KAZHDAN-LUSZTIG R-POLYNOMIALS OF PERMUTATIONS CONTAINING A 231 OR 312 PATTERN

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ABSTRACT. We consider the Kazhdan-Lusztig R-polynomials, $R_{u,v}(q)$, indexed by permutations u, v of S_n, where u contains a pattern of type 231 or 312 and v is obtained from u by applying a transposition and an appropriate 3-cycle. We prove, using combinatorial techniques, that these polynomials are given by a closed product formula. The case of permutations which contain a pattern of type 321 follows from the others.

1. INTRODUCTION

The theory of the Kazhdan-Lusztig R-polynomials arises from the Hecke algebra associated to a Coxeter group W (see e.g.[4], Chap.7) and was introduced by Kazhdan and Lusztig ([5], Sect.2) with the aim of proving the existence of another family of polynomials, the so-called Kazhdan-Lusztig polynomials. The R-polynomials, as the Kazhdan-Lusztig polynomials, are indexed by pairs of elements of W and they are related to the Bruhat order of W. Most of the importance of these polynomials comes from their applications in different contexts, such as representation theory, topology and the algebraic geometry of Schubert varieties (see e.g.[6], [11] and [1]). The importance of the R-polynomials stems mainly from the fact that they allow the computation of the Kazhdan-Lusztig polynomials. Although the explicit calculation of the R-polynomials is easier than that of the Kazhdan-Lusztig polynomials, one encounters hard problems in finding closed formulas for them, even when W is the symmetric group. In recent years purely combinatorial ways to compute the R-polynomials have been found, (see, e.g., [3]). In particular these ways allow combinatorial reasoning and techniques to be applied to the problem of finding explicit formulas for the polynomials of some classes of permutations. Such an approach is used in the paper [2] and consists in the investigation of the R-polynomials associated to pairs of permutations (u, v), where v is obtained from u by applying special permutations. As we prove in section 3 of this paper, from [2] derive closed formulas for pairs of permutations (u, v), where v is obtained from u by swapping two elements or by applying a 3-cycle. Results of the same nature, which are not consequence of the those in [2], are contained in [10] where it is proved that there exists a product formula for pairs (u, v) such that v is obtained from u by applying two nested transpositions. In this paper we consider permutations u in the n-th symmetric group, for which we have fixed five positions $1 \leq i < a < b < c < j \leq n$ and corresponding at the internal positions a, b, c, u contains a pattern of type 231 (resp. of type 312). We show that

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the R-polynomial indexed by the pair (u,v), where v is obtained by applying to u a suitable 3-cycle nested in the transposition (i,j) is given by an explicit product formula.
The organization of the paper is the following. In the next section we recall some basic definitions, notation, and results, both of an algebraic and combinatorial nature that will be used afterwards. In the third section we prove some explicit formulas for the R-polynomials of pairs (u,v), where v is obtained from u by applying respectively, a transposition, a 3-cycle, two disjoint transpositions, or a transposition and a disjoint 3-cycle. In section 4 we exhibit the reduction theorem which leads to a closed formula for the R-polynomials for the class of permutations described above. Finally, in section 5, we pose some open problems.

2. NOTATION AND PRELIMINARIES

In this section we collect some definitions, notation, and results that will be used in the rest of the paper. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1,2,3,\dots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1,2,3,\dots,a\}$, where $[0] \stackrel{\text{def}}{=} \emptyset$. Given n, $m \in \mathbf{P}$, $n \leq m$, we let $[n,m] = [m] \setminus [n-1]$. We write $S = \{a_1,\dots,a_r\}_<$ to mean that $S = \{a_1,\dots,a_r\}$ and $a_1 < \cdots < a_r$. The cardinality of a set A will be denoted with $|A|$. Given a set X we will let $S(X)$ be the set of all bijections of X in itself and $S_n \stackrel{\text{def}}{=} S([n])$.

If $\sigma \in S(X)$ and $X = \{x_1,\dots,x_n\}_< \subseteq \mathbf{P}$ then we write $\sigma = \sigma_1 \dots \sigma_n$ to mean that $\sigma(x_i) = \sigma_i$, for $i = 1,\dots,n$, and we call this *complete notation*. If $\sigma \in S_n$ then we will also write σ on *disjoint cycle form*, (see, e.g.[12], Sect 1.3) and we will not usually write the 1-cycles of σ. For example, if $\sigma = 365492187$ then $\sigma = (1,3,5,9,7)(2,6)$. Given $\sigma,\tau \in S_n$ then $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example, $(1,2)(1,4) = (1,4,2)$.

We define next statistics on a permutation $\sigma \in S_n$

$$D(\sigma) \stackrel{\text{def}}{=} \{s_i = (i,i+1) \in [n] \times [n] : \sigma(i) > \sigma(i+1)\}$$

$D(\sigma)$ is called the *descent set of* σ. An element of $D(\sigma)$ is also called *right descent*, this is because we can consider a left-handed property and define the *left-descent set of* σ as follows

$$D_L(\sigma) \stackrel{\text{def}}{=} D(\sigma^{-1}).$$

$$inv(\sigma) \stackrel{\text{def}}{=} |Inv(\sigma)| = |\{(i,j) \in [n] \times [n] : i < j, \sigma(i) > \sigma(j)\}|$$

the number $inv(\sigma)$ is usually known as inversions of σ, (this number corresponds to the length of σ regarding S_n as a Coxeter group) and the set $Inv(\sigma)$ is the set of the inversions of σ. For example, if $\sigma = 13524$ then $inv(\sigma) = |\{(2,4),(3,4),(3,5)\}| = 3$, $D(\sigma) = \{(3,4)\}$ and $D_L(\sigma) = \{(2,3)(4,5)\}$.

We will always assume that S_n is partially ordered by (strong) *Bruhat order*. We recall (see, e.g., [4], Sect 5.9) that this means that if $u,v \in S_n$, $u \leq v$ iff $\exists t_1,\dots,t_r \in T$ where $T = \{(a,b) \in S_n : 1 \leq a < b \leq n\}$ is the set of transpositions, for $r \in \mathbf{N}$ such that:

(i) $v = ut_1t_2\dots t_r$
(ii) $inv(ut_1 \dots t_{i+1}) > inv(ut_1 \dots t_i)$ for $i = 0,\dots,r-1$.

A well known and useful characterization of the Bruhat order of S_n is the following.

For $u \in S_n$ and $i \in [n]$, let $\{u^{i,1},\dots,u^{i,i}\}_< \stackrel{\text{def}}{=} \{u(1),\dots,u(i)\}$.

Theorem 2.1. *Let $u, v \in S_n$. Then $u \leq v$ iff $u^{i,j} \leq v^{i,j}$ for every $1 \leq j \leq i \leq n-1$.*

We refer the reader to [7], Chap.1 for a proof.
We recall that there exists a maximal element under this order, (see [4] chap. 5) this is the permutation $w_0 \stackrel{\text{def}}{=} n(n-1)(n-2)\dots 321$.
The element w_0 acts, by multiplication, on the elements of S_n as follows: $\forall u \in S_n$, we have that $w_0 u = n+1-u(1)\dots n+1-u(n)$ and $uw_0 = u(n)\dots u(1)$. Moreover the multiplication by w_0 is an antiautomorphism of Bruhat order:

Proposition 2.2. *Let $u, v \in S_n$. Then the following conditions are equivalent:*

(a) $u < v$
(b) $u^{-1} < v^{-1}$
(c) $vw_0 < uw_0$
(d) $w_0 v < w_0 u$

This follows from Theorem 2.1, (see e.g. [7]).
We now introduce the family of *R-polynomials of S_n* by the next theorem definition:

Theorem 2.3. *There exists a unique family of polynomials $\{R_{x,w}(q)\}_{x,w\in S_n} \subseteq \mathbf{Z}[q]$ such that:*

(i) $R_{x,w}(q) = 0$, *if $x \not\leq w$;*
(ii) $R_{x,w}(q) = 1$, *if $x = w$;*
(iii) $R_{x,w}(q) = \begin{cases} R_{xs,ws}(q), & \text{if } s \in D(x) \\ (q-1)R_{x,ws}(q) + qR_{xs,ws}(q), & \text{if } s \notin D(x) \end{cases}$
if $x < w$ and $s \in D(w)$.

See [4], Sect. 7.5 for a proof.
This theorem gives an inductive procedure to compute the R-polynomials of S_n since $inv(ws) < inv(w)$. We note that there is also a left version of Theorem 2.3.
We introduce the family of $\tilde{R}_{u,v}(t)$, which gives a combinatorial interpretation of the R-polynomial of S_n,

Proposition 2.4. *Let $u, v \in S_n$; then there exists a unique polynomial $\tilde{R}_{u,v}(q) \subseteq \mathbf{N}[q]$ such that*

$$R_{u,v}(q) = q^{(inv(v)-inv(u))/2}\tilde{R}_{u,v}(q^{1/2} - q^{-1/2}).$$

It is a consequence of Theorem 2.3 that:

Theorem 2.5. *Let $u, v \in S_n$ be such that $u \leq v$.Then, for every $s \in D(v)$, we have that*

$$\tilde{R}_{u,v}(t) = \begin{cases} \tilde{R}_{us,vs}(t), & \text{if } s \in D(u) \\ \tilde{R}_{us,vs}(t) + t\tilde{R}_{u,vs}(t), & \text{if } s \notin D(u). \end{cases}$$

Theorem 2.5 gives an inductive procedure to compute $\tilde{R}_{u,v}(t)$ since $inv(v(i,i+1)) = inv(v) - 1$, assuming by definition that $\tilde{R}_{u,v}(t) = 0$ if $u \neq v$ and $\tilde{R}_{u,v}(t) = 1$ if $u = v$.
We note that Proposition 2.4 permits to work with the $\tilde{R}_{u,v}(t)$ since every result can be easily translated in terms of R-polynomials, with the advantage that they have positive coefficients instead of integer coefficients as the R-polynomials have.
In the following proposition we give a fundamental property of the $\tilde{R}$-polynomials.

Proposition 2.6. *Let $u, v \in S_n$; then*
$$\tilde{R}_{u,v}(t) = \tilde{R}_{u^{-1},v^{-1}}(t) = \tilde{R}_{w_0 v, w_0 u}(t) = \tilde{R}_{v w_0, u w_0}(t).$$

The above result can be proved using properties of Hecke algebra and 2.4 (see [4], Proposition 7.6). The left version of Theorem 2.5 follows easily from 2.6:
Let $u, v \in S_n$ such that $u \leq v$. Then, for every $s \in D_L(v)$, we have
$$\tilde{R}_{u,v}(t) = \begin{cases} \tilde{R}_{su,sv}(t), & \text{if } s \in D_L(u) \\ \tilde{R}_{su,sv}(t) + t\tilde{R}_{u,sv}(t), & \text{if } s \notin D_L(u). \end{cases}$$

Finally we note that a general closed formula for the R-polynomials does not exist; for example,
$$\tilde{R}_{12345,54321}(t) = t^2(1 + 5t^2 + 10t^4 + 6t^6 + t^8)$$
and
$$\tilde{R}_{123456,654321}(t) = t^3(1 + 9t^2 + 39t^4 + 57t^6 + 36t^8 + 10t^{10} + t^{12}),$$
and these factors are irreducible over the field of rational numbers.

However, there are several general classes of permutations for which explicit formulas do exist,(see, e.g. [2] and [9]); some of them are related to the pairs of permutations (u, v) in which v is obtained from u by applying a particular permutation, in the next section we survey some results in this direction, while now we give a closed product formula for the R-polynomials of permutations which are smaller than a transposition (i, j) under the Bruhat order, that will be used in the proof of Theorem 4.3:

Theorem 2.7. *Let $u, v \in S_n$ be such that $u \leq v \leq (i, j)$ for some $i, j \in [n]$, $i \neq j$. Then*
$$\tilde{R}_{u,v}(t) = t^a(t^2 + 1)^{inv(v) - inv(u) - a/2},$$
for some $a \in \mathbf{N}$

[See [8], Corollary 4.2]. The exponent a that appears in this theorem, is related to reduced expressions for u and v, and is given explicitly in [8], Theorem 4.1.

We end this section by giving the two results of [2] from which we derive the explicit formulas of the next section. To introduce the first theorem, we need to recall the definition of *restriction of a permutation*.
Let $u \in S_n$ and $i, j \in [n]$, $i \leq j$. We define the *restriction of u to $[i, j]$* to be the unique permutation $u_{[i,j]} \in S([i, j])$ such that
$$u^{-1}(u_{[i,j]}(i)) < u^{-1}(u_{[i,j]}(i + 1)) < \cdots < u^{-1}(u_{[i,j]}(j))$$
In other words to find the restriction of a permutation u to $[i, j]$ we can write u^{-1} and consider the set $\{u^{-1\ j,i}, u^{-1\ j,i+1}, \ldots, u^{-1\ j,j}\}_< = \{u^{-1}(i), u^{-1}(i+1) \ldots u^{-1}(j)\}$, then $u_{[i,j]} = [u(u^{-1\ j,i}), u(u^{-1\ j,i+1}), \ldots, u(u^{-1\ j,j})]$.

For example, consider the permutation $u = 4267351$ then $u_{[1,5]} = 42351$ since $u^{-1} = 7251634$ it results $\{u^{-1\ 5,1}, u^{-1\ 5,2}, \ldots, u^{-1\ 5,5}\}_< = \{1, 2, 5, 6, 7\}$, hence $u_{[1,5]}(1) = 4$, $u_{[1,5]}(2) = 2$, $u_{[1,5]}(3) = 3$, $u_{[1,5]}(4) = 5$, $u_{[1,5]}(5) = 1$.

Theorem 2.8. *Let $u, v \in S_n, u \leq v$. Suppose that there exist $1 = i_1 \leq i_2 < i_3 < \cdots < i_k \leq n$ such that $u^{-1}(i_j, i_{j+1}] = v^{-1}(i_j, i_{j+1}]$, $\forall j = 0, \ldots, k$, (where $i_0 \stackrel{\text{def}}{=} 0, i_{k+1} \stackrel{\text{def}}{=} n$). Then*

$$\tilde{R}_{u,v}(t) = \prod_{j=0}^{k} \tilde{R}_{u_{(i_j,i_{j+1}]},v_{(i_j,i_{j+1}]}}(t) \tag{1}$$

We give now the result of [2] from which follows the closed product formulas for the irreducible pairs (u, v) that we will indicate as *2-cycle* or *3-cycle* depending on the cycle type of $u^{-1}v$; it is based on the enumerations of increasing subsequences of a permutation:

Theorem 2.9. *Let $u \in S_n$ and $w \in C_{i,j}(u)$ for some $i, j \in [n]$. Then*

$$\tilde{R}_{u,w^{-1}u}(q) = q^{k(w)-1}(q^2+1)^{\frac{1}{2}(d-k(w)+1)}$$

where $C_{i,j}(u) \stackrel{\text{def}}{=} \{(u(i_1), u(i_2), \ldots, u(i_s)) : s \in [n], i = i_1 < i_2 < \cdots < i_s = j$ and $u(i_1) < u(i_2) < \cdots < u(i_s)\}$, $d \stackrel{\text{def}}{=} inv(wu) - inv(u)$ and $k(w)$ is the length of the cycle w.

See [2], Sect.4.

3. Irreducible pairs of permutations under their R-polynomials factorization

In this section we introduce a classification of the pairs of permutations for which we present, in this paper, explicit product formulas for the associated R-polynomials.
Because of Theorem 2.8, we say that a pair (u, v) of permutations is *irreducible* if the $\tilde{R}_{u,v}(q)$ satisfies the (1) only with the condition that there exists a unique $j \in [0, n+1]$ such that $\tilde{R}_{u_{(i_j,i_{j+1}]},v_{(i_j,i_{j+1}]}}(t) \neq 1$, i.e only one factor of the product is non trivial. Otherwise we say that (u, v) is a reducible pair; clearly it is enough to consider irreducible pairs.

The pairs of permutations of the following Corollary 3.1 and Proposition 3.3 are irreducible, as well as the ones of Theorem 3.4 and of the main results of this paper.

Corollary 3.1 (2-cycles). *Let $u \in S_n$ be such that $u(i) < u(j)$ for some $i, j \in [n]$, $i < j$. If $v = u(i, j)$, then*

$$\tilde{R}_{u,v}(t) = t(t^2+1)^{\frac{1}{2}(inv(v)-inv(u)-1)}$$

.

Proof. It is an immediate consequence of Theorem 2.9 since $v = (u(i), u(j))u = u(i, j)$ and $(u(i), u(j)) \in C_{i,j}(u)$. □

Next one is a technical lemma which is a consequence of Theorem 2.5. We will apply it in the proof of Proposition 3.3 and in the following sections.

Lemma 3.2. *Let $u, v \in S_n$, $u \leq v$ be such that $(k, l) \in Inv(u) \cap Inv(v)$ and $u(i) = v(i), \forall i \in [k+1, l-1]$.*
Suppose that $min\{u(k), v(k)\} \geq max\{u(l), v(l)\}$. Then there exists an index $r \in [k, l-1]$ such that:

$$\tilde{R}_{u,v}(t) = \tilde{R}_{x,y}(t)$$

where $x = u\sigma$, $y = v\sigma$ and $\sigma = (k, k+1, k+2, \ldots, r-1, r, l, l-1, l-2, \ldots, r+1)$.

Proof. We have:

$$\begin{array}{ll} u &= u(1) \\ v &= v(1) \end{array} \cdots \boxed{\begin{array}{c} u(k) \\ v(k) \end{array}} \begin{array}{c} u_{k+1} \\ u_{k+1} \end{array} \cdots \begin{array}{c} u_{l-1} \\ u_{l-1} \end{array} \boxed{\begin{array}{c} u(l) \\ v(l) \end{array}} \cdots \begin{array}{c} u(n) \\ v(n) \end{array}$$

We can suppose that $u_{k+1} < \cdots < u_{l-1}$ by Theorem 2.5.

Let $U \stackrel{\text{def}}{=} max\{v(l), u(l)\}$ and

$$r \stackrel{\text{def}}{=} max\{m \in [k+1, l-1] : u_m < U\}$$

Then $u_{k+1} < \cdots < u_r < U < u_{r+1} < \cdots < u_{l-1}$ and the result follows by applying Theorem 2.5.

□

We say that an inversion which satisfies the hypotheses of Lemma 3.2 "behaves" as descent in the sense of Theorem 2.5.

We examine the situation in which $v \in S_n$ is obtained from a permutation u by applying a 3-cycle, there are several possibilities to consider. We write only the positions in which u and v are different, i.e. the ones that we "rotate" in u to find v, and we indicate with "$\ldots$" the others. .

Proposition 3.3 (3-cycles). *Let $u \in S_n$, $a, b, c \in [n]$, be such that $a < b < c$. If (u, v) is one of the next pairs:*

$$\begin{array}{lcl} u = \ldots \mathbf{a} \ldots \mathbf{b} \ldots \mathbf{c} \ldots & \textit{and} & v = \ldots \mathbf{c} \ldots \mathbf{a} \ldots \mathbf{b} \ldots \\ u = \ldots \mathbf{a} \ldots \mathbf{b} \ldots \mathbf{c} \ldots & \textit{and} & v = \ldots \mathbf{b} \ldots \mathbf{c} \ldots \mathbf{a} \ldots \\ u = \ldots \mathbf{a} \ldots \mathbf{c} \ldots \mathbf{b} \ldots & \textit{and} & v = \ldots \mathbf{c} \ldots \mathbf{b} \ldots \mathbf{a} \ldots \\ u = \ldots \mathbf{b} \ldots \mathbf{a} \ldots \mathbf{c} \ldots & \textit{and} & v = \ldots \mathbf{c} \ldots \mathbf{b} \ldots \mathbf{a} \ldots \end{array}$$

Then

$$\tilde{R}_{u,v}(t) = t^2(t^2+1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$$

Moreover in the other cases, $\tilde{R}_{u,v}(t) = 0$, since $u \not\leq v$.

Proof. If $u = \ldots a \ldots b \ldots c \ldots$ this means that $w = (a, b, c) \in C_{a,c}(u)$; when $v = \ldots c \ldots a \ldots b \ldots$ we can apply Theorem 2.9 since $v = w^{-1}u$. This implies that $\tilde{R}_{u,v}(t) = t^2(t^2+1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$ being $k(w) = 3$.
If $v = \ldots b \ldots c \ldots a \cdots = (a, b, c)u$ then we cannot apply directly the same result. By Proposition 2.6 the pair (vw_0, uw_0) has the same R-polynomial of the pair (u, v). We have $\begin{array}{l} vw_0 = \\ uw_0 = \end{array} \cdots \begin{array}{c} a \\ c \end{array} \cdots \boxed{\begin{array}{c} c \\ b \end{array}} \cdots \boxed{\begin{array}{c} b \\ a \end{array}} \cdots$ and if we define $i = (vw_0)^{-1}(c) = (uw_0)^{-1}(b)$, $j = (vw_0)^{-1}(b) = (uw_0)^{-1}(a)$ then $(i, j) \in Inv(uw_0) \cap Inv(vw_0)$. We can apply Lemma 3.2, ((i, j) behaves as a descent), so that $\tilde{R}_{u,v}(t) = \tilde{R}_{x,y}(t)$, with

$$\begin{array}{l} x = \\ y = \end{array} \cdots \begin{array}{c} a \\ c \end{array} \cdots \begin{array}{c} i_1 \\ i_1 \end{array} \cdots \begin{array}{c} i_k\ b\ c\ i_{k+1} \\ i_k\ a\ b\ i_{k+1} \end{array} \cdots \begin{array}{c} i_s \\ i_s \end{array} \cdots,$$

which satisfies the conditions of Theorem 2.9.

We consider now $u = \ldots a \ldots c \ldots b \ldots$
If $v = \ldots c \ldots b \ldots a \cdots = (a, c, b)u$ then the pair (u, v) is exactly the pair that we called (vw_0, uw_0) in the previous case and we are done.
If $v = \ldots b \ldots a \ldots c \cdots = (a, b, c)u$, by Theorem 2.1 $u \not\leq v$ and $\tilde{R}_{u,v}(t) = 0$.

Finally if $u = \dots b \dots a \dots c \dots$ and $v = \dots c \dots b \dots a \dots = (a,c,b)u$, we observe that $vw_0 = \dots a \dots b \dots c \dots$ and $uw_0 = \dots c \dots a \dots b \dots$ and we can apply Theorem 2.9 as we have already seen. If $v = (a,b,c)u = \dots a \dots c \dots b \dots$ it is obvious that $u \not\leq v$.

The remaining permutations u to consider are the ones in which c is in the first position, but this implies that for each 3-cycle that we apply to u we obtain a permutation v which is not greater than u, so we have completed the proof.

□

Consider now a pair of permutations (u, v) such that $v = u(i,j)(k,l)$ and $i < k < l < j$. there exists a product formula which is not a consequence of Theorem 2.9 and which is proved in [10], Theorem 4.3:

Theorem 3.4. (**two nested transpositions**) *Let $u \in S_n$, $1 \leq i < k < l < j \leq n$ $u(k) > u(l)$, and suppose that $v = u(i,j)(k,l)$, $u < v$. Then*

$$\tilde{R}_{u,v}(t) = t^4(1+t^2)^{(inv(v)-inv(u)-4)/2}$$

Note that if $u(k) < u(l)$ there is not a closed formula, for example: $\tilde{R}_{123465,563421}(t) = t^2(t^{10} + 7t^8 + 15t^6 + 12t^4 + 5t^2 + 1)$ or $\tilde{R}_{124536,654231}(t) = t^2(t^2 + 1)(t^8 + 6t^6 + 10t^4 + 5t^2 + 1)$

The pairs of the next theorems are reducible, in fact they are obtained from Theorem 2.8 and the formulas of the previous corollary and proposition. We choose these classes because they are related to our investigations on the pairs (u, v) such that the cycle type of $u^{-1}v$ is $(i,j)(k,l)$, or $(i,j)(a,b,c)$, where these cycles are disjoint.

Proposition 3.5. (**two disjoint transpositions**) *Let $u, v \in S_n$ be such that $v = u(i,j)(k,l)$ and $i < j < k < l$. If $u < v$, we have that*

$$\tilde{R}_{u,v}(t) = t^2(t^2+1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$$

Proof. We consider pairs of permutations (u, v) such that $v = u(i,j)(k,l)$ and $i < j < k < l$. We write a pair of this type as follows:

$$\begin{array}{rl} u = & \dots \begin{array}{c} u(i) \\ u(j) \end{array} \dots \begin{array}{c} u(j) \\ u(i) \end{array} \dots \begin{array}{c} u(k) \\ u(l) \end{array} \dots \begin{array}{c} u(l) \\ u(k) \end{array} \dots \\ v = & \end{array}$$

The possible orderings of $u(i), u(j), u(k), u(l)$ such that $u < v$ are, by Theorem 2.1: $u(i) < u(j) < u(k) < u(l)$, $u(i) < u(k) < u(j) < u(l)$, $u(i) < u(k) < u(l) < u(j)$. There are also the orderings symmetric to these by multiplication with w_0: $u(k) < u(l) < u(i) < u(j)$, $u(k) < u(i) < u(l) < u(j)$, $u(k) < u(i) < u(j) < u(l)$. It is clear that the fundamental condition for $u < v$ is that $u(i) < u(j)$ and $u(k) < u(l)$ as in the case of 2-cycles.

We assume to be in one of the previous orderings for $u(i), u(j), u(k), u(l)$. We apply Theorem 2.8 to $x = u^{-1}$ and $y = v^{-1}$. It is obvious that $x^{-1}[1,j] = y^{-1}[1,j]$ and $x^{-1}[j+1,n] = y^{-1}[j+1,n]$. Hence $\tilde{R}_{u,v}(t) = \tilde{R}_{x,y}(t) = \tilde{R}_{x_{[1,j]},y_{[1,j]}}(t) \times \tilde{R}_{x_{[j+1,n]},y_{[j+1,n]}}(t)$. By definition of restriction of x and y, we have: $x_{[1,j]} = [u^{-1}(u^{j,1}), \dots, u^{-1}(u^{j,i})]$, $y_{[1,j]} = [v^{-1}(u^{j,1}), \dots, v^{-1}(u^{j,i})]$, since $x^{-1} = u$, $y^{-1} = v$ and $\{u(1), \dots, u(i), \dots u(j)\} = \{u^{j,1}, u^{j,2}, \dots, u^{j,j}\}_<$. It is clear that there exist $1 < s_1 < s_2 < j$ such that $y_{[1,j]} = x_{[1,j]}(s_1, s_2)$ and then by Corollary 3.1 $\tilde{R}_{x_{[1,j]},y_{[1,j]}}(t) = t(t^2+1)^{\frac{1}{2}(d-1)}$, where $d = inv(y_{[1,j]}) - inv(x_{[1,j]})$. The same argument applied to $x_{[j+1,n]}, y_{[j+1,n]}$, gives the result. □

Essentially we have broken x and y in two pairs of permutations in which the second permutation is obtained from the first by applying a transposition; we can do the same for the next result.

Proposition 3.6. (**a transposition and a disjoint 3-cycle**) *Let $u, v \in S_n$ be such that $v = u(i,j)(a,b,c)$ and $i < j < a < b < c$. If $u < v$, then*

$$\tilde{R}_{u,v}(t) = t^3(t^2+1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$$

Proof. In Proposition 3.3 we analyzed the possibilities for the relative order of $u(a)$, $u(b)$, $u(c)$, to be $u < v$ when $v = u\tau$, and τ is the three cycle (a,b,c) or its inverse. (Note that in 3.3 we used a different notation: a,b,c indicated the values of three fixed elements and not the positions as in this case). Here we refer to that result and assume to be under one of those conditions and to add the hypothesis that $u(i) < u(j)$. As before we apply Theorem 2.8 to $x = u^{-1}$ and $y = v^{-1}$ since $x^{-1}[1,j] = y^{-1}[1,j]$ and $x^{-1}[j+1,n] = y^{-1}[j+1,n]$.
Then $\tilde{R}_{u,v}(t) = \tilde{R}_{x,y}(t) = \tilde{R}_{x_{[1,j]},y_{[1,j]}}(t) \times \tilde{R}_{x_{[j+1,n]},y_{[j+1,n]}}(t)$. Now we have $\tilde{R}_{x_{[1,j]},y_{[1,j]}}(t) = t(t^2+1)^{\frac{1}{2}(d-1)}$, while, by Proposition 3.3 the other factor is now $\tilde{R}_{x_{[j+1,n]},y_{[j+1,n]}}(t) = t^2(t^2+1)^{\frac{1}{2}(d_1-1)}$, where $d_1 = inv(y_{[j+1,n]}) - inv(x_{[j+1,n]})$. □

It is clear that in a similar way we can compute the R-polynomials of pairs (u,v), $u < v$, in which v is obtained from u by applying k disjoint transpositions or mixed 3-cycles and 2-cycles but all disjoints, we do not write explicitly these formulas. More interesting are the irreducible classes and in the next section we consider the case of a 3-cycle nested in a transposition. Before ending this section we note that by applying Proposition 2.4 each result on $\tilde{R}_{u,v}(t)$ gives easily the corresponding $R_{u,v}(q)$.

4. MAIN RESULT

In this section we prove our main result, it is a closed product formula for the R-polynomials of a pair of permutations (u,v), where v is obtained from u applying a 3-cycle nested in a transposition (i,j) more precisely $v = u(i,j)(a,b,c)$ if $u(a) > u(c) > u(b)$ or $v = u(i,j)(a,c,b)$ if $u(b) > u(a) > u(c)$, where $i < a < b < c < j$. In other words we consider permutations u which contain a pattern of type 231 or 312 (this means that there exist $1 < i_1 < i_2 < i_3 < n$ such that $u(i_2) < u(i_3) < u(i_1)$, respectively $u(i_3) < u(i_1) < u(i_2)$ between the position (i,j) and to obtain v we apply the cycle that send the elements of this pattern in their natural order. We prove that even the case $u(a) > u(b) > u(c)$ is covered by this theorem. The proof is based on Theorem 4.2 which permits to reduce the computation of $R_{u,v}(t)$ to the R-polynomial of a three-cycle and the transposition $(1,n)$.

Lemma 4.1. *Let $u \in S_n$ and $1 \leq i < a < b < c < j \leq n$.*

(1) *If $u(a) = max\{u(a), u(b), u(c)\}$ then*

$$u < u(i,j)(a,b,c) \iff u(i) < min\{u(b), u(c)\} \quad \textit{and} \quad u(a) < u(j)$$

(2) *If $u(c) = min\{u(a), u(b), u(c)\}$ then*

$$u < u(i,j)(a,c,b) \iff u(i) < u(c) \quad \textit{and} \quad max\{u(a), u(b)\} < u(j)$$

Proof. It is easy to check the sufficient condition.
We prove the other part assuming $u < v$, where $v = u(i,j)(a,b,c)$ or $v = u(i,j)(a,c,b)$.

In both cases immediately we have that $u(i) < u(j)$; in fact if we suppose that $u(j) < u(i)$ and, under the notation of Theorem 2.1 we take the i-th rearrangement of $\{u(1), u(2), \dots, u(i)\}$= $\{u^{i,1}, u^{i,2}, \dots, u^{i,s}, u(i), u^{s+2,i} \dots\}_{\leq}$, where $u^{m,s+1}$ is the position occupied by $u(i)$ in the m-th rearrangement, and of $\{v(1) = u(1), v(2) = u(2), \dots, v(i) = u(j)\}$ =$\{u^{i,1}, u^{i,2}, \dots, u(j), u^{i,s}, u^{s+2,i}, \dots\}_{\leq}$, we will have $u(i) = u^{i,s+1} \not\leq v^{i,s+1} = u_{i,s}$, which implies $u \not\leq v$, a contradiction.

We prove the part 1) of the statement for the case that $u(a) > u(c) > u(b)$, and

$$u = \dots u(i) \dots u(a) \dots u(b) \dots u(c) \dots u(j) \dots$$

$$v = \dots u(j) \dots u(b) \dots u(c) \dots u(a) \dots u(i) \dots$$

We have to show that $u(i) < u(b)$ and that $u(a) < u(j)$. Suppose that $u(a) < u(i)$, the a-th rearrangements will be:

$$\{u(1), \dots, u(i), \dots, u(a)\} = \{u^{a,1}, \dots, u^{a,t}, \mathbf{u(a)}, u^{a,t+2}, \dots, u(i), \dots\}_{\leq}$$

$$\{u(1), \dots, u(j), \dots u(b)\} = \{u^{a,1}, \dots, u(b), \dots, \mathbf{u^{a,t}}, u^{a,t+2}, \dots, u(i), \dots\}_{\leq},$$

where $u^{m,t+1}$ is the position occupied by $u(a)$ in the m-th rearrangement. It follows that $u \not\leq v$, since $u(a) = u^{a,t+1} \not\leq v^{a,t+1} = u^{a,t}$.

To conclude this part suppose now that $u(b) < u(i) < u(a)$ and consider again the a-th rearrangement of u and v:

$$\{u^{a,1}, \dots, u^{a,s-1}, \mathbf{u(i)}, u^{a,s+1}, \dots, u^{a,t}, u(a), u^{a,t+2}, \dots\}_{\leq}$$

$$\{u^{a,1}, \dots, u(b), \dots, \mathbf{u^{a,s-1}}, u^{a,s+1}, \dots, u^{a,t}, u^{a,t+2}, \dots, u(j), \dots\}_{\leq}$$

We have again a contradiction because $u^{a,s} = u(i) \not\leq v^{a,s} = u^{a,s-1}$; moreover from the same rearrangement we can derive that $u(a) < u(j)$ and since $u(b) < u(c) < u(a)$ we are done. The case that $u(a) > u(b) > u(c)$, where u and v are as above, is identical but, in addition, we need to verify that $u(i) < u(c)$. Suppose that $u(c) < u(i)$ and consider the b-th rearrangement of u and v:

$$\{u(1), \dots, u(i), \dots u(a) \dots u(b)\} =$$
$$\{u^{b,1}, \dots, u^{b,s}, \mathbf{u(i)}, u^{b,s+2}, \dots, u^{b,r}, u(b), u^{b,r+2}, \dots, u^{b,t}, u(a), u^{b,t+2}, \dots\}_{\leq}$$
$$\{u(1), \dots, u(j), \dots u(b) \dots u(c)\} =$$
$$\{u^{b,1}, \dots, u(c), \dots, \mathbf{u^{b,s}}, u^{b,s+2}, \dots, u^{b,r}, u(b), u^{b,r+2}, \dots, u^{b,t} \dots, u(j), \dots\}_{\leq}$$

where $u^{b,s+1} = u(i) \not\leq v^{b,s+1} = u^{b,s}$. Hence we have proved that if $u < v$ then $u(i) < min\{u(b), u(c)\}$ and $u(a) < u(j)$ under the hypothesis that $u(a) = max\{u(a), u(b), u(c)\}$.

To prove part 2) of this Lemma we use the fact that the mapping $w \mapsto w^{-1}$ is an automorphism of Bruhat order, as stated in Proposition 2.2: $u \leq v \leftrightarrow u^{-1} \leq v^{-1}$. Let $u(c) < u(a) < u(b)$ and

$$u = \dots u(i) \dots u(a) \dots u(b) \dots u(c) \dots u(j) \dots$$

$$v = \dots u(j) \dots u(b) \dots u(c) \dots u(a) \dots u(i) \dots$$

Suppose that $u < v$ and for simplicity of notation define $u_s = u(s)$ for every $s \in [n]$, then $u^{-1}(u_s) = s$. The problem is to establish the natural ordering of elements of the set $\{u_a, u_b, u_c, u_i, u_j\}$; from the hypothesis we now that $u_c < u_a < u_b$ and, from the initial part of this proof, that $u_i < u_j$. Depending on this ordering is the position of the elements a, b, c, i, j on the inverses of u, v:

$\begin{matrix} u^{-1} \\ v^{-1} \end{matrix} = \dots \begin{matrix} c \\ a \end{matrix} \dots \begin{matrix} a \\ b \end{matrix} \dots \begin{matrix} b \\ c \end{matrix} \dots$ and we want to determine the position of the columns $\begin{smallmatrix} i \\ j \end{smallmatrix}$ and $\begin{smallmatrix} j \\ i \end{smallmatrix}$. Since u^{-1} and v^{-1} are under the conditions of part 1), we can conclude by symmetry that $u_i < u_c$ and $u_b < u_j$; in fact, as example if $u_c < u_i$ then

$$\begin{matrix} u^{-1} \\ v^{-1} \end{matrix} = \dots \begin{matrix} c \\ a \end{matrix} \dots \begin{matrix} i \\ j \end{matrix} \dots \begin{matrix} a \\ b \end{matrix} \dots \begin{matrix} b \\ c \end{matrix} \dots$$

and by the arguments used before it follows that $u^{-1} \nleq v^{-1}$. Since the remaining case is similar to this, we can consider the proof completed.

□

Now we prove the main result of this section:

Theorem 4.2. *Let* $u \in S_n$ *be such that* $u(1) = 1, u(n) = n$. *Consider* $1 < a < b < c < n$ *and suppose that*

$$v = \begin{cases} u(1,n)(a,b,c), & \text{if } u(a) = max\{u(a), u(b), u(c)\} \\ u(1,n)(a,c,b), & \text{if } u(c) = min\{u(a), u(b), u(c)\} \end{cases}$$

If $D(u) \cap D(v) = D_L(u) \cap D_L(v) = \emptyset$, *then* $\tilde{R}_{u,v}(t) = \tilde{R}_{x,(1,n)}(t)$, *where* x *is a* 3*-cycle.*

Proof. We consider the case that $u(b) < u(c) < u(a)$ so $v = u(1,n)(a,b,c)$, explicitly:

$$u = 1u(2)\dots u(a-1)u(a)u(a+1)\dots u(b-1)u(b)u(b+1)\dots u(c-1)u(c)u(c+1)\dots u(n-1)n$$

$$v = nu(2)\dots u(a-1)u(b)u(a+1)\dots u(b-1)u(c)u(b+1)\dots u(c-1)u(a)u(c+1)\dots u(n-1)1$$

Since $D(u) \cap D(v) = \emptyset$ we have that:
$a-1 \notin D(u)$, in fact if $a-1 \in D(u)$ then $a-1 \in D(v)$ because $u(a-1) > u(a) > u(b)$; and similarly that $a \notin D(v)$; $b-1 \notin D(v)$; $b \notin D(u)$; $c-1 \notin D(v)$; $c \notin D(u)$.

It is obvious that $\{1, n-1\} \subseteq D(v)$ and since by our assumptions u is not the identity, $D(u)$ must contains at least the descent a, i.e. $\{a\} \subseteq D(u)$, so we conclude that $\{1, n-1\} \subseteq D(v) \subseteq \{1, a-1, b, c, n-1\}$ and $\{a\} \subseteq D(u) \subseteq \{a, b-1, c-1\}$. We now show that if $a-1, c \in D(v)$ then $D_L(u) \cap D_L(v) \neq \emptyset$, so by our assumptions we conclude that $a-1, c \notin D(v)$.

Suppose that $a-1 \in D(v)$.
This means that $u(a-1) > u(b)$; we define $p \stackrel{\text{def}}{=} max\{m \in [a-2] : u(m) < u(b)\}$, so $1 < u(2) < \dots < u(p) < u(b) < u(p+1)$ that is equivalent to $u(i) = v(i) = i$ for every $i \in [p]$, $u(b) = v(a) = p+1$ and $u(p+1) = v(p+1) = p+2$. It follows that $u^{-1}(p+1) = b > u^{-1}(p+2) = p+1$ and $v^{-1}(p+1) = a > v^{-1}(p+2) = p+1$, therefore $(p+1, p+2) \in D_L(u) \cap D_L(v)$.

We now treat the case that $c \in D(v)$.
We have $u(a) > u(c+1)$, let $A \stackrel{\text{def}}{=} max\{m \in [c+1, n-1] : u(m) < u(a)\}$, so $u(c+1) < \dots < u(A) < u(a) < u(A+1)$ then $u(A+1) = v(A+1) = A+1$, $u(a) = v(c) = A$ and $u(A) = v(A) = A-1$. Clearly $(A-1, A) \in D_L(u) \cap D_L(v)$ since $u^{-1}(A-1) = A > u^{-1}(A) = a$ and $v^{-1}(A-1) > v^{-1}(A) = c$.

We conclude that under our hypothesis $D(v) = \{1, n-1\}$ or $D(v) = \{1, b, n-1\}$. Before starting the proof for these two cases, we observe that $u(m) = m$, for every $m \in [a-1] \cup [c+1, n-1]$.

(i) $\boldsymbol{D(v) = \{1, n-1\}}$
We have immediately that $v = (1, n)$. Therefore it is $v(a) = u(b) = a$, $v(b) = u(c) = b$ and $v(c) = u(a) = c$. So $u = (a, c, b)$.

(ii) $\boldsymbol{D(v) = \{1, b, n-1\}}$ and $\boldsymbol{D(u) = \{a, b-1\}}$
Every step of the proof involve "operations on the pair of permutations (u, v)" which preserve the R-polynomial since are applications of Theorem 2.5 or Proposition 2.6. For this reason we will not indicate the polynomials, but only the resulting permutations, writing the smallest one on the first row.

We recall that:

$$u = 12\dots a-1u(a)u(a+1)\dots u(b-1)u(b)u(b+1)\dots u(c-1)u(c)c+1\dots(n-1)n$$

$$v = n2\dots a-1u(b)u(a+1)\dots u(b-1)u(c)u(b+1)\dots u(c-1)u(a)c+1\dots(n-1)1$$

By our assumptions it must be $u(a) = c$, $u(c) = c-1$ and $u(b) = a$, in fact looking at u we see that $u(b+1) < \cdots < u(c-1) < u(c)$ while looking at v it is $u(b+1) < u(c)$; moreover $u(c) < u(a)$ and $a-1 < u(b) < u(a+1)$. Now, being $u(b-1) > u(b)$, we have to consider the relative natural order of $u(b-1)$ and $u(b+1)$. We first suppose that $u(b-1) > u(b+1)$ and we introduce the next parameters: $p \overset{\text{def}}{=} max\{m \in [a+1, b-2] : u(m) < u(b+1)\}$ and $s \overset{\text{def}}{=} max\{m \in [b+1, c-1] : u(m) < u(b-1)\}$. Therefore $u(b) < u(a+1) < \cdots < u(t) < u(b+1) < u(t+1) < \cdots < u(b-2) < \cdots < u(s) < u(b-1) < u(s+1) < \cdots < u(c) < u(a)$, which implies that $u(m) = m$ for $m \in [a+1, t]$, $u(b+1) = t+1$, $u(m) = m+1$ for $m \in [t+1, b-2]$, $u(s) = s-2$, $u(b-1) = s-1$, $u(s+1) = s$, $u(c) = c-1$. We represent these two permutations from the position a to the position c, in the next figure:

$$\begin{matrix} u & = \\ v & = \end{matrix}\ \begin{matrix} c & a+1 \\ a & a+1 \end{matrix} \cdots \begin{matrix} t \\ t \end{matrix} \boxed{\begin{matrix} t+2 \\ t+2 \end{matrix}} \cdots \boxed{\begin{matrix} s-1 \\ s-1 \end{matrix}} \begin{matrix} a \\ c-1 \end{matrix} \boxed{\begin{matrix} t+1 \\ t+1 \end{matrix}} \cdots \boxed{\begin{matrix} s-2 \\ s-2 \end{matrix}} \begin{matrix} s & s+1 & c-1 & c+1 \\ s & s+1 & c & c+1 \end{matrix}$$

and $(t+1, t+2)$, $(s-2, s-1)$ are left common descents for u and v.
It follows that we can definitively suppose $u(b-1) < u(b+1)$. This implies $u(b-1) = b-1$, $u(b+1) = b, \dots, u(c-1) = c-2$ then

$$\begin{matrix} u & = & 1 & 2 & & a-1 & c & a+1 & & b-1 & a & b & b+1 & & c-2 & c-1 & c+1 & & n-1 & n \\ v & = & n & 2 & \cdots & a-1 & a & a+1 & \cdots & b-1 & c-1 & b & b+1 & \cdots & c-2 & c & c+1 & \cdots & n-1 & 1 \end{matrix}$$

(As a remark we note that if $b+1 = c-1$ we have that $u(b+1) < u(c-1) = b$ and $u(c) = b+1$, thus the block from the position b+1 to c-2 does not appear.)

We apply proposition 2.6 to the pair (u, v) as follows. First we calculate

$$\begin{matrix} u^{-1} & = & 1 & 2 & & a-1 & b & a+1 & & b-1 & b+1 & & c-1 & c & a & & c+1 & n-1 & n \\ v^{-1} & = & n & 2 & \cdots & a-1 & a & a+1 & \cdots & b-1 & b+1 & \cdots & c-1 & b & c & \cdots & c+1 & n-1 & 1 \end{matrix}$$

Now we recall that if $u = u(1)u(2)\dots u(n)$ then $uw_0 = u(n)\dots u(2)u(1)$. Let $u_1 = v^{-1}w_0$ and $v_1 = u^{-1}w_0$, therefore:

$$\begin{matrix} u_1 & = & 1 \\ v_1 & = & n \end{matrix} \boxed{\begin{matrix} n-1 & & c+1 \\ n-1 & \cdots & c+1 \end{matrix}} \begin{matrix} c & b \\ a & c \end{matrix} \boxed{\begin{matrix} c-1 & & b+1 & b-1 & & a+1 \\ c-1 & \cdots & b+1 & b-1 & \cdots & a+1 \end{matrix}} \begin{matrix} a \\ b \end{matrix} \boxed{\begin{matrix} a-1 & & 2 \\ a-1 & \cdots & 2 \end{matrix}} \begin{matrix} n \\ 1 \end{matrix}$$

Our purpose is to obtain again two permutations which have not common left or right descents by applying Theorem 2.5 to the common descents of u_1 and v_1. There is not a unique way to do this, but there are two main steps:

(S1) **Reordering of common blocks**
We focus our attention on the boxed common blocks of the pair (u_1, v_1) and we multiply these permutations by the suitable descents to reorder each block inside. For the reordering of the block $\boxed{\begin{matrix} n-1 & \cdots & c+1 \\ n-1 & & c+1 \end{matrix}}$ from the position 2 to $n-c$, we multiply by $u_1 \prod_{j=1}^{n-c-2} s_2 s_3 \dots s_{n-(c+j)}$ and $v_1 \prod_{j=1}^{n-c-2} s_2 s_3 \dots s_{n-(c+j)}$, where $s_i = (i, i+1)$. With similar product we reorder the other two boxed blocks and at the end of this step we obtain from (u_1, v_1) the permutations indicated with the prefix S1:

$$\begin{matrix} (S1)u_1 & = & 1 \\ (S1)v_1 & = & n \end{matrix} \boxed{\begin{matrix} c+1 & \cdots & n-1 \\ c+1 & & n-1 \end{matrix}} \begin{matrix} c & b & a+1 & \cdots & b-1 & b+1 & \cdots & c-1 & a \\ a & c & a+1 & & b-1 & b+1 & & c-1 & b \end{matrix} \boxed{\begin{matrix} 2 & \cdots & a-1 \\ 2 & & a-1 \end{matrix}} \begin{matrix} n \\ 1 \end{matrix}$$

First block are the integers $[c+1, n-1]$ which are bigger than all the other integers that appear on the complete notation of $(S1)u_1$ and $(S1)v_1$ from the position $n-c+1$ to the position $n-1$; we move this block between $c+1$-th place and $n-1$-th place multiplying the permutations $(S1)u_1$ and $(S1)v_1$ on the right by

$$(s_{n-c}s_{n-c+1}s_{n-c+2}\dots s_{n-2})(s_{n-c-1}s_{n-c}s_{n-c+1}\dots s_{n-3})\dots(s_2\dots s_c)$$

. Similarly we shift on the left the block containing integers of the interval $[2, a-1]$ to their homonymous positions and we obtain

$$\begin{matrix} u_2 & = & 1 & 2 & \cdots & a-1 & c \\ v_2 & = & n & 2 & & a-1 & a \end{matrix} \boxed{\begin{matrix} b \\ c \end{matrix}} \begin{matrix} a+1 & \cdots & b-1 & b+1 & \cdots & c-1 \\ a+1 & & b-1 & b+1 & & c-1 \end{matrix} \boxed{\begin{matrix} a \\ b \end{matrix}} \begin{matrix} c+1 & n-1 & n \\ c+1 & n-1 & 1 \end{matrix}.$$

(S2) **Final shifting**
We have that $\{a+1, c\} \in D(u_2) \cap D(v_2)$ thus we apply Theorem 2.5 as follows: $u_3 = u_2(s_{c-1}s_{c-2}\dots s_{b+1})(s_{a+1}s_{a+2}\dots s_b)$ and $v_3 = v_2(s_{c-1}s_{c-2}\dots s_{b+1})(s_{a+1}s_{a+2}\dots s_b)$, where we called u_3 and v_3 the resulting permutations which have no common right descents as it is clear from the next illustration:

$$\begin{matrix} u_3 & = & 1 & 2 & \cdots & a-1 & c & a+1 & \cdots & b-1 & a & b & b+1 & \cdots & c-1 & c+1 & n-1 & n \\ v_3 & = & n & 2 & & a-1 & a & a+1 & & b-1 & b & c & b+1 & & c-1 & c+1 & n-1 & 1 \end{matrix}.$$

We observe now that u_3 and v_3 have common left descents by computing:

$$\begin{matrix} u_3^{-1} & = & 1 & 2 & \cdots & a-1 & b & a+1 & \cdots & b-1 & b+1 & b+2 & \cdots & c \\ v_3^{-1} & = & n & 2 & & a-1 & a & a+1 & & b-1 & b & b+2 & & c \end{matrix} \boxed{\begin{matrix} a \\ b+1 \end{matrix}} \begin{matrix} c+1 & n-1 & n \\ c+1 & n-1 & 1 \end{matrix}.$$

We apply again Theorem 2.5 on the common descents which are indicated in the next products: $u_f = (u_3^{-1})s_{c-1}s_{c-2}\dots s_{b+1}$ and $v_f = (v_3^{-1})s_{c-1}s_{c-2}\dots s_{b+1}$

$$\begin{matrix} u_f & = & 1 & 2 & \cdots & a-1 & b & a+1 & \cdots & b-1 & b+1 & a & b+2 & \cdots & n-1 & n \\ v_f & = & n & 2 & & a-1 & a & a+1 & & b-1 & b & b+1 & b+2 & & n-1 & 1 \end{matrix}$$

and we are done.

(iii) $\boldsymbol{D(v) = \{1, b, n-1\}}$ and $\boldsymbol{D(u) = \{a, b-1, c-1\}}$

From these hypothesis follows that $u(b) = a$ and $u(a) = c$, moreover we can suppose that $u(b-1) < u(b+1)$ because $u(b-1) > u(b+1)$ implies that $D_L(u) \cap D_L(v)$ is not empty, as we have already shown in (ii).

We define $t \stackrel{\text{def}}{=} max\{m \in [b+1, c-2] : u(m) < u(c)\}$ so $a = u(b) < u(a+1) < \cdots < u(b-1) < u(b+1) < \cdots < u(t) < u(c) < u(t+1) < \cdots < u(c-1) < u(a) = c$. Then it must be $u(m) = m$, for every $m \in [a+1, b-1] \cup [t+1, c-1]$, $u(m) = m-1$ for every $m \in [b+1, t]$, moreover $u(c) = t$, in other words

$$\begin{array}{l} u = 1\ 2 \cdots a-1\ c\ a+1 \cdots b-1\ a\ b \cdots t-1\ t+1 \cdots c-1\ t\ c+1 \cdots n-1\ n \\ v = n\ 2 \cdots a-1\ a\ a+1 \cdots b-1\ t\ b \cdots t-1\ t+1 \cdots c-1\ c\ c+1 \cdots n-1\ 1 \end{array}.$$

The algorithm shown in (ii) with S1, S2, S3 as fundamental steps is valid in this case too; in fact, as one can observe, there is only the central block which will require some additional products in S1 to be reordered. Therefore following (ii) we start with the next computation:

$$\begin{array}{l} u^{-1} = 1\,2 \cdots a-1\,b\,a+1 \cdots b-1\,b+1 \cdots t-1\,t\,c\,t+1 \cdots c-1\,a\,c+1 \cdots n-1\,n \\ v^{-1} = n\,2 \cdots a-1\,a\,a+1 \cdots b-1\,b+1 \cdots t-1\,t\,b\,t+1 \cdots c-1\,c\,c+1 \cdots n-1\,1 \end{array},$$

and then calculating $u_1 = v^{-1}w_0$, $v_1 = u^{-1}w_0$, we have

$$\begin{array}{l} u_1 = 1\,n-1 \cdots c+1\,c\,c-1 \cdots t+1\,b\,t\,t-1 \cdots b+1\,b-1 \cdots a+1\,a\,a-1 \cdots 2\,n \\ v_1 = n\,n-1 \cdots c+1\,a\,c-1 \cdots t+1\,c\,t\,t-1 \cdots b+1\,b-1 \cdots a+1\,b\,a-1 \cdots 2\,1 \end{array}$$

As in (v)(S1) we multiply (u_1, v_1) for appropriate s_i to put the elements of the blocks between the following interval positions $[2, n-c]$, $[n-a-2, n-1]$, $[n-c+2, n-a-4]$, in their natural order (as indicated in the previous figure). This produces the following pair of permutations:

$$\begin{array}{l} u_2 = 1\,c+1 \cdots n-1\,c\,t+1 \cdots c-1\,b\,a+1 \cdots b-1\,b+1 \cdots t-1\,t\,a\,2 \cdots a-1\ n \\ v_2 = n\,c+1 \cdots n-1\,a\,t+1 \cdots c-1\,c\,a+1 \cdots b-1\,b+1 \cdots t-1\,t\,b\,2 \cdots a-1\ 1 \end{array}.$$

Therefore we can interchange the first and last blocks and also shift on the left the block $\boxed{\begin{array}{l} a+1 \cdots b-1 \\ a+1 \cdots b-1 \end{array}}$. One obtains:

$$\begin{array}{l} u_3 = 1\ 2 \cdots a-1\ c\ a+1 \cdots b-1\ t+1 \cdots c-1\ b\ b+1\ t\ \boxed{a}\ c+1 \cdots n-1\ n \\ v_3 = n\ 2 \cdots a-1\ a\ a+1 \cdots b-1\ t+1 \cdots c-1\ c\ b+1\ t\ \boxed{b}\ c+1 \cdots n-1\ 1 \end{array}$$

We have that $c-1 \in D(u_3) \cap D(v_3)$ and applying Theorem 2.5 to the appropriate descents we obtain

$$\begin{array}{l} u_4 = 1\ 2 \cdots c\ a+1 \cdots b-1\ a\ t+1 \cdots c-1\ b\ b+1\ t\ c+1 \cdots n-1\ n \\ v_4 = n\ 2 \cdots a\ a+1 \cdots b-1\ b\ t+1 \cdots c-1\ c\ b+1\ t\ c+1 \cdots n-1\ 1 \end{array}$$

This was the "final shifting", (S2), then we calculate the inverses of u_4 and v_4: observing that $u_5(b) = t+1, \ldots, u_5(b+c-t) = c-1$, $u_5(b+c-t+1) = b$, $u_5(b+c-t+2) = b+1, \ldots, u(c-1) = t-1, u(c) = t$ it results

$$\begin{array}{l} u_4^{-1} = 1\ 2 \cdots a-1\ b\ a+1\ b-1\ a\ b+c-t+1\ b+c-t+2 \cdots c-1 \\ v_4^{-1} = n\ 2 \cdots a-1\ a\ a+1\ b-1\ b\ b+c-t+1\ b+c-t+2 \cdots c-1 \end{array}$$

$$\begin{array}{l} c\ b+1 \cdots b+c-t-1\ b+c-t\ a\ c+1 \cdots n-1\ n \\ c\ b+1 \cdots b+c-t-1\ b+c-t\ b+c-t+1\ c+1 \cdots n-1\ 1 \end{array}$$

and with the shifting of the underlined blocks on the left we have:

$$\begin{array}{lllllllllllll} u_f & = 1 & 2 & & a-1 & b & a+1 & b-1 & a & b+1 & & b+c-t-1 & b+c-t \\ v_f & = n & 2 & \cdots & a-1 & a & a+1 & b-1 & b & b+1 & \cdots & b+c-t-1 & b+c-t \end{array}$$

$$\begin{array}{llllllll} b+c-t+1 & b+c-t+2 & & c-1 & c+1 & & n-1 & n \\ b+c-t+1 & b+c-t+2 & \cdots & c-1 & c+1 & \cdots & n-1 & 1 \end{array}$$

In other words $u_f = (a, b, b+c-t+1)$ where $b+c-t+1 \in [b+2, c-1]$ and $v_f = (1, n)$ as we wanted prove, for part (1).

We now prove a part of the statement (2) of the theorem, i.e. we consider $u \in S_n$ such that $u(1) < u(c) < u(a) < u(b) < u(n)$ so $v = u(1,n)(a,b,c)$, explicitly:

$$u = 12\ldots a-1u(a)u(a+1)\ldots u(b-1)u(b)u(b+1)\ldots u(c-1)u(c)c+1\ldots(n-1)n$$

$$v = n2\ldots a-1u(b)u(a+1)\ldots u(b-1)u(c)u(b+1)\ldots u(c-1)u(a)c+1\ldots(n-1)1$$

By similar arguments used in part (1) case we have that under our hypothesis $\{b\} \subseteq D(u) \subseteq \{a, b, c-1\}$ and $D(v) = \{1, n-1\}$ or $D(v) = \{1, b-1, n-1\}$.

$\boldsymbol{D(v) = \{1, n-1\}}$

Then $v = (1, n)$, so that $u(c) = a$, $u(a) = b$ and $u(b) = c$. Clearly $u = (a, b, c)$ and we have treated the symmetric case in fact $u^{-1} = (a, c, b)$.

$\boldsymbol{D(v) = \{1, b-1, n-1\}}$ and $\boldsymbol{D(u) = \{b, c-1\}}$

$$u = 12\ldots a-1u(a)\ldots u(b-1)u(b)u(b+1)\ldots u(c-1)u(c)c+1\ldots n-1n$$

$$v = n2\ldots a-1u(c)\ldots u(b-1)u(a)u(b+1)\ldots u(c-1)u(b)c+1\ldots n-1n$$

We suppose that $u(b-1) < u(b+1)$ otherwise $D_L(u) \cap D_L(v)$ is not empty. Under this hypothesis we have: $a-1 < u(c) < u(a) < u(a+1) < \cdots < u(b-1) < u(b+1) < \ldots u(c-1) < u(b) < c+1$ that implies $u(c) = a$, $u(a) = a+1$, $u(a+1) = a+2, \ldots,$ $u(b-1) = b$, $u(b+1) = b+1, \ldots, u(c-1) = c-1$, $u(b) = c$. The structure of u and v is completely determined:

$$\begin{array}{lllllllllllllllll} u & = & 1 & 2 & & a-1 & a+1 & a+2 & & b & c & b+1 & & c-1 & a & c+1 & & n-1 & n \\ v & = & n & 2 & \cdots & a-1 & a & a+2 & \cdots & b & a+1 & b+1 & \cdots & c-1 & c & c+1 & \cdots & n-1 & 1 \end{array}$$

The inverses of these permutations are:

$$\begin{array}{llllllllllllllllll} u^{-1} & = & 1 & 2 & & a-1 & c & a & a+1 & & b-1 & b+1 & & c-1 & b & c+1 & & n-1 & n \\ v^{-1} & = & n & 2 & \cdots & a-1 & a & b & a+1 & \cdots & b-1 & b+1 & \cdots & c-1 & c & c+1 & \cdots & n-1 & 1 \end{array}$$

which are of the type described in (iii) since $u^{-1}(a) > u^{-1}(c) > u^{-1}(b)$ e $v^{-1} = u^{-1}(a, c, b)$, $D(u^{-1}) = \{a, c-1\}$ and $D(v^{-1}) = \{1, a+1, n-1\}$. With the notation introduced in (iii) we have $u_f = (b+c-t, a, b)$ but in this case we have $t = max\{i : u^{-1}(i) < u^{-1}(c)\} = b$ since that $u^{-1}(c) = b$, then $u_f = (c, a, b)$ as one can directly compute. The case that $D(u) = \{b\}$ is included in this with $c = b+1$.

$\boldsymbol{D(v) = \{1, b-1, n-1\}}$ and $\boldsymbol{D(u) = \{a, b, c-1\}}$

It is obvious that $u(c) = a$ and $u(b) = c$. We suppose again that $u(b-1) < u(b+1)$ and we define $s \stackrel{\text{def}}{=} max\{m \in [a+1, b-2] : u(m) < u(a)\}$, then: $u(c) < u(a+1) <$

$u(a+2) < \cdots < u(s) < u(a) < u(s+1) < \cdots < u(b-1) < u(b+1) < \ldots u(c-1) < u(b)$. This implies that

$$\begin{array}{l} u = 1\ 2 \cdots a-1\ s+1\ a+1 \cdots s\ s+2 \cdots b-1\ b\ \ c\ \ b+1 \cdots c-1\ a\ c+1 \cdots n-1\ n \\ v = n\ 2 \cdots a-1\ \ a\ \ a+1 \cdots s\ s+2 \cdots b-1\ b\ s+1\ b+1 \cdots c-1\ c\ c+1 \cdots n-1\ 1 \end{array}$$

We can see that the inverses of this permutations satisfy the hypothesis of (iii), in fact:

$$\begin{array}{l} u^{-1} = 1\ 2 \cdots a-1\ c\ a+1 \cdots s\ a\ s+1 \cdots b-1\ b+1 \cdots c-1\ b\ c+1 \cdots n-1\ n \\ v^{-1} = n\ 2 \cdots a-1\ a\ a+1 \cdots s\ b\ s+1 \cdots b-1\ b+1 \cdots c-1\ c\ c+1 \cdots n-1\ 1 \end{array}$$

We observe that $D(u^{-1}) = \{a, s, c-1\}$, $D(v^{-1}) = \{1, s+1, n-1\}$ and being $a < s < s+1 < c$, using the arguments of (iii), we reduce to the pair of permutations $u_f = (s+c+1-t, a, s+1)$ and $v_f = (1, n)$.

To conclude the proof we consider $u \in S_n$, $1 \leq i < a < b < c < j \leq n$, such that $u(a) > u(b) > u(c)$, as we will see it is a consequence of the previous cases.
If $v = u(1,n)(a,c,b)$ we have that (b,c) is a common inversion of u and v which behaves as a descent, by Lemma 3.2 In fact, we can suppose that the intervals $[2, a-1]$ and $[c+1, n-1]$ are fixed points and in $[a+1, b-1] \cup [c+1, n-1]$ there are not descents, moreover $u(b) = v(c)$:

$$u = 12\ldots a-1u(a)u(a+1)\ldots u(b-1)u(b)u(b+1)\ldots u(c-1)u(c)c+1\ldots(n-1)n$$

$$v = n2\ldots a-1u(c)u(a+1)\ldots u(b-1)u(a)u(b+1)\ldots u(c-1)u(b)c+1\ldots(n-1)1$$

This implies that there exists $r \in [b, c-1]$ such that $\tilde{R}_{u,v}(t) = \tilde{R}_{x,y}(t)$, where $x = 12\ldots a-1u(a)u(a+1)\ldots u(b-1)u(b+1)\ldots u_r u(c)u(b)u_{r+1}c+1\ldots(n-1)n$ and $y = n2\ldots a-1u(c)u(a+1)\ldots u(b-1)u(b+1)\ldots u_r u(b)u(a)u_{r+1}\ldots c+1\ldots(n-1)1$ and this case has been studied.
If $v = u(1,n)(a,b,c)$ (a,b) which is a common inversion of u and v behaves as a descent.

$$u = 12\ldots a-1u(a)u(a+1)\ldots u(b-1)u(b)u(b+1)\ldots u(c-1)u(c)c+1\ldots(n-1)n$$

$$v = n2\ldots a-1u(b)u(a+1)\ldots u(b-1)u(c)u(b+1)\ldots u(c-1)u(a)c+1\ldots(n-1)1$$

and since (a,b) is a common inversion of u and v behaves as a descent, by Lemma 3.2 then we reduce to another studied case.

□

Corollary 4.3. *Let $u \in S_n$, $1 \leq i < a < b < c < j \leq n$ and $u(i) < u(s) < u(j), \forall s \in \{a,b,c\}$.*

$$v = \begin{cases} u(i,j)(a,b,c), & \text{if } u(a) = max\{u(a), u(b), u(c)\} \\ u(i,j)(a,c,b), & \text{if } u(c) = min\{u(a), u(b), u(c)\} \end{cases}$$

Then

$$\tilde{R}_{u,v}(t) = t^5(1+t^2)^{(inv(v)-inv(u)-5)/2}$$

Proof. Suppose that u and v have common right or left descents, then we can apply Theorem 2.5 to these descents and reduce to consider a pair of permutations with no common left or right descents. Moreover for 2.1, we can suppose $i = 1 = u(i)$ and $n = j = u(n)$ without lost of generality. Then we are under the conditions of Theorem 4.2 $\tilde{R}_{u,v}(t) = \tilde{R}_{(x,y,z),(1,n)}(t)$, where (x,y,z) is a generic 3-cycle. Applying the Theorem 3.2 of [8] at this last polynomial the result follows. □

Corollary 4.4. *Let $u \in S_n$, $1 \le i < a < b < c < j \le n$ and v under the conditions of Theorem 4.3. Then $R_{u,v}(q) = (q-1)^5(q^2-q+1)^{(inv(v)-inv(u)-5)/2}$.*

Proof. This follows from Proposition 2.4. □

Remarks and applications

The pairs of permutations (u, v) considered in Corollary 4.3 have all the property that $u^{-1}v = (1, n)x$ where x is a 3-cycle, i.e these products are in the same conjugacy class. We observe that the hypothesis on $u(a), u(b), u(c)$ are essential and that the fact that $u^{-1}v$ is in the conjugacy class of $(1, n)x$ is not sufficient to have the same R-polynomial:
$\tilde{R}_{12435,53241}(t) = t^5(2+t^2)$ and $12435 \circ 53241 = (1, 5)(2, 4, 3)$. $2 = u(a) \neq max\{2, 3, 4\}$ as in the next example
$\tilde{R}_{12345,54231}(t) = t^3(t^6 + 5t^4 + 6t^2 + 1)$ and $12345 \circ 54231 = (1, 5)(2, 4, 3)$.
$\tilde{R}_{12435,54321}(t) = t^3(t^6 + 5t^4 + 6t^2 + 1)$ and $12435 \circ 54321 = (1, 5)(2, 3, 4)$, but $3 = u(c) \neq min\{2, 3, 4\}$.

However, under some additional conditions on u, we obtain in the following proposition a closed product formulas which is a consequence of the main result of this paper and the one of [10]. Such a result is an example of the possible applications of these formulas and of the idea that every explicit formulas for the R-polynomials could give formulas for different irreducible classes of permutations, and not uniquely for the reducible classes.

Proposition 4.5. *Let $u, v \in S_n$ and $1 \le i < a < b < c < j \le n$ be such that one of the following conditions is satisfied:*

i) *u contains a pattern of type 132 at the positions a, b, c; $\forall s \in [a+1, b-1]$, $u(s) \notin [u(a), u(c)]$ and $u < v = u(i, j)(a, c, b)$*
ii) *u contains a pattern of type 312 at the positions a, b, c; $\forall s \in [b+1, c-1]$, $u(s) \notin [u(b), u(c)]$ and $u < v = u(i, j)(a, c, b)$*
iii) *u contains a pattern of type 213 at the positions a, b, c; $\forall s \in [b+1, c-1]$, $u(s) \notin [u(a), u(c)]$ and $u < v = u(i, j)(a, b, c)$*
iv) *u contains a pattern of type 231 at the positions a, b, c; $\forall s \in [a+1, b-1]$, $u(s) \notin [u(a), u(b)]$ and $v < u(i, j)(a, b, c)$*

Then

$$\tilde{R}_{u,v}(t) = t^5(1+t^2)^{\frac{inv(v)-inv(u)-7}{2}}(t^2+2)$$

Proof. In order to lighten notation, we write all permutations, in their complete notation, only between the positions i, j; moreover, as in the proof of Theorem 4.3, we apply at each step, excepted the final part, operations on the pair of permutations which preserves the $\tilde{R}$ polynomials, by Theorem 2.5. For this reason we write only the permutations and not their polynomials.

$$\begin{array}{rcl} u &=& u(i) \quad\;\;\; u(a) \quad\;\;\; u(b) \quad\;\;\; u(c) \quad\;\;\; u(j) \\ v &=& u(j) \cdots u(c) \cdots u(a) \cdots u(b) \cdots u(i) \end{array}$$

We prove part i).
From the hypotheses follows that $\forall s \in [a+1, b-1]$, either $u(s) < u(a)$ or $u(s) > u(c)$.

We assume that $u(a-1) < \cdots < u(b-1)$ and $u(b+1) < \cdots < u(c-1)$, by Theorem 2.5. Define $t \stackrel{\text{def}}{=} max\{m \in [a+1, b-1] : u(m) < u(a)\}$, it follows that $u(a+1) < \cdots < u(t) < u(a) < u(c) < u(t+1)$.

We compute (u_1, v_1) where $u_1 = us_a s_{a+1} \ldots s_t$, $u_1 = us_a s_{a+1} \ldots s_t$. More explicitly

$$\begin{aligned} u_1 &= u(i) \cdots u(a-1)u(a+1) \cdots u(t)u(a)u(t+1) \cdots u(b-1)u(b)u(b+1) \cdots u(c-1)u(c)u(c+1) \cdots u(j) \\ v_1 &= u(j) \cdots u(a-1)u(a+1) \cdots u(t)u(c)u(t+1) \cdots u(b-1)u(a)u(b+1) \cdots u(c-1)u(b)u(c+1) \cdots u(i) \end{aligned}$$

We reverse these permutations:

$$\begin{aligned} v_1 w_0 &= u(i) \cdots u(c+1)u(b)u(c-1) \cdots u(b+1)u(a)u(b-1) \cdots u(t+1)\, u(c)\, u(t) \cdots u(a+1)u(a-1) \cdots u(j) \\ u_1 w_0 &= u(j) \cdots u(c+1)u(c)u(c-1) \cdots u(b+1)u(b)u(b-1) \cdots u(t+1)\, u(a)\, u(t) \cdots u(a+1)u(a-1) \cdots u(i) \end{aligned}$$

By definition of t we have that $(v_1 w_0, u_1 w_0)$ has the same $\tilde{R}$-polynomial of the next pair:

$$\begin{aligned} u_2 &= u(i) \cdots u(c+1)u(b)u(c-1) \cdots u(b+1)u(a)u(c)u(b-1) \cdots u(a+1)u(a-1) \cdots u(j) \\ v_2 &= u(j) \cdots u(c+1)u(c)u(c-1) \cdots u(b+1)u(b)u(a)u(b-1) \cdots u(a+1)u(a-1) \cdots u(i) \end{aligned}$$

We reverse again to obtain:

$$\begin{aligned} u_3 &= u(i) \cdots u(a-1)u(a+1)\; u(b-1)u(a)u(b)u(b+1) \cdots u(c-1)u(c)u(c+1) \cdots u(j) \\ v_3 &= u(j) \cdots u(a-1)u(a+1)\; u(b-1)u(c)u(a)u(b+1) \cdots u(c-1)u(b)u(c+1) \cdots u(i) \end{aligned}$$

Here we have defined $u_3 = v_2 w_0$ and $v_3 = u_2 w_0$.

Now we compute $\tilde{R}_{u_3,v_3}(t)$ by applying Theorem 2.5 to the descent $s_{b-1} \in D(v_3)$; since $s_{b-1} \notin D(u_3)$ we have: $\tilde{R}_{u_3,v_3}(t) = \tilde{R}_{u_3 s_{b-1}, v_3 s_{b-1}}(t) + t\tilde{R}_{u_3, v_3 s_{b-1}}(t)$ The pair $(u_3 s_{b-1}, v_3 s_{b-1})$ satisfies the hypotheses of Theorem 4.3, in fact $u_3 s_{b-1}$ contains a pattern 312 at the positions $b-1, b, c$ and $v_3 s_{b-1} = u_3 s_{b-1}(i,j)(b-1, b, c)$; whereas the pair $(u_3, v_3 s_{b-1})$ is under the conditions of Theorem 3.4, since $u_3(b-1) > u_3(b)$ and $v_3 s_{b-1} = u_3(i,j)(b-1, b)$.
Therefore

$$\tilde{R}_{u_3,v_3}(t) = t^5(1+t^2)^{\frac{inv(v_3 s_{b-1}) - inv(u_3 s_{b-1}) - 5}{2}} + t(t^4(1+t^2)^{\frac{inv(v_3 s_{b-1}) - inv(u_3) - 4}{2}}) = t^5(1+t^2)^{\frac{inv(v_3) - inv(u_3) - 7}{2}}(2+t^2)$$

The last equality follows from $inv(u_3) = inv(u_3 s_{b-1}) - 1$ and $inv(v_3) = inv(u_3 s_{b-1}) + 1$, since s_{b-1} is a descent of v_3, but not of u_3. The final result is a consequence of the fact that $\tilde{R}_{u,v}(t) = \tilde{R}_{u_3,v_3}(t)$ implies that $inv(v) - inv(u) = inv(v_3) - inv(u_3)$.

Now we consider to be under the conditions ii).
We assume that the values between the positions $b+1$ and $c-1$ are in increasing order. Since $\forall s \in [b+1, c-1]$, $u(s) \notin [u(b), u(c)]$ we have that either $u(s) < u(b)$

or $u(s) > u(c)$. Let $t \stackrel{\text{def}}{=} max\{m \in [b+1, c-1] : u(m) < u(b)\}$, it follows that $u(b+1) < \cdots < u(t) < u(b) < u(c) < u(t+1)$. We calculate (u_1, v_1) where $u_1 = us_{c-1} \dots s_{t+1}$, $v_1 = vs_{c-1} \dots s_{t+1}$.

$$\begin{array}{ll} u_1 &= u(i) \\ v_1 &= u(j) \end{array} \cdots \begin{array}{c} u(a) \\ u(c) \end{array} \cdots \begin{array}{c} u(b)u(b+1) \\ u(a)u(b+1) \end{array} \cdots \begin{array}{c} u(t)u(c)u(t+1) \\ u(t)u(b)u(t+1) \end{array} \cdots \begin{array}{c} u(j) \\ u(i) \end{array}$$

Now we multiply by w_0:

$$\begin{array}{ll} v_1w_0 &= u(i) \\ u_1w_0 &= u(j) \end{array} \cdots \begin{array}{c} u(t+1)u(b)u(t) \\ u(t+1)u(c)u(t) \end{array} \cdots \begin{array}{c} u(b+1)u(a) \\ u(b+1)u(b) \end{array} \cdots \begin{array}{c} u(c) \\ u(a) \end{array} \cdots \begin{array}{c} u(j) \\ u(i) \end{array}$$

And again by definition of t we can move the column $\begin{array}{c} u(b) \\ u(c) \end{array}$ to the right to obtain:

$$\begin{array}{ll} u_2 &= u(i) \\ v_2 &= u(j) \end{array} \cdots \begin{array}{c} u(t+1)u(t) \\ u(t+1)u(t) \end{array} \cdots \begin{array}{c} u(b+1)u(b)u(a) \\ u(b+1)u(c)u(b) \end{array} \cdots \begin{array}{c} u(c) \\ u(a) \end{array} \cdots \begin{array}{c} u(j) \\ u(i) \end{array}$$

We reverse again:

$$\begin{array}{ll} u_3 &= u(i) \\ v_3 &= u(j) \end{array} \cdots \begin{array}{c} u(a) \\ u(c) \end{array} \cdots \begin{array}{c} u(b)u(c)u(b+1) \\ u(a)u(b)u(b+1) \end{array} \cdots \begin{array}{c} u(j-1)u(j) \\ u(j-1)u(i) \end{array}$$

At this point if we apply Theorem 2.5 to the descent $(b-1, b) \in D(v_3)$ we obtain two summands which are under the conditions of Theorem 3.4 and 4.3 as in i).

We prove case iii).

As before we let $t \stackrel{\text{def}}{=} max\{m \in [b+1, c-1] : u(m) < u(a)\}$, it follows that $u(b+1) < \cdots < u(t) < u(a) < u(c) < u(t+1)$.

$$\begin{array}{ll} u &= u(i)u(i+1) \\ v &= u(j)u(i+1) \end{array} \cdots \begin{array}{c} u(a) \\ u(b) \end{array} \cdots \begin{array}{c} u(b) \\ u(c) \end{array} \cdots \begin{array}{c} u(t)\, u(t+1) \\ u(t)\, u(t+1) \end{array} \cdots \begin{array}{c} u(c) \\ u(a) \end{array} \cdots \begin{array}{c} u(j-1)u(j) \\ u(j-1)u(i) \end{array}$$

With the same computation of the previous cases, we obtain the pair (u_1, v_1), which is explicitly:

$$\begin{array}{ll} u_1 &= u(i)u(i+1) \\ v_1 &= u(j)u(i+1) \end{array} \cdots \begin{array}{c} u(a) \\ u(b) \end{array} \cdots \begin{array}{c} u(b)\, u(b+1) \\ u(c)\, u(b+1) \end{array} \cdots \begin{array}{c} u(t)\, u(c)\, u(t+1) \\ u(t)\, u(a)\, u(t+1) \end{array} \cdots \begin{array}{c} u(j-1)u(j) \\ u(j-1)u(i) \end{array}$$

Then we multiply to the right by w_0:

$$\begin{array}{ll} v_1w_0 &= u(i)u(j-1) \\ u_1w_0 &= u(j)u(j-1) \end{array} \cdots \begin{array}{c} u(t+1)\, u(a)\, u(t) \\ u(t+1)\, u(c)\, u(t) \end{array} \cdots \begin{array}{c} u(b+1)\, u(c) \\ u(b+1)\, u(b) \end{array} \cdots \begin{array}{c} u(b) \\ u(a) \end{array} \cdots \begin{array}{c} u(i+1)u(j) \\ u(i+1)u(i) \end{array}$$

By definition of t, we can move the column $\begin{array}{c} u(a) \\ u(c) \end{array}$ and it results:

$$\begin{array}{ll} u_2 &= u(i)u(j-1) \\ v_2 &= u(j)u(j-1) \end{array} \cdots \begin{array}{c} u(t+1)\, u(t) \\ u(t+1)\, u(t) \end{array} \cdots \begin{array}{c} u(b+1)\, u(a)\, u(c) \\ u(b+1)\, u(c)\, u(b) \end{array} \cdots \begin{array}{c} u(b) \\ u(a) \end{array} \cdots \begin{array}{c} u(i+1)u(j) \\ u(i+1)u(i) \end{array}$$

Now define $u_3 = v_2w_0$ and $v_3 = u_2w_0$. To end the computation we apply again Theorem 2.5 to the descent s_{b-1} of v_3; then we split $\tilde{R}_{u_3,v_3}(t)$ in two summands which are $\tilde{R}_{u_3,v_3s}$ that satisfies the hypotheses of Theorem 3.4, since $u_3(b-1) > u_3(b)$ and $v_3 = u_3(i,j)(b-1,b)$ and $\tilde{R}_{u_3s,v_3s}(t)$ which is under the conditions of Theorem 4.3, since u_3s_{b-1} contains a pattern 231 at the positions $a, b-1, b$ and

$v_3 s_{b-1} = u_3 s_{b-1}(i,j)(a,b-1,b)$.

Finally, we treat case iv).
To prove this case it is enough to observe that if (u,v) satisfies iv) then (vw_0, uw_0) satisfies the conditions of iii). In fact, if we define $u_1 = vw_0$, $v_1 = uw_0$, $\bar{a} = u_1^{-1}(u(a))$, $\bar{b} = u_1^{-1}(u(c))$ and $\bar{c} = u_1^{-1}(u(b))$, then $\forall s \in [\bar{b}+1, \bar{c}-1]$, $u_1(s) \notin [u_1(\bar{a}), u_1(\bar{c})]$, since $u_1(\bar{a}) = u(a)$ and $u_1(\bar{c}) = u(b)$. For an easier visualization we write explicitly u_1, v_1:

$$\begin{aligned} u_1 &= u(i)u(j-1) \cdots u(a) \cdots u(c) \cdots u(c) \cdots u(i+1)u(j) \\ v_1 &= u(j)u(j-1) \cdots u(c) \cdots u(b) \cdots u(a) \cdots u(i+1)u(i) \end{aligned}$$

and it is clear that u_1 contains a pattern 213 at the positions $\bar{a}, \bar{b}, \bar{c}$.
This concludes the proof. ☐

Corollary 4.6. *Let $u, v \in S_n$ be such that one of the conditions of the previous proposition is satisfied. Then*

$$R_{u,v}(q) = (q-1)^5 (q^2-q+1)^{\frac{inv(v)-inv(u)-7}{2}} (q^2+1)$$

Proof. It is an application of Proposition 2.4. ☐

5. Open problems and conjectures

In this last section we propose a natural prosecution of the investigation on the irreducible classes.

One problem is to discover if Theorem 2.9 permits to solve also the problem of irreducible class of 4-cycles, while the problem of two permutations "which differ" for 4-cycles nested in a transpositions is suggested by this paper and also by [10].

In these works we followed the idea that we can summarize in the "natural reordering of a pattern": suppose that a permutation u contains a pattern of type 21 (Theorem 3.4) or 312 or 231 (Corollary 4.4) and v is obtained from u as we have described, by applying the cycle which put the elements of the pattern in their natural order in u, then the R-polynomial associated to (u,v) factors nicely. It is now useful to recall the general definition of pattern of permutation:
A permutation $w = w_1 w_2 \dots w_n \in S_n$ **contains a pattern** $v \in S_k$, $k \leq n$, if there exists a sequence $w_{i_1} w_{i_2} \dots w_{i_k}$ with the same relative order as $v = v_1 v_2 \dots v_k$; usually we write $v_1, v_2, \dots, v_k$ a pattern v in w. See, e.g [13].
For example the permutation $w = 7245163$ contains a pattern 1234 which is 2456.

We say that a permutation w *contains a pattern v between two fixed positions i, j*, if there exists a sequence $w_{i_1} w_{i_2} \dots w_{i_k}$ with the same relative order as $v = v_1 v_2 \dots v_k$ and $i < i_1 < i_2 < \dots i_k < j$

Now we can express the "law of natural reordering" for pattern of length 4:

Conjecture 5.1. *Let $1 \leq i < a < b < c < d < j \leq n$ and $u, v \in S_n$, $u < y$, be one of the following pairs of permutations.*

1. *u contains a pattern of type 4123 between the positions i, j, $v = u(i,j)(a,b,c,d)$*
2. *u contains a pattern of type 3142 between the positions i, j, $v = u(i,j)(a,b,d,c)$*
3. *u contains a pattern of type 2413 between the positions i, j, $v = u(i,j)(a,c,d,b)$*

(4) *u contains a pattern of type 2341 between the positions i, j, $v = u(i,j)(a,d,c,b)$ where a, b, c, d are the indexes to which corresponds the fixed pattern. Then*

$$\tilde{R}_{u,v}(t) = t^6(1+t^2)^{(inv(v)-inv(u)-6)/2}$$

This has been tested using Maple C.A.S up to $n = 8$.

In the next proposition we observe that if the above conjecture is true then there other cases which will be covered:

Proposition 5.2. *Let $1 \leq i < a < b < c < d < j \leq n$ and $x, y \in S_n$, $x < y$, be one of the following pairs of permutations:*

(1) *x contains a pattern of type 4312 between the positions i, j and $y = x(i,j)(a,b,c,d)$*
(2) *x contains a pattern of type 3421 between the positions i, j and $y = x(i,j)(a,d,c,b)$*

where a, b, c, d are the indexes to which corresponds the fixed pattern. Then

$$\tilde{R}_{x,y}(q) = \tilde{R}_{u,v}(q)$$

where (u, v) is the pair of case (2) of Conjecture 5.1

Proof. If x contains a pattern of type 4312 between the positions i, j this means, by definition, that $x(i) < x(c) < x(d) < x(b) < x(a) < x(j)$ and $y = x(i,j)(a,b,c,d)$ then we consider the usual visualization of this paper:

$$\begin{array}{c} x \\ y \end{array} = \begin{array}{c} \ldots x(i) \ldots x(a) \ldots x(b) \ldots x(c) \ldots x(d) \ldots x(j) \ldots \\ \ldots x(j) \ldots x(b) \ldots x(c) \ldots x(d) \ldots x(a) \ldots x(i) \ldots \end{array}$$

We have that $(a, b) \in Inv(x) \cap Inv(y)$ and $x(a) > y(a) = x(b) > y(b)$; therefore we are under the conditions of Lemma 3.2: there exists $r \in [a+1, b-1]$ such that $\tilde{R}_{x,y}(q) = \tilde{R}_{\sigma,\tau}(q)$, where σ and τ are the following permutations:

$$\begin{array}{c} \sigma \\ \tau \end{array} = \begin{array}{c} \ldots x(i) \ldots x(a+1) \ldots x(r)\; x(b)\; x(a)\; x(r+1) \ldots x(c) \ldots x(d) \ldots x(j) \\ \ldots x(j) \ldots x(a+1) \ldots x(r)\; x(c)\; x(b)\; x(r+1) \ldots x(d) \ldots x(a) \ldots x(i) \end{array}$$

Now $(r+2, c) \in Inv(\sigma) \cap Inv(\tau)$ and again Lemma 3.2 can be applied. This implies that there exists $s \in [r+3, c-1]$ such that $\tilde{R}_{\sigma,\tau}(q) = \tilde{R}_{\alpha,\beta}(q)$, where

$$\begin{array}{c} \alpha \\ \beta \end{array} = \begin{array}{c} \ldots x(i) \ldots x(a+1) \ldots x(r)\; x(b)\; x(r+1) \ldots x(s)\; x(c)\; x(a)\; x(s+1) \ldots x(d) \ldots x(j) \\ \ldots x(j) \ldots x(a+1) \ldots x(r)\; x(c)\; x(r+1) \ldots x(s)\; x(d)\; x(b)\; x(s+1) \ldots x(a) \ldots x(i) \end{array}$$

We observe that α contains a pattern of type 3142 corresponding to the indexes $r+1, s+1, s+2, d$ and $\beta = \alpha(i,j)(r+1, s+1, d, s+2)$, as we wanted proof. By a similar application of Lemma 3.2 the case of x that contains a pattern of type 3421 follows. □

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