space of all $\mathbb{R}$-valued locally Lipschitz functions on Ω, while $\mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$ is the family of all those Lipschitz functions $f\colon \mathrm{X} \to \mathbb{R}$ whose support $\mathrm{spt}(f)$ is bounded. Given any $f \in \mathrm{LIP}_{\mathrm{loc}}(\mathrm{X})$, we define the functions $\mathrm{lip}(f), \mathrm{lip}_a(f)\colon \mathrm{X} \to [0, +\infty)$ as

$$\mathrm{lip}(f)(x) := \varlimsup_{y \to x} \frac{\left|f(y) - f(x)\right|}{d(y, x)}, \quad \mathrm{lip}_a(f)(x) := \varlimsup_{y,z \to x} \frac{\left|f(y) - f(z)\right|}{d(y, z)}$$

whenever $x \in \mathrm{X}$ is an accumulation point, and $\mathrm{lip}(f)(x), \mathrm{lip}_a(f)(x) := 0$ elsewhere. We call $\mathrm{lip}(f)$ and $\mathrm{lip}_a(f)$ the *local Lipschitz constant* and the *asymptotic Lipschitz constant* of f, respectively.

We denote by $L^0(\mathfrak{m})$ the family of all real-valued Borel functions on X, considered up to $\mathfrak{m}$-a.e. equality. For any given exponent $p \in [1, \infty]$, we indicate by $L^p(\mathfrak{m}) \subset L^0(\mathfrak{m})$ and $L^p_{\mathrm{loc}}(\mathfrak{m}) \subset L^0(\mathfrak{m})$ the spaces of all p-integrable functions and locally p-integrable functions, respectively. Given an open set $\Omega \subset \mathrm{X}$ and any $E \subset \Omega$, we write $E \Subset \Omega$ to specify that E is bounded and $\mathrm{dist}(E, \mathrm{X} \setminus \Omega) > 0$.

1.1 Functions of bounded variation

In the framework of general metric measure spaces, the definition of *function of bounded variation*—which is typically abbreviated to 'BV function'—has been originally introduced in [38] and is based upon a relaxation procedure. Let us recall it:

Definition 1.1 (*Function of bounded variation*) Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure space. Fix any function $f \in L^1_{\mathrm{loc}}(\mathfrak{m})$. Given any open set $\Omega \subset \mathrm{X}$, we define the *total variation of f on Ω* as

$$|Df|(\Omega) := \inf \left\{ \varliminf_{n\to\infty} \int_\Omega \mathrm{lip}(f_n)\,\mathrm{d}\mathfrak{m} \;\middle|\; (f_n)_n \subset \mathrm{LIP}_{\mathrm{loc}}(\Omega),\; f_n \to f \text{ in } L^1_{\mathrm{loc}}(\mathfrak{m}\llcorner\Omega) \right\}. \tag{1.1}$$

Then f is said to be *of bounded variation*—briefly, $f \in \mathrm{BV}(\mathrm{X})$—if $f \in L^1(\mathfrak{m})$ and $|Df|(\mathrm{X}) < +\infty$.

We can extend the function $|Df|$ defined in (1.1) to all Borel sets via Carathéodory construction:

$$|Df|(B) := \inf \big\{ |Df|(\Omega) \;\big|\; \Omega \subset \mathrm{X} \text{ open}, B \subset \Omega \big\} \quad \text{for every } B \subset \mathrm{X} \text{ Borel}.$$

This way we obtain a finite Borel measure $|Df|$ on X, which is called the *total variation measure* of f.

Proposition 1.2 (Basic properties of BV functions) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a metric measure space. Let* $f, g \in L^1_{\mathrm{loc}}(\mathfrak{m})$. *Let* $B \subset \mathrm{X}$ *be Borel and* $\Omega \subset \mathrm{X}$ *open. Then the following properties hold:*

(i) LOWER SEMICONTINUITY. *The function* $|D\cdot|(\Omega)$ *is lower semicontinuous with respect to the* $L^1_{\mathrm{loc}}(\mathfrak{m}\llcorner\Omega)$*-topology: namely, given any sequence* $(f_n)_n \subset L^1_{\mathrm{loc}}(\mathfrak{m})$ *such that* $f_n \to f$ *in the* $L^1_{\mathrm{loc}}(\mathfrak{m}\llcorner\Omega)$*-topology, it holds* $|Df|(\Omega) \leq \varliminf_n |Df_n|(\Omega)$.
(ii) SUBADDITIVITY. *It holds that* $\big|D(f+g)\big|(B) \leq |Df|(B) + |Dg|(B)$.
(iii) COMPACTNESS. *Let* $(f_n)_n \subset L^1_{\mathrm{loc}}(\mathfrak{m})$ *be a sequence satisfying* $\sup_n |Df_n|(\mathrm{X}) < +\infty$. *Then there exist a subsequence* $(n_i)_i$ *and some* $f_\infty \in L^1_{\mathrm{loc}}(\mathfrak{m})$ *such that* $f_{n_i} \to f_\infty$ *in* $L^1_{\mathrm{loc}}(\mathfrak{m})$.

It follows from item (i) of Proposition 1.2 that the space $\mathrm{BV}(\mathrm{X})$ is a Borel subset of $L^1(\mathfrak{m})$.

Remark 1.3 Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure space. Fix $f \in \mathrm{BV}(\mathrm{X})$ and $m > 0$. Then

$$f \wedge m \in \mathrm{BV}(\mathrm{X}) \quad \text{and} \quad \big|D(f\wedge m)\big|(\mathrm{X}) \leq |Df|(\mathrm{X}). \tag{1.2}$$

Indeed, pick any $(f_n)_n \subset \mathrm{LIP}_{\mathrm{loc}}(\mathrm{X})$ such that $f_n \to f$ in $L^1_{\mathrm{loc}}(\mathfrak{m})$ and $\int \mathrm{lip}(f_n)\,\mathrm{d}\mathfrak{m} \to |Df|(\mathrm{X})$. Therefore, it holds that the sequence $(f_n \wedge m)_n \subset \mathrm{LIP}_{\mathrm{loc}}(\mathrm{X})$ satisfies $f_n \wedge m \to f \wedge m$ in $L^1_{\mathrm{loc}}(\mathfrak{m})$ and $\mathrm{lip}(f_n \wedge m) \leq \mathrm{lip}(f_n)$ for all $n \in \mathbb{N}$. We thus conclude that

$$\big|D(f\wedge m)\big|(\mathrm{X}) \leq \varliminf_{n\to\infty} \int \mathrm{lip}(f_n \wedge m)\,\mathrm{d}\mathfrak{m} \leq \varliminf_{n\to\infty} \int \mathrm{lip}(f_n)\,\mathrm{d}\mathfrak{m} = |Df|(\mathrm{X}),$$

which yields the statement. □

We conclude this subsection by briefly recalling an alternative (but equivalent) approach to the theory of BV functions on abstract metric measure spaces, which has been proposed in [18,19].

A *derivation* over a metric measure space $(\mathrm{X}, d, \mathfrak{m})$ is a linear map $\boldsymbol{b}\colon \mathrm{LIP}_{\mathrm{bs}}(\mathrm{X}) \to L^0(\mathfrak{m})$ such that the following properties are satisfied:

(i) LEIBNIZ RULE. $\boldsymbol{b}(fg) = \boldsymbol{b}(f)\,g + f\,\boldsymbol{b}(g)$ for every $f, g \in \mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$.

(ii) Weak locality. There exists a non-negative function $G \in L^0(\mathfrak{m})$ such that

$$\left|\boldsymbol{b}(f)\right| \leq G\,\mathrm{lip_a}(f) \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } f \in \mathrm{LIP_{bs}}(\mathrm{X}).$$

The least function G (in the $\mathfrak{m}$-a.e. sense) having this property is denoted by $|\boldsymbol{b}|$.

The space of all derivations over $(\mathrm{X}, d, \mathfrak{m})$ is denoted by $\mathrm{Der}(\mathrm{X})$. The *support* $\mathrm{spt}(\boldsymbol{b}) \subset \mathrm{X}$ of a derivation $\boldsymbol{b} \in \mathrm{Der}(\mathrm{X})$ is defined as the essential closure of the set $\{|\boldsymbol{b}| \neq 0\}$. Given any $\boldsymbol{b} \in \mathrm{Der}(\mathrm{X})$ with $|\boldsymbol{b}| \in L^1_{loc}(\mathfrak{m})$, we say that $\mathrm{div}(\boldsymbol{b}) \in L^p$ for some $p \in [1, \infty]$ provided there exists a (necessarily unique) function $\mathrm{div}(\boldsymbol{b}) \in L^p(\mathfrak{m})$ such that $\int \boldsymbol{b}(f)\,\mathrm{d}\mathfrak{m} = -\int f\,\mathrm{div}(\boldsymbol{b})\,\mathrm{d}\mathfrak{m}$ for every $f \in \mathrm{LIP_{bs}}(\mathrm{X})$. The space of all derivations $\boldsymbol{b} \in \mathrm{Der}(\mathrm{X})$ with $|\boldsymbol{b}| \in L^\infty(\mathfrak{m})$ and $\mathrm{div}(\boldsymbol{b}) \in L^\infty$ is denoted by $\mathrm{Der_b}(\mathrm{X})$.

Theorem 1.4 (Representation formula for $|Df|$ via derivations) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a metric measure space. Let* $f \in \mathrm{BV}(\mathrm{X})$ *be given. Then for every open set* $\Omega \subset \mathrm{X}$ *it holds that*

$$|Df|(\Omega) = \sup\left\{ \int_\Omega f\,\mathrm{div}(\boldsymbol{b})\,\mathrm{d}\mathfrak{m} \;\middle|\; \boldsymbol{b} \in \mathrm{Der_b}(\mathrm{X}),\ |\boldsymbol{b}| \leq 1\ \mathfrak{m}\text{-a.e.},\ \mathrm{spt}(\boldsymbol{b}) \Subset \Omega \right\}.$$

For a proof of the above representation formula, we refer to [18, Theorem 7.3.4].

1.2 Sets of finite perimeter

The study of *sets of finite perimeter* on abstract metric measure spaces has been initiated in [38] (where, differently from here, the term 'Caccioppoli set' is used). In this subsection we report the definition of set of finite perimeter and its basic properties, more precisely the ones that are satisfied on any metric measure space (without any further assumption).

Definition 1.5 (*Set of finite perimeter*) Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure space. Fix any Borel set $E \subset \mathrm{X}$. Let us define

$$\mathsf{P}(E, B) := |D\mathbb{1}_E|(B) \quad \text{for every Borel set } B \subset \mathrm{X}.$$

The quantity $\mathsf{P}(E, B)$ is called *perimeter of* E *in* B. Then the set E *has finite perimeter* provided

$$\mathsf{P}(E) := \mathsf{P}(E, \mathrm{X}) < +\infty.$$

The finite Borel measure $\mathsf{P}(E, \cdot)$ on X is called the *perimeter measure* associated to E.

Remark 1.6 Given a Borel set $E \subset \mathrm{X}$ satisfying $\mathfrak{m}(E) < +\infty$, it holds that E has finite perimeter if and only if $\mathbb{1}_E \in \mathrm{BV}(\mathrm{X})$. □

Proposition 1.7 (Basic properties of sets of finite perimeter) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a metric measure space. Let* $E, F \subset \mathrm{X}$ *be sets of finite perimeter. Let* $B \subset \mathrm{X}$ *be Borel and* $\Omega \subset \mathrm{X}$ *open. Then:*

(i) Locality. *If* $\mathfrak{m}\big((E \Delta F) \cap \Omega\big) = 0$, *then* $\mathsf{P}(E, \Omega \cap B) = \mathsf{P}(F, \Omega \cap B)$. *In particular, it holds that* $\mathsf{P}(E, \cdot) = \mathsf{P}(F, \cdot)$ *whenever* $\mathfrak{m}(E \Delta F) = 0$.
(ii) Lower semicontinuity. *The function* $\mathsf{P}(\cdot, \Omega)$ *is lower semicontinuous with respect to the* $L^1_{loc}(\mathfrak{m}\llcorner_\Omega)$*-topology: namely, if* $(E_n)_n$ *is a sequence of Borel subsets of* Ω *such that the convergence* $\mathbb{1}_{E_n} \to \mathbb{1}_E$ *holds in* $L^1_{loc}(\mathfrak{m}\llcorner_\Omega)$ *as* $n \to \infty$, *then* $\mathsf{P}(E, \Omega) \leq \underline{\lim}_n \mathsf{P}(E_n, \Omega)$.
(iii) Subadditivity. *It holds that* $\mathsf{P}(E \cup F, B) + \mathsf{P}(E \cap F, B) \leq \mathsf{P}(E, B) + \mathsf{P}(F, B)$.

vi) COMPLEMENTATION. *It holds that* $P(E, B) = P(E^c, B)$.
(v) COMPACTNESS. *Let* $(E_n)_n$ *be a sequence of Borel subsets of* X *with* $\sup_n P(E_n) < +\infty$. *Then there exist a subsequence* $(n_i)_i$ *and a Borel set* $E_\infty \subset \mathrm{X}$ *such that* $\mathbb{1}_{E_{n_i}} \to \mathbb{1}_{E_\infty}$ *in the* $L^1_{\mathrm{loc}}(\mathfrak{m})$*-topology as* $i \to \infty$.

1.3 Fine properties of sets of finite perimeter in PI spaces

The first aim of this subsection is to recall the definition of PI space and its main properties; we refer to [31] for a thorough account about this topic. Thereafter, we shall recall the definition of *essential boundary* and the main properties of sets of finite perimeter in PI spaces—among others, the isoperimetric inequality, the coarea formula, and the Hausdorff representation of the perimeter measure. Finally, we will discuss the class of *isotropic PI spaces*, which plays a central role in the rest of the paper.

Definition 1.8 (*Doubling measure*) A metric measure space $(\mathrm{X}, d, \mathfrak{m})$ is said to be *doubling* provided there exists a constant $C_D \geq 1$ such that

$$\mathfrak{m}\big(B_{2r}(x)\big) \leq C_D\, \mathfrak{m}\big(B_r(x)\big) \quad \text{for every } x \in \mathrm{X} \text{ and } r > 0.$$

The least such constant C_D is called the *doubling constant* of $(\mathrm{X}, d, \mathfrak{m})$.

Remark 1.9 Let $(\mathrm{X}, d, \mathfrak{m})$ be a doubling metric measure space. Then $\mathrm{spt}(\mathfrak{m}) = \mathrm{X}$. Indeed, it holds that $\mathfrak{m}\big(B_r(x)\big) > 0$ for every $x \in \mathrm{X}$ and $r > 0$, otherwise $\mathfrak{m}$ would be the null measure. Moreover, the metric space (X, d) is proper (i.e., bounded closed subsets of X are compact). □

Doubling spaces do not have a definite dimension (not even locally), but still are 'finite-dimensional'—in a suitable sense. In light of this, it makes sense to consider the *codimension-one Hausdorff measure* $\mathcal{H}$, defined below via Carathéodory construction, which takes into account the local change of dimension of the underlying space.

Definition 1.10 (*Codimension-one Hausdorff measure*) Let $(\mathrm{X}, d, \mathfrak{m})$ be a doubling metric measure space. Given any set $E \subset \mathrm{X}$ and any parameter $\delta > 0$, we define

$$\mathcal{H}_\delta(E) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mathfrak{m}\big(B_{r_i}(x_i)\big)}{2\, r_i} \;\middle|\; (x_i)_i \subset \mathrm{X},\ (r_i)_i \subset (0, \delta],\ E \subset \bigcup_{i \in \mathbb{N}} B_{r_i}(x_i) \right\}.$$

Then we define the *codimension-one Hausdorff measure* $\mathcal{H}$ on $(\mathrm{X}, d, \mathfrak{m})$ as

$$\mathcal{H}(E) := \lim_{\delta \searrow 0} \mathcal{H}_\delta(E) \quad \text{for every set } E \subset \mathrm{X}.$$

Both $\mathcal{H}_\delta$ and $\mathcal{H}$ are Borel regular outer measures on X.

Definition 1.11 (*Ahlfors-regularity*) Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure space. Let $k \geq 1$ be fixed. Then we say that $(\mathrm{X}, d, \mathfrak{m})$ is *k-Ahlfors-regular* if there exist two constants $\tilde{a} \geq a > 0$ such that

$$a r^k \leq \mathfrak{m}\big(B_r(x)\big) \leq \tilde{a} r^k \quad \text{for every } x \in \mathrm{X} \text{ and } r \in \big(0, \mathrm{diam}(\mathrm{X})\big). \tag{1.3}$$

It can be readily checked that any Ahlfors-regular space $(\mathrm{X}, d, \mathfrak{m})$ is doubling, with $C_D = 2^k \tilde{a}/a$.

Definition 1.12 (*Weak* (1, 1)-*Poincaré inequality*) A metric measure space $(\mathrm{X}, d, \mathfrak{m})$ is said to satisfy a *weak* (1, 1)-*Poincaré inequality* provided there exist constants $C_P > 0$ and $\lambda > 1$ such that for any function $f \in \mathrm{LIP_{loc}}(\mathrm{X})$ and any upper gradient g of f it holds that

$$\fint_{B_r(x)} |f - f_{x,r}| \, \mathrm{d}\mathfrak{m} \leq C_P \, r \fint_{B_{\lambda r}(x)} g \, \mathrm{d}\mathfrak{m} \quad \text{for every } x \in \mathrm{X} \text{ and } r > 0,$$

where $f_{x,r} := \mathfrak{m}\big(B_r(x)\big)^{-1} \int_{B_r(x)} f \, \mathrm{d}\mathfrak{m}$ stands for the mean value of f in the ball $B_r(x)$.

Lemma 1.13 (Poincaré inequality for BV functions) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a proper metric measure space satisfying a weak* (1, 1)*-Poincaré inequality. Let* $f \in L^1_{loc}(\mathfrak{m})$ *be such that* $|Df|(\mathrm{X}) < +\infty$*. Then it holds that*

$$\fint_{B_r(x)} |f - f_{x,r}| \, \mathrm{d}\mathfrak{m} \leq C_P \, r \, |Df|\big(B_{\lambda r}(x)\big) \quad \textit{for every } x \in \mathrm{X} \textit{ and } r > 0, \tag{1.4}$$

where the constants C_P *and* λ *are chosen as in Definition 1.12.*

Proof A standard diagonalisation argument provides us with a sequence $(f_n)_n \subset \mathrm{LIP_{loc}}\big(B_{\lambda r}(x)\big)$ such that $f_n \to f$ in $L^1_{loc}(\mathfrak{m}\llcorner_{B_{\lambda r}(x)})$ and $|Df|\big(B_{\lambda r}(x)\big) = \lim_n \int_{B_{\lambda r}(x)} \mathrm{lip}(f_n) \, \mathrm{d}\mathfrak{m}$. Given that the local Lipschitz constant $\mathrm{lip}(f_n)$ is an upper gradient of the function f_n, it holds that

$$\fint_{B_r(x)} \big| f_n - (f_n)_{x,r} \big| \, \mathrm{d}\mathfrak{m} \leq C_P \, r \fint_{B_{\lambda r}(x)} \mathrm{lip}(f_n) \, \mathrm{d}\mathfrak{m} \quad \text{for every } n \in \mathbb{N}. \tag{1.5}$$

Since the closure of $B_r(x)$ is a compact subset of $B_{\lambda r}(x)$, we know that $\mathbb{1}_{B_r(x)} f \in L^1(\mathfrak{m}\llcorner_{B_r(x)})$ and $\mathbb{1}_{B_r(x)} f_n \to \mathbb{1}_{B_r(x)} f$ in $L^1(\mathfrak{m}\llcorner_{B_r(x)})$, so that $(f_n)_{x,r} \to f_{x,r}$ as $n \to \infty$. Moreover, for some function $g \in L^1(\mathfrak{m}\llcorner_{B_r(x)})$ we have (up to a not relabelled subsequence) that $\big|f_n(y)\big| \leq g(y)$ for every $n \in \mathbb{N}$ and $\mathfrak{m}$-a.e. $y \in B_r(x)$. We can further assume that $f_n(y) \to f(y)$ for $\mathfrak{m}$-a.e. $y \in B_r(x)$. Given that $\big|f_n(y) - (f_n)_{x,r}\big| \leq g(y) + g_{x,r}$ for every $n \in \mathbb{N}$ and $\mathfrak{m}$-a.e. $y \in B_r(x)$, we deduce (by dominated convergence theorem) that $\fint_{B_r(x)} \big|f_n - (f_n)_{x,r}\big| \, \mathrm{d}\mathfrak{m} \to \fint_{B_r(x)} |f - f_{x,r}| \, \mathrm{d}\mathfrak{m}$ as $n \to \infty$. Therefore, by letting $n \to \infty$ in (1.5) we conclude that the claim (1.4) is verified. □

For the purposes of this paper, we shall only consider the following notion of *PI space* (which is strictly more restrictive than the usual one, where a weak $(1, p)$-Poincaré inequality is required for some exponent p that is possibly greater than 1):

Definition 1.14 (*PI space*) We say that a metric measure space $(\mathrm{X}, d, \mathfrak{m})$ is a *PI space* provided it is doubling and satisfies a weak (1, 1)-Poincaré inequality.

We introduce the concept of essential boundary in a doubling metric measure space and its main features. The discussion is basically taken from [1,2], apart from a few notational discrepancies.

Given a doubling metric measure space $(\mathrm{X}, d, \mathfrak{m})$, a Borel set $E \subset \mathrm{X}$ and a point $x \in \mathrm{X}$, we define the *upper density of* E *at* x and the *lower density of* E *at* x as

$$\overline{D}(E, x) := \varlimsup_{r \searrow 0} \frac{\mathfrak{m}\big(E \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}, \qquad \underline{D}(E, x) := \varliminf_{r \searrow 0} \frac{\mathfrak{m}\big(E \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)},$$

respectively. Whenever upper and lower densities coincide, their common value is called *density of* E *at* x and denoted by $D(E, x)$. We define the *essential boundary* of the set E as

$$\partial^e E := \Big\{ x \in \mathrm{X} \;\Big|\; \overline{D}(E, x) > 0, \; \overline{D}(E^c, x) > 0 \Big\}.$$

It clearly holds that the essential boundary $\partial^e E$ is contained in the topological boundary ∂E. Moreover, we define the set $E^{1/2} \subset \partial^e E$ *of points of density* $1/2$ as

$$E^{1/2} := \{x \in \mathrm{X} \mid D(E, x) = 1/2\}.$$

Finally, we define the *essential interior* E^1 of E as

$$E^1 := \{x \in \mathrm{X} \mid D(E, x) = 1\}.$$

Clearly, it holds that $\partial^e E \cap E^1 = \emptyset$: if $x \in \partial^e E$ then $\underline{D}(E, x) = 1 - \overline{D}(E^c, x) < 1$, so $x \notin E^1$.

Remark 1.15 Let $F \subset E \subset \mathrm{X}$ be given. Then

$$\partial^e F \subset \partial^e E \cup E^1. \tag{1.6}$$

Indeed, fix any $x \in \partial^e F \backslash \partial^e E$. Then $\overline{D}(E, x) \geq \overline{D}(F, x) > 0$, thus accordingly $\overline{D}(E^c, x) = 0$. This forces $D(E, x) = 1 - D(E^c, x) = 1$, so that $x \in E^1$. Hence, the claim (1.6) is proven. □

The following result is well-known. We report here its full proof for the reader's convenience.

Proposition 1.16 (Properties of the essential boundary) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a doubling metric measure space. Let* $E, F \subset \mathrm{X}$ *be sets of finite perimeter. Then the following properties hold:*

(i) *It holds that* $\partial^e E = \partial^e E^c$.
(ii) *We have that*

$$\partial^e (E \cup F) \cup \partial^e (E \cap F) \subset \partial^e E \cup \partial^e F. \tag{1.7}$$

(iii) *If* $\mathfrak{m}(E \cap F) = 0$*, then* $\partial^e E \subset \partial^e F \cup \partial^e (E \cup F)$.
(iv) *If* $\mathfrak{m}(E \cap F) = 0$*, then* $\partial^e E \cup \partial^e F = \partial^e (E \cup F) \cup (\partial^e E \cap \partial^e F)$.

Proof (i) It trivially stems from the very definition of essential boundary.
(ii) First of all, fix $x \in \partial^e (E \cup F)$. Note that $\overline{D}(E \cup F, x) \leq \overline{D}(E, x) + \overline{D}(F, x)$, as it follows from

$$\begin{aligned}\overline{D}(E \cup F, x) &= \varlimsup_{r \searrow 0} \frac{\mathfrak{m}\big((E \cup F) \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} \leq \varlimsup_{r \searrow 0} \left[\frac{\mathfrak{m}\big(E \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} + \frac{\mathfrak{m}\big(F \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}\right] \\ &\leq \varlimsup_{r \searrow 0} \frac{\mathfrak{m}\big(E \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} + \varlimsup_{r \searrow 0} \frac{\mathfrak{m}\big(F \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} = \overline{D}(E, x) + \overline{D}(F, x).\end{aligned}$$

Therefore, the fact that $\overline{D}(E \cup F, x) > 0$ implies either $\overline{D}(E, x) > 0$ or $\overline{D}(F, x) > 0$. Furthermore, we have that $\overline{D}(E^c, x), \overline{D}(F^c, x) \geq \overline{D}(E^c \cap F^c, x) = \overline{D}\big((E \cup F)^c, x\big) > 0$, whence $x \in \partial^e E \cup \partial^e F$.
In order to prove that even the inclusion $\partial^e (E \cap F) \subset \partial^e E \cup \partial^e F$ is verified, it is just sufficient to combine the previous case with item (i):

$$\partial^e (E \cap F) = \partial^e (E \cap F)^c = \partial^e (E^c \cup F^c) \subset \partial^e E^c \cup \partial^e F^c = \partial^e E \cup \partial^e F.$$

Hence, the proof of (1.7) is complete.

(iii) Pick any point $x \in \partial^e E$. First of all, notice that $\overline{D}(E\cup F,x),\overline{D}(F^c,x) \geq \overline{D}(E,x) > 0$. Moreover, it holds that

$$\begin{aligned}\overline{D}\big((E\cup F)^c,x\big)+\overline{D}(F,x) &= \varlimsup_{r\searrow 0}\frac{\mathfrak{m}\big(E^c\cap F^c\cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}+\varlimsup_{r\searrow 0}\frac{\mathfrak{m}\big(F\cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}\\ &\geq \varlimsup_{r\searrow 0}\left[\frac{\mathfrak{m}\big(E^c\cap F^c\cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}+\frac{\mathfrak{m}\big(F\cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}\right]\\ &= \varlimsup_{r\searrow 0}\frac{\mathfrak{m}\big(E^c\cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}=\overline{D}(E^c,x)>0,\end{aligned}$$

whence either $\overline{D}\big((E\cup F)^c,x\big) > 0$ or $\overline{D}(F,x) > 0$. This shows that $x \in \partial^e F\cup\partial^e(E\cup F)$.

(iv) Item (ii) grants that $\partial^e(E\cup F)\cup(\partial^e E\cap\partial^e F)\subset\partial^e E\cup\partial^e F$. Conversely, item (iii) yields

$$\partial^e E\cup\partial^e F\subset\big(\partial^e(E\cup F)\cup\partial^e E\big)\cap\big(\partial^e(E\cup F)\cup\partial^e F\big)=\partial^e(E\cup F)\cup(\partial^e E\cap\partial^e F),$$

thus obtaining the identity $\partial^e E\cup\partial^e F=\partial^e(E\cup F)\cup(\partial^e E\cap\partial^e F)$. □

In the setting of PI spaces, functions of bounded variation and sets of finite perimeters present several fine properties, as we are going to describe.

Theorem 1.17 (Relative isoperimetric inequality on PI spaces [38]) *Let* $(\mathrm{X},d,\mathfrak{m})$ *be a PI space. Then there exists a constant* $C_I > 0$ *such that the* relative isoperimetric inequality *is satisfied: given any set* $E\subset\mathrm{X}$ *of finite perimeter, it holds that*

$$\min\Big\{\mathfrak{m}\big(E\cap B_r(x)\big),\mathfrak{m}\big(E^c\cap B_r(x)\big)\Big\}\leq C_I\left(\frac{r^s}{\mathfrak{m}\big(B_r(x)\big)}\right)^{1/s-1}P\big(E,B_{2\lambda r}(x)\big)^{s/s-1} \tag{1.8}$$

for every $x\in\mathrm{X}$ *and* $r>0$*, where* $s>1$ *is any exponent greater than* $\log_2(C_D)$.

As an immediate consequence of Theorem 1.17, we have that a given set of finite perimeter E in a PI space $(\mathrm{X},d,\mathfrak{m})$ has null perimeter if and only if either $\mathfrak{m}(E)=0$ or $\mathfrak{m}(E^c)=0$.

Theorem 1.18 (Global isoperimetric inequality on Ahlfors regular PI spaces) *Let* $(\mathrm{X},d,\mathfrak{m})$ *be a k-Ahlfors regular PI space, with* $k>1$. *Then there exists a constant* $C_I'>0$ *such that*

$$\min\big\{\mathfrak{m}(E),\mathfrak{m}(E^c)\big\}\leq C_I'\,P(E)^{k/k-1}\quad\text{for every set } E\subset\mathrm{X} \text{ of finite perimeter.} \tag{1.9}$$

Proof As proven in [38], there exists a constant $C_I'>0$ such that

$$\min\Big\{\mathfrak{m}\big(E\cap B_r(x)\big),\mathfrak{m}\big(E^c\cap B_r(x)\big)\Big\}\leq C_I'\,P\big(E,B_{2\lambda r}(x)\big)^{k/k-1} \tag{1.10}$$

for every $x\in\mathrm{X}$ and $r>0$. By letting $r\to+\infty$ in (1.10), we conclude that (1.9) is satisfied. □

Theorem 1.19 (Coarea formula [38]) *Let* $(\mathrm{X},d,\mathfrak{m})$ *be a PI space. Fix* $f\in L^1_{\mathrm{loc}}(\mathfrak{m})$ *and an open set* $\Omega\subset\mathrm{X}$. *Then the function* $\mathbb{R}\ni t\mapsto P\big(\{f>t\},\Omega\big)\in[0,+\infty]$ *is Borel measurable and it holds*

$$|Df|(\Omega)=\int_{-\infty}^{+\infty}P\big(\{f>t\},\Omega\big)\,\mathrm{d}t. \tag{1.11}$$

In particular, if $f\in\mathrm{BV}(\mathrm{X})$*, then* $\{f>t\}$ *has finite perimeter for a.e.* $t\in\mathbb{R}$.

Remark 1.20 Given a PI space $(\mathrm{X}, d, \mathfrak{m})$ and any point $x \in \mathrm{X}$, it holds that the set $B_r(x)$ has finite perimeter for a.e. radius $r > 0$. This fact follows from the coarea formula (by applying it to the distance function from x). Furthermore, it also holds that $\mathcal{H}\big(\partial B_r(x)\big) < +\infty$ for a.e. $r > 0$, as a consequence of [2, Proposition 5.1]. □

A function $f \in \mathrm{BV}(\mathrm{X})$ is said to be *simple* provided it can be written as $f = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{E_i}$, for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and some sets of finite perimeter $E_1, \dots, E_n \subset \mathrm{X}$ having finite $\mathfrak{m}$-measure. It holds that any function of bounded variation in a PI space can be approximated by a sequence of simple BV functions (with a uniformly bounded total variation), as we are going to state in the next well-known result. Nevertheless, we recall the proof of this fact for the sake of completeness.

Lemma 1.21 (Density of simple BV functions) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a PI space and* $K \subset \mathrm{X}$ *a compact set. Fix any* $f \in \mathrm{BV}(\mathrm{X})$ *with* $\mathrm{spt}(f) \subset K$. *Then there exists a sequence* $(f_n)_n \subset \mathrm{BV}(\mathrm{X})$ *of simple functions with* $\mathrm{spt}(f_n) \subset K$ *such that* $f_n \to f$ *in* $L^1(\mathfrak{m})$ *and* $|Df_n|(\mathrm{X}) \leq |Df|(\mathrm{X})$ *for all* $n \in \mathbb{N}$.

Proof Given that $f^m := (f \wedge m) \vee (-m) \to f$ in $L^1(\mathfrak{m})$ as $m \to \infty$ and $|Df^m|(\mathrm{X}) \leq |Df|(\mathrm{X})$ for all $m > 0$ by Remark 1.3, it suffices to prove the statement under the additional assumption that the function f is essentially bounded, say that $-k < f < k$ holds $\mathfrak{m}$-a.e. for some $k \in \mathbb{N}$. Let us fix any $n \in \mathbb{N}$. Given any $i = -kn+1, \dots, kn$, we can choose $t_{i,n} \in \big((i-1)/n, i/n\big)$ such that

$$\frac{\mathsf{P}\big(\{f > t_{i,n}\}\big)}{n} \leq \int_{(i-1)/n}^{i/n} \mathsf{P}\big(\{f > t\}\big)\,\mathrm{d}t. \tag{1.12}$$

Then we define the simple BV function f_n on X as

$$f_n := -k + \frac{1}{n} \sum_{i=-kn+1}^{kn} \mathbb{1}_{\{f > t_{i,n}\}}.$$

It can be readily checked that $|Df_n|(\mathrm{X}) \leq |Df|(\mathrm{X})$. Indeed, notice that

$$|Df_n|(\mathrm{X}) \leq \frac{1}{n} \sum_{i=-kn+1}^{kn} \mathsf{P}\big(\{f > t_{i,n}\}\big) \overset{(1.12)}{\leq} \int_{-k}^{k} \mathsf{P}\big(\{f > t\}\big)\,\mathrm{d}t \overset{(1.11)}{=} |Df|(\mathrm{X}).$$

Furthermore, let us define $E_{i,n} := \{t_{i,n} < f \leq t_{i+1,n}\}$ for every $i = -kn+1, \dots, kn-1$. Moreover, we set $E_{-kn,n} := \{-k < f \leq t_{-kn+1,n}\}$ and $E_{kn,n} := \{t_{kn,n} < f < k\}$. Therefore, it holds that

$$\begin{aligned} f_n &= -k + \frac{1}{n} \sum_{i=-kn+1}^{kn} \sum_{j=i}^{kn} \mathbb{1}_{E_{j,n}} = -k + \frac{1}{n} \sum_{i=-kn+1}^{kn} (i+kn)\,\mathbb{1}_{E_{i,n}} \\ &= -k\,\mathbb{1}_{E_{-kn,n}} + \sum_{i=-kn+1}^{kn} \frac{i}{n}\,\mathbb{1}_{E_{i,n}}, \end{aligned}$$

thus accordingly $|f - f_n| = |f - i/n| \leq 1/n$ on $E_{i,n}$ for all $i = -kn, \dots, kn$. This ensures that

$$\int |f - f_n|\,\mathrm{d}\mathfrak{m} = \sum_{i=-kn}^{kn} \int_{E_{i,n}} |f - i/n|\,\mathrm{d}\mathfrak{m} \leq \frac{\mathfrak{m}(K)}{n} \overset{n}{\longrightarrow} 0.$$

Therefore, we have that $f_n \to f$ in $L^1(\mathfrak{m})$. Since $\mathrm{spt}(f_n) \subset K$ for every $n \in \mathbb{N}$ by construction, the proof of the statement is achieved. □

Remark 1.22 In the proof of Lemma 1.21 we obtained a stronger property: each approximating function f_n (say, $f_n = \sum_{i=1}^{k_n} \lambda_i^n \mathbb{1}_{E_i^n}$) can be required to satisfy $\sum_{i=1}^{k_n} |\lambda_i^n| \, \mathsf{P}(E_i^n) \leq |Df|(\mathrm{X})$. □

The following result states that, in the context of PI spaces, the perimeter measure admits an integral representation (with respect to the codimension-one Hausdorff measure):

Theorem 1.23 (Representation of the perimeter measure) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a PI space. Let* $E \subset \mathrm{X}$ *be a set of finite perimeter. Then the perimeter measure* $\mathsf{P}(E, \cdot)$ *is concentrated on the Borel set*

$$\Sigma_\tau(E) := \left\{ x \in \mathrm{X} \;\middle|\; \varliminf_{r \searrow 0} \min \left\{ \frac{\mathfrak{m}\big(E \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}, \frac{\mathfrak{m}\big(E^c \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} \right\} \geq \tau \right\} \subset \partial^e E, \quad (1.13)$$

where $\tau \in (0, 1/2)$ *is a constant depending just on* C_D*,* C_P *and* λ*. Moreover, the set* $\partial^e E \setminus \Sigma_\tau(E)$ *is* $\mathcal{H}$*-negligible and it holds that* $\mathcal{H}(\partial^e E) < +\infty$*. Finally, there exist a constant* $\gamma > 0$ *(depending on* C_D*,* C_P*,* λ*) and a Borel function* $\theta_E \colon \partial^e E \to [\gamma, C_D]$ *such that* $\mathsf{P}(E, \cdot) = \theta_E \, \mathcal{H}\llcorner_{\partial^e E}$*, namely*

$$\mathsf{P}(E, B) = \int_{B \cap \partial^e E} \theta_E \, \mathrm{d}\mathcal{H} \quad \text{for every Borel set } B \subset \mathrm{X}. \quad (1.14)$$

We shall sometimes consider θ_E *as a Borel function defined on the whole space* X*, by declaring that* $\theta_E := 0$ *on the set* $\mathrm{X} \setminus \partial^e E$*.*

Proof The result is mostly proven in [2, Theorem 5.3]. The fact that the measure $\mathsf{P}(E, \cdot)$ is concentrated on the set $\Sigma_\tau(E)$ is shown in [2, Theorem 5.4]. Finally, the upper bound $\theta_E \leq C_D$ has been obtained in [7, Theorem 4.6]. □

Lemma 1.24 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a PI space. Let* $F \subset E \subset \mathrm{X}$ *be two sets of finite perimeter such that* $\mathsf{P}(E) = \mathsf{P}(F) + \mathsf{P}(E \setminus F)$*. Then* $\mathcal{H}(\partial^e F \setminus \partial^e E) = 0$*.*

Proof By using item (iii) of Proposition 1.7 we deduce that

$$\mathsf{P}(E) = \mathsf{P}(E, \partial^e E) \leq \mathsf{P}(F, \partial^e E) + \mathsf{P}(E \setminus F, \partial^e E) \leq \mathsf{P}(F) + \mathsf{P}(E \setminus F) = \mathsf{P}(E),$$

which forces the identity $\mathsf{P}(F, \partial^e E) = \mathsf{P}(F)$. This implies $(\theta_F \mathcal{H})(\partial^e F \setminus \partial^e E) = \mathsf{P}\big(F, (\partial^e E)^c\big) = 0$ by Theorem 1.23, whence accordingly $\mathcal{H}(\partial^e F \setminus \partial^e E) = 0$, as required. □

The density function θ_E that appears in the Hausdorff representation formula for $\mathsf{P}(E, \cdot)$ might depend on the set E itself (cf. Example 1.27 below for an instance of this phenomenon). On the other hand, the new results that we are going to present in this paper require the density θ_E to be 'universal'– in a suitable sense. The precise formulation of this property is given in the next definition, which has been proposed in [7, Definition 6.1].

Definition 1.25 (*Isotropic space*) Let $(\mathrm{X}, d, \mathfrak{m})$ be a PI space. Then we say that $(\mathrm{X}, d, \mathfrak{m})$ is *isotropic* provided for any pair of sets $E, F \subset \mathrm{X}$ of finite perimeter satisfying $F \subset E$ it holds that

$$\theta_F(x) = \theta_E(x) \quad \text{for } \mathcal{H}\text{-a.e. } x \in \partial^e F \cap \partial^e E. \quad (1.15)$$

In order to provide examples and counterexamples, it will be convenient to consider the metric measure space we are going to construct. Given any $n \in \mathbb{N}$, we define the *n-spider* S_n as

$$S_n := \{o\} \sqcup (R_1 \cup \cdots \cup R_n), \quad \text{where } R_i := \{i\} \times (0, +\infty) \text{ for every } i = 1, \ldots, n. \tag{1.16}$$

We say that o is the *origin* of S_n, while $R_1, \ldots, R_n$ are the *rays* of S_n. We identify o with $(i, 0)$ for every $i = 1, \ldots, n$. It holds that $(S_n, d, \mathfrak{m})$ is an Ahlfors-regular PI space, where d is given by

$$d\big((i,t),(j,s)\big) := \begin{cases} |t-s| & \text{if } \quad i = j, \\ t+s & \text{if } \quad i \neq j \end{cases}$$

and $\mathfrak{m}$ stands for the 1-dimensional Hausdorff measure on (S_n, d).

Lemma 1.26 *Let E be a set of finite perimeter in the n-spider $(S_n, d, \mathfrak{m})$. Let o and $R_1, \ldots, R_n$ be the origin and the rays of S_n, respectively. Then the essential boundary $\partial^e E$ is a finite set and each intersection $E \cap R_i$ is $\mathfrak{m}$-a.e. equivalent to the union of finitely many subintervals of R_i.*

Moreover, calling I the family of all $i \in \{1, \ldots, n\}$ such that $E \cap R_i$ contains (up to $\mathfrak{m}$-null sets) a set of the form $\{i\} \times (0, \varepsilon)$ for some $\varepsilon > 0$, and $k \in \{1, \ldots, n\}$ the cardinality of I, it holds that

$$\mathsf{P}(E, \cdot) = \lambda\, \delta_o + \sum_{p \in \partial^e E \setminus \{o\}} \delta_p, \quad \textit{where} \quad \lambda := \min\{k, n-k\}. \tag{1.17}$$

Proof Possibly replacing E with its complement E^c—an operation which does not affect $\mathsf{P}(E, \cdot)$ nor $\partial^e E$—we can suppose without loss of generality that $k \leq n - k$, thus $\lambda = k$. The first statement readily follows from the fact that $\mathsf{P}(E, \cdot)$ is absolutely continuous with respect to the counting measure on S_n. Indeed, the Ahlfors-regularity of $(S_n, d, \mathfrak{m})$ grants that $\mathcal{H}$ is equivalent to the 0-dimensional Hausdorff measure, whence Theorem 1.23 yields the previous claim. To prove the last statement, first observe that—since each ray R_i can be identified with $(0, +\infty)$—it holds

$$\mathsf{P}(E, \cdot)\llcorner_{R_i} = \sum_{p \in \partial^e E \cap R_i} \delta_p \quad \text{for every } i = 1, \ldots, n.$$

It thus remains to characterise $\mathsf{P}(E, \cdot)\llcorner_{\Omega}$ for some open neighbourhood Ω of o. To this aim, take any $r \in (0, 1/2)$ such that $\{i\} \times (0, r)$ is $\mathfrak{m}$-a.e. contained in E for every $i \in I$ and call $\Omega := B_r(o)$. It is then clear by construction that $\partial^e E \cap \Omega \subset \{o\}$. Therefore, in order to prove (1.17) it just suffices to show that $\mathsf{P}(E, \Omega) = k$. On the one hand, we define the sequence $(g_j)_{j \geq 3} \subset \mathrm{LIP}(\Omega)$ as

$$g_j(x) := \begin{cases} \min\{jt, 1\} & \text{if } \quad x \in \Omega \cap R_i \text{ for some } i \in I \text{ and } x = (i, t), \\ 0 & \text{if } \quad x \in \Omega \setminus \bigcup_{i \in I} R_i. \end{cases}$$

Hence, it holds that $g_j \to \mathbb{1}_E$ in $L^1(\mathfrak{m}\llcorner_{\Omega})$ and $\int_{\Omega} \mathrm{lip}(g_j)\, \mathrm{d}\mathfrak{m} = k$ for all $j \geq 3$, thus $\mathsf{P}(E, \Omega) \leq k$.

On the other hand, fix any sequence $(f_j)_j \subseteq \mathrm{LIP}_{\mathrm{loc}}(\Omega)$ satisfying $f_j \to \mathbb{1}_E$ in $L^1_{\mathrm{loc}}(\mathfrak{m}\llcorner_{\Omega})$ and for which the limit $\lim_j \int_{\Omega} \mathrm{lip}(f_j)\, \mathrm{d}\mathfrak{m}$ exists. Up to a subsequence, we can also assume that $f_j \to \mathbb{1}_E$ in the $\mathfrak{m}\llcorner_{\Omega}$-a.e. sense. Now let $\varepsilon \in (0, r)$ be given. Then for any $j \in \mathbb{N}$ sufficiently large we can find points $x_i \in B_{\varepsilon}(o) \cap R_i$ with $i \in \{1, \ldots, n\}$, such that $\big| f_j(x_i) -$

$1\big| \leq \varepsilon$ for all $i \in I$ and $\big|f_j(x_i)\big| \leq \varepsilon$ for all $i \notin I$. Given that $\big|\mathbb{1}_I(i) - c\big| \leq \big|f_j(x_i) - c\big| + \varepsilon$ for all $i \in \{1, \ldots, n\}$ and $c \in \mathbb{R}$, we see that

$$k = \min_{c\in\mathbb{R}} \sum_{i=1}^{n} \big|\mathbb{1}_I(i) - c\big| \leq n\varepsilon + \min_{c\in\mathbb{R}} \sum_{i=1}^{n} \big|f_j(x_i) - c\big| \leq n\varepsilon + \sum_{i=1}^{n} \big|f_j(x_i) - f_j(o)\big|$$
$$\leq n\varepsilon + \sum_{i=1}^{n} \int_{\Omega\cap R_i} \operatorname{lip}(f_j)\, \mathrm{d}\mathfrak{m} = n\varepsilon + \int_{\Omega} \operatorname{lip}(f_j)\, \mathrm{d}\mathfrak{m}.$$

By first letting $j \to \infty$ and then $\varepsilon \to 0$, we finally conclude that $\lim_j \int \operatorname{lip}(f_j)\, \mathrm{d}\mathfrak{m} \geq k$ and thus accordingly $\mathsf{P}(E, \Omega) = k$. Therefore, the proof of the last statement is completed. □

Example 1.27 The 4-spider $(S_4, d, \mathfrak{m})$ is not isotropic: calling o its origin and R_1, R_2, R_3, R_4 its rays, we know from Lemma 1.26 that $\theta_{R_1}(o) = 1$ and $\theta_{R_1\cup R_2}(o) = 2$. □

We shall also sometimes work with PI spaces satisfying the following property:

Definition 1.28 (*Two-sidedness property*) Let $(\mathrm{X}, d, \mathfrak{m})$ be a PI space. Then we say that $(\mathrm{X}, d, \mathfrak{m})$ has the *two-sidedness property* provided it holds that

$$\mathcal{H}\big(\partial^e E \cap \partial^e F \cap \partial^e (E \cup F)\big) = 0 \quad \text{for any disjoint sets } E, F \subset \mathrm{X} \text{ of finite perimeter.} \tag{1.18}$$

It immediately follows from [7, Proposition 6.2] that every PI space having the two-sidedness property is isotropic, while the converse implication might fail (as shown by the following example).

Example 1.29 The 3-spider $(S_3, d, \mathfrak{m})$ is an isotropic space: indeed, if $E \subset S_3$ is any set of finite perimeter such that $o \in \partial^e E$, then $\theta_E(o) = 1$ by Lemma 1.26. Moreover, the space $(S_3, d, \mathfrak{m})$ does not have the two-sidedness property, as $\partial^e R_1 \cap \partial^e R_2 \cap \partial^e (R_1 \cup R_2) = \{o\}$ and $\mathcal{H}(\{o\}) > 0$.

The same arguments show that, given a radius $r > 0$, the closure X_r of the ball $B_r(o)$ in S_3 (endowed with the restricted distance $d_{\mathrm{X}_r\times\mathrm{X}_r}$ and measure $\mathfrak{m}\llcorner_{\mathrm{X}_r}$) is an isotropic, Ahlfors-regular PI space which does not have the two-sidedness property. □

A sufficient condition for the two-sidedness property to hold is provided by the following result:

Lemma 1.30 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a PI space with the following property:*

$$\mathcal{H}(\partial^e E \setminus E^{1/2}) = 0 \quad \textit{for every set } E \subset \mathrm{X} \textit{ of finite perimeter} \tag{1.19}$$

(or, equivalently, the measure $\mathsf{P}(E, \cdot)$ *is concentrated on* $E^{1/2}$*). Then the space* $(\mathrm{X}, d, \mathfrak{m})$ *has the two-sidedness property.*

Proof Fix two disjoint sets $E, F \subset \mathrm{X}$ of finite perimeter. Given any $x \in E^{1/2} \cap F^{1/2}$, we have

$$D(E \cup F, x) = \lim_{r\searrow 0} \frac{\mathfrak{m}\big(E \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} + \lim_{r\searrow 0} \frac{\mathfrak{m}\big(F \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} = D(E, x) + D(F, x) = 1,$$

thus in particular $x \notin (E \cup F)^{1/2}$. This shows that $E^{1/2} \cap F^{1/2} \cap (E \cup F)^{1/2} = \emptyset$, whence we can conclude that $\mathcal{H}\big(\partial^e E \cap \partial^e F \cap \partial^e (E \cup F)\big) = 0$. This proves the two-sidedness property. □

Example 1.31 (Examples of isotropic spaces) Let us conclude the section by expounding which classes of PI spaces are known to be isotropic (to the best of our knowledge):

(i) Weighted Euclidean spaces (induced by a continuous, strong A_∞ weight).
(ii) Carnot groups.
(iii) $RCD(K, N)$ spaces, with $K \in \mathbb{R}$ and $N < \infty$. In particular, all (weighted) Riemannian manifolds whose Ricci curvature is bounded from below.

Isotropicity of the spaces in (i) and (ii) is shown in [7, Section 7] and [8], respectively. Also, it follows from the rectifiability results in [26,27] that all Carnot groups of step 2 satisfy (1.19), so also the two-sidedness property. About item (iii), it follows from the results in [3,14] that all $RCD(K, N)$ spaces satisfy (1.19), whence they have the two-sidedness property (and thus are isotropic). □

2 Decomposability of a set of finite perimeter

This section is entirely devoted to the decomposability properties of sets of finite perimeter in isotropic PI spaces. An indecomposable set is, roughly speaking, a set of finite perimeter that is connected in a measure-theoretical sense. Section 2.1 consists of a detailed study of the basic properties of this class of sets. In Sect. 2.2 we will prove that any set of finite perimeter can be uniquely expressed as disjoint union of indecomposable sets (cf. Theorem 2.14). The whole discussion is strongly inspired by the results of [4], where the decomposability of sets of finite perimeter in the Euclidean setting has been systematically investigated. Actually, many of the results (and the relative proofs) in this section are basically just a reformulation—in the metric setting – of the corresponding ones in $\mathbb{R}^n$, proven in [4]. We postpone to Remark 2.19 the discussion of the main differences between the case of isotropic PI spaces and the Euclidean one.

2.1 Definition of decomposable set and its basic properties

Let us begin with the definition of decomposable set and indecomposable set in a general metric measure space.

Definition 2.1 (*Decomposable and indecomposable sets*) Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure space. Let $E \subset \mathrm{X}$ be a set of finite perimeter. Given any Borel set $B \subset \mathrm{X}$, we say that E is *decomposable in B* provided there exists a partition $\{F, G\}$ of $E \cap B$ into sets of finite perimeter such that $\mathfrak{m}(F), \mathfrak{m}(G) > 0$ and $\mathsf{P}(E, B) = \mathsf{P}(F, B) + \mathsf{P}(G, B)$. On the other hand, we say that E is *indecomposable in B* if it is not decomposable in B. For brevity, we say that E is *decomposable* (resp. *indecomposable*) provided it is decomposable in X (resp. indecomposable in X).

Observe that the property of being decomposable/indecomposable is invariant under modifications on $\mathfrak{m}$-null sets and that any $\mathfrak{m}$-negligible set is indecomposable.

Remark 2.2 Let $E \subset \mathrm{X}$ be a set of finite perimeter. Let $\{E_n\}_{n\in\mathbb{N}}$ be a partition of E into sets of finite perimeter and let $\Omega \subset \mathrm{X}$ be any open set. Then it holds that:

$$\mathsf{P}(E, \Omega) = \sum_{n=0}^{\infty} \mathsf{P}(E_n, \Omega) \quad \Longleftrightarrow \quad \mathsf{P}(E, \Omega) \geq \sum_{n=0}^{\infty} \mathsf{P}(E_n, \Omega).$$

Indeed, it can be readily checked that $\mathbb{1}_{\bigcup_{n\leq N} E_n} \to \mathbb{1}_E$ in $L^1_{\mathrm{loc}}(\mathfrak{m})$ as $N \to \infty$, whence items (ii) and (iii) of Proposition 1.7 grant that the inequality

$$\mathsf{P}(E,\Omega) \leq \varliminf_{N\to\infty} \mathsf{P}\Big(\bigcup_{n\leq N} E_n, \Omega\Big) \leq \lim_{N\to\infty} \sum_{n=0}^{N} \mathsf{P}(E_n,\Omega) = \sum_{n=0}^{\infty} \mathsf{P}(E_n,\Omega)$$

is always verified. □

Lemma 2.3 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space. Let* $E, F \subset \mathrm{X}$ *be sets of finite perimeter and let* $B \subset \mathrm{X}$ *be any Borel set. Then the following implications hold:*

(i) *If* $P(E \cup F, B) = P(E, B) + P(F, B)$*, then* $\mathcal{H}(\partial^e E \cap \partial^e F \cap B) = 0$.
(ii) *If* $\mathfrak{m}(E \cap F) = 0$ *and* $\mathcal{H}(\partial^e E \cap \partial^e F \cap B) = 0$*, then* $P(E \cup F, B) = P(E, B) + P(F, B)$.

Proof (i) Suppose that $\mathsf{P}(E \cup F, B) = \mathsf{P}(E, B) + \mathsf{P}(F, B)$. A trivial set-theoretic argument yields

$$\begin{aligned}
&(\theta_{E\cup F}\mathcal{H})\big((\partial^e E \cup \partial^e F) \cap B\big)\\
&\quad = (\theta_{E\cup F}\mathcal{H})\big((\partial^e E \setminus \partial^e F) \cap B\big) + (\theta_{E\cup F}\mathcal{H})\big((\partial^e F \setminus \partial^e E) \cap B\big)\\
&\qquad + (\theta_{E\cup F}\mathcal{H})(\partial^e E \cap \partial^e F \cap B)\\
&\quad = (\theta_{E\cup F}\mathcal{H})(\partial^e E \cap B) + (\theta_{E\cup F}\mathcal{H})(\partial^e F \cap B) - (\theta_{E\cup F}\mathcal{H})(\partial^e E \cap \partial^e F \cap B).
\end{aligned}$$

Given that $\theta_{E\cup F}$ is assumed to be null on the complement of $\partial^e(E \cup F)$, we deduce that

$$\begin{aligned}
&(\theta_{E\cup F}\mathcal{H})(\partial^e E \cap B)\\
&\quad = (\theta_{E\cup F}\mathcal{H})\big(\partial^e E \cap \partial^e(E \cup F) \cap B\big) \overset{(1.15)}{=} (\theta_E\mathcal{H})\big(\partial^e E \cap \partial^e(E \cup F) \cap B\big),\\
&(\theta_{E\cup F}\mathcal{H})(\partial^e F \cap B)\\
&\quad = (\theta_{E\cup F}\mathcal{H})\big(\partial^e F \cap \partial^e(E \cup F) \cap B\big) \overset{(1.15)}{=} (\theta_F\mathcal{H})\big(\partial^e F \cap \partial^e(E \cup F) \cap B\big).
\end{aligned}$$

Accordingly, it holds that

$$\begin{aligned}
\mathsf{P}(E \cup F, B) &\overset{(1.14)}{=} (\theta_{E\cup F}\mathcal{H})\big(\partial^e(E \cup F) \cap B\big) \overset{(1.7)}{\leq} (\theta_{E\cup F}\mathcal{H})\big((\partial^e E \cup \partial^e F) \cap B\big)\\
&= (\theta_{E\cup F}\mathcal{H})(\partial^e E \cap B) + (\theta_{E\cup F}\mathcal{H})(\partial^e F \cap B) - (\theta_{E\cup F}\mathcal{H})(\partial^e E \cap \partial^e F \cap B)\\
&\leq (\theta_E\mathcal{H})(\partial^e E \cap B) + (\theta_F\mathcal{H})(\partial^e F \cap B) - (\theta_{E\cup F}\mathcal{H})(\partial^e E \cap \partial^e F \cap B)\\
&\overset{(1.14)}{=} \mathsf{P}(E, B) + \mathsf{P}(F, B) - (\theta_{E\cup F}\mathcal{H})(\partial^e E \cap \partial^e F \cap B)\\
&= \mathsf{P}(E \cup F, B) - (\theta_{E\cup F}\mathcal{H})(\partial^e E \cap \partial^e F \cap B),
\end{aligned}$$

which forces the equality $(\theta_{E\cup F}\mathcal{H})(\partial^e E \cap \partial^e F \cap B) = 0$. Since $\theta_{E\cup F} \geq \gamma'_{E\cup F} > 0$ on $\partial^e(E \cup F)$, we obtain that $\mathcal{H}\big(\partial^e E \cap \partial^e F \cap \partial^e(E \cup F) \cap B\big) = 0$. Moreover, we have that

$$\begin{aligned}
\mathsf{P}(E, B) &= (\theta_E\mathcal{H})\big(\partial^e E \cap \partial^e(E \cup F) \cap B\big) + (\theta_E\mathcal{H})\big((\partial^e E \cap B) \setminus \partial^e(E \cup F)\big)\\
&= (\theta_{E\cup F}\mathcal{H})\big(\partial^e E \cap \partial^e(E \cup F) \cap B\big) + (\theta_E\mathcal{H})\big((\partial^e E \cap B) \setminus \partial^e(E \cup F)\big)\\
&= \mathsf{P}(E \cup F, \partial^e E \cap B) + (\theta_E\mathcal{H})\big((\partial^e E \cap B) \setminus \partial^e(E \cup F)\big)
\end{aligned}$$

and, similarly, that $\mathsf{P}(F,B)=\mathsf{P}(E\cup F,\partial^e F\cap B)+(\theta_F\mathcal{H})\big((\partial^e F\cap B)\setminus\partial^e(E\cup F)\big)$. This yields

$$\begin{aligned}\mathsf{P}(E\cup F,B)&=\mathsf{P}\big(E\cup F,\partial^e(E\cup F)\cap B\big)\leq\mathsf{P}(E\cup F,\partial^e E\cap B)+\mathsf{P}(E\cup F,\partial^e F\cap B)\\&=\mathsf{P}(E,B)+\mathsf{P}(F,B)-(\theta_E\mathcal{H})\big((\partial^e E\cap B)\setminus\partial^e(E\cup F)\big)-(\theta_F\mathcal{H})\big((\partial^e F\cap B)\setminus\partial^e(E\cup F)\big)\\&=\mathsf{P}(E\cup F,B)-(\theta_E\mathcal{H})\big((\partial^e E\cap B)\setminus\partial^e(E\cup F)\big)-(\theta_F\mathcal{H})\big((\partial^e F\cap B)\setminus\partial^e(E\cup F)\big).\end{aligned}$$

Hence, we conclude that $(\theta_E\mathcal{H})\big((\partial^e E\cap B)\setminus\partial^e(E\cup F)\big)=0$ and $(\theta_F\mathcal{H})\big((\partial^e F\cap B)\setminus\partial^e(E\cup F)\big)=0$. Since $\theta_E\geq\gamma'_E>0$ on $\partial^e E$ and $\theta_F\geq\gamma'_F>0$ on $\partial^e F$, we get that $\mathcal{H}\big((\partial^e E\cap B)\setminus\partial^e(E\cup F)\big)=0$ and $\mathcal{H}\big((\partial^e F\cap B)\setminus\partial^e(E\cup F)\big)=0$. In particular, we have $\mathcal{H}\big((\partial^e E\cap\partial^e F\cap B)\setminus\partial^e(E\cup F)\big)=0$. Consequently, we have finally proven that $\mathcal{H}(\partial^e E\cap\partial^e F\cap B)=0$, as required.

(ii) Let us suppose that $\mathfrak{m}(E\cap F)=0$ and $\mathcal{H}(\partial^e E\cap\partial^e F\cap B)=0$. We already know that the inequality $\mathsf{P}(E\cup F,B)\leq\mathsf{P}(E,B)+\mathsf{P}(F,B)$ is always verified. The converse inequality readily follows from our assumptions, item (iv) of Proposition 1.16 and the representation formula for the perimeter measure:

$$\begin{aligned}\mathsf{P}(E,B)+\mathsf{P}(F,B)&=(\theta_E\mathcal{H})(\partial^e E\cap B)+(\theta_F\mathcal{H})(\partial^e F\cap B)\\&=(\theta_E\mathcal{H})\big((\partial^e E\setminus\partial^e F)\cap B\big)+(\theta_E\mathcal{H})(\partial^e E\cap\partial^e F\cap B)\\&\quad+(\theta_F\mathcal{H})\big((\partial^e F\setminus\partial^e E)\cap B\big)+(\theta_F\mathcal{H})(\partial^e F\cap\partial^e E\cap B)\\&=(\theta_{E\cup F}\mathcal{H})\big((\partial^e E\setminus\partial^e F)\cap B\big)+(\theta_{E\cup F}\mathcal{H})\big((\partial^e F\setminus\partial^e E)\cap B\big)\\&=(\theta_{E\cup F}\mathcal{H})\big((\partial^e E\Delta\partial^e F)\cap B\big)\leq(\theta_{E\cup F}\mathcal{H})\big(\partial^e(E\cup F)\cap B\big)\\&=\mathsf{P}(E\cup F,B).\end{aligned}$$

Therefore, it holds that $\mathsf{P}(E\cup F,B)=\mathsf{P}(E,B)+\mathsf{P}(F,B)$, as required. □

Remark 2.4 Item (i) of Lemma 2.3 fails for the non-isotropic space in Example 1.27: it holds that $\mathsf{P}(R_1\cup R_2)=\mathsf{P}(R_1)+\mathsf{P}(R_2)$, but $\partial^e R_1\cap\partial^e R_2=\{o\}$ with $\mathcal{H}(\{o\})>0$. □

In the setting of PI spaces having the two-sidedness property, the fact of being an indecomposable set of finite perimeter can be equivalently characterised as illustrated by the following result, which constitutes a generalisation of [21, Proposition 2.12].

Theorem 2.5 *Let* $(\mathrm{X},d,\mathfrak{m})$ *be a PI space. Then the following properties hold:*

(i) *Let* $E\subset\mathrm{X}$ *be a set of finite perimeter such that*

$$f\in L^1_{\mathrm{loc}}(\mathfrak{m}),\ |Df|(\mathrm{X})<+\infty,\ |Df|(E^1)=0\quad\Longrightarrow\quad\begin{array}{l}f=t\text{ holds }\mathfrak{m}\text{-a.e. on }E,\\\text{for some constant }t\in\mathbb{R}.\end{array}\tag{2.1}$$

Then E *is indecomposable.*

(ii) *Suppose* $(\mathrm{X},d,\mathfrak{m})$ *has the two-sidedness property. Then any indecomposable subset of* X *satisfies* (2.1).

Proof (i) Suppose $E\subset\mathrm{X}$ is decomposable. Choose two disjoint sets of finite perimeter $F,G\subset\mathrm{X}$ having positive $\mathfrak{m}$-measure such that $E=F\cup G$ and $\mathsf{P}(E)=\mathsf{P}(F)+\mathsf{P}(G)$. Then let us consider the function $f:=\mathbb{1}_F\in L^1_{\mathrm{loc}}(\mathfrak{m})$. Notice that $|Df|(\mathrm{X})=\mathsf{P}(F)<+\infty$. Moreover, we know from Lemma 1.24 that $\mathcal{H}(\partial^e F\setminus\partial^e E)=0$, thus accordingly

$$|Df|(E^1)=(\theta_F\mathcal{H})(\partial^e F\cap E^1)\leq(\theta_F\mathcal{H})(\partial^e E\cap E^1)=0.$$

Nevertheless, f is not $\mathfrak{m}$-a.e. equal to a constant on E, whence E does not satisfy property (2.1).

(ii) Fix an indecomposable set $E \subset \mathrm{X}$. Consider any function $f \in L^1_{\mathrm{loc}}(\mathfrak{m})$ such that $|Df|(\mathrm{X}) < +\infty$ and $|Df|(E^1) = 0$. First of all, we claim that

$$\mathsf{P}(E \cap A, E^1) \leq \mathsf{P}(A, E^1) \quad \text{for every set } A \subset \mathrm{X} \text{ of finite perimeter.} \tag{2.2}$$

Indeed, by exploiting the inclusion $\partial^e(E \cap A) \subset \partial^e E \cup \partial^e A$ and the isotropicity of $(\mathrm{X}, d, \mathfrak{m})$ we get

$$\begin{aligned}
\mathsf{P}(E \cap A, E^1) &= (\theta_{E\cap A}\mathcal{H})\big(\partial^e(E \cap A) \cap E^1\big) = (\theta_{E\cap A}\mathcal{H})\big(\partial^e(E \cap A)\cap(\partial^e E \cup \partial^e A)\cap E^1\big)\\
&\leq (\theta_{E\cap A}\mathcal{H})\big(\partial^e(E \cap A) \cap (\partial^e E \cap E^1)\big) + (\theta_{E\cap A}\mathcal{H})\big(\partial^e(E \cap A) \cap \partial^e A \cap E^1\big)\\
&= (\theta_{E\cap A}\mathcal{H})\big(\partial^e(E \cap A) \cap \partial^e A \cap E^1\big) = (\theta_A\mathcal{H})\big(\partial^e(E \cap A) \cap \partial^e A \cap E^1\big)\\
&\leq (\theta_A\mathcal{H})(\partial^e A \cap E^1) = \mathsf{P}(A, E^1),
\end{aligned}$$

whence the claim (2.2) follows. Now let us define the finite Borel measure μ on X as

$$\mu(B) := \int_{\mathbb{R}} \mathsf{P}\big(\{f > t\}, B\big)\,\mathrm{d}t \quad \text{for every Borel set } B \subset \mathrm{X}.$$

Since $|Df|(\Omega) = \mu(\Omega)$ for every open set $\Omega \subset \mathrm{X}$ by Theorem 1.19, we deduce that $|Df| = \mu$ by outer regularity. In particular, it holds that $\int_{\mathbb{R}} \mathsf{P}\big(\{f > t\}, E^1\big)\,\mathrm{d}t = |Df|(E^1) = 0$, which in turn forces the identity $\mathsf{P}\big(\{f > t\}, E^1\big) = 0$ for a.e. $t \in \mathbb{R}$. Calling $E_t^+ := E \cap \{f > t\}$ for all $t \in \mathbb{R}$, we thus infer from (2.2) that $\mathsf{P}(E_t^+, E^1) = 0$ for a.e. $t \in \mathbb{R}$, so that in particular $\mathcal{H}(\partial^e E_t^+ \cap E^1) = 0$ for a.e. $t \in \mathbb{R}$. Also, we have $\mathcal{H}\big(\partial^e E_t^+ \cap \partial^e(E\setminus E_t^+) \cap \partial^e E\big) = 0$ for a.e. $t \in \mathbb{R}$ by (1.18), whence

$$\begin{aligned}
&\mathcal{H}\big(\partial^e E_t^+ \cap \partial^e(E\setminus E_t^+)\big)\\
&\overset{(1.6)}{\leq} \mathcal{H}\big(\partial^e E_t^+ \cap \partial^e(E\setminus E_t^+) \cap \partial^e E\big) + \mathcal{H}\big(\partial^e E_t^+ \cap \partial^e(E\setminus E_t^+) \cap E^1\big)\\
&= \mathcal{H}\big(\partial^e E_t^+ \cap \partial^e(E\setminus E_t^+) \cap E^1\big) \leq \mathcal{H}(\partial^e E_t^+ \cap E^1) = 0
\end{aligned}$$

holds for a.e. $t \in \mathbb{R}$. Therefore, item (ii) of Lemma 2.3 grants that $\mathsf{P}(E) = \mathsf{P}(E_t^+) + \mathsf{P}(E\setminus E_t^+)$ for a.e. $t \in \mathbb{R}$. Being E indecomposable, we deduce that for a.e. $t \in \mathbb{R}$ we have that either $\mathfrak{m}(E_t^+) = 0$ or $\mathfrak{m}(E\setminus E_t^+) = 0$. Define $E_t^- := E \cap \{f < t\}$ for all $t \in \mathbb{R}$. Pick a negligible set $N \subset \mathbb{R}$ such that

$$\text{either } \mathfrak{m}(E_t^+) = 0 \text{ or } \mathfrak{m}(E_t^-) = 0 \quad \text{for any } t \in \mathbb{R}\setminus N. \tag{2.3}$$

Let us define $t_-, t_+ \in \mathbb{R}$ as follows:

$$\begin{aligned}
t_- &:= \sup\big\{t \in \mathbb{R}\setminus N \;\big|\; \mathfrak{m}(E_t^-) = 0\big\},\\
t_+ &:= \inf\big\{t \in \mathbb{R}\setminus N \;\big|\; \mathfrak{m}(E_t^+) = 0\big\}.
\end{aligned}$$

We claim that $\mathfrak{m}(E_{t_-}^-) = \mathfrak{m}(E_{t_+}^+) = 0$. Indeed, given any sequence $(t_n)_n \subset \mathbb{R}\setminus N$ such that $t_n \nearrow t_-$ and $\mathfrak{m}(E_{t_n}^-) = 0$ for all $n \in \mathbb{N}$, we have that $E_{t_-}^- = \bigcup_n E_{t_n}^-$ and accordingly $\mathfrak{m}(E_{t_-}^-) = 0$. Similarly for $E_{t_+}^+$. In light of this observation, we see that $t_- \leq t_+$, otherwise we would have $E = E_{t_-}^- \cup E_{t_+}^+$ and thus $\mathfrak{m}(E) \leq \mathfrak{m}(E_{t_-}^-) + \mathfrak{m}(E_{t_+}^+) = 0$. We now argue by contradiction: suppose $t_- < t_+$. Then it holds that $\mathfrak{m}(E_t^-), \mathfrak{m}(E_t^+) > 0$ for every $t \in (t_-, t_+)\setminus N$ by definition of $t_\pm$. This leads to a contradiction with (2.3). Then one has $t_- = t_+$, so that $\mathfrak{m}\big(E \cap \{f \neq t_-\}\big) = \mathfrak{m}(E_{t_-}^-) + \mathfrak{m}(E_{t_+}^+) = 0$. This means that $f = t_-$ holds $\mathfrak{m}$-a.e. on E, which finally shows that E satisfies property (2.1). □

Remark 2.6 In item (ii) of Theorem 2.5, the additional assumptions on $(\mathrm{X}, d, \mathfrak{m})$ cannot be dropped. For instance, let us consider the space described in Example 1.29. Calling E the indecomposable set $R_1 \cup R_2$, it holds $E^1 = E \setminus \{o\}$, thus $\mathbb{1}_{R_1} \in \mathrm{BV}(\mathrm{X})$ satisfies $|D\mathbb{1}_{R_1}|(E^1) = 0$, but it is not $\mathfrak{m}$-a.e. constant on E. This shows that E does not satisfy (2.1).

Corollary 2.7 *Let $(\mathrm{X}, d, \mathfrak{m})$ be a PI space. Let $\Omega \subset \mathrm{X}$ be an open, connected set of finite perimeter. Then Ω is indecomposable.*

Proof Let $f \in L^1_{\mathrm{loc}}(\mathfrak{m})$ satisfy $|Df|(\mathrm{X}) < +\infty$ and $|Df|(\Omega^1) = 0$. Being Ω open, it holds $\Omega^1 \supset \Omega$, whence $|Df|(\Omega) = 0$. Given any $x \in \Omega$, we can choose a radius $r > 0$ such that $B_{\lambda r}(x) \subset \Omega$ and accordingly $|Df|\big(B_{\lambda r}(x)\big) = 0$, where $\lambda \geq 1$ is the constant appearing in the weak (1, 1)-Poincaré inequality. Consequently, Lemma 1.13 tells us that $\fint_{B_r(x)} |f - f_{x,r}| \,\mathrm{d}\mathfrak{m} = 0$, thus in particular f is $\mathfrak{m}$-a.e. constant on $B_r(x)$. This shows that f is locally $\mathfrak{m}$-a.e. constant on Ω. Since Ω is connected, we deduce that f is $\mathfrak{m}$-a.e. constant on Ω. Therefore, we finally conclude that Ω is indecomposable by using item (i) of Theorem 2.5. □

Lemma 2.8 *Let $(\mathrm{X}, d, \mathfrak{m})$ be an isotropic PI space. Fix a set $E \subset \mathrm{X}$ of finite perimeter and a Borel set $B \subset \mathrm{X}$. Suppose that $\{F, G\}$ is a Borel partition of E such that $\mathsf{P}(E, B) = \mathsf{P}(F, B) + \mathsf{P}(G, B)$. Then it holds that $\mathsf{P}(A, B) = \mathsf{P}(A \cap F, B) + \mathsf{P}(A \cap G, B)$ for every set $A \subset E$ of finite perimeter.*

Proof First of all, note that $\mathcal{H}\big((\partial^e F \cup \partial^e G) \cap (\partial^e E)^c \cap B\big) \leq \mathcal{H}(\partial^e F \cap \partial^e G \cap B) = 0$ by item (iv) of Proposition 1.16 and item (i) of Lemma 2.3. This forces the identity

$$\mathcal{H}\Big(\big(\partial^e E \Delta (\partial^e F \cup \partial^e G)\big) \cap B\Big) = 0. \tag{2.4}$$

Now fix any set $A \subset E$ of finite perimeter. By using again the property (1.7) we see that

$$\partial^e A \cap B \subset \big(\partial^e (A \cap F) \cup \partial^e (A \cap G)\big) \cap B. \tag{2.5}$$

On the other hand, we claim that

$$(\partial^e A)^c \cap B \subset \Big(\big(\partial^e (A \cap F) \cup \partial^e (A \cap G)\big)^c \cup (\partial^e E)^c\Big) \cap B. \tag{2.6}$$

Indeed, pick any $x \in (\partial^e A)^c$, thus either $D(A, x) = 0$ or $D(A^c, x) = 0$. In the former case we deduce that $D(A \cap F, x), D(A \cap G, x) \leq D(A, x) = 0$, so that $x \notin \partial^e (A \cap F) \cup \partial^e (A \cap G)$. In the latter case we have $D(E^c, x) \leq D(A^c, x) = 0$, whence $x \notin \partial^e E$. This shows the validity of (2.6).

Moreover, notice that $(\partial^e A)^c \cap (\partial^e F)^c \subset \big(\partial^e (A \cap F)\big)^c$ and $(\partial^e A)^c \cap (\partial^e G)^c \subset \big(\partial^e (A \cap G)\big)^c$ hold by property (1.7), thus accordingly we have that

$$(\partial^e A)^c \cap (\partial^e F \cup \partial^e G)^c \cap B \subset \big(\partial^e (A \cap F) \cup \partial^e (A \cap G)\big)^c \cap B. \tag{2.7}$$

By combining (2.4), (2.5), (2.6) and (2.7), we deduce that

$$\mathcal{H}\Big(\big(\partial^e A \Delta \big(\partial^e (A \cap F) \cup \partial^e (A \cap G)\big)\big) \cap B\Big) = 0. \tag{2.8}$$

Since $\mathsf{P}(E, B) = \mathsf{P}(F, B) + \mathsf{P}(G, B)$, we know from item (i) of Lemma 2.3 that $\mathcal{H}(\partial^e F \cap \partial^e G \cap B) = 0$. Property (1.7) ensures that $\partial^e (A \cap F) \cap \partial^e (A \cap G) \subset \partial^e A \cap (\partial^e F \cap \partial^e G)$, which together with the identities $\mathcal{H}(\partial^e F \cap \partial^e G \cap B) = 0$ and (2.8) yield $\mathcal{H}\big(\partial^e (A \cap F) \cap \partial^e (A \cap G) \cap B\big) = 0$. Therefore, item (ii) of Lemma 2.3 gives $\mathsf{P}(A, B) = \mathsf{P}(A \cap F, B) + \mathsf{P}(A \cap G, B)$, thus proving the statement. □

Corollary 2.9 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space. Fix* $E \subset \mathrm{X}$ *of finite perimeter and* $\Omega \subset \mathrm{X}$ *open. Suppose that* $(E_n)_n$ *is a Borel partition of* E *such that* $\mathsf{P}(E, \Omega) = \sum_{n=0}^{\infty} \mathsf{P}(E_n, \Omega)$. *Then*

$$\mathsf{P}(F, \Omega) = \sum_{n=0}^{\infty} \mathsf{P}(F \cap E_n, \Omega) \quad \textit{for every Borel set } F \subset E \textit{ with } \mathsf{P}(F) < +\infty.$$

Proof Fix any $N \in \mathbb{N}$. By repeatedly applying Lemma 2.8 we obtain that

$$\mathsf{P}(F, \Omega) = \sum_{n=0}^{N} \mathsf{P}(F \cap E_n, \Omega) + \mathsf{P}\Big(\bigcup_{n>N} F \cap E_n, \Omega\Big) \geq \sum_{n=0}^{N} \mathsf{P}(F \cap E_n, \Omega).$$

By letting $N \to \infty$ we deduce that $\mathsf{P}(F, \Omega) \geq \sum_{n=0}^{\infty} \mathsf{P}(F \cap E_n, \Omega)$, which gives the statement thanks to Remark 2.2. □

Proposition 2.10 (Stability of indecomposable sets) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space. Fix a set* $E \subset \mathrm{X}$ *be of finite perimeter. Let* $(E_n)_n$ *be an increasing sequence of indecomposable subsets of* X *such that* $E = \bigcup_n E_n$. *Then* E *is an indecomposable set.*

Proof We argue by contradiction: suppose there exists a Borel partition $\{F, G\}$ of the set E such that $\mathfrak{m}(F), \mathfrak{m}(G) > 0$ and $\mathsf{P}(E) = \mathsf{P}(F) + \mathsf{P}(G)$. Given that we have $\lim_n \mathfrak{m}(F \cap E_n) = \mathfrak{m}(F)$ and $\lim_n \mathfrak{m}(G \cap E_n) = \mathfrak{m}(G)$, we can choose an index $n \in \mathbb{N}$ so that $\mathfrak{m}(F \cap E_n), \mathfrak{m}(G \cap E_n) > 0$. By Lemma 2.8 we know that $\mathsf{P}(E_n) = \mathsf{P}(F \cap E_n) + \mathsf{P}(G \cap E_n)$. Being $\{F \cap E_n, G \cap E_n\}$ a Borel partition of E_n, we get a contradiction with the indecomposability of E_n. This gives the statement. □

Lemma 2.11 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space and* $E \subset \mathrm{X}$ *a set of finite perimeter. Fix two Borel sets* $B, B' \subset \mathrm{X}$. *Suppose that* $E \subset B \subset B'$ *and that* E *is indecomposable in* B. *Then it holds that the set* E *is indecomposable in* B'.

Proof We argue by contradiction: suppose that there exists a Borel partition $\{F, G\}$ of E such that $\mathfrak{m}(F), \mathfrak{m}(G) > 0$ and $\mathsf{P}(E, B') = \mathsf{P}(F, B') + \mathsf{P}(G, B')$. Then item (i) of Lemma 2.3 implies that

$$\mathcal{H}(\partial^e F \cap \partial^e G \cap B) \leq \mathcal{H}(\partial^e F \cap \partial^e G \cap B') = 0,$$

whence $\mathsf{P}(E, B) = \mathsf{P}(F, B) + \mathsf{P}(G, B)$ by item (ii) of the same lemma. This is in contradiction with the fact that E is indecomposable in B, thus the statement is proven. □

2.2 Decomposition theorem

The aim of this subsection is to show that any set of finite perimeter in an isotropic PI space can be uniquely decomposed into indecomposable sets.

Remark 2.12 Let $\{a_i^n\}_{i,n\in\mathbb{N}} \subset (0, +\infty)$ be a sequence that satisfies $\lim_n a_i^n = a_i$ for every $i \in \mathbb{N}$ and $\lim_j \overline{\lim}_n \sum_{i=j}^{\infty} a_i^n = 0$. Then $\sum_{i=0}^{\infty} a_i = \lim_n \sum_{i=0}^{\infty} a_i^n$. Indeed, for every $j \in \mathbb{N}$ we have that

$$\begin{aligned}\sum_{i=0}^{j} a_i &= \lim_{n\to\infty} \sum_{i=0}^{j} a_i^n \leq \underline{\lim}_{n\to\infty} \sum_{i=0}^{\infty} a_i^n \leq \overline{\lim}_{n\to\infty} \sum_{i=0}^{\infty} a_i^n = \overline{\lim}_{n\to\infty} \Bigg[\sum_{i=0}^{j} a_i^n + \sum_{i>j} a_i^n\Bigg] \\ &= \sum_{i=0}^{j} a_i + \overline{\lim}_{n\to\infty} \sum_{i>j} a_i^n,\end{aligned}$$

whence by letting $j \to \infty$ we conclude that $\sum_{i=0}^{\infty} a_i \le \underline{\lim}_n \sum_{i=0}^{\infty} a_i^n \le \overline{\lim}_n \sum_{i=0}^{\infty} a_i^n \le \sum_{i=0}^{\infty} a_i$, which proves the claim. □

Proposition 2.13 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space. Let* $E \subset \mathrm{X}$ *be a set of finite perimeter. Fix* $\bar{x} \in \mathrm{X}$ *and* $r > 0$ *such that* $B_r(\bar{x})$ *has finite perimeter. Then there is a unique (in the* $\mathfrak{m}$*-a.e. sense) at most countable partition* $\{E_i\}_{i \in I}$ *of* $E \cap B_r(\bar{x})$*, into indecomposable subsets of* $B_r(\bar{x})$*, such that* $\mathsf{P}(E_i) < +\infty$*,* $\mathfrak{m}(E_i) > 0$ *for every* $i \in I$ *and* $\mathsf{P}\big(E, B_r(\bar{x})\big) = \sum_{i \in I} \mathsf{P}\big(E_i, B_r(\bar{x})\big)$*. Moreover, the sets* $\{E_i\}_{i \in I}$ *are maximal indecomposable sets, meaning that for any Borel set* $F \subset E \cap B_r(\bar{x})$ *with* $\mathsf{P}(F) < +\infty$ *that is indecomposable in* $B_r(\bar{x})$ *there is a (unique)* $i \in I$ *such that* $\mathfrak{m}(F \setminus E_i) = 0$*.*

Proof EXISTENCE. Fix an exponent $s > \max\big\{1, \log_2(C_D)\big\}$ and any $\alpha \in \big(1, \frac{s}{s-1}\big)$. For brevity, call $\Omega := B_r(\bar{x})$. For simplicity, let us set

$$\mu(B) := \mathfrak{m}(B)^{1/\alpha} \quad \text{for every Borel set } B \subset \Omega.$$

Let us denote by $\mathcal{P}$ the collection of all Borel partitions $(E_i)_{i \in \mathbb{N}}$ of $E \cap \Omega$ (up to $\mathfrak{m}$-null sets) such that $\big(\mathfrak{m}(E_i)\big)_{i \in \mathbb{N}}$ is non-increasing, $\sum_{i=0}^{\infty} \mathsf{P}(E_i, \Omega) \le \mathsf{P}(E, \Omega)$, and $\sum_{i=0}^{\infty} \mathsf{P}(E_i) \le \mathsf{P}(E) + \mathsf{P}(\Omega)$. Note that the family $\mathcal{P}$ is non-empty, as it contains the element $(E \cap \Omega, \emptyset, \emptyset, \ldots)$. Let us call

$$M := \sup \left\{ \sum_{i=0}^{\infty} \mu(E_i) \;\middle|\; \{E_i\}_{i \in \mathbb{N}} \in \mathcal{P} \right\}. \tag{2.9}$$

Choose any $\big((E_i^n)_{i \in \mathbb{N}}\big)_n \subset \mathcal{P}$ such that $\lim_n \sum_{i=0}^{\infty} \mu(E_i^n) = M$. Since $\mathsf{P}(E_i^n) \le \mathsf{P}(E) + \mathsf{P}(\Omega) < +\infty$ for every $i, n \in \mathbb{N}$, we know by the compactness properties of sets of finite perimeter that we can extract a (not relabelled) subsequence in n in such a way that the following property holds: there exists a sequence $(E_i)_{i \in \mathbb{N}}$ of Borel subsets of $E \cap \Omega$ such that $\mathbb{1}_{E_i^n} \to \mathbb{1}_{E_i}$ in $L^1(\mathfrak{m}\llcorner_\Omega)$, thus

$$\lim_{n \to \infty} \mu(E_i^n) = \mu(E_i) \quad \text{for every } i \in \mathbb{N}. \tag{2.10}$$

Given any $i, j \in \mathbb{N}$ such that $i \neq j$, we also have that $\mu(E_i \cap E_j) = \lim_n \mu(E_i^n \cap E_j^n) = 0$, thus accordingly $\mathfrak{m}(E_i \cap E_j) = 0$. Moreover, by lower semicontinuity of the perimeter we see that

$$\begin{aligned}
\sum_{i=0}^{\infty} \mathsf{P}(E_i, \Omega) &= \lim_{j \to \infty} \sum_{i=0}^{j} \mathsf{P}(E_i, \Omega) \le \lim_{j \to \infty} \sum_{i=0}^{j} \underline{\lim}_{n \to \infty} \mathsf{P}(E_i^n, \Omega) \\
&\le \lim_{j \to \infty} \underline{\lim}_{n \to \infty} \sum_{i=0}^{j} \mathsf{P}(E_i^n, \Omega) \le \mathsf{P}(E, \Omega)
\end{aligned}$$

and, similarly, that $\sum_{i=0}^{\infty} \mathsf{P}(E_i) \le \mathsf{P}(E) + \mathsf{P}(\Omega)$ for every $i \in \mathbb{N}$. To prove that $(E_i)_{i \in \mathbb{N}} \in \mathcal{P}$ it only remains to show that $\mathfrak{m}\big((E \cap \Omega) \setminus \bigcup_i E_i\big) = 0$. We claim that

$$\lim_{j \to \infty} \overline{\lim}_{n \to \infty} \sum_{i=j}^{\infty} \mu(E_i^n)^{\alpha} \le \lim_{j \to \infty} \overline{\lim}_{n \to \infty} \sum_{i=j}^{\infty} \mu(E_i^n) = 0. \tag{2.11}$$

Observe that the inequality $\mathfrak{m}(E_i^n) \le \mathfrak{m}(\Omega \setminus E_i^n)$ holds for every $i \ge 1$. Let us define

$$\eta := \frac{1}{\alpha} - \frac{s-1}{s} > 0, \qquad C := C_I \left(\frac{r^s}{\mathfrak{m}(\Omega)} \right)^{1/s-1}.$$

We readily deduce from the relative isoperimetric inequality (1.8) that for all $j \geq 1$ we have

$$\begin{aligned} j\,\mathfrak{m}(E_j^n) &\leq \sum_{i=1}^{j} \mathfrak{m}(E_i^n) \leq C \sum_{i=1}^{j} \mathsf{P}\big(E_i^n, B_{2\lambda r}(\bar{x})\big)^{s/s-1} = C \sum_{i=1}^{j} \mathsf{P}(E_i^n)^{s/s-1} \\ &= C\left[\mathsf{P}(E) + \mathsf{P}(\Omega)\right]^{s/s-1} \sum_{i=1}^{j} \left(\frac{\mathsf{P}(E_i^n)}{\mathsf{P}(E) + \mathsf{P}(\Omega)}\right)^{s/s-1} \\ &\leq C\left[\mathsf{P}(E) + \mathsf{P}(\Omega)\right]^{s/s-1} \sum_{i=1}^{j} \frac{\mathsf{P}(E_i^n)}{\mathsf{P}(E) + \mathsf{P}(\Omega)} \\ &\leq C\left[\mathsf{P}(E) + \mathsf{P}(\Omega)\right]^{s/s-1}. \end{aligned}$$

Furthermore, by using the previous estimate and again (1.14) we obtain that

$$\begin{aligned} \sum_{i=j}^{\infty} \mu(E_i^n) &= \sum_{i=j}^{\infty} \mathfrak{m}(E_i^n)^{1/\alpha} = \sum_{i=j}^{\infty} \mathfrak{m}(E_i^n)^{\eta}\, \mathfrak{m}(E_i^n)^{(s-1)/s} \leq \mathfrak{m}(E_j^n)^{\eta} \sum_{i=j}^{\infty} \mathfrak{m}(E_i^n)^{(s-1)/s} \\ &\leq \frac{C^{\eta}\left[\mathsf{P}(E) + \mathsf{P}(\Omega)\right]^{\eta s/s-1}}{j^{\eta}} \sum_{i=j}^{\infty} \mathfrak{m}(E_i^n)^{(s-1)/s} \\ &\leq \frac{C^{\eta}\left[\mathsf{P}(E) + \mathsf{P}(\Omega)\right]^{\eta s/s-1}}{j^{\eta}}\, C^{(s-1)/s} \sum_{i=j}^{\infty} \mathsf{P}\big(E_i^n, B_{2\lambda r}(\bar{x})\big) \\ &= \frac{C^{1/\alpha}\left[\mathsf{P}(E) + \mathsf{P}(\Omega)\right]^{\eta s/s-1}}{j^{\eta}} \sum_{i=j}^{\infty} \mathsf{P}(E_i^n) \\ &= \frac{C^{1/\alpha}\left[\mathsf{P}(E) + \mathsf{P}(\Omega)\right]^{\eta s/s-1+1}}{j^{\eta}} = \frac{C^{1/\alpha}\left[\mathsf{P}(E) + \mathsf{P}(\Omega)\right]^{s/\alpha(s-1)}}{j^{\eta}}. \end{aligned}$$

Consequently, we deduce that the claim (2.11) is verified. By recalling also (2.10) and Remark 2.12, we can conclude that

$$\begin{aligned} \mu\Big(\bigcup_{i\in\mathbb{N}} E_i\Big)^{\alpha} &= \mathfrak{m}\Big(\bigcup_{i\in\mathbb{N}} E_i\Big) = \sum_{i=0}^{\infty} \mathfrak{m}(E_i) = \sum_{i=0}^{\infty} \mu(E_i)^{\alpha} \\ &= \lim_{n\to\infty} \sum_{i=0}^{\infty} \mu(E_i^n)^{\alpha} = \mu(E\cap\Omega)^{\alpha}. \end{aligned}$$

This forces $\mathfrak{m}\big((E\cap\Omega)\setminus\bigcup_i E_i\big) = 0$ and accordingly $(E_i)_{i\in\mathbb{N}} \in \mathcal{P}$. Hence,

$$\sum_{i=0}^{\infty} \mu(E_i) = \lim_{n\to\infty} \sum_{i=0}^{\infty} \mu(E_i^n) = M, \tag{2.12}$$

in other words $(E_i)_{i\in\mathbb{N}}$ is a maximiser for the problem in (2.9). Finally, we claim that each set E_i is indecomposable in Ω. Suppose this was not the case: then for some $j \in \mathbb{N}$ we would find a partition $\{F, G\}$ of E_j into sets of finite perimeter having positive $\mathfrak{m}$-measure and satisfying the identity $\mathsf{P}(E_j, \Omega) = \mathsf{P}(F, \Omega) + \mathsf{P}(G, \Omega)$. We can relabel the family $\{E_i\}_{i\neq j} \cup \{F, G\}$ as $(F_i)_{i\in\mathbb{N}}$ in such a way that $\big(\mathfrak{m}(F_i)\big)_{i\in\mathbb{N}}$ is a non-increasing sequence. Given that

$$\sum_{i=0}^{\infty} \mathsf{P}(F_i, \Omega) = \sum_{i \neq j} \mathsf{P}(E_i, \Omega) + \mathsf{P}(F, \Omega) + \mathsf{P}(G, \Omega) = \sum_{i=0}^{\infty} \mathsf{P}(E_i, \Omega) \leq \mathsf{P}(E, \Omega),$$

we see that $(F_i)_{i \in \mathbb{N}} \in \mathcal{P}$. On the other hand, given that $\alpha > 1$ and $\mu(F), \mu(G) > 0$ we have the inequality $\mu(F) + \mu(G) > \mu(E_j)$, so that

$$\sum_{i=0}^{\infty} \mu(F_i) = \sum_{i \neq j} \mu(E_i) + \mu(F) + \mu(G) > \sum_{i=0}^{\infty} \mu(E_i) = M.$$

This leads to a contradiction with (2.9), whence the sets E_i are proven to be indecomposable in Ω. Therefore, the family $\{E_i\}_{i \in I}$, where $I := \{ i \in \mathbb{N} : \mathfrak{m}(E_i) > 0 \}$, satisfies the required properties.

Maximality. Let $F \subset E \cap \Omega$ be a fixed Borel set with $\mathsf{P}(F) < +\infty$ that is indecomposable in Ω. Choose an index $j \in I$ for which $\mathfrak{m}(F \cap E_j) > 0$. By Corollary 2.9 we know that

$$\mathsf{P}(F \cap E_j, \Omega) + \mathsf{P}\Big(F \cap \bigcup_{i \neq j} E_i, \Omega\Big) = \mathsf{P}(F \cap E_j, \Omega) + \sum_{i \neq j} \mathsf{P}(F \cap E_i, \Omega) = \mathsf{P}(F, \Omega).$$

Given that F is assumed to be indecomposable in Ω, we finally conclude that $F \cap \bigcup_{i \neq j} E_i$ has null $\mathfrak{m}$-measure, so that $\mathfrak{m}(F \setminus E_j) = 0$. This shows that the elements of $\{E_i\}_{i \in I}$ are maximal.

Uniqueness. Consider any other family $\{F_j\}_{j \in J}$ having the same properties as $\{E_i\}_{i \in I}$. By maximality we know that for any $i \in \mathbb{N}$ there exists a (unique) $j \in \mathbb{N}$ such that $\mathfrak{m}(E_i \Delta F_j) = 0$, thus the two partitions $\{E_i\}_{i \in I}$ and $\{F_j\}_{j \in J}$ are essentially equivalent (up to $\mathfrak{m}$-negligible sets). This proves the desired uniqueness. □

We are now ready to prove the main result of this section:

Theorem 2.14 (Decomposition theorem) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space. Let* $E \subset \mathrm{X}$ *be a set of finite perimeter. Then there exists a unique (finite or countable) partition* $\{E_i\}_{i \in I}$ *of* E *into indecomposable subsets of* X *such that* $\mathfrak{m}(E_i) > 0$ *for every* $i \in I$ *and* $\mathsf{P}(E) = \sum_{i \in I} \mathsf{P}(E_i)$*, where uniqueness has to be intended in the* $\mathfrak{m}$*-a.e. sense. Moreover, the sets* $\{E_i\}_{i \in I}$ *are maximal indecomposable sets, meaning that for any Borel set* $F \subset E$ *with* $P(F) < +\infty$ *that is indecomposable there is a (unique)* $i \in I$ *such that* $\mathfrak{m}(F \setminus E_i) = 0$.

Proof Let $\bar{x} \in \mathrm{X}$ be a fixed point. Choose a sequence of radii $r_j \nearrow +\infty$ such that $\Omega_j := B_{r_j}(\bar{x})$ has finite perimeter for all $j \in \mathbb{N}$. Let us apply Proposition 2.13: given any $j \in \mathbb{N}$, there exists an $\mathfrak{m}$-essentially unique partition $\{E_i^j\}_{i \in I_j}$ of $E \cap \Omega_j$, into sets of finite perimeter that are maximal indecomposable subsets of Ω_j, with $\mathfrak{m}(E_i^j) > 0$ for all $i \in I_j$ and $\mathsf{P}(E, \Omega_j) = \sum_{i \in I_j} \mathsf{P}(E_i^j, \Omega_j)$.

Given any $j \in \mathbb{N}$ and $i \in I_j$, we know from Lemma 2.11 that E_i^j is indecomposable in Ω_{j+1}, thus there exists $\ell \in I_{j+1}$ for which $\mathfrak{m}(E_i^j \setminus E_\ell^{j+1}) = 0$. This ensures that—possibly choosing different $\mathfrak{m}$-a.e. representatives of the sets E_i^j's under consideration—we can assume that:

$$\text{For every } j \in \mathbb{N} \text{ and } i \in I_j \text{ there exists (a unique) } \ell \in I_{j+1} \text{ such that } E_i^j \subset E_\ell^{j+1}. \tag{2.13}$$

Given any $x \in E$, let us define the set $G_x \subset E$

$$G_x := \bigcup \big\{ E_i^j \;\big|\; j \in \mathbb{N},\ i \in I_j,\ x \in E_i^j \big\}.$$

One clearly has that $\mathfrak{m}(G_x) > 0$. Moreover, it readily follows from (2.13) that

$$G_x = \bigcup \left\{ E_i^j \mid j \in \mathbb{N},\ j \geq j_0,\ i \in I_j,\ x \in E_i^j \right\} \quad \text{for every } j_0 \in \mathbb{N}. \tag{2.14}$$

We claim that:

$$\text{For every } x, y \in E \text{ it holds that either } G_x \cap G_y = \emptyset \text{ or } G_x = G_y. \tag{2.15}$$

In order to prove it, assume that $G_x \cap G_y \neq \emptyset$ and pick any $z \in G_x \cap G_y$. Then there exist some indices $j_x, j_y \in \mathbb{N}$, $i_x \in I_{j_x}$ and $i_y \in I_{j_y}$ such that $\{x, z\} \subset E_{i_x}^{j_x}$ and $\{y, z\} \subset E_{i_y}^{j_y}$. Possibly interchanging x and y, we can suppose that $j_y \leq j_x$. Given that $E_{i_x}^{j_x} \cap E_{i_y}^{j_y}$ is not empty (as it contains z), we infer from (2.13) that $E_{i_y}^{j_y} \subset E_{i_x}^{j_x}$. Consequently, property (2.14) ensures that the sets G_x and G_y coincide, thus proving the claim (2.15).

Let us define $\mathcal{F} := \{G_x \,:\, x \in E\}$. It turns out that the family $\mathcal{F}$ is at most countable: the map sending each element (j, i) of $\bigsqcup_{j \in \mathbb{N}} I_j$ to the unique element of $\mathcal{F}$ containing E_i^j is clearly surjective. Then rename $\mathcal{F}$ as $\{E_i\}_{i \in I}$. Observe that $\{E_i\}_{i \in I}$ constitutes a Borel partition of E. Now fix $i \in I$. We can choose $j(i) \in \mathbb{N}$ and $\ell(i, j) \in I_j$ for all $j \geq j(i)$ such that $E_i = \bigcup_{j \geq j(i)} E_{\ell(i,j)}^j$. Let us also call

$$F_i^j := \begin{cases} \emptyset & \text{if} \quad j < j(i), \\ E_{\ell(i,j)}^j & \text{if} \quad j \geq j(i). \end{cases}$$

Therefore, $E_i = \bigcup_{j \in \mathbb{N}} F_i^j$. Given any $j \in \mathbb{N}$, we have $\mathsf{P}(E_i \cap \Omega_j, \Omega_j) = \mathsf{P}(E_i, \Omega_j)$ as Ω_j is open. Then $\mathsf{P}(E_i, \Omega_j) = \mathsf{P}(E_i \cap \Omega_j, \Omega_j) = \mathsf{P}(F_i^j, \Omega_j) \leq \mathsf{P}(E, \Omega_j) \leq \mathsf{P}(E)$ holds for every $j \geq j(i)$, so that

$$\mathsf{P}(E_i) = \lim_{j \to \infty} \mathsf{P}(E_i, \Omega_j) \leq \mathsf{P}(E).$$

This shows that the sets $\{E_i\}_{i \in I}$ have finite perimeter, while the fact that they are indecomposable follows from Proposition 2.10. Now fix any finite subset J of I. Similarly to the estimates above, we see that for every $j \geq \max \left\{ j(i) \,:\, i \in J \right\}$ it holds that

$$\begin{aligned} \sum_{i \in J} \mathsf{P}(E_i, \Omega_j) &= \sum_{i \in J} \mathsf{P}(E_i \cap \Omega_j, \Omega_j) = \sum_{i \in J} \mathsf{P}(F_i^j, \Omega_j) \leq \sum_{\ell \in I_j} \mathsf{P}(E_\ell^j, \Omega_j) \\ &\leq \mathsf{P}(E, \Omega_j) \leq \mathsf{P}(E), \end{aligned}$$

whence $\sum_{i \in J} \mathsf{P}(E_i) = \lim_j \sum_{i \in J} \mathsf{P}(E_i, \Omega_j) \leq \mathsf{P}(E)$. By arbitrariness of $J \subset I$ this yields the inequality $\sum_{i \in I} \mathsf{P}(E_i) \leq \mathsf{P}(E)$, thus accordingly $\mathsf{P}(E) = \sum_{i \in I} \mathsf{P}(E_i)$ by Remark 2.2. Finally, maximality and uniqueness can be proven by arguing exactly as in Proposition 2.13. Therefore, the statement is achieved. □

Definition 2.15 (*Essential connected components*) Let $(\mathrm{X}, d, \mathfrak{m})$ be an isotropic PI space. Let us fix a set $E \subset \mathrm{X}$ of finite perimeter. Then we denote by

$$\mathcal{CC}^e(E) := \{E_i\}_{i \in I}$$

the decomposition of E provided by Theorem 2.14. (We assume the index set is either $I = \mathbb{N}$ or $I = \{0, \ldots, n\}$ for some $n \in \mathbb{N}$.) The sets E_i are called the *essential connected components* of E.

Example 2.16 Although we do not know if the Decomposition Theorem 2.14 holds without the assumption on isotropicity, one can see that the assumption on (1, 1)-Poincaré inequality cannot be relaxed to a $(1, p)$-Poincaré inequality with $p > 1$. As an example of this, one can take a fat Sierpiński carpet $S_{\mathbf{a}} \subset [0, 1]^2$ with a sequence $\mathbf{a} \in \ell^2 \backslash \ell^1$, as defined in [35]. The set $S_{\mathbf{a}}$, equipped with a natural measure $\mathfrak{m}$ and distance d, is a 2-Ahlfors-regular metric measure space supporting a $(1, p)$-Poincaré inequality for all exponents $p > 1$. Nevertheless, given any vertical strip of the form $I_{x,\varepsilon} := (x-\varepsilon, x+\varepsilon) \times [0, 1]$, where $x = \sum_{i=1}^n x_i\, 3^{-i} + 2^{-1}\, 3^{-n}$ with $x_i \in \{0, 1, 2\}$ and $n \in \mathbb{N}$, we have $\mathfrak{m}(I_{x,\varepsilon})/\varepsilon \to 0$ as $\varepsilon \to 0$. Thus, any set of finite perimeter $E \subset S_{\mathbf{a}}$ can be decomposed into the union of $E \cap \big([0, x] \times [0, 1]\big)$ and $E \cap \big([x, 1] \times [0, 1]\big)$. Since the family of coordinates x for which this holds is dense in [0, 1], no set of positive measure in $S_{\mathbf{a}}$ can be decomposed into countably many indecomposable sets. □

Remark 2.17 Given an isotropic PI space and a set $E \subset \mathrm{X}$ of finite perimeter, it holds that

$$\mathcal{H}(\partial^e F \backslash \partial^e E) = 0 \quad \text{for every } F \in \mathcal{CC}^e(E). \tag{2.16}$$

This property is an immediate consequence of Lemma 1.24. □

Proposition 2.18 (Stability of indecomposable sets, II) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space. Fix two indecomposable sets* $E, F \subset \mathrm{X}$. *Suppose that either* $\mathfrak{m}(E \cap F) > 0$ *or* $\mathcal{H}(\partial^e E \cap \partial^e F) > 0$. *Then* $E \cup F$ *is an indecomposable set.*

Proof Denote $\mathcal{CC}^e(E \cup F) = \{G_i\}_{i \in I}$. Choose $i, j \in I$ such that $\mathfrak{m}(E \backslash G_i) = \mathfrak{m}(F \backslash G_j) = 0$, whose existence is granted by the maximality of the connected components of $E \cup F$. If $\mathfrak{m}(E \cap F) > 0$ then $i = j$, whence $\mathcal{CC}^e(E \cup F) = \{E \cup F\}$ and accordingly $E \cup F$ is indecomposable. Otherwise, we have $i \neq j$ and $\mathcal{CC}^e(E \cup F) = \{E, F\}$, so that $\mathcal{H}(\partial^e E \cap \partial^e F) = 0$ by item (i) of Lemma 2.3. □

Remark 2.19 Let us highlight the two main technical differences between the proofs we carried out in this section and the corresponding ones for $\mathbb{R}^n$ that were originally presented in [4]:

(i) There exist isotropic PI spaces X where it is possible to find a set of finite perimeter E whose associated perimeter measure $\mathsf{P}(E, \cdot)$ is not concentrated on $E^{1/2}$. For instance, consider the space described in Example 1.29: it is an isotropic PI space where (1.18) fails, thus in particular property (1.19) is not verified (as a consequence of Lemma 1.30). Some of the results of [4]—which have a counterpart in this paper—are proven by using property (1.19). Consequently, the approaches we followed to prove some of the results of this section provide new proofs even in the Euclidean setting.

(ii) An essential ingredient in the proof of the decomposition theorem [4, Theorem 1] is the (global) isoperimetric inequality. In our case, we only have the relative isoperimetric inequality at disposal, thus we need to 'localise' the problem: first we prove a local version of the decomposition theorem (namely, Proposition 2.13), then we obtain the full decomposition by means of a 'patching argument' (as described in the proof of Theorem 2.14). Let us point out that in the Ahlfors-regular case the proof of the decomposition theorem would closely follows along the lines of [4, Theorem 1] (thanks to Theorem 1.18).

Finally, an alternative proof of the decomposition theorem will be provided in Sect. 4. □

3 Extreme points in the space of BV functions

The aim of Sect. 3.1 is to study the extreme points of the 'unit ball' in the space of BV functions over an isotropic PI space (with a uniform bound on the support). More precisely, given an isotropic PI space $(\mathrm{X}, d, \mathfrak{m})$ and a compact set $K \subset \mathrm{X}$, we will detect the extreme points of the convex set made of all functions $f \in \mathrm{BV}(\mathrm{X})$ such that $\mathrm{spt}(f) \subset K$ and $|Df|(\mathrm{X}) \leq 1$, with respect to the strong topology of $L^1(\mathfrak{m})$; cf. Theorem 3.8. Informally speaking, the extreme points coincide—at least under some further assumptions—with the (suitably normalised) characteristic functions of *simple sets*, whose definition is given in Definition 3.1. In Sect. 3.2 we provide an alternative characterisation of simple sets (cf. Theorem 3.17) in the framework of Alhfors-regular spaces, a key role being played by the concept of *saturation* of a set, whose definition relies upon the decomposition properties treated in Sect. 2.

3.1 Simple sets and extreme points in BV

A set of finite perimeter $E \subset \mathbb{R}^n$ having finite Lebesgue measure is a simple set provided one of the following (equivalent) properties is satisfied:

(i) E is indecomposable and saturated, the latter term meaning that the complement of E does not have essential connected components of finite Lebesgue measure.
(ii) Both E and $\mathbb{R}^n \setminus E$ are indecomposable.
(iii) If $F \subset \mathbb{R}^n$ is a set of finite perimeter such that $\partial^e F$ is essentially contained in $\partial^e E$ (with respect to the $(n-1)$-dimensional Hausdorff measure) and $\mathcal{L}^n(F) < +\infty$, then it holds that $F = E$ (up to $\mathcal{L}^n$-null sets).

We refer to [4, Section 5] for a discussion about the equivalence of the above conditions. In the more general setting of isotropic PI spaces, (the appropriate reformulations of) these three notions are no longer equivalent. The one that well captures the property we are interested in (i.e., the fact of providing an alternative characterisation of the extreme points in BV) is item (iii), which accordingly is the one that we choose as the definition of simple set in our context:

Definition 3.1 (*Simple sets*) Let $(\mathrm{X}, d, \mathfrak{m})$ be a PI space. Let $E \subset \mathrm{X}$ be a set of finite perimeter with $\mathfrak{m}(E) < +\infty$. Then we say that E is a *simple set* provided for every set $F \subset \mathrm{X}$ of finite perimeter with $\mathcal{H}(\partial^e F \setminus \partial^e E) = 0$ it holds $\mathfrak{m}(F) = 0$, $\mathfrak{m}(F^c) = 0$, $\mathfrak{m}(F \Delta E) = 0$, or $\mathfrak{m}(F \Delta E^c) = 0$.

It is rather easy to prove that—under some additional assumptions—the definition of simple set we have just proposed is equivalent to (the suitable rephrasing of) item (ii) above:

Proposition 3.2 (Indecomposability of simple sets) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space. Let us consider a set* $E \subset \mathrm{X}$ *of finite perimeter such that* $\mathfrak{m}(E) < +\infty$. *Then:*

(i) *If E is a simple set, then E and E^c are indecomposable.*
(ii) *Suppose that* $(\mathrm{X}, d, \mathfrak{m})$ *has the two-sidedness property. If E and E^c are indecomposable, then E is a simple set.*

Proof (i) Assume $E \subset \mathrm{X}$ is a simple set. First, we prove by contradiction that E is indecomposable: suppose it is not, thus it can be written as $E = F \cup G$ for some pairwise disjoint sets

F, G of finite perimeter such that $\mathfrak{m}(F), \mathfrak{m}(G) > 0$ and $\mathsf{P}(E) = \mathsf{P}(F)+\mathsf{P}(G)$. By combining item (iv) of Proposition 1.16 with item (ii) of Lemma 2.3, we obtain that

$$\mathcal{H}\big((\partial^e F \cup \partial^e G)\backslash \partial^e E\big) \le \mathcal{H}\big(\partial^e F \cap \partial^e G\big) = 0.$$

In particular, we have that $\mathcal{H}(\partial^e F\backslash\partial^e E) = \mathcal{H}(\partial^e G\backslash\partial^e E) = 0$. Being E simple, we get $F = G = E$, which leads to a contradiction. Then E is indecomposable. In order to show that also E^c is indecomposable, we argue in a similar way: suppose $E^c = F' \cup G'$ for pairwise disjoint sets F', G' of finite perimeter with $\mathfrak{m}(F'), \mathfrak{m}(G') > 0$ and $\mathsf{P}(E^c) = \mathsf{P}(F')+\mathsf{P}(G')$. By arguing as before we obtain that $\mathcal{H}(\partial^e E^c\backslash\partial^e F') = \mathcal{H}(\partial^e E^c\backslash\partial^e F') = 0$. Being $\partial^e E^c = \partial^e E$, we can conclude (again since E is simple) that $F' = G' = E^c$, whence the contradiction. Therefore, E^c is indecomposable.

(ii) Assume that $(\mathrm{X}, d, \mathfrak{m})$ has the two-sidedness property and that E, E^c are indecomposable sets. Take a set $F \subset \mathrm{X}$ of finite perimeter such that $\mathcal{H}(\partial^e F\backslash\partial^e E) = 0$. We know from (1.18) that

$$\mathcal{H}\big(\partial^e(E\cap F)\cap\partial^e(E\backslash F)\cap\partial^e E\big) = 0 = \mathcal{H}\big(\partial^e(E^c\cap F)\cap\partial^e(E^c\backslash F)\cap\partial^e E^c\big).$$

Consequently, we deduce that

$$\mathcal{H}\big(\partial^e(E\cap F)\cap\partial^e(E\backslash F)\big) \le \mathcal{H}(\partial^e(E\cap F)\backslash\partial^e E) = \mathcal{H}(\partial^e F\backslash\partial^e E) = 0,$$
$$\mathcal{H}\big(\partial^e(E^c\cap F)\cap\partial^e(E^c\backslash F)\big) \le \mathcal{H}(\partial^e(E^c\cap F)\backslash\partial^e E^c) = \mathcal{H}(\partial^e F\backslash\partial^e E) = 0.$$

Then item (ii) of Lemma 2.3 yields $\mathsf{P}(E) = \mathsf{P}(E\cap F)+\mathsf{P}(E\backslash F)$ and $\mathsf{P}(E^c) = \mathsf{P}(E^c\cap F)+\mathsf{P}(E^c\backslash F)$. Being E (resp. E^c) indecomposable, we conclude that either $\mathfrak{m}(E\cap F) = 0$ or $\mathfrak{m}(E\backslash F) = 0$ (resp. either $\mathfrak{m}(E^c\cap F) = 0$ or $\mathfrak{m}(E^c\backslash F) = 0$). This implies that $\mathfrak{m}(F) = 0$, $\mathfrak{m}(F^c) = 0$, $\mathfrak{m}(F\Delta E) = 0$ or $\mathfrak{m}(F\Delta E^c) = 0$, thus proving that E is a simple set. □

Remark 3.3 In item (ii) of Proposition 3.2, the additional assumption on the space cannot be dropped. To show it, let us consider the closed unit ball X_1 centered at o of the 3-spider $(S_3, d, \mathfrak{m})$. We claim that the conclusion of item (ii) of Proposition 3.2 fails in $(\mathrm{X}_1, d_{\mathrm{X}_1\times\mathrm{X}_1}, \mathfrak{m}\llcorner_{\mathrm{X}_1})$.

By Example 1.29 we know that the two-sidedness property is not satisfied. Now call R_1', R_2', R_3' the intersections of the rays of S_3 with X_1. It thus holds that $R_1'\cup R_2'$ and $\mathrm{X}_1\backslash(R_1'\cup R_2') = R_3'\cup\{o\}$ are indecomposable, but the set $R_1'\cup R_2'$ is not simple. □

Let $(\mathrm{X}, d, \mathfrak{m})$ be a metric measure space. Let $K \subset \mathrm{X}$ be a compact set. Then we define

$$\mathcal{K}(\mathrm{X}; K) := \big\{ f \in \mathrm{BV}(\mathrm{X}) \;\big|\; \mathrm{spt}(f) \subset K,\ |Df|(\mathrm{X}) \le 1 \big\}.$$

Remark 3.4 It holds that

$$\mathcal{K}(\mathrm{X}; K) \text{ is a convex, compact subset of } L^1(\mathfrak{m}).$$

First of all, its convexity is granted by item (ii) of Proposition 1.2. To prove compactness, fix any sequence $(f_n)_n \subset \mathcal{K}(\mathrm{X}; K)$. Item (iii) of Proposition 1.2 says that $f_{n_i} \to f$ in $L^1_{\mathrm{loc}}(\mathrm{X})$ for some subsequence $(n_i)_i$ and some limit function $f \in L^1_{\mathrm{loc}}(\mathfrak{m})$. Given that $\mathrm{spt}(f_n) \subset K$ for every $n \in \mathbb{N}$, we know that $\mathrm{spt}(f) \subset K$, thus $f \in L^1(\mathfrak{m})$ and $f_{n_i} \to f$ in $L^1(\mathfrak{m})$. Finally, by using item (i) of Proposition 1.2 we conclude that $|Df|(\mathrm{X}) \le \underline{\lim}_i |Df_{n_i}|(\mathrm{X}) \le 1$, whence $f \in \mathcal{K}(\mathrm{X}; K)$. □

Recall that $\mathrm{ext}\,\mathcal{K}(\mathrm{X}; K)$ stands for the set of all extreme points of $\mathcal{K}(\mathrm{X}; K)$; cf. "Appendix A". Furthermore, observe that $|Df|(\mathrm{X}) = 1$ holds for every $f \in \mathrm{ext}\,\mathcal{K}(\mathrm{X}; K)$. In the remaining part of this subsection, we shall study in detail the family ext $\mathcal{K}(\mathrm{X};K)$. Our arguments are strongly inspired by the ideas of the papers [23,24].

Let $(\mathrm{X},d,\mathfrak{m})$ be a PI space. Given a set $E\subset\mathrm{X}$ of finite perimeter satisfying $0<\mathfrak{m}(E)<+\infty$ and $\mathfrak{m}(E^c)>0$ (so that $\mathsf{P}(E)>0$), let us define

$$\Phi_\pm(E):=\pm\frac{\mathbb{1}_E}{\mathsf{P}(E)}\in\mathrm{BV}(\mathrm{X}).$$

Observe that $\big|D\Phi_+(E)\big|(\mathrm{X})=\big|D\Phi_-(E)\big|(\mathrm{X})=1$. For any compact set $K\subsetneq\mathrm{X}$, we define

$$\begin{aligned}\mathcal{F}(\mathrm{X};K)&:=\big\{\Phi_\pm(E)\;\big|\;E\subset K\text{ is a set of finite perimeter with }\mathfrak{m}(E)>0\big\},\\ \mathcal{I}(\mathrm{X};K)&:=\big\{\Phi_\pm(E)\;\big|\;E\subset K\text{ is an indecomposable set with }\mathfrak{m}(E)>0\big\},\\ \mathcal{S}(\mathrm{X};K)&:=\big\{\Phi_\pm(E)\;\big|\;E\subset K\text{ is a simple set with }\mathfrak{m}(E)>0\big\}.\end{aligned}$$

Observe that $\mathcal{S}(\mathrm{X};K),\mathcal{I}(\mathrm{X};K)\subset\mathcal{F}(\mathrm{X};K)\subset\mathcal{K}(\mathrm{X};K)$. Given any function $f\in\mathcal{F}(\mathrm{X};K)$, we shall denote by $E_f\subset\mathrm{X}$ the ($\mathfrak{m}$-a.e. unique) Borel set satisfying either $f=\Phi_+(E_f)$ or $f=\Phi_-(E_f)$. If, in addition, the space $(\mathrm{X},d,\mathfrak{m})$ is isotropic, then $\mathcal{S}(\mathrm{X};K)\subset\mathcal{I}(\mathrm{X};K)$ by item (i) of Proposition 3.2.

Proposition 3.5 *Let $(\mathrm{X},d,\mathfrak{m})$ be a PI space and $K\subsetneq\mathrm{X}$ a compact set. Then the closed convex hull of the set $\mathcal{F}(\mathrm{X};K)$ coincides with $\mathcal{K}(\mathrm{X};K)$.*

Proof We aim to show that any function $f\in\mathcal{K}(\mathrm{X};K)$ can be approximated in $L^1(\mathfrak{m})$ by convex combinations of elements in $\mathcal{F}(\mathrm{X};K)$. Let us apply Lemma 1.21: we can find a sequence $(f_n)_n$ of simple BV functions supported on the set K, say $f_n=\sum_{i=1}^{k_n}\lambda_i^n\,\mathbb{1}_{E_i^n}$, so that $f_n\to f$ in $L^1(\mathfrak{m})$ and $\sum_{i=1}^{k_n}|\lambda_i^n|\,\mathsf{P}(E_i^n)\leq 1$ (recall Remark 1.22). Given that we have $\Phi_{\mathrm{sgn}(\lambda_i^n)}(E_i^n)\in\mathcal{F}(\mathrm{X};K)$ and

$$\frac{f_n}{q}=\sum_{i=1}^{k_n}\frac{|\lambda_i^n|\,\mathsf{P}(E_i^n)}{q}\,\Phi_{\mathrm{sgn}(\lambda_i^n)}(E_i^n),\quad\text{where we set }q:=\sum_{i=1}^{k_n}|\lambda_i^n|\,\mathsf{P}(E_i^n)\in[0,1],$$

we conclude that the functions f_n/q belong to the convex hull of $\mathcal{F}(\mathrm{X};K)$. Given that $\mathcal{F}(\mathrm{X};K)$ is symmetric, we know that its convex hull contains the function 0 and accordingly also all the functions f_n. The statement follows. □

Lemma 3.6 *Let $(\mathrm{X},d,\mathfrak{m})$ be a PI space and let $K\subsetneq\mathrm{X}$ be a compact set. Then it holds that*

$$\big\{\lambda f\;\big|\;\lambda\in[-1,1],\;f\in\mathcal{F}(\mathrm{X};K)\big\}\quad\textit{is strongly closed in }L^1(\mathfrak{m}).$$

Proof Let us call $\mathcal{B}:=\big\{\lambda f\,:\,\lambda\in[-1,1],\;f\in\mathcal{F}(\mathrm{X};K)\big\}$. Fix a sequence $(f_n)_n\subset\mathcal{B}$ converging to some function $f\in L^1(\mathfrak{m})$ in $L^1(\mathfrak{m})$. We aim to show that $f\in\mathcal{B}$ as well. Given any $n\in\mathbb{N}$, we can find $\lambda_n\in[-1,1]$ and a set of finite perimeter $E_n\subset K$ such that $\mathfrak{m}(E_n)>0$ and $f_n=\lambda_n\Phi_+(E_n)$. We subdivide the proof into three different cases:

(1) Suppose $\lim_n\mathsf{P}(E_n)=0$. Then there exists a set of finite perimeter $E\subset K$ such that (up to a not relabelled subsequence) it holds $\mathbb{1}_{E_n}\to\mathbb{1}_E$ in $L^1(\mathfrak{m})$. In particular, $\mathsf{P}(E)\leq\lim_n\mathsf{P}(E_n)=0$ and accordingly $\lim_n\mathfrak{m}(E_n)=\mathfrak{m}(E)=0$. Possibly passing to a further subsequence, we may thus assume that $\mathfrak{m}(E_n)<1/2^n$ for all $n\in\mathbb{N}$. Since the identity $f_k(x)=0$ holds for every $k\geq n>0$ and $x\in K\setminus\bigcup_{m\geq n}E_m$, we deduce that $\lim_k f_k(x)=0$ for every $x\in K\setminus\bigcap_n\bigcup_{m\geq n}E_m$. This implies that $f=0\in\mathcal{B}$, as the set $\bigcap_n\bigcup_{m\geq n}E_m$ is $\mathfrak{m}$-negligible by Borel–Cantelli lemma.

(2) Suppose that $\lim_n \mathsf{P}(E_n) = +\infty$. Then it holds that

$$\varlimsup_{n\to\infty} \int |f_n| \, \mathrm{d}\mathfrak{m} \le \varlimsup_{n\to\infty} \frac{|\lambda_n| \, \mathfrak{m}(E_n)}{\mathsf{P}(E_n)} \le \varlimsup_{n\to\infty} \frac{\mathfrak{m}(K)}{\mathsf{P}(E_n)} = 0,$$

whence accordingly $f = 0 \in \mathcal{B}$.

(3) Suppose $\varliminf_n \mathsf{P}(E_n) > 0$ and $\varlimsup_n \mathsf{P}(E_n) < +\infty$. Then there exist $\lambda \in [-1, 1]$ and $c \in (0, +\infty)$ such that—up to a not relabelled subsequence—one has that $\lambda_n \to \lambda$ and $\lim_n \mathsf{P}(E_n) = c$. We can further assume that $\mathbb{1}_{E_n} \to \mathbb{1}_E$ strongly in $L^1(\mathfrak{m})$, for some set of finite perimeter $E \subset K$. Therefore, we deduce that $f = \lambda \mathbb{1}_E / c$. If $\mathfrak{m}(E) = 0$, then $f = 0 \in \mathcal{B}$. If $\mathfrak{m}(E) > 0$, then we can write f as $\lambda' \, \Phi_+(E)$, where we set $\lambda' := \lambda \, \mathsf{P}(E)/c$. Since $\mathsf{P}(E) \le \lim_n \mathsf{P}(E_n) = c$ by lower semicontinuity of the perimeter, we conclude that $\lambda' \in [-1, 1]$ and accordingly $f \in \mathcal{B}$. □

Theorem 3.7 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a PI space and let* $K \subsetneq \mathrm{X}$ *be a compact set. Then it holds that*

$$\mathrm{ext}\, \mathcal{K}(\mathrm{X}; K) \subset \mathcal{I}(\mathrm{X}; K).$$

Proof By Milman Theorem A.1, Proposition 3.5, and Lemma 3.6, we know that any extreme point of $\mathcal{K}(\mathrm{X}; K)$ can be written as λf for some $\lambda \in [-1, 1]$ and $f \in \mathcal{F}(\mathrm{X}; K)$. Moreover, it is clear that $\lambda f \notin \mathrm{ext}\, \mathcal{K}(\mathrm{X}; K)$ for every $\lambda \in (-1, 1)$ and $f \in \mathcal{F}(\mathrm{X}; K)$, since $\lambda f = \frac{1+\lambda}{2} f + \frac{1-\lambda}{2}(-f)$. This shows that $\mathrm{ext}\, \mathcal{K}(\mathrm{X}; K) \subset \mathcal{F}(\mathrm{X}; K)$. It only remains to prove that if $f = \Phi_\sigma(E) \in \mathrm{ext}\, \mathcal{K}(\mathrm{X}; K)$, then E is indecomposable. We argue by contradiction: suppose the set E is decomposable, so that there exist disjoint Borel sets $F, G \subset E$ such that $\mathfrak{m}(F), \mathfrak{m}(G) > 0$ and $\mathsf{P}(E) = \mathsf{P}(F) + \mathsf{P}(G)$. Therefore, we can write

$$\Phi_\sigma(E) = \frac{\mathsf{P}(F)}{\mathsf{P}(E)} \Phi_\sigma(F) + \frac{\mathsf{P}(G)}{\mathsf{P}(E)} \Phi_\sigma(G).$$

This contradicts the fact that f is an extreme point of $\mathcal{K}(\mathrm{X}; K)$, thus the set E is proven to be indecomposable. Hence, we have that $\mathrm{ext}\, \mathcal{K}(\mathrm{X}; K) \subset \mathcal{I}(\mathrm{X}; K)$, as required. □

Theorem 3.8 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space. Let* $K \subsetneq \mathrm{X}$ *be a compact set. Then:*

(i) *It holds that*

$$\mathcal{S}(\mathrm{X}; K) \subset \mathrm{ext}\, \mathcal{K}(\mathrm{X}; K). \tag{3.1}$$

(ii) *Suppose that* $(\mathrm{X}, d, \mathfrak{m})$ *has the two-sidedness property. Suppose also that* K *has finite perimeter, that* $\mathcal{H}(\partial K \setminus \partial^e K) = 0$ *and that* K^c *is connected. Then* $\mathcal{S}(\mathrm{X}; K) = \mathrm{ext}\, \mathcal{K}(\mathrm{X}; K)$.

Proof (i) Let $f \in \mathcal{S}(\mathrm{X}; K)$ be fixed. Thanks to Choquet Theorem A.2, there exists a Borel probability measure μ on $L^1(\mathfrak{m})$, concentrated on $\mathrm{ext}\, \mathcal{K}(\mathrm{X}; K)$, such that

$$\int f \, \varphi \, \mathrm{d}\mathfrak{m} = \iint g \, \varphi \, \mathrm{d}\mathfrak{m} \, \mathrm{d}\mu(g) \quad \text{for every } \varphi \in L^\infty(\mathfrak{m}). \tag{3.2}$$

We claim that, given any Borel set $B \subset \mathrm{X}$, the functional $\Psi_B \colon L^1(\mathfrak{m}) \to [0, +\infty)$ given by

$$\Psi_B(g) := \mathbb{1}_{\mathrm{BV}(\mathrm{X})}(g) \, |Dg|(B) \quad \text{for every } g \in L^1(\mathfrak{m})$$

is Borel measurable. To prove it, call $\mathcal{D}$ the family of all Borel sets $B \subset \mathrm{X}$ such that Ψ_B is a Borel measurable function. Observe that:

(a) $\mathrm{X} \in \mathcal{D}$ by item (i) of Proposition 1.2.
(b) If $A, B \in \mathcal{D}$ satisfy $A \subset B$, then $|Dg|(B \setminus A) = |Dg|(B) - |Dg|(A)$ for all $g \in \mathrm{BV}(\mathrm{X})$. This implies that $\Psi_{B\setminus A} = \Psi_B - \Psi_A$ is Borel measurable and thus $B \setminus A \in \mathcal{D}$.
(c) Given any increasing sequence $(A_n)_n \subset \mathcal{D}$, we have that $|Dg|\big(\bigcup_n A_n\big) = \lim_n |Dg|(A_n)$ for all $g \in L^1(\mathfrak{m})$ thanks to the continuity from below, whence accordingly $\Psi_{\cup_n A_n}$ is Borel measurable (so that $\bigcup_n A_n \in \mathcal{D}$) as it is the pointwise limit of Φ_{A_n} as $n \to \infty$.

All in all, we have proven that $\mathcal{D}$ is a Dynkin system. Given that the topology of (X, d) is contained in $\mathcal{D}$ (again by item (i) of Proposition 1.2), we conclude that $\mathcal{D}$ coincides with the Borel σ-algebra of X by the Dynkin π-λ Theorem. This proves that Ψ_B is Borel for any $B \subset \mathrm{X}$ Borel, as claimed.

With this said, it makes sense to define the Borel measure ν on X as $\nu := \int |Dg| \, \mathrm{d}\mu(g)$, namely

$$\nu(B) = \int |Dg|(B) \, \mathrm{d}\mu(g) \quad \text{for every Borel set } B \subset \mathrm{X}.$$

Given that $|Dg|(\mathrm{X}) = 1$ for every $g \in \operatorname{ext} \mathcal{K}(\mathrm{X}; K)$, we know that $|Dg|(\mathrm{X}) = 1$ for μ-a.e. $g \in L^1(\mathfrak{m})$ and accordingly ν is a probability measure. Now fix any open set $\Omega \subset \mathrm{X}$ containing $\partial^e E_f$. Thanks to Theorem 1.4, we can find a sequence of derivations $(\boldsymbol{b}_n)_n \subset \mathrm{Der_b}(\mathrm{X})$ such that $|\boldsymbol{b}_n| \leq 1$ in the $\mathfrak{m}$-a.e. sense, $\operatorname{spt}(\boldsymbol{b}_n) \Subset \Omega$ and $\int_\Omega f \operatorname{div}(\boldsymbol{b}_n) \, \mathrm{d}\mathfrak{m} \to |Df|(\Omega)$. Therefore, it holds that

$$|Df|(\Omega) \overset{(3.2)}{=} \lim_{n\to\infty} \iint_\Omega g \operatorname{div}(\boldsymbol{b}_n) \, \mathrm{d}\mathfrak{m} \, \mathrm{d}\mu(g) \leq \int |Dg|(\Omega) \, \mathrm{d}\mu(g) = \nu(\Omega). \tag{3.3}$$

Given that $|Df|$ and ν are outer regular, we can pick a sequence $(\Omega_n)_n$ of open subsets of X containing $\partial^e E_f$ such that $|Df|(\partial^e E_f) = \lim_n |Df|(\Omega_n)$ and $\nu(\partial^e E_f) = \lim_n \nu(\Omega_n)$. By recalling the inequality (3.3), we thus obtain that

$$1 = \frac{\mathsf{P}(E_f, \partial^e E_f)}{\mathsf{P}(E_f)} = |Df|(\partial^e E_f) = \lim_{n\to\infty} |Df|(\Omega_n) \leq \varliminf_{n\to\infty} \nu(\Omega_n) = \nu(\partial^e E_f) = 1.$$

This forces the equality $\int |Dg|(\partial^e E_f) \, \mathrm{d}\mu(g) = \nu(\partial^e E_f) = 1$. Given that $|Dg|(\partial^e E_f) \leq 1$ holds for μ-a.e. $g \in L^1(\mathfrak{m})$, we infer that actually $|Dg|(\partial^e E_f) = 1$ for μ-a.e. $g \in L^1(\mathfrak{m})$. Since $g \in \mathcal{I}(\mathrm{X}; K)$ for μ-a.e. $g \in L^1(\mathfrak{m})$ by Theorem 3.7, it makes sense to consider E_g for μ-a.e. $g \in L^1(\mathfrak{m})$. Therefore, we have that

$$\begin{aligned}(\theta\mathcal{H})(\partial^e E_g \setminus \partial^e E_f) &= \mathsf{P}\big(E_g, (\partial^e E_f)^c\big) = \mathsf{P}(E_g)\left(1 - \frac{\mathsf{P}(E_g, \partial^e E_f)}{\mathsf{P}(E_g)}\right) \\ &= \mathsf{P}(E_g)\big(1 - |Dg|(\partial^e E_f)\big) = 0\end{aligned}$$

holds for μ-a.e. $g \in L^1(\mathfrak{m})$. This implies that $\mathcal{H}(\partial^e E_g \setminus \partial^e E_f) = 0$ for μ-a.e. $g \in L^1(\mathfrak{m})$. Since E_f is a simple set, we deduce that $\mathfrak{m}(E_g \Delta E_f) = 0$ for μ-a.e. $g \in L^1(\mathfrak{m})$. This forces $\mu = t\,\delta_f + (1-t)\,\delta_{-f}$ for some $t \in [0,1]$. Given that μ is concentrated on the symmetric set $\operatorname{ext} \mathcal{K}(\mathrm{X}; K)$, we finally conclude that $f \in \operatorname{ext} \mathcal{K}(\mathrm{X}; K)$, as required. This proves the inclusion (3.1).

(ii) Let $f \in \operatorname{ext} \mathcal{K}(\mathrm{X}; K)$ be fixed. Take a set $F \subset \mathrm{X}$ of finite perimeter with $\mathcal{H}(\partial^e F \setminus \partial^e E_f) = 0$. We claim that either $\mathfrak{m}(F \setminus K) = 0$ or $\mathfrak{m}(F^c \setminus K) = 0$. To prove it, notice that (1.7), (1.18) give

$$\begin{aligned}\mathcal{H}\big(\partial^e(F\setminus K) \cap \partial^e(F^c \setminus K)\big) &\leq \mathcal{H}\big((\partial^e F \cup \partial^e K)\setminus K\big) + \mathcal{H}\big(\partial^e(F\setminus K) \cap \partial^e(F^c\setminus K) \cap \partial K\big) \\ &\leq \mathcal{H}\big(\partial^e F \setminus \partial^e E_f\big) + \mathcal{H}\big(\partial^e(F\setminus K) \cap \partial^e(F^c\setminus K) \cap \partial^e K\big) = 0.\end{aligned}$$

Accordingly, item (ii) of Lemma 2.3 yields $\mathsf{P}(K^c) = \mathsf{P}(F\backslash K) + \mathsf{P}(F^c\backslash K)$. Being K^c indecomposable by Corollary 2.7, we conclude that either $\mathfrak{m}(F\backslash K) = 0$ or $\mathfrak{m}(F^c\backslash K) = 0$, as desired. Now call

$$G := \begin{cases} F & \text{if} \quad \mathfrak{m}(F\backslash K) = 0, \\ F^c & \text{if} \quad \mathfrak{m}(F^c\backslash K) = 0. \end{cases}$$

We aim to prove that either $\mathfrak{m}(G) = 0$ or $\mathfrak{m}(G\Delta E_f) = 0$. Suppose that $\mathfrak{m}(G) > 0$. Observe that

$$\mathbb{1}_{E_f} = \mathbb{1}_{G\cup E_f} - \mathbb{1}_{G\backslash E_f} = \mathbb{1}_{E_f\cap G} + \mathbb{1}_{E_f\backslash G}.$$

Thanks to the two-sidedness property, we also know that

$$\mathcal{H}\big(\partial^e(G\cup E_f)\cap\partial^e(G\backslash E_f)\cap\partial^e E_f^c\big) = 0 = \mathcal{H}\big(\partial^e(E_f\cap G)\cap\partial^e(E_f\backslash G)\cap\partial^e E_f\big).$$

Therefore, item (ii) of Lemma 2.3 grants that

$$\mathsf{P}(E_f)=\mathsf{P}(E_f^c,\partial^e E_f^c)=\mathsf{P}(G\cup E_f,\partial^e E_f)+\mathsf{P}(G\backslash E_f,\partial^e E_f)=\mathsf{P}(G\cup E_f)+\mathsf{P}(G\backslash E_f),$$
$$\mathsf{P}(E_f)=\mathsf{P}(E_f,\partial^e E_f)=\mathsf{P}(E_f\cap G,\partial^e E_f)+\mathsf{P}(E_f\backslash G,\partial^e E_f)=\mathsf{P}(E_f\cap G)+\mathsf{P}(E_f\backslash G).$$

Suppose by contradiction that $\mathfrak{m}(G\backslash E_f) > 0$. Then we have $\mathsf{P}(G\backslash E_f) > 0$ and accordingly

$$f = \begin{cases} \mathsf{P}(G\cup E_f)\,\mathsf{P}(E_f)^{-1}\,\Phi_+(G\cup E_f) + \mathsf{P}(G\backslash E_f)\,\mathsf{P}(E_f)^{-1}\,\Phi_-(G\backslash E_f) & \text{if} \quad f = \Phi_+(E_f), \\ \mathsf{P}(G\cup E_f)\,\mathsf{P}(E_f)^{-1}\,\Phi_-(G\cup E_f) + \mathsf{P}(G\backslash E_f)\,\mathsf{P}(E_f)^{-1}\,\Phi_+(G\backslash E_f) & \text{if} \quad f = \Phi_-(E_f). \end{cases}$$

This contradicts the fact that $f \in \operatorname{ext}\mathcal{K}(\mathrm{X}; K)$, whence $\mathfrak{m}(G\backslash E_f) = 0$. Similarly, suppose by contradiction that $\mathfrak{m}(E_f\backslash G) > 0$. Then we have $\mathsf{P}(E_f\backslash G) > 0$ and accordingly

$$f = \begin{cases} \mathsf{P}(E_f\cap G)\,\mathsf{P}(E_f)^{-1}\,\Phi_+(E_f\cap G) + \mathsf{P}(E_f\backslash G)\,\mathsf{P}(E_f)^{-1}\,\Phi_+(E_f\backslash G) & \text{if} \quad f = \Phi_+(E_f), \\ \mathsf{P}(E_f\cap G)\,\mathsf{P}(E_f)^{-1}\,\Phi_-(E_f\cap G) + \mathsf{P}(E_f\backslash G)\,\mathsf{P}(E_f)^{-1}\,\Phi_-(E_f\backslash G) & \text{if} \quad f = \Phi_-(E_f). \end{cases}$$

This contradicts the fact that $f \in \operatorname{ext}\mathcal{K}(\mathrm{X}; K)$, whence $\mathfrak{m}(E_f\backslash G) = 0$. This yields $\mathfrak{m}(G\Delta E_f) = 0$, thus the set E_f is proven to be simple. We conclude that $f \in \mathcal{S}(\mathrm{X}; K)$, as required. □

3.2 Holes and saturation

The decomposition theorem can be used to define suitable notions of *hole* and *saturation* for a given set of finite perimeter in an isotropic PI space:

Definition 3.9 (*Hole*) Let $(\mathrm{X}, d, \mathfrak{m})$ be an isotropic PI space such that $\mathfrak{m}(\mathrm{X}) = +\infty$. Let $E \subset \mathrm{X}$ be an indecomposable set. Then any essential connected component of $\mathrm{X}\backslash E$ having finite $\mathfrak{m}$-measure is said to be a *hole* of E.

Definition 3.10 (*Saturation*) Let $(\mathrm{X}, d, \mathfrak{m})$ be an isotropic PI space such that $\mathfrak{m}(\mathrm{X}) = +\infty$. Given an indecomposable set $F \subset \mathrm{X}$, we define its *saturation* $\operatorname{sat}(F)$ as the union of F and its holes. Moreover, given any set $E \subset \mathrm{X}$ of finite perimeter, we define

$$\operatorname{sat}(E) := \bigcup_{F\in\mathcal{CC}^e(E)} \operatorname{sat}(F).$$

We say that the set E is *saturated* provided it holds that $\mathfrak{m}\big(E\Delta\operatorname{sat}(E)\big) = 0$.

Observe that an indecomposable set $E \subset \mathrm{X}$ is saturated if and only if it has no holes.

Remark 3.11 Given an isotropic PI space $(\mathrm{X}, d, \mathfrak{m})$ such that $\mathfrak{m}(\mathrm{X}) = +\infty$, it holds that any simple set $E \subset \mathrm{X}$ is indecomposable and saturated. Indeed, item (i) of Proposition 3.2 grants that both E, E^c are indecomposable; since $\mathfrak{m}(E^c) = +\infty$, we also conclude that E has no holes. □

Proposition 3.12 (Main properties of the saturation) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space such that* $\mathfrak{m}(\mathrm{X}) = +\infty$. *Let* $E \subset \mathrm{X}$ *be an indecomposable set. Then the following properties hold:*

(i) *Any hole of* E *is saturated.*
(ii) *The set* $\mathrm{sat}(E)$ *is indecomposable and saturated. In particular,* $\mathrm{sat}\big(\mathrm{sat}(E)\big) = \mathrm{sat}(E)$.
(iii) *It holds that* $\mathcal{H}\big(\partial^e \mathrm{sat}(E) \setminus \partial^e E\big) = 0$. *In particular, one has that* $\mathsf{P}\big(\mathrm{sat}(E)\big) \leq \mathsf{P}(E)$.
(iv) *If* $F \subset \mathrm{X}$ *is a set of finite perimeter with* $\mathfrak{m}\big(E \setminus \mathrm{sat}(F)\big) = 0$, *then* $\mathfrak{m}\big(\mathrm{sat}(E) \setminus \mathrm{sat}(F)\big) = 0$.

Proof

(i) Let F be a hole of E. Denote $\mathcal{CC}^e(E^c) = \{F\} \cup \{G_i\}_{i \in I}$. We know from Remark 2.17 that $\mathcal{H}(\partial^e G_i \cap \partial^e E) = \mathcal{H}(\partial^e G_i) > 0$ for all $i \in I$, thus $E \cup \bigcup_{i \in J} G_i$ is indecomposable for any finite set $J \subset I$ by Proposition 2.18. Therefore, the set $F^c = E \cup \bigcup_{i \in I} G_i$ is indecomposable by Proposition 2.10. Given that $\mathfrak{m}(F^c) = +\infty$, we conclude that F has no holes, as required.
(ii) Let us call $\{F_i\}_{i \in I}$ the holes of E. By arguing exactly as in the proof of item (i), we see that the set $\mathrm{sat}(E) = E \cup \bigcup_{i \in I} F_i$ is indecomposable. Moreover, $\mathcal{CC}^e\big(\mathrm{sat}(E)^c\big) = \mathcal{CC}^e(E^c) \setminus \{F_i\}_{i \in I}$, so that $\mathrm{sat}(E)$ has no holes. In other words, the set $\mathrm{sat}(E)$ is saturated.
(iii) Calling $\{F_i\}_{i \in I}$ the holes of E, we clearly have that $\partial^e \mathrm{sat}(E) \subset \partial^e E \cup \bigcup_{i \in I} \partial^e F_i$ by (1.7). Given that $\mathcal{H}(\partial^e F_i \setminus \partial^e E) = 0$ for all $i \in I$ by Remark 2.17, we conclude that $\mathcal{H}\big(\partial^e \mathrm{sat}(E) \setminus \partial^e E\big) = 0$ as well. Furthermore, observe that the latter identity also yields

$$\mathsf{P}\big(\mathrm{sat}(E)\big) = (\theta_{\mathrm{sat}(E)} \mathcal{H})\big(\partial^e \mathrm{sat}(E)\big) \overset{(1.15)}{=} (\theta_E \mathcal{H})\big(\partial^e \mathrm{sat}(E)\big) \leq (\theta_E \mathcal{H})(\partial^e E) = \mathsf{P}(E).$$

(iv) Let us denote $\mathcal{CC}^e\big(\mathrm{sat}(F)^c\big) = \{F_i\}_{i \in I}$. Given any $i \in I$, we have that F_i is indecomposable, has infinite $\mathfrak{m}$-measure, and satisfies $\mathfrak{m}(E \cap F_i) = 0$. Then there exists a unique set $G_i \in \mathcal{CC}^e(E^c)$ such that $\mathfrak{m}(F_i \setminus G_i) = 0$, thus in particular $\mathfrak{m}(G_i) = +\infty$. This says that the sets $\{G_i\}_{i \in I}$ cannot be holes of E, whence $\bigcup_{i \in I} G_i \subset \mathrm{sat}(E)^c$ and accordingly $\mathfrak{m}\big(\mathrm{sat}(E) \setminus \mathrm{sat}(F)\big) = 0$. □

Lemma 3.13 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be an isotropic PI space such that* $\mathfrak{m}(\mathrm{X}) = +\infty$. *Let* $E \subset \mathrm{X}$ *be a set of finite perimeter. Then it holds that* $\mathcal{H}\big(\partial^e \mathrm{sat}(E) \setminus \partial^e E\big) = 0$.

Proof Given any $F \in \mathcal{CC}^e(E)$, we have $\mathcal{H}\big(\partial^e \mathrm{sat}(F) \setminus \partial^e F\big) = 0$ by item (iii) of Proposition 3.12. Moreover, since $\mathrm{sat}(E) = \bigcup_{F \in \mathcal{CC}^e(E)} \mathrm{sat}(F)$ we know that $\partial^e \mathrm{sat}(E) \subset \bigcup_{F \in \mathcal{CC}^e(E)} \partial^e \mathrm{sat}(F)$ as a consequence of (1.7). Therefore, we deduce that

$$\mathcal{H}\big(\partial^e \mathrm{sat}(E) \setminus \partial^e E\big) \leq \sum_{F \in \mathcal{CC}^e(E)} \mathcal{H}\big(\partial^e \mathrm{sat}(F) \setminus \partial^e E\big) \leq \sum_{F \in \mathcal{CC}^e(E)} \mathcal{H}(\partial^e F \setminus \partial^e E) \overset{(2.16)}{=} 0,$$

thus proving the statement. □

Let us now focus on the special case of Ahlfors-regular, isotropic PI spaces. In this context, simple sets can be equivalently characterised as those sets that are both indecomposable and saturated (cf. Theorem 3.17). In order to prove it, we need some preliminary results:

Proposition 3.14 *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a k-Ahlfors regular, isotropic PI space with* $k > 1$. *Suppose that* $\mathfrak{m}(\mathrm{X}) = +\infty$. *Let* $E \subset \mathrm{X}$ *be an indecomposable set such that* $\mathfrak{m}(E) < +\infty$. *Then there exists exactly one essential connected component* $F \in \mathcal{CC}^e(E^c)$ *satisfying* $\mathfrak{m}(F) = +\infty$.

Proof Let us prove that at least one essential connected component of E^c has infinite $\mathfrak{m}$-measure. We argue by contradiction: suppose $\mathfrak{m}(E_i) < +\infty$ for all $i \in I$, where we set $\mathcal{CC}^e(E) = \{E_i\}_{i\in I}$. In particular, we have that $\mathfrak{m}(E_i^c) = +\infty$ holds for every $i \in I$, whence Theorem 1.18 yields

$$\sum_{i\in I} \mathfrak{m}(E_i)^{k-1/k} \leq C_I' \sum_{i\in I} \mathsf{P}(E_i) = C_I' \, \mathsf{P}(E) < +\infty.$$

By using the Markov inequality we deduce that $J := \big\{i \in I \,:\, \mathfrak{m}(E_i) \geq 1\big\}$ is a finite family, thus the set $\bigcup_{i\in J} E_i$ has finite $\mathfrak{m}$-measure. This leads to a contradiction, as it implies that

$$+\infty = \mathfrak{m}\Big(\bigcup_{i\in I\setminus J} E_i\Big) = \sum_{i\in I\setminus J} \mathfrak{m}(E_i) \leq \sum_{i\in I\setminus J} \mathfrak{m}(E_i)^{k-1/k} \overset{(1.9)}{\leq} C_I' \, \mathsf{P}(E) < +\infty.$$

Hence, there exists $i \in I$ such that $\mathfrak{m}(E_i) = +\infty$. Suppose by contradiction to have $\mathfrak{m}(E_j) = +\infty$ for some $j \in I\setminus\{i\}$. Then $E_j \subset E_i^c$ and accordingly $\mathfrak{m}(E_i^c) = +\infty$, which is not possible as we have that $\min\big\{\mathfrak{m}(E_i), \mathfrak{m}(E_i^c)\big\} \leq C_I' \, \mathsf{P}(E_i)^{k/k-1} < +\infty$ by Theorem 1.18. The statement follows. □

Remark 3.15 The Ahlfors-regularity assumption in Proposition 3.14 cannot be dropped, as shown by the following example. Let us consider the strip $\mathrm{X} := \mathbb{R} \times [0, 1] \subset \mathbb{R}^2$, endowed with the (restricted) Euclidean distance and the 2-dimensional Hausdorff measure, which is an isotropic PI space. Then the square $E := [0, 1]^2 \subset \mathrm{X}$ is an indecomposable set having finite measure, but its complement consists of two essential connected components having infinite measure. □

Remark 3.16 If $(\mathrm{X}, d, \mathfrak{m})$ is a k-Ahlfors regular, isotropic PI space with $k > 1$ and $\mathfrak{m}(\mathrm{X}) = +\infty$, then for any indecomposable set $E \subset \mathrm{X}$ with $\mathfrak{m}(E) < +\infty$ it holds that $\mathfrak{m}\big(\mathrm{sat}(E)\big) < +\infty$.

Indeed, we know that $\mathfrak{m}\big(\mathrm{sat}(E)^c\big) = +\infty$ by Proposition 3.14, whence the set $\mathrm{sat}(E)$ must have finite $\mathfrak{m}$-measure (otherwise we would contradict Theorem 1.18). □

Theorem 3.17 (Simple sets on Ahlfors-regular spaces) *Let* $(\mathrm{X}, d, \mathfrak{m})$ *be a k-Ahlfors regular PI space with* $k > 1$ *and* $\mathfrak{m}(\mathrm{X}) = +\infty$. *Suppose* $(\mathrm{X}, d, \mathfrak{m})$ *has the two-sidedness property. Let* $E \subset \mathrm{X}$ *be a set of finite perimeter with* $\mathfrak{m}(E) < +\infty$. *Then E is simple if and only if it is both indecomposable and saturated.*

Proof Necessity stems from Remark 3.11. To prove sufficiency, suppose that E is indecomposable and saturated. Proposition 3.14 grants that E^c is the unique element of $\mathcal{CC}^e(E)$ having infinite $\mathfrak{m}$-measure, thus in particular E^c is indecomposable. By applying item (ii) of Proposition 3.2, we finally conclude that the set E is simple, as desired. □

4 Alternative proof of the decomposition theorem

We provide here an alternative proof of the Decomposition Theorem 2.14, in the particular case in which the set under consideration is bounded (the boundedness assumption is added for simplicity, cf. Remark 4.6 for a few comments about the unbounded case). The inspiration for this approach is taken from [33]. We refer to "Appendix B" for the language and the results we are going to use in this section.

Let $(\mathrm{X}, d, \mathfrak{m})$ be an isotropic PI space. Given any open set $\Omega \subset \mathrm{X}$ and any set $E \subset \Omega$ having finite perimeter in X, we define the family $\Xi_\Omega(E)$ as

$$\Xi_\Omega(E) := \left\{ F \subset E \text{ of finite perimeter in X} \;\middle|\; \mathsf{P}(E, \Omega) = \mathsf{P}(F, \Omega) + \mathsf{P}(E \backslash F, \Omega) \right\}.$$

Observe that $\Xi_\Omega(E) = \Xi_{\Omega'}(E)$ holds whenever $\Omega, \Omega' \subset \mathrm{X}$ are open sets with $E \Subset \Omega$ and $E \Subset \Omega'$.

Remark 4.1 It holds that E is indecomposable in Ω if and only if $\Xi_\Omega(E)$ is trivial, i.e.,

$$\Xi_\Omega(E) = \left\{ F \subset E \text{ Borel} \;\middle|\; \mathfrak{m}(F) = 0 \text{ or } \mathfrak{m}(E \backslash F) = 0 \right\}.$$

The proof of this fact is a direct consequence of the very definition of indecomposable set. □

Lemma 4.2 *Let $(\mathrm{X}, d, \mathfrak{m})$ be an isotropic PI space. Let $\Omega \subset \mathrm{X}$ be an open set with $\mathcal{H}(\partial\Omega) < +\infty$. Let $E \subset \Omega$ be a set of finite perimeter in X. Then $\Xi_\Omega(E)$ is a σ-algebra of Borel subsets of E. Moreover, if $E \Subset \Omega$, then the assumption $\mathcal{H}(\partial\Omega) < +\infty$ can be dropped.*

Proof Trivially, we have that $E \in \Xi_\Omega(E)$ and $\Xi_\Omega(E)$ is closed under complement. Moreover, fix any two sets $F, G \in \Xi_\Omega(E)$. Since $P(E, \Omega) = \mathsf{P}(F, \Omega) + \mathsf{P}(E \backslash F, \Omega) = \mathsf{P}(G, \Omega) + \mathsf{P}(E \backslash G, \Omega)$, we deduce that $\mathsf{P}(G, \Omega) = \mathsf{P}(F \cap G, \Omega) + \mathsf{P}(G \backslash F, \Omega)$ and $\mathsf{P}(E \backslash G, \Omega) = \mathsf{P}(F \backslash G, \Omega) + \mathsf{P}\big(E \backslash (F \cup G), \Omega\big)$ by Lemma 2.8. Consequently, the subadditivity of the perimeter yields

$$\begin{aligned}
\mathsf{P}(E, \Omega) &\le \mathsf{P}(F \cup G, \Omega) + \mathsf{P}\big(E \backslash (F \cup G), \Omega\big) \\
&\le \mathsf{P}(F \cap G, \Omega) + \mathsf{P}(G \backslash F, \Omega) + \mathsf{P}(F \backslash G, \Omega) + \mathsf{P}\big(E \backslash (F \cup G), \Omega\big) \\
&= \mathsf{P}(G, \Omega) + \mathsf{P}(E \backslash G, \Omega) = \mathsf{P}(E, \Omega).
\end{aligned}$$

This forces the equality $\mathsf{P}(E, \Omega) = \mathsf{P}(F \cup G, \Omega) + \mathsf{P}\big(E \backslash (F \cup G), \Omega\big)$. Given that $F \cup G$ has finite perimeter, we have proved that $F \cup G \in \Xi_\Omega(E)$. This shows that $\Xi_\Omega(E)$ is closed under finite unions. Finally, to prove that $\Xi_\Omega(E)$ is closed under countable unions, fix any $(F_i)_i \subset \Xi_\Omega(E)$. Calling $F := \bigcup_{i \in \mathbb{N}} F_i$, we aim to prove that $F \in \Xi_\Omega(E)$. We denote $F_i' := F_1 \cup \cdots \cup F_i \in \Xi_\Omega(E)$ for all $i \in \mathbb{N}$. Given that $F = \bigcup_{i \in \mathbb{N}} F_i'$, we have $\mathbb{1}_{F_i'} \to \mathbb{1}_F$ and $\mathbb{1}_{E \backslash F_i'} \to \mathbb{1}_{E \backslash F}$ in $L^1_{\mathrm{loc}}(\mathfrak{m}\llcorner\Omega)$. Hence, by lower semicontinuity and subadditivity of the perimeter we can conclude that

$$\begin{aligned}
\mathsf{P}(E, \Omega) &\le \mathsf{P}(F, \Omega) + \mathsf{P}(E \backslash F, \Omega) \le \varliminf_{i \to \infty} \mathsf{P}(F_i', \Omega) + \varliminf_{i \to \infty} \mathsf{P}(E \backslash F_i', \Omega) \\
&\le \varliminf_{i \to \infty} \big(\mathsf{P}(F_i', \Omega) + \mathsf{P}(E \backslash F_i', \Omega)\big) = \mathsf{P}(E, \Omega),
\end{aligned}$$

which forces $\mathsf{P}(E, \Omega) = \mathsf{P}(F, \Omega) + \mathsf{P}(E \backslash F, \Omega)$. Notice also that $\mathbb{1}_{F_i'} \to \mathbb{1}_F$ in $L^1_{\mathrm{loc}}(\mathfrak{m})$, whence

$$\begin{aligned}
\mathsf{P}(F) &\le \varliminf_{i \to \infty} \mathsf{P}(F_i') = \varliminf_{i \to \infty} \big(\mathsf{P}(F_i', \Omega) + \mathsf{P}(F_i', \partial\Omega)\big) \le \mathsf{P}(E, \Omega) + \varliminf_{i \to \infty} (\theta_{F_i'} \mathcal{H})(\partial\Omega) \\
&\le \mathsf{P}(E, \Omega) + C_D\, \mathcal{H}(\partial\Omega) < +\infty.
\end{aligned}$$

This says that the set F has finite perimeter in X, thus F belongs to $\Xi_\Omega(E)$, as desired.

To prove the last statement, let us assume that $E \Subset \Omega$. By exploiting Remark 1.20 and the boundedness of E, we can find an open ball $B \subset \mathrm{X}$ such that $E \Subset B$ and $\mathcal{H}(\partial B) < +\infty$, thus accordingly the family $\Xi_\Omega(E) = \Xi_B(E)$ is a σ-algebra by the previous part of the proof. □

Remark 4.3 Let $(\mathrm{X}, d, \mathfrak{m})$ be an isotropic PI space and $\Omega \subset \mathrm{X}$ an open set. Then we claim that

$$\Xi_\Omega(G) = \big\{F \in \Xi_\Omega(E) \,\big|\, F \subset G\big\} \quad \text{for every } E \subset \Omega \text{ Borel with } \mathsf{P}(E) < +\infty \text{ and } G \in \Xi_\Omega(E).$$

We separately prove the two inclusions. Fix $F \in \Xi_\Omega(G)$. Since $\mathsf{P}(E, \Omega) = \mathsf{P}(G, \Omega) + \mathsf{P}(E \backslash G, \Omega)$, we know from Lemma 2.8 that $\mathsf{P}(E \backslash F, \Omega) = \mathsf{P}(G \backslash F, \Omega) + \mathsf{P}(E \backslash G, \Omega)$. Therefore, we have that

$$\begin{aligned}\mathsf{P}(E, \Omega) &= \mathsf{P}(G, \Omega) + \mathsf{P}(E \backslash G, \Omega) = \mathsf{P}(F, \Omega) + \mathsf{P}(G \backslash F, \Omega) + \mathsf{P}(E \backslash G, \Omega)\\ &= \mathsf{P}(F, \Omega) + \mathsf{P}(E \backslash F, \Omega),\end{aligned}$$

thus proving that $F \in \Xi_\Omega(E)$. Conversely, let us fix any set $F' \in \Xi_\Omega(E)$ such that $F' \subset G$. Given that $\mathsf{P}(E, \Omega) = \mathsf{P}(F', \Omega) + \mathsf{P}(E \backslash F', \Omega)$, we conclude that $\mathsf{P}(G, \Omega) = \mathsf{P}(F', \Omega) + \mathsf{P}(G \backslash F', \Omega)$ again by Lemma 2.8. This shows that $F' \in \Xi_\Omega(G)$, which yields the sought conclusion. □

Lemma 4.4 *Let $(\mathrm{X}, d, \mathfrak{m})$ be an isotropic PI space. Let $\Omega \subset \mathrm{X}$ be an open set. Let $E \subset \Omega$ be a set of finite perimeter in X. Then for any finite partition $\{E_1, \ldots, E_n\} \subset \Xi_\Omega(E)$ of the set E it holds that $\mathsf{P}(E, \Omega) = P(E_1, \Omega) + \cdots + P(E_n, \Omega)$.*

Proof Recall that $\mathsf{P}(E, \Omega) = \mathsf{P}(E_i, \Omega) + \mathsf{P}(E \backslash E_i, \Omega)$ for all $i = 1, \ldots, n$, thus by repeatedly applying Lemma 2.8 we obtain that

$$\mathsf{P}(E, \Omega) = \mathsf{P}(E_1, \Omega) + \mathsf{P}(E_2 \cup \cdots \cup E_n, \Omega) = \cdots = \mathsf{P}(E_1, \Omega) + \cdots + \mathsf{P}(E_n, \Omega).$$

Therefore, the statement is achieved. □

Theorem 4.5 *Let $(\mathrm{X}, d, \mathfrak{m})$ be an isotropic PI space. Let $\Omega \subset \mathrm{X}$ be an open set. Then the measure space $(E, \Xi_\Omega(E), \mathfrak{m}_{\llcorner E})$ is purely atomic for every bounded set $E \Subset \Omega$ of finite perimeter.*

Proof We can assume without loss of generality that $\Omega = B_r(\bar{x})$ for some $\bar{x} \in \mathrm{X}$ and $r > 0$. For the sake of brevity, let us denote $\mathbb{M}_F := (F, \Xi_\Omega(F), \mathfrak{m}_{\llcorner F})$ for every $F \in \Xi_\Omega(E)$. It follows from Remark 4.3 that $\Xi_\Omega(E)_{\llcorner F} = \Xi_\Omega(F)$ and that the atoms of $\mathbb{M}_F$ coincide with the atoms of $\mathbb{M}_E$ that are contained in F. Accordingly, in order to prove that $\mathbb{M}_E$ is purely atomic, it suffices to show that $\mathbb{M}_F$ is atomic for any set $F \in \Xi_\Omega(E)$ with $\mathfrak{m}(F) > 0$. We argue by contradiction: suppose $\mathbb{M}_F$ is non-atomic. Let us fix any $\varepsilon > 0$. Corollary B.5 grants that there exists a finite partition $\{F_1, \ldots, F_n\} \subset \Xi_\Omega(F)$ of F such that $\mathfrak{m}(F_i) \leq \min\big\{\varepsilon, \mathfrak{m}(F \backslash F_i)\big\}$ for all $i = 1, \ldots, n$. Let us apply Theorem 1.17: calling C the quantity $C_I\left(r^s/\mathfrak{m}(B_r(\bar{x}))\right)^{1/s-1}$, one has that

$$\left(\frac{\mathfrak{m}(F_i)}{C}\right)^{s-1/s} \leq \mathsf{P}\big(F_i, B_{2\lambda r}(\bar{x})\big) = \mathsf{P}(F_i, \Omega) \quad \text{for every } i = 1, \ldots, n.$$

Since $\sum_{i=1}^n \mathsf{P}(F_i, \Omega) = \mathsf{P}(F, \Omega)$ holds by Lemma 4.4, we deduce from the previous inequality that

$$\frac{\mathfrak{m}(F)}{C^{s-1/s}\,\varepsilon^{1/s}} \leq \frac{1}{C}\sum_{i=1}^{n}\mathfrak{m}(F_i)\left(\frac{\mathfrak{m}(F_i)}{C}\right)^{-1/s} = \sum_{i=1}^{n}\left(\frac{\mathfrak{m}(F_i)}{C}\right)^{s-1/s} \leq \mathsf{P}(F,\Omega). \tag{4.1}$$

By letting $\varepsilon \searrow 0$ in (4.1) we get that $\mathsf{P}(F,\Omega) = +\infty$, which yields a contradiction. Therefore, we conclude that the measure space $\mathbb{M}_F$ is non-atomic, as required. □

Alternative proof of Theorem 2.14 for E bounded. Maximality and uniqueness can be proven as in Proposition 2.13, thus we can just focus on the existence part of the statement. The measure space $(E, \Xi_X(E), \mathfrak{m}_{\llcorner E})$ is purely atomic by Theorem 4.5, thus there exists an at most countable family of pairwise disjoint atoms $\{E_i\}_{i\in I} \subset \Xi_X(E)$ such that $\mathfrak{m}\big(E\setminus\bigcup_{i\in i} E_i\big) = 0$ by Remark B.2. Moreover, we deduce from Remark 4.3 that each set E_i is an atom of $(E_i, \Xi_X(E_i), \mathfrak{m}_{\llcorner E_i})$, which is clearly equivalent to saying that $\Xi_X(E_i)$ is trivial (in the sense of Remark 4.1). Accordingly, the set E_i is indecomposable for every $i \in I$. Finally, Lemma 4.4 grants that $\mathsf{P}(E) = \sum_{i\in I}\mathsf{P}(E_i)$. □

Remark 4.6 Let us briefly outline how to prove the decomposition theorem via Theorem B.3 in the general case (i.e., when E is possibly unbounded). More specifically, we show that the existence part of Proposition 2.13 (under the additional assumption that $\partial B_r(\bar{x})$ has finite $\mathcal{H}$-measure) can be deduced from Theorem B.3, whence Theorem 2.14 follows (thanks to Remark 1.20).

Our aim is to show that $\big(E\cap\Omega, \Xi_\Omega(E\cap\Omega), \mathfrak{m}_{\llcorner E\cap\Omega}\big)$ is purely atomic, where we set $\Omega := B_r(\bar{x})$. We argue by contradiction: suppose $(F, \Xi_\Omega(F), \mathfrak{m}_{\llcorner F})$ is non-atomic for some $F \in \Xi_\Omega(E\cap\Omega)$. Then Corollary B.5, Theorems 1.17 and 1.23 ensure that for any $\varepsilon > 0$ there exist a finite partition $\{F_1,\ldots,F_{n_\varepsilon}\} \subset \Xi_\Omega(F)$ of F and a constant $c > 0$ such that $\mathfrak{m}(F_1),\ldots,\mathfrak{m}(F_{n_\varepsilon}) \leq \varepsilon$ and

$$c\,\mathfrak{m}(F_i)^{s-1/s} \leq \mathsf{P}(F_i,\Omega) + C_D\,\mathcal{H}\big(\Sigma_\tau(F_i)\cap\partial\Omega\big) \quad \text{for every } i = 1,\ldots,n_\varepsilon, \tag{4.2}$$

where the set $\Sigma_\tau(F_i)$ is defined as in (1.13). Given any $\ell \in \mathbb{N}$ such that $\ell\,\tau > 1$, it is clear that the set $\bigcap_{i\in S}\Sigma_\tau(F_i)$ is empty whenever we choose $S \subset \{1,\ldots,n_\varepsilon\}$ of cardinality greater than ℓ. Therefore, we deduce from (4.2) and the identity $\sum_{i=1}^{n_\varepsilon}\mathsf{P}(F_i,\Omega) = \mathsf{P}(F,\Omega)$ that

$$c\,\frac{\mathfrak{m}(F)}{\varepsilon^{1/s}} = c\sum_{i=1}^{n_\varepsilon}\frac{\mathfrak{m}(F_i)}{\varepsilon^{1/s}} \leq c\sum_{i=1}^{n_\varepsilon}\mathfrak{m}(F_i)^{s-1/s} \leq \mathsf{P}(F,\Omega) + C_D\,\mathcal{H}(\partial\Omega)\,\ell.$$

Finally, by letting $\varepsilon \searrow 0$ we conclude that $\mathsf{P}(F,\Omega) = +\infty$, which leads to a contradiction. □

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Appendix A: Extreme points

Let V be a normed space. Let $K \neq \emptyset$ be a convex, compact subset of V. Then we shall denote by $\operatorname{ext} K$ the set of all *extreme points* of K, namely of those points $x \in K$ that cannot be written as $x = ty + (1-t)z$ for some $t \in (0, 1)$ and some distinct $y, z \in K$. The Krein–Milman theorem states that K coincides with the closed convex hull of $\operatorname{ext} K$; cf. [34]. Furthermore, it actually holds that $\operatorname{ext} K$ is the 'smallest' set having this property:

Theorem A.1 (Milman [37]) *Let V be a normed space. Let $\emptyset \neq K \subset V$ be convex and compact. Suppose that the closed convex hull of a set $S \subset K$ coincides with K. Then* $\operatorname{ext} K$ *is contained in the closure of S.*

Another fundamental result in functional analysis and convex analysis is the following celebrated strengthening of the Krein–Milman theorem:

Theorem A.2 (Choquet [39]) *Let V be a normed space. Let $\emptyset \neq K \subset V$ be convex and compact. Then for any point $x \in K$ there exists a Borel probability measure μ on V (depending on x), which is concentrated on* $\operatorname{ext} K$ *and satisfies*

$$L(x) = \int L(y)\, \mathrm{d}\mu(y) \quad \textit{for every } L\colon V \to \mathbb{R} \textit{ linear and continuous.}$$

Remark A.3 In the above result, the measure μ is concentrated on $\operatorname{ext} K$. For completeness, we briefly verify that $\operatorname{ext} K$ is a Borel subset of V: the set $K \setminus \operatorname{ext} K$ can be written as $\bigcup_n C_n$, where

$$C_n := \left\{ \frac{y+z}{2} \;\middle|\; y, z \in K,\ \|y - z\|_V \geq 1/n \right\} \quad \text{for every } n \in \mathbb{N}.$$

Given that each set C_n is a closed subset of V, we conclude that $\operatorname{ext} K$ is Borel. □

Appendix B: Lyapunov vector-measure theorem

In the theory of vector measures, an important role is played by the following theorem (due to Lyapunov): the range of a non-atomic vector measure is closed and convex; cf., for instance, [20]. For our purposes, we need a simpler version of this theorem (just for scalar measures). For the reader's convenience, we report below (see Theorem B.3) an elementary proof of this result.

Let us begin by recalling the definition of atom in a measure space (see also [12]):

Definition B.1 (Atom) Let $(\mathrm{X}, \mathcal{A}, \mu)$ be a measure space. Then a set $A \in \mathcal{A}$ with $\mu(A) > 0$ is said to be an *atom* of μ provided for any set $A' \in \mathcal{A}$ with $A' \subset A$ it holds that either $\mu(A') = 0$ or $\mu(A \setminus A') = 0$. The measure space $(\mathrm{X}, \mathcal{A}, \mu)$ is called *non-atomic* if there are no atoms, *atomic* if there exists at least one atom, and *purely atomic* if every measurable set of positive μ-measure contains an atom.

Remark B.2 Given a purely atomic measure space $(\mathrm{X}, \mathcal{A}, \mu)$ and a set $E \in \mathcal{A}$ such that $\mu(E) > 0$, there exists an at most countable family $\{A_i\}_{i \in I} \subset \mathcal{A}$ of pairwise disjoint atoms of μ, which are contained in E and satisfy $\mu\big(E \setminus \bigcup_{i \in I} A_i\big) = 0$; cf. [32, Theorem 2.2]. □

Recall that a measure space $(\mathrm{X}, \mathcal{A}, \mu)$ is *semifinite* provided for every set $E \in \mathcal{A}$ with $\mu(E) > 0$ there exists $F \in \mathcal{A}$ such that $F \subset E$ and $0 < \mu(F) < +\infty$.

Theorem B.3 (Non-atomic measures have full range) *Let* $(\mathrm{X}, \mathcal{A}, \mu)$ *be a semifinite, non-atomic measure space. Then for every constant* $\lambda \in \big(0, \mu(\mathrm{X})\big)$ *there exists* $A \in \mathcal{A}$ *such that* $\mu(A) = \lambda$.

Proof First of all, let us prove the following claim:

$$\begin{aligned}&\text{Given any set } A \in \mathcal{A} \text{ with } \mu(A) > 0 \text{ and any } \varepsilon > 0,\\ &\text{there exists } B \in \mathcal{A} \text{ such that } B \subset A \text{ and } 0 < \mu(B) < \varepsilon.\end{aligned} \tag{B.1}$$

In order to prove it, fix a subset $A' \in \mathcal{A}$ of A with $0 < \mu(A') < +\infty$ (whose existence follows from the semifiniteness assumption) and any $k \in \mathbb{N}$ such that $k > \mu(A')/\varepsilon$. Since μ admits no atoms, we can find a partition $B_1, \ldots, B_k \in \mathcal{A}$ of A' such that $\mu(B_i) > 0$ for every $i = 1, \ldots, k$. Hence, there must exist $i = 1, \ldots, k$ such that $\mu(B_i) < \varepsilon$, otherwise we would have that

$$\mu(A') = \mu(B_1) + \cdots + \mu(B_k) \geq k\,\varepsilon > \mu(A').$$

Therefore, the set $B := B_i$ satisfies $B \subset A' \subset A$ and $0 < \mu(B) < \varepsilon$. This proves the claim (B.1).

We recursively build a sequence $(A_n)_n \subset \mathcal{A}$. The set A_1 is any element of $\mathcal{A}$ with $0 < \mu(A_1) < \lambda$, which can be found thanks to (B.1). Now let us suppose to have already defined $A_1, \ldots, A_{n-1}$ for some natural number $n \geq 2$ with the following properties: $A_1, \ldots, A_{n-1} \in \mathcal{A}$ are pairwise disjoint sets that satisfy $\mu(A_1), \ldots, \mu(A_{n-1}) > 0$ and $\sum_{i=1}^{n-1} \mu(A_i) < \lambda$. We set

$$\mathcal{F}_n := \left\{ B \in \mathcal{A} \;\middle|\; B \subset \mathrm{X} \setminus \bigcup_{i=1}^{n-1} A_i,\ 0 < \mu(B) < \lambda - \sum_{i=1}^{n-1} \mu(A_i) \right\}.$$

Property (B.1) grants that $\mathcal{F}_n$ is non-empty, thus in particular $s_n := \sup\big\{\mu(B) \,\big|\, B \in \mathcal{F}_n\big\} > 0$. Let A_n be any element of $\mathcal{F}_n$ such that $\mu(A_n) \geq s_n/2$. Notice that $A_1, \ldots, A_n \in \mathcal{A}$ are pairwise disjoint sets of positive μ-measure for which $\mu(A_1) + \cdots + \mu(A_n) < \lambda$.

Now let us call $A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. We argue by contradiction: suppose that $\mu(A) \neq \lambda$. Given that $\mu(A) = \lim_n \sum_{i=1}^{n} \mu(A_i) \leq \lambda$, this means that $\mu(A) < \lambda$. We know from (B.1) that there exists a set $B \in \mathcal{A}$ with $B \subset \mathrm{X} \setminus A$ and $0 < \mu(B) < \lambda - \mu(A)$. Since $\sum_{n=1}^{\infty} \mu(A_n) < \lambda < +\infty$, we can pick some $n \geq 1$ for which $\mu(A_n) < \mu(B)/2$. On the other hand, one has that $B \subset \mathrm{X} \setminus A \subset \mathrm{X} \setminus \bigcup_{i=1}^{n-1} A_i$ and $0 < \mu(B) < \lambda - \mu(A) \leq \lambda - \sum_{i=1}^{n-1} \mu(A_i)$, whence accordingly $B \in \mathcal{F}_n$. Consequently, it must hold that $\mu(A_n) \geq s_n/2 \geq \mu(B)/2$, which leads to a contradiction. We conclude that $\mu(A) = \lambda$, which finally yields the statement. □

Remark B.4 Given a semifinite, non-atomic measure space $(\mathrm{X}, \mathcal{A}, \mu)$ and a set $E \in \mathcal{A}$, it holds that $(E, \mathcal{A}_{\llcorner E}, \mu_{\llcorner E})$ is semifinite and non-atomic as well, where the restricted σ-algebra $\mathcal{A}_{\llcorner E}$ is defined as $\mathcal{A}_{\llcorner E} := \big\{A \cap E \,:\, A \in \mathcal{A}\big\}$. In particular, one can readily deduce from Theorem B.3 that for any $\lambda \in \big(0, \mu(E)\big)$ there exists $A \in \mathcal{A}_{\llcorner E}$ such that $\mu(A) = \lambda$. □

Corollary B.5 *Let* $(\mathrm{X}, \mathcal{A}, \mu)$ *be a finite, non-atomic measure space. Then for every* $\varepsilon > 0$ *there exists a partition* $\{A_1, \ldots, A_n\} \subset \mathcal{A}$ *of* X *such that* $\mu(A_i) \leq \min\big\{\varepsilon, \mu(A_i^c)\big\}$ *for all* $i = 1, \ldots, n$.

Proof Fix any $\varepsilon' > 0$ such that $\varepsilon' < \varepsilon$ and $\varepsilon' < \mu(\mathrm{X})/2$. We proceed in a recursive way: first of all, choose a set $A_1 \in \mathcal{A}$ with $\mu(A_1) = \varepsilon'$, whose existence is granted by

Theorem B.3. Now we can pick a set $A_2 \in \mathcal{A}_{\llcorner A_1^c}$ such that $\mu(A_2) = \varepsilon'$ (recall Remark B.4). After finitely many steps, we end up with pairwise disjoint measurable sets $A_1, \dots, A_{n-1}$ such that $\mu\big(\mathrm{X}\backslash(A_1 \cup \dots \cup A_{n-1})\big) < \varepsilon'$. Let us define $A_n := \mathrm{X}\backslash(A_1 \cup \ldots \cup A_{n-1}) \in \mathcal{A}$. Therefore, the sets $A_1, \dots, A_n$ do the job. □

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