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A refined Lusin type theorem for gradients

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ABSTRACT

We prove a refined version of the celebrated Lusin type theorem for gradients by Alberti, stating that any Borel vector field f coincides with the gradient of a C^1 function g , outside a set E of arbitrarily small Lebesgue measure. We replace the Lebesgue measure with any Radon measure μ , and we obtain that the estimate on the L^p norm of Dg does not depend on $\mu(E)$, if the value of f is μ -a.e. orthogonal to the decomposability bundle of μ . We observe that our result implies the 1-dimensional version of the flat chain conjecture by Ambrosio and Kirchheim on the equivalence between metric currents and flat chains with finite mass in $\mathbb{R}^n$ and we state a suitable generalization for k -forms, which would imply the validity of the conjecture in full generality.

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1. Introduction

In this note, we improve the *Lusin type theorem for gradients* by Alberti [1, Theorem 1] and its generalization for Radon measures [11, Theorem 2.1], by proving the following result. We recall that the decomposability bundle of a Radon measure μ on $\mathbb{R}^n$, is a Borel map $V(\mu, \cdot) : \mathbb{R}^n \rightarrow \bigcup_{k \leq n} Gr(k, n)$, taking as values vector subspaces of $\mathbb{R}^n$, see

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Section 2 for the definition. Given a Borel map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we denote by f_V the projection of f on $V(\mu, \cdot)$, that is, the map $f_V : x \mapsto P_{V(\mu, x)}f(x)$, where, for every subspace L of $\mathbb{R}^n$, P_L denotes the orthogonal projection onto L . Similarly $f_{V^\perp}$ denotes the projection onto the orthogonal complement of $V(\mu, \cdot)$.

Theorem 1.1. *Let μ be a Radon measure on $\mathbb{R}^n$, let $\Omega \subset \mathbb{R}^n$ be an open set such that $\mu(\Omega) < \infty$ and let $f : \Omega \rightarrow \mathbb{R}^n$ be a Borel map. There exists a constant $C(n)$ such that for every $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ and a function $g : \Omega \rightarrow \mathbb{R}$ of class C^1 such that,*

$$\mu(\Omega \setminus K) < \varepsilon,$$

$$Dg(x) = f(x) \quad \forall x \in K,$$

$$\|Dg\|_{L^p(\Omega, \mu)} \leq C\varepsilon^{\frac{1}{p}-1} \|f_V\|_{L^p(\Omega, \mu)} + (1 + \varepsilon) \|f_{V^\perp}\|_{L^p(\Omega, \mu)} \quad \text{for every } p \in [1, \infty].$$

Remark 1.2.

- (i) The main result of [1] corresponds to the choice in which μ is the Lebesgue measure. In this case $V(\mu, \cdot) \equiv \mathbb{R}^n$, so that the second addendum in the estimate for $\|Dg\|_{L^p(\Omega, \mu)}$ vanishes.
- (ii) Our result improves also Theorem 2.1 of [11], providing an estimate for $\|Dg\|_{L^p(\Omega, \mu)}$ which remains bounded when $\varepsilon \rightarrow 0$, if $f(x)$ is orthogonal to $V(\mu, x)$, for μ -a.e. $x \in \Omega$.
- (iii) In the latter case, since the Lipschitz constant of g does not depend on ε , one could be tempted to conclude that a stronger conclusion holds. That is, there exists a Lipschitz function g such that $D_{f(x)}g(x) = 1$, for μ -a.e. $x \in \Omega$. However, this conclusion is, in general, false. In fact, the natural choice for g could be a subsequential limit as $\varepsilon \rightarrow 0$ of the functions g_ε (depending on ε) produced by Theorem 1.1, but being the limit g only Lipschitz, its directional derivative in direction $f(x)$ could not exist, because $f(x) \notin V(\mu, x)$ for μ -a.e. x , see [3, Theorem 1.1 (ii)]. Even in the case in which $D_{f(x)}g(x)$ exists at μ -a.e. x , the conclusion above would still not be legitimate because this directional derivative is not closable from the space of Lipschitz functions to $(L^\infty(\Omega, \mu), w^*)$, see Theorem 1.3 of [2].
- (iv) While the sharpness of the bound $C\varepsilon^{\frac{1}{p}-1} \|f_V\|_{L^p(\Omega, \mu)}$ is discussed in [1, Remark 4, Proposition 5], it is natural to ask whether $(1 + \varepsilon) \|f_{V^\perp}\|_{L^p(\Omega, \mu)}$ can be replaced by $\|f_{V^\perp}\|_{L^p(\Omega, \mu)}$ in the conclusion of Theorem 1.1. The answer is negative, as shown by the following example in $\Omega = (0, 2)^2 \subset \mathbb{R}^2$:

$$\mu = \mathcal{H}^1 \llcorner ((0, 2) \times \{1\}) + \delta_{(1,1)}, \quad f(x, y) = \begin{cases} (0, 1) & \text{if } x \in (0, 2) \setminus \{1\}, y = 1 \\ (0, 2) & \text{if } (x, y) = (1, 1) \\ (0, 0) & \text{otherwise.} \end{cases}$$

In fact, let us fix $\varepsilon \in (0, 1)$ and any K, g satisfying the first two conclusions of Theorem 1.1. Since $\mu(\{D_2g = 1\}) > 2 - \varepsilon$ and $D_2g(1, 1) = 2$, it holds

$$\|Dg\|_{L^p(\Omega, \mu)}^p \geq 2 - \varepsilon + \int_{x_0}^1 |D_2g(x, 1)|^p dx + 2^p \geq 2 - \varepsilon + (1 - x_0) \left(\frac{3}{2}\right)^p + 2^p,$$

where $x_0 := \min\{x \in (0, 1) : D_2g(t, 1) \geq \frac{3}{2} \forall t \in (x, 1]\}$, which is strictly less than 1 by continuity of D_2g . On the other hand

$$\|f\|_{L^p(\Omega, \mu)}^p = 2 + 2^p,$$

thus

$$\|Dg\|_{L^p(\Omega, \mu)}^p - \|f\|_{L^p(\Omega, \mu)}^p \geq -\varepsilon + (1 - x_0) \left(\frac{3}{2}\right)^p$$

which is positive for p sufficiently large.

In Section 4, we state a Conjecture which is a suitable generalization of Theorem 1.1 for k -forms and we show that its validity implies the truth of the Ambrosio-Kirchheim flat chain conjecture, see [5], modifying a strategy outlined in [10]. In particular, we observe that Theorem 1.1 implies the validity of the conjecture in dimension $k = 1$.

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2. Notation and preliminaries

We denote by $\mathcal{L}^n$ the standard Lebesgue measure on $\mathbb{R}^n$ and by $\mathcal{H}^s$ the s -dimensional Hausdorff measure on $\mathbb{R}^n$.

For any (possibly vector-valued) Radon measure μ on $\mathbb{R}^n$ we denote by $|\mu|$ its total variation measure and by $\mathbb{M}(\mu) := |\mu|(\mathbb{R}^n)$. For any Borel set $E \subset \mathbb{R}^n$, we denote by $\mu \llcorner E$ the restriction of μ to E , that is, the measure $A \mapsto \mu(E \cap A)$, for every Borel set A .

If $e \in \mathbb{S}^{n-1}$ and $\alpha \in [0, \frac{\pi}{2}]$, we denote by $C(e, \alpha)$ the convex cone with axis e and opening angle α , namely

$$C(e, \alpha) := \{x \in \mathbb{R}^n : x \cdot e \geq \cos \alpha |x|\}.$$

We recall the following definition; see §2.6, §6.1 and Theorem 6.4 of [3].

Definition 2.1 (*Decomposability bundle*). Given a positive Radon measure μ on $\mathbb{R}^n$ its *decomposability bundle* is any Borel map $V(\mu, \cdot)$ taking values in the set $\bigcup_{k \leq n} Gr(k, n)$, that is, in the union of the Grassmannians of vector subspaces of $\mathbb{R}^n$ of any dimension, with the following property. For μ -a.e. $x \in \mathbb{R}^n$, a vector $v \in \mathbb{R}^n$ belongs to $V(\mu, x)$ if and only if there exists a vector-valued measure T with $\operatorname{div} T = 0$ such that

$$\lim_{r \rightarrow 0} \frac{\mathbb{M}((T - v\mu) \llcorner B(x, r))}{\mu(B(x, r))} = 0.$$

During the proof of the main result, we are going to use the notion of C -null sets and two lemmas proved in [3]; we recall them for the reader's convenience.

The notion of $C(e, \alpha)$ -null sets formalizes the idea of sets that are not seen by Lipschitz curves with tangent vector in $C(e, \alpha)$.

Definition 2.2 (*C -null sets*). Given a cone $C = C(e, \alpha)$ in $\mathbb{R}^n$, a set $E \subset \mathbb{R}^n$ is C -null if, for any Lipschitz curve $\gamma: [0, 1] \rightarrow \mathbb{R}^n$ satisfying $\dot{\gamma}(t) \in C$ for $\mathcal{L}^1$ -a.e. $t \in [0, 1]$, it holds

$$\mathcal{H}^1(E \cap \gamma([0, 1])) = 0.$$

If the tangent bundle of a measure μ has trivial intersection with a cone C , then there exists a C -null set of full μ -measure, as stated by the following lemma.

Lemma 2.3 ([3, Lemma 7.5]). Let a Borel set B in $\mathbb{R}^n$, a cone $C = C(e, \alpha)$ in $\mathbb{R}^n$ and a measure μ be given such that $V(\mu, x) \cap C = \{0\}$ for μ -a.e. $x \in B$. Then there exists a C -null Borel set F contained in B such that $\mu(B \setminus F) = 0$.

For a $C(e, \alpha)$ -null compact set K , the “width” of K seen by Lipschitz curves in the directions of $C(e, \alpha)$ is null by definition. It is then natural to expect the existence of smooth functions of arbitrarily small size which grow with derivative 1 in the direction e in K and are almost constant in the other directions, depending on the angle α . This is precisely the content of the following result.

Lemma 2.4 ([3, Lemma 4.12]). Let be given a cone $C = C(e, \alpha)$ in $\mathbb{R}^n$ and a C -null compact set K in $\mathbb{R}^n$. Then, for every $\varepsilon > 0$, there exists a smooth function $w: \mathbb{R}^n \rightarrow \mathbb{R}$ such that, for every $x \in \mathbb{R}^n$, the following hold:

- (i) $0 \leq w(x) \leq \varepsilon$;
- (ii) $0 \leq D_e w(x) \leq 1$, and $D_e w(x) = 1$ if $x \in K$, where $D_e w(x)$ is the directional derivative of w in the direction e at point x ;
- (iii) $|d_Z w(x)| \leq \frac{1}{\tan \alpha}$, where $Z := e^\perp$, $d_Z w(x)$ is the derivative of w at x with respect to the subspace Z and $|d_Z w(x)|$ is its operator norm.

3. Proof of the main result

Thanks to [11, Theorem 2.1], in order to prove Theorem 1.1, it is sufficient to prove the following result.

Theorem 3.1. *Let μ be a Radon measure on $\mathbb{R}^n$, let $\Omega \subset \mathbb{R}^n$ be an open set such that $\mu(\Omega) < \infty$ and let $f: \Omega \rightarrow \mathbb{R}^n$ be a Borel function satisfying $f(x) \in V(\mu, x)^\perp$ for μ -a.e. $x \in \Omega$. For every $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ and a function $g: \Omega \rightarrow \mathbb{R}$ of class C^1 such that*

$$\begin{aligned} \mu(\Omega \setminus K) &< \varepsilon, \\ Dg(x) &= f(x) \quad \forall x \in K, \\ \|Dg\|_{L^p(\Omega, \mu)} &\leq (1 + \varepsilon)\|f\|_{L^p(\Omega, \mu)}, \quad \text{for every } p \in [1, \infty]. \end{aligned}$$

The proof is based on the following lemmas.

Lemma 3.2 (quantitative version of Lusin's theorem). *Let μ be a Radon measure on $\mathbb{R}^n$, let $\Omega \subset \mathbb{R}^n$ be an open set such that $\mu(\Omega) < \infty$ and let $f: \Omega \rightarrow \mathbb{R}^n$ be a Borel function. Then, for every $\varepsilon > 0$, there exists $h \in C_c(\Omega, \mathbb{R}^n)$ such that*

$$\begin{aligned} \mu(\{x \in \Omega: h(x) \neq f(x)\}) &< \varepsilon, \\ \|h\|_{L^p(\Omega, \mu)} &\leq (1 + \varepsilon)\|f\|_{L^p(\Omega, \mu)} \quad \forall p \in [1, +\infty]. \end{aligned}$$

Remark 3.3. Notice that h depends only on f and ε and does not depend on p .

Proof of Lemma 3.2. Of course, we can assume $f \not\equiv 0$, otherwise there is nothing to prove, and $\varepsilon < 1$. Moreover, we can assume that $f \in L^p(\Omega, \mu)$ for some $p \in [1, +\infty]$, otherwise the thesis is just the classical Lusin's theorem for Radon measures [9, 2.3.5 - 2.3.6], up to performing a cut-off of the resulting continuous function in a small neighborhood of $\partial\Omega$ (see part 3 of this proof).

For every $t > 0$ we call

$$E(f, t) := \{x \in \Omega: |f(x)| \geq t\}.$$

- **1 - Truncation.** We first observe that, since $f \in L^p(\Omega, \mu)$ for some $p \in [1, +\infty]$, there exists $t > 0$ such that either $0 < \mu(E(f, t)) < \frac{\varepsilon}{3}$ or

$$\mu(E(f, t)) \geq \frac{\varepsilon}{3} \quad \text{and} \quad \mu(E(f, s)) = 0 \quad \forall s > t. \quad (3.1)$$

In either case, we call $E := E(f, t)$ and we construct f_1 by projecting f on the sphere of radius t , that is, we define

$$f_1(x) = \begin{cases} f(x) & \text{if } |f(x)| < t \\ t \frac{f(x)}{|f(x)|} & \text{if } |f(x)| \geq t. \end{cases}$$

Clearly $\|f_1\|_{L^\infty(\Omega, \mu)} = t$ and, since $|f_1(x)| \leq |f(x)|$ for every $x \in \Omega$, we have

$$\|f_1\|_{L^p(\Omega, \mu)} \leq \|f\|_{L^p(\Omega, \mu)} \quad \forall p \in [1, +\infty].$$

Moreover $f_1 = f$ μ -a.e. in case (3.1) holds, whereas

$$\mu(\{x \in \Omega: f_1(x) \neq f(x)\}) \leq \mu(E) < \frac{\varepsilon}{3}, \quad (3.2)$$

in the other case.

- **2 - Continuous approximation.** Choosing $0 < \eta < \varepsilon \cdot \min\{\mu(E), \frac{\varepsilon}{3}\}$, by Lusin's theorem for Radon measures [9, 2.3.5 - 2.3.6] there exists a continuous function $f_2: \Omega \rightarrow \mathbb{R}$ such that the set $F := \{x \in \Omega: f_1(x) \neq f_2(x)\}$ satisfies

$$\mu(F) < \eta.$$

Up to projecting on $B_t(0)$, we can assume $\|f_2\|_{C(\Omega)} \leq t$, thus

$$\|f_2\|_{L^\infty(\Omega, \mu)} \leq \|f_2\|_{C(\Omega)} \leq t = \|f_1\|_{L^\infty(\Omega, \mu)} \leq \|f\|_{L^\infty(\Omega, \mu)}. \quad (3.3)$$

We now estimate the difference $\|f_2\|_{L^p(\Omega, \mu)} - \|f\|_{L^p(\Omega, \mu)}$ uniformly for $p \in [1, +\infty)$. We have

$$\int_{\Omega} |f_2(x)|^p d\mu(x) \leq \int_{\Omega \setminus (F \cup E)} |f(x)|^p d\mu(x) + t^p \mu(F \cup E),$$

while

$$\int_{\Omega} |f(x)|^p d\mu(x) \geq \int_{\Omega \setminus (F \cup E)} |f(x)|^p d\mu(x) + t^p \mu(E).$$

By these estimates, using $\mu(F) < \varepsilon \mu(E)$ and the relation (where the first inequality is due the fact that incremental ratios of the function $s \mapsto s^{\frac{1}{p}}$ are decreasing since it is concave, while the second one follows by $p \geq 1$)

$$(a + (1 + \varepsilon)b)^{\frac{1}{p}} - (a + b)^{\frac{1}{p}} \leq b^{\frac{1}{p}} [(1 + \varepsilon)^{\frac{1}{p}} - 1] \leq \varepsilon b^{\frac{1}{p}} \quad \forall a, b \geq 0, \quad \forall p \in [1, +\infty),$$

we obtain

$$\|f_2\|_{L^p(\Omega, \mu)} - \|f\|_{L^p(\Omega, \mu)} \leq \varepsilon t \mu(E)^{\frac{1}{p}} \leq \varepsilon \|f\|_{L^p(\Omega, \mu)} \quad \forall p \in [1, +\infty]. \quad (3.4)$$

Gathering (3.3) and (3.4) we get

$$\|f_2\|_{L^p(\Omega, \mu)} \leq (1 + \varepsilon) \|f\|_{L^p(\Omega, \mu)} \quad \forall p \in [1, +\infty]. \quad (3.5)$$

We moreover record that the choice of η , (3.2) and the initial assumption $\varepsilon < 1$ imply

$$\mu(\{x \in \Omega: f_2(x) \neq f(x)\}) < \frac{2}{3}\varepsilon. \quad (3.6)$$

- **3 - Cut-off.** Since Ω is open and μ is a Radon measure, there exists $\rho > 0$ such that

$$\mu(\{x \in \Omega: \text{dist}(x, \partial\Omega) \leq \rho\}) < \frac{\varepsilon}{3}. \quad (3.7)$$

We choose any cut-off function $\chi \in C_c(\Omega)$ such that $\chi(x) = 1$ if $\text{dist}(x, \partial\Omega) \geq \rho$ and we define $h := \chi f_2$. By $|h| \leq |f_2|$, (3.5), (3.6) and (3.7), h is the desired function. $\square$

Remark 3.4. In fact, $\|h\|_{L^p(\Omega, \mu)} \leq \|f\|_{L^p(\Omega, \mu)}$ can be obtained in the conclusion of Lemma 3.2 if either μ has no atoms or f is such that

$$\forall \eta > 0 \quad \exists t > 0: \quad \eta > \mu(E(f, t)) > 0.$$

- In the latter case, if in the step 1 of the proof we choose $\lambda > 1$ such that $\mu(E(f, \lambda t)) > 0$, then the projection f_1 of f on $B_t(0)$ satisfies

$$\|f\|_{L^p(\Omega, \mu)}^p - \|f_1\|_{L^p(\Omega, \mu)}^p \geq (\lambda^p - 1)t^p \mu(E(f, \lambda t)) \quad \forall p \in [1, +\infty).$$

If we choose the set F in the second step sufficiently small, one easily gets $\|f_2\|_{L^p(\Omega, \mu)}^p \leq \|f\|_{L^p(\Omega, \mu)}^p$ in the second step as well.

- Similarly, if μ has no atoms, we can fix a set $G \subset E(f, t)$ of small measure and construct f_1 by projecting f on the sphere of radius λt in G , for some $\lambda < 1$. This produces

$$\|f\|_{L^p(\Omega, \mu)}^p - \|f_1\|_{L^p(\Omega, \mu)}^p \geq (1 - \lambda^p)t^p \mu(G) \quad \forall p \in [1, +\infty).$$

As above, a suitable choice of the set F in the second step allows to conclude.

We recall that $C(e, \alpha)$ is the convex cone with axis e and angle α , see section 2.

Lemma 3.5. Let μ be a Radon measure on an open set $\Omega \subset \mathbb{R}^n$, let $B \subset\subset \Omega$ be an open ball and assume that there exists $e \in \mathbb{S}^{n-1}$ and $\alpha > 0$ such that $V(\mu, x) \cap C(e, \alpha) = \{0\}$ for μ -a.e. $x \in B$. Let $\delta > 0$ and let $f: \Omega \rightarrow \mathbb{R}^n$ be a Borel function such that

$$|f(x) - \|f\|_{L^\infty(B, \mu)} e| \leq \delta \quad \text{for every } x \in B. \quad (3.8)$$

Then, for every $\eta > 0, \gamma > 0$ there exist an open set $U \subset B$ and a function $w: \Omega \rightarrow \mathbb{R}$ of class C^1 such that

$$\begin{aligned} \sup_B |w| &\leq \eta, \\ \mu(B \setminus U) &\leq \gamma, \\ |Dw(x) - f(x)| &\leq 4\delta + \frac{\|f\|_{L^\infty(B, \mu)}}{\tan \alpha} \quad \text{for every } x \in U, \\ |Dw(x)| &\leq \|f\|_{L^\infty(B, \mu)} \left(1 + \frac{1}{\tan \alpha}\right) \quad \text{for every } x \in B. \end{aligned}$$

Proof. By Lemma 2.3, there exists a $C(e, \alpha)$ -null Borel set $F \subset B$ such that $\mu(B \setminus F) = 0$. Hence we can find a compact subset $K \subset F$ such that $\mu(B \setminus K) < \gamma$. Let w_0 be the smooth function constructed in Lemma 2.4, with $\varepsilon = \frac{\eta}{\|f\|_{L^\infty(B, \mu)} + \delta}$. Combining properties (ii) and (iii) of Lemma 2.4, we deduce that

$$|Dw_0(x) - e| < \frac{1}{\tan \alpha}, \quad \text{for every } x \in K$$

and

$$|Dw_0(x)| < 1 + \frac{1}{\tan \alpha}, \quad \text{for every } x \in \mathbb{R}^n.$$

By (3.8) the function $w := \|f\|_{L^\infty(B, \mu)} w_0$ satisfies:

$$\begin{aligned} \sup_B |w| &\leq \eta, \\ |Dw(x) - f(x)| &\leq |Dw(x) - \|f\|_{L^\infty(B, \mu)} e| + \|\|f\|_{L^\infty(B, \mu)} e - f(x)\| \\ &\leq \frac{\|f\|_{L^\infty(B, \mu)}}{\tan \alpha} + \delta \quad \forall x \in K, \\ |Dw(x)| &\leq \|f\|_{L^\infty(B, \mu)} \left(1 + \frac{1}{\tan \alpha}\right) \quad \forall x \in B. \end{aligned}$$

Since w is of class C^1 , we can take a sufficiently small open tubular neighborhood U of K in B so that, for every $x \in U$, denoting y a point in K which minimizes the distance to x , we have

$$\begin{aligned} |Dw(x) - f(x)| &\leq |Dw(x) - Dw(y)| + |Dw(y) - f(y)| + |f(y) - f(x)| \\ &\leq \delta + \sup_{y \in K} |Dw(y) - f(y)| + \text{osc}_B(f) \\ &\leq 4\delta + \frac{\|f\|_{L^\infty(B, \mu)}}{\tan \alpha}. \quad \square \end{aligned}$$

Lemma 3.6 (approximation of continuous functions with compact support). Let μ be a Radon measure on $\mathbb{R}^n$, let $\Omega \subset \mathbb{R}^n$ be an open set such that $\mu(\Omega) < \infty$ and let $h: \Omega \rightarrow \mathbb{R}^n$ be a continuous function with compact support in Ω satisfying $h(x) \in V(\mu, x)^\perp$ for μ -a.e. $x \in \Omega$. Then for every $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ and a function $g: \Omega \rightarrow \mathbb{R}$ of class C^1 such that

$$\begin{aligned}\mu(\Omega \setminus K) &< \varepsilon, \\ \text{supp } g &\subseteq \text{supp } h, \\ Dg(x) &= h(x) \quad \forall x \in K, \\ \|Dg\|_{C^0(\Omega)} &\leq (1 + \varepsilon)\|h\|_{C^0(\Omega)}.\end{aligned}$$

Proof. Of course we can assume $\varepsilon < 1$ and $h \neq 0$. By the classical Lusin's theorem for Radon measures [9, 2.3.5 - 2.3.6], we find a compact set C such that the map $x \mapsto V(\mu, x)$ is continuous on C , that $h(x) \in V(\mu, x)^\perp$ for every $x \in C$ and

$$\mu(\Omega \setminus C) < \frac{\varepsilon}{4}. \quad (3.9)$$

The idea is to construct a series of C^1 functions g_n such that $h - \sum_{k=0}^n Dg_k$ converges uniformly to 0 on a compact set $K \subset \Omega$ with large measure and define $g := \sum_{k=0}^{+\infty} g_k$. More precisely we are going to inductively define a decreasing sequence of open sets V_n with $V_1 \subset \text{supp } h$ and a sequence of functions $g_n \in C_c^1(V_n)$ with the following properties:

$$\mu(\Omega \setminus V_n) \leq (1 - 2^{-n-1})\frac{\varepsilon}{2} \quad \forall n \geq 0, \quad (3.10)$$

$$\left\| h - \sum_{k=0}^n Dg_k \right\|_{C^0(V_n)} \leq \left(\frac{\varepsilon}{2}\right)^n \|h\|_{C^0(\Omega)} \quad \forall n \geq 0, \quad (3.11)$$

$$\|Dg_n\|_{C^0(\Omega)} \leq \left(1 + \frac{\varepsilon}{2}\right) \left(\frac{\varepsilon}{2}\right)^n \|h\|_{C^0(\Omega)} \quad \forall n \geq 0, \quad (3.12)$$

and finally we will fix a suitable compact set $K \subseteq \bigcap_{n \in \mathbb{N}} V_n$ on which $h = \sum_{k=1}^{+\infty} g_k$. As base of the induction, we set

$$g_0 = 0, \quad V_0 := \Omega.$$

- *Step 1: First approximation.*

Let us choose $\delta > 0, 0 < \alpha < \frac{\pi}{2}$ such that

$$4\delta + \frac{\|h\|_{C^0(\Omega)}}{\tan \alpha} \leq \frac{\varepsilon}{2} \|h\|_{C^0(\Omega)}. \quad (3.13)$$

Let $x \in C$; we distinguish two cases.

- If $h(x) = 0$, by continuity there exists $r_x > 0$ such that

$$|h(y)| \leq \frac{\delta}{2} \quad \forall y \in B_{r_x}(x).$$

- If $h(x) \neq 0$, we set

$$e = \frac{h(x)}{|h(x)|}.$$

By continuity of V and h , and using that $h(x) \in V(\mu, x)^\perp$ for every $x \in C$, we can find $r_x > 0$ such that $B_{r_x}(x) \subset \subset \{h \neq 0\}$ and

$$V(\mu, y) \cap C(e, \alpha) = \emptyset, \quad \forall y \in C \cap B_{r_x}(x), \quad (3.14)$$

$$\langle h(y); e \rangle > (1 - \delta)|h(y)| \quad \forall y \in B_{r_x}(x). \quad (3.15)$$

By classical covering arguments, see for instance [8, Theorem 1.29], there exists a finite set of points $x_1, \dots, x_N \in C$ and radii $r_i \in (0, r_{x_i})$ for $i = 1, \dots, N$ such that the closed balls $\overline{B}_{r_i}(x_i) \subset \Omega$ are mutually disjoint and

$$\mu \left(C \setminus \bigcup_{i=1}^N B_{r_i}(x_i) \right) < 2^{-5}\varepsilon.$$

Let us call $B_i := B_{r_i}(x_i)$. We can choose $0 < d < \min\{r_i : i = 1, \dots, N\}$ so that, denoting $\tilde{B}_i := B_{r_i-d}(x_i)$, it holds

$$\mu \left(C \setminus \bigcup_{i=1}^N \tilde{B}_i \right) < 2^{-4}\varepsilon.$$

We fix

$$\eta \leq \delta d.$$

Now, for every fixed $i = 1, \dots, N$, we construct a function $w_i : \Omega \rightarrow \mathbb{R}$ as follows.

- If $h(x_i) = 0$, we set $U_i := B_i$ and we fix $w_i \equiv 0$. We clearly have

$$\begin{aligned} |Dw_i(y) - h(y)| &< \delta, \quad \forall y \in C \cap B_i, \\ |Dw_i(y)| &\leq \|h\|_{C^0(\Omega)} \quad \forall y \in B_i. \end{aligned}$$

- If $h(x_i) \neq 0$, then by (3.14) and (3.15) we can apply Lemma 3.5 to the measure $\mu \llcorner (C \cap B_i)$ in order to find, choosing $\gamma_i < \frac{\varepsilon}{2^i N}$, open sets $U_i \subset B_i$ and functions $u_i : \Omega \rightarrow \mathbb{R}$ of class C^1 satisfying

$$\begin{aligned} \sup_{B_i} |u_i| &\leq \eta, \\ \mu(C \cap \tilde{B}_i \setminus U_i) &< \gamma_i, \\ |Du_i(x) - h(x)| &\leq 4\delta + \frac{\|h\|_{C^0(\Omega)}}{\tan \alpha} \quad \text{for every } x \in U_i, \\ |Du_i(x)| &\leq \|h\|_{C^0(\Omega)} \left(1 + \frac{1}{\tan \alpha}\right) \quad \text{for every } x \in B_i. \end{aligned}$$

We now consider a cutoff function $\chi \in C_c(B_i, [0, 1])$ such that

$$\begin{cases} \chi(x) = 1 & \text{if } x \in \tilde{B}_i, \\ |D\chi|(x) \leq \frac{2}{d}. & \text{for all } x \in B_i \end{cases}$$

and define the function

$$w_i := \chi u_i.$$

Thus w_i satisfies

$$\begin{aligned} \sup_{B_i} |w_i| &\leq \eta, \\ |Dw_i(x) - h(x)| &\leq 4\delta + \frac{\|h\|_{C^0(\Omega)}}{\tan \alpha} \quad \text{for every } x \in U_i \cap \tilde{B}_i, \\ |Dw_i(x)| &\leq |\chi(x)| |Dw_i(x)| + |D\chi(x)| |u_i(x)| \leq \|h\|_{C^0(\Omega)} \left(1 + \frac{1}{\tan \alpha}\right) + 2\delta \\ &\quad \text{for every } x \in B_i. \end{aligned}$$

Thus the function $g_1: \Omega \rightarrow \mathbb{R}$, defined by $g_1(x) := \sum_{i=1}^N \tilde{w}_i(x)$ is compactly supported in $\bigcup_{i=1}^N B_{r_i}(x_i)$ and satisfies

$$\begin{aligned} |Dg_1(x) - h(x)| &\leq 4\delta + \frac{\|h\|_{C^0(\Omega)}}{\tan \alpha} \stackrel{(3.13)}{\leq} \frac{\varepsilon}{2} \|h\|_{C^0(\Omega)} \quad \text{for every } x \in \bigcup_{i=1}^N (U_i \cap \tilde{B}_i), \\ |Dg_1(x)| &\leq \|h\|_{C^0(\Omega)} \left(1 + \frac{1}{\tan \alpha}\right) + 2\delta \leq \left(1 + \frac{\varepsilon}{2}\right) \|h\|_{C^0(\Omega)} \quad \text{for every } x \in \Omega, \\ \text{supp } g_1 &\subseteq \bigcup_{i=1}^N B_{r_i}(x_i) \subset \{h \neq 0\} \subset \text{supp } h. \end{aligned}$$

Letting

$$V_1 := \bigcup_{i=1}^N (U_i \cap \tilde{B}_i),$$

we notice that

$$|h(x) - Dg_1(x)| \leq \frac{\varepsilon}{2} \|h\|_{C^0(\Omega)} \quad \text{for every } x \in V_1,$$

$$\mu(C \setminus V_1) < 2^{-3}\varepsilon \stackrel{(3.9)}{\implies} \mu(\Omega \setminus V_1) \leq (1 - 2^{-2})\frac{\varepsilon}{2}.$$

• *Step 2: Iteration.*

We now assume that g_n, V_n are defined and satisfy (3.10), (3.11) and (3.12): we want to construct g_{n+1}, V_{n+1} .

We apply the same arguments of Step 1 with $h - \sum_{k=0}^n Dg_k$ in place of h and V_n in place of Ω ; moreover we choose the new radii r_i in such a way that the new balls $B_i, \tilde{B}_i$ are contained in V_n and satisfy

$$\mu\left(C \cap V_n \setminus \bigcup_{i=1}^N \tilde{B}_i\right) < 2^{-n-4}\varepsilon.$$

We thus construct the open sets $U_i \subset V_n \cap B_i$, then V_{n+1} the union of $U_i \cap \tilde{B}_i$, and the function $g_{n+1} \in C_c^1(V_n)$ as in the previous step, satisfying the following properties:

$$\mu(\Omega \setminus V_{n+1}) \leq (1 - 2^{-n-2})\frac{\varepsilon}{2},$$

$$\text{supp } g_{n+1} \subset V_n,$$

$$\left| h(x) - \left(\sum_{k=0}^n Dg_k(x) \right) - Dg_{n+1}(x) \right| \leq \frac{\varepsilon}{2} \left\| h(x) - \sum_{k=0}^n Dg_k(x) \right\|_{C^0(\Omega)}$$

$$\stackrel{(3.11)}{\leq} \left(\frac{\varepsilon}{2} \right)^{n+1} \|h\|_{C^0(\Omega)} \quad \forall x \in V_{n+1},$$

$$|Dg_{n+1}(x)| \leq \left(1 + \frac{\varepsilon}{2} \right) \left\| h(x) - \sum_{k=0}^n Dg_k(x) \right\|_{C^0(V_n)} \stackrel{(3.11)}{\leq} \left(1 + \frac{\varepsilon}{2} \right) \left(\frac{\varepsilon}{2} \right)^n \|h\|_{C^0(\Omega)}$$

$$\forall x \in V_n.$$

• *Step 3: Convergence and estimates.*

By (3.10) it holds $\mu(\Omega \setminus \bigcap_{n \in \mathbb{N}} V_n) \leq \frac{\varepsilon}{2}$. Hence there exists a compact set $K \subset \bigcap_{n \in \mathbb{N}} V_n$ such that

$$\mu(\Omega \setminus K) < \varepsilon.$$

We define $g := \sum_{i=1}^{\infty} g_i$ and we observe that the series converges in the strong topology of $C^1(\Omega)$. Indeed each $\text{supp } g_n \subseteq V_{n-1} \subset \Omega$ and, by (3.11), we get

$$\sum_{n=1}^{\infty} \|Dg_n\|_{C^0(\Omega)} \leq \left(1 + \frac{\varepsilon}{2} \right) \|h\|_{C^0(\Omega)} \sum_{n=0}^{\infty} \left(\frac{\varepsilon}{2} \right)^n$$

$$= \left(1 + \frac{\varepsilon}{2}\right) \frac{1}{1 - \frac{\varepsilon}{2}} \|h\|_{C^0(\Omega)} \stackrel{(\varepsilon < 1)}{<} (1 + \varepsilon) \|h\|_{C^0(\Omega)}.$$

This estimate implies the strong convergence of the series $\sum_{n=1}^{+\infty} g_n$ to g in $C_c^1(\Omega)$ and it moreover proves

$$\|Dg\|_{C^0(\Omega)} \leq (1 + \varepsilon) \|h\|_{C^0(\Omega)}.$$

Finally, letting $n \rightarrow +\infty$ in (3.11), the convergence of the series $\sum_{n=1}^{+\infty} g_n$ to g in $C_c^1(\Omega)$ implies

$$h(x) = Dg(x) \quad \forall x \in K. \quad \square$$

Proof of Theorem 1.1. By Lemma 3.2, there exists $h \in C_c(\Omega)$ such that

$$\begin{aligned} \mu(\{x \in \Omega: h(x) \neq f(x)\}) &< \frac{\varepsilon}{2}, \\ \|h\|_{L^p(\Omega, \mu)} &\leq \left(1 + \frac{\varepsilon}{2}\right) \|f\|_{L^p(\Omega, \mu)} \quad \forall p \in [1, +\infty]. \end{aligned}$$

If $h \equiv 0$, then $g \equiv 0$ is the desired function; otherwise we fix $N \in \mathbb{N}$ such that

$$\eta := \frac{\|h\|_{C^0(\Omega)}}{N} < \frac{\varepsilon}{4} \inf_{p \in [1, +\infty]} \left\{ \frac{\|h\|_{L^p(\Omega, \mu)}}{\mu(\Omega)^{\frac{1}{p}}} \right\}, \quad (3.16)$$

and such N exists since the above infimum is strictly positive (by continuity of the map $p \mapsto \|h\|_{L^p(\Omega, \mu)} / \mu(\Omega)^{\frac{1}{p}}$). We define N functions $h_1, \dots, h_N \in C_c(\Omega, \mathbb{R}^n)$ as follows:

$$h_i(x) = \begin{cases} 0 & \text{if } |h(x)| \leq (i-1)\eta \\ h(x) - (i-1)\eta \frac{h(x)}{|h(x)|} & \text{if } (i-1)\eta < |h(x)| \leq i\eta \\ \eta \frac{h(x)}{|h(x)|} & \text{if } |h(x)| > i\eta. \end{cases}$$

Hence the functions h_i have the following properties:

$$\begin{aligned} \text{supp } h_N &\subset\subset \text{supp } h_{N-1} \cdots \subset\subset \text{supp } h_1 = \text{supp } h, \\ 0 &\leq h_i \leq \eta \quad \forall i = 1, \dots, N, \\ h &= h_1 + \cdots + h_N, \end{aligned}$$

Now for every $i = 1, \dots, N$ we apply Lemma 3.6 to h_i , obtaining a function $g_i \in C_c^1(\Omega)$ and a compact set $K_i \subset \text{supp } h_i$ which satisfies the conclusions of the lemma with $\frac{\varepsilon}{2N}$ in place of ε . Let $C := \bigcap_{i=1}^N K_i$.

We define $g = \sum_{i=1}^N g_i$ and we claim that g satisfies the conclusions of the theorem. First of all we have $f = Dg$ on $C \cap \{h = f\}$ and

$$\mu(\Omega \setminus (C \cap \{h = f\})) \leq \mu(\Omega \setminus C) + \mu(\{h \neq f\}) < \varepsilon.$$

By interior approximation, we can find a compact set $K \subset (C \cap \{h = f\})$ such that $\mu(\Omega \setminus K) < \varepsilon$.

It remains to estimate the L^p -norms of Dg : to this aim, we first claim that

$$|Dg(x)| \leq |h(x)| + 2\eta \quad \forall x \in \Omega. \quad (3.17)$$

Indeed, first of all clearly $|Dg(x)| \leq |h(x)| + 2\eta$ for every $x \in \Omega \setminus \text{supp } h_1$ since $\text{supp } g \subseteq \text{supp } h = \text{supp } h_1$ by construction. Moreover, for every $x \in \text{supp } h_i \setminus \text{supp } h_{i+1}$ it holds

$$|h(x)| \geq (i-1)\eta, \quad (3.18)$$

while $g_j(x) = 0$ for every $j > i$ (since $\text{supp } g_j \subset \text{supp } h_j$) implies

$$|Dg(x)| \stackrel{(\text{supp } g_j \subset \text{supp } h_j)}{\leq} \sum_{j=1}^i |Dg_j(x)| \leq i \left(1 + \frac{\varepsilon}{2N}\right) \eta \stackrel{(\varepsilon < 1)}{\leq} (i+1)\eta. \quad (3.19)$$

(3.18) and (3.19) prove (3.17), that in turn implies

$$\|Dg\|_{L^p(\Omega, \mu)} \leq \|h\|_{L^p(\Omega, \mu)} + 2\eta(\mu(\Omega))^{\frac{1}{p}} \stackrel{(3.16)}{<} \left(1 + \frac{\varepsilon}{2}\right) \|h\|_{L^p(\Omega, \mu)} \quad \forall p \in [1, +\infty),$$

concluding the proof. $\square$

4. Flat chain conjecture

The question on which differential operator admits a Lusin type theorem as in [1] has recently attracted interest, see [6]. We now focus on the case where the gradient is replaced by the exterior derivative acting on $(k-1)$ -forms.

We conjecture that the following version of Theorem 3.1 holds, where f is replaced by a k -form. Here V^k is the equivalent of the decomposability bundle for k -vectors, see [4]. We denote by $I(h, n)$ the set of multi-indexes of length h in $\mathbb{R}^n$, i.e. the set of arrays $(i_1, \dots, i_h)$ with $1 \leq i_1 < i_2 < \dots < i_h \leq n$.

Conjecture 4.1. *Let μ be a Radon measure on $\mathbb{R}^n$, let $\Omega \subset \mathbb{R}^n$ be an open set such that $\mu(\Omega) < \infty$ and let $\omega: \Omega \rightarrow \Lambda^k(\mathbb{R}^n)$ be a Borel k -form satisfying*

$$\langle \omega(x), \tau \rangle = 0 \quad \text{for every } \tau \in V^k(\mu, x), \text{ for } \mu\text{-a.e. } x \in \Omega.$$

For every $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ and a $(k-1)$ -form $\phi = \sum_{I \in I(k-1, n)} \phi_I dx_I$ in Ω of class C^1 such that

$$\begin{aligned}\mu(\Omega \setminus K) &< \varepsilon, \\ d\phi(x) &= \omega(x) \quad \forall x \in K, \\ \text{Lip}(\phi_I) &\leq C(n)\|\omega\|_{L^\infty(\Omega, \mu)} \quad \text{for every } I \in I(k-1, n).\end{aligned}$$

We note that a positive solution of the above conjecture would immediately imply the validity of the Ambrosio-Kirchheim flat chain conjecture about the equivalence of metric currents in $\mathbb{R}^n$ and Federer-Fleming flat chains with finite mass, see [5]. In particular, since Theorem 1.1 implies Conjecture 4.1 in the case $k = 1$, it also implies the (already known) 1-dimensional version of the flat chain conjecture, see also [12,2,10]. The conjecture was also proved in dimension $k = n$ in [7]. Here we show the implication between the two conjectures, referring to [10] for the relevant notation. We only remind the reader that $\tilde{T}$ represents the classical current associated with a metric current T (of finite mass).

Proof of the flat chain conjecture assuming the validity of Conjecture 4.1. Towards a proof by contradiction of the flat chain conjecture, assume that there exists a non trivial metric k -current $T = \tau\mu$ in $\mathbb{R}^n$ such that $\tilde{T}$ is not a flat chain. By subtracting the flat chain $(P_{V^k}\tau)\mu$ (this current is a flat chain because of [4, Theorem 1.2]), we may assume that $\tau(x) \in V^k(\mu, x)^\perp$ for μ -a.e. x . Here both the projection $P_{V^k} : \Lambda_k(\mathbb{R}^n) \rightarrow \Lambda_k(\mathbb{R}^n)$ on V^k and the orthogonal complement $(V^k)^\perp$ are meant with respect to the Euclidean scalar product on the space of k -vectors.

As shown in the proof of Theorem 4.4 of [10], μ is singular with respect to Lebesgue measure, and by Lemma 4.2 of [10], there exists a sequence of vectors $v_j \in \mathbb{R}^n$ such that $v_j \rightarrow 0$ and, denoting, for every $w \in \mathbb{R}^n$, $\tau_w : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the map $x \mapsto x + w$, the currents $\tilde{T}$ and $(\tau_{v_j})_\# \tilde{T}$, are mutually singular for every $j \in \mathbb{N}$. In particular, denoting

$$\mathbb{M}_e(T) := \sup\{\langle T, \omega \rangle : \omega \in C_c^\infty(\Omega, \Lambda^k(\mathbb{R}^n)), |\omega(x)| \leq 1, \forall x \in \Omega\},$$

where $|\omega(x)|$ denotes the Euclidean norm of the k -covector $\omega(x)$, we have

$$\mathbb{M}_e(\tilde{T} - (\tau_{v_j})_\# \tilde{T}) = 2\mathbb{M}_e(\tilde{T}) \quad \text{for every } j \in \mathbb{N}.$$

For every $j \in \mathbb{N}$, let ω_j be a Borel k form such that $|\omega_j|_{L^\infty(\Omega, \mu)} \leq 1$ and

$$\langle \tilde{T} - (\tau_{v_j})_\# \tilde{T}, \omega_j \rangle = 2\mathbb{M}_e(\tilde{T}).$$

In particular for every $j \in \mathbb{N}$,

$$\langle \omega_j, \tau \rangle = 0 \quad \text{for every } \tau \in V^k(\mu, x), \text{ for } \mu\text{-a.e. } x.$$

Fix $\varepsilon > 0$ sufficiently small and let ϕ_j be the corresponding $(k-1)$ -forms obtained applying Conjecture 4.1 with $\mu = |\tilde{T} - (\tau_{v_j})_\# \tilde{T}|$. Possibly subtracting a constant, we can

assume that $\phi_j(0) = 0$, for every j , so that we can find a $C(n)$ -Lipschitz $(k-1)$ -form ϕ_∞ such that, up to non-relabelled subsequences,

$$\phi_j \rightarrow \phi_\infty \quad \text{locally uniformly.} \quad (4.1)$$

We deduce that for every $j \in \mathbb{N}$

$$\begin{aligned} 2\mathbb{M}_e(\tilde{T}) &= \langle \tilde{T} - (\tau_{v_j})_\# \tilde{T}, \omega_j \rangle = \langle \tilde{T}, \omega_j \rangle - \langle (\tau_{v_j})_\# \tilde{T}, \omega_j \rangle \\ &= \langle \tilde{T}, \omega_j \rangle - \langle \tilde{T}, (\tau_{v_j})^\# \omega_j \rangle \leq \langle \tilde{T}, d\phi_j \rangle - \langle \tilde{T}, (\tau_{v_j})^\# d\phi_j \rangle - C(n) \binom{n}{k-1} \varepsilon \\ &= T(1, \phi_j) - T(1, \phi_j \circ \tau_{v_j}) - C(n) \binom{n}{k-1} \varepsilon. \end{aligned}$$

Thanks to the continuity of metric currents, the equi-Lipschitz continuity of the ϕ_j 's, and (4.1), the quantity $T(1, \phi_j) - T(1, \phi_j \circ \tau_{v_j})$ tends to 0 as $j \rightarrow +\infty$ and we reach a contradiction, if ε is chosen sufficiently small. $\square$

Remark 4.2.

- Thanks to Proposition 3.2 and Proposition 3.3 of [10], in order to prove the flat chain conjecture in full generality, it would suffice to prove Conjecture 4.1 for closed forms ω .
- In the previous proof we did not need the full validity of the equality

$$d\phi(x) = \omega(x) \quad \forall x \in K,$$

but only that, for a given k -vector field $\tau(x) \in V^k(\mu, x)^\perp$, the $(k-1)$ -form ϕ yields

$$\langle d\phi(x), \tau(x) \rangle \geq \langle \omega(x), \tau(x) \rangle \quad \forall x \in K.$$

Declaration of competing interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability

No data was used for the research described in the article.

References

- [1] G. Alberti, A Lusin type theorem for gradients, *J. Funct. Anal.* 100 (1991) 110–119.

- [2] G. Alberti, D. Bate, A. Marchese, On the closability of differential operators, *J. Funct. Anal.* 289 (7) (2025) 111029, <https://doi.org/10.1016/j.jfa.2025.111029>, issn: 0022-1236, 1096-0783.
- [3] G. Alberti, A. Marchese, On the differentiability of Lipschitz functions with respect to measures in the Euclidean space, *Geom. Funct. Anal.* 26 (1) (2016) 1–66.
- [4] G. Alberti, A. Marchese, On the structure of flat chains with finite mass, Preprint: <https://arxiv.org/abs/2311.06099>, 2023.
- [5] L. Ambrosio, B. Kirchheim, Currents in metric spaces, *Acta Math.* 185 (1) (2000) 1–80.
- [6] L. De Masi, C. Gasparetto, Non-rigidity of the absolutely continuous part of $\mathcal{A}$ -free measures, *J. Funct. Anal.* 289 (10) (2025).
- [7] G. De Philippis, F. Rindler, On the structure of $\mathcal{A}$ -free measures and applications, *Ann. Math.* (2) 184 (3) (2016) 1017–1039.
- [8] L.C. Evans, R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Revised, Textbooks in Mathematics, CRC Press, Boca Raton, FL, ISBN 978-1-4822-4238-6, 2015, pp. xiv+299.
- [9] H. Federer, *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York, Inc., New York, 1969, xiv+676.
- [10] A. Marchese, A. Merlo, A simple proof of the 1-dimensional flat chain conjecture, Preprint: <https://cvgmt.sns.it/paper/6587/>, 2024.
- [11] A. Marchese, A. Schioppa, Lipschitz functions with prescribed blowups at many points, *Calc. Var. Partial Differ. Equ.* 58 (2019) 1–33.
- [12] A. Schioppa, Metric currents and Alberti representations, *J. Funct. Anal.* 271 (11) (2016) 3007–3081.