Dirichlet problem for pluriholomorphic functions of two complex variables

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Abstract. In this paper the Dirichlet problem for pluriholomorphic functions of two complex variables is investigated. The key point is the relation between pluriholomorphic functions and pluriharmonic functions. The link is constituted by the Fueter-regular functions of one quaternionic variable. Previous results about the boundary values of pluriharmonic functions and new results on L^2 traces of regular functions are applied to obtain a characterization of the traces of pluriholomorphic functions.

Keywords. pluriholomorphic functions, pluriharmonic functions, quaternionic regular functions

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1. Introduction

We consider some boundary value problems in two complex variables on a class of pseudoconvex domains containing the unit ball B. The class consists of domains Ω that satisfy a $L^2(\partial\Omega)$-estimate (cf. §3.1). We conjecture that the estimate always holds on a strongly pseudoconvex domain in $\mathbb{C}^2$.

We relate two boundary value problems on Ω by means of quaternionic ψ-regular functions, a variant of Fueter-regular functions (see §2 for precise definitions) studied by many authors (see for instance [13, 16, 18]). We are interested in the Dirichlet problems for pluriholomorphic functions and for pluriharmonic functions. Pluriholomorphic functions are solutions of the system $\frac{\partial^2 g}{\partial\bar{z}_i\partial\bar{z}_j} = 0$ for $1 \le i, j \le 2$ (see e.g. [6, 7, 8]). The Dirichlet problem for this system is not well posed and the homogeneous problem has infinitely many independent solutions (see also [1, 2, 3, 4]). As noted in [8], the Dirichlet problem for pluriharmonic functions has a different character, related to strong ellipticity: the solution, if

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it exists, is unique and the system can be splitted into equations for the real and imaginary parts of g.

The key point is that if $f = f_1 + f_2 j$ is ψ-regular, then f_1 is pluriholomorphic (and harmonic) if and only if f_2 is pluriharmonic. Then we can apply the results on the traces of pluriharmonic functions given in [5] and [15] and obtain a characterization of the traces of pluriholomorphic functions.

We begin by giving an application of an existence principle in Functional Analysis proved by Fichera in the 50's (cf. [9, 10] and [5]§12). We obtain a result on the boundary values of class $L^2(\partial\Omega)$ of ψ-regular functions (Theorem 3.3): every function f_1 which belongs to the class $L^2(\partial\Omega)$ together with its normal derivative $\overline{\partial}_n f_1$ is the first complex component of a ψ-regular function on Ω, of class $L^2(\partial\Omega)$. On the unit ball B, where computation of L^2-estimates can be more precise, the result is optimal. We show that the condition on the normal derivative cannot be relaxed and therefore the operation of ψ-regular conjugation is not bounded in the harmonic Hardy space $h^2(B)$.

In §4 we apply the preceding theorem to show that every domain that satisfies the $L^2(\partial\Omega)$-estimate is pseudoconvex.

In §5 we give the application of Theorem 3.3 to the Dirichlet problem for pluriholomorphic functions. We generalize some results obtained by Detraz [6] and Dzhuraev [7] on the unit ball (cf. also [1, 2, 3, 4, 8]). We show that if Ω satisfies the $L^2(\partial\Omega)$-estimate, a function $h \in L^2(\partial\Omega)$ with $\overline{\partial}_n h \in L^2(\partial\Omega)$ is the trace of a harmonic pluriholomorphic function on Ω if and only if it satisfies an orthogonality condition (see Corollary 5.1 for the precise statement). On the unit ball, this condition can be expressed in terms of spherical harmonics (Proposition 5.3).

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2. Notations and definitions

Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ be a bounded domain with C^∞-smooth boundary in $\mathbb{C}^2$. We assume $\rho \in C^\infty$ on $\mathbb{C}^2$ and $d\rho \neq 0$ on $\partial\Omega$. For every complex valued function $g \in C^1(\overline{\Omega})$, we can define on a neghbourhood of $\partial\Omega$ the radial derivatives

$$\partial_n g = \sum_k \frac{\partial g}{\partial z_k}\frac{\partial \rho}{\partial \bar{z}_k}\frac{1}{|\partial\rho|} \quad \text{and} \quad \overline{\partial}_n g = \sum_k \frac{\partial g}{\partial \bar{z}_k}\frac{\partial \rho}{\partial z_k}\frac{1}{|\partial\rho|},$$

where $|\partial\rho|^2 = \sum_{k=1}^n \left|\frac{\partial\rho}{\partial z_k}\right|^2$. By means of the Hodge $*$-operator and the Lebesgue surface measure $d\sigma$, we can also write $\overline{\partial}_n g d\sigma = *\overline{\partial} g_{|\partial\Omega}$. Let L be the tangential Cauchy-Riemann operator

$$L = \frac{1}{|\partial\rho|}\left(\frac{\partial\rho}{\partial\bar{z}_2}\frac{\partial}{\partial\bar{z}_1} - \frac{\partial\rho}{\partial\bar{z}_1}\frac{\partial}{\partial\bar{z}_2}\right).$$

A function $g \in C^1(\partial\Omega)$ is a CR-function if and only if $Lg = 0$ on $\partial\Omega$.

We will denote by $Phol(\Omega)$ the space of *pluriholomorphic* functions on Ω (cf. [6, 7, 8]). They are $C^2(\Omega)$ solutions of the system

$$\frac{\partial^2 g}{\partial \bar{z}_i \partial \bar{z}_j} = 0 \quad \text{on } \Omega \quad (1 \leq i, j \leq 2).$$

We can identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$. A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is *(left) regular* on Ω (in the sense of Fueter) if

$$\mathcal{D}f = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

Given the "structural vector" $\psi = (1, i, j, -k)$, f is called *(left) ψ-regular* on Ω if

$$\mathcal{D}'f = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

We refer to the papers of Sudbery[19], Shapiro and Vasilevski[18] and Nōno[13] for the theory of regular functions. In complex components, ψ-regularity is equivalent to the equations

$$\frac{\partial f_1}{\partial \bar{z}_1} = \frac{\partial \overline{f_2}}{\partial z_2}, \quad \frac{\partial f_1}{\partial \bar{z}_2} = -\frac{\partial \overline{f_2}}{\partial z_1}.$$

Note that every holomorphic map (f_1, f_2) on Ω defines a ψ-regular function $f = f_1 + f_2 j$ and that the complex components are both holomorphic or both non-holomorphic. Every regular or ψ-regular function is harmonic and if Ω is pseudoconvex, every complex harmonic function is the complex component of a ψ-regular function on Ω.

3. L^2-solutions and ψ-regular functions

3.1. L^2 boundary estimate. Now we suppose that on Ω the following $L^2(\partial\Omega)$-estimate is satisfied: there exists a positive constant C such that

$$|(f, Lg)| \leq C\|\partial_n f\|\|\overline{\partial}_n g\| \tag{*}$$

for every complex harmonic functions f, g on Ω, of class C^1 on $\overline{\Omega}$. Here (f, g) denotes the $L^2(\partial\Omega)$-product and $\|f\|$ the $L^2(\partial\Omega)$-norm.

Let B be the unit ball of $\mathbb{C}^2$ and $S = \partial B$. The space $L^2(S)$ is the sum of the pairwise orthogonal spaces $\mathcal{H}_{p,q}$, whose elements are the harmonic homogeneous polynomials of degree p in z_1, z_2 and q in $\bar{z}_1, \bar{z}_2$ (cf. for example Rudin[17]§12.2). The spaces $\mathcal{H}_{p,q}$ can be identified with the spaces of the restrictions of their elements to S (*spherical harmonics*).

Proposition 3.1. *On the unit ball B of $\mathbb{C}^2$ the estimate (*) is satisfied with constant $C = 1$.*

Proof. It suffices to prove the estimate for a pair of polynomials $f \in \mathcal{H}_{s,t}$, $g \in \mathcal{H}_{p,q}$, since the orthogonal subspaces $\mathcal{H}_{p,q}$ are eigenspaces of the operators ∂_n and $\overline{\partial}_n$. We can restrict ourselves to the case $s = p+1 > 0$ and $q = t+1 > 0$, since otherwise the product (f, Lg) is zero. We have

$$|(f, Lg)|^2 \leq \|f\|^2 \|Lg\|^2 = \|f\|^2 (L^* Lg, g) = \|f\|^2 (-\overline{L}Lg, g) = \|f\|^2 (p+1)q \|g\|^2$$

since the $L^2(S)$-adjoint L^* is equal to $-\overline{L}$ (cf. [17, §18.2.2]) and $\overline{L}L = -(p+1)qId$ when $q > 0$. On the other hand,

$$\|\partial_n f\| \|\overline{\partial}_n g\| = (p+1)q \|f\| \|g\|.$$

and the estimate is proved. □

Remark 3.2. We will prove in §4 that the estimate (*) implies the pseudoconvexity of Ω. We conjecture that in turn the estimate is always valid on a (strongly) pseudoconvex domain in $\mathbb{C}^2$.

3.2. An existence principle.

We recall an existence principle in Functional Analysis proved by Fichera in the 50's (cf. [9, 10] and [5]§12).

Let M_1 and M_2 be linear homomorphisms from a vector space V over the real (complex) numbers into the Banach spaces B_1 and B_2, respectively. Let Ψ_1 be a linear functional defined on B_1. Then a necessary and sufficient condition for the existence of a linear functional Ψ_2 defined on B_2 such that

$$\Psi_1(M_1(v)) = \Psi_2(M_2(v)) \quad \forall\ v \in V$$

is that there exists a constant K such that for all $v \in V$,

$$\|M_1(v)\| \leq K \|M_2(v)\|.$$

Moreover, the following estimate holds:

$$\inf_{\Psi_0 \in \mathcal{N}} \|\Psi_2 + \Psi_0\| \leq K \|\Psi_1\|,$$

where $\mathcal{N} = \{\Psi_0 \in B_2^* \mid \Psi_0(M_2(v)) = 0\ \forall v \in V\}$.

3.3. Application to ψ-regular functions.

We apply the existence principle to the following setting. Let V be the space $Harm^1(\Omega)$ of complex valued harmonic functions on Ω, of class C^1 on $\overline{\Omega}$. We consider the Hilbert space

$$W_n^1(\partial\Omega) = \{f \in L^2(\partial\Omega) \mid \partial_n f \in L^2(\partial\Omega)\}$$

w.r.t. the product

$$(f, g)_{W_n^1} = (f, g) + (\partial_n f, \partial_n g)$$

and the conjugate space

$$\overline{W}^1_n(\partial\Omega) = \{f \in L^2(\partial\Omega) \mid \overline{\partial}_n f \in L^2(\partial\Omega)\}$$

with product

$$(f,g)_{\overline{W}^1_n} = (f,g) + (\overline{\partial}_n f, \overline{\partial}_n g).$$

Here we identify $f \in L^2(\partial\Omega)$ with its harmonic extension on Ω. For every $\alpha > 0$, a function $f \in C^{1+\alpha}(\partial\Omega)$ belongs to $W^1_n(\partial\Omega)$ and to $\overline{W}^1_n(\partial\Omega)$. By means of the identification of $L^2(\partial\Omega)$ with its dual, we get dense, continuous injections $W^1_n(\partial\Omega) \subset L^2(\partial\Omega) = L^2(\partial\Omega)^* \subset W^1_n(\partial\Omega)^*$.

Let A be the closed subspace of $L^2(\partial\Omega)$ whose elements are conjugate CR-functions. It was shown by Kytmanov in [11]§17.1 that the set of the harmonic extensions of elements of A is the kernel of ∂_n.

Let $B_1 = \left(W^1_n(\partial\Omega)/A\right)^*$ and $B_2 = L^2(\partial\Omega)$. Let $M_1 = \pi \circ L$, $M_2 = \overline{\partial}_n$, where π is the quotient projection $\pi : L^2 \to L^2/A = \left(L^2/A\right)^* \subset B_1$.

For every $g \in L^2(\partial\Omega)$, let $g^\perp$ denote the component of g in $A^\perp$. A function $h_1 \in W^1_n(\partial\Omega)$ defines a linear functional $\Psi_1 \in B_1^* = W^1_n(\partial\Omega)/A$ such that $\Psi_1(\pi(g)) = (g^\perp, h_1)_{L^2}$ for every $g \in L^2(\partial\Omega)$. If h is a CR function on $\partial\Omega$,

$$(L\phi, \bar{h}) = \frac{1}{2}\int_{\partial\Omega} h\overline{\partial}(\phi dz) = 0 \Rightarrow (L\phi)^\perp = L\phi.$$

Then $\Psi_1(M_1(\phi)) = (L\phi, h_1)$.

By the previous principle, the existence of $h_2 \in L^2(\partial\Omega)$ such that

$$\int_{\partial\Omega} \bar{h}_1 L\phi d\sigma = \int_{\partial\Omega} \bar{h}_2 \overline{\partial}_n \phi d\sigma \quad \forall\ \phi \in Harm^1(\Omega)$$

is equivalent to the existence of $C > 0$ such that

$$\|\pi(L\phi)\|_{(W^1_n(\partial\Omega)/A)^*} \leq C\|\overline{\partial}_n\phi\|_{L^2(\partial\Omega)} \quad \forall\ \phi \in Harm^1(\Omega). \tag{**}$$

The functional $\pi(L\phi) \in L^2/A = \left(L^2/A\right)^* \subset B_1$ acts on $\pi(g) \in L^2/A$ in the following way:

$$\pi(L\phi)(\pi(g)) = (g^\perp, L\phi)_{L^2} = (g, L\phi)_{L^2}$$

since $L\phi \in A^\perp$.

We get then the following result.

Theorem 3.3. *Assume that the boundary $\partial\Omega$ is connected and estimate (*) is satisfied. Given $f_1 \in \overline{W}^1_n(\partial\Omega)$, there exists $f_2 \in L^2(\partial\Omega)$ such that $f = f_1 + f_2 j$ is the trace of a ψ-regular function on Ω. The function f_2 is unique up to a CR function. Moreover, f_2 satisfies the estimate*

$$\inf_{f_0} \|f_2 + f_0\|_{L^2(\partial\Omega)} \leq C\|f_1\|_{\overline{W}^1_n(\partial\Omega)},$$

where the infimum is taken among the CR functions $f_0 \in L^2(\partial\Omega)$.

Proof. From (*) we get

$$\sup_{\|\pi(g)\|_{W^1_n(\partial\Omega)/A}\le 1} |(g, L\phi)| \le C\|\overline{\partial}_n\phi\|_{L^2(\partial\Omega)} \quad \forall\ \phi \in Harm^1(\Omega)$$

which is the same as estimate (**). From the existence principle applied to $h_1 = \bar{f}_1 \in W^1_n(\partial\Omega)$, we get $f_2 = -h_2 \in L^2(\partial\Omega)$ such that

$$\int_{\partial\Omega} f_1 L\phi d\sigma = -\int_{\partial\Omega} \bar{f}_2\overline{\partial}_n\phi d\sigma \quad \forall\ \phi \in Harm^1(\Omega).$$

Therefore

$$\frac{1}{2}\int_{\partial\Omega} f_1\overline{\partial}\phi\wedge d\zeta = -\int_{\partial\Omega} \bar{f}_2 * \overline{\partial}\phi \quad \forall\ \phi \in Harm^1(\Omega)$$

and the result follows from the $L^2(\partial\Omega)$-version of Theorem 5 in [16], that can be proved as in [16] using the results given in [18, §3.7]. The estimate given by the existence principle is

$$\inf_{f_0\in\mathcal{N}} \|f_2 + f_0\|_{L^2(\partial\Omega)} \le C\|\Psi_1\|_{W^1_n/A} \le C\|h_1\|_{W^1_n(\partial\Omega)} = C\|f_1\|_{\overline{W}^1_n(\partial\Omega)},$$

where $\mathcal{N} = \{f_0 \in L^2(\partial\Omega) \mid (\overline{\partial}_n\phi, f_0)_{L^2(\partial\Omega)} = 0\,\forall\phi \in Harm^1(\Omega)\}$ is the subspace of CR-functions in $L^2(\partial\Omega)$ (cf. [11]§17.1 and [5]§23). ☐

If $\Omega = B$, then the space $W^1_n(S)/A$ is a Hilbert space also w.r.t. the product

$$(\pi(f), \pi(g))_{W^1_n/A} = (\partial_n f, \partial_n g).$$

This is a consequence of the estimate $\|g^\perp\|_{L^2(S)} \le \|\partial_n g\|_{L^2(S)}$, which holds for every $g \in W^1_n(S)$: if $g = \sum_{p\ge 0,q\ge 0} g_{p,q}$ is the orthogonal decomposition of g in $L^2(S)$, then

$$\|\partial_n g\|^2 = \sum_{p>0,q\ge 0} \|p g_{p,q}\|^2 \ge \sum_{p>0,q\ge 0} \|g_{p,q}\|^2 = \|g^\perp\|^2.$$

Then

$$\|\pi(g)\|^2_{W^1_n/A} = \|g^\perp\|^2_{L^2} + \|\partial_n g\|^2_{L^2} \le 2\|\partial_n g\|^2_{L^2}$$

and therefore $\|\pi(g)\|_{W^1_n/A}$ and $\|\partial_n g\|_{L^2}$ are equivalent norms on $W^1_n(S)/A$.

We can repeat the arguments of the previous proof and get the following:

Theorem 3.4. *Given $f_1 \in \overline{W}^1_n(S)$, there exists $f_2 \in L^2(S)$ such that $f = f_1 + f_2 j$ is the trace of a ψ-regular function on B. The function f_2 is unique up to a CR function. Moreover, f_2 satisfies the estimate*

$$\inf_{f_0} \|f_2 + f_0\|_{L^2(S)} \le \|\overline{\partial}_n f_1\|_{L^2(S)}.$$

Remark 3.5. On the unit ball B of $\mathbb{C}^2$, the estimate which is obtained from (**) by taking the $L^2(S)$-norm also in the left-hand side is no longer valid (take for example $\phi \in \mathcal{H}_{k-1,1}(S)$). The necessity part of the existence principle gives that there exists $f_1 \in L^2(S)$ for which does not exist any $L^2(S)$ function f_2 such that $f_1 + f_2 j$ is the trace of a ψ-regular function on B. Then the operation of ψ-regular conjugation is not bounded in the harmonic Hardy space $h^2(B)$.

Note that this is different from pluriharmonic conjugation (cf. [20]) and in particular from the one-variable situation, which can be obtained by intersecting the domains with the complex plane $\mathbb{C}_j$ spanned by 1 and j. In this case f_1 and f_2 are real-valued and $f = f_1 + f_2 j$ is the trace of a holomorphic function on $\Omega \cap \mathbb{C}_j$ w.r.t. the variable $\zeta = x_0 + x_2 j$.

A function $f_1 \in L^2(S)$ with the required properties is $f_1 = z_2(1-\bar{z}_1)^{-1}$. In fact, it can be computed that $\|f_1\|_{L^2(S)} = 1$, but $\overline{\partial}_n f_1 = \bar{z}_1 z_2 (1-\bar{z}_1)^{-2}$ is not of class $L^2(S)$ and so $f_1 \notin \overline{W}^1_n(S)$. The function $f = f_1 + f_2 j$, with $f_2 = \frac{1}{2}\bar{z}_2^2(1-z_1)^{-2}$, is a ψ-regular function on B. The second component f_2 is not of class $L^2(S)$ and the same is true for every function $f_2' = f_2 + f_0$, f_0 holomorphic on B. In fact, let $f_2^r(z) = f_2(rz)$ for every $r \in (0,1)$, and the same notation for f_0^r and ${f_2'}^r$, then

$$f_2^r = \frac{1}{2} r \bar{z}_2^2 (1 - r z_1)^{-2} = \sum_{k=1}^{\infty} \frac{k}{2} r \bar{z}_2^2 (r z_1)^{k-1}$$

is orthogonal in $L^2(S)$ to the functions holomorphic in a neighbourhood of $\overline{B}$. Then

$$\|{f_2'}^r\|^2_{L^2(S)} = \|f_2^r + f_0^r\|^2_{L^2(S)} \geq \|f_2^r\|^2_{L^2(S)}$$

is unbounded w.r.t. r, and so $f_2' \notin L^2(S)$.

4. L^2-estimate and pseudoconvexity

We now show that estimate (*), via Theorem 3.3, implies the pseudoconvexity of Ω. We adapt the proof given by Nōno in [14] of a result proved by Laufer in [12].

Proposition 4.1. *If the domain Ω satisfies estimate (*), then it is a domain of holomorphy.*

Proof. If Ω is not a domain of holomorphy, there exists an open domain Ω', in which Ω is strictly contained, such that every $h \in \mathcal{O}(\Omega)$ extends holomorphically to Ω'. Let $\zeta^0 \in \Omega' \setminus \overline{\Omega}$ and set $f_1(z) = |z - \zeta^0|^{-2}$. The function f_1 is harmonic in Ω, of class C^∞ on $\overline{\Omega}$. Theorem 3.3 gives $f_2 \in L^2(\partial\Omega)$ whose harmonic extension on Ω satisfies

$$\overline{\partial} f_2 = -\frac{\partial f_1}{\partial z_2} d\bar{z}_1 + \frac{\partial f_1}{\partial z_1} d\bar{z}_2 = |z - \zeta^0|^{-4}((\bar{z}_2 - \bar{\zeta}_2^0) d\bar{z}_1 - (\bar{z}_1 - \bar{\zeta}_1^0) d\bar{z}_2)$$

on Ω. Let $h(z_1, z_2) = (z_1 - \zeta_1^0) f_2 + |z - \zeta^0|^{-2}(\bar{z}_2 - \bar{\zeta}_2^0)$. An easy computation shows that h is holomorphic on Ω. But $h(\zeta_1^0, z_2) = |z_2 - \zeta_2^0|^{-2}(\bar{z}_2 - \bar{\zeta}_2^0) = (z_2 - \zeta_2^0)^{-1}$ and therefore h cannot be holomorphically extended to Ω', giving a contradiction. $\square$

5. Traces of pluriholomorphic functions

We give an application of Theorem 3.3 to pluriholomorphic functions. The key point is that if $f = f_1 + f_2 j$ is ψ-regular, then f_1 is pluriholomorphic if and only if f_2 is pluriharmonic. Then we can apply the results on the traces of pluriharmonic functions given in [5] and [15] in order to obtain a characterization of the traces of pluriholomorphic functions (cf. [6, 7]).

Let $Harm_0^1(\Omega) = \{\phi \in C^1(\overline{\Omega}) \mid \phi \text{ is harmonic on } \Omega,\ \overline{\partial}_n \phi \text{ is real on } \partial\Omega\}$. This space of harmonic functions can be characterized by means of the Bochner-Martinelli operator of the domain Ω (cf. [15]).

Corollary 5.1. *Assume that Ω has connected boundary and satisfies the condition (*). Let $h \in \overline{W}_n^1(\partial\Omega)$. Then h is the trace of a harmonic pluriholomorphic function on Ω if and only if the following orthogonality condition is satisfied:*

$$\int_{\partial\Omega} h \overline{\partial} \phi \wedge d\zeta = 0 \quad \forall \phi \in Harm_0^1(\Omega). \tag{***}$$

Proof. From Theorem 3.3 we get $f_2 \in L^2(\partial\Omega)$ such that $f = h + f_2 j$ is the trace of a ψ-regular function on Ω. From Theorem 4 in [16] it follows that

$$\int_{\partial\Omega} h \overline{\partial} \phi \wedge d\zeta = -2 \int_{\partial\Omega} \bar{f}_2 \overline{\partial}_n \phi d\sigma \quad \forall\ \phi \in Harm^1(\Omega).$$

Therefore the orthogonality condition for h is equivalent to the pluriharmonic trace condition for $\bar{f}_2$. But the pluriharmonicity of the harmonic extension of $\bar{f}_2$ is equivalent to that of f_2 and to the pluriholomorphicity of the harmonic extension of h. $\square$

Remark 5.2. If Ω has a pluriholomorphic defining function ρ (as in the case of the unit ball B), then $h \in Phol(\Omega) \cap C^1(\overline{\Omega})$ implies that Lh is CR on $\partial\Omega$, since $Lh = \overline{\rho}_2 \dfrac{\partial h}{\partial \bar{z}_1} - \overline{\rho}_1 \dfrac{\partial h}{\partial \bar{z}_2}$ is holomorphic on Ω. In particular, if $h \in Phol(\Omega) \cap C^2(\overline{\Omega})$, then $LLh = 0$ on $\partial\Omega$.

Proposition 5.3. *On B condition (***) is equivalent to*

$$\int_S hL(sP_{s,t} + t\overline{P_{s,t}}) d\sigma = 0 \quad \forall P_{s,t} \in \mathcal{H}_{s,t}, \forall s, t > 0.$$

If $h \in C^1(S)$ and Lh is a CR-function on S, then h satisfies the condition (***).

Proof. In [15] it was shown that $Harm^1_0(B)$ is the space $\mathrm{Fix}(N_0) = \{\phi \in C^1(\overline{B}) : \phi$ is harmonic in B and $N_0(\phi) = \phi\}$, where N_0 is the real linear projection defined for $P_{s,t} \in \mathcal{H}_{s,t}$ by

$$N_0(P_{s,t}) = \begin{cases} \dfrac{s}{s+t} P_{s,t} + \dfrac{t}{s+t} \overline{P_{s,t}}, & \text{for } t > 0 \\ P_{s,t}, & \text{for } t = 0. \end{cases}$$

If $s = 0$ or $t = 0$, $LN_0(P_{s,t}) = 0$ and this proves the first part. If Lh is a CR-function, then to get (***) it must be shown that $(h, \overline{L}(s\overline{P_{s,t}} + tP_{s,t})) = 0$ for every $s > 0$, $t > 0$. For any $s > 0$, $\overline{L}$ is an isomorphism between $\mathcal{H}_{s,t}$ and $\mathcal{H}_{s-1,t+1}$. Then if $s, t > 0$, there exists Q such that $(h, \overline{L}(s\overline{P_{s,t}} + tP_{s,t})) = -(Lh, s\overline{P_{s,t}} + tP_{s,t}) = -(Lh, \overline{L}Q) = 0$ since Lh is CR on S.

It follows from Proposition 5.3 and the preceding remark that on the unit ball B the harmonic assumption for a pluriholomorphic function with trace $h \in C^1(S)$ can be removed. In particular, we get a result proved in [6] (cf. Proposition 6): h extends to a pluriholomorphic function on B if and only if $LLh = 0$ on S. Moreover, if $h \in Phol(B) \cap C^{1+\alpha}(\overline{B})$, $\alpha > 0$, then the harmonic extension $\tilde{h}$ of $h_{|S}$ on B is pluriholomorphic on B, since $h \in \overline{W}^1_n(S)$. Then $h = \tilde{h} + (|z|^2 - 1)g$, with g holomorphic, continuous on $\overline{B}$. The last assertion is a consequence of the following proposition.

Proposition 5.4. *If Ω has a pluriholomorphic defining function ρ and $h \in Phol(\Omega) \cap C^1(\overline{\Omega})$ vanishes on $\partial\Omega$, then there exists a holomorphic function $g \in C^0(\overline{\Omega})$ such that $h = \rho g$.*

Proof. Let $g \in C^0(\overline{\Omega})$ such that $h = \rho g$. Then $\bar{\rho}_i$ and $\dfrac{\partial h}{\partial \bar{z}_i} = \bar{\rho}_i g + \rho \dfrac{\partial g}{\partial \bar{z}_i}$ are holomorphic on Ω. We set

$$\tilde{g} = \begin{cases} \dfrac{1}{\bar{\rho}_1} \dfrac{\partial h}{\partial \bar{z}_1} & \text{where } \bar{\rho}_1 \neq 0, \\ \dfrac{1}{\bar{\rho}_2} \dfrac{\partial h}{\partial \bar{z}_2} & \text{where } \bar{\rho}_2 \neq 0. \end{cases}$$

Then there exists a neighbourhood V of $\partial\Omega$ such that $\tilde{g}$ is holomorphic on $V \cap \Omega$. Therefore $\tilde{g}$ extends holomorphically on Ω. Moreover, $\overline{\partial}(h - \rho\tilde{g}) = 0$ where $\bar{\rho}_1 \neq 0, \bar{\rho}_2 \neq 0$ and $h - \rho\tilde{g}$ vanishes on $\partial\Omega$. Then $h = \rho\tilde{g}$ on Ω by continuity and $g = \tilde{g}$ is holomorphic.

Remark 5.5. The boundary of a domain Ω with a pluriholomorphic defining function ρ is a quadric hypersurface or a hyperplane. The function ρ has the form

$$\rho = a_1|z_1|^2 + a_2|z_2|^2 + 2Re(\beta\bar{z}_1 z_2 + \alpha_1\bar{z}_1 + \alpha_2\bar{z}_2) + b$$

for some real a_1, a_2, b and complex $\alpha_1, \alpha_2, \beta$. Then, if Ω is bounded, it is indeed biholomorphic to the unit ball.

Example 5.6. As an example of a function $h \notin C^1(S)$ to which the criterion of Proposition 5.3 can be applied we can take $h = \bar{z}_2(1 - z_1)^{-1}$. This function is of class $\overline{W}^1_n(S)$ but h and $Lh \notin C^0(S)$. h satisfies the criterion and is pluriholomorphic on B. The function f_2 which exists according to Theorem 3.4 is, up to a CR-funciton, the pluriharmonic function $f_2 = \log(1 - \bar{z}_1)$, with squared norm $\|f_2\|^2_{L^2(S)} = \pi^2/6 - 1 < 1 = \|\overline{\partial}_n h\|^2_{L^2(S)}$.

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