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1 Introduction

We address the problem of computing an optimal schedule for maximizing the average coverage of a wireless sensor network. We show that a time-discretized version of the problem achieves the same optimal solution as one where nodes are allowed to wake up at any arbitrary time. We refer the reader to [1, 2] for further information regarding the problem.

2 Development

Notation: we consider a topology with a set $\mathcal{N}$ of nodes. The number of nodes is $N = |\mathcal{N}|$. The epoch E is divided into s slots. The awake interval has duration d slots, and is such that

$$s = I \cdot d$$

where I is an integer. Thus, the epoch is divided into I awake intervals.

Slots are ordered and identified by their position $0 \leq k \leq s-1$. We use a subscript k to identify quantities related to slot k. We assume that operations on slot indices are done modulo s, so that index s is equal to 0, index $s+1$ is equal to 1, index -1 is equal to $s-1$, and so on.

Node n has a wake-up time $0 \leq w_n \leq s-1$. An optimal schedule is an assignment to the wake-up times of all nodes such that the integral of the coverage over the epoch is maximized. We say that a schedule is *aligned* if the nodes are scheduled at integer multiples of the awake interval:

Definition 1 (Aligned nodes)**.** *A node n is* aligned *if and only if*

$$w_n \text{ mod } d = 0.$$

The line that identifies the slots at which aligned nodes are scheduled is called an *alignment boundary*. The slots between two consecutive alignment boundaries is called an *alignment region*.

Definition 2 (Aligned schedule)**.** *A schedule W is* aligned *if all nodes in the schedule are aligned.*

If the scheduled is aligned, then all nodes are scheduled at the alignment boundaries, that is, they pairwise either completely overlap in time, or they don't overlap at all. We are going to prove the following.

Theorem 1. *There exists an optimal aligned schedule.*

To prove the theorem, observe that we can partition the set $\mathcal{N}$ of nodes into those that are already aligned, and those which are not:

Definition 3 (Aligned partition)**.** *The set of nodes $\mathcal{N}$ is partitioned into two sets:*

$$\mathcal{A} = \{n \in \mathcal{N} \mid n \textit{ is aligned } \},$$
$$\mathcal{B} = \{n \in \mathcal{N} \mid n \textit{ is not aligned } \} = \mathcal{N} - \mathcal{A}$$

Definition 4 (Nodes per slot)**.** *For every slot k, let $\mathcal{A}_k$ be the set of aligned nodes that are awake at slot k, and let $\mathcal{B}_k$ be the set of non-aligned nodes that are awake at slot k:*

$$\mathcal{A}_k = \{n \in \mathcal{A} \mid k - d + 1 \leq w_n \leq k\}$$
$$\mathcal{B}_k = \{n \in \mathcal{B} \mid k - d + 1 \leq w_n \leq k\}$$

It turns out that the set of non-aligned nodes at the slots across an alignment boundary are the same.

Lemma 1 (Same nodes across alignment)**.** *Let k be a slot at the beginning of an alignment boundary, i.e., k mod $d = 0$. Then, $\mathcal{B}_{k-1} = \mathcal{B}_k$.*

Proof. By way of contradiction, assume $n \in \mathcal{B}_k$ and $n \notin \mathcal{B}_{k-1}$. By Definition 4, since $n \in \mathcal{B}_k$,

$$k - d + 1 \leq w_n \leq k. \tag{1}$$

Similarly, since $n \notin \mathcal{B}_{k-1}$,

$$k - 1 - d + 1 = k - d \nleq w_n \vee w_n \nleq k - 1. \tag{2}$$

Since $k - d + 1 \leq w_n$, and since $k - d \leq k - d + 1$, it is also $k - d \leq w_n$. Therefore, by Equation 2, it must be $w_n \nleq k - 1$. Hence, since the order is total, $w_n > k - 1$, which is equivalent to $w_n \geq k$. Since, by Equation 1, $w_n \leq k$, by antisymmetry, $w_n = k$. By assumption, k mod $d = 0$, which implies w_n mod $d = 0$. Therefore, by Definition 1, n is aligned. Hence, by Definition 3, $n \notin \mathcal{B}$. Finally, by Definition 4, $n \notin \mathcal{B}_k$, which contradicts the hypothesis.

The proof is similar if we assume $n \in \mathcal{B}_{k-1}$ and $n \notin \mathcal{B}_k$. Therefore, $\mathcal{B}_{k-1} = \mathcal{B}_k$. ☐

The aligned nodes in slots that belong to the same alignment region are, of course, the same.

Lemma 2 (Same nodes in alignment region)**.** *Let k and k' be two slots in the same alignment region, i.e., such that $dm \leq k, k' \leq d(m+1)-1$, for some integer m. Then $\mathcal{A}_k = \mathcal{A}_{k'}$.*

Proof. Assume $n \in \mathcal{A}_k$. By hypothesis, $k \geq dm$, thus $k - d \geq d(m-1)$ and therefore $k - d + 1 > d(m-1)$. By Definition 4, $w_n \geq k - d + 1$, so, by transitivity, $w_n > d(m-1)$. Similarly, $w_n \leq k$, and since by hypothesis $k \leq d(m+1) - 1$, it is also $w_n < d(m+1)$. Therefore

$$d(m-1) < w_n < d(m+1).$$

Since n is aligned, it must be $w_n \bmod d = 0$, so it must be $w_n = dm$. Now, by hypothesis, $dm \leq k'$, and therefore, $w_n \leq k'$. Similarly, since by hypothesis $k' \leq d(m+1) - 1$, by rearranging the terms, $k' - d + 1 \leq dm$. Hence, $k' - d + 1 \leq w_n$. Therefore, by Definition 4, $n \in \mathcal{A}_{k'}$.

Symmetrically, one shows that if $n \in \mathcal{A}_{k'}$, then $n \in \mathcal{A}_k$. Therefore $\mathcal{A}_k = \mathcal{A}_{k'}$. □

On the other hand, the sets of aligned nodes across an alignment boundary are disjoint.

Lemma 3. *Let k be a slot at the beginning of an alignment boundary, i.e., $k \bmod d = 0$. Then, $\mathcal{A}_{k-1} \cap \mathcal{A}_k = \emptyset$.*

Proof. I'm not yet sure I will use this result in the following, so for now the proof is left to the reader. □

We are going to compute the gain (positive or negative) in covered area that is obtained by shifting the schedule of *all* the non-aligned nodes by one slot to the left or to the right. To do so, we must compute the coverage before and after the shift. Let $z : \mathcal{N} \rightarrow 2^{\mathbb{R}^2}$ be the function that to each node associates the subset of $\mathbb{R}^2$ that is sensed by the node. Let $A : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function that to a subset of $\mathbb{R}^2$ gives the corresponding area. Let $a_k \subseteq \mathbb{R}^2$ be the area covered by the nodes in $\mathcal{A}_k$, and $b_k \subseteq \mathbb{R}^2$ the area covered by the nodes in $\mathcal{B}_k$:

$$\begin{aligned} a_k &= \bigcup_{n \in \mathcal{A}_k} z(n) \\ b_k &= \bigcup_{n \in \mathcal{B}_k} z(n) \end{aligned} \tag{3}$$

Corollary 1. *Let k and k' be two slots in the same alignment region, i.e., such that $dm \leq k, k' \leq d(m+1) - 1$, for some integer m. Then $a_k = a_{k'}$.*

Proof. The proof follows from Lemma 2, since $\mathcal{A}_k = \mathcal{A}_{k'}$. □

The coverage for each slot can be computed as the area covered by the nodes in $\mathcal{A}_k$, plus the area covered by the nodes in $\mathcal{B}_k$, less the area covered by both. The total coverage S is simply the sum over all slots:

$$\begin{aligned} S &= \sum_{k=0}^{s-1} A(a_k) + A(b_k) - A(a_k \cap b_k) \\ &= \sum_{k=0}^{s-1} A(a_k) + \sum_{k=0}^{s-1} A(b_k) - A(a_k \cap b_k) \end{aligned} \tag{4}$$

The gain due to a shift of the non-aligned nodes to the right can be computed by shifting the b_k and leaving the a_k unaltered. We obtain:

$$G^+ = S_{\text{after}} - S_{\text{before}} \tag{5}$$

$$= \sum_{k=0}^{s-1} A(a_k) + \sum_{k=0}^{s-1} A(b_{k-1}) - A(a_k \cap b_{k-1}) - \tag{6}$$

$$\sum_{k=0}^{s-1} A(a_k) - \sum_{k=0}^{s-1} A(b_k) - A(a_k \cap b_k) \tag{7}$$

$$= \sum_{k=0}^{s-1} A(b_{k-1}) - A(a_k \cap b_{k-1}) - \sum_{k=0}^{s-1} A(b_k) - A(a_k \cap b_k) \tag{8}$$

by changing k into $k+1$ in the first sum,

$$= \sum_{k=-1}^{s-2} A(b_k) - A(a_{k+1} \cap b_k) - \sum_{k=0}^{s-1} A(b_k) - A(a_k \cap b_k) \tag{9}$$

by considering operations modulo s in the first sum and by rearranging the summands,

$$= \sum_{k=0}^{s-1} A(b_k) - A(a_{k+1} \cap b_k) - \sum_{k=0}^{s-1} A(b_k) - A(a_k \cap b_k) \tag{10}$$

$$= \sum_{k=0}^{s-1} -A(a_{k+1} \cap b_k) - \sum_{k=0}^{s-1} -A(a_k \cap b_k) \tag{11}$$

$$= \sum_{k=0}^{s-1} A(a_k \cap b_k) - A(a_{k+1} \cap b_k) \tag{12}$$

Given this expression, we may define the *gain* of slot k for a right shift as

$$g_k^+ = A(a_k \cap b_k) - A(a_{k+1} \cap b_k), \tag{13}$$

that is, the gain is given by the area overlap between the non-aligned and the aligned nodes before the shift, minus the area overlap of the same non-aligned nodes with the aligned nodes in the new slot, after the shift. The total gain can therefore be expressed as:

$$G^+ = \sum_{k=0}^{s-1} g_k^+. \tag{14}$$

Slots which are not near an alignment boundary give no gain, as shown next.

Lemma 4. *Let k be a slot such that k mod $d \leq d-2$ (i.e., k is not to the immediate left of an alignment boundary). Then, $g_k^+ = 0$.*

Proof. Let m be such that $dm \leq k \leq d(m+1)-1$. Let $k' = k+1$. Then, obviously, $dm \leq k'$. In addition, $k' \leq d(m+1)$. Since k mod $d \leq d-2$, it must be $1 \leq k'$ mod $d \leq d-1$. Hence, k' mod $d \neq 0$. Therefore it must be $k' \leq d(m+1)-1$. Consequently, by Corollary 1, $a_k = a_{k+1}$. By formula 13, $g_k^+ = 0$. □

By the previous lemma, the significant terms in formula 14 are only those that correspond to slots to the immediate left of an alignment boundary. Recalling that $I = s/d$, we can therefore rewrite the formula as:

$$G^+ = \sum_{i=0}^{I-1} g_{di-1}^+. \tag{15}$$

Similarly, we can define the gain for a left shift of the non-aligned nodes. We have:

$$g_k^- = A(a_k \cap b_k) - A(a_{k-1} \cap b_k), \tag{16}$$

and therefore

$$G^- = \sum_{k=0}^{s-1} g_k^-. \tag{17}$$

By arguments similar to the ones above, one shows that $g_k^- = 0$ for all slots which are not to the immediate right of an alignment boundary. Hence, one can rewrite formula 17 as:

$$G^- = \sum_{i=0}^{I-1} g_{di}^-. \tag{18}$$

We are going to show that shifting right or shifting left give gains that are equal, but of opposite sign. We first show it for gains of adjecent slots.

Lemma 5 (Local gains are opposite)**.** *Let $di = k$ be a slot marking the beginning of an alignment region. Then,*

$$g_{di}^- = -g_{di-1}^+.$$

Proof. The proof consists of the following series of equalities:

$$g_{di}^- = g_k^- \tag{19}$$
$$= A(a_k \cap b_k) - A(a_{k-1} \cap b_k) \tag{20}$$

Since $k = di$, $k \bmod d = 0$, therefore, by Lemma 1, $\mathcal{B}_{k-1} = \mathcal{B}_k$. Hence, by formula 3, $b_{k-1} = b_k$. Therefore,

$$= A(a_k \cap b_{k-1}) - A(a_{k-1} \cap b_{k-1}) \tag{21}$$

$$= -(A(a_{k-1} \cap b_{k-1}) - A(a_k \cap b_{k-1})) \tag{22}$$

$$= -g^+_{k-1} \tag{23}$$

$$= -g^+_{di-1} \tag{24}$$

□

Corollary 2 (Gains are opposite)**.**

$$G^+ = -G^-.$$

Proof. The proof follows easily by matching corresponding terms in the expressions of G^+ and G^-. □

Our last result shows that by shifting the non-aligned nodes in one direction we obtain a change in the covered area which is equal, but of opposite sign, to the change obtained by shifting the non-aligned nodes in the opposite direction. The next result shows that if shifting does not result in any new node getting aligned, then an additional shift in the same direction will give the same gain as the previous shift in that direction.

Lemma 6 (Shift again)**.** *Let W_1 and W_2 be two schedules obtained by shifting the non-aligned nodes $\mathcal{B}$ to the right. Assume $\mathcal{A}_1 = \mathcal{A}_2$. Then, $G_1^+ = -G_2^- = G_2^+$.*

Proof. Since $\mathcal{A}_1 = \mathcal{A}_2$, the node partition does not change after the shift. Thus, shifting back the non-aligned nodes must bring the coverage to its previous value. Hence, $G_1^+ = -G_2^-$. By Corollary 2, $-G_2^- = G_2^+$. □

The result is similar if we shift to the left, and the partition does not change. We are now in a position to prove the main theorem.

Proof of Theorem 1. Let a non-aligned schedule be given. We will construct an aligned schedule which has equal or better coverage.

Compute G^+ and G^-, and shift the non-aligned nodes in the direction of positive gain. If $G^+ = G^- = 0$, then shift to the right. Continue to shift in the same direction until at least one non-aligned node becomes aligned. By Lemma 6, since we started in the direction of positive gain, we will continue to shift in the direction of positive gain. The process goes on until at least one node becomes aligned. At this point, we repartition the nodes, and proceeds recursively with other shifts, until all nodes are aligned. At each step after some nodes have become aligned, since the partitions change, we only shift the nodes that are not yet aligned, while the newly aligned nodes remain fixed. Thus all gains change, and we might as well change shift direction. In the process, we have moved only in the direction of positive or zero gain, so the new schedule has equal or better coverage than the starting one.

Let now W be an optimal schedule. If W is aligned then the theorem is proved. Otherwise, construct a new schedule W' using the procedure described above. By construction, W' is also optimal, and is aligned. Therefore, there exists an optimal aligned schedule. □

References

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