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On finite d-maximal groups

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Abstract

Let d be a positive integer. A finite group is called d-maximal if it can be generated by precisely d elements, whereas its proper subgroups have smaller generating sets. For $d \in \{1, 2\}$, the d-maximal groups have been classified up to isomorphism and only partial results have been proved for larger d. In this work, we prove that a d-maximal group is supersolvable and we give a characterisation of d-maximality in terms of so-called *maximal* (p, q)*-pairs*. Moreover, we classify the maximal (p, q)-pairs of small rank obtaining, as a consequence, the classification of the isomorphism classes of 3-maximal finite groups.

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1 | INTRODUCTION

Let G be a finite group and let $\mathrm{d}(G)$ denote its minimum number of generators. Let d be a positive integer.

Definition 1.1. A finite group G is said to be *d-maximal* if $\mathrm{d}(G) = d$ and, for every proper subgroup H of G, one has $\mathrm{d}(H) < d$.

The only 1-maximal groups are the cyclic groups of prime order, while the 2-maximal groups — also called *minimal non-cyclic* — have been classified by Miller and Moreno [9]: up to isomorphism, if G is a minimal non-cyclic group, then G is an elementary abelian p-group of rank 2, the quaternion group Q_8, or there are distinct primes p and q such that $G = P \rtimes Q$ is a semidirect product of a cyclic group P of order p with a cyclic q-group Q and $Q/C_Q(P)$ has order q. The structure of d-maximal p-groups has been investigated by Laffey [7]. Adapting an argument of J. G. Thompson, he proved that, if p is an odd prime and P is a d-maximal p-group, then P has class at most 2 and the Frattini subgroup of P has exponent p and coincides with its derived subgroup; in particular, $|P| \leqslant p^{2d-1}$. The situation for $p = 2$ turned out to be much more intricate. In 1996, Minh [10] constructed a 4-maximal 2-group of class 3 and order 2^8. Nowadays groups of order 2^8 can be examined using a computer. There are 20 241 groups G of order 2^8 with $\mathrm{d}(G) = 4$, and only two of them are 4-maximal with nilpotency class 3. All the known d-maximal 2-groups are of class at most 3 and the following question is open.

Question 1.2. Are there d-maximal 2-groups of arbitrary large nilpotency class?

Clearly a nilpotent d-maximal group must be a p-group. In this paper, we are interested in d-maximal groups in the more general situation where G is not nilpotent. Using the classification of the finite non-abelian simple groups, we prove that a finite d-maximal group is solvable and its order is divisible by at most two different primes, as the next result shows.

Theorem 1.3. *Let G be a non-nilpotent d-maximal group. Then there exist distinct primes p and q such that the derived subgroup P of G is a Sylow p-subgroup of G and G/P is a cyclic q-group. Moreover, if Q is a Sylow q-subgroup of G, then $Q/C_Q(P)$ has order q.*

In light of the previous result, it is natural to investigate the structure of finite p-groups that can occur as the derived subgroup of a non-nilpotent d-maximal group. We recall that a power automorphism of a finite group is an automorphism sending every subgroup to itself. If G is an elementary abelian p-group, then a power automorphism of G is just scalar multiplication by some element of $\mathbb{F}_p^\times$.

Definition 1.4. Let p and q be prime numbers. A *maximal* (p,q)-pair of rank d is a pair (P,α) where P is a finite p-group, $\alpha \in \mathrm{Aut}(P)$ has prime order q dividing $p-1$, and the following properties are satisfied:

(a) the minimal number of generators of every subgroup of P is at most $\mathrm{d}(P) = d$;
(b) the image of α in $\mathrm{Aut}(P/\Phi(P))$ is a non-trivial power automorphism;
(c) if H is a proper subgroup of P with $\mathrm{d}(H) = \mathrm{d}(P)$, then either $\alpha(H) \neq H$ or the image of α in $\mathrm{Aut}(H/\Phi(H))$ is not a non-trivial power automorphism.

We reformulate Theorem 1.3 in terms of maximal pairs.

Theorem 1.5. *A finite group G is d-maximal if and only if one of the following occurs:*

(1) *the group G is a d-maximal p-group;*
(2) *there exist a maximal (p,q)-pair (P,α) of rank $d-1$ and a cyclic q-group $\langle\beta\rangle$ such that G is isomorphic to $P \rtimes \langle\beta\rangle$ and, for every $x \in P$, one has $\alpha(x) = \beta(x)$.*

The Miller and Moreno classification of minimal non-cyclic groups can be essentially reformulated to saying that, if (P,α) is a maximal (p,q)-pair of rank 1, then P has order

p. In Section 5, we classify the maximal (p,q)-pairs (P,α) of rank 2, proving in particular that either P has exponent p and order at most p^3 or $(p,q)=(3,2)$, in which case there is a unique exceptional example with P of order 81 and class 3. This result, combined with a recent classification of the 3-maximal p-groups [1], allows us to give in Section 5.1 the full classification of the finite 3-maximal groups. In Section 6, we classify the maximal (p,q)-pairs (P,α) of rank 3: in this case, P has class at most 3 and order at most p^6, and if $|P|=p^6$, then $(p,q)=(3,2)$.

The behaviour of maximal pairs of small rank suggests the following question.

Question 1.6. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ with the property that $|P| \leqslant p^{f(d)}$, whenever (P,α) is a maximal (p,q)-pair of rank d?

It follows from Lemmas 3.1 and 3.2 that, if (P,α) is a maximal (p,q)-pair of rank d and P has class at most c, then $|P| \leqslant p^{cd}$; the previous question is thus equivalent to the following one.

Question 1.7. Does there exist a function $g : \mathbb{N} \to \mathbb{N}$ with the property that P has class at most $g(d)$, whenever (P,α) is a maximal pair of rank d?

We are not aware of examples of maximal pairs (P,α) with P of class greater than 3. This motivates the next problem.

Question 1.8. Is it possible to construct maximal pairs (P,α) with P of arbitrarily large nilpotency class?

The following is our main contribution to the solution of the previous questions. It implies, in particular, that the derived length of a d-maximal group of odd order is at most 3.

Theorem 1.9. *Let (P,α) be a maximal (p,q)-pair. If $q>2$, then P has class at most 2.*

The proof of Theorem 1.9 is given in Section 4.2, and involves results on maximal pairs (P,α) where P is regular (the definition of regularity is given in Section 4.1). The class of regular p-groups is not only easier to study, but also a reasonable family to restrict to. Indeed, as soon as $p \geqslant 2d$ and (P,α) is a maximal pair of rank d, the group P is regular (see Lemma 4.3).

Notation. We use standard group theory notation and write

- $\mathrm{Z}(G)$ for the centre of G,
- $\Phi(G)$ for the Frattini subgroup of G,
- $(\gamma_i(G))_{i\geqslant 1}$ for the lower central series of G.

If p is a prime number, n a non-negative integer and P a finite p-group, we write $\Omega_n(P)$ and $\mho_n(P)$ for the following subgroups:

$$\Omega_n(P) = \langle x \in P \mid x^{p^n} = 1\rangle \quad \text{and} \quad \mho_n(G) = \langle x^{p^n} \mid x \in G\rangle.$$

2 | MAXIMAL GROUPS AND MAXIMAL PAIRS

In this section, we translate the problem of classifying d-maximal groups into that of classifying maximal pairs, as defined in the Introduction.

2.1 | The structure of d-maximal groups

The following theorem is [8, Cor. 4]. Its proof uses several different properties of the finite simple groups and requires their classification.

Theorem 2.1. *Let G be a finite group. Let $D = \max_{S\in\mathrm{Syl}(G)} \mathrm{d}(S)$, where S runs among the Sylow subgroups of G. Then, $\mathrm{d}(G) \leqslant D+1$. If $\mathrm{d}(G) = D+1$, then there exists an odd prime p and a quotient of G isomorphic to a semidirect product of an elementary abelian p-group P of rank D with a cyclic group $\langle\alpha\rangle$, where α acts on P as a non-trivial power automorphism.*

The next result describes the d-maximal groups with trivial Frattini subgroup. Note that, in the second case of Proposition 2.2, the subgroup A will necessarily act on the elementary abelian group P by scalar multiplication by elements of $\mathbb{F}_p^\times$, and therefore, its order q will have to divide $p-1$, yielding, in particular, that p is odd.

Proposition 2.2. *Let G be a d-maximal finite group such that $\Phi(G) = 1$. Then there exists a prime number p such that one of the following holds.*

(1) *The group G is an elementary abelian p-group of rank d.*
(2) *The group G is isomorphic to a semidirect product $P \rtimes A$, where P is an elementary abelian p-group of rank $d-1$ and A is a central prime-order subgroup of $\mathrm{GL}_{d-1}(\mathbb{F}_p)$.*

Proof. If G is nilpotent, then G is a direct product of elementary abelian groups and (1) follows easily from d-maximality. Let $D = \max_{S\in\mathrm{Syl}(G)} \mathrm{d}(S)$ and observe that $D < d$. From Theorem 2.1, we obtain a normal subgroup N of G such that G/N is isomorphic to a semidirect product $P \rtimes \langle\alpha\rangle$ where P is elementary abelian of rank $d-1$ and α acts on P as a non-trivial power automorphism. In particular, $p \neq 2$ and $\mathrm{d}(G/N) = d$. We claim that $N = 1$. If this were not the case, since $\Phi(G) = 1$, there would exist a maximal subgroup M of G such that $G = MN$, and thus,

$$\mathrm{d}(M) \geqslant \mathrm{d}(M/(M\cap N)) = \mathrm{d}(G/N) = d,$$

which is impossible. Let $\sigma \in \langle\alpha\rangle$. If σ acts non-trivially on P, then $\mathrm{d}(P \rtimes \langle\sigma\rangle) = d$, which gives $\langle\sigma\rangle = \langle\alpha\rangle$. Since some Sylow subgroup of $\langle\alpha\rangle$ must act non-trivially on P, we conclude that α has prime-power order, say q^t. Moreover, α^q must act trivially on P. Since $\alpha^q \in \Phi(\langle\alpha\rangle)$, we conclude that $\alpha^q \in \Phi(G) = 1$. The proof is complete. $\square$

The following two results are well known. The first is a direct consequence of the Schur–Zassenhaus theorem (see [11, 9.3.5]), whereas the second is [6, Thm. 1.6.2].

Lemma 2.3. *Let G be a finite group. If p divides $|G|$, then p divides $|G : \Phi(G)|$.*

Lemma 2.4. *Let G be a finite group, φ an automorphism of G and N a normal φ-invariant subgroup whose order is coprime to the order of φ. Then $\mathrm{C}_{G/N}(\varphi) = \mathrm{C}_G(\varphi)N/N$.*

Proposition 2.5. *Let G be a d-maximal finite group and assume that G is not a p-group. Then G is isomorphic to a semidirect product $P \rtimes \langle \alpha \rangle$, where P is a p-group for some odd prime p, and $\alpha \in \mathrm{Aut}(P)$ has prime-power order q^t for some q dividing $p-1$. Moreover, $\mathrm{d}(P) = d-1$, and α^q centralises P.*

Proof. Let G be a non-nilpotent d-maximal group. Since $d = \mathrm{d}(G) = \mathrm{d}(G/\Phi(G))$, we have that $\overline{G} = G/\Phi(G)$ is d-maximal and Frattini-free. Proposition 2.2 ensures the existence of an elementary abelian p-group $\overline{P}$ and $\overline{\alpha} \in \mathrm{Aut}(\overline{P})$ of prime order q dividing $p-1$ such that $\overline{G}$ is isomorphic to the semidirect product $\overline{P} \rtimes \langle \overline{\alpha} \rangle$. As a consequence of Lemma 2.3, there exist integers $n \geqslant d-1$ and $t \geqslant 1$ such that $|G| = p^n q^t$. Since the Sylow p-subgroup $\overline{P}$ of $\overline{G}$ is normal, so is the Sylow p-subgroup P of G. Therefore, G can be written as a semidirect product $P \rtimes Q$, where Q is a Sylow q-subgroup. Since $Q/\Phi(Q)$ is isomorphic to $(G/P)/\Phi(G/P)$, it follows from $(\Phi(G)P)/P \subseteq \Phi(G/P)$ that $Q/\Phi(Q)$ is a quotient of the cyclic group $G/(\Phi(G)P)$. So, $Q = \langle \alpha \rangle$ for some $\alpha \in Q$, and $\mathrm{d}(P) = d-1$. By d-maximality, α^q induces the identity on $P/\Phi(P)$. From Lemma 2.4, we conclude that α^q must act trivially on the whole of P. □

Remark 2.6. Let $D = \max_{S \in \mathrm{Syl}(G)} \mathrm{d}(S)$. The inequality $d(G) > D$ plays a crucial role in our proof that a non-nilpotent d-maximal finite group G must be solvable. However, this inequality alone is not sufficient to deduce the solvability. Consider, for example, the direct product $\mathrm{Alt}(5) \times H$, where H is the semidirect product $(\mathbb{F}_{29})^2 \rtimes \langle \alpha \rangle$ with α of order 7 in $\mathbb{F}_{29}^{\times}$.

2.2 | Maximal (p, q)-pairs

Let G be a non-nilpotent d-maximal group and let P and α be as in Proposition 2.5. In particular, α^q generates a central subgroup of G contained in $\Phi(G)$. It follows that the quotient $G/\langle \alpha^q \rangle$ is again d-maximal and of order $p^n q$, for some positive integer n. Theorem 1.5 states that the study of these quotients is essentially equivalent to the investigation of maximal pairs.

Proof of Theorem 1.5. It follows from Proposition 2.5 that a d-maximal group G satisfies either (1) or (2). Conversely, assume $G = P \rtimes \langle \beta \rangle$ is as described in (2) with $|\beta| = q^t$. This implies that $\mathrm{d}(G) = d$. Let now H be a proper subgroup of G. If H is contained in P, then $\mathrm{d}(H) < d$ by property (a) of maximal pairs of rank $d-1$. So, assume that H is not contained in P, and let Q be a Sylow q-subgroup of H. If $|Q| < q^t$, then H is the direct product of $(H \cap P)$ and Q, and therefore, $\mathrm{d}(H) = \max(\mathrm{d}(H \cap P), \mathrm{d}(Q)) < d$. Finally, assume $|Q| = q^t$. Then, by the Sylow theorems, there exists $g \in G$ with $Q^g = \langle \beta \rangle$ and $H^g = (H^g \cap P)\langle \beta \rangle$. In particular, $H^g \cap P$ is β-invariant, and property (c) of maximal pairs of rank $d-1$ gives that $\mathrm{d}(H) = \mathrm{d}(H^g) < d$. □

2.3 | Actions through characters

In this section, let A be a finite group and let p be an odd prime. Let $\chi : A \to \mathbb{Z}_p^{\times}$ be a character. We define actions *through characters* and present some related results that we will apply in the study of maximal (p, q)-pairs.

Definition 2.7. The group A is said to act on a group G *through* χ if, for each $a \in A$ and $g \in G$, one has $g^a = g^{\chi(a)}$.

Remark 2.8. Let (P, α) be a maximal (p, q)-pair and $A = \langle \alpha \rangle$. Then, as a consequence of property (b) of maximal pairs, there exists a non-trivial character $\chi : A \to \mathbb{Z}_p^\times$ such that A acts on $P/\Phi(P)$ through χ. Moreover, it follows from property (c) that if H is a proper α-invariant subgroup of P with $\mathrm{d}(H) = \mathrm{d}(P)$ and such that A acts on $H/\Phi(H)$ through a character χ_H, then necessarily $\chi_H = 1$.

The following result is straightforward.

Lemma 2.9. *Let (P, α) and (Q, β) be maximal (p, q)-pairs of ranks d and e, respectively. Then the following hold.*

1. *If N is an α-invariant normal subgroup of P contained in $\Phi(P)$ and $\overline{\alpha} \in \mathrm{Aut}(P/N)$ is induced by α, then $(P/N, \overline{\alpha})$ is also a maximal (p, q)-pair of rank d.*
2. *If $\langle \alpha \rangle$ and $\langle \beta \rangle$ act on $P/\Phi(P)$ and $Q/\Phi(Q)$ through the same character, then $(P \times Q, (\alpha, \beta))$ is a maximal (p, q)-pair of rank $d + e$.*

The following results are taken from [12, Sec. 2] and use the same notation. In order, they are [12, Lemma 2.5], [12, Lem. 2.6], [12, Cor. 2.12] and [12, Cor. 2.13].

Lemma 2.10. *Let P be a finite p-group that is also an A-group and assume that the induced action of A on $P/\gamma_2(P)$ is through χ. Then, for all integers $i \geqslant 1$, the induced action of A on $\gamma_i(P)/\gamma_{i+1}(P)$ is through χ^i.*

Lemma 2.11. *Let P_1 and P_2 be finite p-groups that are also A-groups, and assume that A acts on P_1 through χ. Moreover, let $\phi : P_1 \to P_2$ be a surjective homomorphism respecting the action of A, that is, for all $a \in A$ and $g \in P_1$, one has that $\phi(g^a) = \phi(g)^a$. Then A acts on P_2 through χ.*

Lemma 2.12. *Let P be a finite abelian p-group on which A acts through χ. Assume that $A = \langle \alpha \rangle$ has order 2 and write*

$$P^+ = \{x \in P \mid \alpha(x) = x\} \quad \text{and} \quad P^- = \{x \in P \mid \alpha(x) = x^{-1}\}.$$

Then $P = P^+ \oplus P^-$.

Lemma 2.13. *Let P a finite p-group on which A acts through χ. Let N be a normal A-invariant subgroup of P such that the restriction of α to N equals the inversion map $x \mapsto x^{-1}$. Assume, moreover, that also the automorphism of P/N that is induced by α is equal to the inversion map. Then, α is the inversion map on P and P is abelian.*

3 | GENERAL RESULTS ON MAXIMAL PAIRS

Until the end of Section 3, let (P, α) denote a maximal (p, q)-pair of rank d and $A = \langle \alpha \rangle$. Moreover, let $\chi : A \to \mathbb{Z}_p^\times$ be the character through which A acts on $P/\Phi(P)$ as in Remark 2.8.

Lemma 3.1. *The following hold:*

(1) *one has* $\Phi(P) = \gamma_2(P)$*;*
(2) *for each* $i \geqslant 1$*, the quotient* $\gamma_i(P)/\gamma_{i+1}(P)$ *is elementary abelian;*
(3) *the induced action of* A *on* $\gamma_i(P)/\gamma_{i+1}(P)$ *is through* χ^i.

Proof. We start by proving (1). Applying Lemma 2.9(1) to $N = \gamma_2(P)$, we assume without loss of generality that P is abelian of exponent dividing p^2. Then pth powering is a homomorphism $P \to \mho_1(P)$, and therefore, it follows from Lemma 2.11 that A acts on $\mho_1(P)$ through χ. In order not to contradict property (c) of maximal pairs, the group P has to be equal to $\Omega_1(P)$, that is, P has exponent p. Now (2) immediately follows from (1), whereas (3) is the combination of (1) with Lemma 2.10. □

The following result follows directly from Lemma 3.1 and Remark 2.8.

Lemma 3.2. *Let* (P, α) *be a maximal pair of rank* d *and let* i *be a positive integer. Then the following hold:*

(1) *one has* $\mathrm{d}(\gamma_i(P)/\gamma_{i+1}(P)) \leqslant d$*;*
(2) *if* $i \geqslant 2$ *and* $\mathrm{d}(\gamma_i(P)/\gamma_{i+1}(P)) = d$*, then* $\chi^i = 1$.

The following definition is taken from [12, Sec. 2.3].

Definition 3.3. Let G be a finite p-group and let H be a subgroup of G. A positive integer j is called a *jump of* H in G if $H \cap \gamma_j(G) \neq H \cap \gamma_{j+1}(G)$.

Lemma 3.4. *Let* H *be an* A*-invariant subgroup of* P *for which the* p*th powering map is an endomorphism. Then, for each jump* ℓ *of* $\mho_1(H)$*, there exists a jump* i *of* H *such that* $i < \ell$ *and* $i \equiv \ell \bmod q$.

Proof. Let ℓ be a jump of $\mho_1(H)$ and let $y \in \mho_1(H) \setminus \{1\}$ be such that $y \in \gamma_\ell(P) \setminus \gamma_{\ell+1}(P)$. Since pth powering is an endomorphism of H, the subgroup $\mho_1(H)$ equals the set of pth powers of elements of H. Let $x \in H$ be such that $x^p = y$ and let i be the unique positive integer such that $x \in \gamma_i(P) \setminus \gamma_{i+1}(P)$. Then, i is a jump of H and $i < \ell$ thanks to Lemma 3.1(2). Moreover, the pth powering map induces a surjective homomorphism $\langle x\rangle\gamma_{i+1}(P)/\gamma_{i+1}(P) \to \langle y\rangle\gamma_{\ell+1}(P)/\gamma_{\ell+1}(P)$. It follows from Lemmas 2.11 and 3.1(3) that the induced action of A on $\langle y\rangle\gamma_{\ell+1}(P)/\gamma_{\ell+1}(P)$ is both through χ^i and χ^ℓ. As the order of α is q, this implies that $i \equiv \ell \bmod q$. □

Corollary 3.5. *Let* i *be a positive integer. Then* $\mho_1(\gamma_i(P))$ *is contained in* $\gamma_{i+q}(P)\gamma_{2i}(P)$.

Proof. Write $\overline{P} = P/\gamma_{2i}(P)$ and use the bar notation for the subgroups of $\overline{P}$. Then, $\gamma_i(\overline{P}) = \overline{\gamma_i(P)}$ is abelian, and therefore, pth powering on $\gamma_i(\overline{P})$ is an endomorphism. It follows from Lemma 3.4 that $\mho_1(\gamma_i(\overline{P}))$ is contained in $\gamma_{i+q}(\overline{P})$, and so, we derive that $\gamma_i(P)$ is contained in $\gamma_{i+q}(P)\gamma_{2i}(P)$. □

Corollary 3.6. *The group* $\mho_1(\gamma_2(P))$ *is contained in* $\gamma_4(P)$.

Proposition 3.7. *Assume that* P *has class 3. Then* P *does not have exponent* p.

Proof. For a contradiction, assume that P has exponent p. If M is a complement of $\gamma_2(P) \cap Z(P)$ in $\gamma_3(P)$, then Lemma 2.9 yields that $\overline{P} = P/M$ also belongs to a maximal pair and satisfies $Z(\overline{P}) = \gamma_2(\overline{P}) \cap Z(\overline{G})$. We assume thus, without loss of generality, that $\gamma_2(P) \cap Z(P) = \gamma_3(P)$ and, additionally, that $|\gamma_3(P)| = p$. Write $|\gamma_2(P) : \gamma_3(P)| = p^m$ and $C = C_P(\gamma_2(P))$. From the non-degeneracy of the map $P/C \times \gamma_2(P)/\gamma_3(P) \to \gamma_3(P)$, we derive that $|P : C| = p^m$. It follows that

$$|C| = \frac{|P|}{p^m} = \frac{|P : \gamma_2(P)| \cdot |\gamma_2(P) : \gamma_3(P)| \cdot p}{p^m} = p^{d+1}.$$

Not to contradict property (a) of maximal pairs, the commutator subgroup of C has to be nontrivial and so, being normal in P, we derive $\gamma_3(P) \subseteq \gamma_2(C)$. Note now that $\gamma_3(C) \subseteq [C, \gamma_2(P)] = 1$ and so C has class 2. As the commutator map $C \times C \to \gamma_2(C)$ is bilinear, we conclude that there exist $x, y \in C \setminus \gamma_2(P)$ such that $[x, y] \in \gamma_3(C)$. This is a contradiction to $\chi \neq 1$. □

4 | THE STRUCTURE OF REGULAR PAIRS

In the wide world of p-groups, the subclass of regular groups is somewhat tamer, sharing, in some sense, a number of properties with abelian groups. In this section, we study the effect of assuming regularity on a p-group P that belongs to a maximal (p, q)-pair (P, α). Moreover, we use regularity to prove general results on maximal pairs.

4.1 | Regularity

Let p be a prime number and let P be a finite p-group. Then, P is said to be *regular* if, for every $x, y \in P$, one has

$$(xy)^p \equiv x^p y^p \bmod \mho_1(\gamma_2(\langle x, y \rangle)).$$

The following lemma collects the properties of regular groups we will make use of. We refer the interested reader to [5, Sec. III.10] for more on regularity.

Lemma 4.1. *Let p be a prime number and P a finite p-group. Let, moreover, ℓ and k be non-negative integers and M and N be normal subgroups of P. Then, the following hold.*

(1) *If the class of P is at most $p - 1$, then P is regular.*
(2) *If the exponent of P is p, then P is regular.*
(3) *If the order of P is smaller than p^p, then P is regular.*
(4) *If $|P : \mho_1(P)| < p^p$, then P is regular.*
(5) *If P is regular, then $[\mho_\ell(M), \mho_k(N)] = \mho_{\ell+k}([M, N])$.*
(6) *If P is regular, then $\mho_k(P) = \{x^{p^k} \mid x \in P\}$ and $\Omega_\ell(P) = \{x \in P \mid x^{p^\ell} = 1\}$.*
(7) *If P is regular, then $|\mho_k(P)| = |P : \Omega_k(P)|$.*

Proof. In order, these can be found in Satz 10.2(a)-(d), Satz 10.13, Satz 10.8(a), Satz 10.5, Satz 10.7(a) and Satz 10.13 from [5, Ch. III]. □

Definition 4.2. A maximal (p, q)-pair (P, α) is called *regular* if P is regular.

As Lemma 4.1 together with the following lemma shows, regular pairs are very common among maximal (p, q)-pairs.

Lemma 4.3. *Let (P, α) be a maximal (p, q)-pair of rank d. If $p \geqslant 2d$, then P is regular.*

Proof. By Proposition 3.7, the quotient $P/\mho_1(P)$ has class at most 2 and this implies that $|P : \mho_1(P)| \leqslant p^{2d-1}$. Indeed, if the central $\gamma_2(P)$ had order p^d, we could easily construct an elementary abelian subgroup containing $\gamma_2(P)$ with index p, contradicting property (a). We derive that, if $p \geqslant 2d$, then $|P : \mho_1(P)| \leqslant p^{p-1}$ and P is regular by Lemma 4.1(4). $\square$

The next lemma is a stronger version of Corollary 3.5 for regular pairs.

Lemma 4.4. *Let (P, α) be a regular maximal (p, q)-pair and let $i > 0$ be an integer. Then, $\mho_1(\gamma_i(P)) \subseteq \gamma_{i+q}(P)\gamma_{4i}(P)$.*

Proof. Thanks to the regularity assumption, the pth powering map induces an endomorphism on $\gamma_i(P)/\mho_1(\gamma_{2i}(P))$. From Lemma 3.4 and Corollary 3.5, we conclude that $\mho_1(\gamma_i(P)) \subseteq \gamma_{i+q}(P)\mho_1(\gamma_{2i}(P)) \subseteq \gamma_{i+q}(P)\gamma_{4i}(P)$. $\square$

Proof of Theorem 1.9. We assume that P has class at least 3 and show that $q = 2$. As a consequence of Lemma 2.9, we assume without loss of generality that P has class 3. If $p = 3$, we have that $q = 2$, so we assume, additionally, that $p > 3$. Then, by Lemma 4.1(1), the group P is regular. Applying Proposition 3.7 and Lemma 4.4 with $i = 1$, we obtain that $\{1\} \neq \mho_1(P) \subseteq \gamma_{q+1}(P)$. In particular, $q + 1 \leqslant 3$ and so $q = 2$. $\square$

The derived length of odd order d-maximal groups is at most 3. The following restriction on their order follows.

Proposition 4.5. *Let G be a d-maximal group of odd order. If p is a prime and G is a p-group, then $|G| \leqslant p^{2d-1}$. Otherwise, there exist distinct primes p and q and integers $n \leqslant 2d - 3$ and $t \geqslant 1$ such that $|G| = p^n q^t$.*

Proof. If G is a p-group, then the class of G is at most 2, and, $\gamma_2(G)$ being elementary abelian, $|G| \leqslant p^{2d-1}$ follows. Otherwise, let (P, α) be as in Theorem 1.5. From Theorem 1.9, we know that the class of P is at most 2. Now, the number q being odd, the equality $|P| = |P : \gamma_2(P)||\gamma_2(P)|$ together with Lemma 3.2 provides $n \leqslant d - 1 + d - 2$, as desired. $\square$

4.2 | Regular pairs

We now focus exclusively on regular pairs. Because of this, until the end of this section, let (P, α) be a maximal regular (p, q)-pair of rank d. The results proven here are not only interesting for their own sake, but will be also applied in the study of maximal pairs of small rank.

Lemma 4.6. *Assume that P has class 3. Then, $\mho_1(P) = \gamma_3(P)$.*

Proof. Thanks to Theorem 1.9 and Lemma 4.4, we know that $\mho_1(P)$ is contained in $\gamma_3(P)$ and, by Proposition 3.7, that $\mho_1(P) \neq 1$. If $\mho_1(P)$ were properly contained in $\gamma_3(P)$, modding out by $\mho_1(P)$ would contradict Proposition 3.7, so we conclude that $\gamma_3(P) = \mho_1(P)$. □

Lemma 4.7. *Let $c \geqslant 3$ be the class of P. Then, $\mho_1(\gamma_{c-2}(P)) = \gamma_c(P)$.*

Proof. We work by induction on c and note that the case $c = 3$ is given by Lemma 4.6. Assume now that $c > 3$ and that the result holds for $c - 1$, in other words that $\gamma_{c-1}(P) = \mho_1(\gamma_{c-3}(P))\gamma_c(P)$. The subgroup $\gamma_c(P)$ being central, Lemma 4.1(5) yields the following:

$$\begin{aligned}\mho_1(\gamma_{c-2}(P)) &= \mho_1([P, \gamma_{c-3}(P)]) = [P, \mho_1(\gamma_{c-3}(P))]\\ &= [P, \mho_1(\gamma_{c-3}(P))\gamma_c(P)] = [P, \gamma_{c-1}(P)] = \gamma_c(P).\end{aligned}$$

This concludes the proof. □

Proposition 4.8. *Assume that the class of P is at least 3. Then, $\mho_1(P) = \gamma_3(P)$.*

Proof. Let c denote the class of P: we work by induction on c. The base of the induction is given by Lemma 4.6, so we assume that the result holds for $c - 1$, that is, that $\gamma_3(P) = \mho_1(P)\gamma_c(P)$. We assume also, without loss of generality, that $|\gamma_c(P)| = p$ and, for a contradiction, that $\gamma_c(P)$ is not contained in $\mho_1(P)$, that is, that $\mho_1(P) \cap \gamma_c(P) = 1$. It follows from Theorem 1.9 and Lemma 4.4 that $\mho_1(\gamma_{c-2}(P)) = 1$. However, the subgroup $\gamma_c(P)$ being central, Lemma 4.1(5) and Lemma 4.7 yield

$$\begin{aligned}\{1\} = \mho_1(\gamma_{c-2}(P)) &= \mho_1([P, \gamma_{c-3}(P)]) = [P, \mho_1(\gamma_{c-3}(P))]\\ &= [P, \mho_1(\gamma_{c-3}(P))\gamma_c(P)] = [P, \gamma_{c-1}(P)] = \gamma_c(P).\end{aligned}$$

Contradiction. □

Corollary 4.9. *Assume that the class of P is at least 3 and let i and j be positive integers. Then, the following hold.*

(1) $\mho_1(\gamma_i(P)) = \gamma_{i+2}(P)$.
(2) *If at least one of i and j is odd, then $[\gamma_i(P), \gamma_j(P)] = \gamma_{i+j}(P)$.*
(3) *If $i = 2k + 1$, then $|\gamma_i(P) : \gamma_{i+2}(P)| \leqslant p^d$.*

Proof.

(1) We work by induction on i and note that the claim holds for $i = 1$ thanks to Proposition 4.8. Assume now that $i > 1$ and that $\mho_1(\gamma_{i-1}(P)) = \gamma_{i+1}(P)$. It follows from Lemma 4.1(5) that

$$\mho_1(\gamma_i(P)) = \mho_1([P, \gamma_{i-1}(P)]) = [P, \mho_1(\gamma_{i-1}(P))] = [P, \gamma_{i+1}(P)] = \gamma_{i+2}(P).$$

(2) Without loss of generality, assume that i is odd and write $i = 2k + 1$. It follows from Corollary 4.9 and Lemma 4.1(5) that

$$[\gamma_i(P), \gamma_j(P)] = [\mho_k(P), \gamma_j(P)] = \mho_k([P, \gamma_j(P)]) = \mho_k(\gamma_{j+1}(P)) = \gamma_{j+1+2k}(P) = \gamma_{i+j}(P).$$

(3) Since $i > 1$, Point (1) yields that $\gamma_i(P)/\gamma_{i+2}(P)$ is elementary abelian. Not to contradict property (a), the number of generators of the last quotient is at most d.

□

Lemma 4.10. *Let a be a positive integer and assume that $|\gamma_{1+2a}(P) : \gamma_{2+2a}(P)| = p$. Then, the class of P is $1 + 2a$.*

Proof. As a consequence of Corollary 4.9, for any index i, one has $\mho_a(\gamma_i(P)) = \gamma_{i+2a}(P)$. Moreover, P being regular, we have $|\Omega_a(\gamma_i(P)| = |\gamma_i(P) : \mho_a(\gamma_i(P))| = |\gamma_i(P) : \gamma_{i+2a}(P)|$. In particular, we derive

$$\frac{|P|}{|\gamma_2(P)\Omega_a(P)|} = \frac{|P| \cdot |\Omega_a(\gamma_2(P))|}{|\gamma_2(P)| \cdot |\Omega_a(P)|} = \frac{|P| \cdot |\gamma_2(P)| \cdot |\gamma_{1+2a}(P)|}{|\gamma_2(P)| \cdot |P| \cdot |\gamma_{2+2a}(P)|} = \frac{|\gamma_{1+2a}(P)|}{|\gamma_{2+2a}(P)|} = p.$$

It follows from Corollary 4.9 and Lemma 4.1(5) that

$$\begin{aligned}\gamma_{2+2a}(P) &= \mho_a(\gamma_2(P)) = \mho_a([P,P]) = \mho_a([P, \gamma_2(P)\Omega_a(P)]) \\ &= [P, \mho_a(\gamma_2(P))] = [P, \gamma_{2+2a}(P)] = \gamma_{3+2a}(P)\end{aligned}$$

and thus $\gamma_{2+2a}(P) = 1$. □

5 | MAXIMAL PAIRS OF RANK 2

In this section, we classify the maximal (p, q)-pairs of rank 2 and, as a consequence, the finite 3-maximal groups. To this end, until the end of Section 5, let (P, α) be a maximal (p, q)-pair of rank 2.

Proposition 5.1. *The group P has maximal class.*

Proof. We fix $i \geqslant 2$ and show that $|\gamma_i(P) : \gamma_{i+1}(P)| \leqslant p$. Since $d = 2$, we know that $|\gamma_i(P) : \gamma_{i+1}(P)| \leqslant p^2$. Assume for a contradiction that $|\gamma_i(P) : \gamma_{i+1}(P)| = p^2$. Note that $\gamma_i(P)/\gamma_{i+2}(P)$ is abelian and, thanks to Corollary 3.5, its exponent divides p. Then, since $P/\gamma_2(P)$ is a two-dimensional vector space over $\mathbb{F}_p$ and the commutator map induces a surjective homomorphism $\wedge^2(P/\gamma_2(P)) \to \gamma_2(P)/\gamma_3(P)$, we have that $i > 2$. In particular, $\gamma_{i-1}(P)/\gamma_{i+1}(P)$ is abelian, of order at least p^3, and, by Corollary 3.5, of exponent p. This gives a contradiction to property (a) of maximal pairs. □

Lemma 5.2. *If $p > 3$, then P has order dividing p^4.*

Proof. Write $|P| = p^n$ and assume, for a contradiction, that $n \geqslant 5$. Thanks to Proposition 5.1, the group P has maximal class and thus a unique quotient $\overline{P}$ of order p^5 and class $4 \leqslant p-1$. Thanks to Lemma 4.1(1), the group $\overline{P}$ is regular, and therefore, Lemma 4.10 yields that $\overline{P}$ has class 3. Contradiction. □

Proposition 5.3. *Assume that $p > 3$. Then, P is isomorphic to one of the following:*

(1) *an elementary abelian group of order p^2;*
(2) *an extraspecial group of order p^3 and exponent p.*

Proof. We first prove that $|P| \leqslant p^3$. For a contradiction, suppose that $|P| \geqslant p^4$. Then, Lemma 5.2 yields that $|P| = p^4$ and, by Proposition 5.1, the class of P is 3. It follows from Lemma 4.1(1) that P is regular and from Proposition 4.8 that $\mho_1(P) = \gamma_3(P)$. Moreover, Theorem 1.9 ensures that $q = 2$. It is easily seen that $C = \mathrm{C}_P(\gamma_2(P))$ is abelian of order p^3. The rank of P being 2, this implies that $\mho_1(C) = \gamma_3(P)$, and so, C is different from $M = \Omega_1(P)$, which is also a maximal subgroup of P (see Lemma 4.1(7)). Since both C and M contain $\Phi(P)$, both subgroups are A-invariant. Write now $\overline{M} = M/\gamma_3(P)$ and note that $\overline{M}$ is abelian and A-invariant. Then, Lemma 2.12 implies that $\overline{M} = \overline{M}^+ \oplus \overline{M}^-$ where both summands have order p. Let N be the unique subgroup of M mapping to $\overline{M}^-$ in $\overline{M}$. Since A acts on $\gamma_3(P)$ through $\chi^3 = \chi$, we derive from Lemma 2.13 that N is an elementary abelian subgroup of order p^2 on which A acts through χ. This gives a contradiction to property (c) of maximal pairs and $d = 2$.

We have proved that $|P| \leqslant p^3$ and so $|P|$ is p^2 or p^3. If $|P| = p^2$, then clearly, P is elementary abelian; assume therefore that $|P| = p^3$. Thanks to Lemma 4.4, the exponent of P is equal to p and, the rank of P being 2, the group P is non-abelian. □

Proposition 5.4. *Assume that $p = 3$. Then, P is isomorphic to one of the following:*

(1) *an elementary abelian group of order 9;*
(2) *an extraspecial group of order 27 and exponent 3;*
(3) *the group* `SmallGroup(81,10)`.

Proof. The claim is easily verified when $|P| \leqslant 27$, we assume therefore that $|P| \geqslant 81$. The remaining part of the proof is computational and has been checked by all three authors in the computer algebra systems `GAP` [4] and `SageMath` [13].

Thanks to Proposition 5.1, we know that P has maximal class. There exist precisely four groups of order $3^4 = 81$ and maximal class up to isomorphism: these are the groups `SmallGroup(81,7)`, `SmallGroup(81,8)`, `SmallGroup(81,9)` and `SmallGroup(81,10)` in the `SmallGroup` library of `GAP` [3]. Each of these groups has an automorphism α of order 2 that induces scalar multiplication by -1 on the Frattini quotient. For each of these groups other than `SmallGroup(81,10)`, the subgroup generated by the elements of order 3 has order at least 27: this ensures that the group has a subgroup of order p^2 on which α acts as scalar multiplication by -1, contradicting property (c). On the contrary, the subgroup of `SmallGroup(81,10)` that is generated by the elements of order 3 is equal to the derived subgroup of `SmallGroup(81,10)`, from which it is not difficult to deduce that (`SmallGroup(81,10)`, α) is a maximal pair of rank 2 yielding the 3-maximal group `SmallGroup(162,22)`.

If we now move to the groups of order $3^5 = 243$, we find that `SmallGroup(243,26)` is the unique 3-group, up to isomorphism, of maximal class and order 243 that possesses an automorphism β of order 2 that induces scalar multiplication by -1 on the Frattini quotient. However, the quotient of `SmallGroup(243,26)` by its centre is isomorphic to `SmallGroup(81,9)` and thus not isomorphic to `SmallGroup(81,10)`. As a consequence of Lemma 2.9, we derive that `SmallGroup(243,26)` is not part of any maximal pair and our classification is therefore complete. □

5.1 | The classification of 3-maximal groups

Combining [1, Thm. 1.11, Prop. 4.3] (used for (1) and (2)) with Theorem 1.5, and Propositions 5.3 and 5.4 (used for (3)), we obtain the list of 3-maximal finite groups. Specifically, a finite group G is 3-maximal if and only if one of the following occurs.

(1) There exists an odd prime p such that G is a p-group. Moreover, G is isomorphic to one of the following groups:
  - (i) an elementary abelian group of order p^3;
  - (ii) the group of order p^4 defined by

$$\langle a, b, c \mid a^{p^2} = b^p = c^p = [a, b] = [a, c] = 1, [c, b] = a^p \rangle.$$

(2) The group G is a 2-group. More precisely, G is isomorphic to one of the following groups:
  - (i) an elementary abelian group of order 8;
  - (ii) the direct product $C_2 \times Q_8$;
  - (iv) the central product $C_4 * Q_8 = C_4 * D_8$;
  - (iv) `SmallGroup(32,32)`.

(3) There exist an odd prime p and a positive integer t such that G is a semidirect product $P \rtimes \langle \alpha \rangle$ where P is a p-group, α has order q^t for some prime q that divides $p - 1$, $\alpha^q \in \mathrm{Z}(G)$ and $G/\langle \alpha^q \rangle$ is isomorphic to one of the following:
  - (i) a semidirect product $P \rtimes C_q$, where P is elementary abelian of order p^2;
  - (ii) a semidirect product $P \rtimes C_q$ with P extraspecial of exponent p and order p^3;
  - (iii) `SmallGroup(162,22)`.

6 | MAXIMAL PAIRS OF RANK 3

In order to gather more evidence in the direction of answering the questions from the Introduction, in this section, we completely classify the maximal (p, q)-pairs of rank 3.

Lemma 6.1. *Let (P, α) be a maximal (p, q)-pair of rank 3. If P has nilpotency class 2, then P has order p^4, exponent p and $\gamma_2(P)$ of order p.*

Proof. For a contradiction, let (P, α) be a maximal (p, q)-pair with $\mathrm{d}(P) = 3$ and $\gamma_2(G)$ central of order p^2. The group P is regular by Lemma 4.1(1), and it has exponent p thanks to Lemma 4.4. Define $V = P/\mathrm{Z}(P)$ and $W = \gamma_2(P)$, with $\dim W = 2$. Then, the commutator map induces a surjective homomorphism $\phi : \wedge^2 V \to W$, showing, in particular, that V has dimension 3 (otherwise $\dim \wedge^2 V = 1$). It follows that ϕ has a one-dimensional kernel, spanned by $g\,\mathrm{Z}(P) \wedge h\,\mathrm{Z}(P)$, say. Then, the subgroup generated by g, h and $\gamma_2(G)$ is an abelian group of order p^4 and exponent p.

This gives a contradiction to property (a). We have proved that $|\gamma_2(p)| = p$, and thus, P has order p^4. □

Lemma 6.2. *Let (P, α) be a maximal (p, q)-pair of rank 3. If $p > 3$ and $|P| = p^5$, then the following hold.*

1. *The group $\gamma_2(P)$ is isomorphic to $C_p \times C_p$.*
2. *The group $\gamma_3(P)$ is isomorphic to C_p.*
3. *The group $\mathrm{C}_P(\gamma_2(P))$ is isomorphic to $C_{p^2} \times C_p \times C_p$.*
4. *The order of $\Omega_1(P)$ is equal to p^4.*
5. *One has $\gamma_2(\Omega_1(P)) = \gamma_2(P)$.*
6. *One has $q = 2$.*

Proof. Assume that $|P| = p^5$ and that $p > 3$. Then, P has class 3 by Lemma 6.1 and it is regular by Lemma 4.1(1). In particular, we have that $|\gamma_2(P)| = p^2$ and $|\gamma_3(P)| = p$ and, thanks to Theorem 1.9, that $q = 2$. Moreover, $\gamma_2(P)$ has exponent p by Corollary 3.6, and thus, (1)–(2)-(6) are proved. Set now $C = \mathrm{C}_P(\gamma_2(P))$ and note that C is maximal in P. Then, it holds that $[P, C] = \gamma_2(P)$ and also that $[C, [P, C]] = [C, \gamma_2(P)] = 1$. By the Three Subgroups Lemma, we have $[P, [C, C]] = 1$, yielding that $[C, C]$ is contained in $\gamma_3(P)$. Then, the commutator map induces a bilinear map $C/\gamma_2(G) \times C/\gamma_2(G) \to \gamma_3(G)$, yielding that either $\langle\alpha\rangle$ acts on $\gamma_3(P)$ through $\chi^2 = 1$ or $[C, C] = 1$. Since $\chi \neq 1$, we derive that C is abelian. Not to contradict property (a), we have therefore that $\exp(C) \neq p$ and, as a consequence of Proposition 4.8, that $\mho_1(C) = \gamma_3(P) = \mho_1(P)$. This proves (3) and Lemma 4.1(7) takes care of (4). We conclude by proving (5). To this end, let $M = \Omega_1(P)$ and, for a contradiction, assume that $\gamma_2(M) \subsetneq \gamma_2(P)$. If M is abelian, then we have a contradiction to property (a), so, since $\gamma_3(P) = \gamma_2(P) \cap \mathrm{Z}(P)$ and $\gamma_2(M)$ is normal in P, it holds that $\gamma_2(M) = \gamma_3(P)$. Define now $\overline{P} = P/\gamma_3(P)$ and use the bar notation for the subgroups of $\overline{P}$. The automorphism α induces an automorphism $\overline{\alpha}$ of $\overline{P}$ and it follows from Lemma 2.12 that $\overline{M} = \overline{M}^+ \oplus \overline{M}^-$. Let now N be a subgroup of P that contains $\gamma_3(P)$ and such that $\overline{N} = \overline{M}^-$. Since $\gamma_3(P) = \gamma_3(P)^-$, it follows from Lemma 2.13 that $N = N^-$ and N is abelian. Since N is contained in M, this yields a contradiction to property (c) of maximal pairs. □

Proposition 6.3. *Let (P, α) be a maximal (p, q)-pair of rank 3. If $p > 3$ and $|P| = p^5$, then $q = 2$ and P is uniquely determined up to isomorphism. Indeed, P is isomorphic to*

$$X = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^p = x_5, x_2^p = x_3^p = x_4^p = x_5^p = 1, [x_2, x_3] = x_4, [x_2, x_4] = x_5,$$

$$[x_1, x_2] = [x_1, x_3] = [x_1, x_4] = [x_1, x_5] = [x_2, x_5] = [x_3, x_4] = [x_3, x_5] = [x_4, x_5] = 1\rangle,$$

where the following hold:

- *the group $Y = \langle x_1, x_3, x_4\rangle$ is a maximal abelian subgroup of X;*
- *$\langle x_2\rangle$ is a complement of Y in X;*
- *one has $\langle x_5\rangle = \mho_1(X) = \mathrm{Z}(X)$.*

Proof. The result could be deduced from the list of finite groups of order p^5, given by Bender in [2]. In that paper, the groups of order p^5 are divided in 54 families, and the only one satisfying the conditions obtained in the previous lemma is the unique group in family 23. We prefer to give a direct proof. Let $C = C_P(\gamma_2(P))$ and $M = \Omega_1(P)$. It follows from Lemma 6.2, that

$$\gamma_2(P) \subseteq M \cap C \cong C_p \times C_p \times C_p,$$

so we may choose x_2, x_3, x_4, x_5 so that

$$M = \langle x_2, x_3, x_4, x_5 \rangle, \quad M \cap C = \langle x_3, x_4, x_5 \rangle, \quad \gamma_2(P) = \langle x_4, x_5 \rangle, \quad \gamma_3(P) = \langle x_5 \rangle.$$

Since $[M, M] = \gamma_2(P)$ and $[M, M] = [x_2, C \cap M]$, we may choose x_3, x_4 so that $[x_2, x_3] = x_4$ and $[x_2, x_4] = x_5$. Now let $y \in C \setminus M$. Since $\gamma_2(P) = [x_2, M \cap C]$, there exists $x \in M \cap C$ such that $[yx, x_2] = 1$. This implies $x_1 = yx \in Z(P)$. Then $x_1^p = y^p \in P^p = \gamma_3(P)$, so it is not restrictive to assume $x_1^p = x_5$. □

Remark 6.4. Let $P = X$ be the group described in Proposition 6.3. Then the map that sends x_4 to x_4 and $x_i \to x_i^{-1}$ if $i \in \{1, 2, 3, 5\}$ can be extended to an automorphism α of P of order 2. We verify that (P, α) is a maximal $(p, 2)$-pair of rank 3. Suppose that H is a proper subgroup of P with $\mathrm{d}(H) \geqslant \mathrm{d}(P) = 3$. Then $p^3 \leqslant |H| \leqslant p^4$. If $|H| = p^4$, then H is a maximal subgroup of P, and therefore, $\Phi(P) = \langle x_4, x_5 \rangle \subseteq H$. Moreover, either $H = \Omega_1(P)$ or $\exp(H) = p^2$. In any case, x_5 belongs to $\Phi(H)$. So, $\mathrm{d}(H) \leqslant 3$ and if $\mathrm{d}(H) = 3$, then $\Phi(H) = \langle x_5 \rangle$. In the latter case, since $x_4 \in H$ and $\alpha(x_4) = x_4$, the map α does not induce a non-trivial power automorphism of $H/\Phi(H)$. Finally, suppose that H is elementary abelian of order p^3 and that α induces a non-trivial power automorphism on H. It must be that H is contained in $\Omega_1(P)$ and $x_4 \notin H$. This is impossible, because H would be a maximal subgroup of $\Omega_1(P)$ and it would contain $\Phi(\Omega_1(P)) = \langle x_4, x_5 \rangle$.

Proposition 6.5. *Let (P, α) be a maximal (p, q)-pair of rank 3. If $p > 3$, then the order of P is at most p^5.*

Proof. Assume for a contradiction that $|P| = p^6$ and that $p > 3$. Then P has class at least 3 by Lemma 6.1 and, since P has rank 3, the index $|\gamma_3(P) : \gamma_4(P)|$ is either p or p^2. The group P is regular thanks to Lemma 4.1(1). Since in the first case, Lemma 4.10 yields that P has class 3 and that $|\gamma_2(P) : \gamma_3(P)| = p^2$: this contradicts Lemma 6.1 combined with Lemma 2.9. We have thus proved that P has class 3 and that $|\gamma_3(P)| = p^2$. Observe now that the surjective homomorphism $\wedge^2(P/\gamma_2(P)) \to \gamma_2(P)/\gamma_3(P)$ that is induced by the commutator map has a non-trivial kernel. We fix $g\gamma_2(P) \wedge h\gamma_2(P) \neq 0$ in such kernel and define $M = \langle g, h \rangle \gamma_2(P)$. Then, M has order p^5 and $\gamma_2(M)$ is contained in $\gamma_3(P)$. Since $\chi \neq 1$, it follows that $\gamma_2(M) = 1$, contradicting the fact that $|\,\mathrm{C}_P(\gamma_2(P)) : \gamma_2(P)| = p$. □

Remark 6.6. The example described in Proposition 6.3 occurs also when $p = 3$. There exists, however, another non-isomorphic maximal (3,2)-pair with P of order 3^5, namely P is the direct product $C_3 \times X$, where X is isomorphic to `SmallGroup(81,10)`. This is indeed a consequence of Lemma 2.9 and Proposition 5.4. A computational check through the `SmallGroup` library of `GAP` reveals that there is also a unique possibility of order 3^6: if $\tilde{P}$ is equal to `SmallGroup(729,148)`, then $\tilde{P}$ has an automorphism $\tilde{\alpha}$ of order 2, with the property that the semidirect product $\tilde{P} \rtimes \langle \tilde{\alpha} \rangle$, which is isomorphic to `SmallGroup(1458,805)`, is a 4-maximal group. This information allows us to prove that there exists no maximal (3,2)-pair (P, α) of rank 3 with $|P| \geqslant 3^7$. For this purpose, it suffices to exclude the possibility $|P| = 3^7$. Assume by contradiction that such a group P exists. Then $\tilde{P}$ would be an epimorphic image of P. Since $\tilde{P}$ has nilpotency class 3, the class of $\tilde{P}$ would be either 3 or 4. In the first case, $|\gamma_3(P)| = 3 \cdot |\gamma_3(\tilde{P})| = 3^3$, but this is impossible. So, $|\gamma_4(P)| = 3$ and $P/\gamma_4(P) \cong \tilde{P}$. There are 1023 groups P with $|P| = 3^7$ satisfying $|\gamma_2(P)| = |\Phi(P)| = 3^4$ and $|\gamma_4(P)| = 3$, but a computational check shows that none of them satisfies $P/\gamma_3(P) \cong \tilde{P}$.

Remark 6.7. When $d > 3$, there exist maximal pairs (P, α) of rank d, with P of class 2, but $\gamma_2(P)$ is non-cyclic. For example, there are three maximal pairs (P, α) or rank 4, up to isomorphism, such that $|P| = 3^6$, and $\gamma_2(P) \cong C_3 \times C_3$. These are `SmallGroup(1458,1540)`, `SmallGroup(1458,1576)` and `SmallGroup(1458,1613)`.

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