UNIVERSITY OF TRENTO
Department of Mathematics

PhD Thesis-Cycle XXXVI

EDGE-COLORINGS AND FLOWS IN CLASS 2 GRAPHS

CANDIDATE:
Gloria Tabarelli

SUPERVISOR:
Prof. Giuseppe Mazzuoccolo

ACKNOWLEDGEMENTS

First of all a big thank to Giuseppe Mazzuoccolo, for having made this journey possible, accepting me as his PhD student. For your time, patience, good advice and kindness. It has been a great pleasure to have you as a supervisor, thank you!

Thank to Peter Schuster for your help and presence in these years, and to Francesca Mantese, for your interest and disposability.

I also would like to thank Edita Máčajová, for having accepted me as a visitor student in Bratislava, Martin Škoviera, Jozef Rajník, František Kardoš and all the combinatorial group of Comenius University for their kind and warm welcome in Bratislava. I really enjoyed that time, thank you all.

Thank you Davide and Jean Paul, for your kindness and advices.

Thank you Lorenzo, Isaak, Yulai, Domenico, Federico, Jozef, Giulia, Simone, Simona, Arrigo, Gloria, Tommaso, Marién, Anita, Marco, Anamari, Francesca: it has been a real pleasure to spend time with you at conferences.

Finally, I would like to thank all the people that have been part of my journey in mathematics, from its very beginning almost 10 years ago. In this time, I experienced joys, sorrows, true friendships, truly nice and really hard times. All of this has been possible because of any single one of you, and will always remain in the memories book of my life. Thank you from the bottom of my heart for having written this chapter with me.

INTRODUCTION

This dissertation is focused on problems that are unsolved for Class 2 graphs. To be Class 2, for a graph, is opposite to be Class 1, and qualifies the non-existence of an edge-coloring of the graph with a specific number of colors.
Indeed, a classical result in graph theory, namely Vizing's Theorem, ensures that a simple graph with maximum degree Δ admits a proper edge-coloring either with Δ or with $\Delta + 1$ colors. In the former case the graph is said to be Class 1, in the latter Class 2. To be Class 1 or Class 2 for a graph is somehow significant in graph theory since many open and outstanding conjectures are completely solved for Class 1 graphs, while still unsolved for, maybe some classes, of Class 2 ones.
In this thesis we consider problems that are strongly related to 4 of these outstanding conjectures, namely the Cycle Double Cover Conjecture, the Berge-Fulkerson Conjecture, the Petersen Coloring Conjecture and the Tutte's 5-flow Conjecture. The former two of them are about covering the edge-set of a graph with families of cycles or perfect matchings, respectively. They are both implied by the Petersen Coloring Conjecture, which is about coloring a graph using another graph. Finally, the Tutte's 5-flow Conjecture is about flows in graphs and proposes a general upper bound for the so-called flow number of a graph. All these conjectures are still unsolved only for the Class 2 cubic graphs.

In Chapter 1 we introduce the edge-coloring problem in graphs and we survey some major results about it. We also consider Class 2 regular graphs, focusing especially on cubic ones, and we list some of their main features and classic results that appear in literature about them. After this, in Section 1.4, we state the problems we work on in the thesis, highlighting their strong relation with the mentioned conjectures.

The problem considered in Chapter 2 concerns directly Cycle Double Cover and Berge-Fulkerson Conjectures. We try to determine properties of a possible minimum counterexample to these conjectures. In both cases we want to establish if such possible counterexamples have a high cyclic-edge-connectivity. This is a problem that deserves attention since it is conjectured (see [53, 59]) that highly cyclically-edge-connected Class 2 cubic graphs do not exist. To our aims, we apply a technique developed in [60] and what we obtain are new strong restrictions on the structure of such possible minimum counterexamples.

Chapter 3 is devoted to the Petersen Coloring Conjecture, and, more in general, to H-colorings of cubic and r-regular graphs. Generalizing results of [68, 34, 35] we prove that, in a specific sense, the Petersen graph is the only graph that could possibly appear in the statement of the Petersen Coloring Conjecture, meaning that it is the only graph that could possibly color all bridgeless cubic graphs. We also consider a generalization of the problem posed by the Petersen Coloring Conjecture for r-regular graphs, $r > 3$. We provide evidence that, when $r > 3$, there is no possibility to have an analogous of the Petersen Coloring Conjecture. In other words, we prove that there is not an r-regular (multi)graph that colors all r-regular (multi)graphs. Our evidence has been subsequently strengthened in [58], where it is proved that even considering subclasses of r-regular graphs, there is no possibility to color all of them with a single graph.

The topic of Chapter 4 is d-dimensional flows, which are a generalization in higher dimension of the flows considered by the Tutte's 5-flow Conjecture. As we will highlight, while the case $d \geqslant 3$ has already been considered in literature, the case $d = 2$, i.e. the one of two-dimensional flows, has not. The main purpose of our work is to state an analogous of the Tutte's 5-flow Conjecture in dimension 2. More specifically, we want to propose a general upper bound for the so-called 2-dimensional flow number of a cubic graph. When dealing with Class 1 cubic graphs, we manage to give a tight upper bound, while for Class 2 ones, again, we have an upper bound, but we cannot prove that it is tight. Another main focus of our work about 2-dimensional flows concerns lower bounds for the 2-dimensional flow number of cubic graphs. Indeed, determining lower bounds for this parameter turns out to be a hard problem, even for very small graphs like the Petersen graph. However, with geometric arguments, we manage to give a non-trivial lower bound for the 2-dimensional flow number of a cubic graph in terms of its odd-girth.

The last problem we deal with in this dissertation is palette index, in Chapter 5. The palette index of a graph G is the minimum number of palettes that can appear around the vertices of G in a proper edge-coloring of the graph. Determining the palette index of a regular graph G is a trivial problem when G

is Class 1, while it turns out to be challenging when G is Class 2. However, despite being an edge-coloring problem, at first sight, this problem seems deeply different from the others considered in this thesis. Instead, we will see that it can be stated in terms of H-colorings in the context of hypergraphs. Results that appeared in literature about palette index concern the determination of the palette index of some particular classes of graphs, such as for example trees, complete graphs, bipartite graphs and others, while results on general graphs do not appear. We manage to prove a sufficient condition for a general graph to have palette index larger its minimum degree. We also provide a characterization of graphs having palette index at most 3 in terms of decompositions into Class 1 regular subgraphs.

We conclude the dissertation with Chapter 6, where we summarize the main results and open problems of the previous chapters, focusing attention on new possible research lines.

LIST OF CONTRIBUTIONS

[P1] E. Máčajová, G. Mazzuoccolo, and G. Tabarelli. Cycle Separating Cuts in possible counterexamples to the Cycle Double Cover and the Berge-Fulkerson Conjectures. 2024. Submitted.

[P2] G. Mazzuoccolo, G. Tabarelli, and J.P. Zerafa. On the existence of graphs which can colour every regular graph. *Discret. Appl. Math.*, 337:246–256, 2023. https://doi.org/10.1016/j.dam.2023.05.006.

[P3] D. Mattiolo, G. Mazzuoccolo, J. Rajník, and G. Tabarelli. On d-dimensional nowhere-zero r-flows on a graph. *Eur. J. Math.*, 9(101), 2023. https://doi.org/10.1007/s40879-023-00694-1.

[P4] D. Mattiolo, G. Mazzuoccolo, J. Rajník, and G. Tabarelli. A lower bound for the complex flow number of a graph: a geometric approach. *J. Graph Theory.*, 1–18, 2024. https://doi.org/10.1002/jgt.23075.

[P5] D. Mattiolo, G. Mazzuoccolo, J. Rajník, and G. Tabarelli. Geometric descriptions of d-dimensional flows of a graph. 2023. Submitted.

[P6] D. Mattiolo, G. Mazzuoccolo, and G. Tabarelli. Graphs with large palette index. *Discr. Math.*, 345(5), 2022. https://doi.org/10.1016/j.disc.2022.112814.

[P7] D. Labbate, D. Mattiolo, G. Mazzuoccolo, F. Romaniello, and G. Tabarelli. A Characterization of Graphs with Small Palette Index. *Symmetry*, 15(154), 2023. https://doi.org/10.3390/sym15010154.

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1

INTRODUCTION AND BACKGROUND

In this dissertation we will consider a bunch of problems in graph theory that are difficult for a specific class of graphs, namely the cubic Class 2 regular graphs. The main reason to do so is that in graph theory there are several outstanding conjectures for which proving them to be true for this class of graphs would imply them to be true in general. In this chapter we report some major results about cubic Class 2 regular graphs that appear in the literature and some of their known features and properties. We conclude presenting some important conjectures that are still open for this class of graphs and relating these conjectures to the problems we will consider in the next chapters.

1.1 CLASS 1 AND CLASS 2 REGULAR GRAPHS

Depending on the context we consider, a graph G may not admit parallel edges nor loops, i.e. be *simple*, admit parallel edges but no loops, i.e. be a *multigraph* or it may admit both parallel edges and loops, i.e. be a *pseudograph*. Moreover G could be *directed* or *undirected*. We will clearly specify time by time to which class of graphs we are referring to. In this first chapter we consider undirected multigraphs.

The leading concept we will deal with all along the dissertation is the one of proper edge-coloring of a graph. More precisely, a *proper edge-coloring* of a graph G is an assignment of colors to the edges of G in such a way that any pair of edges that are incident to a common vertex receive different colors. If a proper edge-coloring uses k distinct colors, we say that it is a proper k-edge-coloring. If a graph admits a proper k-edge-coloring we say that it is k-*edge-colorable*. From now on, we will only consider proper edge-colorings, so that we will just write "edge-coloring" instead of "proper edge-coloring".

The minimum integer k such that G admits a k-edge-coloring is said to be the *edge-chromatic number* or *chromatic index* of G and it is denoted by $\chi(G)$. Clearly, if Δ is the maximum degree of G, then $\chi(G) \geqslant \Delta$. Moreover, by a classical result of Vizing [89], a graph G admits an edge-coloring with $\Delta + \mu$ colors, where μ denotes the multiplicity of G (i.e. the maximum number of parallel edges joining two distinct vertices of the graph). This means that $\chi(G) \leqslant \Delta + \mu$.

Definition 1.1. *A graph* G *is said to be* Class 1 *if* $\chi(G) = \Delta$, Class 2 *if* $\chi(G) > \Delta$.

If G is simple ($\mu = 1$), by Vizing's theorem we have either $\chi(G) = \Delta$ or $\chi(G) = \Delta + 1$, so that, according to Definition 1.1, a simple graph G is Class 1 if $\chi(G) = \Delta$ and Class 2 if $\chi(G) = \Delta + 1$.

Definition 1.1 partitions the class of graphs with maximum degree Δ into two subclasses, according to their edge-chromatic number. This partition is useful especially when considering *regular graphs*, that are graphs where every vertex has the same degree. Here and in what follows, if every vertex of a graph G has degree $r \geqslant 0$, we will say that G is an r*-regular graph*. Recall that a *perfect matching* of a graph G is a set M of edges of G such that every vertex of G is incident to exactly one of the edges of M. By Definition 1.1, an r-regular graph is Class 1 if and only if it admits an r-edge-coloring, and this means it can be decomposed into r edge-disjoint perfect matchings, each one of them corresponding to one of the r colors in the r-edge-coloring. Hence, Class 2 r-regular graphs are the ones that cannot be decomposed into r edge-disjoint perfect matchings.

The first question that arises at this point is whether, for each $r \geqslant 0$, Class 1 and Class 2 r-regular graphs exist. If $r = 0$ or $r = 1$ every r-regular graph G is clearly Class 1, since it has either no edges or its edges form a perfect matching themselves. For each $r \geqslant 2$ there are both Class 1 and Class 2 r-regular graphs. Indeed, for example, the complete bipartite graph $K_{r,r}$ is Class 1 and r-regular for each r. On the other side, a Class 2 r-regular graph can be constructed, for each $r \geqslant 2$, by applying Theorem 1.2, a classical result by Tutte, [88]. Since we will use this construction in Chapter 5, we describe it in some detail. If G is a graph and $S \subseteq V(G)$, the graph $G - S$ is the graph obtained from G by deleting the vertices in S and all the edges adjacent to a vertex in S. Moreover, we denote by $q(G)$ the number of odd connected components of a graph G.

Theorem 1.2 (Tutte's theorem)**.** *A graph* G *has a perfect matching if and only if, for every* $S \subseteq V(G)$, $q(G - S) \leqslant |S|$.

Observe that in any r-regular graph G, $r \geqslant 2$, a sufficient condition to be Class 2 is to not have a perfect matching. To construct an example of an r-regular Class 2 graph, let $k = \lfloor \frac{r}{2} \rfloor$. Consider k copies of $K_{r,r}$ and delete an edge e in each of them. Connect the vertices that were previously adjacent to e in each copy to a new vertex v by an edge. Denote this new graph by G_v. If r is even, then G_v is r-regular and it has an odd number of vertices, that is $2rk + 1$. Hence, it comes directly from the definition of perfect matching that it cannot have one. However, to conclude the same, we could apply Theorem 1.2 with $S = \emptyset$, and observe that $1 = q(G) > |S| = 0$. If r is odd, consider r copies of G_v, and connect the vertex v in each copy to a new vertex u by an edge. The graph so obtained is r-regular, and, by applying Theorem 1.2 with $S = \{u\}$ we have $r = q(G - S) > |S| = 1$, so that it does not have a perfect matching.

As a remark, we want to emphasize that the previous construction relies on a sufficient condition for a graph to be Class 2, while this condition is far from being also necessary. Indeed there are many regular graphs admitting perfect matchings that are also Class 2, as we will see in the rest of this dissertation.

Hence, we have easily seen that both Class 1 and Class 2 r-regular graphs exist for $r \geqslant 2$. The other natural question that arises is how, given some r-regular graph G, we can determine whether it is Class 1 or Class 2. If $r = 2$, the answer is straightforward, since 2-regular graphs are either cycles or disjoint union of cycles. In this case, if the graph contains an odd cycle, the graph is Class 2, for otherwise it is Class 1. However, if $r \geqslant 3$ this question seems not to have an easy answer. Indeed, it has been proved (see [40]) that this problem is NP-complete, even if we just consider $r = 3$. In Section 1.2 we give more details on this.

Another question that arises when dealing with Class 1 and Class 2 regular graphs is quantitative: for any given r, which is the proportion of Class 1 r-regular graphs with respect to Class 2 ones? To answer this question, one can observe that, when r is even, every r-regular graph G of odd order is Class 2, so that in this case there are “many" Class 2 graphs. In the odd regular case, there cannot be graphs of odd order, and hence it is not that obvious to sketch an answer. The cubic case ($r = 3$) has been considered in the literature and it has been proved (see [73]) that almost all cubic graphs are Class 1. In particular, if

C_n denotes the probability to be Class 1 for a cubic graph of order $2n$, as a consequence of the main result in [73], it holds that $\lim_{n\to\infty} C_n = 1$. Hence, in the class of cubic graphs, Class 2 ones are very rare. The proof in [73] is probabilistic, but Class 2 cubic graphs are very rare also in a strictly numerical sense. Indeed, in [12], all (weakly) *non-trivial* Class 2 cubic graphs up to 34 vertices have been generated. With (weakly) *non-trivial* we mean a specific subset of Class 2 cubic graphs that are considered interesting for reasons we will highlight in Section 1.3. As we see in Table 1, for each order, there are almost no (weakly non-trivial) Class 2 cubic graphs among all cubic graphs of that order. We stress that for orders greater than 36, there is no known algorithm that can generate all cubic graphs, and in particular Class 2 ones, of a given order, since their number is huge.

Order	*Cubic graphs*	*Class 2 non-trivial cubic graphs*	*Ratio*
4	1	0	0
6	2	0	0
8	6	0	0
10	21	1	4.8×10^{-2}
12	94	0	0
14	540	0	0
16	4 207	0	0
18	42 110	2	4.7×10^{-5}
20	516 344	6	1.6×10^{-5}
22	7 373 924	31	4.2×10^{-6}
24	118 573 592	155	1.3×10^{-6}
26	2 103 205 738	1 297	6.2×10^{-7}
28	40 634 185 402	12 517	3.1×10^{-7}
30	847 871 397 424	139 854	1.6×10^{-7}
32	18 987 149 095 005	1 764 950	9.3×10^{-8}
34	454 032 821 688 754	3 833 587	8.4×10^{-9}

Table 1: Number of (connected) cubic graphs and (weakly) non-trivial Class 2 cubic graphs for each order up to 34. Column 4 reports, for each order, the ratio between the number of Class 2 non-trivial cubic graphs and the number of cubic graphs. Data in Columns 2 and 3 are taken from [74] and [12], respectively.

As mentioned in the introduction of the chapter, cubic graphs, and especially cubic Class 2 graphs, are of particular interest in graph theory, since many long-lasting and important conjectures are still open for this class of graphs. We devote the rest of the chapter to explore this class of graphs in more detail.

1.2 CLASS 2 CUBIC GRAPHS ARE DIFFICULT TO COLOR

In this section we see in detail that determining whether a cubic graph is Class 1 or not is NP-complete. For a wide survey and results about the theory of NP-completeness we refer to [30], while here we just give a quick and informal introduction to it, which is enough for our aims. The reader which is familiar with the theory of NP-completeness may skip the next paragraph.

For our purposes, a *problem* is a general question to be answered, where there usually are several *parameters*, left unspecified. In order to state a problem we need a general description of its parameters and the list of properties that the *solution* is required to satisfy. When particular values are specified for all the parameters of a problem, we have an *instance* of such a problem. Once an instance of a problem is given, we want to know whether the problem has a solution for that given instance, and, if this is the case, to explicitly know it. A general sequence of steps, i.e. an *algorithm*, solves a problem if, applied to any instance of that problem, is guaranteed to always produce a solution to the problem for that given instance, if there is one. We are interested in finding the most *efficient* algorithm to solve a problem, and, since time requirements are a dominant factor to determine whether an algorithm is useful or not, an algorithm can be considered efficient if it is fast. The measure of the time requirements of an algorithm is its *time complexity function*, that receives in input the *size* of the problem instance and calculates the biggest amount of time needed by the algorithm to solve an instance of the problem of that size. A *polynomial time algorithm* is an algorithm whose time complexity function can be bounded from above by a polynomial function. If this is not the case, with a slight abuse of terminology, we say that we are dealing with *exponential time* algorithms. A problem can be considered "well solved" when a polynomial time algorithm is known for it. Otherwise, if it so hard that no polynomial time algorithm can possibly solve it, a problem is said to be *intractable*. The main causes of intractability of a problem are two. The first occurs when the problem is so difficult that an exponential time is needed to discover a solution, and the second occurs when the solution itself is so extensive that it cannot be described with an expression having a polynomial lenght in the size of the instance. This second type of intractability is significant and it is important to recognize it when it occurs, but in what follows we focus only on the first type of intractability. In this first type of intractability we can further distinguish two subclasses of problems. The *undecidable* ones, that are the ones for which no algorithm at all, and in particular no polynomial time algorithm, can be given for solving them. Otherwise, we speak of *decidable* problems. In this class there are problems which can be solved in polynomial time using what is called a *nondeterministic algorithm*. A nondeterministic algorithm can be viewed as composed of two separate stages. The first one is a guessing stage and the second is a checking stage. In the guessing stage the algorithm merely guesses a solution to the problem, while the checking stage is an algorithm that checks whether the guessed solution is actually a solution to the problem or not. A nondeterministic algorithm operates in polynomial time if the algorithm at the checking stage is a polynomial time algorithm. We say that a problem is in the class NP if there exists a polynomial time nondetermistic algorithm that solves it. A problem is said to be in the class P if there is a polynomial time algorithm that solves it. Since every polynomial time algorithm that solves a problem in P may be used as the checking stage for a nondeterministic algorithm for the same problem, it follows that the class P is contained in the class NP, i.e. $P \subseteq NP$. However, there are many reasons to believe that this inclusion is proper, one of them is that a nondeterministic algorithm has the possibility to check an arbitrary large number of possible solution in polynomial time. This leads to suspect that nondeterministic algorithms are more powerful than standard algorithms. Whether $P \neq NP$ or not is an open difficult problem, namely one of the Millennium Problems proposed by the Clay Mathematics Institute. However, if we work under the hyphotesis that $P \neq NP$, the class $NP - P$ is of particular interest. To study it, we would like to be able to compare problems in terms of how difficult they are. To do this we introduce the idea of *reduction* of a problem to another. We say that a problem P_1 reduces to a problem P_2 if there is a polynomial time algorithm that transforms any instance of P_1 in an instance of P_2, such that an instance has a solution for P_1 if and only if the transformed instance has a solution for P_2. Since the reduction of P_1 to P_2 happens in polynomial time, we can say that P_2 is *at least as hard as* P_1. We say that a problem P is NP-complete if $P \in NP$ and for any other problem $P' \in NP$, it holds that P' reduces to P. Hence, NP-complete problems can be viewed as the hardest problems in NP. It is possible to prove that, if for two problems P_1, $P_2 \in NP$, P_1 is NP-complete and P_1 reduces to P_2, then also P_2 is NP-complete. Hence, once we know that a problem is NP-complete, we can prove that other problems are NP-complete simply by proving they are NP and by reducing the known NP-complete problem to one of them. This is, as said above in a very informal and quick way, all that we need about NP-completeness for our purposes.

In particular, we need that to prove that a problem is NP-complete, it suffices to show that it is in NP and to reduce a known NP-complete problem to it. This is what is done in [40] with the edge-chromatic number problem.

In [40], the NP-complete problem known as 3-SAT is reduced to the problem of finding a 3-edge-coloring of a cubic graph. The problem 3-SAT is defined as follows. Let $U = \{u_1, ..., u_m\}$ be a set of Boolean variables. Let $u_i \in U$, and let the negation of u_i, which is a Boolean variable which is true if and only if u_i is false, be $\overline{u}_i$. We call u_i and $\overline{u}_i$ *literals* over U. A *clause* $C = \{l_1, l_2, l_3\}$ is a set of three literals over U. We say that C is *satisfied* if there exists a truth assignment to the variables of U such that at least one of l_1, l_2 and l_3 has the value "true". An *instance* of 3-SAT is a set of clauses $\mathcal{C} = \{C_1, ...C_t\}$ and the problem is to determine if $\mathcal{C}$ is *satisfiable*, that is if there exists a truth assignment to $u_1, ..., u_m$ such that all the clauses in $\mathcal{C}$ are simultaneously satisfied.

Example 1.3. *Let* $U = \{u_1, u_2, u_3, u_4\}$ *and let* $C_1 = \{u_1, \overline{u_2}, u_3\}$ *and* $C_2 = \{u_1, \overline{u_3}, \overline{u_4}\}$*. Then* $\mathcal{C} = \{C_1, C_2\}$ *is an instance of 3-SAT and it is satisfiable. Indeed, any truth assignment to the variables of* U *such that* u_1 *receives the value "true" makes both* C_1 *and* C_2 *satisfied. If, instead,* $C_1 = \{u_1, u_1, u_1\}$, $C_2 = \{u_3, u_3, u_3\}$ *and* $C_3 = \{\overline{u_1}, \overline{u_3}, \overline{u_3}\}$*, then* $\mathcal{C} = \{C_1, C_2, C_3\}$ *is an instance of 3-SAT which is not satisfiable. Indeed, to satisfy* C_1 *and* C_2*, both* u_1 *and* u_3 *must receive the value "true", but this makes* C_3 *not satisfied.*

It is possible to reduce the so called problem SAT to 3-SAT. The problem SAT is basically the same as 3-SAT, but each clause has an arbitrary (finite) number of literals. In [18], it is proved that SAT is an NP-complete problem, and hence also 3-SAT is NP-complete, since it is also in NP.

The problem of finding the chromatic index of a cubic graph, the chromatic index problem from now on, is clearly in NP, since every edge-coloring of a cubic graph (instance) can be verified in polynomial time. Here we show in detail how the polynomial-time reduction of 3-SAT to this problem, given in [40], is performed.

The idea is to start from a given instance $\mathcal{C}$ of 3-SAT and to associate to it a certain cubic graph G which is 3-edge-colorable if and only if $\mathcal{C}$ is satisfiable. To construct G we need three types of components, namely *inverting*, *variable-setting* and *satisfaction-testing* ones. These components will be glued together according to the structure of $\mathcal{C}$.

An inverting component is depicted in Figure 1, together with its symbolic representation, which will be used in what follows. An inverting component I has the property that in any 3-edge-coloring of I, one of the pairs of the dangling edges $\{a_1, a_2\}$ or $\{b_1, b_2\}$ must receive the same color, while the remaining 3 dangling edges must receive pairwise different colors.

A variable-setting component is depicted in Figure 2. As the name suggests, this component is associated to a variable u_i appearing in $\mathcal{C}$. In particular, if u_i and $\overline{u_i}$ appear n times in $\mathcal{C}$, then, the associated variable-setting component to u_i, is made by gluing together $2n$ inverting components as in Figure 2, in such a way that it has n *output pairs* $\{p_{1,j}, p_{2,j}\}$, each of them associated to the j-th appearence of u_i or $\overline{u_i}$ in $\mathcal{C}$. In any 3-edge-coloring of a variable-setting component, for each output pair $\{p_{1,j}, p_{2,j}\}$, the dangling edges $p_{1,j}, p_{2,j}$ must receive the same color.

Finally, a satisfaction-testing component is associated to a clause C_i of $\mathcal{C}$, and it is constructed by gluing together 3 inverting components where a vertex has been suppressed, as depicted in Figure 3. This component admits a 3-edge-coloring if and only if in at least one of the three input pairs $\{q_{1,j}, q_{2,j}\}$, the dangling edges $q_{1,j}$ and $q_{2,j}$ receive the same color. We say that a pair of dangling edges in a component *represents the value* T (true) if, in a 3-edge-coloring of the component, its edges receive the same color. Otherwise the pair is said to represent the

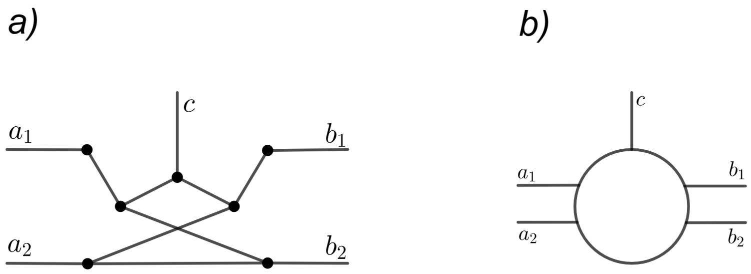


Figure 1: *(a)* An inverting component.
(b) Its symbolic representation.

value F (false). Using this convention, it is clear that a clause is satifiable if and only if its associated satisfaction-testing component is 3-edge-colorable.

We are now in a position to exhibit the polynomial-time reduction from the problem 3-SAT to the chromatic index problem. Consider an instance $\mathcal{C} = \{C_1, C_2, C_3\}$ of 3-SAT. Construct from it a cubic graph G as follows. For each variable u_i appearing in $\mathcal{C}$ take the associated variable-setting component U_i. Take also, for each clause $C_j \in \mathcal{C}$, its associated satisfaction-testing component S_j. Let $l_{j,k}$ denote the k-th literal of C_j, for $k, j \in \{1, 2, 3\}$. If $l_{j,k} = u_i$, then identify the k-th input pair of S_j with the output pair of U_i which is associated to the considered appeareance of u_i. If $l_{j,k} = \overline{u_i}$ then insert an inverting component between the k-th input pair of S_j and the corresponding output pair of U_i. Denote the cubic graph with dangling edges so obtained by H (see Figure 4 for an example). The graph G is obtained by considering two copies of H and identifying the remaining dangling edges in corresponding pairs. Clearly, G is obtained from $\mathcal{C}$ using a polynomial-time algorithm, and moreover, using the properties of the components described above, it is possible to verify that G is 3-edge-colorable if and only if $\mathcal{C}$ is satisfiable.

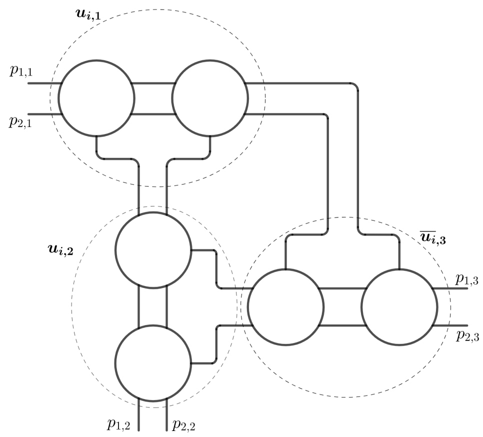


Figure 2: The variable setting component associated to the variable u_i, when it appears three times, two as u_i (labeled as $u_{i,1}$ and $u_{i,2}$) and one as $\overline{u_i}$ (labeled as $\overline{u_i}_{,3}$).

1.3 SOME FEATURES OF CUBIC CLASS 2 GRAPHS

As highlighted in Sections 1.1 and 1.2, Class 2 cubic graphs are difficult both to find, since they are very few, and to recognize, since it is NP-complete to determine the chromatic index of a graph. In this section we get more in touch with the Class 2 cubic graphs, reporting some of their known features and giving some examples of significant graphs in this class.

It is very easy to see that a cubic graph containing a bridge (i.e. an edge whose removal disconnects the graph) is Class 2. This can be proved by observing that, since the number of vertices of odd degree in a graph is even, each of the two components separated by a bridge must have an odd number of vertices. But this means that a bridge must be contained in every perfect matching of the graph, so that the graph cannot be Class 1.
The first bridgeless Class 2 cubic graph was found in 1898 by J. Petersen [70], as a counterexample to a conjecture of Tait, which claimed that all bridgeless cubic graphs are 3-edge-colorable. This graph is the Petersen graph and it is depicted in Figure 5. From now on, we will denote it either by its name or by P. Actually the Petersen graph appeared for the first time 12 years ear-

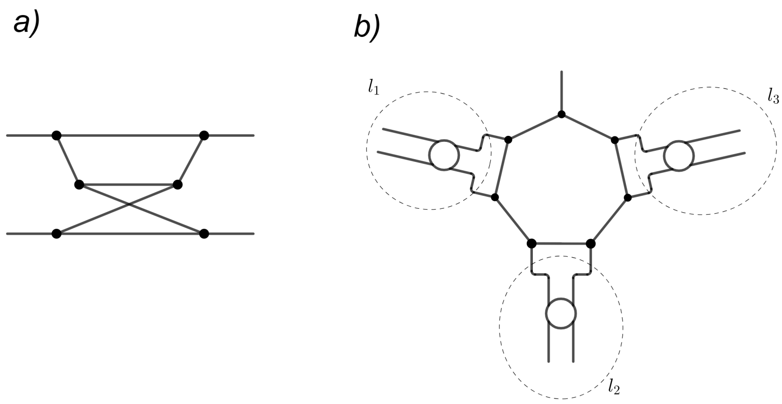


Figure 3: *(a)* An inverting component with a vertex suppressed.
(b) The satisfaction-testing component associated to a clause $C = \{l_1, l_2, l_3\}$.

lier in a different context, precisely in a paper by A. B. Kempe, [50], where he noticed it is related to the Desargues configuration. During time, P revealed itself to be very significant in graph theory: it is a counterexample for many conjectures and it has many symmetry properties. For the interested reader, the book [39] furnishes a wide compendium of the properties of the Petersen graph, relating them to the various parts of graph theory where P plays a significant role. In [20], B. Descartes wrote:

> I have often tried to find other cubic graphs which cannot be three-coloured. I do think that the right way to attack the Four-Colour Theorem is to classify the exceptions to Tait's Conjecture and see if any correspond to graphs in the plane. I did find some, but they were mere trivial modifications of the Petersen graph, obtained by detaching the three edges meeting at some vertex from one another so that the vertex becomes three vertices, and joining these three by additional edges and vertices so as to obtain another cubic graph. [...] I wondered if there could be any other exceptions to Tait's Conjecture, besides the Petersen graph. [...] I did eventually discover one.

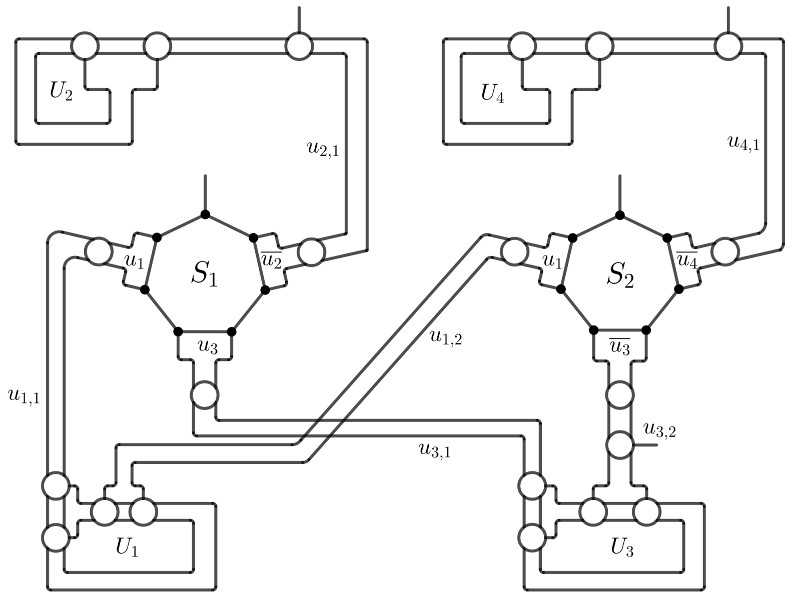


Figure 4: The graph H, associated to the instance $\mathcal{C}$ of 3-SAT in Example 1.3.

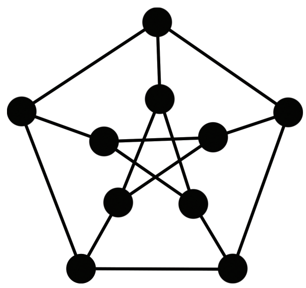


Figure 5: The Petersen graph

These words, which date back to 1948, highlight that finding bridgeless cubic Class 2 graphs significantly different from the Petersen graph, have been a difficult challenge. Indeed, up to 1975, only four bridgeless Class 2 cubic graphs were found. This lead M. Gardner to call them *snarks*, referring to the elusive object of the poem *The hunting of the snark* by Lewis Carroll [29].

In particular, after Petersen graph was discovered, in 1946 two new snarks were found, the Blanuša snarks (see Figure 6 and Figure 7), both having 18 vertices [7].

Also, in 1948, a snark on 210 vertices, the Descartes snark, was discovered by B. Descartes in [20], and in 1973 a snark on

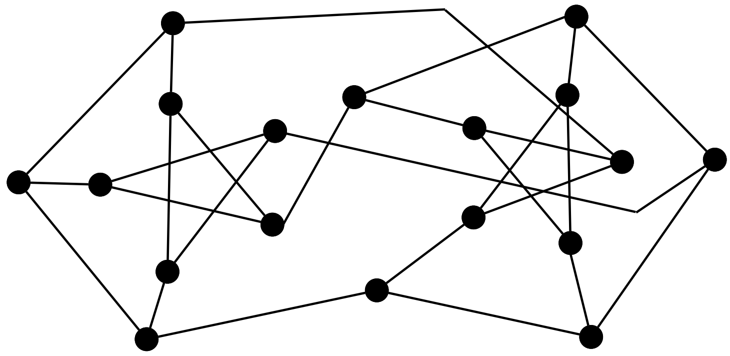


Figure 6: First Blanusa snark

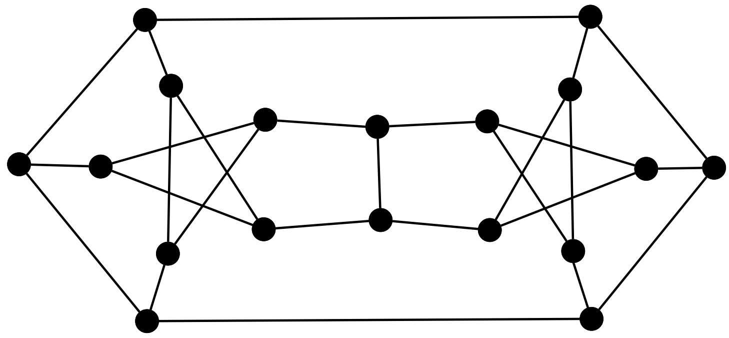


Figure 7: Second Blanusa snark

50 vertices, the Szekeres snark, was found [84]. Until 1975 only these mentioned snarks were known, while in that year Isaacs [44] proved the existence of two infinite families of snarks. One of these is the family of the so-called *flower snarks*, which, according to [39], was already discovered by Grinberg in 1972, but never published. The Isaac snark I_n, for each odd n, $n \geqslant 3$, is a graph on $4n$ vertices. I_3 and I_5 are depicted in Figure 8.

The other infinite family of snarks discovered by Isaacs is the one of the so-called BDS snarks, which also includes the two Blanuša snarks, the Szekeres and the Descartes snarks. Isaacs discovered also another snark on 30 vertices, the Double star snark.

After that, many other snarks and infinite families of snarks have been discovered. Among the others we mention the Goldberg snarks [33], another infinite family of snarks found in 1981. Surveys on snarks can be found in e. g. [17, 91, 92].
The family of flower snarks was constructed by Isaacs applying the technique of the *dot product* (see Figure 9).

Definition 1.4. *A dot product of two connected cubic graphs L and R is a cubic graph, denoted by $L{\cdot}R$, constructed from L and R as follows:*

(1) remove a pair of adjacent vertices x and y from L

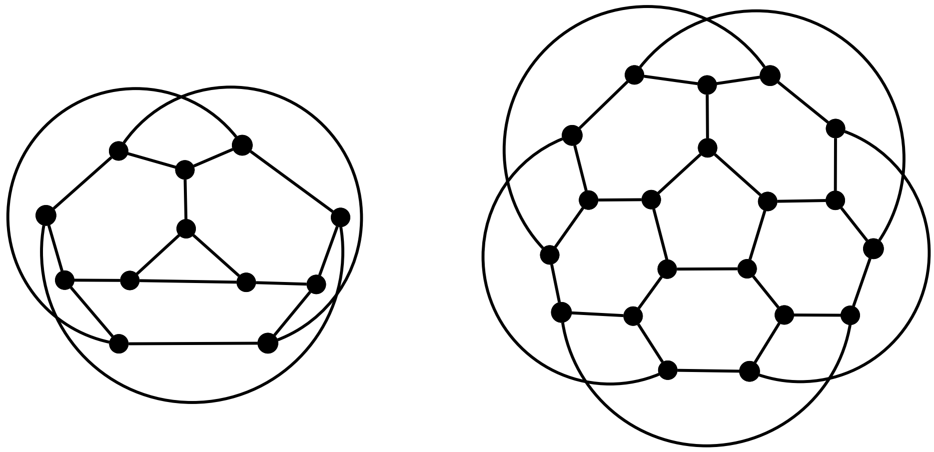


Figure 8: The Isaacs snarks I_3 and I_5

(2) *remove two independent edges* ab *and* cd *from R*

(3) *join the previous neighbours of* x *with* a *and* b, *and the previous neighbours of* y *with* c *and* d, *or the previous neighbours of* x *with* c *and* d, *and the ones of* y *with* a *and* b

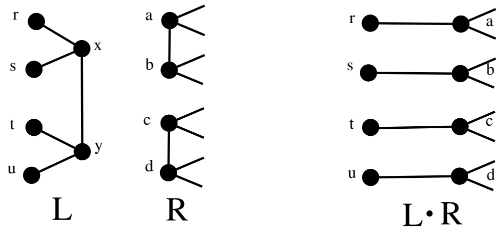


Figure 9: The dot product of two cubic graphs L and R.

The dot product of two cubic graphs is not unique, it depends on the choice of vertices and edges in L and R and by the possible choice in point (3) of the definition.

In [44], it is proved that if L and R are snarks, then their dot product $L \cdot R$ is a snark. This makes the dot product a useful tecnique to generate snarks.

Another useful and more general tecnique to generate snarks is the one known as *superposition*, which is due to Kochol and appears in [56]. This tecnique is based on the idea to replace, in a snark, its edges by snarks and its vertices by arbitrary cubic graphs, in a way to obtain a new snark. This tecnique allows to generate snarks with large girth (which, in a graph G, is the length of a smallest cycle of G). As remarked in [25], the superposition technique to construct new snarks of Kochol, is equivalent to a technique proposed independently by Fiol in [23] and [24]. Other tecniques to construct snarks can be found, e.g. in [69, 82, 83, 22].

The aforementioned techniques to construct new snarks required quite a long time to be discovered. However snarks have some properties, that, as mentioned by Descartes in [20], are well-known for a long time, which allow to construct snarks in a very straightforward way. This happens, for example, when a cubic graph has a 2-edge-cut or a non-trivial 3-edge-cut (a k-edge-cut is a set of k edges whose removal disconnects the graph. It is non-trivial when at least one edge remains in all the generated connected components). It can be easily seen that if G is a snark and it has a 2-edge-cut or a non-trivial 3-edge-cut, then at least one of the connected components (properly completed to a cubic graph) obtained by the removal of the edge-cut is a snark. This means that G is a snark that can be somehow *reduced* to a smaller snark, and hence can be considered *trivial*.

Another well known property of snarks is that removing cycles of length 3 or 4 does not affect the edge-uncolorability, so that also snarks having such cycles can be considered *trivial*.

Actually, it has become standard (see e.g. [69]) to consider a snark to be *non-trivial* when it has girth at least 5 and it is cyclically 4-edge-connected. For a graph, to be cyclically k-edge-connected, means that any edge-cut with less than k-edges, disconnects the graph leaving at least one component that does not contain a cycle.
We remark that many of the known families of snarks are non-trivial in this sense. Apart for the reasons explained above, one of the main reasons that led to consider a snark non-trivial if it has these additional properties, is that it turns out that for some open and important conjectures (see Section 1.4), a possible minimum counterexample must be a non-trivial snark in the above sense. In Section 1.1, we mentioned the concept of *weakly non-trivial snark* as it is in [12]: it is intended to be a cyclically 4-edge-connected snark with girth at least 4. Anyway, in this dissertation, and according to what is done e.g. in [13, 82] and others, we keep the concept of *snark* as wide as possible, and we say that a snark is simply a bridgeless Class 2 cubic graph. When additional properties such as cyclic-edge-connectivity or girth are required, we will clearly specify it.

Definition 1.5. *A* snark *is a connected, bridgeless, Class 2 cubic graph.*

After this general introduction to Class 2 cubic graphs, the next section is devoted to introduce some important and open problems in which snarks are deeply involved.

1.4 OPEN CONJECTURES AND PROBLEMS IN THE NEXT CHAPTERS

Some classical conjectures that are unsolved for snarks regard *graph coverings* of two main types, with cycles or with matchings. Two of them are the *Cycle Double Cover Conjecture* and the *Berge-Fulkerson Conjecture*, which are both considered in Chapter 2 and presented in Section 1.4.1 and 1.4.2, respectively. Also the *Petersen Coloring Conjecture* is presented in Section 1.4.3. This conjecture is about edge-colorings, is unsolved for snarks and serves as introduction to H-colorings, which we will study in Chapter 3. Moreover, it implies both Cycle Double Cover and Berge-Fulkerson conjectures. In Section 1.4.4 we present another important conjecture, the *Tutte* 5*-flow Conjecture*, which is unsolved for snarks and serves as an introduction to the problem of d-dimensional flows on graphs, which is considered in Chapter 4. Finally, in Section 1.4.5, we introduce a more recent parameter which is related to edge-colorings of graphs, the *palette index*. In Chapter 5 we focus on the problem of determining the palette index of regular graphs, which is also unsolved only for Class 2 regular graphs.

1.4.1 *Cycle Double Cover Conjecture*

One of the main conjectures about coverings with cycles is the *Cycle Double Cover Conjecture*. By a *cycle* in a graph G we mean a connected subgraph of G where every vertex has even degree. A *cycle cover* of a graph G is a collection $\mathcal{C}$ of cycles of G such that every edge of G belongs to at least one cycle of $\mathcal{C}$. We say that a cycle cover $\mathcal{C}$ of a graph G is *uniform* if every edge of G belongs to the same number k of cycles of $\mathcal{C}$. If this is the case, $\mathcal{C}$ is a k-cycle cover. A 2-cycle cover is also called a *cycle double cover*. The Cycle Double Cover Conjecture is due to Seymour [79] and Szekeres [84], and is stated for graphs that could possibly have loops.

Conjecture 1.6 (Cycle Double Cover Conjecture-CDC)**.** *Every bridgeless graph admits a cycle double cover.*

First observe that being bridgeless is a necessary condition for a graph G to have a cycle cover. Indeed no bridge can belong to an even subgraph.

Now we show that a minimum possible counterexample G to Conjecture 1.6 is a cyclically 4-edge-connected snark. By minimum, in this case, we mean that G is the graph with the smallest number of edges that does not admit a cycle double cover.

Clearly G is connected, bridgeless and has no loops by minimality (if ℓ is a loop of G, $G - \{\ell\}$ has a cycle double cover that can be extended to G by adding the cycle ℓ twice).

Observe that if G has a vertex of degree 1 it would have a bridge. Moreover, assume that v is a vertex of G of degree 2 and u, w are its neighbours. Remove the vertex v and create a new edge uw: this results in a smaller graph G', that admits a cycle double cover. But then it is possible to reconstruct a cycle double cover of G by deleting the edge uw of the two cycles passing through it and adding to them the edges uv and vw. So that G cannot have vertices of degree smaller than 3.

The following lemma is due to Fleischner [26].

Lemma 1.7 (Splitting lemma)**.** *Let G be a connected bridgeless graph. Suppose $v \in V(G)$ such that $d_G(v) \geqslant 3$ and x, y, z are three edges incident with v. Form the graphs $G_{x,y}$ and $G_{x,z}$ by splitting away*[1] *the pairs $\{x, y\}$ and $\{x, z\}$, respectively, and assume x and z belong to different blocks if v is a cut vertex*[2] *of G. Then either $G_{x,y}$ or $G_{x,z}$ is connected and bridgeless.*

Lemma 1.7 allows to prove that G cannot have vertices of degree greater than 3, and hence is a cubic graph. Indeed, by previous discussion G has no vertex of degree smaller than 3. Assume that there exists a vertex v with degree strictly greater than 3. It follows from Lemma 1.7 that it is possible to find two edges e_1 and e_2 incident to v with the following property: by deleting e_1 and e_2 and adding a new edge joining the ends of e_1 and e_2 distinct from v, one obtains a bridgeless graph G'. G' has a cycle double cover and to construct a cycle double cover for G it is enough, for each one of the two cycles C_i containing the

1 The indicated splitting operations was introduced by Fleischner and works as follows. Let G be a connected graph and $v \in V(G)$ with $d_G(v) \geqslant 3$. If $x = vv_1$ and $y = vv_2$ are two edges incident with v, then splitting away the pair $\{x, y\}$ of edges from the vertex v results in a new graph $G_{x,y}$ obtained from G by deleting the edges x and y, and adding a new edge v_1v_2.

2 A cut vertex in a connected graph G is a vertex v such that $G - \{v\}$ is not connected.

added edge e, to construct new cycles in G as $C_i - \{e\} \cup \{e_1, e_2\}$. This is a contradiction, and hence we can conclude that G is cubic.

Finally we can prove that G is cyclically 4-edge-connected. Indeed, assume that G has an edge-cut of size 2 given by the edges ab and cd, with vertices a and c belonging to the same connected component of $G - \{ab, cd\}$. Remove them and add the edges ac and bd. The obtained connected, bridgeless cubic graphs have a cycle double cover since they are smaller than G, and so both the edges ac and bd lies in exactly two cycles in each component. It is then straightforward to reconstruct a cycle double cover for G: consider one cycle C_1 that contains ac in one component and one cycle C_2 containing bd in the other one. Construct a new cycle in G as $C_1 - \{ac\} \cup C_2 - \{bd\} \cup \{ab, cd\}$. Repeat the procedure for the other cycles containing ac and bd respectively and a cycle double cover for G is obtained. This is a contradiction, so G cannot contain any cycle separating 2-edge-cut.

Assume by contradiction that G has a non-trivial 3-edge-cut given by the edges aa', bb' and cc', with a, b, c in the same connected component of $G - \{aa', bb', cc'\}$. By joining a, b and c to a new vertex v and a', b', c' to a new vertex v', two cubic bridgeless graphs G' and G'' with fewer edges than G are obtained. Both G' and G'' admit a cycle double cover: observe that if C_1 and C_2 are the two cycles containing the edge av, then, up to renaming cycles, C_1 contains bv and not cv and C_2 contains cv and not bv. Hence there is a cycle C_3 containing bv and cv. The same holds for three cycles C'_1, C'_2 and C'_3 in a cycle double cover of G''. Using the same technique as before to extend cycles to G, it is straightforward to reconstruct a cycle double cover for G, which results in a contradiction.

Moreover, G cannot be Class 1. Indeed, consider a 3-edge-coloring c of G with colors 1,2 and 3. Observe that the edge-induced subgraph by a pair $\{x, y\}$ of colors of c, is a union of cycles of G. Then the collection of cycles given by the pairs $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ is a cycle double cover for G.

As a final remark, in [43] it is proved that G has girth at least 12.

Hence we have the following proposition.

Proposition 1.8. *A possible minimum counterexample to Conjecture 1.6 is a cyclically* 4*-edge-connected snark with girth at least* 12*.*

Proposition 1.8 reduces the Cycle Double Cover Conjecture to a subclass of snarks, and hence gives a strong reason to the importance that has been given to the study of snarks during time.

In Chapter 2 we apply a technique proposed in [60] to the study of a minimum possible counterexample to Conjecture 1.6, and we obtain some new restrictions on its structure.

1.4.2 *Berge-Fulkerson Conjecture*

Berge-Fulkerson conjecture is about coverings of graphs with perfect matchings. It first appeared in a paper of Fulkerson [28] and it is also attributed to Berge [78]. It suggests the following.

Conjecture 1.9. *(Berge-Fulkerson, 1971) Every bridgeless cubic graph* G *admits six perfect matchings such that every edge of* G *belongs to exactly two of them.*

Conjecture 1.9 is equivalent to another conjecture of Berge, which appeared in [78].

Conjecture 1.10 (Berge Conjecture, 1979)**.** *Every bridgeless cubic graph* G *admits* 5 *perfect matchings such that every edge of* G *belongs to at least one of them.*

It is clear that Conjecture 1.9 implies Conjecture 1.10, since it suffices to remove one of the six perfect matching in a Berge-Fulkerson cover to obtain a Berge cover with five perfect matchings. As proved in [64], Conjecture 1.10 implies Conjecture 1.9, so that they are equivalent.

Conjecture 1.9 has numerous connections to other problems in graph theory as well as relations with other geometric structures (see [60] for more details and references about these connections). In particular, as we will mention in Chapter 3, this conjecture, if true, would imply other weaker conjectures about coverings with matchings in cubic graphs.

Until now only partial results have been obtained about this conjecture: it has been verified for some explicitly defined classes of cubic graphs, see for example [27, 37, 38, 14, 64].

In particular, as we expect, the conjecture is true for Class 1 cubic graphs. Indeed every color in a 3-edge-coloring of a Class

1 cubic graph induces a perfect matching: to construct a family of six perfect matchings satysfying Berge-Fulkerson condition it is enough to repeat each of these induced perfect matchings twice.

Hence, as it happens for the Cycle Double Cover Conjecture, also Berge-Fulkerson Conjecture is reduced to snarks. An attempt to attack the conjecture is based on another conjecture of Jaeger and Swart which states that every snark has cyclic edge connectivity at most 6 [48]. Indeed it is not difficult to prove that a possible counterexample to Conjecture 1.9 must be a cyclically 4-edge-connected snark. Moreover, in [60], the following proposition is proved.

Proposition 1.11. *A possible minimum counterexample to Conjecture 1.9 is a cyclically* 5*-edge-connected snark with girth at least* 5*.*

Hence, if the conjecture of Jaeger and Swart is true, a possible counterexample to Berge-Fulkerson conjecture can only have cyclic edge connectivity 5 or 6, and so, proving that it must have cyclic edge connectivity at least 7 would imply the Berge-Fulkerson conjecture.

It is considered safe enough to rely on this strategy of proof since there are other conjectures on Hamiltonian graphs that lead to believe that highly cyclically edge connected cubic graphs are Class 1. One of these conjectures, due to C. Thomassen (see [53, 59]), suggests that for large enough κ, every cyclically κ-edge-connected cubic graph is Hamiltonian, and hence 3-edge-colorable. Jaeger and Swart conjecture is of the same style, but it weakens *Hamiltonian* to *3-edge-colorable*. There exists an example of a cubic graph, the Coxeter graph, which is cyclically 7-edge-connected but not Hamiltonian: hence in Thomassen's conjecture κ must be at least 8. If Thomassen's conjecture is valid for $\kappa = 8$, clearly there cannot exist cyclically 8-edge-connected snarks, but since there are no counterexamples to Jaeger and Swart conjecture of cyclic edge connectivity 7, it is believed that the conjecture is valid also for cyclically 7-edge-connected cubic graphs.

In Chapter 2 we rely on this proof strategy and we focus on trying to prove that a minimum possible counterexample to the Berge-Fulkerson Conjecture is a cyclically 6-edge-connected snark. We obtain some new restrictions on the structure of a possible cyclically 5-edge-connected minimum counterexample to the conjecture.

1.4.3 *Petersen Coloring Conjecture*

The Petersen Coloring Conjecture implies both the Berge-Fulkerson Conjecture and the Cycle Double Cover Conjecture [79, 84, 94], and it is one of the most trying and arduous conjectures in graph theory.

We give some terminology in order to introduce it. Let H be an arbitrary graph: an H-*coloring* of G is a proper edge-coloring $f : E(G) \to E(H)$ of G with edges of H, such that for each vertex $u \in V(G)$, there exists a vertex $v \in V(H)$ with $f(\partial_G u) = \partial_H v$, where $\partial_G u$ denotes the set of edges adjacent to u in the graph G. If G admits an H-colouring, then we write $H \prec G$ and we say that the graph H colors the graph G. Let P denote the well-known Petersen graph. One of the most important conjectures in graph theory is the Petersen Coloring Conjecture by Jaeger, [46].

Conjecture 1.12 (Petersen Coloring Conjecture)**.** *For any bridgeless cubic graph* G, $P \prec G$.

As it happens for Conjecture 1.6 and Conjecture 1.9, also the Petersen Coloring Conjecture is easy for Class 1 cubic graphs and it is still unsolved for snarks. Indeed, if G is a cubic Class 1 graph, take a 3-edge-coloring c of G with colors $\{1, 2, 3\}$, consider a vertex $u \in V(P)$ and set $\partial_P u = \{e_1, e_2, e_3\}$. Let $h \colon \{1, 2, 3\} \to \partial_P u$ such that $h(i) = e_i$ for each $i \in \{1, 2, 3\}$. Then $f = h \circ c$ is a P-coloring of G. In the case of snarks, it has been proved that, for a graph G, to have a Petersen Coloring is equivalent to have *normal* 5*-edge-coloring* [47]. A normal k-edge-coloring of a cubic graph is a k-edge-coloring such that the set of colors assigned to any edge and to the four edges adjacent to it, has cardinality either 3 or 5. In literature there are several results regarding normal edge-colorings in cubic Class 2 graphs, see for example [1, 36, 75, 65, 66]. In particular, in [65], it is proved that any simple cubic graph, not necessarily bridgeless, has a normal 7-edge-coloring, which is the best general upper bound that is known up to now (and it is also optimal for the class of simple cubic graphs).

In Chapter 3, generalising results contained in [68, 34, 35], we prove that P is the only possible graph that can color all the bridgeless cubic graphs, and some related results of the same type regarding other weaker conjectures.

1.4.4 *Tutte 5-flow Conjecture*

We introduce some definitions in order to state Tutte's 5-flow conjecture. The first one is that of an *orientation* D of a graph G, that is an assignment of a direction to each edge.

If D is an orientation on a graph G then for each vertex v of G it is natural to define $D_-(v)$ as the class of edges whose tail is incident to v and $D_+(v)$ as the class of edges whose head is incident to v.

A *flow* on a graph G is a pair (D, f) where D is an orientation of G and $f\colon E(G) \to \mathbb{Z}$ is a function such that for every vertex $v \in V(G)$ the Kirchhoff's law holds, that is

$$\sum_{e \in D_-(v)} f(e) = \sum_{e \in D_+(v)} f(e)$$

For an integer $k \geqslant 2$, a *nowhere-zero k-flow* on a graph G is a flow (D, f) on G such that $f\colon E \to \mathbb{Z}$, $-(k-1) \leqslant f(e) \leqslant k-1$ and $f(e) \neq 0$ for every edge e of G. The *flow number* of a graph G is the minimum integer k such that G admits a nowhere-zero k-flow, and it is denoted by $\phi(G)$.

Tutte's 5-flow conjecture [86] states the following.

Conjecture 1.13 (Tutte 5-flow Conjecture)**.** *Every bridgeless graph has a nowhere-zero 5-flow.*

Hence, Tutte's conjecture indicates a possible general upper bound for the flow number of a bridgeless graph G, namely $\phi(G) \leqslant 5$.

In the direction to prove this conjecture, in the late seventies Jaeger [45] and Kilpatrick [52] proved that every bridgeless graph admits a nowhere-zero 8-flow: this result has been improved by Seymour in 1981 [77]. He proved that every bridgeless graph admits a nowhere-zero 6-flow, so that what we have up to now is that $\phi(G) \leqslant 6$ for every bridgeless graph G.

Instead, the study of possible minimum counterexamples to Conjecture 1.13 led to the proof, in 1988, that a minimum counterexample to the conjecture must be 3-edge-connected [46], while more recent results due to Kochol [54, 55] ensures that a possible minimum counterexample, if it exists, must be a cyclically 6-edge-connected snark with girth at least 11. This means that also Conjecture 1.13 is reduced to snarks, and adds a strong motivation to the study of the structure of snarks. Moreover, in [86] and [87] it is proved that, for a cubic graph G,

- G is Class 1 if and only if it has a nowhere-zero 4-flow
- G is biparite if and only if it has a nowhere-zero 3-flow

Hence the flow number can distinguish Class 1 from Class 2 cubic graphs.

There is a well known and well studied generalization of the concept of nowhere-zero k-flow on a graph. This is the one of *circular nowhere-zero* r*-flow*. More precisely, if G is a graph, and $r \geqslant 2$ is a real number then a circular nowhere-zero r-flow on G is a pair (D, f) where D is an orientation of G and $f\colon E(G) \to \mathbb{R}$ is a real valued function such that $f(e) \in [1, r-1]$ for all edges $e \in E(G)$ and the Kirchhoff's law holds at every vertex of G. The *circular flow number* of a graph G is the infimum of the real numbers r such that G admits a circular nowhere-zero r-flow, and it is denoted by $\phi_c(G)$. As a consequence of a result in [32], it holds that $\phi_c(G) \in \mathbb{Q}$ and it is a minimum for every bridgeless graph G. It clearly holds that $\phi_c(G) \leqslant \phi(G)$ for every bridgeless graph G, so that Conjecture 1.13 proposes an upper bound also for $\phi_c(G)$. Moreover, a result in [81], implies the following, for a cubic graph G:

- G is bipartite if and only if $\phi_c(G) = 3$
- G is Class 1 and non-bipartite if and only if $\phi_c(G) = 4$
- G is Class 2 if and only if $\phi_c(G) > 4$

As a remark, we point out that in [57], it is proved that for every rational number $p \in (4, 5]$, there are infinitely many snarks with circular flow number p, so that every possible circular flow value is actually realized by some snark.

Hence, similarly to the flow-number, the circular flow number is a parameter of a cubic graph that distinguishes if the graph is Class 1 or Class 2.

In Chapter 4, supported by some evidence, we will propose a general upper bound for a generalization of the concept of circular flow number, the 2-dimensional circular flow number. In the case of Class 1 graphs we have an upper bound which is also optimal, while, in the case of snarks we guess an upper bound, and we cannot say whether it is optimal or not. In this sense, also the problem of determining the 2-dimensional flow of a graph, is difficult for Class 2 graphs.

1.4.5 *Palette index*

Consider a k-edge-coloring of a graph G. We define the *palette* of a vertex $v \in V(G)$, with respect to the coloring c of G, to be the set $P_c(v) = \{c(e) \colon e \in E(G) \text{ and } e \text{ is incident to } v\}$. The *palette index* $\check{s}(G)$ of a graph G is the minimum number of distinct palettes, taken over all edge-colorings, occurring among the vertices of the graph.

This parameter was formally introduced in [42] and several results appear since then, see [9, 11, 15, 31, 41, 80]. All that papers mainly consider the computation of the palette index in some special classes of graphs, such as trees, complete graphs, bipartite complete graphs, $3-$ and $4-$regular graphs and some others.

It is an easy consequence of the definition that a graph has palette index equal to 1 if and only if it is a Class 1 regular graph. Hence, the palette index of a graph is a parameter that can distinguish Class 1 and Class 2 regular graphs. It has been proved that there are no regular graphs with palette index 2. Hence, when we focus on regular Class 2 graphs, we know that they have palette index at least 3. The only other known general fact for this problem is that, by Vizing theorem [89], it holds that $\check{s}(G) \leqslant r+1$ for a Class 2 r-regular graph G. However, in the case of cubic graphs, the problem of finding the palette index is completely solved by the following theorem.

Theorem 1.14 ([42]). *Let G be a connected cubic graph.*

- *G is 3-edge-colorable if and only if $\check{s}(G) = 1$;*
- *G is not 3-edge-colorable with a 1-factor if and only if $\check{s}(G) = 3$;*
- *G is not 3-edge-colorable without a 1-factor if and only if $\check{s}(G) = 4$.*

In Chapter 5 we consider this problem and we prove a sufficient condition for a general graph to have palette index larger than its minimum degree. As a consequence of this result, we construct, for every r odd, a family of r-regular graphs with palette index reaching the maximum admissible value, $r+1$ and also the first known family of simple graphs whose palette index grows quadratically with respect to their maximum degree. Moreover, we provide a characterization of graphs having palette index at most 3 in terms of decompositions in regular Class 1 subgraphs.

2 CYCLE SEPARATING CUTS IN POSSIBLE COUNTEREXAMPLES TO THE CYCLE DOUBLE COVER AND THE BERGE-FULKERSON CONJECTURES

This chapter is based on contribution [P1].

The Cycle Double Cover Conjecture (see [79, 84]) and the Berge-Fulkerson Conjecture (see [28, 78]) share a compelling similarity rooted in their common focus on the concept of edge coverings. While the former asserts that every bridgeless graph can be covered by cycles in such a way that each edge belongs to precisely two of them, the latter posits the existence of six perfect matchings, each covering every edge exactly twice in a bridgeless cubic graph. Although the Cycle Double Cover Conjecture is stated for general bridgeless graphs, a well-known reduction, via the splitting lemma [26], brings it to the family of bridgeless cubic graphs. As we already mentioned in Section 1.4.1 and Section 1.4.2, the existence of the required covering is trivially guaranteed for both conjectures if the cubic graph is Class 1, hence the relevant family of graphs to be studied is that of Class 2 ones.

In this chapter, we adapt the techniques employed in [60] to study the effect of cyclic edge-connectivity on potential minimum counterexamples for the Cycle Double Cover and the Berge-Fulkerson Conjectures. Our primary focus is twofold: first, we explore the structure of a possible minimum cyclically 4-edge-connected counterexample for the Cycle Double Cover Conjecture. Second, we direct our attention to the Berge-Fulkerson Conjecture, investigating the cyclically 5-edge-connected case (recall that in [60] it is showed that a minimum possible counterexample to the Berge-Fulkerson Conjecture is cyclically 5-edge-connected). Despite the inability to exclude the existence of a minimum counterexample with the given cyclic edge-connectivity, we provide, in both cases, new strong restrictions on its structure.

2.1 NOTATION

We introduce notation and auxiliary results that we will use in the following sections to prove our main results.

A *multipole* is a pair (V, E) consisting of a set of vertices V and a set of edges E. Each edge possesses two ends, each of which may be incident with a vertex. If an edge has both ends incident with a vertex, it is called a *proper edge*. If exactly one of the ends of an edge is incident with a vertex, the edge is called a *dangling edge*. Finally, if none of them is incident with a vertex, the edge is called an *isolated edge*. An end of an edge which is not incident with a vertex is called a *semiedge*.

A k-*pole* is a multipole with precisely k semiedges. An *ordered* k-*pole* is a k-pole with a linear ordering of its semiedges.

In the following, when colorings of the edges of a k-pole are considered, it is always implicitly assumed that if we assign a color to a given edge, then the same color is also assigned to all (possible) semiedges of that edge; viceversa, if we claim that a semiedge has a certain color then the same holds for its edge as well. Then, from now on, we indifferently say that a color is assigned either to an edge or to a semiedge.

Let us recall that a *circuit* is a 2-regular connected graph, while a *cycle* is a graph with at least one edge and every vertex of even degree: when we limit our attention to cubic graphs, a cycle is nothing but a vertex disjoint union of circuits, while it could have vertices of larger (even) degree in the general case. The main parameter considered along this chapter is the cyclic-edge-connectivity. A graph G is *cyclically* k-*edge-connected* if it does not contain an edge-cut S such that $|S| < k$ and $G - S$ contains at least two components containing cycles. The *cyclic connectivity* of a graph G is the greatest k such that G is cyclically k-edge-connected. In the rest of the chapter, we often say *minimum counterexample*, instead of *possible minimum counterexample* to one of the considered conjectures, for brevity. However, we do not intend in any way to suggest that such counterexamples should actually exist or not.

2.2 ON 4-EDGE-CUTS IN A MINIMUM COUNTEREXAMPLE TO CDC-CONJECTURE

In a minimum counterexample to the Cycle Double Cover Conjecture there cannot be 2-valent vertices and 2- or 3-edge-cuts, and this follows by a standard argument which uses minimality (it is presented in Proposition 1.8). Hence, we focus on 4-edge-cuts in a minimum counterexample to the Cycle Double Cover Conjecture, and to do so, we apply the same technique and we use notations similar to ones used by E. Máčajová and G. Mazzuoccolo in [60]. Note that here we consider graphs that are not necessarily cubic, but we can repeat all arguments also in the cubic case as we will remark at the end of this section.

Definition 2.1. *Let* $H = (V, E)$ *be a multipole. A* CDC-coloring *of* H *is a function* φ *which assigns to every element in* E *a* 2*-subset of the set of colors* $\{1, 2, \ldots, t\}$ *for some integer* t*, in such a way that any color occurs an even number of times along the edges incident to a vertex.*

It is straightforward that a CDC-coloring of a graph G is equivalent to the existence of a set of cycles of G covering each edge of G twice. Moreover, the edge-induced subgraph of G by a color in $\{1, 2, ..., t\}$ is a cycle in G.

Remark 2.2. *A stronger version of the Cycle Double Cover Conjecture suggests that it is possible to assume* t *at most* 5 *(see [16] and also [94, 93] for a comprehensive survey on cycle covers). Clearly, it is not guaranteed that a minimum counterexample for the stronger version coincides with a potential minimum counterexample for the general version. However, our arguments never require more than five colors, so the structural restrictions of a minimum counterexample for one conjecture extend to the other as well.*

Now, we consider the behavior of a CDC-coloring on the four semiedges of a 4-pole in more detail. Let H be an ordered 4-pole. Each color of a CDC-coloring occurs an even number of times on the semiedges of H. Now we show that in every CDC-coloring of H, the pairs of colors in the semiedges can be expressed as an overlap of two colorings of the semiedges, each of which uses at most two colors and such that each color appears in at most one of these two colorings. An edge-coloring of the semiedges of an ordered 4-pole where ALL semiedges receive the same color is said to be of *type A*, while, for $i = 2, 3, 4$, an edge-coloring of the semiedges of an ordered 4-pole with two colors such that the first semiedge has the same color as the i-th one is said to be of *type* T_i.

Let φ be a CDC-coloring of an ordered 4-pole H. If all the four semiedges have the same pair of colors in φ, H is said to be of *type AA*. Otherwise, if one color appears in all semiedges, then φ is of type AT_i for some $i \in \{2,3,4\}$. Finally, since every color occurs on an even number of edges, the only remaining cases to consider are the cases when four colors are present in φ on the four semiedges and each of them appears on exactly two semiedges. In this case, we can partition the four colors in two subsets such that the two colors in the first subset give a coloring of type T_i and the other two colors a coloring of type T_j. Such CDC-coloring will be denoted as of *type* T_iT_j, for some $i, j \in \{2,3,4\}$.

Proposition 2.3. *Each CDC-coloring of the semiedges of an ordered 4-pole is of type* XY *where* $X, Y \in \{A, T_2, T_3, T_4\}$.

From now on, we do not distinguish a CDC-coloring from another by the specific set of colors used for the semiedges, but only by the type of colorings. Moreover a CDC-coloring of type XY and a CDC-coloring of type YX for $X, Y \in \{A, T_2, T_3, T_4\}$ are always considered of the same type. Hence, we have exactly 10 types of CDC-colorings of an ordered 4-pole, namely AA, AT_i for $i \in \{2,3,4\}$, and T_iT_j for $i, j \in \{2,3,4\}$.

We denote by $\mathcal{C}$ the set of these 10 types of CDC-colorings, and we denote by $\mathcal{C}(H)$ the set of admissible types of CDC-colorings for a given ordered 4-pole H. A priori $\mathcal{C}(H)$ is one of the 2^{10} elements of the power set of $\mathcal{C}$, but instead of directly working with subset of $\mathcal{C}$, we prefer to construct an auxiliary graph M and to identify $\mathcal{C}(H)$ with a suitable subgraph of M.

The graph M has four vertices, denoted by A, T_2, T_3 and T_4, and every vertex is connected to every other vertex and to itself by a loop. Vertices of M correspond to the four possible types of edge-colorings of the semiedges of a 4-pole as introduced before. Each of the ten edges (here and later we always refer to loops as edges with two semiedges incident to the same vertex) corresponds to a different type of CDC-coloring. More precisely, the one obtained by the composition of the two edge-colorings of its semiedges. Six copies of the graph M are depicted in Figure 10. In the leftmost copy, vertices are labeled, and the same arrangement of vertices is implicitely assumed in the subsequent copies. We associate a subgraph of M, denoted by H^*,

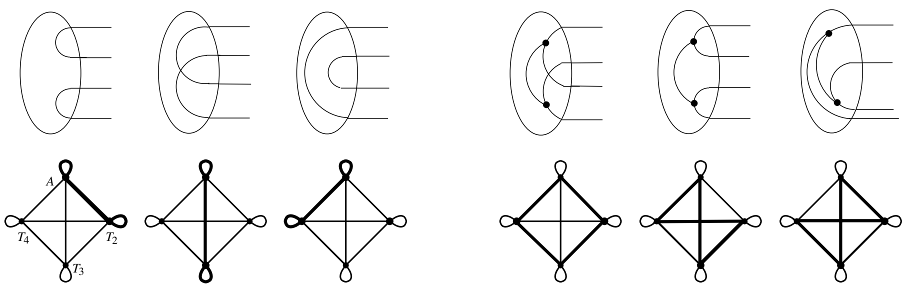


Figure 10: The subgraphs of M associated to some ordered acyclic 4-poles.

to every ordered 4-pole H in the following way. Consider the set $\mathcal{C}(H)$ and recall that each of its elements corresponds to an edge of M. Then, we define H^* as the subgraph of M induced by all edges which correspond to an element of $\mathcal{C}(H)$.

We show in Figure 10 six possible ordered acyclic 4-poles (indeed all the possible cubic 4-poles) which will be useful in our main proof and their associated subgraphs of M. In particular, note that acyclic poles with no vertex are associated to a dumbbell subgraph of M, while the three acyclic poles with two vertices are associated to a 4-cycle. It is worth noting that this point constitutes one of the main obstruction compared to what was obtained in [60] to rule out the existence of 4-edge-cuts in a minimal counterexample to the Berge-Fulkerson Conjecture. In that case, each dumbbell subgraph of M was associated to an acyclic 4-pole, while here we have different types of admissible subgraphs which impose less restrictive conditions in what follows.

2.2.1 *Bichromatic chains*

In this section we introduce a variation of the tool known as *Kempe chain*. This tool was firstly introduced by A.B. Kempe in 1879 [51], in his famous attempt to prove the Four Color Theorem, and then widely used subsequently in literature. In Section 2.3.1 we propose another variation and use of this tool of the late 19^{th}-century.

Let φ be a CDC-coloring of a 4-pole H. Let s be a semiedge of H and denote by c_1 one of the two colors in $\varphi(s)$ and by c_2 a color not in $\varphi(s)$ (here we also admit that c_2 is a color in $\{1, 2, \dots, t\}$ unused in φ). Consider the subgraph H' of H induced by all the edges e such that $\varphi(e)$ contains exactly one of

c_1 and c_2. Let K be the connected component of H' which contains the semiedge s. Clearly, K contains at least two semiedges, let s' denote one different from s. Every path BC in K containing s and s' will be called a c_1-c_2-*bichromatic chain* . Note that it may happen that only one of the colors c_1 and c_2 appears in a c_1-c_2-bichromatic chain. Starting from φ, we can obtain a new CDC-coloring of H by performing a *color switch along the* c_1-c_2-*bichromatic chain* BC, that is an interchange of the two colors c_1 and c_2 for all edges (or semiedges) of BC. Note also that in case of a cubic 4-pole, K itself is always a path beginning and ending with dangling edges and then BC is uniquely determined.

Now, we prove some necessary conditions for a subgraph of M associated to an ordered 4-pole.

Lemma 2.4. *Let* H *be an ordered* 4*-pole. Then, the subgraph* H^* *of* M *has no vertex of degree* 1 *and no vertex whose only incident edge is a loop.*

Proof. Let X, Y be two arbitrary elements, possibly the same, of the set $\{A, T_2, T_3, T_4\}$. Consider the vertex of M corresponding to the element X and assume, by contradiction, that XY is the unique edge of M in H^* incident to X (note that if $X = Y$ then XY is a loop). Consider a CDC-coloring φ of H of type XY.

If $Y = A$, without loss of generality, we can assume that φ assigns color 1 to all the four semiedges of the ordered 4-pole H. Choose a color which does not appear in the four semiedges, say 2. Whereas, if $Y = T_i$ for $i \in \{2, 3, 4\}$, assume, without loss of generality, that the two colors which defines the edge-coloring Y are 1 and 2. In both cases, consider a 1-2-bichromatic chain BC in H starting from the first semiedge and ending in another semiedge. By a color switch along BC we obtain a CDC-coloring of type XZ, where, in all cases, Z is different from Y. Hence, XY and XZ are two distinct edges of H^* incident X, that is X is not a vertex of degree 1 in H^* and it is not incident uniquely to the loop XX. □

Lemma 2.5. *Let* H *be an ordered* 4*-pole. If the subgraph* H^* *of* M *contains a loop* XX *with* $X \in \{A, T_2, T_3, T_4\}$*, then at least one of the following holds:*

- *the two edges* XY *and* YY *belong to* H^* *for some* $Y \neq X$*;*
- *the edges* XY, XZ *and* YZ *belong to* H^* *for two distinct* Y, Z *different from* X*.*

Proof. Let X be an element of the set $\{A, T_2, T_3, T_4\}$ and assume XX is an edge of H^*. By Lemma 2.4, XX cannot be the unique edge of H^* incident to X, then H^* contains a further edge XY incident to X, where $Y \neq X$. Consider a CDC-coloring φ of H of type XX.

If $X = A$, without loss of generality, we can assume that φ assigns color 1 and 2 to all the four semiedges of the ordered 4-pole H. Consider two colors, say 3 and 4, which do not appear in the four semiedges.

Consider a 1-3-bichromatic chain BC_1 and a 2-4-bichromatic chain BC_2 in H both starting from the first semiedge. By a color switch along BC_1 we obtain a CDC-coloring of type XY ($Y \neq X$) while by a color switch along BC_2 we obtain a CDC-coloring of type XZ ($Z \neq X$). Since the pairs of colors involved in the two bichromatic chains are different, then we can perform the color switches along BC_1 and BC_2 at the same time, thus obtaining a CDC-coloring of type YZ. If $Y = Z$, then XY and YY belong to H^*, otherwise XY, XZ and YZ belong to H^* and the assertion follows in this case.

Now we consider the case $X = T_i$ for some $i \in \{2, 3, 4\}$. Without loss of generality, we can assume that φ assigns color $1, 2, 3$ and 4 to the four semiedges of the ordered 4-pole H and the first and the i^{th} semiedges receive colors 1 and 3. Consider a 1-2-bichromatic chain BC_1 and a 3-4-bichromatic chain BC_2 in H starting from the first semiedge and ending in another semiedge. By a color switch along BC_1 we obtain a CDC-coloring of type XY ($Y \neq X$) while by a color switch along BC_2 we obtain a CDC-coloring of type XZ ($Z \neq X$). As before, since the pairs of colors involved in the two bichromatic chains are different, then we can perform the color switches along BC_1 and BC_2 at the same time, thus obtaining the assertion. □

We consider a minimum counterexample G for the CDC-Conjecture. As already observed, G must be cyclically 4-edge-connected. Here, we show that if G admits a 4-edge-cut S separating two circuits of G, then we can uniquely characterize the two subgraphs of M associated to the two 4-poles separated by S.

Theorem 2.6. *Let G be a possible minimum counterexample to the Cycle Double Cover Conjecture and let S be a 4-edge-cut separating two circuits of G. Denote by G_1 and G_2 the two 4-poles separated by*

S. Then, for a suitable choice of the three distinct values of i, j, k *in* $\{2, 3, 4\}$*, the edge-sets of* G_1^* *and* G_2^* *are equal to*

$$\{AA, AT_k, AT_j, T_kT_j\} \text{ and } \{T_iT_i, T_jT_j, T_kT_k, T_iT_k, T_iT_j\}.$$

Proof. Assume there exists a counterexample to the Cycle Double Cover Conjecture, and let G be one of minimum order. It is well-known that we can assume G of minimum degree 3 and without cycle-separating 2- and 3-edge-cuts.

Consider the two subgraphs G_1^* and G_2^* of M. First observe that for $i \in \{1, 2\}$, G_i^* has at least one edge. Indeed, if G_i^* contains no edge, this implies that G_i does not admit a CDC-coloring. By glueing together G_i and an arbitrary acyclic 4-pole, we obtain a bridgeless graph smaller than G which does not admit a CDC-coloring, a contradiction by minimality of G.

Furthermore, since G is a counterexample to the CDC Conjecture, then G_1^* and G_2^* must be edge-disjoint. Otherwise, G_1 and G_2 admit a CDC-coloring of the same type and, up to permutation of colors, we can glue such CDC-colorings together to obtain a CDC-coloring of G.

CLAIM 1: G_i^* share at least one edge with each of the subgraphs associated to the six acyclic ordered 4-poles depicted in Figure 10. Proof of Claim 1: If this is not the case, let H be the acyclic ordered 4-pole such that H^* and G_i^* are edge-disjoint. The graph obtained by glueing together H and G_i is longer a counterexample and it is smaller than G, a contradiction.

CLAIM 2: G_i^* cannot contain all edges of a subgraph associated to one of the acyclic ordered 4-pole depicted in Figure 10. Proof of Claim 2: Suppose there exists an acyclic ordered 4-pole H whose edge-set is a subset of the edge-set of G_i^*. The graph obtained by glueing together H and G_j, $j \neq i$, is a counterexample smaller than G, a contradiction.

Now, we divide the proof in two cases according that the loop AA of M belongs to one of the two subgraphs G_1^* and G_2^* or not.

Case I - AA does not belong to G_i^* for $i = 1, 2$. For every $j \in \{2, 3, 4\}$ exactly one of the two edges AT_j and T_jT_j should belong to G_1^* and the other one to G_2^*, otherwise the dumbbell subgraph with vertices in A and T_j and G_i^* are edge-disjoint for at least one i, contradiction by Claim 1. Moreover, both G_1^* and G_2^* have degree different from 1 in A by Lemma 2.4, then the three edges AT_j belong to the same subgraph, without loss of generality say G_1^* and it follows that all three loops T_jT_j belong to G_2^*. At least one between G_1^* and G_2^* contains at most one

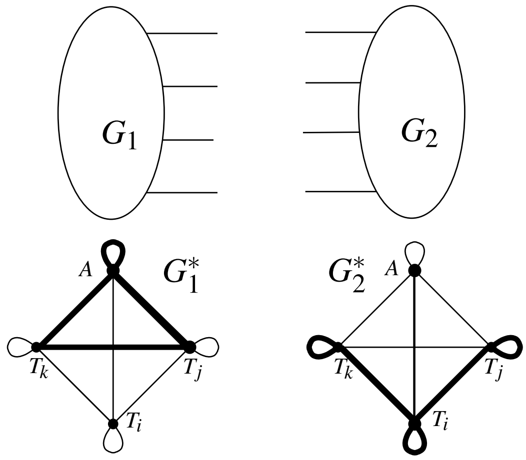


Figure 11: The case described in the statement of Theorem 2.6.

of the three edges T_jT_k for $j \neq k$: in any case, there is either a degree 1 vertex or an isolated loop in one of the two subgraphs, a contradiction by Lemma 2.4.

Case II - AA belongs to one of the two subgraphs (say G_1^*). Since AA is a loop, one of the two cases in Lemma 2.5 occurs, but by Claim 2 the first one is not possible. Then, there exist $j, k \in \{2, 3, 4\}$, with $j \neq k$, such that AT_j, AT_k, T_jT_k are also edges of G_1^*. Let i be the unique element of $\{2, 3, 4\}$ different from j and k. The edge AT_i is not in G_2^* since G_2^* cannot have a degree 1 vertex in A. Hence, all the three loops T_jT_j, T_iT_i and T_kT_k belong to G_2^* by Claim 1.

Finally, both T_iT_j and T_iT_k belong to G_2^* otherwise G_2^* has an isolated loop. The edge AT_i is the unique edge of M which does not belong to one of the two subgraphs so far. We have already proved that it cannot belong to G_2^* and, similarly, it cannot belong to G_1^* otherwise G_1^* has a degree 1 vertex in T_i, contradiction by Lemma 2.4. □

In order to prove that a minimum counterexample cannot admit a 4-edge-cut, it is thus sufficient to establish that a 4-pole of either of the two types described in Theorem 2.6 cannot exist. We suspect that neither can exist, and we leave it as a conjecture.

Conjecture 2.7. *No 4-pole has the set of CDC-coloring equal to*

$$\{AA, AT_k, AT_j, T_kT_j\} \text{ or } \{T_iT_i, T_jT_j, T_kT_k, T_iT_k, T_iT_j\},$$

respectively, for any choice of the three distinct values of i, j, k *in* $\{2, 3, 4\}$.

Since all the acyclic 4-poles considered in the proof are cubic, our argument can be repeated if we consider a minimum cubic counterexample. In this case, we can prove that if such multipoles would exist, then they cannot be 3-edge-colorable.

Next lemma analyzes some general necessary properties of the graph H^* when H is a 3-edge-colorable cubic 4-pole.

Lemma 2.8. *Let H be a 3-edge-colorable cubic 4-pole. Then the followings occur:*

- *H^* has an edge of type AX for some $X \in \{A, T_2, T_3, T_4\}$*
- *if H is connected, the minimum degree of H^* is at least 1.*

Proof. Consider a 3-edge-coloring α of the edges of H with colors $1, 2, 3$. Clearly the three 2-factors C_{ij} each of which induced by the edges of H colored i and j, $i \neq j$, form a CDC-coloring φ of H. Observe that the colors of α never appear all together on the semiedges of H, so that there exists at least one 2-factor C_{ij} that encounters all the semiedges of H, so that φ must be of type AX for some $X \in \{A, T_2, T_3, T_4\}$.

If H is connected, then for every $i \in \{2, 3, 4\}$ consider a path Q_i starting from the first semiedge of H and ending in the i-th semiedge. Then the family of cycles given by $\{C_{12} \Delta Q_i,\ C_{13} \Delta Q_i,\ C_{23} \Delta Q_i,\ Q_i\}$, where $A \Delta B$ indicates the symmetric difference of the sets A and B, form a CDC-coloring of H of type T_iX for some $X \in \{A, T_2, T_3, T_4\}$. □

The following statement is a corollary of Lemma 2.8 and Theorem 2.6.

Corollary 2.9. *If a cubic minimum counterexample to the CDC-Conjecture is not cyclically 5-edge-connected, then both the components G_1 and G_2 separated by a 4-edge-cut must be non 3-edge-colorable 4-poles containing at least one circuit.*

Finally, also considering Remark 2.2, we can summarize all previous considerations in the following statement.

Theorem 2.10. *Let G be a graph and let S be a 4-edge-cut separating two circuits of G. Denote by G_1 and G_2 the two 4-poles separated by S. Assume G satisfies (at least) one of the followings.*

1. *G is a minimum counterexample to the CDC-Conjecture;*
2. *G is a minimum cubic counterexample to the CDC-Conjecture;*
3. *G is a minimum counterexample to the 5-CDC-Conjecture;*
4. *G is a minimum cubic counterexample to the 5-CDC-Conjecture.*

Then, for a suitable choice of the three distinct values of i, j, k *in* $\{2, 3, 4\}$*, the edge-sets of* G_1^* *and* G_2^* *are equal to*

$$\{AA, AT_k, AT_j, T_kT_j\} \text{ and } \{T_iT_i, T_jT_j, T_kT_k, T_iT_k, T_iT_j\}.$$

2.3 5-EDGE-CUTS IN A POSSIBLE COUNTEREXAMPLE TO BF-CONJECTURE

In this section we employ the same technique as in the previous one to the study of cyclic 5-edge-cuts in a possible minimum counterexample to the Berge-Fulkerson Conjecture. As the Berge-Fulkerson Conjecture, in its basic form, is stated for cubic graphs, multipoles studied here will be cubic.

We will use the following definition.

Definition 2.11. *Let* $H = (V, E)$ *be a cubic multipole. A* Berge-Fulkerson coloring*, BF-coloring for short, is a function* φ *which assigns to every element in* E *a 2-subset of the set of colors* $\{1, 2, 3, 4, 5, 6\}$*, in such a way that the subsets assigned to any two adjacent edges are disjoint.*

Clearly, a BF-coloring of a cubic graph G is equivalent to the existence of six perfect matchings of G covering each edge of G twice. Moreover, each of the six color classes induces a perfect matching of G.

From now on, we will refer to a 5-pole, implicitly understanding it to be both ordered and cubic.

Let H be a 5-pole. Then each of the six colors of a BF-coloring of H appears an odd number of times on the semiedges of H. Hence, a color can occur 1, 3, or 5 times. If there is one color appearing 5 times, any other color must appear only once: we denote this type of BF-coloring by (12345).

In all other cases, two colors appear three times and each of the remaining four colors appears once. We will say that a BF-coloring is of type $(xy)(x'y')$, for $x < y$ and $x' < y'$ belonging to $\{1, 2, 3, 4, 5\}$, if one of the two colors which appear three times on the semiedges is *n*ot present on the semiedges in positions x and y, and the other one is not present in positions x' and y'. Along the chapter, we do not distinguish between the types $(xy)(x'y')$ and $(x'y')(xy)$. Moreover, if $(x, y) = (x', y')$, we will denote the coloring of type of $(xy)(xy)$ simply by $[xy]$.

Hence, there are 56 distinct types of BF-colorings: 45 of type $(xy)(x'y')$, with either $x \neq x'$ or $y \neq y'$, 10 of type $[xy]$ and, in addition, the type (12345).

In all types of BF-colorings, we will call *l*onely a color of φ which appears exactly once along the semiedges of the 5-pole. With a slight abuse of terminology, we will also use the term lonely position/semiedge, when we refer to a semiedge which receives (at least one) a lonely color. For a given ordered 5-pole, the set of admissible types of BF-colorings can be represented as a subgraph of an auxiliary graph N. The set of vertices of N is $\{(xy)|x, y \in \{1, 2, 3, 4, 5\}, x < y\} \cup \{(12345)\}$. The type $(xy)(x'y')$ of a BF-coloring will be represented by an edge between the two vertices (xy) and $(x'y')$ of N. The type (12345) will be represented by a loop over the vertex (12345) and, similarly, a type $[xy]$ is represented by a loop over the vertex (xy).

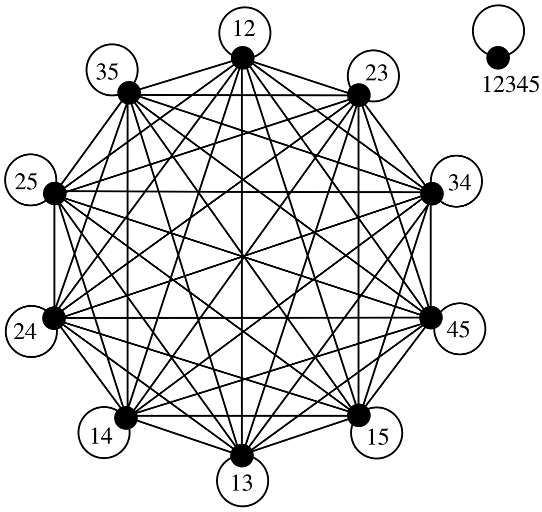


Figure 12: The auxiliary graph N. We omit the parenthesis to make the figure more simple.

Given a 5-pole H, denote by $\mathcal{C}(H)$ the set of all admissible types of BF-colorings for H. Clearly each type in $\mathcal{C}(H)$ can be represented as an edge in the auxiliary graph N and we can denote by H^* the subgraph of N induced by all these edges. A priori H^* could be any subgraph of N: hence, for a given H, H^* is one of the 2^{56} possible subgraphs of N. This number can actually be significantly reduced by using Kempe switches, similarly as in [60].

2.3.1 *Kempe switches*

Let φ be a BF-coloring of an ordered cubic 5-pole H. Let s be a semiedge of H and denote by c_1 one of the two colors in $\varphi(s)$ and by c_2 one of the four colors not in $\varphi(s)$. Consider the subgraph of H induced by all edges e such that $\varphi(e) \cap \{c_1, c_2\}$ is not empty. Let K be the connected component of such a subgraph which contains the semiedge s. Clearly, K is a path and then it contains exactly two semiedges, let s' denote the one different from s. Moreover, every edge of K in φ receives exactly one of the two colors c_1 and c_2 and these colors alternate along the edges of the path K. The component K will be called a c_1-c_2-Kempe chain. Starting from φ, we can obtain a different BF-coloring of H by performing a Kempe switch, that is an interchange of the two colors c_1 and c_2 along all edges of K.

Observe that the BF-coloring so obtained is of different type with respect to φ if and only if either exactly one of c_1 and c_2 is a lonely color in φ or if both of them are not lonely and there is exactly one semiedge receiving both c_1 and c_2 in φ.

Notice also that, given a BF-coloring φ on a 5-pole H, it could be possible to perform more than one Kempe switch. For example, consider a BF-coloring of type $(xy)(x'y')$, and denote by 1 and 4 the colors appearing three times on the semiedges, by 2 and 3 the colors in positions x and y, respectively, and by 5 and 6 the colors in positions x' and y', respectively. It is then possible to consider two independent Kempe chains with colors 1 and c with $c \in \{2, 3\}$ and 4 and c' with $c' \in \{5, 6\}$, and performing both the Kempe switches along these chains results in another BF-coloring of H.

We remark that the exact size of the set of all types of BF-colorings that can be obtained by performing Kempe switches starting from φ in a pole H, depends on the specific BF-coloring φ and on the structure of the ordered 5-pole. However, previous observations ensure that it always contains at least one element different from φ.

Now we can formalize the possibility to perform Kempe switches in terms of types of BF-colorings, i.e. in terms of edges of the auxiliary graph N. Let φ be a BF-coloring of a 5-pole H. Observe that for two distinct colors c_1 and c_2 there exists a c_1-c_2-Kempe chain beginning and ending on the dangling edges if and only if the symmetric difference of the set of dangling edges colored c_1 and the set of dangling edges colored c_2 is non-empty (this is equivalent to the fact that it has at least two elements). Let us call such a pair of colors *switchable*. Note that, for two colors, being switchable or not does not depend on φ but only on the type of φ.

Let H be a 5-pole, let α be an edge of H^* and let $c_1, c_2, \dots, c_6$ be the six colors representing α. Further, let $i, j, k, m \in \{1, 2, \dots, 6\}$ be any four pairwise distinct indices such that the colors c_i and c_j are switchable and c_k and c_m are switchable, and consider the corresponding c_i-c_j- and the c_k-c_m-Kempe chains in H. Then the edge-set of H^* contains at least:

- an edge β corresponding to a type of BF-coloring resulting from a Kempe switch on a c_i-c_j-Kempe chain K_1 in H, and
- an edge γ corresponding to a type of BF-coloring resulting from a Kempe switch on a c_k-c_m-Kempe chain K_2 in H, and
- the edge δ corresponding to the type of BF-coloring resulting by performing both Kempe switches on K_1 and K_2.

We will refer to this property of the edge α of H^* by saying that, from α, we can *perform one or two Kempe switches*. Notice that, if we know $\alpha \in E(H^*)$, we cannot determine the specific type of β, γ and δ without the exact knowledge of the particular 5-pole and of the specific BF-coloring φ corresponding to α, but we can assure that at least a β and a γ and the corresponding δ belong to $E(H^*)$.

Definition 2.12. *Let M be a generic subgraph of the auxiliary graph N. We say that M is* closed under one or two Kempe switches *if for every $\alpha \in E(M)$ there exists β, γ and δ in $E(M)$ obtainable via Kempe switches as described above.*

2.3.2 *BF-colorings of some small 5-poles*

In this subsection we highlight the types of admissible BF-colorings of some particular poles that we need later for our purpose. Notice that, the only acyclic 5-poles are the poles P and Q depicted in Figure 13. In the figure, they are represented with their induced subgraphs of N, P^* and Q^* respectively.

In Figure 14 the 5-pole C, whose vertex-induced subgraph is a 5-cycle, is depicted, together with its associated subgraph C^* of N. From now on, with a slight abuse of terminology, we will denote such a pole simply as a *5-cycle*. Observe that reordering

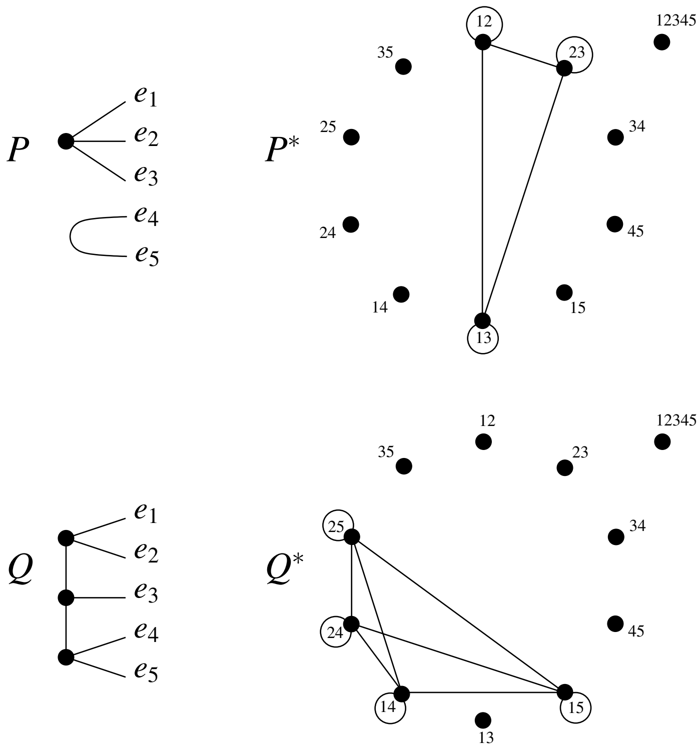


Figure 13: The acyclic 5-poles P and Q and their associated graphs P^* and Q^*.

the semiedges of these 5-poles gives raise to isomorphic subgraphs P^*, Q^* and C^* of N, but with different vertex sets.

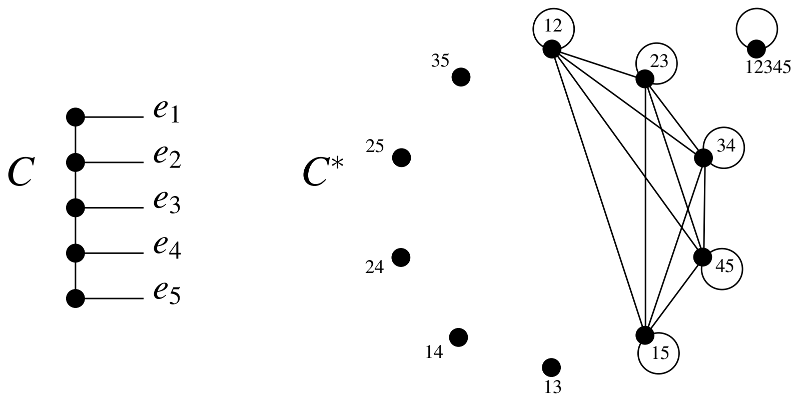


Figure 14: An ordered 5-pole C and its associated subgraph C^*.

2.3.3 *Structural restrictions in a minimum possible counterexample to the BF-Conjecture*

Now we consider a minimum possible counterexample G to the BF-Conjecture. In [60] it is proved that G should be a cyclically 5-edge-connected non 3-edge-colorable cubic graph. Here, we show that if G admits a cyclic 5-edge-cut S, then the coloring types of the two 5-poles correspond to one of 13 possibilities out of more than 2^{111}. The number of possibilities is given by the combinations with repetition by choosing 2 items from 2^{56} objects: this number is equal to $2^{111} + 2^{55}$.

Let R and L be the two 5-poles arising from the edge-cut S, and let R^* and L^* their associated subgraphs of N, respectively. Clearly, R^* and L^* must be edge-disjoint, for otherwise R and L would admit BF-colorings φ_1 and φ_2 of the same type and we could glue them together, thus obtaining a BF-coloring on G. Moreover, to be admissible, R^* and L^* must satisfy the following necessary conditions.

- L^* and R^* are closed under one or two Kempe switches (see Definition 2.12).

- Both R^* and L^* do not have as a subgraph any of the associated subgraphs A^* to an acyclic 5-pole A. Indeed, assume that R^* does admit such a subgraph. Replacing R with A would lead to a smaller counterexample to the BF-Conjecture, since A^* would remain edge-disjoint with L^*.

- Both R^* and L^* are not edge-disjoint from all the associated subgraphs A^* to an acyclic 5-pole A. Indeed, if the complement of R^* admits such a subgraph, replacing L with A would lead to a smaller counterexample to the BF-Conjecture, since A^* would remain edge-disjoint with R^*.

In what follows we provide a brief description of the program we implemented to identify pairs of types of 5-poles which can occur in a smallest counterexample to Berge-Fulkerson Conjecture.

First, we identified all Kempe closed subsets of K. But as $|K| = 56$, the number of its subsets is 2^{56} which is too big number to consider all of them. Therefore we proceeded as follows: We decomposed K into four disjoint subsets:

- $A = \{(12345)\}$

- $B = \{(ij)(km); |\{i,j,k,m\}| = 4\}$

- $C = \{(ij)(im); |\{i,j,m\}| = 3\}$

- $D = \{(ij)(ij); i \neq j\}$

It can be easily seen that $|A| = 1$, $|B| = 15$, $|C| = 30$, $|D| = 10$.

For all subsets C' of C, we checked if C' is minimal with respect to permutation of five dangling edges. Then for all subsets D' of D we checked if starting from a coloring in $C' \cup D'$ and perfoming Kempe switch or Kempe double switch whether each of the resulting colou rings is either outside $C \cup D$ or in $C' \cup D'$.

If yes, we verified if does not contain a subset corresponding to P^* or its complement. Further we checked if $C' \cup D'$ is minimal under permutation of five dangling edges. If the answer was yes, we proceeded to identifying suitable subsets of $A \cup B$.

For all subsets A' of A and $B'$of B we checked whether every in coloring of $A' \cup B'$ all Kempe switches and Kemple double switches are in $A' \cup B'$ or outside $A \cup B$. The same for the sets B and C.

Finally we checked whether all subset $A' \cup B' \cup C' \cup D'$ is closed under Kempe swithches and does not contain Q^* or its complement.

We alltogether obtained 883 sets of types of coloring that fulfil all these requirements. Among them, after all permutation of dangling edges, we checked that there are exactly 13 disjoint pairs. These pairs are depicted in Figures 15 and 16.

Observe that, in cases from I to XII, R^* (grey edges) is a copy of K_5 with loops on vertices plus the (12345) coloring, that is the graph induced by the admissible types of BF-colorings of a 5-cycle. Hence, by minimality of G, in these cases, we can assume that the pole R is a 5-cycle.

In the remaining part of the section we study 3-edge-colorability of the 5-poles R and L. In the following lemma we state some general necessary conditions for the associated graph H^*, when H is a 3-edge-colorable 5-pole.

In the next proposition, we extend the notion of lonely colors to a 3-edge-coloring of a 5-pole in the natural way, that is a color is lonely if it appears exactly one time along the semiedges in the considered 3-edge-coloring.

Lemma 2.13. *A 3-edge-colorable 5-pole H which has a 3-edge-coloring with lonely colors in positions x and y admits a BF-coloring of each of the following types:* $[xy]$, $[x_1y]$, $[xy_1]$, $(xy)(x_1y)$, $(xy)(xy_1)$ *and* $(x_1y)(xy_1)$, *for some* $x_1 \neq x, y$ *and* $y_1 \neq y, x$.

Proof. Observe that if the pole H admits two different 3-edge-colorings c and c' with lonely colors in positions x,y and x',y',

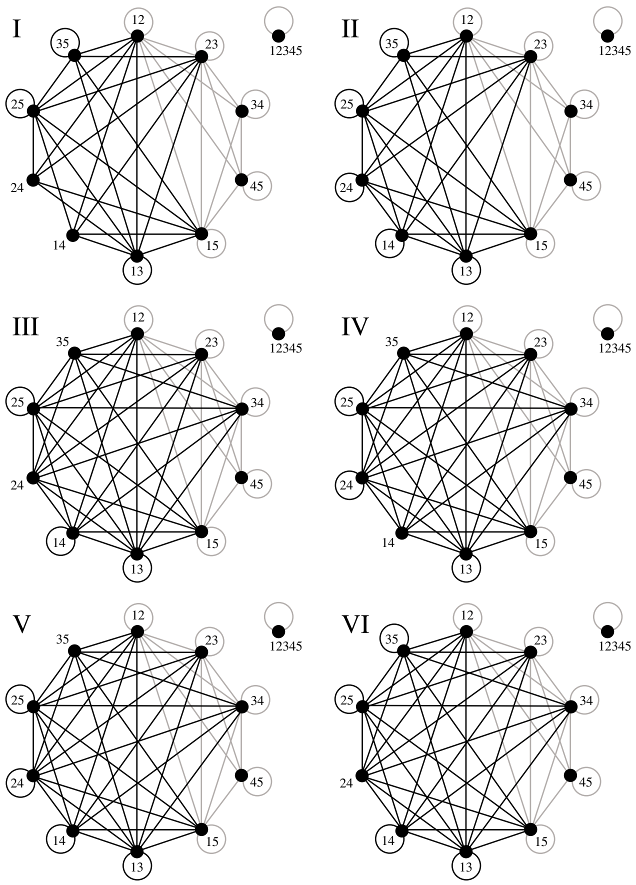


Figure 15: Possible pairs of subgraphs R^* and L^*, cases I to VI

respectively, then it admits BF-colorings of types $[xy]$, $[x'y']$ and $(xy)(x'y')$. Indeed, the first two ones are given by an overlap of two copies of c and c', respectively, and the third one is given by the overlap of a copy of c and a copy of c'.

Consider a 3-edge-coloring c of H with lonely colors in positions x and y, denote by x and y respectively also their colors and denote by z the remaining color of c. Consider a Kempe chain with colors x and z and exchange the colors along it: since the chain ends in a semiedge in position x_1, $x_1 \neq x, y$, a 3-edge-coloring of H with lonely colors in positions x_1 and y is obtained. The same procedure with a Kempe chain of colors y and z proves that H admits a 3-edge-coloring with lonely colors in positions x and y_1, where $y_1 \neq y, x$ since the Kempe chain cannot end nor in x neither in y. Hence H admits BF-colorings of all types $[xy]$, $[x_1y]$, $[xy_1]$, $(xy)(x_1y)$, $(xy)(xy_1)$ and $(x_1y)(xy_1)$. □

As a consequence of Lemma 2.13, we obtain that, for a 3-edge-colorable 5-pole H, in the associated subgraph H^* there are at

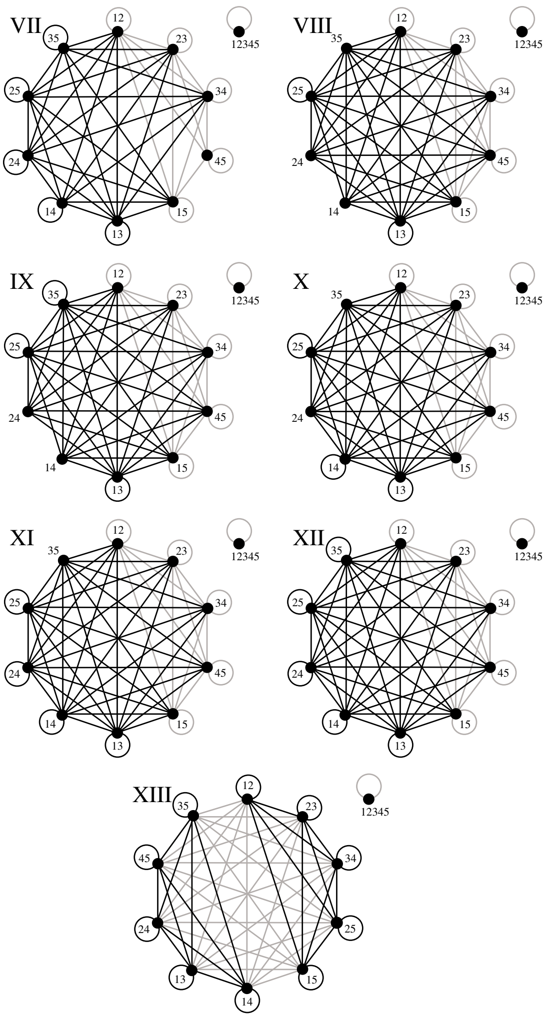


Figure 16: Possible pairs of subgraphs R^* and L^*, cases VII to XIII. We rearrange the labels of the vertices in case XIII for the sake of readability.

least 3 distinct loops. Moreover, if a position i, $i \in \{1, \ldots, 5\}$ appears in one of such loops, then it must appear at least another time in a different loop. Hence, if we consider the subgraphs L^* (black edges) in cases from I to XIII (always refer to Figures 15 and 16) to be associated to a 3-edge-colorable 5-pole L, we have that all cases are ruled out by previous observations on loops but II, VII, XII and XIII. Cases II and VII do not satisfy the necessary condition in Lemma 2.13, since there is at least a pair of loops $[xy]$ and $[x_1y]$ for which $(xy)(x_1y)$ is not an edge . Hence, we conclude that, if the pole L is 3-edge-colorable, the only possibilities for L^* are cases XII and XIII. Finally, also the subgraph R^* of case XIII cannot be associated to a 3-edge-colorable pole R, since it does not contain any loop.

We summarize what have been discussed in the following theorem.

Theorem 2.14. *If* G *is a cyclically* 5*-edge-connected minimum counterexample to the Berge-Fulkerson Conjecture and it is not cyclically* 6*-edge-connected, then every cycle separating 5-edge-cut separates two non acyclic 5-poles* L *and* R *such that, up to interchanging the roles of* L *and* R*,*

- *both* L *and* R *admit a BF-coloring and the BF-colorings for* L *and* R *are exactly the ones of the subgraphs* L^* *and* R^* *of cases from I to XIII depicted in Figures 15 and 16;*
- *in cases from I to XII the pole* R *is a* 5*-cycle, so that* G *has girth exactly* 5*;*
- *in cases from I to XI,* L *is a non* 3*-edge-colorable pole;*
- *in case XIII,* R *is a non* 3*-edge-colorable pole.*

3 H-COLORINGS

This chapter is based on contribution [P2].

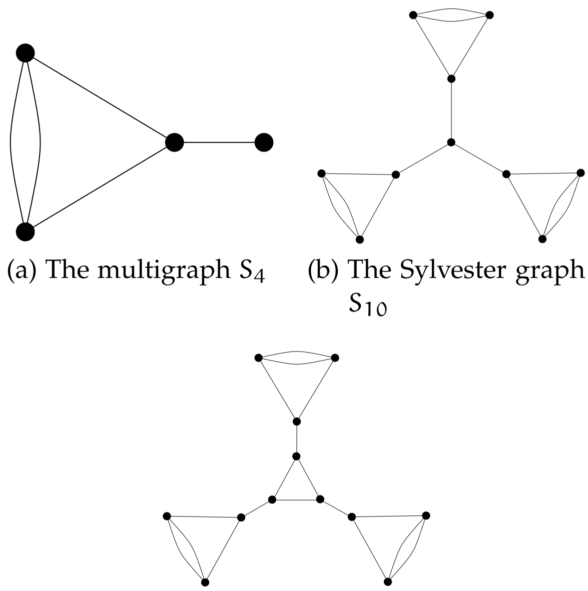
We consider undirected graphs that can be either simple or multigraphs. We recall and refine some notations we have already mentioned in Chapter 1, for the sake of completeness. Let $U \subseteq V(G)$. We denote the set consisting of all the edges having exactly one endvertex in U by $\partial_G U$, and when it is obvious which graph G we are referring to we just write ∂U. When U consists of only one vertex, say u, we write ∂u, instead of $\partial\{u\}$, for simplicity. Let H be an arbitrary graph: an H-*coloring* of G is a proper edge-coloring $f : E(G) \to E(H)$ of G with edges of H, such that for each vertex $u \in V(G)$, there exists a vertex $v \in V(H)$ with $f(\partial_G u) = \partial_H v$. If G admits an H-coloring, then we write $H \prec G$ and we say that the graph H colors the graph G. We will see in Chapter 5 that the concept of H-coloring of a graph can be generalized and, in this generalized form, will describe in a compact way the one of palette index of a graph. As mentioned in Chapter 1, one of the most important conjectures in graph theory is Conjecture 1.12, the Petersen Coloring Conjecture [46], which we recall here.

Conjecture (Petersen Coloring Conjecture)**.** *For any bridgeless cubic graph* G, $P \prec G$.

As already remarked in Chapter 1, Conjecture 1.12 implies several other relevant conjectures in the field of graph theory such as Conjecture 1.9, the Berge-Fulkerson Conjecture, and Conjecture 1.6, the Cycle Double Cover Conejcture. The Berge-Fulkerson Conjecture implies several other weaker conjectures and results on bridgeless cubic graphs, one among the others is the Fan-Raspaud Conjecture [21] (see also [61]), which claims that every bridgeless cubic graph admits three perfect matchings with empty intersection. A result implied by Conjecture 1.9 is the S_4-Theorem, [49], which states the following.

Theorem 3.1 (S_4-Theorem)**.** *For every bridgeless cubic graph* G*, there exist two perfect matchings such that the deletion of their union leaves a bipartite subgraph of* G*.*

Theorem 3.1 has been recently proved in [49], and it corresponds to the so–called S_4-Conjecture proposed in 2013 by Mazzuoccolo (see [63]). We remark that in [67], Mazzuoccolo and Zerafa showed that Theorem 3.1 is equivalent to saying that for every bridgeless cubic graph G, $S_4 \prec G$, where S_4 is the subcubic multigraph portrayed in Figure 17a.



(a) The multigraph S_4

(b) The Sylvester graph S_{10}

(c) The multigraph S_{12}

Figure 17: The multigraphs S_4, S_{10} and S_{12}

In the same spirit of Conjecture 1.12, Mkrtchyan [68] also proposed the following two conjectures for cubic graphs, for which connectivity conditions are relaxed. In fact, the following two conjectures are stated for cubic graphs which are not necessarily bridgeless.

Conjecture 3.2 (S_{12}-Conjecture)**.** *For each cubic graph G admitting a perfect matching, $S_{12} \prec G$.*

Conjecture 3.3 (S_{10}-Conjecture)**.** *For each cubic graph G, $S_{10} \prec G$.*

The multigraph S_{10} is also referred to as the *Sylvester graph* and is depicted together with the multigraph S_{12} in Figure 17 (see also [34]).

Mkrtchyan proved the following theorem (Theorem 2.4 in [68]).

Theorem 3.4 (Mkrtchyan, 2012 [68])**.** *If H is a connected bridgeless cubic graph with $H \prec P$, then $H \simeq P$.* [1]

1 Here and in the rest of the chapter, the symbol $\simeq$ denotes a graph isomorphism

Consequently, the following holds.

Corollary 3.5 (Mkrtchyan, 2012 [68])**.** *If* H *is a connected bridgeless cubic graph such that* $H \prec G$ *for every bridgeless cubic graph* G*, then* $H \simeq P$.

In other words, the previous result says that we cannot replace the Petersen graph in Conjecture 1.12 with any other connected bridgeless cubic graph. Nevertheless, if we choose H from the larger class of connected cubic graphs (not necessarily bridgeless), there are other possible candidates. In particular, if we minimise the assumptions on the graph H by considering the class of connected graphs (not even cubic), then another candidate is given by the graph S_4.

Theorem 3.12 is one of the main results of this chapter, and it is a generalisation of Theorem 3.4: it is obtained by removing any restriction on the degree of the vertices of the graph H in an H-coloring of the Petersen graph. Analogously, Corollary 3.13 is the natural generalisation of Corollary 3.5, but, in order to explain its statement, we need to introduce the following terminology. Let G be a multigraph having three degree 3 vertices and a further vertex of arbitrary degree. Denote this set of four vertices by X. If the induced multisubgraph G[X] is isomorphic to S_4, then we say that G *exposes* S_4 and that G[X] is an *exposed copy* of S_4 in G. Observe that both S_{10} and S_{12} expose (three times) S_4.

Indeed, as a consequence of Theorem 3.12 we prove that the unique graphs that can color every bridgeless cubic graph are exactly P and all graphs which expose S_4 (see Corollary 3.13). In a similar way, Corollary 3.16 and Corollary 3.17 in Section 3.2 are related to Conjecture 3.3 and Conjecture 3.2, respectively.

All the above mentioned conjectures deal with the question asking whether there exists a connected graph H such that $H \prec G$ for any G in a given class of cubic graphs. Table 2 summarizes the possibilities for the eventual existence of such a graph H that we obtain as consequences of the results presented in Section 3.2. It is divided according to the cases when H is assumed to be a simple graph or a multigraph. In this table, we consider three classes of graphs (that may be multigraphs) to be colored by some connected graph H: (i) bridgeless cubic graphs, (ii) cubic graphs admitting a perfect matching, and (iii) cubic graphs.

If the graph H that colors all the graphs in each of the corresponding classes exists, then the only possibilities are the ones presented in the table.

Cubic graphs	H *simple graph*	H *with parallel edges*
bridgeless	$H \simeq P$ (Theorem 3.12)	$H_f \simeq S_4$ (Theorem 3.12)
with a perfect matching	$\nexists$ (Remark 3.20)	$H \simeq S_{10}$ or $H \simeq S_{12}$ (Corollary 3.19)
any	$\nexists$ (Remark 3.20)	$H \simeq S_{10}$ (Corollary 3.18)

Table 2: Possibilities for the eventual existence of an H-coloring for different classes of cubic graphs

In Section 3.3 we partially answer the question dealing with whether there exists a graph H such that $H \prec G$ for any r-regular graph G, for $r > 3$, in a given class. The results obtained are summarised in Table 3.

r-regular graphs, $r > 3$	H (multi)graph
simple graph	$\nexists$ for r even (Theorem 3.26)
multigraph	$\nexists$ for any r (Theorem 3.23)

Table 3: Non existence of an H-coloring for r-regular simple graphs and multigraphs

3.1 NOTATION AND TECHNICAL LEMMAS

Before continuing, we need some further definitions and notation which we introduce in order to focus our study only on the relevant part of H in a given H-coloring f of some graph G. First, observe that, if there is no pair of distinct vertices v and w of H such that $\partial_H w = \partial_H v$, then an H-coloring f of G naturally induces the map $f_V : V(G) \to V(H)$ defined for every vertex u of $V(G)$ as $f_V(u) = v$, where v is the unique vertex of H such that $f(\partial_G u) = \partial_H v$. In what follows, the irrelevant part of H shall arise due to the vertices $v \in V(H)$ for which $v \notin \mathrm{Im}(f_V)$.

Lemma 3.6. *Let* G *be a connected graph and let* $f : E(G) \to E(H)$ *be an* H*-coloring of* G*. Then, the induced subgraph* $H[\mathrm{Im}(f)]$ *of* H *is connected.*

Proof. Observe that by definition of H-coloring, if e_1 and e_2 are two adjacent edges of G, then $f(e_1)$ is adjacent to $f(e_2)$ in

$H[\mathrm{Im}(f)]$. The result follows immediately by the connectivity assumption on G. □

By the previous lemma, from now on we can assume that H is connected, since only the edges of one connected component belong to the image of any H-coloring of a connected graph G. Note that if H is connected then the map f_V is well defined for any given H-coloring f, except if H is the graph tK_2 on two vertices and with t parallel edges between them. Moreover, it is straightforward that a graph G admits a tK_2-coloring if and only if G is t-regular and t-edge-colorable and consequently, if and only if it admits a $K_{1,t}$-coloring, where $K_{1,t}$ is the star on $t+1$ vertices. Hence, it is not restrictive assuming $|V(H)| > 2$ in what follows.

Let H and G be connected graphs such that $H \prec G$ and $|V(H)| > 2$. Let f be an H-coloring of G and consider the map f_V. We denote by H_f, the edge-induced subgraph $H[\mathrm{Im}(f)]$ and with a slight abuse of terminology we shall refer to the graph H_f as the image of the H-coloring f. Note that in general $\mathrm{Im}(f_V) \subseteq V(H_f)$, since an edge uv of H_f must have at least one of its end-vertices u and v in $\mathrm{Im}(f_V)$, but not necessarily both of them. Every vertex of H_f which does not belong to $\mathrm{Im}(f_V)$ is said to be *unused*.

Starting from the graph H_f, we can obtain a large variety of connected graphs, say H', such that G admits an H'-coloring. A first easy procedure is obtained by considering an arbitrary connected graph H' having H_f as an induced subgraph with the further property that $d_{H'}(v) = d_{H_f}(v)$ for every $v \in \mathrm{Im}(f_V)$. A more general way is obtained by eventually splitting in advance unused vertices of H_f in arbitrary graphs (see Figure 18 for a possible example, where splitting of vertices is also portrayed). Finally, we remark that if H_f has no unused vertex (that is, $H_f = H$), then no connected graph H' different from H can be obtained as a combination of previous operations.

Definition 3.7. *Let G and H be connected graphs such that $|V(H)| > 2$ and $H \prec G$. Let f be an H-coloring of G and let f_V be the induced map on the vertices of G. We define the graph $\tilde{H}_f$ as the graph obtained from H_f by splitting every unused vertex u of H_f into $d_{H_f}(u)$ vertices of degree 1. We refer to the graph $\tilde{H}_f$ as the* splitted image *of f.*

In what follows, with a slight abuse of notation, we shall always refer in the same way to a vertex u in $\mathrm{Im}(f_V)$ independently to whether we are considering it in H, H_f or $\tilde{H}_f$. For simplicity, the functions corresponding to an H_f-coloring and

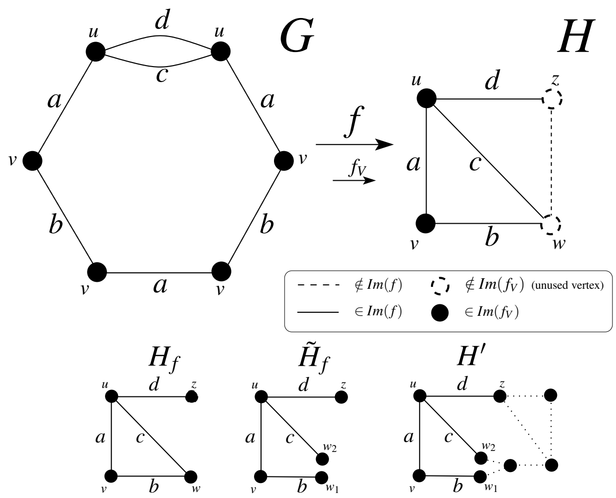


Figure 18: H, H_f, $\tilde{H}_f$ and a possible example for H'

an $\tilde{H}_f$-coloring of some graph G are both denoted by f as well. An unused vertex v is referred to in the same way both in H and in H_f, whilst the vertices of $\tilde{H}_f$ obtained by splitting v are referred to as *the vertices arising from* v. Finally, we remark that since G is connected, every two distinct vertices in H_f are the endvertices of a path whose inner vertices all belong to $Im(f_V)$. Consequently, $\tilde{H}_f$ is connected by Lemma 3.6.

In what follows we make use of some results contained in Lemma 2.2 in [35]. We reproduce only the part of the lemma that we shall need in the sequel, even if in a slightly more general form. Moreover, we add and prove statement *(d)*.

Lemma 3.8. *Let* G *and* H *be graphs. Assume that* H *is connected with* $H \prec G$*, and let* f *be an* H*-coloring of* G*.*

(a) If M *is any matching of* H*, then* $f^{-1}(M)$ *is a matching of* G*.*

(b) $\chi'(G) \leqslant \chi'(H)$ *(where* χ' *denotes the chromatic index of a graph).*

(c) If M *is a perfect matching of* H*, then* $f^{-1}(M)$ *is a perfect matching of* G*.*

(d) If G *is connected, let* X *be an edge-cut of* H_f *such that* $H_f - X$ *does not contain any isolated vertex. Then* $f^{-1}(X)$ *is an edge-cut of* G*.*

Proof. Statements *(a)*, *(b)*, *(c)* follow from [35], so it suffices to prove statement *(d)*.

(d) Let $\overline{G} = G - f^{-1}(X)$ and $\overline{H} = H_f - X$. Consider $\overline{f}\colon E(\overline{G}) \to E(\overline{H})$, the restriction of f to $\overline{G}$. Since $X = f(f^{-1}(X))$, the function $\overline{f}$ is an $\overline{H}$-coloring of $\overline{G}$, and since $H_f - X$ does not contain any isolated vertex, it holds that $\overline{H}_{\overline{f}} = \overline{H}$. Suppose that $f^{-1}(X)$ is not an edge-cut of G, for contradiction. This means that $\overline{G}$ is connected. However, by Lemma 3.6, $\overline{H}_{\overline{f}} = \overline{H}$ is connected, contradicting X being an edge-cut of H_f. □

Before we continue, we prove the following lemma which gives statement *(c)* of Lemma 3.8 as a corollary. This lemma shall also be used in Section 3.3.2.

Lemma 3.9. *Let G and H be graphs, and assume H to be connected. Let f be an H-coloring of G and let f_V be the induced map on the vertices of G. If M is a matching of H such that every vertex $v \in$ Im(f_V) is matched in M, then $f^{-1}(M)$ is a perfect matching of G.*

Proof. Since M is a matching of H, by Lemma 3.8, $f^{-1}(M)$ is a matching of G, so it suffices to show that $f^{-1}(M)$ covers all the vertices of G. For each $u \in V(G)$, $f_V(u) \in \mathrm{Im}(f_V)$, and so there exists a unique edge $e \in M$ such that e is incident to the vertex $f_V(u)$ in H. This means that for every vertex $u \in V(G)$, there exists exactly one edge in $\partial_G u$ which is colored by an edge in M, implying that $f^{-1}(M)$ is a perfect matching of G, as required. □

3.2 H-COLORINGS OF CUBIC GRAPHS

Before proving the main result of this section (Theorem 3.12) we need some further technical results for the case when G is cubic.

Remark 3.10. *Consider an H-coloring f of a connected cubic graph G. For every vertex $u \in V(\tilde{H}_f)$ exactly one of the following holds:*

- *u has degree 1 in $\tilde{H}_f$ and either it is itself an unused vertex in H_f or it arises from an unused vertex of H_f; or*
- *u has degree 3 in $\tilde{H}_f$ and it is a vertex of H which belongs to Im(f_V).*

Lemma 3.11. *Let H be a connected graph. Let f be an H-coloring of the Petersen graph P. If $e = uv$ is a bridge in $\tilde{H}_f$, then exactly one of u and v has degree 1 in $\tilde{H}_f$.*

Proof. Let f_V be the map induced by f on the vertices of P and, for contradiction, suppose that both u and v belong to $\mathrm{Im}(f_V)$, which results in both vertices having degree 3 in $\tilde{H}_f$, by Remark 3.10. Hence, all edges in $\partial_H u$ (and $\partial_H v$) belong to $\mathrm{Im}(f)$, that is they belong to the edge-set of $\tilde{H}_f$. In particular, the edge $e = uv$ belongs to $\mathrm{Im}(f)$. Let ℓ_1 and ℓ_2 be the two edges incident to u in $\tilde{H}_f$ other than uv, and let r_1 and r_2 be the other two edges incident to v in $\tilde{H}_f$. Since e is an edge-cut and a matching of $\tilde{H}_f$, by Lemma 3.8, $f^{-1}(e)$ is an edge-cut and a matching of P. The only matchings of the Petersen graph which are also edge-cuts are perfect matchings of P. Consequently, $f^{-1}(e)$ is a perfect matching of P, say M, which can be chosen arbitrarily due to the symmetry of the Petersen graph (in a more precise terminology we remark that the Petersen graph is 3-arc-transitive, see for example [4]). The complement of M in the Petersen graph consists of two disjoint 5-cycles. Without loss of generality, by following the notation used in Figure 19, we can assume that:

- each edge $u_i v_i$ has color e;
- $f_V(u_i) = u$, for every vertex u_i of the outer 5-cycle; and
- $f_V(v_i) = v$, for every vertex v_i of the inner 5-cycle.

It follows that all the edges in the outer 5-cycle (similarly, inner 5-cycle) should be alternately mapped to ℓ_1 and ℓ_2 (respectively, r_1 and r_2) by f. However, this is not possible since these two cycles have odd length. Hence, since $e \in \mathrm{Im}(f)$ implies that at least one of u and v belongs to $\mathrm{Im}(f_V)$, by Remark 3.10 we conclude that exactly one of the vertices u and v belongs to $\mathrm{Im}(f_V)$ and, the one which does not, has degree 1. $\square$

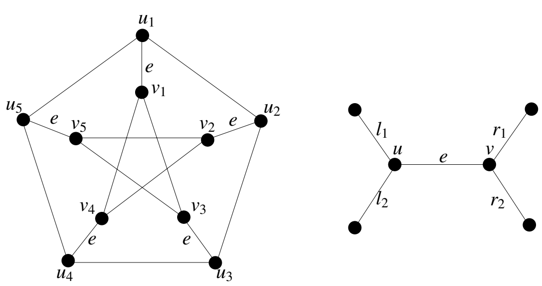


Figure 19: Steps from Lemma 3.11

Theorem 3.12. *Let* H *be a connected graph such that* $H \prec P$ *and let* f *be an* H*-coloring of* P*. Then, either* $H = H_f \simeq P$ *or* $H_f \simeq S_4$.

Proof. First, assume that $\tilde{H}_f$ is cubic. By Remark 3.10, $\tilde{H}_f = H_f$, and by Lemma 3.11, it follows that it is bridgeless, and so, by Theorem 3.4, $H = H_f \simeq P$. Hence, we can assume that $\tilde{H}_f$ is not cubic, and so, by Remark 3.10, it admits a vertex v whose degree is 1 and is adjacent to a vertex u whose degree is 3. Let $e = uv$, and let the other two edges in $\tilde{H}_f$ incident to u be denoted by a and b. By Lemma 3.11, the edges a and b cannot share a further endvertex other than u, otherwise $\tilde{H}_f$ is isomorphic to the connected 3-edge-colorable graph on four vertices with two degree 1 vertices and two degree 3 vertices, thus implying that the Petersen graph P is 3-edge-colorable by Lemma 3.8, a contradiction. Therefore, a and b share exactly one endvertex (u), and we let w and z be the two distinct vertices in $V(\tilde{H}_f) \setminus \{v\}$ such that $a = uw$ and $b = uz$.

Without loss of generality, assume that the spoke u_1v_1 of P is colored by $e = uv$. Since v does not belong to $\mathrm{Im}(f_V)$, $f_V(u_1) = f_V(v_1) = u$, and so we can assume further that $f(u_1u_5) = f(v_1v_4) = a$ and $f(u_1u_2) = f(v_1v_3) = b$, as in Figure 20. The case when $f(u_1u_5) = f(v_1v_3) = a$ and $f(u_1u_2) = f(v_1v_4) = b$ is equivalent by the symmetry of P. Since P is not 3-edge-colorable, u cannot be the only vertex in $\mathrm{Im}(f_V)$. Hence, at least one of w and z must also have degree equal to 3 in $\tilde{H}_f$. By Lemma 3.11, since $\tilde{H}_f$ cannot admit a bridge with both of its endvertices having degree 3, the vertices w and z must both have degree 3 in $\tilde{H}_f$.

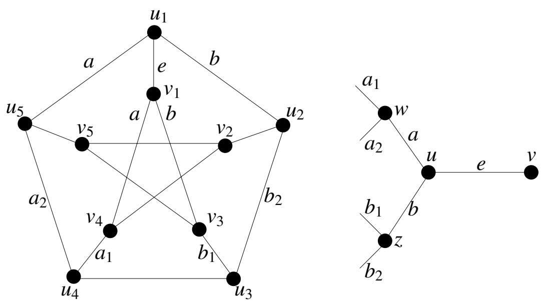


Figure 20: The edges colored a_1, a_2, b_1, b_2 in P

Claim A. u_1v_1 is the unique edge with color e.

Proof of Claim A. Suppose there is another edge m in P which is colored by e. Either m is at distance 1 from u_1v_1 or it is at distance 2. Then, there exists C, a 5-cycle or a 6-cycle, respectively, of P passing through both u_1v_1 and m. Hence, the other edges in C are colored by a or b, since they are all incident with some edge which is colored by e.

If C is a 6-cycle, the vertices of C that are not incident with u_1v_1 or m, have two of their incident edges colored by a and b, implying that they are mapped by f_V into u, and thus their third incident edges, say l_1 and l_2 respectively, are also colored by e. This is a contradiction, since l_1 and l_2 are edges of P incident to a common vertex. If C is a 5-cycle, there exists a vertex of C having two of its incident edges colored by a and b, implying that it is mapped by f_V into u and thus its third incident edge is also colored by e. But in this case, there exists a 5-cycle C' of P whose edges are incident to some edge colored by e, implying that all edges of C' must be colored by a and b. This is a contradiction since C' is an odd cycle. ∎

Let a_1 and a_2 be the two edges in $\tilde{H}_f - a$ which are incident to the vertex w, and let b_1 and b_2 be the two edges in $\tilde{H}_f - b$ which are incident to the vertex z. Since no edge but u_1v_1 has color e in P, all the edges incident to an edge with color a (similarly, b) receive colors a_1 and a_2 (respectively, b_1 and b_2). Hence, without loss of generality we can assume:

- $f(u_4v_4) = a_1$ and $f(u_4u_5) = a_2$; and
- $f(u_3v_3) = b_1$ and $f(u_2u_3) = b_2$,

as in Figure 20.

Claim B. $a_1 = b_1$ and $a_2 = b_2$.
Proof of Claim B. Due to the edge u_3u_4 in P, there exists an edge g in $\tilde{H}_f$ such that a_1, a_2, g are incident with a common vertex, and b_1, b_2, g are incident with a common vertex. Moreover, since $f(v_1v_4) = f(u_1u_5) = a$, we have $f(v_2v_4) = a_2$ and $f(u_5v_5) = a_1$. Similarly, $f(v_3v_5) = b_2$ and $f(u_2v_2) = b_1$. This means that a_1 and b_2 share a common vertex in $\tilde{H}_f$, and similarly, a_2 and b_1 share a common vertex in G. Since the vertices of $\tilde{H}_f$ can have degree 1 and 3, the only way how the above statements can be satisfied is by having $\{a_1, a_2\} = \{b_1, b_2\}$. In particular, since $f(u_5v_5) = a_1$ and $f(v_3v_5) = b_2$, b_2 must be equal to a_2, proving our claim. ∎

By Claim B, $\tilde{H}_f \simeq S_4$, and since $\tilde{H}_f$ has a unique vertex of degree 1, it cannot be obtained by splitting unused vertices of some other graph, and so, $\tilde{H}_f = H_f$, as required (in Figure 21 an S_4-coloring of P is represented). □

As before, since the Petersen graph is bridgeless and cubic, the following holds.

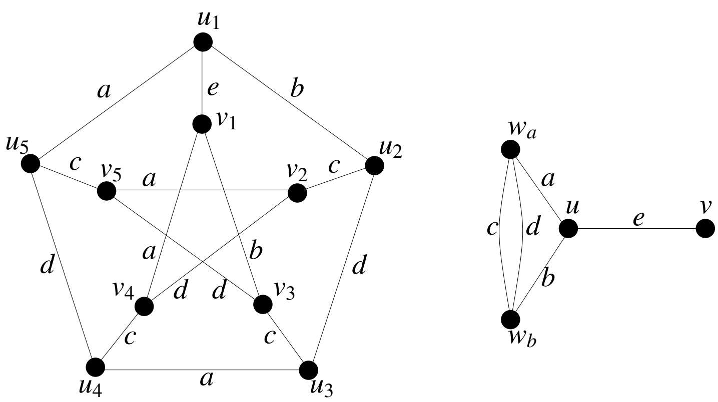


Figure 21: An S_4-coloring of P

Corollary 3.13. *If there exists a connected graph* H *coloring all bridgeless cubic graphs, then either* $H \simeq P$ *or* H *exposes* S_4*.*

To conclude this section we provide a generalisation of the following two theorems, proved in [68] and [35].

Theorem 3.14 (Mkrtchyan, 2013 [68])**.** *Let* H *be a connected cubic graph with* $H \prec S_{10}$*. Then* $H \simeq S_{10}$*.*

Theorem 3.15 (Hakobyan & Mkrtchyan, 2019 [35])**.** *Let* H *be a connected cubic graph with* $H \prec S_{12}$*. Then, either* $H \simeq S_{10}$ *or* $H \simeq S_{12}$*.*

More specifically, in the same way as Theorem 3.12 generalises Theorem 3.4, the next corollaries generalise previous results by removing the regularity assumption on the graph H.

Corollary 3.16. *Let* H *be a connected graph with* $H \prec S_{10}$*. Then,* $H \simeq S_{10}$*.*

Corollary 3.17. *Let* H *be a connected graph with* $H \prec S_{12}$*. Then, either* $H \simeq S_{10}$ *or* $H \simeq S_{12}$*.*

Both these corollaries are a direct consequence of Theorem 3.14 and Theorem 3.15. Indeed, let H be a connected graph and let f be an H-coloring of S_{10} (similarly, S_{12}). Suppose $\tilde{H}_f$ is not cubic: then $\tilde{H}_f$ can be extended to infinitely many connected cubic graphs by the procedure described just above Definition 3.7. All of them color S_{10} (respectively, S_{12}), a contradiction to Theorem 3.14 (respectively, Theorem 3.15). Hence, $\tilde{H}_f$ is cubic and the statements respectively follow by Theorem 3.14 and Theorem 3.15, once again.

As before, once we recall that S_{12} has a perfect matching, two other corollaries follow from Corollary 3.16 and Corollary 3.17.

Corollary 3.18. *If there exists a connected graph* H *coloring all cubic graphs, then* $H \simeq S_{10}$.

Corollary 3.19. *If there exists a connected graph* H *coloring all cubic graphs with a perfect matching, then either* $H \simeq S_{10}$ *or* $H \simeq S_{12}$.

To obtain all the results displayed in Table 2, we need just a final remark. It is possible to construct cubic graphs with a perfect matching having a subgraph as in Figure 22. In [65] it is shown that such cubic graphs have a normal 7-edge-coloring (see Section 1.4.3), but they don't have any normal k-edge-coloring with $k < 7$, and hence they cannot admit a Petersen coloring (see Section 1.4.3 again). Hence, considering also our Theorem 3.12 we can state the following remark.

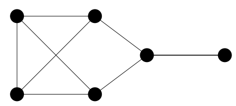


Figure 22: The subgraph mentioned in [65]

Remark 3.20. *A connected simple graph* H *that colors any cubic graph* G *with a perfect matching (and hence any cubic graph) does not exist.*

3.3 H-COLORINGS IN r-REGULAR GRAPHS, FOR $r > 3$

In this section we analyse whether there exists a connected graph H such that every r-regular graph G admits an H-coloring, for each $r > 3$. Clearly, the answer could depend on the class of graphs from where we choose the graph G: the bigger the class, the more unlikely it is that the same graph H would color all of them.

In Section 3.3.1 we consider the case of G being a multigraph. On the other hand, in Section 3.3.2 we restrict our attention to the subclass of simple regular graphs. In the former case, we are able to give a complete negative answer, whilst in the latter one we give a negative answer for G having even degree, and we leave the odd case as an open problem (see Problem 3.27).

3.3.1 H-colorings in r-regular multigraphs, for $r > 3$

In this section we show that, for every even $r > 3$, there is no graph H such that $H \prec G$ for every r-regular multigraph G. We note that H is not necessarily simple and can be a multigraph.

In each of the multigraphs S_4, S_6 and S_{12}, portrayed in Figure 23, there is a unique way how one can pair all the vertices of each multigraph such that the vertices in each pair are adjacent. Consequently, these three multigraphs each admit a unique perfect matching up to which parallel edges are chosen, shown in bold in Figure 23. Notwithstanding whether we are referring to S_4, S_6 or S_{12}, in Section 3.3.1, we shall refer to this perfect matching in each of these multigraphs by M. For every $k \geqslant 0$, let $S_4 + kM$ (similarly, $S_6 + kM$ or $S_{12} + kM$) be the $(k+3)$-regular multigraph obtained from S_4 (respectively, S_6 or S_{12}) after adding k edges parallel to every edge in M. When $k = 0$, $S_4 + 0M$, $S_6 + 0M$ and $S_{12} + 0M$ are assumed to be S_4, S_6 and S_{12}, respectively.

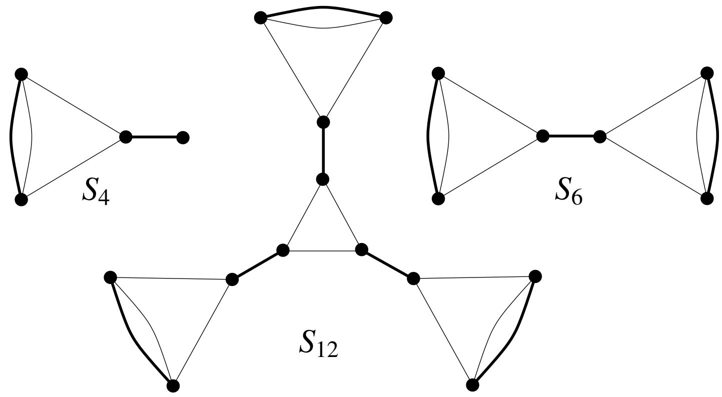


Figure 23: The chosen perfect matching M for the multigraphs S_4, S_6 and S_{12}

In analogy with the already introduced definition of an *exposed copy of* S_4 we define in detail what an *exposed copy of* $S_4 + kM$ is, for some $k \geqslant 0$. Let G be a multigraph having three vertices of degree $k+3$ and a further vertex of arbitrary degree. Denote this set of vertices by X. If the induced multisubgraph $G[X]$ is isomorphic to $S_4 + kM$, then we say that G *exposes* $S_4 + kM$ and that $G[X]$ is an *exposed copy* of $S_4 + kM$ in G.

In the next proposition we show that for any $r > 3$ there exists an r-regular multigraph G that admits only G-colorings.

Proposition 3.21. *Let* H *be a connected graph with* $H \prec S_{12} + kM$, *for some* $k \geqslant 1$. *Then,* $H \simeq S_{12} + kM$.

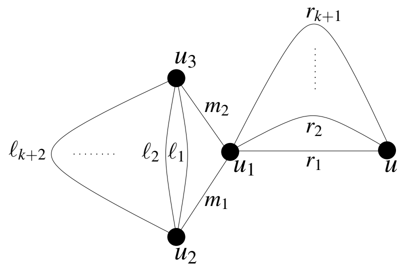


Figure 24: General labelling of an exposed copy of $S_4 + kM$ in G

Proof. Let f be an H-coloring of G, where $G = S_{12} + kM$. Let z^1, z^2, z^3 be the three vertices of G which induce a 3-cycle (without parallel edges). Let Z_1, Z_2, Z_3 be the three disjoint exposed copies of $S_4 + kM$ in G, such that $z^i \in Z_i$, for each $i \in \{1, 2, 3\}$. Additionally, in each copy of Z_i, we label the vertices by u^i, v^i and w^i, where u^i is the unique vertex adjacent to z^i in Z_i. The $k+2$ edges with endvertices v^i and w^i are labeled by $\ell^i_1, ..., \ell^i_{k+2}$, whilst the $k+1$ edges with endvertices u^i and z^i by $r^i_1, ..., r^i_{k+1}$. Finally, the two edges u^iv^i and u^iw^i are denoted by m_1 and m_2, respectively. In what follows, when we refer to a generic copy of $S_4 + kM$ in G we will omit the superscripts in the labelling of vertices and edges of G (see Figure 24), and in their images under the action of f_V.

We first show that for each copy of Z_i in G, the following holds. By the definition of H-coloring, $f(\ell_1), f(\ell_2), \dots, f(\ell_{k+2})$, $f(m_1), f(m_2)$ are $k+4$ distinct edges in H since the edges $\ell_1, \ell_2, \dots, \ell_{k+2}, m_1, m_2$ are distinct and pairwise adjacent in G. Hence, $f_V(v) \neq f_V(w)$. Indeed, if by contradiction $f_V(v)$ and $f_V(w)$ are equal, say to $x \in V(H)$, then all the edges $f(\ell_1), f(\ell_2), \dots, f(\ell_{k+2})$, $f(m_1), f(m_2)$ are incident to x since $\ell_1, \ell_2, \dots, \ell_{k+2}, m_1, m_2$ are exactly all the edges incident to v and w in G. These add up to $k+4$ edges, meaning that $d_H(x) = k+4$. However, $d_G(v) = k+3$, that is, $d_H(f_V(v)) \neq d_G(v)$, a contradiction. It follows that $f(\ell_1), f(\ell_2), \dots, f(\ell_{k+2})$ are parallel edges in H with endvertices $f_V(v) = v'$ and $f_V(w) = w'$. In particular, $f(m_1)$ must be incident to v' and $f(m_2)$ must be incident to w'. Moreover, since m_1 and m_2 are adjacent edges in G, $f(m_1)$ and $f(m_2)$ are adjacent edges in H. Denote by $u' \in V(H) - \{v', w'\}$ their common endvertex. Consequently, $f_V(u) = u'$, and the edges $f(r_1), \dots, f(r_{k+1})$ are incident to u' in H but not to v' and w'.

We now prove that $f_V(z^i) \neq f_V(u^i)$ for each $i \in \{1,2,3\}$. Without loss of generality, suppose that $f_V(z^1) = f_V(u^1) = u'^1$, for contradiction. Since $r_1^1, ... r_{k+1}^1$ are all incident with both m_1^1 and m_2^1, it must be that $\{f(z^1z^2), f(z^1z^3)\} = \{f(m_1^1), f(m_2^1)\}$. This means that $f_V(z^2) \in \{u'^1, v'^1, w'^1\}$. If $f_V(z^2) = u'^1$, exactly one of $f(r_1^2), ..., f(r_{k+1}^2)$ is an element of $\{f(m_1^1), f(m_2^1)\} \setminus \{f(z^1z^2)\}$, implying that $f_V(u^2) = u'^1$ as well. Since $\{f(m_1^2), f(m_2^2)\} \subset \{f(z^1z^2)\} \cup \{f(r_1^1), ... f(r_{k+1}^1)\}$, we have that $f_V(u^2) = f_V(v^2)$, or $f_V(w^2) = f_V(v^2)$, a contradiction, since u'^2, v'^2, w'^2 are pairwise distinct. Therefore, $f_V(z^2) \in \{v'^1, w'^1\}$. If, without loss of generality, $f_V(z^2) = v'^1$, then $f_V(u^2) \in \{v'^1, w'^1\}$. Consequently, $\{f_V(v^2), f_V(w^2)\} = \{u'^1, v'^1, w'^1\} \setminus \{f_V(u^2)\}$, a contradiction, since the edges $f(r_1^2), \dots, f(r_{k+1}^2)$ are not incident to v'^1 and w'^1. Consequently, $f_V(z^1) \neq f_V(u^1)$, as required.

Hence, up to now we have proved that the induced multisubgraph $H[f_V(V(Z_i))]$ is an exposed copy of $S_4 + kM$ in H. From now on, we denote $H[f_V(V(Z_i))]$ by Z_i', for each $i \in \{1,2,3\}$.

Claim A. Z_1', Z_2', Z_3' are pairwise edge-disjoint.
Proof of Claim A. Without loss of generality, suppose that $E(Z_1') \cap E(Z_2') \neq \emptyset$, for contradiction. Since f_V maps the vertices of Z_i having degree $k+3$ into vertices of degree $k+3$ in H, either $Z_1' = Z_2'$, or $H\left[(E(Z_1') \cup E(Z_2')\right] \simeq S_6 + kM$. First, assume that $Z_1' = Z_2'$. In this case, $f_V(z^1) = f_V(z^2)$, and, without loss of generality, we assume that $f(r_j^1) = f(r_j^2)$ for every $j \in \{1, 2, \dots, k+1\}$. Moreover, all the edges $f(z^1z^2)$, $f(z^2z^3)$ and $f(z^1z^3)$ must be pairwise distinct in H, and each of them must be incident to $f_V(z^1)$ (which is equal to $f_V(z^2)$). None of the edges $f(z^1z^2)$, $f(z^2z^3)$ and $f(z^1z^3)$ coincide with $f(r_j^1)$ (which is equal to $f(r_j^2)$) for any $j \in \{1, 2, \dots, k+1\}$, since z^1z^2, z^1z^3 and z^2z^3 are all incident to at least one of r_j^1 and r_j^2 in G. However, this means that $d_H(f_V(z^1)) > k+3$, a contradiction. Therefore, we must have the other case, that is, $H[E(Z_1) \cup E(Z_2)] \simeq S_6 + kM$. However, since H is connected, if $H[E(Z_1) \cup E(Z_2)] \simeq S_6 + kM$, then $H \simeq S_6 + kM$, meaning that either $Z_1' = Z_3'$ or $Z_2' = Z_3'$, a contradiction once again. ∎

Hence, H contains three edge-disjoint exposed copies of $S_4 + kM$. Let $W = \{z^1z^2, z^2z^3, z^1z^3\}$. Observe that $f(z^1z^2)$, $f(z^2z^3)$ and $f(z^1z^3)$ are pairwise distinct and pairwise adjacent in H, so that the possibilities for the edge-induced subgraph $H[f(W)]$ by the edges of W in H are: a 3-cycle ($H[f(W)] \simeq C_3$), a single vertex of degree 3, say z', which is adjacent to three distinct neighbours ($H[f(W)] \simeq K_{1,3}$), or a single vertex of degree 3, say z', having two distinct neighbours.

Claim B. The only possibility for $H[f(W)]$ is a 3-cycle, that is, $H[f(W)] \simeq C_3$.

Proof of Claim B. Indeed, in both the other cases there are at least two pairs of edges, say, the pair $\{f(z^1z^2), f(z^2z^3)\}$ and the pair $\{f(z^1z^2), f(z^1z^3)\}$, such that the unique endvertex of the edges in each pair is the vertex z' in H. This means that z^1 and z^2 are mapped into z'. Since Z_1' and Z_2' are edge-disjoint in H and the edge $f(z^1z^2)$ must be incident to all the edges $\{f(r_1^i), \dots, f(r_{k+1}^i) : i = 1, 2\}$ of Z_1' and Z_2' , the vertex z', which belongs to $Im(f_V)$, has degree at least $2(k+1)+1$, a contradiction, since G is $(k+3)$-regular and $2k+3 > k+3$ for $k \geqslant 1$. ■

Moreover, for any $j \in \{1, 2, \dots, k+1\}$, $f(r_j^1)$ must be incident to $f(z^1z^2)$ and $f(z^1z^3)$, $f(r_j^2)$ with $f(z^1z^2)$ and $f(z^2z^3)$, and, $f(r_j^3)$ with $f(z^2z^3)$ and $f(z^1z^3)$. Combining Claim A and Claim B with these last necessary properties we deduce that H is isomorphic to G. □

We are now in a position to prove the main result of this section.

Theorem 3.22. *For each $r > 3$, there is no connected graph H coloring all r-regular multigraphs admitting a perfect matching.*

Proof. Suppose such a graph H exists. For each fixed $r > 3$, choose $G = S_{12} + (r-3)M$, where M is the perfect matching in S_{12} as in Figure 23. Since $S_{12} + (r-3)M$ is r-regular and $H \prec S_{12} + (r-3)M$, by Proposition 3.21 we have that H must be $S_{12} + (r-3)M$. Now, let G_r be an r-regular multigraph admitting an $(S_{12} + (r-3)M)$-coloring f. Since $r > 3$, $S_{12} + (r-3)M$ contains (at least) two disjoint perfect matchings, say M_1 and M_2, and consequently, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are two disjoint perfect matchings of G_r, by Lemma 3.8. Hence, in order to find an r-regular multigraph with a perfect matching and without an $(S_{12} + (r-3)M)$-coloring, it suffices to exhibit an r-regular multigraph admitting a perfect matching but without two disjoint perfect matchings, for every $r > 3$. Examples of such multigraphs are constructed in [72] and called poorly matchable (see also [62]). The assertion follows. □

In the previous theorem, we consider G as an r-regular multigraph admitting a perfect matching. Clearly, the result holds in the larger class of r-regular multigraphs.

Theorem 3.23. *For each $r > 3$, there is no connected graph H coloring all r-regular multigraphs.*

3.3.2 *H-colorings in r-regular simple graphs, for $r > 3$*

In this section our aim is to show that, for every even $r > 3$, there is no connected graph H such that $H \prec G$ for every simple r-regular graph G. We remark that, as in previous section, H is not necessarily simple and can be a multigraph.

Before proceeding, let $\mathcal{K}_t^r$ denote the family of r-regular multigraphs of order t, whose vertices are pairwise adjacent. Note that a graph G in $\mathcal{K}_t^r$ admits a t-clique as a spanning (simple) subgraph of G.

Lemma 3.24. *Let H be a connected graph. For every $r \geqslant 1$, if the complete graph K_{2r+1} admits an H-coloring, then $H \in \mathcal{K}_t^{2r}$, where t is an odd integer and no vertex of H is unused.*

Proof. Let $f : E(K_{2r+1}) \to E(H)$ be an H-coloring of K_{2r+1}, and let f_V be the induced map on the vertices of K_{2r+1}. Let v_1 and v_2 be two distinct vertices in $\mathrm{Im}(f_V)$. Note that these two vertices exist since $|\mathrm{Im}(f_V)| = 1$ would imply that K_{2r+1} is $2r$-edge-colorable. We claim that $v_1v_2 \in E(H)$. Let u_1 and u_2 be two (distinct) vertices in $V(K_{2r+1})$ such that $f_V(u_i) = v_i$, for each $i \in \{1, 2\}$. Since u_1 is adjacent to u_2, $f(u_1u_2)$ is incident to both v_1 and v_2, implying that $v_1v_2 \in E(H)$. This proves our claim.

Consequently, there exists an integer $t \in \{2, 3, , \ldots, 2r + 1\}$ such that H contains a complete graph K_t as a subgraph and whose vertex set is $\mathrm{Im}(f_V) \subseteq V(H)$. For simplicity, we shall refer to this subgraph as K_t. Next, we claim that t must be odd. For, suppose not, and assume that t is even. Let M be a matching of H that is also a perfect matching of K_t. Consequently, M covers all the vertices of $\mathrm{Im}(f_V)$, since $V(K_t) = \mathrm{Im}(f_V)$. However, by Lemma 3.9, $f^{-1}(M)$ is a perfect matching of K_{2r+1}, a contradiction, since K_{2r+1} does not admit a perfect matching. Therefore, t must be odd.

We next claim that H contains a simple spanning subgraph isomorphic to a t-clique, that is, $\mathrm{Im}(f_V) = V(H)$. For, suppose not. Then, there exists an edge $xy \in E(H)$, such that $x \in Im(f_V)$ and $y \notin Im(f_V)$. Let M' be a matching of H with $|M'| = \frac{t-1}{2}$ such that M' covers all the vertices of $Im(f_V)$ except x. Let $N = M' \cup \{xy\}$. The set of edges N is a matching of H which covers all the vertices in $Im(f_V)$. However, by Lemma 3.9, this implies that $f^{-1}(N)$ is a perfect matching of K_{2r+1}, a contradiction once again. Therefore, H contains a complete graph of odd order as a simple spanning subgraph. □

Let $r > 1$ and let K'_{2r+1} be the complete graph on $2r+1$ vertices minus an edge. Let J_{2r} be the graph obtained by considering r copies of K'_{2r+1} such that all the vertices of degree $2r-1$ in these copies are adjacent to a new vertex u, resulting in a $2r$-regular simple graph. We refer to the vertex u as the central vertex of J_{2r}, and the r copies of K'_{2r+1} are denoted by $R_1, \ldots, R_r$.

Lemma 3.25. *Let $r > 1$ and let H be a graph such that $H \prec J_{2r}$. Then, $H \notin \mathcal{K}_t^{2r}$, for all possible t.*

Proof. Suppose that there exists a graph $H \in \mathcal{K}_t^{2r}$ such that $H \prec J_{2r}$, for contradiction. Let f be an H-coloring of J_{2r} and let f_V be the induced map on the vertices of J_{2r}.

Let u_1, u_2 be two vertices of J_{2r} adjacent to u and belonging to R_1 and R_2, respectively. Consider a cycle C of H (possibly of length 2) which contains the two edges $f(uu_1)$ and $f(uu_2)$ incident to $f_V(u)$. The preimage $f^{-1}(E(C))$ is a 2-regular subgraph of J_{2r}. Indeed, if f is an H-coloring of a graph G and F is a k-regular subgraph of H containing at least one vertex of $Im(f_V)$, then $f^{-1}(E(F))$ induces a k-regular subgraph of G. Moreover, one of the connected components of $f^{-1}(E(C))$ is a cycle passing through u and containing the two edges uu_1 and uu_2, a contradiction since J_{2r} does not have such a cycle. □

By the previous two lemmas, there exist no graph which colors both K_{2r+1} and J_{2r}, implying our last result.

Theorem 3.26. *For every $r > 1$, there is no connected graph H coloring all $2r$-regular simple graphs.*

The following open problem is suggested in [P2] in order to have a complete answer to the general question asked in Section 3.3, that is, whether there exists a graph H such that for every r-regular graph G, G admits an H-coloring, for each $r > 3$. In order to fully answer this question, by Theorem 3.23 and Theorem 3.26, it suffices to consider the following.

Problem 3.27. *Let $r > 1$ be odd. Determine whether there exists a connected graph H coloring all r-regular simple graphs.*

The question whether there exists a graph H in some class that colors any graph G in some other class has been addressed in the cubic case considering various classes for both H and G, for example, the class of bridgeless cubic graphs or the class of cubic graphs having a perfect matching. The same could be done in the case when G is assumed to be an r-graph. Let us recall that an r-graph is a connected r-regular graph such that $|\partial X| \geqslant r$ for every odd subset X of the vertex set. Hence, in [P2], we also suggest the following.

Problem 3.28. *Let $r > 3$. Determine whether there exists an r-graph H coloring all (simple) r-graphs.*

To conclude this chapter, we mention that Problem 3.28 has recently received a negative answer in [58]. To prove this negative answer, the authors of [58] define, for any given $r \geqslant 3$, $\mathcal{H}_r$ to be an inclusion-wise minimal set of connected r-graphs, such that for every connected r-graph G there is a graph $H \in \mathcal{H}_r$ such that $H \prec G$. They also prove, in Corollary 3.8 of [58], that $\mathcal{H}_r$ is unique for each $r \geqslant 3$. After that, they obtain Corollary 3.12, which states the following.

Corollary. *Either $\mathcal{H}_3 = \{P\}$ or $\mathcal{H}_3$ is an infinite set. Moreover, if $r \geqslant 4$, then $\mathcal{H}_r$ is an infinite set.*

This corollary is for multigraphs, while the negative answer is also given in the case of simple r-graphs. This is done with Corollary 3.20 of [58], which states the following.

Corollary. *Let $r \geqslant 3$ and let $\mathcal{H}'_r$ be a set of connected r-graphs such that every connected simple r-graph can be colored by an element of $\mathcal{H}'_r$. Then*

- *If the Petersen Coloring Conjecture is false, then $\mathcal{H}'_3$ is an infinite set.*
- *If $r \geqslant 4$, then $\mathcal{H}'_r$ is an infinite set.*

The results of [58] mean that an infinite number of r-graphs is needed to color all the (simple) r-graphs for any fixed $r > 3$. Furthermore, if $r = 3$, they prove that either the Petersen graph P colors all the 3-graphs (3-graphs coincides with bridgeless cubic graphs), or an infinite number of 3-graphs is needed to color all the (simple) 3-graphs. Hence, if the Petersen Coloring

Conjecture is true, bridgeless cubic graphs would be, also from this point of view, a very particular and singular class of regular graphs.

4

d-DIMENSIONAL FLOWS

This chapter is based on contributions [P3], [P4] and [P5].

As anticipated in Chapter 1, in this chapter we focus on a generalization of the circular flow number of a graph, i.e. the d-*dimensional flow number* of a graph G. We give some introductory definitions and a literature background.

Let $r \geqslant 2$ be a real number and d a positive integer, a d-*dimensional nowhere-zero* r-*flow* on a graph G, denoted (r, d)-NZF on G, is an orientation of G together with an assignment $\varphi \colon E(G) \to \mathbb{R}^d$ such that, for all $e \in E(G)$, the (Euclidean) norm of $\varphi(e)$ lies in the interval $[1, r-1]$ and, for every vertex, the sum of the inflow and outflow is the zero vector in $\mathbb{R}^d$. The d-*dimensional flow number* of a bridgeless graph G, denoted by $\phi_d(G)$, is defined as the infimum of the real numbers r such that G admits an (r, d)-NZF. Note that, by Seymour's 6-flow theorem [77] we have that $\phi_d(G) \leqslant 6$ for every d. Actually, $\phi_d(G)$ is a minimum: due to the above upper bound, it suffices to consider only the set of feasible d-dimensional nowhere-zero r-flows with $r \leqslant 6$, which can be represented as a compact subset of $\mathbb{R}^{d \cdot |E(G)|}$. Moreover, the function that assigns to every feasible flow the maximum norm among its components, that are elements of $\mathbb{R}^d$, is continuous.

In the above definitions it is not restrictive to assume any graph G to be connected. So we only consider connected graphs in the rest of the chapter.

The notion of (r, d)-NZF includes some parameters already considered in the literature. First of all, the 1-dimensional case, that is $\phi_1(G)$, is nothing but the classical circular flow number of a graph (see [32] and also Section 1.4.4 for the classical definition). According to our notation, Conjecture 1.13, the Tutte's 5-flow Conjecture, can be stated as follows.

Conjecture (Tutte's 5-flow Conjecture)**.** *Let G be a bridgeless graph. Then, $\phi_1(G) \leqslant 5$.*

An upper bound for ϕ_d is also conjectured for $d \geqslant 3$. Indeed, Jain suggested (see [96]) that every bridgeless graph admits a nowhere-zero flow with flow values taken on the unitary sphere S^2, that is the set of unit vectors of $\mathbb{R}^3$. Clearly, such a conjecture can be stated in our terminology as follows.

Conjecture 4.1 (S^2-flow Conjecture)**.** *Let* G *be a bridgeless graph. Then,* $\phi_d(G) = 2$ *for every* $d > 2$*.*

Let us remark that we will use the term cycle in its largest acception of subgraph with all vertices of even degree, i.e. *even subgraph*. Such a use is quite common in this context and permits to simplify the presentation. It is already observed in [85] that $\phi_7(G) = 2$ for every bridgeless graph G. This is a consequence of a covering result by Bermond, Jackson and Jaeger [6], claiming that every bridgeless graph G has seven cycles such that every edge of G is contained in exactly four of them. Moreover, the Berge-Fulkerson conjecture (Conjecture 1.9, see [28, 78]), if it holds true, implies that every bridgeless cubic graph has six cycles such that every edge is in exactly four of them. As noted in [85], this would imply that $\phi_6(G) = 2$ for any bridgeless cubic graph G. In a similar way, if the conjecture of Celmins [16] and Preissmann [71] on the existence, for every bridgeless graph, of five cycles covering each edge twice is true (known as the 5-Cycle Double Cover Conjecture, see also [93]), then $\phi_5(G) = 2$ for any bridgeless graph G.

Now, it is natural to ask what is a general upper bound for the 2-dimensional case. Indeed, Conjecture 1.13 and Conjecture 4.1 do not address the case $d = 2$. As far as we know, such a question was not considered in the literature before than in [P3], and one of the main goals of the work in [P3] is proposing a general upper bound for $\phi_2(G)$. This is done in Section 4.1, with Corollary 4.4, Theorem 4.5 and Problem 4.17.

Let us note that a 2-dimensional nowhere-zero r-flow can be viewed as a generalization of a 1-dimensional nowhere-zero flow where flow values are taken in the complex field $\mathbb{C}$. For this reason, from now on, we will call a 2-dimensional flow also a *complex flow* and we will write $\phi_{\mathbb{C}}(G)$ instead of $\phi_2(G)$ to denote the 2-dimensional flow number of a graph G, and we will call it the *complex flow number of* G. This notion is already considered in [85] in relation with Conjecture 4.1. Among other results, it is proved that $\phi_2(G) = 2$ if G is 6-edge-connected, but no discussion about a general upper bound for ϕ_2 is proposed by the author. Some other results on 2-dimensional nowhere-zero r-flows are obtained in [90], where the special case of flow values taken in the 2-dimensional unit sphere S^1 is considered.

The task of establishing good lower bounds for $\phi_C(G)$ is the hardest one in the study of this parameter. In Section 4.2, which presents the results contained in [P4], we make use of geometric and combinatorial arguments to prove a non-trivial lower bound for $\phi_C(G)$ in terms of the length of a shortest odd cycle (the *odd-girth*) of a bridgeless cubic graph G. This result is a consequence of Theorem 4.8, which exactly determines the complex flow number for every wheel graph W_n on $n+1$ vertices.

Finally, in Section 4.3 we introduce the work of [P5]. In particular, motivated by some results of Thomassen [85], we provide a geometric description of some d-dimensional flows on a graph G, and we prove that the existence of a suitable cycle double cover of G is equivalent, for G, to admit such a geometrically constructed (r, d)-NZF. This geometric approach allows us to provide upper bounds for $\phi_{d-2}(G)$ and $\phi_{d-1}(G)$, assuming that G admits an (oriented) d-cycle double cover.

4.1 POSSIBLE UPPER BOUNDS FOR $\phi_C(G)$

In previous section we mentioned the 5-Cycle Double Cover Conjecture by Celmins and Preissman. This is one of the many variations on the Cycle Double Cover Conjecture (see [94] and [93] for a comprehensive survey). Here we will consider one of them, which is also one of the strongest formulations, known as the *Oriented 5-Cycle Double Cover Conjecture*. In order to introduce it, we need to recall some terminology.

If G is a graph and O is an orientation of the edges of G, we denote by $O(G)$ the directed graph so obtained and, for every edge $e \in E(G)$, we denote by $O(e)$ its orientation with respect to O. A subgraph H of $O(G)$ is a *directed cycle* of $O(G)$ if for each vertex v of H, the indegree of v equals the outdegree of v.

The collection $\mathcal{C} = \{O_1(C_1), \dots, O_k(C_k)\}$ of directed cycles of a graph G is said to be an *oriented cycle double cover* of G if every edge e of G belongs to exactly two cycles C_i and C_j and the directions of $O_i(C_i)$ and $O_j(C_j)$ are opposite on e.

If we would like to stress the number of cycles in $\mathcal{C}$ we will write that $\mathcal{C}$ is an oriented k-cycle double cover of G.

The Oriented 5-Cycle Double Cover Conjecture, which is due to Archdeacon and Jaeger [2, 46], states the following.

Conjecture 4.2 (Oriented 5-cycle double cover Conjecture)**.** *Each bridgeless graph* G *has an oriented* 5*-cycle double cover.*

Now, we show that if Conjecture 4.2 holds true, then we can deduce a general upper bound for the parameter $\phi_{\mathbb{C}}$. We shall obtain such a relation by the following more general result.

Theorem 4.3. *Let* G *be a bridgeless graph and* $k \in \{2, 3, 4, 5\}$*. If* G *admits an oriented* k*-cycle double cover, then*

- $\phi_{\mathbb{C}}(G) = 2$*, if* $k \leqslant 3$*;*
- $\phi_{\mathbb{C}}(G) \leqslant 1 + \sqrt{2}$*, if* $k = 4$*;*
- $\phi_{\mathbb{C}}(G) \leqslant \tau^2$*, if* $k = 5$*.*

where τ *denotes the Golden Ratio* $\frac{1+\sqrt{5}}{2}$ [1].

Proof. Let $\mathcal{C} = \{O_1(C_1), \dots, O_k(C_k)\}$ be an oriented k-cycle double cover of G. We construct a complex flow on G as follows. Choose an arbitrary orientation O of G and k elements $p_1, \dots, p_k$ in $\mathbb{R}^2$. For every $i \in \{1, \dots, k\}$, we add a flow value equal to p_i to all edges $e \in C_i$ such that $O_i(e) = O(e)$, while we add $-p_i$ to all edges $e \in C_i$ such that $O_i(e) \neq O(e)$.

Observe that this procedure generates a complex flow, where every edge which belongs to $C_i \cap C_j$ receives one of the two vectors $\pm(p_i - p_j)$. In order to obtain an $(r, 2)$-NZF, we need the norm of each flow value $p_i - p_j$ to be at least one. Then, we choose $p_1, \dots, p_k$ pointing at the k vertices of a regular k-gon of side length 1. If $k \in \{2, 3\}$, since $|p_i - p_j| = 1$ for every $i \neq j$, then $\phi_{\mathbb{C}}(G) = 2$. If $k = 4$, since $|p_i - p_j|$ for every $i \neq j$ is either 1 or $\sqrt{2}$, then $\phi_{\mathbb{C}}(G) \leqslant 1 + \sqrt{2}$. Finally, if $k = 5$, the diagonals of a regular pentagon are in the golden ratio to its sides. Hence $|p_i - p_j|$ is equal to either 1 or τ for every $i \neq j$, then $\phi_{\mathbb{C}}(G) \leqslant \tau^2 (= 1 + \tau)$. □

Note that our choice of the vectors $p_1, \dots, p_k$ in each of the three cases of the proof of Theorem 4.3 is known to be optimal in order to minimize the ratio between the maximum and the minimum length of k points in the Euclidean plane (see [5]).

Corollary 4.4. *The Oriented* 5*-Cycle Double Cover Conjecture (Conjecture 4.2) implies* $\phi_{\mathbb{C}}(G) \leqslant \tau^2$ *for every bridgeless graph* G*.*

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1 To our knowledge, the Greek letter τ represented the Golden Ratio for hundreds of years, up to the early 20th century. This ancient notation is used along the chapter for the sake of a better distinction from the flow number.

In Section 4.1.1, we will discuss the problem of finding a graph G such that $\phi_C(G)$ is close to τ^2.

The upper bound of τ^2 is obtained by assuming true a well-known conjecture. Now, we complete this section by proving a general upper bound for $\phi_C(G)$ as a consequence of the proof of the 6-flow theorem of Seymour.

Theorem 4.5. *If* G *is a bridgeless graph, then* $\phi_C(G) \leqslant 1 + \sqrt{5}$.

Proof. In the proof of the 6-flow theorem [77, p. 133] Seymour showed that each bridgeless graph G has an integer 2-flow φ_2 and an integer 3-flow φ_3 such that $\varphi_2(e) \neq 0$ or $\varphi_3(e) \neq 0$ for each edge $e \in E(G)$. Let φ be a complex flow on $O(G)$, for an arbitrary orientation O, such that $\varphi(e) = (\varphi_2(e), \varphi_3(e))$ for each $e \in E(O(G))$. Since φ_2 and φ_3 are 2-flow and 3-flow, respectively, we have $\sqrt{\varphi_2(e)^2 + \varphi_3(e)^2} \leqslant \sqrt{1^2 + 2^2} = \sqrt{5}$. Also, one of the values $\varphi_2(e)$ and $\varphi_3(e)$ is nonzero, so $\sqrt{\varphi_2(e)^2 + \varphi_3(e)^2} \geqslant 1$. Thus, φ is indeed a $(1 + \sqrt{5}, 2)$-NZF of G. □

4.1.1 *Complex flows on cubic graphs*

In the case of nowhere-zero circular flows (i.e. 1-dimensional flows) it is well known that every bridgeless graph has a nowhere-zero r-flow if and only if every bridgeless cubic graph has a nowhere-zero r-flow. Following the same proof, one can get the following result.

Proposition 4.6. *For all positive integers* d *and real numbers* $r \geqslant 2$, *the following statements are equivalent:*

- *every bridgeless graph has a* d*-dimensional nowhere-zero* r*-flow;*
- *every bridgeless cubic graph has a* d*-dimensional nowhere-zero* r*-flow.*

By Proposition 4.6, there is a fixed constant k such that $\phi_C(G) \leqslant k$ for all bridgeless graphs G if and only if the same holds for all bridgeless cubic graphs. Hence, the problem of studying d-dimensional flows of bridgeless graphs can be restricted to the one of studying d-dimensional flows on cubic graphs.

Recall that Thomassen [85] proved that a cubic graph is bipartite if and only if it has an S^1-flow, that is a complex nowhere-zero 2-flow. In particular, up to a rotation, one can assume that the flow values are the three cube roots of unity, that is the complex solutions of the equation $z^3 = 1$.

As a further step in studying the complex flow numbers of cubic graphs we consider those being 3-edge-colorable. Observe that any 3-edge-colorable cubic graph has an oriented 4-cycle double cover (see for instance [94]), hence the following proposition follows from Theorem 4.3.

Proposition 4.7. *Let* G *be a 3-edge-colorable cubic graph. Then* $\phi_{\mathbb{C}}(G) \leqslant 1 + \sqrt{2}$.

The above inequality is the best possible as one can directly check that $\phi_{\mathbb{C}}(K_4) = 1 + \sqrt{2}$. However, we can obtain it as a special case (i.e. $n = 3$) of the following more general result which gives an exact value for $\phi_{\mathbb{C}}(W_n)$, where W_n is the wheel graph of order $n + 1$.

Since the proof of Theorem 4.8 is quite long and technical, we devote Section 4.2 to it.

Theorem 4.8. *Let* W_n *be the wheel graph of order* $n + 1$*, for* $n \geqslant 3$*. Then*

$$\phi_{\mathbb{C}}(W_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{n}{n-1}) & \text{if } n \equiv 1, 3 \mod 6 \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{n+1}{n}) & \text{if } n \equiv 5 \mod 6. \end{cases}$$

The next lemma is an immediate consequence of Theorem 4.8.

Lemma 4.9. *Let* G *be a cubic graph containing a chordless cycle* C *of length* k*. Then* $\phi_{\mathbb{C}}(G) \geqslant \phi_{\mathbb{C}}(W_k)$.

Proof. Suppose to the contrary that $\phi_{\mathbb{C}}(G) < \phi_{\mathbb{C}}(W_k)$. Then G has an $(r, 2)$-NZF φ with $r < \phi_{\mathbb{C}}(W_k)$. Contract all the vertices of G that are not in C to a unique vertex v. The obtained graph is W_k and φ induces on W_k an $(r', 2)$-NZF with $r' \leqslant r < \phi_{\mathbb{C}}(W_k)$, a contradiction. □

Lemma 4.9 combined with the values given in Theorem 4.8, gives the following.

Corollary 4.10. *Let* G *be a cubic graph with odd-girth equal to* g*. Then* $\phi_{\mathbb{C}}(G) \geqslant \phi_{\mathbb{C}}(W_g)$.

Using Corollary 4.10 we prove the following

Proposition 4.11. *Let n be odd and let P_n be the prism graph of order $2n$. Then $\phi_{\mathbb{C}}(P_n) = \phi_{\mathbb{C}}(W_n)$.*

Proof. Since P_n has odd-girth n, we have $\phi_{\mathbb{C}}(P_n) \geqslant \phi_{\mathbb{C}}(W_n)$.

Moreover, each flow on W_n can be easily extended to a flow on P_n using the same vectors: for every 4-cycle $uu'v'v$ where uu' and vv' are spokes of P_n, we set the flow from u to v to be the same as the flow from v' to u'. Thus we have $\phi_{\mathbb{C}}(P_n) = \phi_{\mathbb{C}}(W_n)$. □

Also, Corollary 4.10, together with Proposition 4.7, implies the following result.

Proposition 4.12. *Let G be a 3-edge-colorable cubic graph with a triangle. Then $\phi_{\mathbb{C}}(G) = 1 + \sqrt{2}$.*

Up to now, the unique infinite classes of non-bipartite cubic graphs for which we are able to determine the exact value of $\phi_{\mathbb{C}}$ are the ones considered in Proposition 4.11 and Proposition 4.12.

4.1.2 *Flow-triangulations*

In the rest of the section we provide upper bounds on the complex flow number of certain cubic graphs. To make our descriptions of complex flows more compact, we show that they can be equivalently represented in a geometric way. The main idea of this approach is that, by the Kirchhoff's law, the three vectors assigned to three edges incident with the same vertex correspond to a triangle in the Euclidean plane. Thus we can represent a complex flow as a suitable collection of triangles.

By a *triangle* we mean a subset of the Euclidean plane consisting of its three sides and interior points. Let s_1 and s_2 be sides of triangles T_1 and T_2, respectively. We say that s_1 and s_2 are *attachable* if we can translate T_1 to T_1' in such a way that the image of s_1 coincides with s_2 and T_1' and T_2 have no common internal points. In other words, attachable sides need to be parallel, of the same length and they need to have their triangles on mutually opposite sides. An *r-flow triangulation* of a bridgeless cubic graph G is a collection $\mathcal{T}$ containing a triangle T_v for each vertex v of G such that

(i) for each $v \in V(G)$, each edge incident to v corresponds to a unique side of T_v;

(ii) lengths of sides of all triangles from $\mathcal{T}$ lie in the interval $[1, r-1]$;

(iii) for each edge $uv \in E(G)$, the sides of the triangles T_u and T_v corresponding to uv are attachable.

Proposition 4.13. *Let* G *be a bridgeless cubic graph. Then* G *has an* r*-flow triangulation if and only if* G *has an* $(r, 2)$*-flow.*

Proof. We start with the only if part. Let O be an arbitrary orientation of the edges of G. We construct a complex flow on G as follows. Consider an oriented edge uv of $O(G)$ and let a and b be the vectors corresponding to the attachable sides of triangles T_u and T_v, respectively, which are oriented in such a way that T_u is on the right side of a and T_v is on the left side of b. Due to the definition of attachable sides, the vectors a and b have the same direction, so they are equal. We set to a the flow value of the edge uv. Note that if we orient uv in the opposite direction, it receives the opposite vector, thus we do not need any specific orientation of G.

We prove that this assignment is an $(r, 2)$-NZF. Consider a vertex v and orient all three edges incident with v as incoming. The vectors assigned to these edges form a triangle T_v and since all of them have T_v on the left side, they sum up to zero.

Now for the if part, assume that G has an $(r, 2)$-NZF. For each vertex v of G, let e_1, e_2 and e_3 be the oriented edges of $O(G)$ incident with v. For each $i \in \{1, 2, 3\}$, let a_i be the flow value of e_i, if v is the tail of e_i, and let a_i be the opposite of flow value of e_i otherwise. Then, the vectors a_1, a_2 and a_3 sum up to zero. Moreover, we can arrange them to form an oriented triangle T_v that is on the left side of each of a_1, a_2 and a_3.

We prove that the triangles T_v for each $v \in V(G)$ form an r-flow triangulation. Properties (i) and (ii) are trivially satisfied. Let uv be an oriented edge of $O(G)$ with flow value a. Since u is the tail and v is the head of uv, the triangle T_u lies on the left side of a and T_v lies on the right. Thus the sides of T_u and T_v corresponding to a are attachable. Hence Property (iii) also holds. □

For a bridgeless cubic graph G, finding the representation of a complex flow through a flow triangulation is, in general, only a reformulation of the original problem. However, in the following examples we present flow triangulations in some "nice" way. The term nice can be understood in several ways, but perhaps the most basic one requires that the intersection of every two different triangles T_1 and T_2, if not empty, consists either of one vertex, or of two coinciding sides s_1 and s_2 of T_1 and T_2, respectively. In the latter case, s_1 and s_2 correspond to the same edge of G and the set of all such edges induces a connected spanning subgraph of G. Examples of such nice flow triangulations of K_4 and $K_{3,3}$ are depicted in Figure 25. In all our figures, the graph is grey with its vertices placed in their corresponding triangles. Bold sides are of length 1 and dashed ones are always the sides with maximum length. Nevertheless, we do not know whether such a "nice" flow triangulation exists for every complex flow.

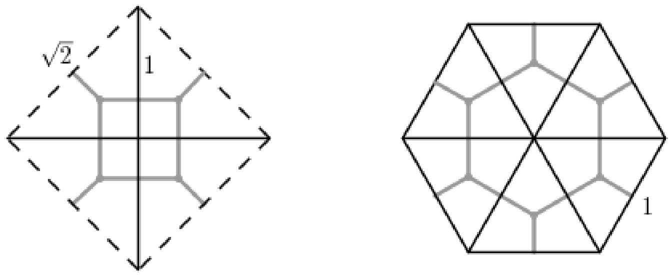

Figure 25: A $(1+\sqrt{2})$-flow triangulation of K_4 (left) and $K_{3,3}$ (right).

4.1.3 *Possible upper bounds for the complex flow number of some Class 2 cubic graphs*

We have already seen an upper bound on the complex flow number of 3-edge-colorable cubic graphs in Proposition 4.7. As usual, in order to prove a general bound on $\phi_{\mathbb{C}}(G)$ for every bridgeless cubic graph G, the hard case is when G is not 3-edge-colorable. Therefore, we are naturally interested in the complex flow number of the Petersen graph, which is the smallest such graph. Let us say that determining this value appears to be a hard problem. Here we propose an upper bound by constructing a suitable flow triangulation.

Proposition 4.14. *The complex flow number of the Petersen graph is at most* $1+\sqrt{7/3}$.

Proof. Throughout this proof, we take all the indices modulo 3. Consider, in the real Euclidean plane, an equilateral triangle $p_1p_2p_3$ with side length 1. For $i \in \{1,2,3\}$, let q_ip_i be the reflection of $p_{i-1}p_i$ through p_i and let q'_1, q'_2 and q'_3 be the points such that $q_1 q_3' q_2 q_1' q_3 q_2'$ is a regular hexagon. By adding the segments $q_i' p_{i+1}$ and $q_i' p_{i+2}$, for each $i \in \{1, 2, 3\}$, we obtain 10 triangles as depicted in Figure 26. The solid, dash-dotted and dashed lines have lengths 1, $\sqrt{4/3}$ and $\sqrt{7/3}$, respectively. It is easy to check that these triangles form a $(1+\sqrt{7/3})$-flow triangulation of the Petersen graph. $\square$

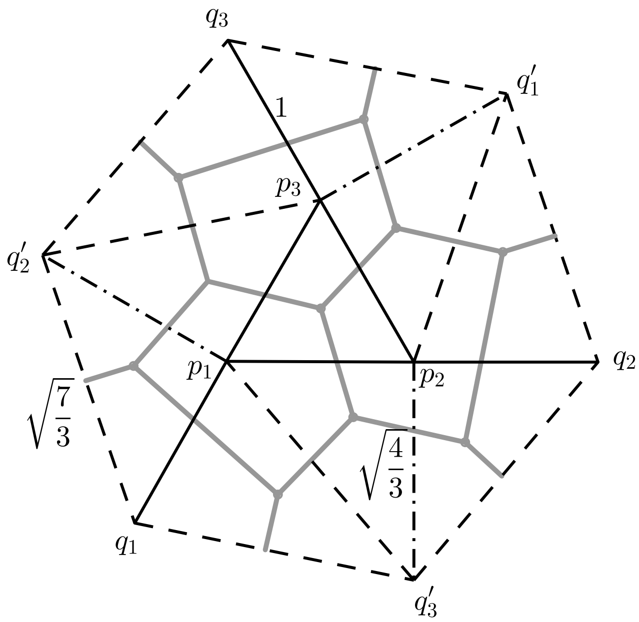

Figure 26: A $(1+\sqrt{7/3})$-flow triangulation of the Petersen graph.

Supported by computational results we believe that this is the exact complex flow number of the Petersen graph. Since we currently have no tools for proving such high lower bounds on complex flow numbers, we propose the following conjecture.

Conjecture 4.15. *The complex flow number of the Petersen graph is* $1+\sqrt{7/3}$.

As we mentioned in Section 1.4.4, the 1-dimensional flow number can distinguish Class 1 cubic graphs, which have 1-dimensional flow number at most 4, from the bridgeless Class 2 ones, having 1-dimensional flow number greater than 4 (see for instance [86]). However, the complex flow number does not serve for this purpose. One of the counterexamples is the Isaacs snark J_5 (see Figure 27) for which we show that $\phi_{\mathbb{C}}(J_5) < 1+\sqrt{2} = \phi_{\mathbb{C}}(K_4)$. We have found an $(r, 2)$-NZF of J_5 for $r = 1 + 1.387893647$ with the help of a computer.

Proposition 4.16. $\phi_{\mathbb{C}}(J_5) \leqslant 2.387893647 < 1+\sqrt{2}$.

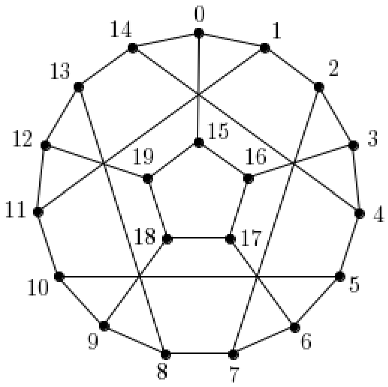

Figure 27: Isaacs snark J_5

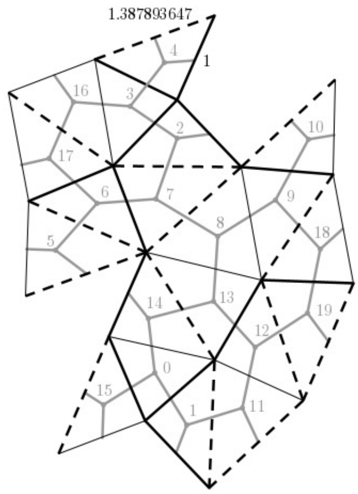

Figure 28: A complex flow on J_5

Figure 28 depicts an approximation of the flow triangulation corresponding to the found flow. We emphasise only the sides with minimum (bold) and maximum (dashed) length.

The Petersen graph is the worst case for many other problems in this area. Surprisingly, this seems not to be the case here. Indeed if we replace every vertex of P with a triangle, denoting the resulting graph by P_Δ, we are not able to extend our $(1+\sqrt{7/3}, 2)$-NZF on P to a $(1+\sqrt{7/3}, 2)$-NZF on P_Δ. The best $(r, 2)$-NZF flow on P_Δ we have up to now is for $r \approx 2.59$, also found by a computer.

We wonder if $\tau^2 \approx 2.618$ is the upper bound on the complex flow number of all bridgeless graphs and also whether this bound is reached by some graph. Therefore, we propose the following problems.

Problem 4.17. *Determine if* $\phi_C(G) \leqslant \tau^2$ *for every bridgeless graph* G.

Problem 4.18. *Establish the existence (or not) of a bridgeless cubic graph* G *with* $\phi_C(G) = \tau^2$.

We would also like to note that flow triangulations can be represented in a topological way. For instance, the $(1+\sqrt{7/3})$-flow triangulation of the Petersen graph can be described as a dual of P embedded on a torus. Similarly, the aforementioned flow triangulations for K_4, $K_{3,3}$ and J_5 can be also obtained from embeddings on some orientable surfaces. However, since we need to measure Euclidean distance, we avoid mentioning other surfaces, where the notion of distance is not clear.

Also, we noted that it is not clear if every complex flow on a cubic graph can be represented through a nice flow triangulation. We do not know the answer even for bipartite cubic graphs, which are perhaps the most simple family of cubic graphs for this problem, since each 2-flow triangulation consists of equilateral triangles with side length 1. Therefore, we leave it as a further open problem.

4.2 COMPLEX FLOW NUMBER OF W_n

This section is devoted to prove Theorem 4.8, which gives a non-trivial lower bound for the complex flow number of the wheel graphs.

For every integer $n \geqslant 3$, let W_n be the wheel graph with $n+1$ vertices and consider the orientation of its edges as in Figure 29. More precisely, the n vertices of the external cycle of W_n are labeled with $v_0, v_1, ..., v_{n-1}$ and the central vertex with u. All edges uv_j and $v_{j-1}v_j$ in the chosen orientation of W_n are directed towards v_j (here and in what follows indices are taken modulo n).

Let φ be a $(\lambda+1, 2)$-NZF of W_n. Set

$$\varphi(uv_j) = z_j \in \mathbb{C}, j \in \{0, ..., n-1\},$$

$$\varphi(v_j v_{j+1}) = p_j \in \mathbb{C}, j \in \{0, ..., n-1\}.$$

In particular, since φ is a $(\lambda+1, 2)$-NZF of W_n, the norm of each flow value is a real number which lies in the interval $[1, \lambda]$, i.e. $1 \leqslant |p_j|, |z_j| \leqslant \lambda$ holds. Moreover, the relation

$$z_j = p_j - p_{j-1} \tag{1}$$

holds for every $j = 0, ..., n-1$. Relation (1) suggests that the knowledge of all values p_j is sufficient to reconstruct the entire flow. Hence, we can represent any $(\lambda+1, 2)$-NZF of W_n as a cyclic sequence (i.e. the first element of the sequence is considered to succeed the last one) of n points $(p_0, \ldots, p_{n-1})$ in the complex plane. We often need to refer to the vectors having ends in two consecutive points of the sequence. In particular, we denote by $p_{j-1}p_j$ the vector in the complex plane having its tail in the point p_{j-1} and its head in the point p_j. With a slight abuse of notation, we will sometimes denote the vector $p_{j-1}p_j$ by z_j, stressing, when necessary "vector z_j".

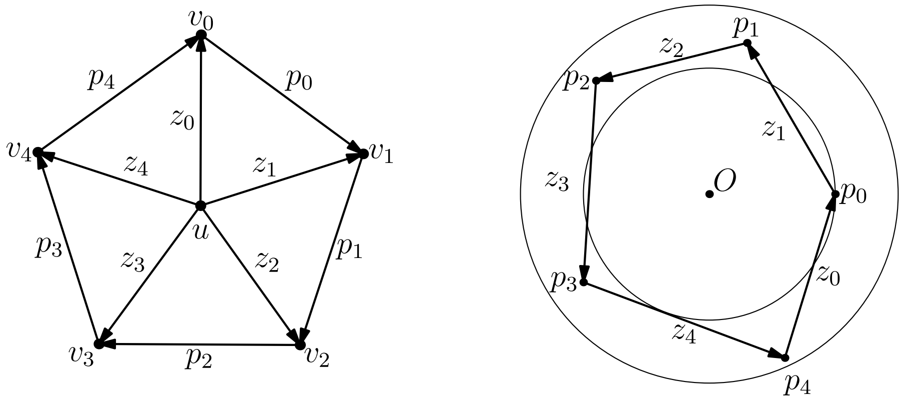

Figure 29: A representation of a complex flow of W_5.

On the other hand, an arbitrary cyclic sequence $(p_0, \ldots, p_{n-1})$ of n points represents a $(\lambda+1, 2)$-NZF of W_n(see Figure 29) if:

- each point p_j belongs to the circular crown between circumferences centered in the origin and of radius 1 and λ, denoted by $\mathcal{C}_I$ and $\mathcal{C}_E$ respectively;
- the norm of each vector z_j lies in the interval $[1, \lambda]$.

In what follows, by using such a representation, we first exhibit a complex flow of W_n for each odd n and then we prove its optimality. In what follows we will consider only odd values of n, since, for even values of n, W_n has a $(3,1)$-NZF, and, therefore, by Proposition 1 in [85], $\phi_{\mathbb{C}}(W_n) = 2$.

4.2.1 *An upper bound for the complex flow number of W_n*

Let n be an odd number and set $t = \lfloor \frac{n}{6} \rfloor$. We distinguish three cases according to the congruence of n modulo 6. We furnish a geometric description of each case and then we formally give the sequences of points representing the flows. Figure 30 represents an example of the described sequences for each possible odd congruence class modulo 6.

For $n \equiv 5 \pmod 6$, we consider points p_j as the vertices of a regular star polygon $\{\frac{n}{t+1}\}$ (following the standard Schläfli notation, see [19]) inscribed in $\mathcal{C}_I$. The length of each side of the polygon is equal to $2\sin(\frac{\pi}{6} \cdot \frac{n+1}{n})$. For $n \equiv 1 \pmod 6$, we construct points p_j on $\mathcal{C}_I$ as follows: starting from p_0 and moving in clockwise direction, we have p_1 at distance 1 from p_0. All other points are obtained by moving on $\mathcal{C}_I$ in anticlockwise direction, each point at distance $2\sin(\frac{\pi}{6} \cdot \frac{n}{n-1})$ from the previous one. The distance between p_{n-1} and p_0 results to be also $2\sin(\frac{\pi}{6}\cdot\frac{n}{n-1})$. For $n \equiv 3 \pmod 6$, we construct points p_j on $\mathcal{C}_I$ except p_1 which belongs to $\mathcal{C}_E$. Starting from $p_0 \in \mathcal{C}_I$ and moving in clockwise direction, we have $p_1 \in \mathcal{C}_E$ at distance 1 from p_0. Then, $p_2 \in \mathcal{C}_I$ is at distance 1 from p_1, again in clockwise direction. All other points are obtained following $\mathcal{C}_I$ in anticlockwise direction, each at distance $2\sin(\frac{\pi}{6}\cdot\frac{n}{n-1})$ from the previous one. Once again also the distance between p_{n-1} and p_0 is $2\sin(\frac{\pi}{6}\cdot\frac{n}{n-1})$.

Hence, the sequences of points result to be the followings.

- if $n \equiv 5 \pmod 6$,
$$p_j = e^{ij\left(\frac{\pi}{3}\cdot\frac{n+1}{n}\right)}, \forall j : 0 \leqslant j \leqslant n-1,$$
- if $n \equiv 1 \pmod 6$,
$$p_0 = e^{i\frac{\pi}{3}} \text{ and } p_{j+1} = e^{ij\left(\frac{\pi}{3}\cdot\frac{n}{n-1}\right)}, \forall j : 0 \leqslant j \leqslant n-2,$$
- if $n \equiv 3 \pmod 6$,
$$p_0 = e^{2i\left(\frac{\pi}{6}\cdot\frac{2n-3}{n-1}\right)}, p_1 = 2\sin\left(\frac{\pi}{6}\cdot\frac{n}{n-1}\right) e^{i\left(\frac{\pi}{6}\cdot\frac{2n-3}{n-1}\right)} \text{ and}$$
$$p_{j+2} = e^{ij\left(\frac{\pi}{3}\cdot\frac{n}{n-1}\right)}, \forall j : 0 \leqslant j \leqslant n-3.$$

For each n, we denote by λ^* the maximum between the distances of two consecutive points of the corresponding sequence and the norms of the points p_j's.
Observe that, if $n \equiv 1,3 \mod 6$, then $\lambda^* = 2\sin(\frac{\pi}{6}\cdot\frac{n}{n-1})$, while if $n \equiv 5 \mod 6$, $\lambda^* = 2\sin(\frac{\pi}{6}\cdot\frac{n+1}{n})$. Hence, for each odd n, we have constructed a $(\lambda^*+1, 2)$-NZF of W_n and λ^*+1 gives an upper bound for $\phi_{\mathbb{C}}(W_n)$.

Remark 4.19. *For each $n \geqslant 3$, $\phi_{\mathbb{C}}(W_n) \leqslant \phi_{\mathbb{C}}(W_3) \leqslant 1+\sqrt{2}$.*

We will make use of this remark along the proof of Theorem 4.8 in order to guarantee a general upper bound for $\phi_{\mathbb{C}}(W_n)$ which will be sufficiently small for our aims.

4.2.2 *A lower bound for the complex flow number of W_n*

In this section we prove that, for each odd n, the value of λ^* determined by the corresponding sequence described in Section 4.2.1, is indeed also a lower bound for $\phi_{\mathbb{C}}(W_n)$.

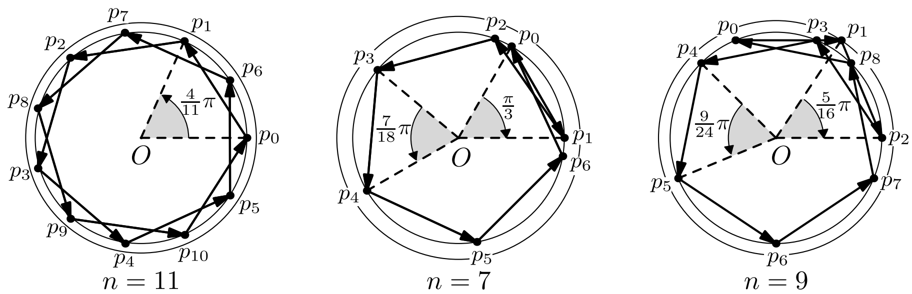

Figure 30: Three sequences corresponding to, from left to right, optimal flows of W_{11}, W_7 and W_9.

In other words, let φ be an optimal $(\lambda+1, 2)$-NZF of W_n, n odd, then we already proved in Section 4.2.1 that $\lambda \leqslant \lambda^*$, and we want to show that $\lambda = \lambda^*$.

From now on we can assume $\lambda \leqslant \sqrt{2}$ due to Remark 4.19. Let $(p_0, \dots, p_{n-1})$ be the associated cyclic sequence of points p_j. As already remarked, we denote the vector $p_{j-1}p_j$ by z_j for each $j \in \{0, \dots, n-1\}$, where all indices are taken modulo n. In particular, we have $\max_j\{|z_j|, |p_j|\} = \lambda$ since φ is optimal. If $0 \neq \theta \in (-\pi, \pi)$ denotes the amplitude of an angle and $\theta > 0$ ($\theta < 0$), then the positive (negative) rotation is by definition in anticlockwise (clockwise) direction. Similarly, if $p_{j-1} = |p_{j-1}|e^{i\alpha_{j-1}}$ and $p_j = |p_j|e^{i\alpha_j}$ are two consecutive points in the cyclic sequence then the vector z_j is said to be positively (negatively) oriented, or simply positive (negative), if $\alpha_j - \alpha_{j-1}$ is positive (negative). Note that $\alpha_{j-1} \neq \alpha_j$ since $\lambda \leqslant \sqrt{2}$ (< 2) (see Remark 4.19).

First of all, let us define some geometric transformations of the cyclic sequence $(p_0, \dots, p_{n-1})$ that will be largely used in what follows. Let $\theta \in \mathbb{R}$ and $h, k \in \{0, \dots, n-1\}$. Define $\rho_{h,k}(\theta)$ as the transformation which rotates all points $p_h, p_{h+1}, \dots, p_k$ around the origin by an angle of θ and fixes all the others. Note that we are considering a cyclic sequence, hence k could be less than h and $\rho_{h,k}(\theta) \neq \rho_{k,h}(\theta)$. Indeed, if $h < k$, we have

$$\begin{aligned}\rho_{h,k}(\theta)(p_0, \dots, p_h, \dots, p_k, \dots, p_{n-1}) = \\ = (p_0, \dots, p_{h-1}, p_h e^{i\theta}, \dots, p_k e^{i\theta}, p_{k+1}, \dots, p_{n-1}),\end{aligned}$$

and

$$\begin{aligned}\rho_{k,h}(\theta)(p_0, \dots, p_h, \dots, p_k, \dots, p_{n-1}) = \\ = (p_0 e^{i\theta}, \dots, p_h e^{i\theta}, p_{h+1}, \dots, p_{k-1}, p_k e^{i\theta}, \dots, p_{n-1} e^{i\theta}).\end{aligned}$$

Claim 4.20. *Let $(p_0, \ldots, p_{n-1})$ be a cyclic sequence of points representing a $(\lambda+1, 2)$-NZF of W_n. Then, for any $h \neq k \in \{0, 1, \ldots, n-1\}$, there is an angle θ such that the sequence $\rho_{h,k-1}(\theta)(p_0, \ldots, p_{n-1}) = (p'_0, \ldots, p'_{n-1})$ satisfies $|p_j| = |p'_j|$ for all j and $|z_j| = |z'_j|$ for all $j \notin \{h, k\}$. Moreover, the angle θ can be chosen so that*

(a) If z_h and z_k have the same orientation, $|z_h| < \lambda$ and $|z_k| > 1$, then $\lambda > |z'_h| > |z_h|$ and $1 < |z'_k| < |z_k|$.

(b) If z_h and z_k have opposite orientations and $|z_h|, |z_k| < \lambda$, then $\lambda > |z'_h| > |z_h|$ and $\lambda > |z'_k| > |z_k|$.

(c) If z_h and z_k have opposite orientations and $|z_h|, |z_k| > 1$, then $1 < |z'_h| < |z_h|$ and $1 < |z'_k| < |z_k|$.

For $j \in \{h, k\}$, if $|z'_j| > |z_j|$ ($|z'_j| < |z_j|$), we say that the transformation lengthens (shortens) z_j (by an arbitrary small factor).

Proof. From the definition of $\rho_{h,k-1}(\theta)$, it follows directly that $|p_j| = |p'_j|$ for all j and $|z_j| = |z'_j|$ for all $j \notin \{h, k\}$. For Case *(a)* we choose θ positive or negative according to the common orientation of the vectors z_h and z_k. In Cases *(b)* and *(c)*, say that z_h is positive and z_k is negative, we choose θ positive or negative, respectively. In all the cases the absolute value of θ can be chosen arbitrary small to ensure arbitrary small scale factor and then $1 < |z'_h|, |z'_k| < \lambda$. □

For our aims, we also need to define $\sigma_{h,k}(\theta)$ as the transformation which rotates the point p_h around the point p_k by an angle θ and fixes any other point of the sequence.

$$\sigma_{h,k}(\theta)(p_0, \ldots, p_h, \ldots, p_{n-1}) = \\ = (p_0, \ldots, (p_h - p_k)e^{i\theta} + p_k, \ldots, p_{n-1}).$$

The main idea of the proof is choosing time by time an optimal flow φ of W_n satisfying additional minimality assumptions (explained later in details). We will show that if such a φ does not correspond to one of the three sequences (up to isometries) described in Section 4.2.1, then we can modify it to obtain a new sequence which contradicts the minimality assumptions on φ.

In the rest of the proof, we need to distinguish two cases.

CASE I: each vector z_j has norm less than λ.

The first case we consider is when $|z_j| < \lambda$ for every $j \in \{0, 1, ..., n-1\}$. In this case, consider the optimal comples flows of W_n having the minimum number, say m_1, of vectors z_j with $|z_j| = 1$. Among them, choose φ as one with the minimum number, say m_2, of points p_j with $|p_j| = \lambda$ (i.e. $p_j \in \mathcal{C}_E$). Moreover, without loss of generality, we can assume that φ has at least one of the vectors z_j which is positive, otherwise we can simply consider $-\varphi$.

First of all, we prove that by our choice of φ the relation $|z_j| = 1$ follows for every index j and that all vectors z_j are positive.

Suppose by contradiction that there exists an index h such that $|z_h| > 1$. By assumption $|z_h| < \lambda$. If $m_1 > 0$, then there exists k such that $|z_k| = 1$. According to Claim 4.20, we can lenghten z_k and shorten or lengthen z_h (according to its orientation) constructing a sequence of points having less than m_1 vectors of norm 1, a contradiction. Then, we can assume $m_1 = 0$. Note that since φ is optimal, there exists $l \in \{0, 1, ..., n-1\}$ such that $p_l \in \mathcal{C}_E$. Construct a new sequence by setting $p'_l = (1-\varepsilon)p_l$ and $p'_j = p_j$ for all $j \neq l$. It is possible to choose $\varepsilon > 0$ sufficiently small such that $1 < |p'_l|, |z'_l|, |z'_{l+1}| < \lambda$, so that the new sequence still represents a complex flow of W_n. However, the new sequence has no vector z_j of norm 1 like the original sequence, but less than m_2 points belonging to $\mathcal{C}_E$, a contradiction again. Then, we have that $|z_j| = 1$, for every index j.

Assume there exist two indices h, k such that z_h is positive and z_k is negative. Applying Claim 4.20 we can lengthen both of them to reduce the number of vectors having norm 1, a contradiction with the choice of φ.

Then, all vectors z_j are positive and with $|z_j| = 1$. Now we show that, for each odd n, a sequence of points p_j with such properties corresponds to a $(\lambda+1, 2)$-flow having $\lambda > \lambda^*$. This leads to a contradiction since φ is chosen to be optimal. Indeed, let $\alpha_j > 0$ be the angle subtended by the vector z_j. It holds that $\sum_{j=0}^{n-1} \alpha_j = 2a\pi$ for some positive integer a. Moreover, since $\lambda \leqslant \sqrt{2} < \tau$ (Golden Ratio) the angle α_j is at least $2\arcsin(\frac{1}{2\lambda})$ which is the angle obtained with $p_{j-1}, p_j \in \mathcal{C}_E$. Hence, we have

$$\arcsin\left(\frac{1}{2\lambda}\right) \leqslant \frac{a}{n}\pi.$$

We look for the minimum possible λ which realizes previous inequality. It is clearly obtained when the equality holds. Moreover, since $\lambda > 1$, $\arcsin(\frac{1}{2\lambda}) < \frac{\pi}{6}$ holds, that is $a < \frac{n}{6}$. So, λ is minimum and larger than 1 for $a = \lfloor \frac{n}{6} \rfloor$. Hence, if $n = 6t + h$, we have that $\arcsin(\frac{1}{2\lambda}) = \frac{t}{n}\pi$ and so $\lambda = \frac{1}{2\sin(\frac{t}{n}\pi)}$.

For $n > 1$, it follows that the relations

$$\lambda \geqslant \frac{1}{2\sin(\frac{n-1}{n} \cdot \frac{\pi}{6})}$$

and

$$\lambda^* \leqslant 2\sin\left(\frac{n}{n-1} \cdot \frac{\pi}{6}\right)$$

hold. Moreover, direct computations show that the real function

$$\sin\left(\frac{\pi x}{6}\right)\sin\left(\frac{\pi}{6x}\right)$$

has maximum equal to $\frac{1}{4}$ reached only for $x = 1$. Then, $\lambda > \lambda^*$ holds for every odd $n > 1$, a contradiction.

CASE II: there is a vector z_j of norm λ.

Now we can assume that for some $k \in \{0, ..., n-1\}$, it holds $|z_k| = \lambda$.

Without loss of generality assume that the vector z_k is positive. Consider the set of optimal complex flows of W_n having the minimum number, say $m_1 > 0$, of vectors z_j with $|z_j| = \lambda$. Among all such optimal flows, we choose φ in such a way that it has the minimum number, say m_2, of points p_j with $|p_j| = \lambda$, that is with the minimum number of points which belong to $\mathcal{C}_E$.

Claim 4.21. $|z_j|, |p_j| \in \{1, \lambda\}$ *for every* $j \in \{0, \dots, n-1\}$*. In particular,* $|z_j| = \lambda$ *if and only if* z_j *is positive (and then* $|z_j| = 1$ *if and only if* z_j *is negative).*

Proof. First we prove that if the vector z_j is positive then $|z_j| = \lambda$. By contradiction suppose there exists $h \in \{0, \dots, n-1\}$ such that $|z_h| < \lambda$ and z_h is positive. By Claim 4.20 we can shorten z_k and lengthen z_h yielding an optimal flow having less than m_1 vectors with norm λ, a contradiction.

In a similar way we prove that if the vector z_j is negative then $|z_j| = 1$. By contradiction assume there exists $h \in \{0, \dots, n-1\}$ such that $|z_h| > 1$ and z_h is negative. Following Claim 4.20 we shorten z_k and z_h, obtaining again a contradiction as in the previous case on the choice of φ.

Hence, for every z_j we have $|z_j| = \lambda$ if z_j is positive, while $|z_j| = 1$ if z_j is negative.

We complete the proof of the claim by showing that there is no index h such that $1 < |p_h| < \lambda$. If this is the case, then we will construct a new sequence of points p'_j by applying a suitable transformation of the original sequence which leads to a contradiction. If z_h and z_{h+1} are both positive, we set $p'_h = (1-\varepsilon)p_h$ and $p'_j = p_j$ for all $j \neq h$, where $\varepsilon > 0$ is chosen sufficiently small in such a way that $1 < |p'_{h-1}p'_h| < \lambda$ and $1 < |p'_h p'_{h+1}| < \lambda$. The new sequence corresponds to an optimal flow with $m_1 - 2$ vectors z'_j with norm λ, a contradiction. If z_h and z_{h+1} are both negative, we set $p'_h = (1+\varepsilon)p_h$ and $p'_j = p_j$ for all $j \neq h$, where $\varepsilon > 0$ is chosen sufficiently small in such a way that $1 < |p'_{h-1}p'_h| < \lambda$ and $1 < |p'_h p'_{h+1}| < \lambda$. Then we shorten z'_k and z'_h as in Claim 4.20 and we obtain an optimal flow with $m_1 - 1$ vectors with norm λ, a contradiction. If z_h is positive and z_{h+1} is negative, then we transform the original sequence by using $\sigma_{h,h+1}(\theta)$, where θ is sufficiently small and it is positive (resp. negative) if the angle $\angle p_{h-1}p_h p_{h+1}$ is non-negative (resp. negative). Vice versa, if z_h is negative and z_{h+1} is positive, then we transform the original sequence by using $\sigma_{h,h-1}(\theta)$, where θ is sufficiently small and it is negative (resp. positive) if the angle $\angle p_{h-1}p_h p_{h+1}$ is non-negative (resp. negative). In all cases, the resulting sequence of points defines a complex nowhere-zero flow on W_n with less than m_1 vectors having norm λ, a contradiction. This completes the proof of Claim 4.21. □

By Claim 4.21 we have only eight different types of vectors z_j in φ. Indeed, z_j is completely defined up to rotations once we have its direction (and then its norm by Claim 4.21) and the norm of p_{j-1} and p_j, which is in $\{1, \lambda\}$. Then, a vector z_j can be denoted by XY^*(see Figure 31), where $X, Y \in \{I, E\}$ and $* \in \{+, -\}$ are chosen in the following way.

- $X = I$ if $|p_{j-1}| = 1$ and $X = E$ if $|p_{j-1}| = \lambda$;
- $Y = I$ if $|p_j| = 1$ and $Y = E$ if $|p_j| = \lambda$;
- $* = +$ or $* = -$ if z_j is positive or negative, respectively.

Claim 4.22. *There exists an index* $j \in \{0, \ldots, n-1\}$ *such that* $|p_j| = 1$.

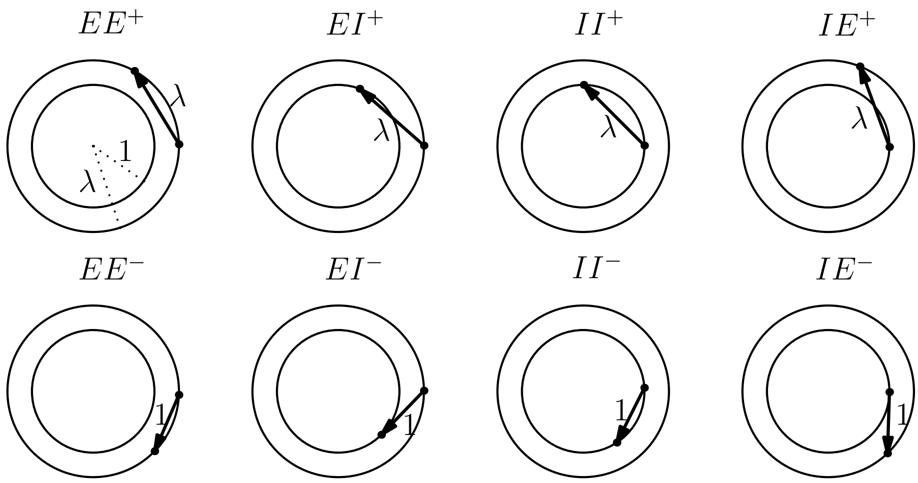

Figure 31: The eight types of vectors z_j in a representation of the chosen optimal flow φ of W_n.

Proof. By contradiction, if $|p_j| = \lambda$ for every index j, then all vectors z_j are of type either EE^+ or EE^-. Moreover, z_k is of type EE^+ and at least one of them is of type EE^-. Otherwise, $|p_j| = \lambda$ and $|z_j| = \lambda$ for all j, that is impossible in an optimal flow. In particular, there must be an index h with z_h of type EE^+ and z_{h+1} of type EE^-. The sequence is cyclic so we surely find the sequence EE^+, EE^-. Hence we can construct a sequence of points p'_j which defines a flow with less than m_1 vectors of maximum length λ by applying the transformation $\sigma_{h,h+1}(\theta)$, for a sufficiently small $\theta > 0$. □

By Claim 4.22, without loss of generality, we can assume $|p_0| = 1$ from now on.

Claim 4.23. *All positive vectors z_j are of type* II^+.

Proof. First we prove that φ has no vector z_j of type IE^+. By contradiction, assume z_h is of type IE^+. There are four possibilities for the vector z_{h+1}, namely EE^+, EE^-, EI^+ and EI^-. Since we have $\lambda \leqslant \sqrt{2} < \Phi$ for every odd n, the mutual position of the two vectors z_h and z_{h+1} in each case is like the ones represented in Figure 32. In all these cases, by applying $\sigma_{h,h-1}(\theta)$ for a sufficiently small $\theta > 0$ we obtain a new sequence of points p'_j which corresponds to an optimal flow with either less than m_1 vectors of norm λ (if z_{h+1} is positive) or m_1 vectors of norm λ but less than m_2 points on $\mathcal{C}_E$ (if z_{h+1} is negative), a contradiction in both cases.

Moreover, φ has no vector of type EE^+. Indeed, note that the angle subtended at the centre by a vector of type EE^+ on $\mathcal{C}_E$ is

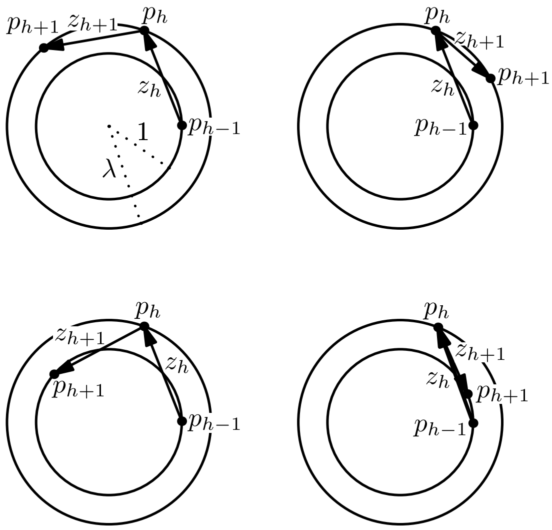

Figure 32: Mutual position of z_h and z_{h+1}.

equal to $\frac{\pi}{3}$. Then, it is the same angle subtended at the centre by a vector of type II^+ on $\mathcal{C}_I$. If z_h is of type EE^+ (note that $h \geqslant 2$ since $p_0 \in \mathcal{C}_I$), then we can construct a new sequence of points $(p'_0, ..., p'_{n-1})$ in the following way:

- $p'_0 = p_0$
- $|p'_0 p'_1| = 1$ and $p'_1 \in \mathcal{C}_I$
- $p'_{j-1}p'_j$ is of the same type as $p_{j-2}p_{j-1}$ for $2 \leqslant j \leqslant h$
- $p'_{j-1}p'_j$ is of the same type as $p_{j-1}p_j$ for $h < j \leqslant n-1$.

Since we replaced a vector $p_{h-1}p_h$ of type EE^+ with a vector $p'_0p'_1$ of type II^+ which subtends the same angle, while maintaining all the other vectors of the same type, the new sequence of points has less than m_1 vectors having norm λ, a contradiction.

Finally, we prove that φ has no vectors of type EI^+. Indeed, if z_h is of type EI^+ with $h > 0$, then z_{h-1} cannot be positive because both vectors of type EE^+ and IE^+ are already excluded. Then, it could be either of type IE^- or EE^-. Since $\lambda \leqslant \sqrt{2}$ the mutual position of the points p_{h-2}, p_{h-1} and p_h is like the ones in Figure 33.

In both these cases, by applying $\sigma_{h-1,h-2}(\theta)$ for a sufficiently small $\theta < 0$, we obtain a new configuration of points p'_j which corresponds to an optimal flow with less than m_1 vectors z'_j of norm λ, a contradiction. This completes the proof of Claim 4.23. $\square$

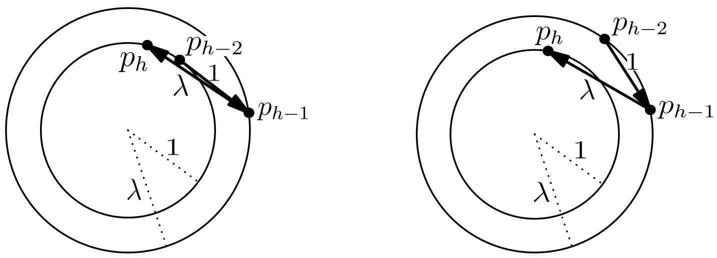

Figure 33: Mutual position of the points p_{h-2}, p_{h-1} and p_h.

In what follows we will make use of the measure of some angles depicted in Figure 34. We denote by 2α and 2β the angles subtended at the centre by a chord of length 1 on $\mathcal{C}_E$ and of length λ on $\mathcal{C}_I$, respectively. The following relations hold.

$$\alpha = \arcsin\left(\frac{1}{2\lambda}\right), \quad \beta = \arcsin\left(\frac{\lambda}{2}\right).$$

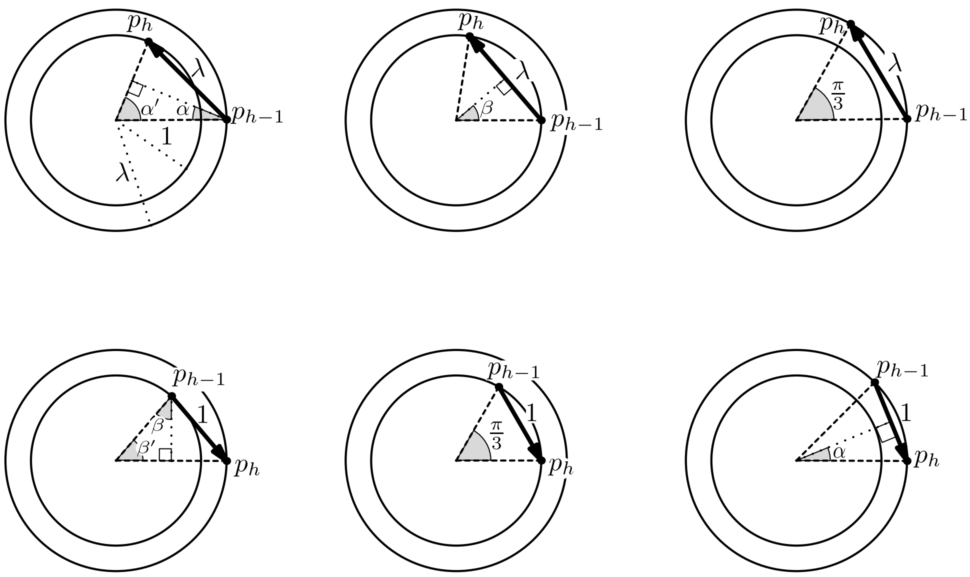

Figure 34: The angles subtended at the centre by all different types of vectors having norm 1 and λ.

Since $1 < \lambda \leqslant \sqrt{2}$, it follows

$$\arcsin\left(\frac{\sqrt{2}}{4}\right) \leqslant \alpha < \frac{\pi}{6}, \quad \frac{\pi}{6} < \beta \leqslant \frac{\pi}{4}.$$

Moreover, we prove the inequality $\alpha + \beta > \frac{\pi}{3}$ which will be used in what follows.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta =$$

$$= \sqrt{\left(1 - \frac{1}{4\lambda^2}\right)\left(1 - \frac{\lambda^2}{4}\right)} - \frac{1}{4}.$$

Since $\alpha+\beta$ is not larger than $\frac{5\pi}{3}$, then $\alpha+\beta > \frac{\pi}{3}$ if and only if $\sqrt{\left(1-\frac{1}{4\lambda^2}\right)\left(1-\frac{\lambda^2}{4}\right)} - \frac{1}{4} < \frac{1}{2}$.
This inequality easily leads to $4(\lambda^2-1)^2 > 0$ which is always satisfied.

Finally, we denote by $\alpha' = \frac{\pi}{2} - \alpha$ and $\beta' = \frac{\pi}{2} - \beta$ the complement angles of α and β, respectively.

Claim 4.24. *No vector* z_j *is of type* EE^-.

Proof. By Claim 4.23 and since $p_0 \in \mathcal{C}_I$, to prove this claim it suffices to show that the two ordered sequences of three consecutive vectors of types IE^-, EE^-, EI^- and IE^-, EE^-, EE^- cannot appear in φ.

We first prove that a subsequence of type IE^-, EE^-, EI^- cannot appear. Assume that the points p_j corresponding to the subsequence IE^-, EE^-, EI^- are p_j, p_{j+1}, p_{j+2} and p_{j+3} as in Figure 35. Observe that the angle subtended at the centre by $p_j p_{j+3}$ is

$$\beta' + 2\alpha + \beta' = 2\beta' + 2\alpha = 2\left(\frac{\pi}{2} - \beta\right) + 2\alpha = \pi - 2(\beta - \alpha) < \pi \tag{2}$$

where the last inequality holds since $\beta > \alpha$ for every $\lambda \in [1, \sqrt{2}]$.

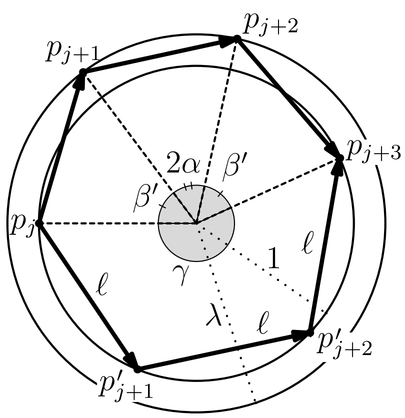

Figure 35: Configuration of points p_j, p_{j+1}, p_{j+2} and p_{j+3} corresponding to the subsequence of types IE^-, EE^-, EI^-.

Replace p_{j+1} and p_{j+2} by $p'_{j+1}, p'_{j+2} \in \mathcal{C}_I$, respectively, in such a way that $p_j p'_{j+1}, p'_{j+1} p'_{j+2}$ and $p'_{j+2} p_{j+3}$ are all positive vectors with $|p_j p'_{j+1}| = |p'_{j+1} p'_{j+2}| = |p_{j+2} p'_{j+3}| = \ell$ (see Figure 35). Let us prove that $1 < \ell < \lambda$. Indeed, we have

$$\gamma = 2\pi - (2\alpha + 2\beta') = \pi - 2\alpha + 2\beta$$

Recalling that $\alpha+\beta > \frac{\pi}{3}$ and $\beta > \frac{\pi}{6}$, we obtain $2\alpha+4\beta > \pi$, that is $6\beta > \pi - 2\alpha + 2\beta = \gamma$ and so $\frac{\gamma}{3} < 2\beta$. Hence, $\ell < \lambda$. Moreover, since $\gamma > \pi$, and so $\frac{\gamma}{3} > \frac{\pi}{3}$, by (2), we have also that $\ell > 1$. Hence, the new sequence of points corresponds to an optimal flow with the same number m_1 of vectors of norm λ, but less than m_2 points on $\mathcal{C}_E$, a contradiction.

In a very similar way we prove that the subsequence of types IE^-, EE^-,EE^- cannot appear in a representation of φ. Again, assume that the points p_j corresponding to the subsequence IE^-, EE^-, EE^- are p_j, p_{j+1}, p_{j+2} and p_{j+3}.

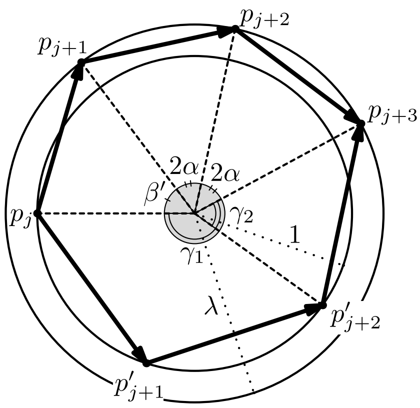

Figure 36: Configuration of points p_j, p_{j+1}, p_{j+2} and p_{j+3} corresponding to the subsequence of types IE^-, EE^-, EE^-.

Again we replace the points p_{j+1} and p_{j+2} of the sequence by two new points $p'_{j+1}, p'_{j+2} \in \mathcal{C}_I$ in such a way that $|p_j p'_{j+1}| = |p'_{j+1} p'_{j+2}| = |p'_{j+2} p_{j+3}| = \ell$, as shown in Figure 36. We prove that $1 < \ell < \lambda$. Denote by γ_1 the angle subtended at the centre by $p_j p'_{j+2}$ and by γ_2 the angle subtended by $p'_{j+2} p_{j+3}$. Then, ℓ is such that $\beta' + 4\alpha + \gamma_1 + \gamma_2 = 2\pi$ holds.

In order to prove $\ell < \lambda$, it suffices to show that the sum of the angles obtained with $|p_j p'_{j+1}| = |p'_{j+1} p'_{j+2}| = |p'_{j+2} p_{j+3}| = \lambda$, that is $\beta' + 4\alpha + 4\beta + \alpha'$, is strictly larger than 2π. Since

$$\beta' + 4\alpha + 4\beta + \alpha' = \frac{\pi}{2} - \beta + 4\alpha + 4\beta + \frac{\pi}{2} - \alpha = \pi + 3(\alpha+\beta),$$

it follows that $\beta' + 4\alpha + 4\beta + \alpha' > 2\pi$ if and only if $\alpha + \beta > \frac{\pi}{3}$, which is already proved to be satisfied.

In order to prove $\ell > 1$, it suffices to show that the sum of the angles obtained with $|p_j p'_{j+1}| = |p'_{j+1} p'_{j+2}| = |p'_{j+2} p_{j+3}| = 1$, that is $\beta' + 4\alpha + 2\frac{\pi}{3} + \beta'$, is strictly smaller than 2π. Since

$$\beta' + 4\alpha + 2\frac{\pi}{3} + \beta' = \pi - 2\beta + 4\alpha + \frac{2}{3}\pi,$$

it follows that $\beta' + 4\alpha + 2\frac{\pi}{3} + \beta' < 2\pi$ if and only if $2\alpha - \beta < \frac{\pi}{6}$.

Recalling that $\alpha + \beta > \frac{\pi}{3}$ and $\alpha < \frac{\pi}{6}$, we have $2\alpha - \beta = 3\alpha - (\alpha + \beta) < \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$.

Hence, the new sequence of points corresponds to an optimal flow having $m_1 - 1$ vectors of norm λ, a contradiction once again. This completes the proof of Claim 4.24.

□

Now we are going to rotate some points of the sequence $(p_0, ..., p_{n-1})$ associated with φ to obtain a new sequence denoted by $(q_0, ..., q_{n-1})$, in such a way that for every vector $z_h = p_{h-1}p_h$ there exists a unique vector $w_k = q_{k-1}q_k$ such that w_k is obtained by a suitable rotation of the vector z_h around the origin. By definition, the sequence of types of the vectors w_j is a permutation of the sequence of the types of the vectors z_j. Hence, the values of m_1 and m_2 do not change for this new sequence.

By previous claims such a sequence can contain only the following four types of vectors: IE^-, EI^-, II^- and II^+. Moreover, if a vector of type IE^- appears in the sequence, then it is necessarily followed by a vector of type EI^-. We choose $(q_0, ..., q_{n-1})$ in such a way that $p_0 \equiv q_0$ and all pairs IE^-, EI^-, if present, appear at the beginning of the sequence. They are followed by all vectors of type II^-, if present, and finally by all vectors of type II^+ (note that at least one positive vector appears by our assumptions). The sequence of types can be described in general by the following ordered sequence.

$$(IE^-, EI^-, ..., IE^-, EI^-, II^-, \dots, II^-, II^+, ..., II^+)$$

Now we prove that some specific subsequences cannot appear in the sequence associated to $(q_0, \dots, q_{n-1})$.

Claim 4.25. *The subsequences of consecutive vectors of types*

(a) IE^-, EI^-, IE^-

(b) IE^-EI^-, II^-

(c) II^-, II^-

cannot appear in the ordered sequence of types associated to $(q_0, ..., q_n)$.

Proof. (a) We argue similarly to what we did in the proof of Claim 4.24. Assume that the four consecutive points corresponding to the subsequence IE^-, EI^-, IE^- are q_j, q_{j+1}, q_{j+2} and q_{j+3}.

We obtain a new sequence by replacing the two points q_{j+1}, q_{j+2} by the points $q'_{j+1}, q'_{j+2} \in \mathcal{C}_I$ such that $|q_j q'_{j+1}| = |q'_{j+1} q'_{j+2}|$ and $1 < |q'_{j+2} q_{j+3}| < \lambda$ (see Figure 37). Let us show that such a choice is admissible.

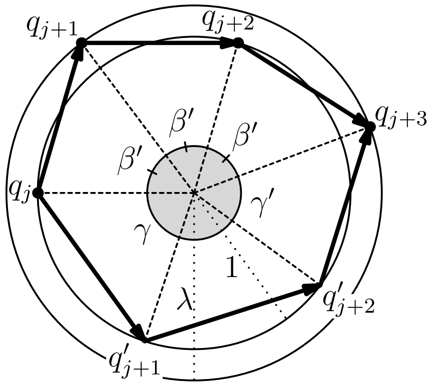

Figure 37: Configuration of points q_j, q_{j+1}, q_{j+2} and q_{j+3} corresponding to the subsequence of types IE^-, EI^-, IE^-.

Denote by γ and γ' the angles subtended at the centre by $q_j q'_{j+2}$ and $q'_{j+2} q_{j+3}$, respectively. Let $|q'_{j+2} q_{j+3}| > 1$, which is possible since $3\beta' < \pi$, so that $\gamma' > \beta'$. Hence, $\gamma = 2\pi - 3\beta' - \gamma' < 2\pi - 4\beta' = 4\arcsin(\frac{\lambda}{2}) = 4\beta$.

Since $\beta' < \pi/3$, $2\pi - 4\beta' > \frac{2}{3}\pi$. Note that, $\gamma' = \beta' + \varepsilon$, for a certain $\varepsilon > 0$. We choose ε sufficiently small in such a way that $\gamma = 2\pi - 3\beta' - \gamma' = 2\pi - 4\beta' - \varepsilon > \frac{2}{3}\pi$. So $|q_j q'_{j+1}| > 1$. Moreover, $\gamma = 2\pi - 3\beta' - \gamma' < 2\pi - 4\beta' = 4\beta$ implies that $|q_j q'_{j+1}| < \lambda$.

The new sequence of points has m_1 vectors of norm λ, but less than m_2 points which belongs to $\mathcal{C}_E$, a contradiction.

(b) Assume that the four consecutive points corresponding to the subsequence IE^-, EI^-, II^- are q_j, q_{j+1}, q_{j+2} and q_{j+3}. Denote by γ the explement angle of the angle subtended at the centre by $q_j q_{j+3}$. We obtain a new sequence by replacing the two points q_{j+1}, q_{j+2} by the points $q'_{j+1}, q'_{j+2} \in \mathcal{C}_I$ such that $|q_j q'_{j+1}| = |q'_{j+1} q'_{j+2}| = |q'_{j+2} q_{j+3}|$ (see Figure 38). Since $\beta' < \frac{\pi}{3}$, we have $2\beta' + \frac{\pi}{3} < \pi$, then $|q_j q'_{j+1}| > 1$.

Moreover, since $\lambda > 1$, $\gamma = 2\pi - (\frac{\pi}{3} + 2\beta') = \frac{2}{3}\pi + 2\beta < 6\beta$. Hence, $|q_j q'_{j+1}| < \lambda$.

The new sequence of points corresponds to an optimal flow having m_1 vectors of maximum norm λ, but less than m_2 points q_i on $\mathcal{C}_E$, a contradiction.

(c) Let w_j and w_{j+1} be the last two vectors of type II^- in the sequence, that is w_{j+2} is of type II^+. We obtain a new sequence by replacing the two points $q_{j+1}, q_{j+2} \in \mathcal{C}_I$ by the points

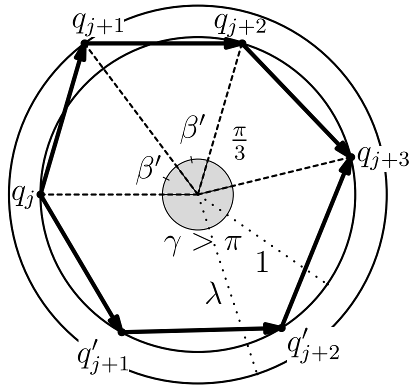

Figure 38: Configuration of points q_j, q_{j+1}, q_{j+2} and q_{j+3} corresponding to the subsequence of types IE^-, EI^-, II^-.

q'_{j+1}, q'_{j+2} such that $q'_{j+1} = (1+\varepsilon)q_{j+1}$ and $q'_{j+2} = q_{j+2}e^{i\theta}$, where $\varepsilon > 0$ and $\theta > 0$ are chosen sufficiently small and in such a way that $|q'_{j+1}q'_{j+2}| = 1$ and $1 < |q'_{j+2}q_{j+3}| < \lambda$. The new sequence of points corresponds to an optimal flow having less than m_1 vectors of maximum norm λ, a contradiction. This completes the proof of Claim 4.25. □

By previous claims there exists an optimal flow of W_n such that the types of its vectors respect one of the following three sequences.

(i) $(IE^-, EI^-, II^+, ..., II^+)$

(ii) $(II^-, II^+, ..., II^+)$

(iii) $(II^+, ..., II^+)$

For each given n odd, the maximum among all values $|z_j|$ and $|p_j|$ is completely determined once we know which of the three sequences we are considering. The exact values in each of these cases are summarized in Table 4. We give here an example of direct computation of the values in the last column of the table. The remaining values are computed similarly. In this particular sequence, every vector z_j has the same norm λ, while all points p_j belong to $\mathcal{C}_I$. Hence, the angle subtended at the centre by each z_j on $\mathcal{C}_I$ is exactly $2\beta = 2\arcsin(\frac{\lambda}{2})$. Hence, for some integer $k > 0$, we have $2n \arcsin\frac{\lambda}{2} = 2k\pi$, that is

$$\lambda = 2\sin\left(\frac{k}{n}\pi\right).$$

We are looking for the minimum possible $\lambda > 1$. Then, k is chosen as the smallest integer such that $\frac{k}{n}\pi > \frac{\pi}{6}$, that is $k = \lceil\frac{n}{6}\rceil$.

Set $n = 6t + h$, for $h = 1, 3, 5$. We obtain $\lambda = 2\sin\left(\frac{\pi(t+1)}{6t+h}\right) = 2\sin\left(\frac{\pi}{6}\cdot\frac{6t+6}{6t+h}\right) = 2\sin\left(\frac{\pi}{6}\cdot\frac{n+(6-h)}{n}\right)$.

Finally, comparing for each congruence of n the three possible values for λ, it turns out that the minimum λ is obtained with configuration (ii) if $n \equiv 1 \mod 6$, configuration (i) if $n \equiv 3 \mod 6$ and configuration (iii) if $n \equiv 5 \mod 6$. Observe that such optimal configurations are exactly the ones presented in Section 4.2.1.

	$IE^-, EI^-, II^+, \dots, II^+$	$II^-, II^+, \dots, II^+$	$II^+, \dots, II^+$
$n \equiv 1 \mod 6$	$2\sin(\frac{\pi}{6}\cdot\frac{n+2}{n-1})$	$\mathbf{2\sin(\frac{\pi}{6}\cdot\frac{n}{n-1})}$	$2\sin(\frac{\pi}{6}\cdot\frac{n+5}{n})$
$n \equiv 3 \mod 6$	$\mathbf{2\sin(\frac{\pi}{6}\cdot\frac{n}{n-1})}$	$2\sin(\frac{\pi}{6}\cdot\frac{n+4}{n-1})$	$2\sin(\frac{\pi}{6}\cdot\frac{n+3}{n})$
$n \equiv 5 \mod 6$	$2\sin(\frac{\pi}{6}\cdot\frac{n+4}{n-1})$	$2\sin(\frac{\pi}{6}\cdot\frac{n+2}{n-1})$	$\mathbf{2\sin(\frac{\pi}{6}\cdot\frac{n+1}{n})}$

Table 4: Exact values for the maximum norm in configurations (i), (ii) and (iii), according to the congruence of n modulo 6. In bold the value λ^* for each of the three cases.

Recall that, if n is even, then $\phi_C(W_n) = 2$. Therefore, Theorem 4.8, which we recall here, follows.

Theorem 4.26. *Let W_n be the wheel graph of order $n+1$, for $n \geqslant 3$. Then,*

$$\phi_C(W_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 + 2\sin(\frac{\pi}{6}\cdot\frac{n}{n-1}) & \text{if } n \equiv 1,3 \mod 6, \\ 1 + 2\sin(\frac{\pi}{6}\cdot\frac{n+1}{n}) & \text{if } n \equiv 5 \mod 6. \end{cases}$$

4.2.3 *A general lower bound for ϕ_C*

The value $\phi_C(W_n)$, for each odd n, gives a general non-trivial lower bound for $\phi_C(G)$, where G is a bridgeless cubic graph, in terms of its odd-girth. This is a straightforward consequence of the following standard observation. If C is an odd cycle of minimum length in G, then C is chordless. Contract all vertices of G not belonging to C to a unique vertex, thus obtaining a wheel graph whose complex flow number cannot be more than the complex flow number of G. Then, by Theorem 4.8 we deduce the following general result.

Theorem 4.27. *Let* G *be a non-bipartite cubic graph and let* g *be its odd-girth. Then,*

$$\phi_{\mathbb{C}}(G) \geqslant \begin{cases} 1 + 2\sin(\frac{\pi}{6} \cdot \frac{g}{g-1}) & \text{if } g \equiv 1,3 \mod 6, \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{g+1}{g}) & \text{if } g \equiv 5 \mod 6. \end{cases}$$

Let us remark that lower bounds in Theorem 4.27 are tight due to the prism graph P_n of order $2n$. Indeed, it is easy to see that each complex $\phi_{\mathbb{C}}(W_n)$-flow on W_n can be extended to a complex $\phi_{\mathbb{C}}(W_n)$-flow on P_n by a symmetry argument (see Proposition 4.11).

4.3 A GEOMETRIC DESCRIPTION OF d-DIMENSIONAL FLOWS

The aim of this section is to give a geometric description of d-dimensional nowhere-zero r-flows in finite graphs and to prove more connections with the (Oriented) Cycle Double Cover Conjecture.

For brevity, from now on, an (r, d)-NZF using only vectors from a set X will be called an X-flow.

Motivated by some results of Thomassen [85], in what follows we provide a geometric interpretation of (r, d)-NZFs. More precisely, in [85] the following proposition is proved (see Proposition 1 in [85]).

Proposition 4.28. *Let* G *be a graph. Then (a) and (b) below are equivalent, and they imply (c) where*

(a) G *has a nowhere zero 3-flow.*

(b) G *has an* R_3*-flow.*

(c) G *has an* S^1*-flow.*

If G *is cubic the three statements are equivalent, and* G *satisfies (a), (b), (c) if and only if* G *is bipartite.*

In particular, it is shown that a bridgeless graph G has a $(3, 1)$-NZF if and only if G has an R_3-flow, i.e. a flow with elements in $R_3 = \{z \in \mathbb{C} \colon z^3 = 1\}$. This is equivalent to having an S^1-flow, when G is cubic. Moreover, it is also shown that a bridgeless graph G has a 3-cycle double cover if and only if G has a T-flow, i.e. a flow with elements in $T = \{(1, 1, 0), (0, 1, 1), (1, 0, 1), (1, -1, 0), (0, 1, -1), (1, 0, -1)\}$. A natural generalization of T to a suitable set of elements T_d in $\mathbb{R}^d$ gives the equivalence between the existence of a T_d-flow on G and a d-cycle double cover of G. We also consider oriented cycle double covers and we show that, for every $d \geqslant 3$, a graph G has an oriented d-cycle double cover if and only if it admits a flow with values in the set H_d, which is a slight variation of the set T_d. We also give an alternative description of H_d in terms of the line graph of a crown graph. Finally, using a geometric argument, we give an upper bound on $\phi_{d-1}(G)$ and $\phi_{d-2}(G)$, assuming that G admits a d-cycle double cover (not necessarily oriented) and an oriented d-cycle double cover, respectively.

4.3.1 H_d-*flows*

Let R_k be the set of the k-th roots of unity, that is the solutions to the complex equation $z^k = 1$. It is straightforward to see that if a graph admits an R_3-flow then it admits an S^1-flow. DeVos [96] suggested that the converse could also be true. Thomassen [85] showed that this is true for cubic graphs, but does not hold in general.

Tutte's 3-flow Conjecture claims that every 4-edge-connected graph has a $(3, 1)$-NZF. Together with Thomassen's result this implies that every 4-edge-connected graph has an S^1-flow. In our terminology this is equivalent to saying that $\phi_{\mathbb{C}}(G) = 2$ for every 4-edge-connected graph G.

In this section, we would like to look at an (r, d)-NZF from a slightly different point of view. We modify a bit the notation in order to obtain an easier generalization to higher dimensions. Instead of considering flow values in R_3, we consider flow values in $H = R_3 \cup -R_3$, which is clearly equivalent because of the possibility of reorienting any edge in the opposite direction if needed. The points of H are the vertices of a regular hexagon in the complex plane, and so they are all points of S^1. Note that, up to rigid movements and a normalization, every nowhere-zero flow having as flow values the six vertices of an arbitrary regular hexagon centered in the origin of $\mathbb{R}^d$, can be transformed into an H-flow (and then in an R_3-flow).

For every $d \geqslant 3$, we consider the following subsets of $\mathbb{R}^d$:

$$\Sigma_d = \{(x_1, \dots, x_d) \in \mathbb{R}^d : \sum_{i=1}^{d} x_i = 0, \sum_{i=1}^{d} x_i^2 = 2\},$$

$$H_d = \Sigma_d \cap \mathbb{Z}^d.$$

The set Σ_d is a sphere of dimension $d-2$ embedded in $\mathbb{R}^d$. Thus we have the following.

Remark 4.29. *A graph admits a Σ_d-flow if and only if it admits an S^{d-2}-flow.*

Since H_d is the set of points of Σ_d having integer coordinates, we can describe H_d as the set of points of $\mathbb{R}^d$ having exactly one coordinate equal to 1, exactly one equal to -1 and all remaining coordinates equal to 0. For instance, we have

$$H_3 = \{(1,-1,0),(-1,1,0),(1,0,-1),(-1,0,1),(0,1,-1),(0,-1,1)\}$$

It is easy to check that such six points are the vertices of a regular hexagon embedded in $\mathbb{R}^3$. Then, as already observed, a graph admits an R_3-flow if and only if it admits an H_3-flow.

The notation introduced above permits to prove the following more general equivalence.

Theorem 4.30. *A graph G admits an H_d-flow if and only if G admits an oriented d-cycle double cover.*

Proof. Let $\mathcal{C} = \{O_1(C_1), \dots, O_d(C_d)\}$ be an oriented d-cycle double cover of G. Choose an arbitrary orientation O for the graph G and consider $e \in E(G)$. Since $\mathcal{C}$ is an oriented d-cycle double cover of G there exist exactly two different indices $h, k \in \{1, 2, ..., d\}$ such that $e \in C_h \cap C_k$ and $O_h(e) = O(e) \neq O_k(e)$. We assign to the edge e the d-tuple $\varphi(e)$ having 1 in the entry h, -1 in the entry k and 0 in all other entries. Note that, for $i \in \{1, \dots, d\}$, the i-th component of φ defines a nowhere-zero 2-flow on C_i with respect to O. Hence, φ is an H_d-flow on G with respect to the chosen orientation O.

For the converse, let φ be an H_d-flow on G with respect to the orientation O of G. Construct an oriented d-cycle double cover of G as follows. For each $i \in \{1, ..., d\}$, let C_i be the subgraph of G induced by the edges $e \in E(G)$ such that $\varphi(e) \neq 0$ in the i-th entry. Note that C_i is a cycle of G. Indeed, since φ is an H_d-flow on G, for every vertex $v \in V(G)$ there is an even number, eventually 0, of edges incident to v with a non-zero value of φ in the i-th entry. Construct an orientation O_i on C_i by assigning $O_i(e) = O(e)$ on every edge e of C_i such that $\varphi(e) = 1$, and letting $O_i(e)$ be opposite to $O(e)$ otherwise. By construction of O_i, for every vertex $v \in V(C_i)$, the indegree of v with respect to O_i is equal to the outdegree of v with respect to O_i, since φ is an H_d-flow on G with respect to the orientation O. Hence, $O_i(C_i)$ is a directed cycle of G. Moreover, note that, for every edge $e \in E(G)$, $\varphi(e)$ has exactly two non-zero entries and that such entries have opposite values. Therefore, the collection $\mathcal{C} = \{O_1(C_1) \dots, O_d(C_d)\}$ is an oriented d-cycle double cover of G. □

The following result is a consequence of Proposition 4.28 and previous results.

Corollary 4.31. *Let* G *be a graph. The following assertions are equivalent:*

1. G *has a nowhere-zero* 3*-flow;*
2. G *has an* R_3*-flow;*
3. G *has an* H_3*-flow;*
4. G *has an oriented* 3*-cycle double cover.*

Hence, in our terminology, Thomassen shows that the existence of an H_3-flow for a graph G implies the existence of a Σ_3-flow for a graph G. Notice that, from our definition of H_3, this implication is straightforward. In [85] it is shown that the converse is not true unless G is a cubic graph. Indeed, as already remarked, Thomassen presented examples of graphs admitting a Σ_3-flow but without an H_3-flow.

The Petersen graph P is an example of a graph that admits a Σ_4-flow but, since it is not Class 1, without an oriented 4-cycle double cover, and hence without an H_4-flow. Indeed, observe that P admits a Σ_4-flow since it admits an S^2-flow (Remark 4.29), which is depicted in Figure 39. In this figure, the vectors z_1, z_2, z_3, z_4, and z_5 are unit vectors. They are arranged to form a pentagon and a star lying in parallel planes and having their vertices on the unitary sphere.

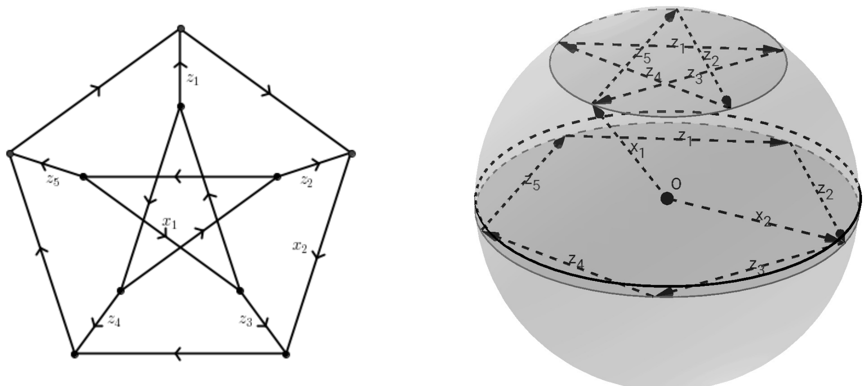

Figure 39: An S^2-flow on the Petersen Graph.

By previous considerations it seems natural to ask whether there exists a graph G having a Σ_5-flow but without an H_5-flow. We remark that such graph might not even exist as, by Theorem 4.30, it would be a counterexample to the Oriented 5-Cycle Double Cover Conjecture.

4.3.2 *A description of* H_d *as vertex set of a line graph*

Note that H_4 can be viewed as the set of vertices of a cuboctahedron (see [19]) embedded in $\mathbb{R}^4$. Analogously, H_5 is the set of vertices of a runcinated 5-cell in $\mathbb{R}^5$ (see [95] for a definition), and more in general polytopes of dimension $d-1$ embedded in $\mathbb{R}^d$ can be obtained, for every set H_d, by connecting two elements of H_d with an edge if and only if their difference is still an element of H_d. For every $d \geqslant 4$, all these polytopes can be viewed as a sort of generalization of the regular hexagon: indeed, as it happens in the regular hexagon, all their edges have the same length and such length is equal to the distance between the origin and each vertex.

In terms of graphs, we prove that the graph G_d corresponding to the $(d-1)$-dimensional polytope described above is nothing but the line graph of the crown graph $Cr(2d)$ on $2d$ vertices. Recall that the crown graph $Cr(2d)$ can be described as a complete bipartite graph from which the edges of a perfect matching have been removed.

Proposition 4.32. *Consider the graphs* G_d *and* $Cr(2d)$ *described above. Then* $G_d \cong L(Cr(2d))$.

Proof. By definition of G_d we have $V(G_d) = H_d$ and $E(G_d) = \{ab \mid a, b \in H_d,\ \pm(a-b) \in H_d\}$. Denote the vertices of $Cr(2d)$ by $u_1, \ldots, u_d$, $v_1, \ldots v_d$, where the vertices u_i are in an independence class of $Cr(2d)$ and the vertices v_i in the other one. Let $E(Cr(2d)) = \{u_i v_j \mid i \neq j\}$ and use the same notation for the edges of $Cr(2d)$ and the vertices of its line graph.

Associate to any $u_i v_j \in V(L(Cr(2d)))$ the d-tuple with 1 in the i-th entry, -1 in the j-th entry and 0 in all the other entries. This association gives a bijective map between the vertices of $L(Cr(2d))$ and $H_d = V(G_d)$. Observe that a vertex $u_i v_j \in V(L(Cr(2d)))$ is adjacent to any other vertex $u_l v_m$ such that either $i = l$ or $j = m$. Moreover, the edge-set of G_d can be also described as the set of (unordered) pairs (a, b) such that $a, b \in H_d$ and a and b are equal in exactly one non-zero entry. Hence the bijective map described above induces a bijection also between $E(L(Cr(2d)))$ and $E(G_d)$. □

4.3.3 *Upper bounds for* $\phi_{d-1}(G)$ *and* $\phi_{d-2}(G)$

At the beginnning of this chapter, we noted that if a graph G admits a 5-cycle double cover, then $\phi_5(G) = 2$. If we make a stronger hypothesis on G, that is that G admits an oriented 5-cycle double cover, then we obtain that $\phi_4(G) = 2$. Indeed, by Theorem 4.30, this is equivalent, for G, to have an H_5-flow. Hence, G admits also a Σ_5-

flow, and by Remark 4.29, also an S^3-flow. In particular, it holds that $\phi_4(G) = 2$.

Hence the following holds true.

Proposition 4.33. *The Oriented* 5*-Cycle Double Cover Conjecture (Conjecture 4.2) implies the* S^2*-flow Conjecture (Conjecture 4.1) for* $d \geqslant 4$.

More in general, arguing as above, it holds that if a graph G has an oriented d-cycle double cover, it has an S^{d-2}-flow, implying that $\phi_{d-1}(G) = 2$.

If we assume G to only admit a d-cycle double cover, not necessarily oriented, we obtain the following upper bound on $\phi_{d-1}(G)$.

Proposition 4.34. *For every* $d \geqslant 3$*, if a graph* G *has a* d*-cycle double cover* $\mathcal{C}$*, then* $\phi_{d-1}(G) \leqslant 1 + \sqrt{d/(d-2)}$.

Proof. We can assume that $\mathcal{C} = \{C_1, \ldots, C_d\}$. We now consider d points $a_1, ..., a_d \in \mathbb{R}^{d-1}$. More precisely, for $i \in \{1, \ldots, d-1\}$, let $a_i \in \mathbb{R}^{d-1}$ have the value $d\sqrt{d} - 2\sqrt{d} - 1$ in its i-th entry and $-\sqrt{d} - 1$ in all the other entries. Let $a_d \in \mathbb{R}^{d-1}$ have $d-1$ in all its entries. Let O be an orientation of G and, for each $i \in \{1, \ldots, d\}$, fix an eulerian orientation O_i on C_i.

For all $i \in \{1, \ldots, d\}$, add the flow value a_i, resp. $-a_i$, to all edges $e \in C_i$ such that $O_i(e) = O(e)$, resp. $O_i(e) \neq O(e)$.

Note that every edge of G is contained in exactly two members of $\mathcal{C}$. Then, every edge of G receives a vector with a norm $|a_i + a_j| = (d-1)\sqrt{2(d-2)}$ or $|a_i - a_j| = (d-1)\sqrt{2d}$, for $1 \leqslant i \neq j \leqslant d$. After normalizing, we obtain a $(1 + \sqrt{d/(d-2)}, d-1)$-NZF on G. □

Observe that the upper bound of Proposition 4.34 approaches 2 as d grows. We give a similar result for ϕ_{d-2}.

In [5], the authors claim that very likely the ratio between the maximum and the minimum distance for a set of d points in R^{d-2} is larger than $\sqrt{4/3}$. That was proved to be false by Seidel in 1969. Example 2.3 and Example 2.4 in [76] are proved to be optimal in order to minimize such a ratio. By using Seidel's examples we obtain the following.

Proposition 4.35. *For every* $d \geqslant 3$*, if a graph* G *admits an oriented* d*-cycle double cover* $\mathcal{C}$*, then*

$$\phi_{d-2}(G) \leqslant \begin{cases} 1 + \sqrt{\frac{d}{d-2}} & \text{if } d \text{ even,} \\ 1 + \sqrt{\frac{d^2-1}{d^2-2d-1}} & \text{if } d \text{ odd.} \end{cases}$$

.

Proof. If $d = 3$ the statement follows from Corollary 4.31.

Let $d \geqslant 4$ and let $\mathcal{C} = \{C_1, \ldots, C_d\}$. Set $d_1 \leqslant d_2$ such that $d_1 + d_2 = d - 2$ and $|d_1 - d_2| \leqslant 1$. Let U_1 and U_2 be two orthogonal and complementary subspaces of $\mathbb{R}^{d-2}$ passing through the origin, and having dimensions d_1 and d_2, respectively. For $i = 1, 2$, consider a regular d_i-dimensional simplex F_i in U_i and centered in the origin. Choose F_1 and F_2 having the same side length equal to $\sqrt{d-2}$ if d even, and $\sqrt{d - \frac{1}{d} - 2}$, if d odd.

Denote by $P_1, P_2, ..., P_d$ the union of the vertices of F_1 and F_2.

As in the proof of Proposition 4.34, let O be an orientation of G and, for each $i \in \{1, \dots, d\}$, fix an eulerian orientation O_i on C_i. For all $i \in \{1, \dots, d\}$, add the flow value P_i, resp. $-P_i$, to all edges $e \in C_i$ such that $O_i(e) = O(e)$, resp. $O_i(e) \neq O(e)$.

If d is even, see also Example 2.3 in [76], the distance between two points P_i and P_j is either $\sqrt{d}$ or $\sqrt{d-2}$. If d is odd, see also Example 2.4 in [76], the distance between two points P_i and P_j is either $\sqrt{d - \frac{1}{d}}$ or $\sqrt{d - \frac{1}{d} - 2}$. Thus, in the former case the ratio between maximum and minimum norm of flow values is $\sqrt{d/d-2}$ and in the latter case is $\sqrt{(d^2-1)/(d^2-2d-1)}$. □

4.4 2-DIMENSIONAL FLOWS WITH p-NORMS

This section is based on a recent collaboration with Prof. Sascha Kurz, from University of Bayreuth and my supervisor, Prof. Giuseppe Mazzuoccolo.

The idea underlying this work is to consider 2-dimensional flows on graphs with respect to norms other than the Euclidean norm. In particular, the norms we consider are the so-called p-norms. We briefly recall their definition in a 2-dimensional real vector space. Let $p \geqslant 1$ be a real number, and let $x = (x_1, x_2)$ be a 2-dimensional vector. Then $\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p}$ is the p-norm of x. Clearly, the 2-norm coincides with the Euclidean norm. We also consider the so-called infinity norm of x, which is $\|x\|_\infty = \max\{|x_1|, |x_2|\}$ (as widely accepted, we will denote it as the p-norm with $p = \infty$). A 2-dimensional r-flow of a graph G with respect to a p-norm is defined as a $(r, 2)$-NZF of G, where the norm of each flow value is the corresponding p-norm instead of the Euclidean one. We will denote the 2-dimensional flow number of a graph G with respect to a p-norm as $\varphi_2^p(G)$.

In this context we manage to prove some non-trivial lower bounds for $\varphi_2^p(P)$, for each p, see Proposition 4.36. Moreover in the case of $p = \infty$ we are able to give the exact value of $\varphi_2^\infty(P)$, see Proposition 4.37.

Proposition 4.36. *Let* P *denote the Petersen graph, and let* $p \geqslant 1$ *a real number. Then* $\varphi_2^p(P) \geqslant 1 + \frac{3}{2}\sqrt[p]{\frac{1}{2}}$.

Proof. Assume by contradiction that there exists a 2-dimensional flow φ of P such that $\varphi = (\varphi_1(e), \varphi_2(e))$ for each edge $e \in E(P)$, with $\|\varphi(e)\|_p \geqslant 1$ and $\varphi_i(e) \in \left(-\frac{3}{2}\sqrt[p]{\frac{1}{2}}, \frac{3}{2}\sqrt[p]{\frac{1}{2}}\right)$ for $i = 1, 2$.

We say an edge $e \in E(P)$ to be *good* with respect to φ_i if $|\varphi_i(e)| \in \left[\sqrt[p]{\frac{1}{2}}, \frac{3}{2}\sqrt[p]{\frac{1}{2}}\right)$, *bad* otherwise. Observe that an edge e can be good with respect to both φ_1 and φ_2, but it cannot be bad with respect to both φ_1 and φ_2, for otherwise

$$\|\varphi(e)\|_p = \sqrt[p]{|\varphi_1(e)|^p + |\varphi_2(e)|^p} < \sqrt[p]{\left(\sqrt[p]{\frac{1}{2}}\right)^p + \left(\sqrt[p]{\frac{1}{2}}\right)^p} = 1.$$

Denote by B_i the subgraph of P induced by the bad edges with respect to φ_i and by G_i the one induced by the good edges with respect to φ_i, $i = 1, 2$. By previous observation at least one between B_1 and B_2 has at most 7 (less than $\frac{|E(P)|}{2}$) edges, say B_1.

Claim 1: B_i is a spanning subgraph of P and $\Delta(B_i) \leqslant 2$, for $i = 1, 2$.

Proof. Observe that $\Delta(G_i) \leqslant 2$, because the sum of three real numbers all with absolute value in the interval $\left[\sqrt[p]{\frac{1}{2}}, \frac{3}{2}\sqrt[p]{\frac{1}{2}}\right)$ cannot give 0 as a result, making the Kirkoff's law impossible to be satisfied by φ around a vertex of P. Hence B_i is spanning, for otherwise $\Delta(G_i) = 3$ and $\Delta(B_i) \leqslant 2$, for otherwise $\Delta(G_j) = 3$ when $j \neq i$. □

Claim 2: A path of length 2 cannot be a connected component of B_i, for $i = 1, 2$.

Proof. Consider the situation depicted in Figure 40, where the edges e and f are assumed to be bad with respect to φ_i, and all the other depicted edges are good with respect to ϕ_i. Then, $|\varphi_i(e)|, |\varphi_i(f)| <$

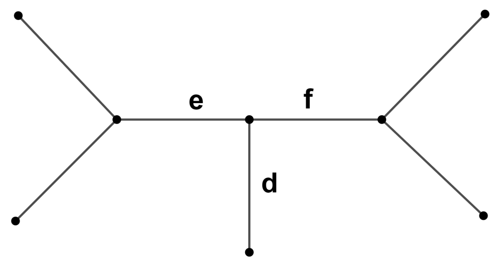

Figure 40: A path of length two with two bad edges

$\frac{1}{2}\sqrt[p]{\frac{1}{2}}$, since they are the difference of two values both in the interval $[\sqrt[p]{\frac{1}{2}}, \frac{3}{2}\sqrt[p]{\frac{1}{2}})$. Hence the edge d cannot be good with respect to φ_i, since it is the sum of two values with modulo strictly less than $\frac{1}{2}\sqrt[p]{\frac{1}{2}}$, a contradiction. □

Claim 3: $E(B_i)$ cannot contain a perfect matching of P, for $i = 1, 2$.

Proof. For otherwise, also $E(G_j)$, with $j \neq i$, would contain a perfect matching M of P. Without loss of generality we can assume $\varphi_j(e) \geqslant 0$ for every edge e of M and fixed j, otherwise we can reorient edges with negative values. Since every perfect matching of P is an edge cut of P, we have that the sum of the inflow and the outflow on each of the two connected components separated by M must be 0. Observe that, with respect to one of these two components, say C, we have three possible cases: all the edges of M are directed towards C, exactly 4 edges of M are directed towards C or exactly 3 edges of M are directed towards C. In the first case, the inflow to C is at least $5\sqrt[p]{\frac{1}{2}}$, and there is no outflow from C, a contradiction. In the second case the inflow to C is at least $4\sqrt[p]{\frac{1}{2}}$ while the outflow is at most $\frac{3}{2}\sqrt[p]{\frac{1}{2}}$, a contradiction. In the third case, the inflow is at least $3\sqrt[p]{\frac{1}{2}}$, while the outflow is at most $3 \cdot \frac{3}{2}\sqrt[p]{\frac{1}{2}}$, a contradiction once again. Hence we have the statement. □

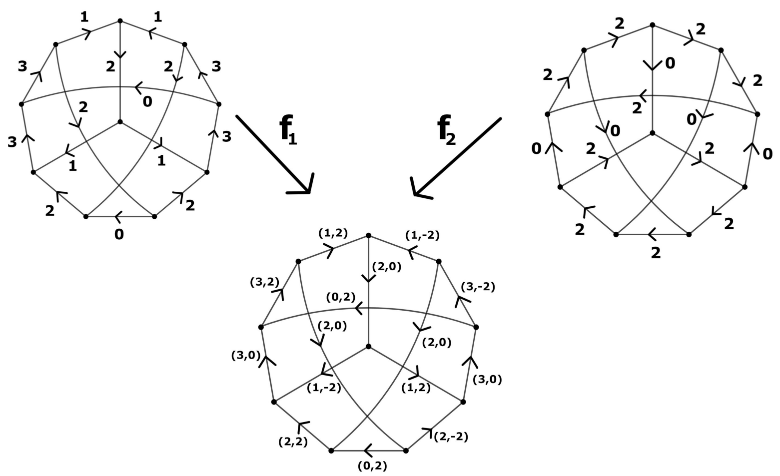

Figure 41: A $(1+\frac{3}{2}, 2)$-NZF of P with respect to the ∞-norm, constructed by components with the flows f_1 and f_2. The maximum of the ∞-norm of the values of $f = (f_1, f_2)$ is 3, while the minimum is 2, so that the ratio is $\frac{3}{2}$.

By Claim 1, $|E(B_1)| \geqslant 5$, and, by Claim 3, we have that $|E(B_1)| > 5$. Hence, since we have already observed that $|E(B_1)| \leqslant 7$, we have either $|E(B_1)| = 6$ or $|E(B_1)| = 7$. If B_1 contains cycles, they would be of length at least 5 (which is the girth of P) and this would prevent B_1 from being spanning, since $|E(B_1)| \leqslant 7$. Hence, B_1 is a spanning forest. Moreover, again by Claim 1, since $\Delta(B_1) \leqslant 2$, B_1 is a disjoint union of paths, with no one of them of length 2, by Claim 2. All these considerations mean that B_1 can contain either 3 paths of length 1 and one path of length 3, or 2 paths of length 1 and 1 path of length 5. But in these two remaining cases, B_1 would contain a perfect matching of P, a contradiction with Claim 3. This concludes our proof. □

Proposition 4.37. *Let* P *be the Petersen graph. Then* $\varphi_2^\infty(P) = 1 + \frac{3}{2}$.

Proof. A $(1+\frac{3}{2}, 2)$-NZF of P with respect to the ∞-norm is constructed by components f_1 and f_2 in Figure 41. The proof that this flow is optimal works as the proof of Proposition 4.36. In particular we assume by contradiction that there exists a 2-dimensional flow φ of P such that $\varphi = (\varphi_1(e), \varphi_2(e))$ for each edge $e \in E(P)$, with $\|\varphi(e)\|_\infty \geqslant 1$ and $\varphi_i(e) \in \left(-\frac{3}{2}, \frac{3}{2}\right)$ for $i = 1, 2$. We define an edge e to be good with respect to φ_i if $|\varphi_i(e)| \in \left[1, \frac{3}{2}\right)$ and bad otherwise. After this, the proof repeats exactly the reasoning of the one of Proposition 4.36. □

5

PALETTE INDEX

This chapter is based on contributions [P6] and [P7].

We devote this chapter to the palette index of graphs, which is a parameter related to colorings of the edges. As it happens for the other problems considered in this thesis, also the determination of the palette index is a difficult problem for Class 2 graphs, while it turns out to be elementary for Class 1 regular graphs. We recall the definition and some basic properties of the palette index of a graph, some of which are already mentioned in Section 1.4.5. This is done in order to make the chapter self-contained. Let c be an edge-coloring of a graph G. We define the *palette* of a vertex $v \in V(G)$ with respect to c to be the set $P_c(v) = \{c(e)\colon e \in E(G) \text{ and } e \text{ is incident to } v\}$ (note that if v is an isolated vertex of G then $P_c(v)$ is the empty set for each edge-coloring c). The *palette index* $\check{s}(G)$ of a graph G is the minimum number of distinct palettes, taken over all edge-colorings, occurring among the vertices of the graph. This parameter was introduced in 2014 [42] and several results have appeared since then, see [9, 11, 15, 31, 41, 80]. All mentioned contributions mainly consider the computation of the palette index in some special classes of graphs, such as trees, complete graphs, complete bipartite graphs, $3-$ and $4-$regular graphs and some others. Furthermore, it turns out that the palette index can be used to model some problems related to the self-assembly of DNA structure, see [10].

It is an easy consequence of the definition of palette index that a graph has palette index equal to 1 if and only if it is a k-regular and Class 1 graph. Moreover, it is proved in [42] that no regular graph has palette index equal to 2. The situation is less understood when we ask for r-regular graphs with a large palette index. The case of cubic graphs (i.e. $r = 3$) is completely solved by Theorem 1.14 in [42], which we recall here.

Theorem. *Let G be a connected cubic graph.*

- *G is 3-edge-colorable if and only if $\check{s}(G) = 1$;*
- *G is not 3-edge-colorable with a 1-factor if and only if $\check{s}(G) = 3$;*
- *G is not 3-edge-colorable without a 1-factor if and only if $\check{s}(G) = 4$.*

Moreover, it is a trivial observation that $\check{s}(G) \leqslant r+1$ for an r-regular graph G: indeed, every $(r+1)$-edge-coloring (such a coloring does exist by Vizing's theorem [89], see Section 1.1) has at most

$r+1$ distinct palettes. Theorem 1.14 shows an infinite family of cubic graphs with palette index 4, and an example of a 4-regular graph with palette index 5 is constructed in [11]. The general question about the existence of an r-regular graph G with the maximum possible palette index, i.e. $\check{s}(G) = r+1$, is open for every $r > 4$. In [P6], we give a complete answer to this question for every r odd.

We devote Section 5.1 to the general problem of finding graphs with large palette index. In particular, we present the main contribute of [P6], which is Theorem 5.2, a general result which gives a sufficient condition for an arbitrary graph G to have $\check{s}(G)$ larger than its minimum degree. In the special case that G is an odd regular graph, Theorem 5.2 furnishes a very strong restriction for the value of the palette index of G, as we can see in Corollary 5.3. In Section 5.1.1 we make use of such result to construct, for every positive integer k, a family of $(2k+1)$-regular graphs with $\check{s}(G) = 2k+2$, while in Section 5.1.2, we move our attention to the non-regular case. As far as we know, no family of graphs with palette index which grows faster than $\Delta\log(\Delta)$ was known before our work in [P6], where Δ denotes the maximum degree of G. In [3], a family of multigraphs whose palette index is expressed by a quadratic polynomial in Δ is presented. Problem 5.1 in [3] asks for the construction of a family of graphs without multiple edges with the same property. We give a complete answer to such a problem. Indeed, as a by-product of our result for odd regular graphs, we show a family of simple graphs having palette index which grows quadratically in Δ.

In the second part of the chapter, which corresponds to Section 5.2, in the very same spirit as in Section 5.1, we focus on the problem of relating the palette index of a graph G to some structural properties of G. We introduce a description of the set of palettes induced by an edge-coloring on a graph in terms of an associated hypergraph H. This description allows to describe the palette index of a graph in terms of H-colorings, as will be highlighted in Section 5.2. Moreover, this description will be very practical in proving our main results in Section 5.2.1, where we present a complete characterization of graphs having palette index at most 3 in terms of the existence of some graph decompositions into Class 1 regular subgraphs.

5.1 GRAPHS WITH LARGE PALETTE INDEX

In this section we prove a sufficient condition for a graph to have palette index larger than its minimum degree. Before going to the main result we give some preliminary definitions. Let G be a graph, we denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degree of G, respectively. Moreover, for every vertex v of G we denote by $d_G(v)$ the degree of v in G.

A subgraph K of G is an *even subgraph* of G if every vertex has even degree in K. A *spanning even subgraph* of G is an even subgraph K of G such that $V(K) = V(G)$.

Let c be an edge-coloring of a graph G with colors in the set $\mathcal{C}$ and generating palettes $P_1, \ldots, P_t$. We consider a map $\phi_c \colon \mathcal{P}(\mathcal{C}) \to (\mathbb{Z}_2^t, +)$, associated to c, where $\mathcal{P}(\mathcal{C})$ denotes the power set of $\mathcal{C}$ and $(\mathbb{Z}_2^t, +)$ denotes the elementary abelian group of order 2^t whose elements are all strings of length t with values 0 and 1, and + denotes the addition modulo 2. For every subset A of $\mathcal{C}$ we define $\phi_c(A) = (p_1, \ldots, p_t)$, where we set

$$p_i = \begin{cases} 0 & \text{if } |P_i \cap A| \text{ even,} \\ 1 & \text{if } |P_i \cap A| \text{ odd,} \end{cases}$$

for every $i \in \{1, \ldots, t\}$.

In other words, the map ϕ_c establishes the parity of the cardinality of the intersection between A and every palette of c.

For every subset A of $\mathcal{C}$ we consider the subgraph G_A induced by all edges with a color in A in the edge-coloring c. The following remark is straightforward.

Remark 5.1. *Let* c *be an edge-coloring of* G *and let* A *be a subset of the set of colors. The subgraph* G_A *of* G *is an even subgraph of* G *if and only if* $\phi_c(A) = (0, \ldots, 0)$.

We are now in position to prove our main result.

Theorem 5.2. *Let* G *be a graph such that* $\Delta(G) \geqslant 2$ *and* G *has no spanning even subgraph without isolated vertices. Then,* $\check{s}(G) > \delta(G)$.

Proof. First of all, we observe that if $\delta(G) = 0$ the relation $\check{s}(G) > 0$ trivially holds for every graph G. Hence, we can assume $\delta(G) > 0$ from now on. Let us prove that the number of colors in any edge-coloring of G is larger than $\delta(G)$. If G is not regular, this follows immediately since the number of colors is at least $\Delta(G) > \delta(G)$. If G is an r-regular graph with $r > 1$, then r is odd, otherwise G itself is a spanning even subgraph without isolated vertices. In this case G has not a perfect matching M, otherwise the complement of M would be a spanning even subgraph of minimum degree $r - 1$, a contradiction. Then, G cannot admit an r-edge-coloring.

By contradiction we assume that G admits an edge-coloring c with colors in $\mathcal{C}$ which induces t palettes and $t \leqslant \delta(G)$. As already proved, we have $|\mathcal{C}| > \delta(G)$. Now we prove the existence of a subset of $\mathcal{C}$ which induces a spanning even subgraph K of G with $\delta(K) \geqslant 2$, thus obtaining a contradiction.

Let A be a largest subset of $\mathcal{C}$ such that G_A is an even subgraph of G. Note that such a subset does exist since the empty set induces an even subgraph of G.

We only need to prove that $\delta(G_A) \geqslant 2$. Assume that this is not the case, then G_A has an isolated vertex v. Let P_j be the palette of v in the edge-coloring c. Since $d_{G_A}(v) = 0$, $P_j \cap A = \emptyset$ holds.

Clearly $|P_j| \geqslant \delta(G)$ holds and we proved that $|\mathcal{C}| > \delta(G)$. Hence, we can construct a subset R_j of $\mathcal{C}$ by choosing $\delta(G)$ arbitrary elements from P_j and an additional color in $\mathcal{C}$, denoted by α. We have the following two cases:

1. $\alpha \notin A$ (i.e. $A \cap R_j = \emptyset$);

2. $\alpha \in A$ (i.e. $A \cap R_j = \{\alpha\}$).

The number of non-empty subsets of R_j is $2^{\delta(G)+1} - 1$, while the possible images of these subsets under the action of ϕ_c are at most 2^t, that is the order of the elementary abelian group $\mathbb{Z}_2^t$. Since we assumed $t \leqslant \delta(G)$, there exist two distinct subsets I_1 and I_2 of R_j such that $\phi_c(I_1) = \phi_c(I_2)$.

Consider the symmetric difference $I = I_1 \triangle I_2 \subseteq R_j$. The following holds:

$$\phi_c(I) = \phi_c(I_1 \triangle I_2) = \phi_c(I_1) + \phi_c(I_2) = (0, \dots, 0),$$

where $+$ denote the addition in the group $\mathbb{Z}_2^t$.

Finally we obtain a contradiction by proving that the set $A \triangle I$ is a set of colors larger than A which induces an even subgraph of G.

If $A \cap R_j = \emptyset$, then A and I are disjoint sets and I is not empty, then $A \triangle I = A \cup I$ is larger than A.

If $A \cap R_j = \{\alpha\}$, then we prove that I contains at least two elements different from α. Indeed, note that I contains at least one element of P_j. Otherwise, $I = \{\alpha\}$ and, if P is a palette that contains α, $I_1 \cap P$ cannot have the same parity of $I_2 \cap P$, contradicting $\phi_c(I_1) = \phi_c(I_2)$. Therefore, since I contains an even number of elements of P_j by $\phi_c(I) = (0, \dots, 0)$, we have that I contains at least two elements of P_j. Moreover, $A \cap I \subseteq \{\alpha\}$ holds since α is the unique possible element of A which could belong to I. Then, $|A \triangle I| = |A| + |I| - 2|A \cap I| > |A|$ where the last inequality holds since $|I| > 2$ whenever $A \cap I = \{\alpha\}$. Then, $A \triangle I$ is larger than A again. Finally,

$$\phi_c(A \triangle I) = \phi_c(A) + \phi_c(I) = (0, \dots, 0),$$

since $\phi_c(A) = (0, \dots, 0)$ by assumption and $\phi_c(I) = (0, \dots, 0)$ as already proved. Then, $A \triangle I$ induces an even subgraph of G and it is larger than A, a contradiction again. It follows that G_A is a spanning even subgraph of G without isolated vertices and the assertion is proved. □

5.1.1 *Regular graphs with palette index as large as possible*

In this section we construct families of r-regular graphs having palette index equal to $r+1$.

First of all, we deduce the following easy consequence of Theorem 5.2 in the case of odd regular graphs.

Corollary 5.3. *For every positive integer k, let G be a $(2k+1)$-regular graph with no spanning even subgraph without isolated vertices. Then, $\check{s}(G) = 2k+2$.*

Proof. As already observed, the palette index of a $(2k+1)$-regular graph cannot be larger than $2k+2$. Then, it suffices to prove that $\check{s}(G) > 2k+1$. The relation $\delta(G) = \Delta(G) = 2k+1 > 1$ holds and G has no spanning even subgraph without isolated vertices. Then G has palette index larger than $2k+1$ by Theorem 5.2. □

Corollary 5.3 gives a sufficient condition for an odd regular graph to have maximum possible palette index. Theorem 1.14 says that this sufficient condition is also necessary in the cubic case (i.e. $k = 1$). Indeed, when $k = 1$, the non-existence of a spanning even subgraph without isolated vertices is equivalent to the non-existence of a perfect matching in G.

In [P6], we wonder if the same holds for $k > 1$ and we leave it as an open problem.

Problem 5.4. *Prove (or disprove) that if G is a $(2k+1)$-regular graph such that $\check{s}(G) = 2k+2$, then G has no spanning even subgraph without isolated vertices.*

We complete this section by showing, for every integer k, a family of $(2k+1)$-regular graphs which satisfy the condition of Corollary 5.3, and thus having palette index $2k+2$.

It suffices to consider a $(2k+1)$-regular graph G with a vertex v such that every edge incident to v is a bridge of G (see Figure 42, where the graph depicted has the same structure of the Class 2 regular graphs constructed in Section 1.1). Clearly, G cannot have a spanning even subgraph K with $\delta(K)$ at least 2, since K should contain at least two of the $2k+1$ bridges incident to v, but an even subgraph cannot contain any bridge of G.

Then, we can state the following proposition which gives an answer, for odd regular graphs, to an open problem in [11], where Bonvicini and Mazzuoccolo wondered about the existence of an r-regular graph with palette index $r+1$ for every $r > 4$. Observe that the graphs we propose as a solution to the question may be both simple or multigraphs, depending only on how it is choosen to complete the graph.

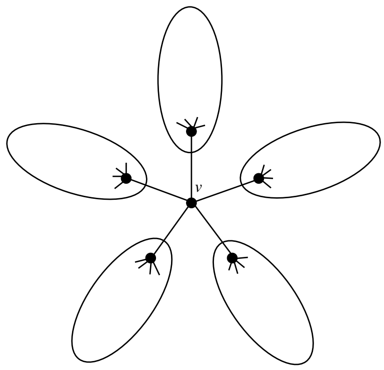

Figure 42: The structure of a 5-regular graph with palette index 6.

Proposition 5.5. *For every positive integer* k*, let* G *be a* $(2k+1)$*-regular graph admitting a vertex* v *such that every edge incident to* v *is a bridge of* G*. Then,* $\check{s}(G) = 2k+2$*.*

The problem of establishing the existence of a $2k$-regular graph with palette index $2k+1$, for $k > 2$, instead, remains open.

5.1.2 *Simple graphs with palette index growing quadratically with respect to their maximum degree*

In [3], the authors present a family of multigraphs whose palette index is expressed by a quadratic polynomial in Δ. They leave the construction of a family of simple graphs with the same property as an open problem. We use our main result in previous section to obtain such a family of graphs in a very straightforward way.

For every positive integer i, let G_i be a $(2i+1)$-regular graph with $\check{s}(G_i) = 2i+2$ (see Corollary 5.5). Moreover, for every positive integer k, let H_k be the disjoint union of $G_1, G_2, \ldots, G_k$.

Clearly, $\Delta(H_k) = \Delta(G_k) = 2k+1$ holds. Moreover, every connected component of H_k has vertices of degree different from the degree of the vertices in every other component. Hence, it follows

$$\check{s}(H_k) = \sum_{i=1}^{k} \check{s}(G_i) = \sum_{i=1}^{k} (2i+2) = k^2 + 3k = \frac{\Delta^2 + 4\Delta - 5}{4}.$$

It is not hard to construct examples of graphs with the same property and also connected. Starting from H_k, we add a new extra vertex which is declared to be adjacent to exactly one vertex in each connected component of H_k; the choice of the vertex in each component is unrelevant. The graph so obtained is connected and it has maximum degree one more than H_k. Moreover, it is an easy check that its palette index is larger than $\check{s}(H_k)$. Then, we have an infinite family of simple graphs whose palette index grows quadratically with respect to their maximum degrees.

5.2 GRAPHS WITH SMALL PALETTE INDEX

Along the section we will make use both of graphs and hypergraphs. Since we never mentioned the concept of hypergraph before, we just recall that an *hypergraph* is a pair (V, E), where V is the set of vertices and E the set of hyperedges. A hyperedge $h \in E$, is a subset of V of arbitrary cardinality. According to this definition, a graph is nothing but a hypergraph having all edges of cardinality one or two. Every time we refer to a hypergraph, we mean that it may admit both parallel hyperedges (i.e. hyperedges on the same subset of vertices) and loops (i.e. hyperedges of cardinality one), whereas we consider only graphs without loops, i.e. multigraphs. For other notation not explicitely defined here, we refer to [8].

Recall that a *decomposition* of a graph G is a family $\{H_i\}_{i\in I}$ of subgraphs of G such that $E(H_i) \neq \emptyset$ for every $i \in I$, $\bigcup_{i\in I} E(H_i) = E(G)$ and $E(H_i) \cap E(H_j) = \emptyset$ for every $i \neq j \in I$.

We denote by $\mathcal{C} = \{1, 2, ..., k\}$ the set of colors and by $\mathcal{P}_c = \{P_1, P_2, ..., P_t\}$ the set of distinct palettes that an edge-coloring c induces among the vertices of G. Observe that the empty palette belongs to the set $\mathcal{P}_c$ if and only if G has some isolated vertices. For each color $i \in \mathcal{C}$ we define $E_i = \{e \in E(G) \mid c(e) = i\}$, and for each palette $P_j \in \mathcal{P}_c$ we define $V_j = \{v \in V(G) \mid P_c(v) = P_j\}$. Finally, if $\emptyset \neq X \subseteq \mathcal{C}$, we denote by $G[X]$ the subgraph of G induced by all the edges $e \in E(G)$ such that $c(e) \in X$. Then the following remark is straightforward.

Remark 5.6. *Let $\emptyset \neq X \subseteq \mathcal{C}$. The following statements are equivalent:*

- $G[X]$ *is an $|X|$-regular Class 1 subgraph of* G
- *for all $v \in V(G[X])$, $X \subseteq P_c(v)$.*

The following definition will be largely used in what follows.

Definition 5.7. *A k-edge-coloring c of G is* $\check{s}$-minimal *if its associated set of palettes $\mathcal{P}_c$ has cardinalty $\check{s}$ and there is no k'-edge-coloring c' of G with an associated set of palettes such that $|\mathcal{P}_{c'}| = |\mathcal{P}_c|$ and $k' < k$.*

In other words, an edge-coloring of a graph G is $\check{s}$-minimal if it has the minimum number of colors among all edge-colorings minimizing the number of palettes. Such a parameter was firstly considered in [3]. Moreover, it is remarked in [42] that an $\check{s}$-minimal edge-coloring could need a number of colors larger than the chromatic index of the graph.

As mentioned, it will be practical in what follows to associate a hypergraph to an edge-coloring of a graph G. Hence, let c be a k-edge-coloring of a graph G. We define H_G^c as the hypergraph having

$\mathcal{P}_c$ as set of vertices and k hyperedges $h_1, h_2, ..., h_k$, where $h_i = \{P \in \mathcal{P}_c \mid i \in P\}$ (see at the top of Figure 43 for an example). Moreover, when (almost) all hyperedges of H_G^c have size 1 or 2, we will depict them as loops and edges of a multigraph, for the sake of clarity (see at the bottom of Figure 43).

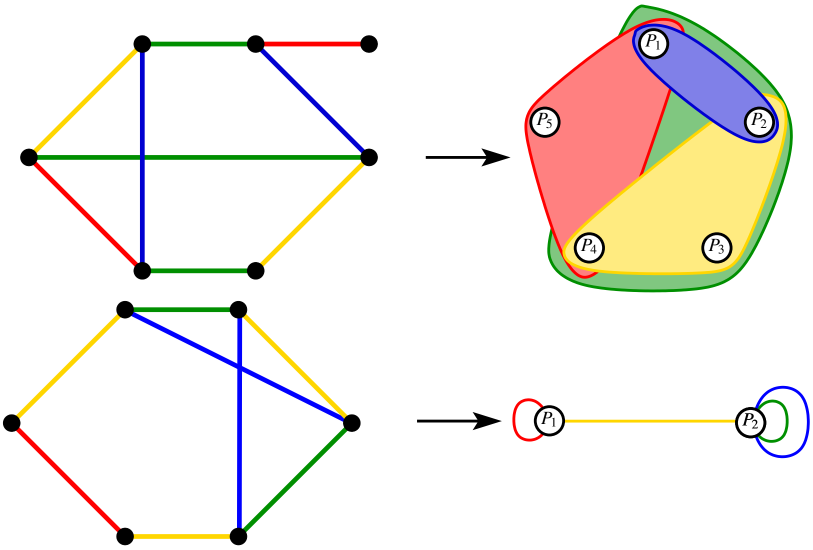

Figure 43: Edge-colorings and their associated hypergraphs

Observe that if c is an $\check{s}$-minimal edge-coloring of G, and $h_\alpha, h_\beta \in E(H_G^c)$, then there must exist a vertex $v \in V(H_G^c)$ such that both h_α and h_β are incident with v. For otherwise, if h_α and h_β are independent hyperedges in H_G^c, there does not exist a palette $P \in \mathcal{P}_c$ with $\{\alpha, \beta\} \subseteq P$. Hence, the subgraph of G induced by $\{e \in E(G) \mid c(e) \in \{\alpha, \beta\}\}$ is 1-regular. Therefore, it is possible to recolor the edges that receive color β in c with α, thus constructing a new edge-coloring of G with the minimum number of palettes $\check{s}$ and less than k colors, contradicting the $\check{s}$-minimality of c. Hence we have the following

Remark 5.8. *If c is an $\check{s}$-minimal edge-coloring of G, then any two hyperedges $h_i, h_j \in E(H_G^c)$ are incident with some common vertex.*

We conclude this section by showing an interesting relation between the hypergraph H_G^c and the concept of H-coloring of a graph G (see also Chapter 3 and Chapter 1).

Observe that there is no obstruction to define $\partial_H(v)$ as the set of all hyperedges incident with the vertex v of a hypergraph H. Then, we can also assume H to be a hypergraph in the definition of H-coloring. Although we are not aware of any paper where this more general formulation is considered, it naturally works in the same way.

In our context, it is straightforward that the map $f\colon E(G) \to E(H_G^c)$, such that $f(e) = h_{c(e)}$, is an H_G^c-coloring of G. Then, an alternative description of the palette index of a graph directly follows.

Proposition 5.9. *The palette index $\check{s}(G)$ of a graph G is equal to the order of a smallest hypergraph H such that G admits an H-coloring.*

5.2.1 *Graphs with palette index at most 3*

In this section we provide a complete characterization of graphs having palette index at most 3. First of all, in Theorem 5.10, we characterize graphs having palette index equal to 2 and then, in Theorem 5.11, we characterize graphs with palette index at most 3. Both characterizations are given with respect to the existence of a suitable decomposition of the graph G in Class 1 regular subgraphs. Note that the number of such subgraphs depends on the graph G itself and, in particular, it is not uniquely determined by its palette index. In what follows we consider families of graphs with given palette index. Therefore, we need to consider decompositions in "at most" a certain number of subgraphs H_i's, where by at most we mean that not all H_i's always appear in the decomposition of a certain G in the considered family. In order to improve the readability of the following proofs, we always describe each subgraph H_i, but we omit to specify every time in what instances such an H_i is actually present in the decomposition.

Theorem 5.10. *Let G be a graph, and let Δ and δ denote its maximum and minimum degree respectively. Then, $\check{s}(G) = 2$ if and only if $\Delta > \delta$ and G can be decomposed into at most two Class 1 regular subgraphs H_0 and H_1 such that*

- H_0 *is spanning and δ-regular;*
- H_1 *is $(\Delta - \delta)$-regular.*

Proof. Let G be a graph with palette index 2. We show that G has the required decomposition.

Since regular graphs with palette index 2 do not exists (see [42]), G is not regular and therefore $\Delta > \delta$. Let $c\colon E(G) \to \{1, 2, \ldots, k\}$ be an $\check{s}$-minimal edge-coloring of G, with $\mathcal{P}_c = \{P_1, P_2\}$. Its associated hypergraph H^c_G is of order two and may admit only two types of hyperedges: hyperedges both incident with P_1 and P_2, and loops incident with a unique palette. Since c is $\check{s}$-minimal, by Remark 5.8, all loops must be incident with the same palette, say P_1. If there is no hyperedge in H^c_G having cardinality two, then P_2 is the empty palette, meaning that G has some isolated vertices, that is $\delta = 0$. Since $\mathcal{C} \neq \emptyset$, we define $H_1 = G[\mathcal{C}]$. Since $V(G[\mathcal{C}]) = V_1$ and $\mathcal{C} = P_1$, by Remark 5.6, H_1 is a Δ-regular Class 1 subgraph of G. Clearly $E(H_1) = E(G)$ so that $\{H_1\}$ is a decomposition of G as required in the statement.

If H_G^c has hyperedges $h_1, \ldots, h_t$, $t > 0$, of cardinality two with vertices P_1 and P_2, without loss of generality we denote by $h_{t+1}, \ldots, h_k$ the remaining loops on P_1. Observe that $k > t$, for otherwise, we would have that $P_1 = P_2$, so that $\check{s}(G) = 1$ and $|V(H_G^c)| = 1$ since c is $\check{s}$-minimal. The hypergraph H_G^c is depicted in Figure 44. Note that $P_2 = \{1, 2, ..., t\} \subset P_1 = \{1, 2, ..., k\}$.

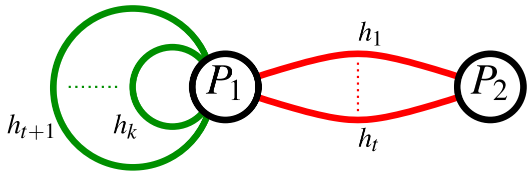

Figure 44: The hypergraph H_G^c associated to a $\check{s}$-minimal edge-coloring of a graph with $\check{s}(G)$=2.

Hence $V(G[P_2]) = V_1 \cup V_2 = V(G)$ and for all $v \in V(G)$, $P_2 \subset P_c(v)$. Set $H_0 = G[P_2]$. By Remark 5.6, the subgraph H_0 is a spanning t-regular Class 1 subgraph of G. Moreover, $t = \delta$. Let $X = \mathcal{C} \setminus P_2 = \{t+1, ..., k\}$ and set $H_1 = G[X]$. It follows $V(H_1) = V_1$, and, for every $v \in V(G) \cap V_1$, $X \subseteq P_1$. By Remark 5.6, H_1 is a $(k-t)$-regular Class 1 subgraph of G. Moreover, $k = \Delta$ and then $k - t = \Delta - \delta$ holds. Since $\mathcal{C} = P_2 \cup X$, $E(H_0) \cup E(H_1) = E(G)$ follows. Moreover $P_2 \cap X = \emptyset$, so that $\{H_0, H_1\}$ is a decomposition of G as required in the statement.

Conversely, assume to have a decomposition of G into at most two subgraphs H_0 and H_1 as in the statement. Since G is not regular, then $\check{s}(G) \geqslant 2$. It remains to prove that $\check{s}(G) \leqslant 2$. To this aim, observe that, if $\delta > 0$, H_0 is a δ-regular Class 1 subgraph and then it admits a δ-edge-coloring c_0 with colors $\{1, 2, ..., \delta\}$. For the same reason, H_1 is $(\Delta - \delta)$-regular and admits a $(\Delta - \delta)$-edge-coloring c_1 with colors $\{\delta + 1, ..., \Delta\}$. Define the edge-coloring $c\colon E(G) \to \{1, 2, ..., \Delta\}$ as

$$c(e) = \begin{cases} c_0(e) \text{ if } e \in E(H_0), \\ c_1(e) \text{ if } e \in E(H_1). \end{cases}$$

For every vertex $v \in V(G)$, $\partial_G(v) = \partial_{H_0}(v) \cup \partial_{H_1}(v)$. Hence, either $P_c(v) = \{1, 2, ..., \delta\}$ or $P_c(v) = \{1, 2, ..., \delta, \delta + 1, ..., \Delta\}$, i.e. $\check{s}(G) \leqslant 2$. □

Next result gives a complete characterisation of graphs having palette index at most 3.

Theorem 5.11. *Let* G *be a graph. Then* $\check{s}(G) \leqslant 3$ *if and only if there exists a decomposition of* G *in at most four Class 1 regular subgraphs* H_0, H_1, H_2, H_3 *with the following properties:*

- H_0 *is spanning*
- *there exists a partition of* $V(G)$ *in at most three subsets* A_1, A_2, A_3 *such that* $V(H_1) = A_2 \cup A_3$, $V(H_2) = A_1 \cup A_3$ *and either* $V(H_3) = A_3$ *or* $V(H_3) = A_1 \cup A_2$.

Proof. Assume that $\check{s}(G) \leqslant 3$ and let c be an $\check{s}$-minimal coloring of G.

If $\check{s} = 1$, G is Class 1 and regular. If $E(G) = \emptyset$, each vertex is associated with the empty palette. Let $E(G) \neq \emptyset$, then we set $H_0 = G$, $A_1 = V(G)$. Hence $\{H_0\}$ is the required decomposition of G.

If $\check{s} = 2$, then G is non regular with palettes P_1 and P_2. By Theorem 5.10, G admits a decomposition into at most two subgraphs H_0 and H_1. Let $A_1 = A_3 = \emptyset$ and $A_2 = V_1$. Then, $\{H_0, H_1\}$ is the required decomposition.

If $\check{s} = 3$, let $\mathcal{P} = \{P_1, P_2, P_3\}$. Up to the existence of some colors which belongs to all the three palettes, the only possibilities for the associated hypergraph H_G^c are the four depicted in Figure 45.

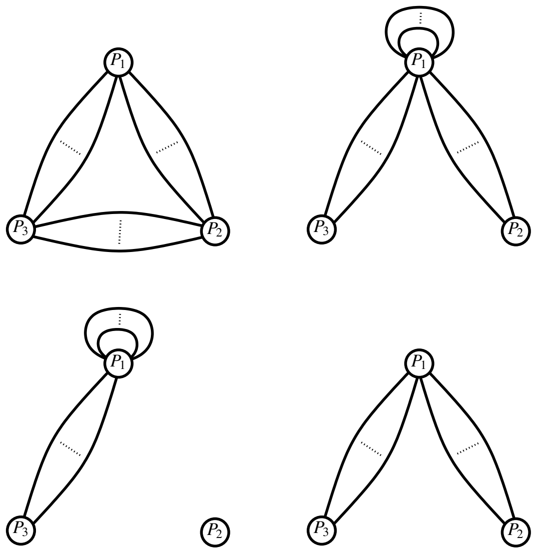

Figure 45: The four possibilities for the associated hypergraph H_G^c when $\check{s} = 3$ (up to relabelling palettes).

We set $A_i = V_i$, for all $i = 1, 2, 3$. Since c is $\check{s}$-minimal, for every $i \neq j \neq k \in \{1, 2, 3\}$ the set of colors lying in $(P_i \cap P_j) \setminus P_k$ and the set of colors lying in $P_k \setminus (P_i \cup P_j)$ cannot be both non-empty. In addition to this, among the three sets $P_k \setminus (P_i \cup P_j)$, $i \neq j \neq k$, at most one can be non-empty. If $P_1 \cap P_2 \cap P_3 \neq \emptyset$, set $H_0 = G[P_1 \cap P_2 \cap P_3]$. Since $P_1 \cap P_2 \cap P_3 \subseteq P_c(v)$ for all $v \in V(G)$, H_0 is regular Class 1 and spanning, by Remark 5.6.

We must now distinguish two cases:

- $P_k \setminus (P_i \cup P_j) = \emptyset$, for all $i \neq j \neq k \in \{1, 2, 3\}$

In this case, if $(P_i \cap P_j) \setminus P_k \neq \emptyset$, set $H_k = G[(P_i \cap P_j) \setminus P_k]$, for all $i \neq j \neq k \in \{1, 2, 3\}$. Note that the non-empty sets $(P_i \cap P_j) \setminus P_k$, together with $P_1 \cap P_2 \cap P_3$, form a partition of $\mathcal{C}$, so that $\{H_0, H_1, H_2, H_3\}$ is a decomposition of G. Observe also that $V(H_k) = A_i \cup A_j$ and for all $v \in A_i \cup A_j$, either $P_c(v) = P_i$ or $P_c(v) = P_j$. This implies that $(P_i \cap P_j) \setminus P_k \subseteq P_c(v)$, so that each H_k is Class 1 and regular by Remark 5.6. Hence $\{H_0, H_1, H_2, H_3\}$ is the required decomposition of G.

- $P_k \setminus (P_i \cup P_j) \neq \emptyset$ for some $i \neq j \neq k$

Without loss of generality, since at most one among the three sets $P_k \setminus (P_i \cup P_j)$ can be non-empty, suppose that $P_3 \setminus (P_1 \cap P_2) \neq \emptyset$. Then we set: $H_1 = G[(P_2 \cap P_3) \setminus P_1]$ if $(P_2 \cap P_3) \setminus P_1 \neq \emptyset$, $H_2 = G[(P_1 \cap P_3) \setminus P_2]$ if $(P_1 \cap P_3) \setminus P_2 \neq \emptyset$ and $H_3 = G[P_3 \setminus (P_1 \cup P_2)]$.

Since $\{(P_3 \setminus (P_1 \cup P_2), (P_1 \cap P_3) \setminus P_2, (P_2 \cap P_3) \setminus P_1, P_1 \cap P_2 \cap P_3)\}$ is a partition of $\mathcal{C}$, we conclude that $\{H_0, H_1, H_2, H_3\}$ is a decomposition of G. Observe also that $V(H_1) = A_2 \cup A_3$, $V(H_2) = A_1 \cup A_3$ and $V(H_3) = A_3$, and that, by Remark 5.6, every H_i, $i = 1, 2, 3$ is regular and Class 1. Hence $\{H_0, H_1, H_2, H_3\}$ is the required decomposition of G.

Vice versa, let $\{H_0, H_1, H_2, H_3\}$ a decomposition of G as in the statement. Let c_i be a minimal edge-coloring of H_i, for each $i = 0, 1, 2, 3$. We define an edge-coloring c of G as follows. Since for every $e \in E(G)$ there is a unique $i \in \{0, 1, 2, 3\}$ such that $e \in E(H_i)$, let $c(e) = c_i(e)$. It suffices to show that c defines at most three distinct palettes on the graph G. By hypothesis A_1, A_2, A_3 is a partition of $V(G)$, then for all $u, v \in A_i$, we have that u and v belong to exactly the same set of subgraphs H_i, implying that $P_c(u) = P_c(v)$. Thus c induces at most three palettes. □

Recall that a graph has palette index 1 if and only if it is regular and Class 1. Then Theorem 5.11 in combination with Theorem 5.10 implicitly permits to describe graphs with palette index exactly equal to three. In particular, a compact description is possible in the regular case.

Corollary 5.12. *Let G be a k-regular graph. Then $\check{s}(G) = 3$ if and only if G is Class 2 and it can be decomposed in three Class 1 $\frac{k-r}{2}$-regular subgraphs, for $0 \leqslant r < k$, and, if $r > 0$, a Class 1 r-regular spanning subgraph.*

Proof. Since G is regular, $\check{s}(G) = 1$ if and only if G is a Class 1 graph. Moreover, again, if since G is regular, $\check{s}(G) \neq 2$ (see [42]). Assume $\check{s}(G) = 3$. Then, G is a Class 2 graph. By Theorem 5.11, G can be decomposed in at most four Class 1 regular subgraphs H_0, H_1, H_2 and H_3, where H_0, if $r > 0$, is a spanning Class 1 r-regular subgraph of G. Let A_1, A_2 and A_3 be a partition of $V(G)$ as in the statement of Theorem 5.11. The case $V(H_3) = A_3$ cannot occur, otherwise every vertex in A_3 belongs to all four subgraphs H_i's, while all vertices in A_1 and A_2 do not. A contradiction by regularity of G and H_i's. Then, $V(H_3) = A_1 \cup A_2$ holds. Denote by d_i the common degree of every vertex in H_i, for $i = 1, 2, 3$. It follows that for every vertex $v \in A_i$,

$$k = d_G(v) = r + \sum_{j \neq i} d_j.$$

Hence the following relation holds for each $i = 1, 2, 3$,

$$k - r = \sum_{j \neq i} d_j.$$

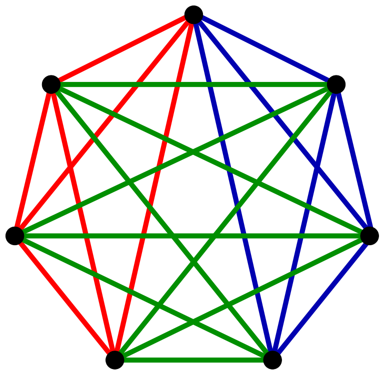

Figure 46: A decomposition of K_7 in three Class 1 cubic graphs.

It follows that H_1,H_2 and H_3 are all $\frac{k-r}{2}$-regular graphs.

Conversely, the graphs in the statement give a decomposition which satisfy the conditions in the statement of Theorem 5.11. Then, $\check{s}(G) \leqslant 3$. Since G is regular and Class 2, the relation $\check{s}(G) < 3$ does not hold and the assertion follows. □

Note that the spanning subgraph H_0 could not appear in the decomposition described in Corollary 5.12 (or if you prefer it can be considered as an empty subgraph). This happens, for instance, in Figure 46 where we present a decomposition of the complete graph K_7 in two copies of K_4 and a copy of $K_{3,3}$.

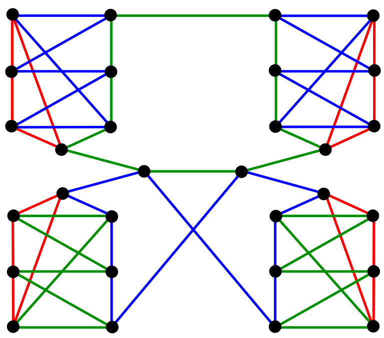

Figure 47: A decomposition of a 4-regular graph in 3 Class 1 2-regular graphs

One could be tempted to think that H_0 can always be chosen as the largest Class 1 regular subgraph of G in order to obtain a decomposition such as the one described in Corollary 5.12. We conclude the section by showing that this is not the case. In other words, the choice of H_0 as the largest Class 1 spanning regular subgraph of G could leave a subgraph of G which does not admit a decomposition in three Class 1 regular subgraphs, while the graph G itself could admit such a decomposition.

We consider the 4-regular graph depicted in Figure 47. As already shown in [11], such graph has a decomposition in three Class 1 2-

regular subgraphs and then it has palette index equal to 3. On the other hand, even if it admits a perfect matching, it does not admit two disjoint perfect matchings. Then, the complement of any perfect matching is a cubic graph without a perfect matching, that is by Theorem 1.14 a cubic graph with palette index equal to 4. It follows that such a cubic graph cannot be decomposed in three Class 1 regular subgraphs as required.

6

FINAL REMARKS AND FURTHER RESEARCH

This final part of the dissertation is devoted to recall the main results presented in the previous chapters, in order to highlight them under the perspective of the general spirit of the thesis. We also recall the main possible open problems arising from our work.

As we saw, in this thesis we consider problems that appear to be hard to solve for Class 2 graphs.

In Chapter 2 our main results are Theorem 2.10 and Theorem 2.14 which give some structural necessary properties of a possible minimum counterexample to the Cycle Double Cover Conjecture and the Berge-Fulkerson Conjecture, respectively. We recall that such counterexamples, if they exist, are snarks, that are particular Class 2 cubic graphs. The technique used in Chapter 2 aims to prove that such possible minimum counterexamples have high cyclic-edge-connectivity, namely 5 and 6 for possible counterexamples to the Cycle Double Cover Conjecture and to the Berge-Fulkerson Conjecture, respectively. However, this technique seems to give no further information on the structure of such possible counterexamples, at least not in a straightforward way. Indeed, it is conjectured (see [48]) that snarks have cyclic-edge-connectivity at most 6, and hence, further progress using this technique would reasonably mean a big step towards a complete proof of such conjectures.

In Chapter 3 we focus on the Petersen Coloring Conjecture, for which, again, Class 2 cubic graphs are the only open case. In Corollary 3.13 we strengthen a result of V. Mkrtchyan, Corollary 3.5, by proving that the Petersen graph is, in a specific sense, the only possible graph that could color every bridgeless cubic graph. Our result has been further strengthened in [58], where the authors prove that either the Petersen graph colors all bridgeless cubic graphs or an infinite number of graphs is needed to color all the bridgeless cubic graphs. We considered also H-colorings of r-regular graphs, and we proved that the Petersen Coloring Conjecture does not admit a generalization in r-regular graphs, for $r > 3$ (see Table 3). As mentioned in Section 3.3.2, further research on this topic could be carried on by trying to answer Problem 3.27, while Problem 3.28 has been completely answered in [58].

In Chapter 4 we consider 2-dimensional flows on graphs and a parameter related to them, the complex flow number of a graph G, namely $\varphi_{\mathbb{C}}(G)$. In particular, as the Tutte's 5-flow Conjecture does for the 1-dimensional case, we propose some possible upper bounds for $\varphi_{\mathbb{C}}(G)$ when G is a bridgeless graph. If G is a cubic Class 1 graph, we prove that $\varphi_{\mathbb{C}}(G) \leqslant 1+\sqrt{2}$ and that $\varphi_{\mathbb{C}}(K_4) = 1+\sqrt{2}$, so that in the case of Class 1 cubic graphs, our upper bound is tight. The same does not hold for the proposed upper bound for general cubic graphs, that is $1+\tau$, where τ denotes the Golden Ratio. The problem to establish whether this upper bound is tight appears to be challenging, and without surprise, regards Class 2 cubic graphs.

We also focus on determining lower bounds for $\varphi_{\mathbb{C}}(G)$, which seems to be a very hard task even for very small graphs like the Petersen graph. In Theorem 4.27 we determine a non-trivial lower bound for $\varphi_{\mathbb{C}}(G)$ when G is a cubic graph, in terms of its odd girth. In collaboration with Prof. S. Kurz, from University of Bayreuth, we tried to develop further research on this topic by considering complex flows with respect to different norms instead of the Euclidean norm. In this context we manage to prove non-trivial lower bounds for the 2-dimensional flow number of the Petersen graph with respect to all the p-norms. However, this approach is still widely inexplored and further research can be carried out on it.

Finally, in Chapter 5 we focus on the palette index of graphs, which is an index related to edge-colorings. As we saw, determining the palette index for Class 1 regular graphs is a trivial task, while for Class 2 ones there are very few general results. The main result of our work is Theorem 5.2, which gives a sufficient condition for a general graph to have palette index larger than its minimum degree. When considering regular graphs, this condition is sufficient, for a graph, to have the maximum admissible value of the palette index, while for a cubic graph, this condition is also necessary. The main open question we propose in Problem 5.4 is whether this sufficient condition is also necessary, for an r-regular graph, $r > 3$, to have the maximum admissible value of the palette index.

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