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# A numerical comparative study of completion methods for pairwise comparison matrices

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ABSTRACT

In the context of some multi-criteria decision-making methods, such as the Analytic Hierarchy Process, an expert is required to compare entities, e.g. alternatives and criteria. However, often, for various reasons, the expert cannot provide judgments on all pairs of entities. For these cases, several completion methods have been proposed in the literature to estimate the missing values of pairwise comparison matrices. In this paper, we study the similarity of eleven completion methods on the basis of numerical simulations and hierarchical clustering. We perform simulations for matrices of different orders considering various numbers of missing comparisons. Finally, the results suggest the existence of a cluster of five extremely similar methods, and a method significantly dissimilar from all the others.

## 1. Introduction

Pairwise comparisons are a pivotal concept in decision analysis and in modern multi-criteria decision-making methods [1] and theories such as the Analytic Hierarchy Process (AHP) and Multi-Attribute Value Theory (MAVT). While there is not an agreement on the optimal number of pairwise comparisons to be elicited from an expert, there is a meeting of minds on considering $(n-1)$ properly chosen comparisons and $n(n-1)/2$ to be the minimum and the maximum number, respectively. On the one hand, most scholars and analysts would frown upon using only $(n-1)$ comparisons since it is desirable to ask some additional questions. For instance, referring to the case of $(n-1)$ comparisons, Keeney and Raiffa [2] stated that "it may be desirable to ask additional questions" and Eisenführ et al. [3] that "it is sensible not to limit ourselves to the determination on $(n-1)$ tradeoffs ... but to deliberately create more than the minimum required number of trade-off measurements". On the other hand, it is also reasonable to assume that not all the possible $n(n-1)/2$ pairwise comparisons shall be elicited: Monte Carlo studies highlighted that half of them are enough to reach sufficient stability of the results [4].

For the above-mentioned reasons, we deduce that, for practical purposes, the extreme cases $(n-1)$ and $n(n-1)/2$ should hold a limited relevance. Hence, it is reasonable to envision that only a limited number of pairwise comparisons be elicited directly from the expert. At this point, the values of the remaining comparisons can be estimated using the existing ones. Such estimations can be also seen as recommendations to guide the expert in the elicitation process or in the revision process of his/her preferences.

When it comes to pairwise comparison matrices, some operations, for instance, inconsistency measurement and priority derivation have been studied from an axiomatic point of view. That is, reasonable properties were proposed in the form of axioms to define well-behaved inconsistency indices [5–8] and prioritization methods [9–11]. Remember that consistency basically means perfect coherence in judgments. On the contrary, it seems that only trivial axioms can be stipulated to formalize the concept of completion method. These axioms could be simple regularity conditions such as invariance under row–column permutations and transposition and the fact that if a matrix could be completed in a consistent way, then any sound completion method should do so. Thus, in the absence of a sufficiently rich axiomatic framework, it is even more important to resort to numerical studies to study the comparative behavior of different methods. Remarkably, at present, there is no evidence on how similar different completion methods are, and whether, from the pragmatic point of view, some of them can be used interchangeably without much variation in the results.

The goal of this study is to analyze completion methods for pairwise comparison matrices from a numerical point of view. We will use simulations to test the methods in some realistic cases. In particular, this analysis will highlight possible differences and similarities between methods. Our choice focused on eleven representative methods: the first three methods are optimization methods based on the minimization of the inconsistency, the next three methods are based on minimizations of biases/errors in judgments, other three methods are algorithms that are

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not based on optimization problems, and the last two are prioritization methods.

The paper is organized as follows. Section 2 presents the preliminary concepts of pairwise comparisons, inconsistency, completion methods, and related notation. Section 3 offers a self-contained presentation of eleven completion methods. The same eleven methods will be analyzed, by means of numerical simulations and hierarchical clustering, in Section 4. The main results will be discussed in Section 5. Finally, Section 6 concludes the paper.

## 2. Preliminaries

Given a non-empty finite set of $n$ alternatives or criteria, in a multi-criteria decision-making problem, a matrix $\mathbf{A} = (a_{ij})_{n\times n}$ with $a_{ij} > 0$ is called a (multiplicative) *pairwise comparison matrix* (PCM) if $a_{ji} = 1/a_{ij}\ \forall i, j$. In this context, the entry $a_{ij}$ represents the ratio between the relative importance of criterion/alternative $i$ over criterion/alternative $j$. A multiplicative PCM $\mathbf{A} = (a_{ij})_{n\times n}$ is said to be *consistent* if

$$a_{ij} = a_{ih}a_{hj} \text{ for } i, j, h = 1, 2, \dots, n, \tag{1}$$

otherwise, it is called *inconsistent*. As mentioned in the introduction, a consistent matrix corresponds to perfectly coherent judgments. Pairwise comparison matrices are the core of Saaty's AHP [12,13], where, in light of our cognitive limits, Saaty recommended to use a 1 to 9 comparison scale [14]. For this reason, unless otherwise stated, in the rest of the manuscript we will assume the restriction $a_{ij} \in [\frac{1}{9}, 9]$. However, it must be remarked that this pragmatic solution is not necessary from the formal point of view and results can be adapted by simply removing the scale restriction. Moreover, Saaty and Ozdemir [15] supported the idea that the number of entities to be compared should not exceed seven. The reason is due to the inconsistency of pairwise judgments among alternatives: when there are too many comparisons, inconsistencies are more spread out and it is difficult to localize and rectify them.

The second well-known type of matrices that helps derive a priority vector from the pairwise preferences is known as *reciprocal preference relations* [16]. A matrix $\mathbf{R} = (r_{ij})_{n\times n}$ is called a *reciprocal preference relation* if $r_{ij} \in [0, 1]$ and $r_{ji} = 1 - r_{ij},\ i, j = 1, 2, \dots, n$. Of course, it follows that $r_{ii} = 0.5\ \forall i$. Reciprocal preference relations are sometimes known as *fuzzy preference relations* [17–19], where the degree of preference between alternatives $i$ and $j$ is represented by $r_{ij}$. The values of $\mathbf{R}$ are interpreted as: $r_{ij} = 0.5$ indicates indifference between alternative $i$ and alternative $j$, $r_{ij} > 0.5$ indicates alternative $i$ is preferred to alternative $j$, $r_{ij} = 1$ indicates alternative $i$ is absolutely preferred to alternative $j$ and $r_{ij} = 0$ has the opposite meaning. A reciprocal preference relation $\mathbf{R} = (r_{ij})_{n\times n}$ is said to be (additively) *consistent* if the additive consistency property proposed by Tanino [16] is satisfied, i.e.,

$$r_{ij} = r_{ih} + r_{hj} - 0.5, \quad i, j, h = 1, \dots, n. \tag{2}$$

Fedrizzi [20] was the first to show that PCMs in the interval $[1/9, 9]$ and reciprocal preference relations in $[0, 1]$ can be seen as equivalent representations of preference. In fact, there exists a transformation function $f : [1/9, 9] \to [0, 1]$ such that

$$r_{ij} = f(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij}) \tag{3}$$

with inverse $a_{ij} = f^{-1}(r_{ij}) = 9^{2r_{ij}-1}$.

Another relevant and more general approach to preference representation was presented by Cavallo and D'Apuzzo [21] and refers to a general group structure. However, we will not deal with this approach: we focus on multiplicative PCMs, and reciprocal preference relations are only used for the third completion method M3 in Section 3. In this case, the transformation function (3) converts multiplicative PCMs into reciprocal preference relations, and again back to multiplicative PCMs after the implementation of method M3.

**Table 1**
Random consistency index $RI_n$ [24].

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| $RI_n$ | 0.8816 | 1.1086 | 1.2479 | 1.3417 | 1.4057 | 1.4499 | 1.4854 |

Expert's preference judgments are not always consistent. For instance, it is extremely rare for a pairwise comparison matrix of order 4 to be consistent using Saaty's scale $\{1/9, \dots, 1/2, 1, 2, \dots, 9\}$ [22]. Thus, it is inevitable that a moderate degree of inconsistency ought to be acceptable. As a result, a variety of consistency indices exist in literature [23]. It is worth noting that the terms *consistency index* and *inconsistency index* have been used interchangeably in the literature. In our study, we consider Saaty's consistency ratio [12,13], defined as

$$CR(\mathbf{A}) = \frac{CI(\mathbf{A})}{RI_n}$$

where

$$CI(\mathbf{A}) = \frac{\lambda_{\max}(\mathbf{A}) - n}{n - 1}$$

is Saaty's consistency index, $\lambda_{\max}(\mathbf{A})$ is the Perron–Frobenius eigenvalue of $\mathbf{A}$, and $RI_n$ is the random consistency index corresponding to matrix order $n$ as stated in Table 1, which is the average $CI$ of random matrices using Saaty's scale $\{\frac{1}{9}, \frac{1}{8}, \dots, \frac{1}{2}, 1, 2, \dots, 8, 9\}$.

To verify the set of acceptable PCMs, Saaty [12] proposed a 10% cut-off rule. That is, matrices with $CR < 0.1$ are acceptable. Otherwise, judgments should be revised until $CR < 0.1$. The higher $CR$, the more inconsistent the judgments in the PCMs. Furthermore, $\lambda_{\max}(\mathbf{A}) \geq n$, with $\lambda_{\max}(\mathbf{A}) = n$ if and only if $\mathbf{A}$ is consistent [13].

There exist many other inconsistency indices in the literature. For example, Koczkodaj's inconsistency index [25,26], denoted as $KI(\mathbf{A})$, was defined as

$$KI(\mathbf{A}) = \max\left\{1 - \min\left\{\frac{a_{ih}a_{hj}}{a_{ij}}, \frac{a_{ij}}{a_{ih}a_{hj}}\right\}, 1 \leq i < h < j \leq n\right\}. \tag{4}$$

### 2.1. Missing comparisons

In practical applications, one or more elements of a PCM could be missing due to several reasons, such as shortage of time, lack of information about a particular problem, a reluctance to make a direct comparison between two alternatives, or skipping some direct comparisons purposefully in order to make a better decision [27,28]. A PCM $\mathbf{A} = (a_{ij})_{n\times n}$ is said to be *incomplete* if some of its entries, denoted with $'*'$ in the following, are missing. Replacing the missing comparisons $'*'$ with variables $x_1, x_2, \dots, x_k \in \mathbb{R}_+$ in the upper triangular part of $\mathbf{A}$ and with their reciprocals in the lower triangular part, an example of the incomplete PCM $\mathbf{A}$ can be expressed in the form:

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}(x_1, x_2, \dots, x_k) = \begin{pmatrix} 1 & a_{12} & x_1 & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & a_{23} & \cdots & x_k \\ \frac{1}{x_1} & \frac{1}{a_{23}} & 1 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{1}{a_{1n}} & \frac{1}{x_k} & \frac{1}{a_{3n}} & \cdots & 1 \end{pmatrix}$$

where $k$ is the number of missing comparisons in the upper triangle of $\mathbf{A}$ [29]. Moreover, the structure of an incomplete PCM can be represented by an undirected graph $G := (V, E)$, where $V = \{1, 2, \dots, n\}$ represents the set of vertices (alternatives), and $E$ represents the set of undirected edges $\{i, j\}$ associated with the known entries of $\mathbf{A}$. Note that the adjacency matrix of graph $G$ can be obtained from matrix $\mathbf{A}$ by replacing the missing comparisons with zeros and the known comparisons with ones (except the diagonal entries which are replaced by zeros). In addition, the undirected graph $G$ is called *connected* if there is a path from any vertex to any other vertex in the graph. The connectedness of $G$ is a crucial property for our numerical simulations.

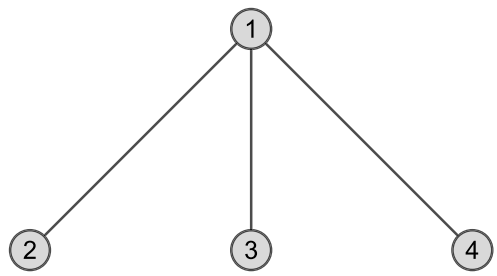


**Fig. 1.** Undirected graph representation of **A**.

**Example 1.** Let us consider a $4 \times 4$ incomplete PCM with three missing comparisons where the missing comparisons are represented by $'*'$ as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 & 9 \\ \frac{1}{3} & 1 & * & * \\ \frac{1}{2} & * & 1 & * \\ \frac{1}{9} & * & * & 1 \end{pmatrix}.$$

The undirected graph representation of this incomplete PCM is shown in Fig. 1.

## 3. Completion methods for incomplete pairwise comparison matrices

In this section, we present eleven selected completion methods called M1, ...,M11. The first nine methods have the common objective of completing incomplete preferences according to some suitable criteria. Hence, they can be considered completion methods in a *narrow* sense. On the contrary, the last two methods, M10 and M11, can be considered completion methods in a *broad* sense, as their goal is to find priority vectors of incomplete PCMs without completing them first. However, the obtained weights can also be used to estimate the values of the missing entries. We ruled out methods which could not satisfy some minimal requirements to make the analysis meaningful. For instance, we excluded the method proposed by Koczkodaj et al. [30] according to which one should complete a PCM in a way that minimizes the value of the inconsistency index $KI$. In fact, this optimization problem often presents infinitely many optimal solutions. For illustrative purposes, one can consider the following incomplete PCM $\mathbf{A}(x)$ with one missing comparison $x$,

$$\mathbf{A}(x) = \begin{pmatrix} 1 & x & 3 & 1 \\ 1/x & 1 & 1/2 & 4 \\ 1/3 & 2 & 1 & 5 \\ 1 & 1/4 & 1/5 & 1 \end{pmatrix}. \tag{5}$$

The analytic form of $KI$ of $\mathbf{A}(x)$ on $[1/9, 9]$, denoted $KI(\mathbf{A}(x))$, is,

$$KI(\mathbf{A}(x)) = \begin{cases} 1 - \frac{x}{6}, & 1/9 \le x < 0.4 \\ \frac{14}{15}, & 0.4 \le x \le 3.75 \\ 1 - \frac{1}{4x}, & 3.75 < x \le 9. \end{cases} \tag{6}$$

Fig. 2 shows that the minimization of $KI(\mathbf{A}(x))$ yields infinitely many optimal solutions, and thus the existence of a pitfall in the application of this method to the completion of preferences.[1]

[1] For sake of truth, there seem to be some recent research in overcoming the non-uniqueness of the optimization problem involving $KI$ (e.g., [31]) by using a lexicographic approach which is, however, different from the minimization of $KI$.

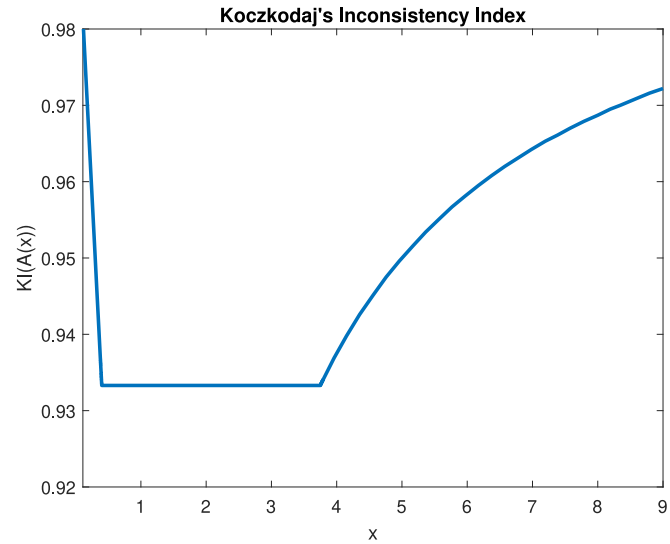


**Fig. 2.** Koczkodaj's inconsistency index $KI(\mathbf{A}(x))$ in (6).

Before describing the eleven completion methods, let us assume that $\mathbf{A} = (a_{ij})_{n \times n}$ is an incomplete PCM of order $n$ with $k$ missing comparisons $(x_1, \dots, x_k) = \mathbf{x}$.

### M1. $\lambda_{\max}$-based optimal completion method

A basic principle which is commonly followed in the estimation of missing comparisons is to proceed in such a way that the estimated entries of the matrix are as coherent as possible with the existing ones. Said so, given the prominence of $CR$ as a measure of inconsistency, the minimization of the Perron–Frobenius eigenvalue function ($\lambda_{\max}$) subject to interval constraints $[1/9, 9]$ has been considered a fair goal of the estimation process [32,33]. That is, missing entries are estimated as the argument solving the following optimization problem.

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}_+^k} \quad & \lambda_{\max}(\mathbf{A}(\mathbf{x})) \\ \text{s.t.} \quad & 1/9 \le x_p \le 9, \ p = 1, 2, \dots, k \end{aligned} \tag{7}$$

where $k$ is the number of missing comparisons and $\mathbf{x} = (x_1, \dots, x_k)$. Given the non-analytic nature of $\lambda_{\max}$, recent literature [33–35] has successfully focused on algorithms to solve (7).

### M2. $c_3$-based optimal completion method

Shiraishi et al. [36] considered the non-positive inconsistency index

$$c_3(\mathbf{A}) = \sum_{1 \le i < h < j \le n} \left( 2 - \frac{a_{ih} a_{hj}}{a_{ij}} - \frac{a_{ij}}{a_{ih} a_{hj}} \right)$$

which attains value 0 if and only if the PCM is consistent and is negative otherwise. That is, its polarity is inverted with respect to $CR$. As shown by Brunelli et al. [37], $c_3$ is proportional to another independently introduced index [38], and this double interpretation seems to support this measure. At this point, Obata et al. [39] proposed the maximization of $c_3$ subject to the interval constraint $[1/9, 9]$, i.e.

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}_+^k} \quad & c_3(\mathbf{A}(\mathbf{x})) \\ \text{s.t.} \quad & 1/9 \le x_p \le 9, \ p = 1, 2, \dots, k. \end{aligned} \tag{8}$$

The objective function $c_3$ is an analytic function of its variables and holds sufficient convexity properties to be easily solved with convex optimization methods [40].

### M3. $\rho$-based optimal completion method

Fedrizzi and Giove [17] considered the framework where preferences are expressed in the unit interval in the form of reciprocal preference relations and the consistency condition (2) as the ground to propose the following inconsistency index,

$$\rho(\mathbf{R}) = \sum_{i,h,j=1}^{n} (r_{ih} + r_{hj} - r_{ij} - 0.5)^2.$$

In light of the good properties of $\rho$, Fedrizzi and Giove [17] proposed its minimization to estimate the missing comparisons in $\mathbf{R}$. That is,

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}_+^k} \quad & \rho(\mathbf{R}(\mathbf{x})) \\ \text{s.t.} \quad & 0 \le x_p \le 1, \ p = 1, 2, \dots, k \end{aligned} \tag{9}$$

where $\mathbf{R} = (r_{ij})_{n\times n}$ is a reciprocal preference relation. Albeit formulated for reciprocal preference relations, this method can be easily applied to PCMs too: a PCM is mapped into its corresponding reciprocal preference relation by means of $f$ as in (3), optimization problem (9) is solved, and then $f^{-1}$ is used to map the result into a complete PCM $\mathbf{A} = (a_{ij})_{n\times n}$ with entries in $[1/9, 9]$.

### M4. A method of $\delta$-based local inconsistency indicator

To estimate the missing comparisons of an incomplete PCM, Ergu and Kou [41] and Ergu et al. [42] considered the consistency condition $a_{ij} = a_{ih}a_{hj} \ \forall i, j, h$ for PCMs. Starting from this condition they formulated a local index,

$$\delta_{ij}(\mathbf{A}) = \frac{1}{n}\sum_{h=1}^{n}(a_{ih}a_{hj} - a_{ij}) \ \ \forall i, j \tag{10}$$

which is equal to 0 if all the indirect comparisons of $i$ with $j$ through $h$ agree with the pairwise comparison $a_{ij}$. If we consider the incomplete PCM $\mathbf{A}(\mathbf{x})$, then we can call $\delta_{ij}(\mathbf{A}(\mathbf{x}))$ its indicator for the $(i, j)$th position. In light of these considerations, Ergu and Kou [41] proposed to solve the following optimization problem to estimate the missing entries:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}_+^k} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\delta_{ij}(\mathbf{A}(\mathbf{x}))\right)^2 \\ \text{s.t.} \quad & 1/9 \le x_p \le 9, \ p = 1, 2, \dots, k. \end{aligned} \tag{11}$$

### M5. $\epsilon$-based least absolute error (LAE) method

Ergu et al. [43] considered the well-known result stating that a PCM $\mathbf{A}$ is consistent if and only if there exists a (weight) vector $\mathbf{w} = (w_1, \dots, w_n)$ such that

$$a_{ij} = \frac{w_i}{w_j} \quad \forall i, j.$$

Moreover, in the consistent case, the $i$th component of $\mathbf{w}$ is related to the entries on the $i$th row of matrix $\mathbf{A}$ by means of the following function,

$$w_i = \left(\prod_{h=1}^{n} a_{ih}\right)^{\frac{1}{n}}.$$

This said, Ergu et al. [43] used this alternative characterization of consistency and proposed the following quantity

$$c_{ij} = \frac{\left(\prod_{h=1}^{n} a_{ih}\right)^{\frac{1}{n}}}{\left(\prod_{h=1}^{n} a_{jh}\right)^{\frac{1}{n}}} \cdot a_{ji}$$

to quantify the inconsistency related to the $(i, j)$th comparison. Following this formulation, and noting that in the consistent case $c_{ij} = 1$, Ergu et al. [44] proposed a local error,

$$\epsilon_{ij}(\mathbf{A}) = c_{ij} - 1 = \frac{\left(\prod_{h=1}^{n} a_{ih}\right)^{\frac{1}{n}}}{\left(\prod_{h=1}^{n} a_{jh}\right)^{\frac{1}{n}}} \cdot a_{ji} - 1 \quad \forall i, j \tag{12}$$

which is equal to 0 if all the indirect comparisons of $i$ with $j$ through $h$ agree with the pairwise comparison $a_{ij}$. If we consider the incomplete PCM $\mathbf{A}(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}_+^k$, the optimal values of the unknown variables are obtained by minimizing the sum of absolute values of the errors $\epsilon_{ij}(\mathbf{A}(\mathbf{x}))$. As a result, the optimization problem of Least Absolute Error (LAE) is written as

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}_+^k} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\epsilon_{ij}(\mathbf{A}(\mathbf{x}))\right| \\ \text{s.t.} \quad & 1/9 \le x_p \le 9, \ p = 1, 2, \dots, k. \end{aligned} \tag{13}$$

### M6. $\epsilon$-based least squares method (LSM)

The optimization problem of Least Squares Method (LSM) [44], which is also based on Eq. (12), can be written as follows by minimizing the sum of the *squares* of each error $\epsilon_{ij}(\mathbf{A}(\mathbf{x}))$ as a function of unknown variables $x_1, \dots, x_k$. That is,

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}_+^k} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\epsilon_{ij}(\mathbf{A}(\mathbf{x}))\right)^2 \\ \text{s.t.} \quad & 1/9 \le x_p \le 9, \ p = 1, 2, \dots, k. \end{aligned} \tag{14}$$

### M7. Connecting paths method

If $\mathbf{A}$ is an incomplete matrix of order $n$ and its associated graph is connected, it is natural to fill in the missing comparison $a_{ij}$ of $\mathbf{A}$ by taking the average of the intensities of all connecting paths that connect $i$ to $j$ [29]. Assume that $a_{ih}$ and $a_{hj}$ are the known comparisons for some index $h$. In the consistent situation, $a_{ij}$ can be written as $a_{ij} = a_{ih}a_{hj}$. The pair $a_{ih}$ and $a_{hj}$, $h \neq i, j$, is called an *elementary connecting path* of $a_{ij}$ [45]. In the general situation, $a_{ij}$ can be estimated by considering a path connecting the two alternatives/criteria $i$ and $j$. Let us denote this path by the index $r$ and its intensity by $CP(a_{ij})_r$:

$$CP(a_{ij})_r : \ a_{ij} = a_{i,h_1} \cdot a_{h_1,h_2} \cdots a_{h_t,j}, \tag{15}$$

where $i, j, h_1, \dots, h_t \in \{1, \dots, n\}, 1 \le t \le n-2$ [46]. According to Harker [29] and Chen and Triantaphyllou [46], the missing comparison $a_{ij}$ of the upper triangular part of incomplete PCM $\mathbf{A}$ could be estimated by taking the geometric mean of the intensities of all connecting paths of $a_{ij}$, i.e.,

$$a_{ij} = \left(\prod_{r=1}^{N} CP(a_{ij})_r\right)^{\frac{1}{N}} \tag{16}$$

where $CP(a_{ij})_r$ represents the intensity of the $r$th connecting path of $a_{ij}$, and $N$ is the total number of connecting paths. Finally, the missing comparisons of the lower triangular part of the matrix $\mathbf{A}$ are computed using reciprocity $a_{ji} = 1/a_{ij}$. This method is also known as *the geometric mean method* for missing comparisons estimation. As we take the interval $[1/9, 9]$ into account, if the estimated value $a_{ij}$ is out of the interval $[1/9, 9]$, readjustment of $a_{ij}$ will be made by considering: if $a_{ij} < 1/9$, then $a_{ij} = 1/9$, and if $a_{ij} > 9$, then $a_{ij} = 9$.

### M8. Alonso et al.'s method

An estimation function for incomplete PCMs [47] was adapted following the method for fuzzy preference relations introduced by Herrera-Viedma et al. [19]. To estimate the missing comparisons based on the iterative Algorithm 1 in [19], two different requirements must be met:

**(a)** Identify which missing comparisons can be estimated at each phase of the algorithm.
For a given incomplete PCM $\mathbf{A} = (a_{ij})_{n\times n}$, let us first consider the following expressions:

i. $V = \{(i,j) \mid i,j \in \{1,\dots,n\} \wedge i \neq j\}$, where $V$ represents the set of pairs of alternatives (pairwise comparisons) without taking the diagonal entries of $\mathbf{A}$ into account;
ii. $MV = \{(i,j) \in V \mid a_{ij}$ is unknown$\}$, where $MV$ represents the set of missing values (missing comparisons) of $\mathbf{A}$;
iii. $EV = V \setminus MV$, where $EV$ represents the set of estimated values (known comparisons) of $\mathbf{A}$ (except the diagonal entries);
iv. The set of intermediate alternatives $j$ when $i \neq k$: $H^1_{ik} = \{j \neq i,k \mid (i,j),(j,k) \in EV\}$; $H^2_{ik} = \{j \neq i,k \mid (j,k),(j,i) \in EV\}$; $H^3_{ik} = \{j \neq i,k \mid (i,j),(k,j) \in EV\}$.

Then, the set of missing comparisons that must be estimated in iteration $h$ of the algorithm (represented by $EMV_h$), the set of all known values (known comparisons) in iteration $h$ (denoted as $KV_h$), and the set of unknown values (missing comparisons) in iteration $h$ (denoted as $UV_h$) are computed as follows:

$$KV_h = EV \cup \left(\bigcup_{\ell=0}^{h-1} EMV_\ell\right); UV_h = MV \setminus \left(\bigcup_{\ell=0}^{h-1} EMV_\ell\right);$$

$$EMV_h = \{(i,k) \in UV_h \mid \exists j \in \{H^1_{ik} \cup H^2_{ik} \cup H^3_{ik}\}\}.$$

Note that $EMV_0 = \emptyset$, $KV_1 = EV$, and $EV$ will be replaced by $KV_h$ to calculate $H^1_{ik}$, $H^2_{ik}$ and $H^3_{ik}$ in iteration $h$ with updated PCM $\mathbf{A}$.

**(b)** Develop an expression for estimating a particular missing comparison $cp'_{ik}$ in iteration $h$:

$$cp'_{ik} = \left(\prod_{\ell\in K}\left(\prod_{j\in H^\ell_{ik}}^{n} ca^{j\ell}_{ik}\right)^{1/\#H^\ell_{ik}}\right)^{1/\#K} \tag{17}$$

where $K = \{\ell \in \{1,2,3\} \mid H^\ell_{ik} \neq \emptyset\}$, $ca^{j1}_{ik} = a_{ij}a_{jk}$, $ca^{j2}_{ik} = a_{jk}/a_{ji}$, and $ca^{j3}_{ik} = a_{ij}/a_{kj}$. Moreover, we restrict the values of $cp'_{ik}$ in the range $[1/9, 9]$: if $cp'_{ik} < 1/9$, then $a_{ik} = 1/9$; else if $cp'_{ik} > 9$, then $a_{ik} = 9$; else $a_{ik} = cp'_{ik}$.

Then, the estimate $cp'_{ik}$ of missing comparisons with $(i,k) \in EMV_h$ in iteration $h$ will be found using the function *estimate_p(i,k)* [47] as follows.

function *estimate_p(i,k)*

0. $K = \emptyset$
1.1. $H^1_{ik} = \{j \neq i,k \mid (i,j),(j,k) \in KV_h\}$; if $(\#H^1_{ik} \neq 0)$ then $K = K \cup \{1\}$
1.2. $H^2_{ik} = \{j \neq i,k \mid (j,k),(j,i) \in KV_h\}$; if $(\#H^2_{ik} \neq 0)$ then $K = K \cup \{2\}$
1.3. $H^3_{ik} = \{j \neq i,k \mid (i,j),(k,j) \in KV_h\}$; if $(\#H^3_{ik} \neq 0)$ then $K = K \cup \{3\}$
2. Calculate $cp'_{ik}$ in Eq. (17)
3. Restrict $cp'_{ik}$ in the range $[1/9, 9]$: if $cp'_{ik} < 1/9$, then $a_{ik} = 1/9$; else if $cp'_{ik} > 9$, then $a_{ik} = 9$; else $a_{ik} = cp'_{ik}$.

end function

Finally, the function *estimate_p(i,k)* calculates all the possible values $cp'_{ik}$ of the alternative $i$ over alternative $k$ using the three aforementioned possible expressions. The general algorithm for the missing comparisons estimation procedure in iteration $h$ is presented in Algorithm 1.

**Algorithm 1** Iterative algorithm in step $h$ [19]

0. $EMV_0 = \emptyset$
1. $h = 1$
2. while $EMV_h \neq \emptyset$ {
3. for every $(i,k) \in EMV_h$ {
4. *estimate_p(i,k)*
5. }
6. $h++$
7. }

The iterative algorithm terminates when $h$ reaches the maximum iteration, i.e., when the set of missing comparisons in iteration $h$ becomes empty (i.e. $EMV_{maxIter} = \emptyset$). Since we only consider connected graphs, the completion of all missing comparisons with this algorithm is successful and the algorithm stops after all the missing comparisons are obtained. For more details, see [47].

### M9. DEMATEL-based optimal completion method

A DEMATEL-based completion method for incomplete PCMs in AHP was proposed by Zhou et al. [48], which enables a decision-maker to provide a direct completion of the missing comparisons in the incomplete matrix. Furthermore, it employs multiplicative PCMs with no minimization process. In this method, the direct-relation matrix is the incomplete matrix itself by replacing all missing comparisons with zero. Generally, the method consists of four basic steps in order to fill in the incomplete PCMs on the interval $[1/9, 9]$:

**Step 1.** Convert the incomplete PCM $\mathbf{A} = (a_{ij})_{n\times n}$ into a direct-relation matrix $\mathbf{D}_r$ and compute the normalized matrix $\mathbf{N}$. The direct-relation matrix $\mathbf{D}_r$ is written as $\mathbf{D}_r = (d_{ij})_{n\times n}$, where $d_{ij} = a_{ij}$ if $a_{ij}$ is a known value and $d_{ij} = 0$ if $a_{ij}$ is an unknown value. The normalized matrix $\mathbf{N}$ can be obtained from $\mathbf{N} = \mathbf{D}_r/m$, where $m$ is the maximum of maximum row sum and maximum column sum of $\mathbf{D}_r$, i.e.,

$$m = \max\left\{\max_i \sum_{j=1}^{n} d_{ij}, \max_j \sum_{i=1}^{n} d_{ij}\right\}.$$

All elements of $\mathbf{N}$ are now in $[0, 1]$.

**Step 2.** Convert the direct-relation matrix $\mathbf{D}_r$ into total-relation matrix $\mathbf{T}_r$ using the formula

$$\mathbf{T}_r = \mathbf{N}(\mathbf{I} - \mathbf{N})^{-1}$$

where $\mathbf{I}$ is the identity matrix of order $n$, and $\mathbf{N}$ is the normalized matrix. Total-relation matrix consists of direct and indirect relations among influential factors, and it provides information on how one factor or criteria influences another. That is, if $R_i$ $(i = 1,\dots,n)$ is the sum of $i$th row of $\mathbf{T}_r$ and $C_i$ $(i = 1,\dots,n)$ is the sum of $i$th column of $\mathbf{T}_r$, then $R_i+C_i$ identifies the relevance of the $i$th influencing factor (i.e., the degree of relationship between each factor/criterion). Moreover, $R_i-C_i$ categorizes the $i$th influential factor/criterion as either a 'cause' $(R_i - C_i > 0)$ or an 'effect' $(R_i - C_i < 0)$ [48,49].

It is worth mentioning that the total-relation matrix $\mathbf{T}_r$ is derived from

$$\begin{aligned}\mathbf{T}_r &= \sum_{p=1}^{\infty} \mathbf{N}^p \\ &= \lim_{p\to\infty} (\mathbf{N} + \mathbf{N}^2 + \cdots + \mathbf{N}^p) \\ &= \lim_{p\to\infty} \mathbf{N}(\mathbf{I} + \mathbf{N} + \mathbf{N}^2 + \cdots + \mathbf{N}^{p-1}) \\ &= \lim_{p\to\infty} \mathbf{N}(\mathbf{I} - \mathbf{N})^{-1}(\mathbf{I} - \mathbf{N})(\mathbf{I} + \mathbf{N} + \mathbf{N}^2 + \cdots + \mathbf{N}^{p-1}) \\ &= \lim_{p\to\infty} \mathbf{N}(\mathbf{I} - \mathbf{N})^{-1}(\mathbf{I} - \mathbf{N}^p) \\ &= \mathbf{N}(\mathbf{I} - \mathbf{N})^{-1}\end{aligned} \tag{18}$$

because $\mathbf{N}^p$ converges to a zero matrix as $p \to \infty$.

**Step 3.** Transform the total-relation matrix $\mathbf{T}_r = (t_{ij})_{n\times n}$ into a complete PCM $\mathbf{C} = (c_{ij})_{n\times n}$ with:

$$c_{ij} = \sqrt{\frac{t_{ij}}{t_{ji}}} \text{ for } i < j.$$

Note that $\mathbf{C}$ satisfies the reciprocal condition $c_{ji} = 1/c_{ij}\ \forall i, j$, and also $c_{ii} = 1\ \forall i$.

**Step 4.** The missing comparisons $a_{ij}$ of the incomplete PCM $\mathbf{A}$ are estimated as $a_{ij} = c_{ij}\ (i, j = 1, 2, \ldots, n)$. Moreover, we restrict the values of missing comparisons to be in the interval $[1/9, 9]$: if $c_{ij} < 1/9$, then $a_{ij} = 1/9$, and if $c_{ij} > 9$, then $a_{ij} = 9$ by keeping the known entries of the incomplete PCM $\mathbf{A}$ unchanged. Now $\mathbf{A}$ will be the completed matrix from the method.

Note that, in Step 2, if $\mathbf{I}-\mathbf{N}$ is singular, the incomplete PCM will not be completed because the total-relation matrix $\mathbf{T}_r$ cannot be computed. In this case, the authors suggested that $\mathbf{T}_r = \mathbf{N}+\mathbf{N}^2+\cdots+\mathbf{N}^p$, where $p$ is a positive integer greater than or equal to 5 (in most cases), can be used to calculate the approximation of the total-relation matrix because the $CR$ of the completed matrix $\mathbf{A}$ decreases slowly as $p$ increases starting at $p = 5$. For more details, see [48].

### M10. Harker's eigenvalue method

Harker's method (eigenvalue method) [27] starts with the construction of an auxiliary matrix $\mathbf{B} = (b_{ij})_{n\times n}$ from an incomplete PCM $\mathbf{A} = (a_{ij})_{n\times n}$:

$$b_{ij} = \begin{cases} 1 + k_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \ \&\ a_{ij} = * \\ a_{ij}, & \text{otherwise} \end{cases}$$

where $k_i$ represents the number of missing entries on the $i$th row of $\mathbf{A}$ and $'*'$ indicates the missing entries. Finally, the method computes the maximum eigenvalue of $\mathbf{B}$ and its corresponding eigenvector $\mathbf{w} = (w_1, \ldots, w_n)^T$. Normalization of the eigenvector $\mathbf{w}$ (i.e., $\sum_{i=1}^n w_i = 1$) is important for its uniqueness. With the obtained weight vectors, the generic missing entry $a_{ij} = *$ is replaced by the ratio between the corresponding weights, i.e. $w_i/w_j$. Moreover, to make it comparable with the other methods, which use the bounds $[1/9, 9]$, we restrict the values of missing comparisons to be in the interval $[1/9, 9]$: if $a_{ij} < 1/9$, then $a_{ij} = 1/9$, and if $a_{ij} > 9$, then $a_{ij} = 9$ by keeping the known entries of the incomplete PCM $\mathbf{A}$ unchanged.

### M11. Incomplete logarithmic least squares method

The method of logarithmic least squares (LLS) for incomplete PCMs is similar to Harker's eigenvalue method, as the goal of both is to find priority vectors for incomplete PCMs. The LLS method for incomplete PCMs [32] determines the weight vector as the solution of the optimization problem

$$\begin{aligned} \min_{(w_1,\ldots,w_n)} \quad & \sum_{\{i,j\}\in E} \left(\log a_{ij} - \log \frac{w_i}{w_j}\right)^2 \\ \text{s.t.} \quad & \sum_{i=1}^n w_i = 1 \\ & \frac{1}{9} \le \frac{w_i}{w_j} \le 9, \quad i, j = 1, \ldots, n \end{aligned} \tag{19}$$

where, we recall, $E$ is the set of pairs of indices for which there is a comparison. Then, also in this case, the missing entries are obtained as ratios between their corresponding weights.

## 4. Numerical simulations and hierarchical clustering

A metric (Manhattan distance) is used to measure the closeness between two completed matrices:

$$D(\mathbf{L}, \mathbf{B}) = \| \log(\mathbf{L}) - \log(\mathbf{B}) \| = \sum_{i=1}^n \sum_{j=1}^n \left| \log(l_{ij}) - \log(b_{ij}) \right| \tag{20}$$

where $\mathbf{L} = (l_{ij})_{n\times n}$ and $\mathbf{B} = (b_{ij})_{n\times n}$ are the completed matrices obtained by applying two different methods to the same incomplete matrix $\mathbf{A} = (a_{ij})_{n\times n}$ of order $n$. Note that formula (20) is related to the formula of Mazurek et al. [50] with the exception of the logarithmic transformations. Although various distance formulas are provided in the literature, it is reasonable to apply the logarithm to each matrix element before computing the distance between two pairwise comparison matrices. This is because, for example, the distance between judgments $\frac{1}{8}$ and $\frac{1}{9}$ should be the same as that between judgments 8 and 9 [20, pp. 233–235]. Notice that log stands for natural logarithm.

The following notation will be used from now on:

- $D$ denotes the distance in Eq. (20);
- $D^*$ denotes the mean distance in Eq. (20) computed over a number of instances.

Since we are interested in calculating the mean distances and analyzing the closeness between the completed matrices, the smaller $D^*$ the more similar the results of two completion methods are.

Random consistent PCMs with a slight perturbation were generated to carry out the numerical simulations for $n = 4, \ldots, 8$ along with various numbers of missing comparisons. The random consistent matrices are modified by means of a random perturbation on $[1/9, 9]$ following a procedure similar to that used by [33]. That is, a consistent PCM $\mathbf{A} = (a_{ij})$ on $[1/9, 9]$ is generated by using $a_{ij} \leftarrow w_i/w_j$, where $(w_1, \ldots, w_n)$ is a randomly generated vector with $w_i \in [1, 9]$. Then modify the consistent matrix using a random perturbation $a_{ij} \leftarrow a_{ij} \cdot \beta$, $\beta \sim \text{Lognormal}(0, \sigma^2)$ with $\sigma = 0.7$, which results in inconsistent PCMs with a consistency ratio reasonably close to Saaty's threshold 0.1, i.e, the distribution of the $CR$ of the perturbed PCMs has expected value very close to 0.1. Once this is done for the upper triangle, the lower triangular part is obtained by reciprocity.

The process for randomly removing comparisons from a complete matrix to create an incomplete matrix of order $n$ is the following [33, 51]. We first remove one or more comparisons at random, independently and using a uniform distribution in the upper triangle, replacing them with unknowns. Then, starting with the upper triangle, we recreate the lower triangle to obtain the reciprocals of the unknowns. As a result, an incomplete PCM will be constructed. A test is performed in order to check that the corresponding undirected graph associated to each incomplete PCM is connected. It is worth noting that the connectedness of the associated undirected graph is a crucial property for the uniqueness of the solution for the optimization-based methods.

The procedure for calculating the mean distances $D^*$ of the completed matrices from different methods applied to the same incomplete matrix on the basis of connected graphs is as follows.

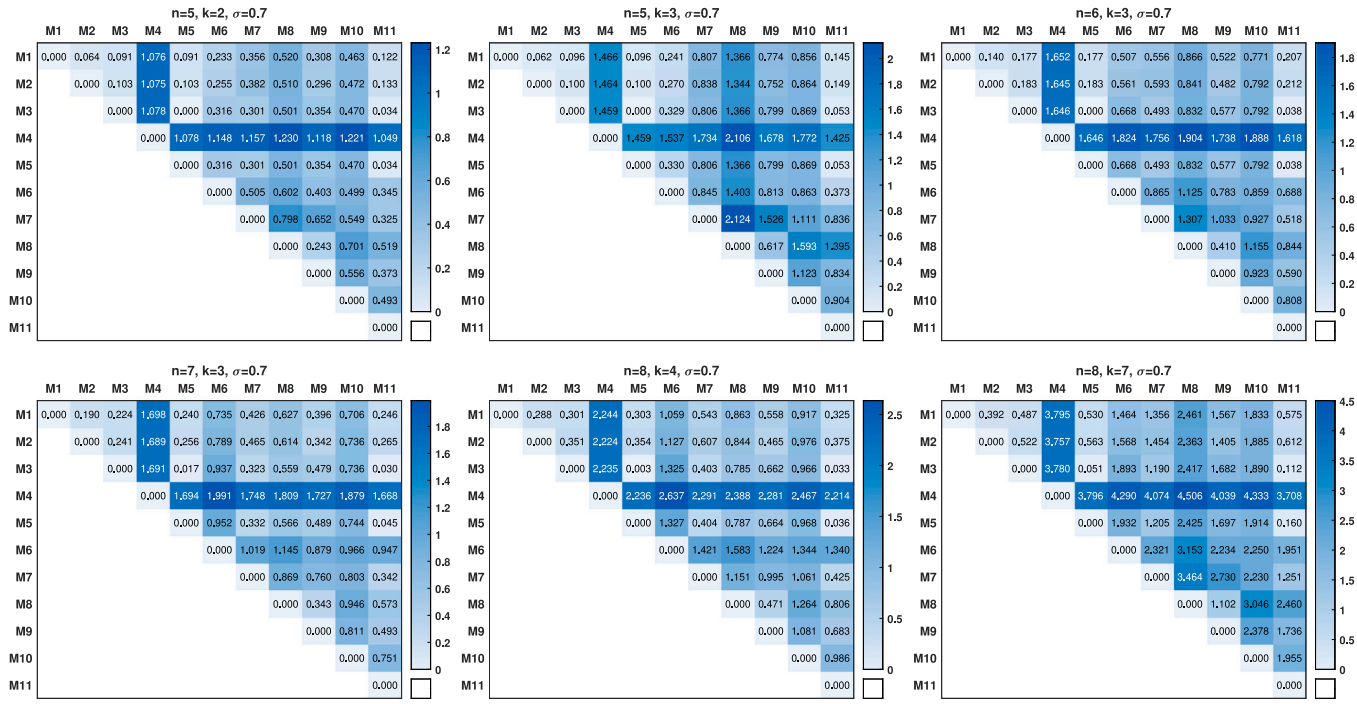


**Fig. 3.** Heatmaps and distances $D^*$ over 1000 simulations using $\sigma = 0.7$ for different values of $n$ and $k$.

(1) Fix the number of missing comparisons $k$ and matrix order $n$;
(2) Generate a random complete matrix (i.e., consistent matrix perturbed with $\sigma = 0.7$);
(3) Make the matrix incomplete at random, as described above;
(4) Select the incomplete matrix if its associated undirected graph is connected, and apply all methods M1–M11 on the same incomplete matrix. If the incomplete matrix is not connected, go to step (2);
(5) Compute, by means of (20), the distances $d_{st}$ between the completed matrices obtained by applying two different methods $M_s$ and $M_t$ on the same incomplete matrix. Distances $d_{st}$ form a matrix $\mathbf{D}$;
(6) Repeat steps (2)–(5) until 1000 matrices $\mathbf{D}$ are obtained;
(7) Calculate the mean distance matrix $\mathbf{D}^*$ among the 1000 matrices $\mathbf{D}$.

In addition, we apply an agglomerative hierarchical clustering algorithm using a single-linkage method [52,53], which is simple and suitable for our goal. The algorithm merges the two clusters that are the most similar, and keeps deleting rows and columns in the distance matrix so that old clusters are joined into new ones on the basis of the minimum distance. Formally, the value at which two clusters are merged corresponds to the minimum distance between their elements, i.e., two clusters $X$ and $Y$ are merged at $\min\{d^*_{st} | s \in X, t \in Y\}$ where $d^*_{st}$ is the distance between elements.

The basic steps of the hierarchical clustering are as follows. Given a distance matrix $\mathbf{D}^* = (d^*_{st})_{11\times 11}$, where the entries of the matrix are the mean distances $D^*$ between the 11 methods: $\mathrm{M}_s$ and $\mathrm{M}_t$ $(s, t = 1, \ldots, 11)$.

(1) Start with $m = 11$ clusters, with each object forming its own (single) cluster. Then $m$ decreases by 1 at each step;
(2) Join the two clusters that are the most similar using the minimum distance;
(3) Update the distance matrix;
(4) Repeat steps (2) and (3) until only one cluster remains.

## 5. Results

The average distances between methods are reported in Fig. 3 by means of heatmaps. The colorbar (on the rightmost side) describes the ranges of values used to color each heatmap cell (from light-blue to dark-blue in ascending order). Larger values are represented by dark-blue and smaller values by light-blue. For instance, in Fig. 3, the cells of the heatmap associated with method M4 contain greater values in the distance matrix and are colored with dark-blue in an L-shaped pattern in each subfigure. Note that the heatmaps only show the values in the upper triangle because the distance matrix is symmetric. In addition, for all the considered distances, the two methods M3 and M5 provide extremely similar results.

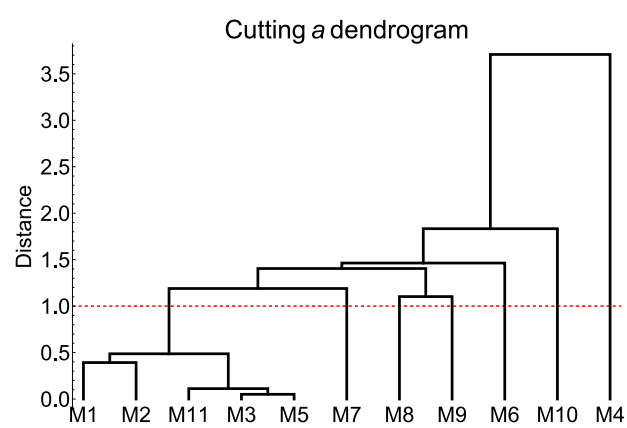


**Fig. 4.** An example of cutting a dendrogram into seven clusters: {M1,M2,M3,M5,M11}, {M7}, {M8,M9}, {M6}, {M10}, {M4}.

Hierarchical clustering of the eleven completion methods is presented in the form of dendrograms. The dendrogram is constructed in a bottom-up hierarchy (known as agglomerative hierarchical clustering). For instance, as shown in Fig. 4, the height of the link that connects M3 and M5 is the smallest, thus they are the most similar; {M3, M5} and M11 are the next closest (or most similar) methods; M1 and M2 are the third most similar methods; the set of methods {M1, M2, M3, M5, M11} is the fourth most similar; {M8, M9} is the fifth cluster of most similar methods, and so on.

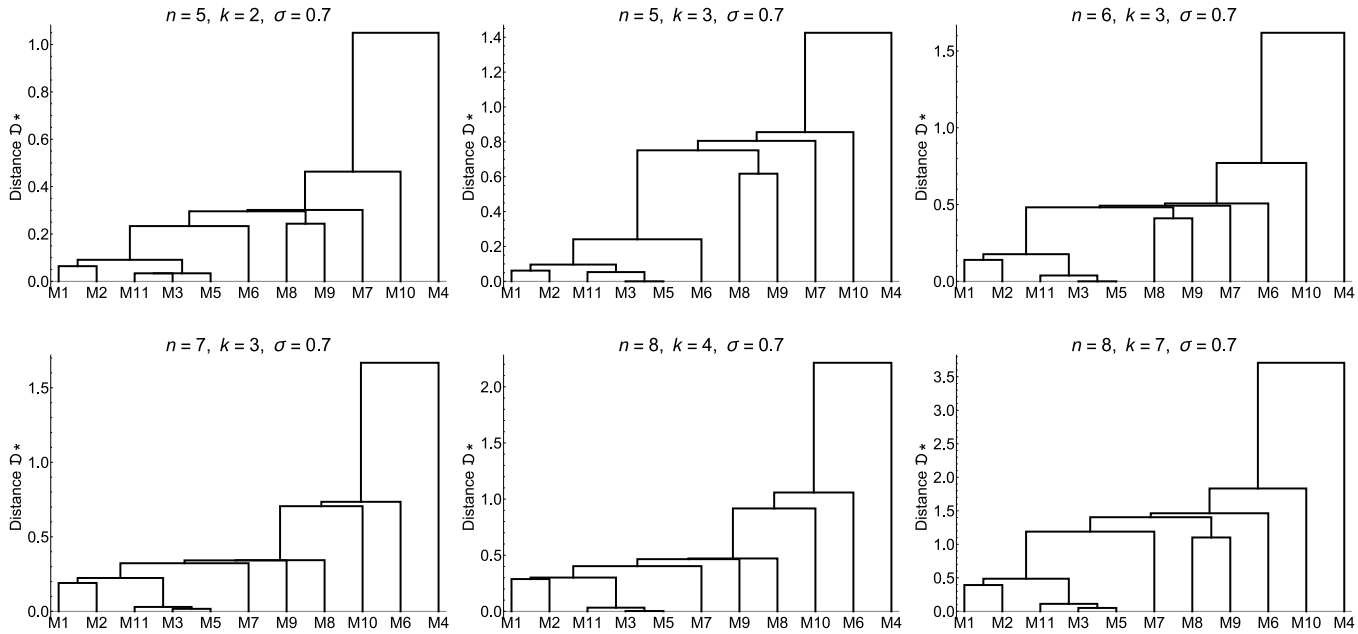


**Fig. 5.** Hierarchical clustering of the eleven completion methods corresponding to Fig. 3.

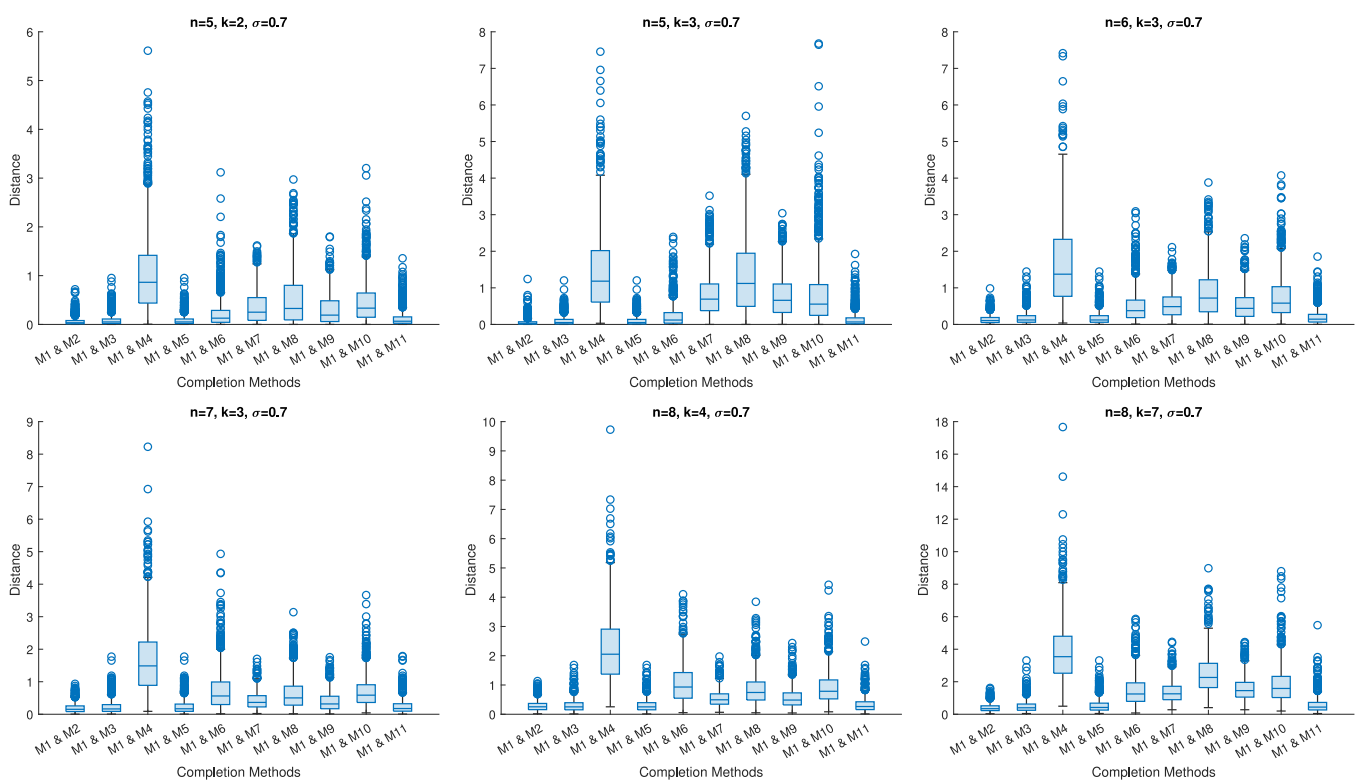


**Fig. 6.** The distribution of 1000 distance $D$ values obtained from the perturbed PCMs with $\sigma = 0.7$.

We performed simulations by further varying the order of the matrix and the number of missing comparisons. If we consider the results reported in Fig. 5, we see that if we cut all dendrograms into a constant of seven clusters, we always find five methods – M1, M2, M3, M5, M11 – in one cluster. That suggests that these five methods have a strong similarity, regardless of the order of the matrix or the number of missing entries, exception made for the very specific case with limited relevance $k = 1$: at $k = 1$ while $n$ increases, the five methods M3, M5, M7, M8 and M11 are the most similar. Conversely, the mean distances associated with method M4 are generally higher. In fact, in all the dendrograms, cluster M4 has the maximum height and this method can be considered an outlier.

We considered that, due to the possible existence of outliers, the median values can significantly differ from the mean distances reported in the heatmaps in Fig. 3. For this reason we analyzed the distributions of distances yielding the mean values reported in Fig. 3. As an example we consider the box plots of the method M1 (which corresponds to the minimization of $\lambda_{\max}$) compared to all the other methods for a varying number of missing comparisons and order of the matrix. That is, in

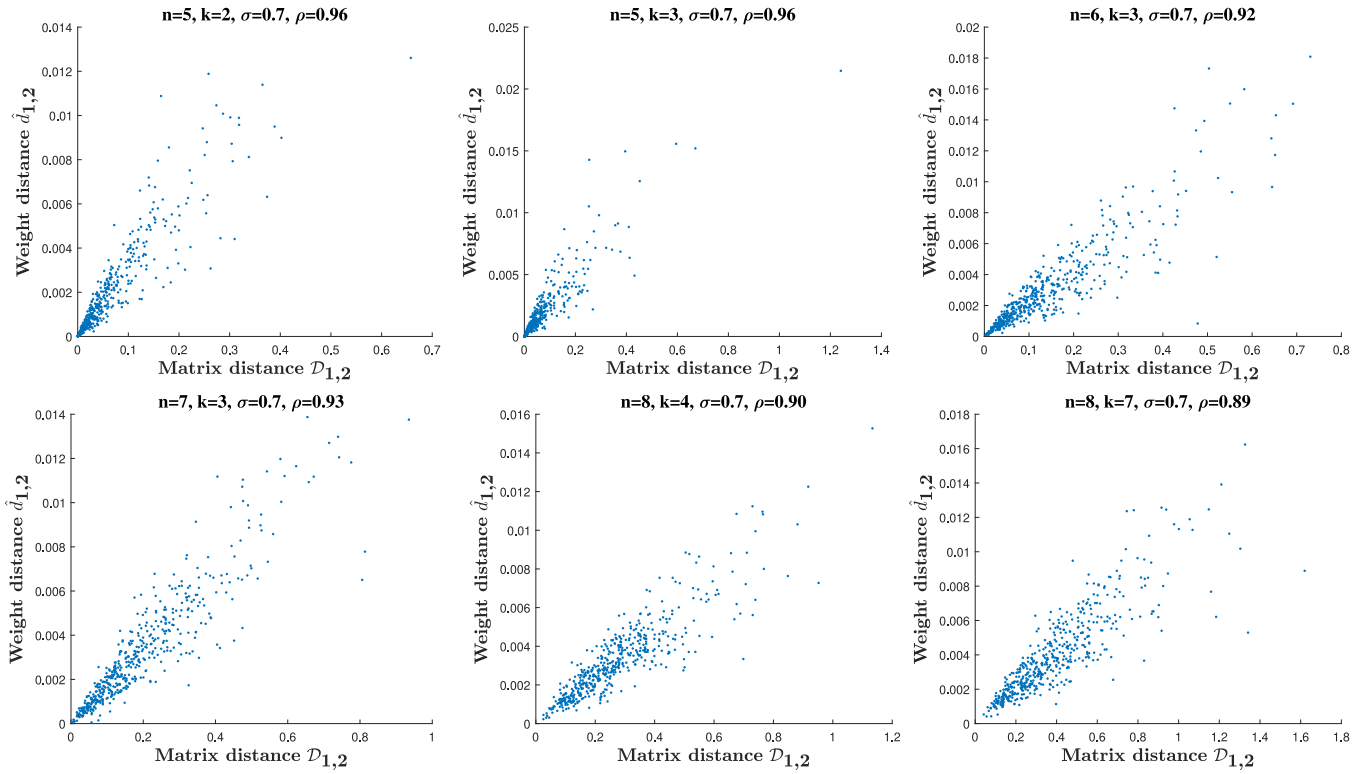


**Fig. 7.** The relation between distances from the completed PCMs ($\mathcal{D}_{1,2}$) and distances from weight vectors ($\hat{d}_{1,2}$) by taking the first 500 matrices out of 1000 perturbed PCMs with $\sigma = 0.7$ for the sake of visualization. The indices $\{1, 2\}$ indicate the methods M1 and M2, respectively. The value of $\rho$ indicates the Spearman rank correlation coefficient.

Fig. 6, we disaggregate the values obtained on the first rows of each heatmap in Fig. 3.

By doing so we note that the distances are not symmetrically distributed, but, conversely, they are right-skewed and therefore the average values reported in Fig. 3 act as upper bounds for the median values. Fig. 6 also shows that, in some specific cases, the two methods M1 and M4 can give very different results. This is actually true for every method compared to M4.

One further question may regard the extent to which the similarity between two completion methods is related with the similarity of the weight vectors extracted from the complete PCMs. We calculated the weight vectors (normalized eigenvectors) from the completed matrices $\mathbf{C}$ using the right eigenvector method:

$$\lambda_{\max}\mathbf{w} = \mathbf{C}\mathbf{w}$$

where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{C}$. Then, the distance between two weight vectors was calculated using the Manhattan distance $\hat{d}_{s,t}$:

$$\hat{d}_{s,t}(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} |u_i - v_i| \tag{21}$$

where $\mathbf{u} = (u_1, \dots, u_n)^T$ and $\mathbf{v} = (v_1, \dots, v_n)^T$ are weight vectors derived from the same completed PCM by applying two different methods $\mathrm{M}_s$ and $\mathrm{M}_t$ $(s, t = 1, 2, \dots, 11)$, whereas the distance between two completed matrices is represented by $\mathcal{D}_{s,t}$ from two different methods $\mathrm{M}_s$ and $\mathrm{M}_t$ (see Eq. (20)). Note that all weight vectors are normalized ($\sum_{i=1}^{n} u_i = 1$ and $\sum_{i=1}^{n} v_i = 1$) and derived from the completed matrices (after completion).

The relationship between the distances based on the completed matrices and the distances from the weight vectors can be seen in Figs. 7 and 8, which represent two extreme cases: M1 and M2 (the most similar), and M1 and M4 (dissimilar). Spearman's rank correlation coefficients indicate that, for both cases, the scatter plots have strong co-monotonic relationships.

We chose to use perturbed PCMs as they are closer to real-world decision makers' preferences, and we believe that, for this reason, an analysis carried out on them is more relevant. Nevertheless, we extended our results to the case of randomly generated matrices in the light of their higher degree of inconsistency, in order to see the nature of the results along with the increasing inconsistency. Fig. 9 reports instances of dendrograms obtained for randomly generated PCMs. In contrast to the results obtained on perturbed matrices, one can see that, as inconsistency increases, M6 (the method based on squared errors) diverges and it becomes an outlier, sometimes to an even greater extent than M4. On the other hand, in spite of their remarkably different formulation, M3 and M5 maintain a very high level of similarity.

Through numerical simulations, we have also verified that all the completion methods result in the same completed (and consistent) matrix when only considering incomplete PCMs corresponding to connected graphs, provided that the initial complete PCMs are all consistent.

All the computations and visualizations of the results were performed using MathWorks MatLab R2021b and Wolfram Mathematica 12.1.

## 6. Discussion and conclusions

The aim of this paper was to study the similarities of different completion methods and highlight possible differences based on numerical simulations and hierarchical cluster analysis. Eleven completion methods were analyzed utilizing a reasonable distance measure, and using matrices of order 4 to 8, with a varying number of missing comparisons. The first six methods are optimization-based, the next three are not optimization based, and the last two derive the priority vector from incomplete PCMs, and eventually complete the PCMs using ratios between some of the obtained weights. The first nine methods were selected based on their common purpose of estimating missing values in an incomplete matrix with respect to a suitable criterion related to

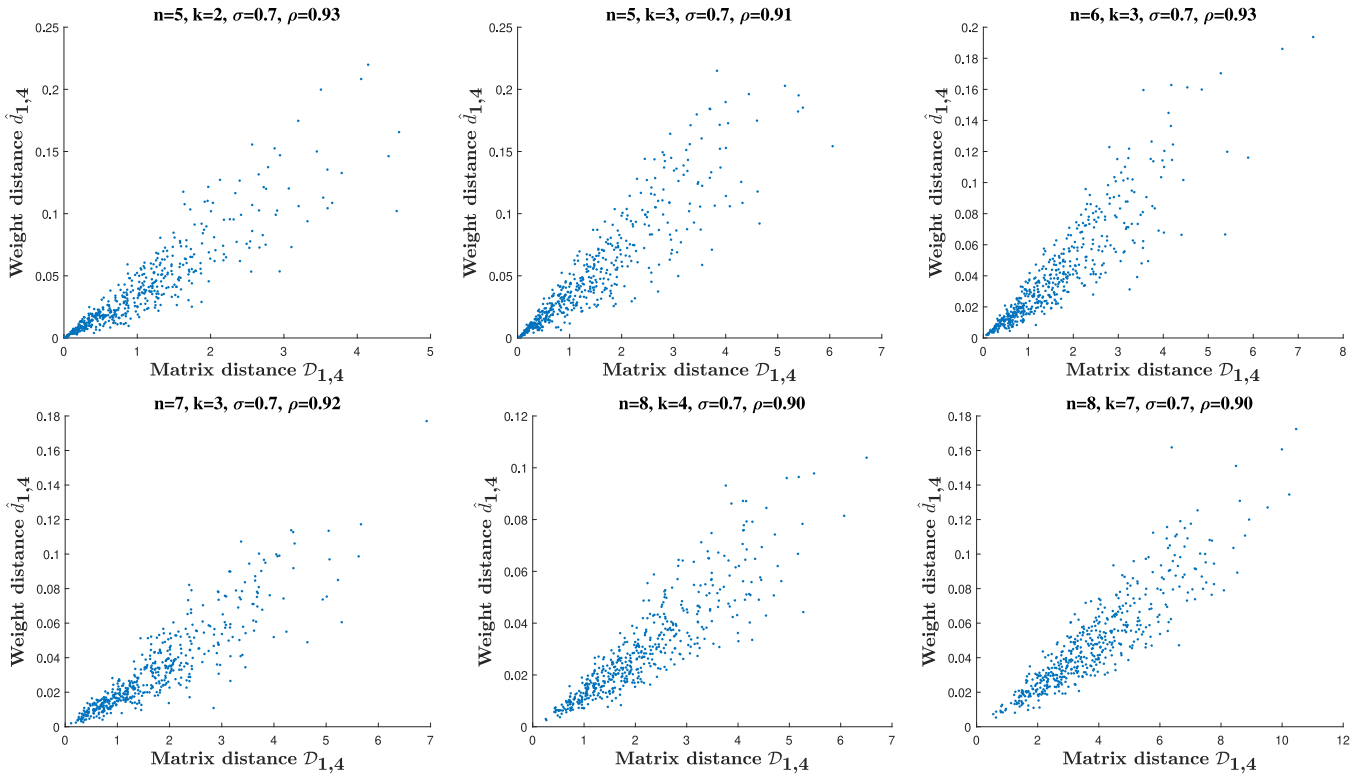


**Fig. 8.** The relation between distances from the completed PCMs ($\mathcal{D}_{1,4}$) and distances from weight vectors ($\hat{d}_{1,4}$) by taking the first 500 matrices out of 1000 perturbed PCMs with $\sigma = 0.7$ for the sake of visualization. The indices $\{1,4\}$ indicate the methods M1 and M4, respectively. The value of $\rho$ indicates the Spearman rank correlation coefficient.

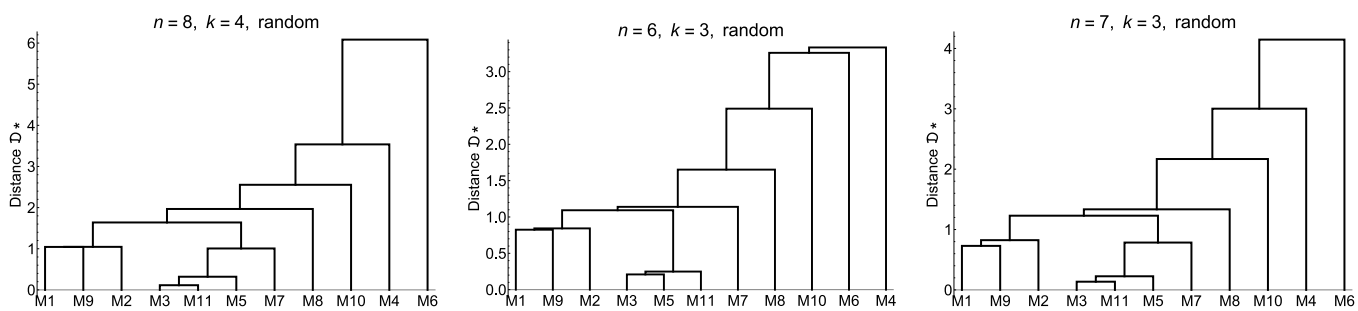


**Fig. 9.** Hierarchical clustering in the case of randomly generated PCMs.

cardinal consistency. The well-known method by Koczkodaj et al. [30], for example, was not considered because it fails to meet some minimal requirements. Likewise, other methods based on ordinal consistency, e.g. [54], were not considered either given their significantly different nature. On the other hand, we do not exclude that further research could compare these methods too.

Inconsistent pairwise comparison matrices close to a reasonable consistency were considered in the numerical simulations. Moreover, the undirected graphs associated with incomplete matrices were forced to be connected: some methods consider the graph's connectedness as a necessary condition for finding a unique solution.

The results of the hierarchical clustering indicate that M1, M2, M3, M5 and M11 are the most similar methods in terms of the minimal diversity among the completed matrices. From a practical perspective, it is safe to say that these methods can be used interchangeably and without significant differences in the output.

One question arises if we want to interpret the discrepancy of the results obtained by using M4. While a precise cause cannot be ascertained, it is reasonable to assume that it could be connected, at least to some extent, to the fact that biases, on which M4 is based, may differ very much from inconsistencies. Consider, for instance, the entry $a_{15} = 5$ of the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1/2 & 3 & 2 & 5 \\ 2 & 1 & 5 & 4 & 2 \\ 1/3 & 1/5 & 1 & 2 & 2 \\ 1/2 & 1/4 & 1/2 & 1 & 4 \\ 1/5 & 1/2 & 1/2 & 1/4 & 1 \end{pmatrix}.$$

If we compare its value with the values obtained by means of non-trivial indirect comparisons $a_{1j}a_{j5}\ \forall j \neq 1,5$ we obtain

$$a_{12}a_{25} - a_{15} = 1 - 5 = -4$$
$$a_{13}a_{35} - a_{15} = 6 - 5 = 1$$
$$a_{14}a_{45} - a_{15} = 8 - 5 = 3$$

which highlight the presence of inconsistencies. Nevertheless, when they are summed up according to Eq. (10) they yield a null bias, as errors exist but they cancel each other out. This discrepancy between the concepts of inconsistency and bias makes the choice of the completion method even more important and a possible direction for future research. We also consider that the reliability of the weight vector

depends on the method with which it is extracted from a PCM, and we note that, for incomplete PCMs this reliability may also depend on the completion method used before the weights are extracted from the complete PCM. An analysis of the completion methods in light of the reliability of the weight vector may be a direction for future research.

## CRediT authorship contribution statement

**Hailemariam Abebe Tekile:** Concept, Simulation, Analysis, Writing or revision of the manuscript. **Matteo Brunelli:** Concept, Simulation, Analysis, Writing or revision of the manuscript. **Michele Fedrizzi:** Concept, Simulation, Analysis, Writing or revision of the manuscript.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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