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Examples on the Non-Uniqueness of the Rank 1 Tensor Decomposition of Rank 4 Tensors

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Abstract: We discuss the non-uniqueness of the rank 1 tensor decomposition for rank 4 tensors of format $m_1 \times \cdots \times m_k$, $k \geq 3$. We discuss several classes of examples and provide a complete classification if $m_1 = m_2 = 4$.

Keywords: tensor rank; rank 1 tensor decomposition; Segre variety; multiprojective space

MSC: 15A69; 14N05; 14N07

1. Introduction

Fix integers $k > 0$ and m_i, $1 \leq i \leq k$. Fix a base field $\mathbb{K}$, say algebraically closed, and let $V_1, \dots, V_k$ be $\mathbb{K}$-vector spaces of dimension $m_1, \dots, m_k$. An element $T \in V_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_k$ is called a tensor of format $(m_1, \dots, m_k)$. Now, assume $T \neq 0$. A rank 1 tensor is a tensor of the form $v_1 \otimes \cdots \otimes v_k$ with $v_i \in V_i$ and $v_i \neq 0$ for all i. The tensor T is said to be *concise* if there are not subspaces $W_i \subseteq V_i$, $i = 1, \dots, k$ with $W_i \neq V_i$ for at least one i and $T \in W_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} W_k$. The tensor rank $\mathrm{rank}(T)$ of T is the minimal integer m such that T is a sum of m rank 1 tensors ([1]). In many applications, it is very important to know if the rank 1 tensor decompositions $T = T_1 + \cdots + T_m$, $m = \mathrm{rank}(T)$, with each T_i a rank 1 tensor, are " unique ", i.e., unique up to the ordering of the $\mathrm{rank}(T)$ rank 1 tensors. The uniqueness is essential to have low rank robust approximations of tensors ([2,3]). There are many criteria to say that a specific T has a unique rank 1 tensor decomposition, starting with the famous Kruskal's criterion ([4–10]). Even the original Kruskal's criterion is sharp ([11]) and each of its extensions has a way to construct examples of non-uniqueness just outside the range of its assumptions. We recommend [10]; as far as we know, it is the more general one, well-written, full of references, and here, the main proofs are combinatorial.

Our work is different. We try to describe all tensors of low rank for which the rank 1 decomposition is not unique. This was done in [12] for rank 2 and rank 3 tensors. In this paper, we introduce a new class of tensors with non-unique rank 1 tensor decomposition with exactly $\mathrm{rank}(T)$ terms, tensors of Type II (they occur only in rank at least 4).

Take $T \in V_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_k$, $T \neq 0$, and any non-zero constant c. Obviously $\mathrm{rank}(T) = \mathrm{rank}(cT)$ and the rank 1 tensor decompositions of T and cT are the same. Thus, it is natural to work with the projective space $\mathbb{P}(V_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_k)$ and consider the rank 1 decompositions of the equivalence class $[T] \in \mathbb{P}(V_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_k)$. Set $r{:} = -1 + \prod_{i=1}^{k} m_i$. Note that $r = \dim \mathbb{P}(V_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_k)$. From now on, we call $\mathbb{P}^r$ the latter projective space and often call its elements " tensors " instead of " equivalent classes of non-zero tensors ". Set $n_i{:} = m_i - 1$, $1 \leq i \leq k$, and $Y{:} = \prod_{i=1}^{k} \mathbb{P}^{n_i}$. All our proofs involve the multiprojective space Y. We recall that $\mathbb{P}^r$ is the target of the Segre embedding ν of Y, i.e., the embedding of Y by the complete linear system $|\mathcal{O}_Y(1, \dots, 1)|$. For any set E in a projective space let $\langle E \rangle$ denote its linear span. For any $q = [T] \in \mathbb{P}^r$ the *solution set* $\mathcal{S}(q)$ is the set of all finite sets $A \subset Y$ such that $\#A = \mathrm{rank}(T)$ and $q \in \langle \nu(A) \rangle$. The solution set $\mathcal{S}(q)$ is exactly the set of all rank 1 tensor decompositions of T with $\mathrm{rank}(T)$ terms, up to an order of the addenda. The set $\mathcal{S}(q)$ has an algebraic structure (it is a constructible set in the Zariski topology) and so it makes sense to consider the integer $\dim \mathcal{S}(q)$ as in many of our results.

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Our aim is to introduce some ways to produce tensors with non-uniqueness. In [12], from rank 3 on the authors met the following type of non-uniqueness.

Definition 1. *Take $q \in \mathbb{P}^r$ concise for Y and $A, B \in \mathcal{S}(q)$ such that $A \neq B$. We say that the triple (q, A, B) is of Type I if there is $q' \in \mathbb{P}^r$ such that $A \neq B$, $A \cap B \neq \emptyset$, $q' \in \langle \nu(A \setminus A \cap B)\rangle \cap \langle \nu(B \setminus A \cap B)\rangle$ and $q \in \langle \{q'\} \cup \nu(A \cap B)\rangle$. We say that (q, A, B) has Type I_x if $x = \#(A \cap B)$. We say that q has Type I (resp. Type I_x) if there is a triple (q, A, B) of Type I (resp. I_x).*

In some cases, even if q has Type I, not all triples (q, A, B) are of Type I (Remark 1). Note that $\mathrm{rank}(q) = \mathrm{rank}(q') + \#(A \cap B)$ if (q, A, B) has Type I with q' as in Definition 1. Type I only occurs from rank 3 on (see [12] (Case (6) of Th. 7.1) for the rank 3 case). For rank 3 non-uniqueness of Type I was the only class which occurs for multiprojective spaces of large dimension. This is not true for tensors of rank at least 4. The main actor of this paper is the following definition.

Definition 2. *Take $q \in \mathbb{P}^r$ concise for Y and take $A, B \in \mathcal{S}(q)$ such that $A \neq B$. We say that the triple (q, A, B) is of Type II or it has Type II non-uniqueness if there are $q_1, q_2 \in \mathbb{P}^r$ (not necessarily concise for Y) and partitions $A = A_1 \sqcup A_2$, $B = B_1 \sqcup B_2$, such that $q \in \langle \{q_1, q_2\}\rangle$, $A_i, B_i \in \mathcal{S}(q_i)$ and $A_i \neq B_i$ for all $i = 1, 2$. We say that q has Type II if there are $A, B \in \mathcal{S}(q)$ such that $A \neq B$ and (q, A, B) has Type II.*

Type II only occurs from rank 4 on. Note that in Definition 2 the rank of q is the sum of the ranks of q_1 and q_2. Proposition 2 describes the multiprojective spaces having a Type II non-uniqueness for rank 4 concise tensors. In Section 5, we provide the examples needed to prove the following results.

Theorem 1. *Fix integers $k \geq 3$ and $n_i > 0$, $1 \leq i \leq k$, such that $n_1 + \cdots + n_k = 5$. Set $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Then there is a concise rank 4 tensor q with $\dim \mathcal{S}(q) > 0$ and (q, A, B) neither of Type I nor of Type II for any $A, B \in \mathcal{S}(q)$.*

Theorem 2. *Fix integers $k \geq 3$ and $n_i > 0$, $1 \leq i \leq k$, such that $n_i \leq 3$ for all i and $n_1 + \cdots + n_k = 6$. Set $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Then there is a concise rank 4 tensor q with $\dim \mathcal{S}(q) > 0$ and (q, A, B) neither of Type I nor of Type II for any $A, B \in \mathcal{S}(q)$.*

Conjecture 1. *We conjecture that if $k \geq 3$ and $n_1 + \cdots + n_k \geq 7$, then all rank 4 tensors q on $Y:= \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $\mathcal{S}(q)$ not a singleton is either of Type I_x, $x \in \{1, 2\}$, or of Type II.*

We do not know, in general, how to prove that if $A, B \in \mathcal{S}(q)$ and $A \cap B = \emptyset$, then (q, A, B) is of Type II. Remark 1 shows that in general not all triples (q, A, B) with q of rank 4 and $A \cap B = \emptyset$ have Type II. We prove the following result.

Theorem 3. *Take $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k \geq 3$, such that $n_1 = n_2 = 3$. Take any concise $q \in \langle \nu(Y)\rangle$ such that $\mathrm{rank}(q) = 4$ and $\#\mathcal{S}(q) > 1$. Then either (q, A, B) is of Type II for all $A, B \in \mathcal{S}(q)$ or there is $x \in \{1, 2\}$ such that (q, A, B) has Type I_x for all $A, B \in \mathcal{S}(q)$ and the set $A \cap B$ only depends on q.*

Proposition 1. *Assume $n_1 \geq \cdots \geq n_k > 0$, $k \geq 3$, such that $n_1 = 3$, $n_2 \leq 2$, $n_3 \leq 2$, $n_i = 1$ for all $i \geq 4$ and $n_2 + \cdots + n_k \geq 3$. Take $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Then there is a concise q with rank 4 such that $\dim \mathcal{S}(q) \geq 4$ and there is $A, B \in \mathcal{S}(q)$ with (q, A, B) of Type I_2.*

Remark 1. *Proposition 1 means that most sets $E \in \mathcal{S}(q)$ do not contain $A \cap B$. A dimensional count shows the existence of $E, F \in \mathcal{S}(q)$ such that $E \cap F = \emptyset$ (this is even true for the related rank 3 case [12] (Example 3.6)). This is one of the main technical problems to prove that q has Type I or Type II. Proposition 6 shows that this problem never occurs for Y and q as in Theorem 3, i.e., with* *the assumptions of Theorem 3 if there are $E, F \in \mathcal{S}(q)$ such that $E \cap F = \emptyset$, then q has Type II and all triples (q, A, B) have Type II.*

Section 7 shows that the results just stated are effective. Remarks 7–9 show how to test if a low rank tensor q has a triple (q, A, B) of Type II. We conclude the section with two open problems.

We work over an algebraically closed base field, but this case is extended to any infinite field (Remark 6). The examples work for finite fields with cardinality not very small, but we do not know in each case the minimal cardinality allowed for a field.

2. Preliminaries

Take a multiprojective space $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Fix $i = 1, \dots, k$. Let $\pi_i : Y \to \mathbb{P}^{n_i}$ denote the projection onto the i-th factor of Y. Set $Y_i := \prod_{h \neq i} \mathbb{P}^{n_h}$. Let $\eta_i : Y \to Y_i$ denote the projection (it is the map that forget the i-th component of any $p = (p_1, \dots, p_k) \in Y$). Let ε_i (resp. $\hat{\varepsilon}_i$) denote the multiindex $(a_1, \dots a_k) \in \mathbb{N}^k$ such that $a_i = 1$ and $a_j = 0$ for all $j \neq i$ (resp. $a_i = 0$ and $a_j = 1$ for all $j \neq i$).

Let $C \subset Y$ be an integral curve. The *multidegree* $(a_1, \dots, a_k) \in \mathbb{N}^k$ of C is defined by the formula $a_i := \deg(\mathcal{O}_C(\varepsilon_i))$, $1 \leq i \leq k$.

Take $A, B \in \mathcal{S}(q)$ such that $A \neq B$. Since $q \notin \langle \nu(E) \rangle$ for any $E \subsetneq A$ and any $E \subsetneq A$, $\nu(A)$ and $\nu(B)$ are linearly independent, i.e., $h^1(\mathcal{I}_A(1, \dots, 1)) = h^1(\mathcal{I}_B(1, \dots, 1)) = 0$. Since $q \in \langle \nu(A) \rangle \cap \langle \nu(B) \rangle$, $h^1(\mathcal{I}_{A \cup B}(1, \dots, 1)) > 0$.

Remark 2. *Take (q, A, B) of Type II. Then $h^1(\mathcal{I}_{A \cup B}(1, \dots, 1)) \geq 2$. Now, assume that q is concise and that it has rank 4. By [12] (Proposition 3.2) we have $\sum_{i=1}^{k} (n_i - 1) \leq 4$. We say that (q, A, B) splits if $\sum_i (n_i - 1) = 4$ (see Proposition 2 for the existence of split triples). If $n_1 \geq \cdots \geq n_k > 0$, (q, A, B) splits if and only if (omitting all $n_i = 1$) either $n_1 = n_2 = 3$ or $n_1 = 3$, $n_2 = n_3 = 2$ or $n_1 = n_2 = n_3 = n_4 = 2$. All Type II triples (q, A, B) of rank 4 which do no not split are obtained by a finite sequence of linear projections (in the sense of Section 2.1) from a Type II concise triple $(\tilde{q}, \tilde{A}, \tilde{B})$ on a bigger projective space $\tilde{Y}$ with the same number of factors, k, of Y.*

Remark 3. *Take (q, A, B) of Type I_x and set $E := A \cap B$. Since E is contained in an element, A, of $\mathcal{S}(q)$, $\nu(E)$ is linearly independent. Since $A \neq E$, $q \notin \langle \nu(E) \rangle$ and hence $\dim(\langle \nu(A) \rangle \cap \langle \nu(B) \rangle) \geq x$.*

Remark 4. *Take (q, A, B) of Type II. Obviously, $h^1(\mathcal{I}_{A \cup B}(1, \dots, 1)) \geq 2$ and $\dim \langle \nu(A) \rangle \cap \langle \nu(B) \rangle \geq 1$.*

Lemma 1. *Take any finite set $A \subset Y$ evincing the rank of a tensor, q. Let k be the number of factors of Y. If $u, v \in A$ and $\pi_i(u) = \pi_i(v)$ for at least $k - 1$ indices i, then $u = v$.*

Proof. Assume $u \neq v$. Since $\eta_j(u) = \eta_j(v)$ for some j, there is $L \subset Y$ such that $\nu(L)$ is a line contained in the j-th ruling of the Segre variety $\nu(Y)$ and $\{u, v\} \subset L$. Since $u, v \in L$ and $u \neq v$, $\nu(L) \subseteq \langle \nu(A) \rangle$. Since $\nu(L)$ is a line, there is $w \in L$ such that $q \in \langle \nu((A \setminus \{u, v\}) \cup \{w\}) \rangle$. Thus, q has rank at most $\#A - 1$, a contradiction. □

Lemma 2. *Assume $k \geq 3$. Take a concise q of rank 4 such that $\#\mathcal{S}(q) > 1$ and take $A, B \in \mathcal{S}(q)$ such that $A \neq B$. Then $\#(A \cap B) \leq 2$.*

Proof. Assume $\#(A \cap B) = 3$. Since $\nu(A)$ and $\nu(B)$ are linearly independent, $\langle \nu(A \cap B) \rangle$ is a hyperplane of $\langle \nu(A) \rangle$ and of $\langle \nu(B) \rangle$. Since $q \in \langle \nu(A) \rangle \cap \langle \nu(B) \rangle$, $q \notin \langle \nu(A') \rangle$ for any $A' \subsetneq A$ and $q \notin \langle \nu(B') \rangle$ for any $B' \subsetneq B$, we get $\langle \nu(A) \rangle = \langle \nu(B) \rangle$. Hence, $\langle \nu(A) \rangle$ is a 3-dimensional space containing at least 5 points of $\nu(Y)$. Since Y is the minimal multiprojective space containing A and $k \geq 3$, [13] gives $k = 3$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. No such q has rank 4 ([1] (Theorem 3.11.1.1)). □

Lemma 3. *Assume $k \geq 3$. Take a concise q such that there are $A, B \in \mathcal{S}(q)$ such that $A \neq B$ and $A \cap B \neq \emptyset$. Set $x := \#(A \cap B)$. Then (q, A, B) is of Type I_x.*

Proof. Set $A' := A \setminus A \cap B$ and $B' := B \setminus B \cap A$. Since q is concise, $q \notin \langle \nu(A \cap B) \rangle$. Thus, $\langle \nu(A') \rangle \cap \langle \nu(B') \rangle \neq \emptyset$, i.e., $h^1(\mathcal{I}_{A' \cup B'}(1, \dots, 1)) > 0$, and there is $q' \in \langle \nu(A') \rangle \cap \langle \nu(B') \rangle$ such that $q \in \langle \{q'\} \cup \nu(A \cap B) \rangle$. Since A evinces a rank, A' evinces the rank of q'. Thus, (q, A, B) is of Type I_x. □

Lemma 4. *Take $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and take any $q \in \langle \nu(Y) \rangle \setminus \nu(Y)$. Then q has rank 2, $\mathcal{S}(q)$ is isomorphic to the complement of a smooth conic in a projective plane and $\cup_{A \in \mathcal{S}(q)} A = Y \setminus C$ where $C \in |\mathcal{O}_Y(1,1)|$ and C is smooth. Fix any $A \in \mathcal{S}(q)$. Then $\pi_{1|A}$ and $\pi_{2|A}$ are injective, $h^1(\mathcal{I}_A(1,0)) = h^1(\mathcal{I}_A(0,1)) = 0$ and Y is the minimal multiprojective space containing A.*

Proof. We have $\langle \nu(Y) \rangle \cong \mathbb{P}^3$ and $\nu(Y)$ is a smooth quadric. Fix a line $L \subset \mathbb{P}^3$ containing q. Since $q \notin \nu(Y)$, Bezout theorem gives $\deg(L \cap \nu(Y)) = 2$. Thus, either L is tangent to $\nu(Y)$ and meets $\nu(Y)$ only at the tangency point or $L \cap \nu(Y)$ is the union of 2 distinct points and $(L \cap \nu(Y)) \in \mathcal{S}(q)$. The set of all $p \in \nu(Y)$ such that the line $\langle \{p, q\} \rangle$ is tangent to $\nu(Y)$ is the polar conic D of q with respect to $\nu(Y)$ and D is smooth. Write $D = \nu(C)$ with $C \in |\mathcal{O}_Y(1,1)|$. Note that $\cup_{A \in \mathcal{S}(q)} A = Y \setminus C$. The set of all lines of $\mathbb{P}^3$ passing through q is a projective plane and the set of all lines through q and meeting the plane section D of $\nu(Y)$ is a smooth conic of this projective plane. Fix $A \in \mathcal{S}(q)$, say $A = \{a, b\}$. Set $L := \langle \{q, \nu(a)\} \rangle$. Since $q \notin \nu(Y)$, $L \nsubseteq \nu(Y)$. Thus, $\pi_{1|A}$ and $\pi_{2|A}$ are injective. Hence, Y is the minimal multiprojective space containing A and $h^1(\mathcal{I}_A(1,0)) = h^1(\mathcal{I}_A(0,1)) = 0$. □

2.1. Linear Projections

We use the following construction, called *linear projection from a point of the i-th factor*. Fix $i \in \{1, \dots, k\}$, $o \in \mathbb{P}^{n_i}$ and a concise $q \in \mathbb{P}^r = \langle \nu(Y) \rangle$. Let Y' be the product of k projective spaces $\mathbb{P}^{m_1}, \dots, \mathbb{P}^{m_k}$ with $m_j = n_j$ if $j \neq i$ and $m_i = n_i - 1$, with the convention that $\mathbb{P}^0$ is a point if $n_i = 1$. Call ν also the Segre embedding of Y'. Call Y'' the multiprojective space with k factors, one of them being a point, with $\mathbb{P}^{n_j}$ as a factor if $j \neq i$ and $\{o\}$ as its i-th factor. Let $\ell_{i,o} : Y \setminus Y'' \to Y'$ be the morphism which is the identity map for all factors $j \neq i$, while on the i-th factor $\ell_{i,o}$ is the linear projection from o. Since $Y'' \subset Y$ we see $\langle \nu(Y'') \rangle$ as a linear subspace of $\langle \nu(Y) \rangle$. Thus, the linear projection μ of $\mathbb{P}^r$ from its linear subspace $\langle \nu(Y'') \rangle$ is well-defined outside $\langle \nu(Y'') \rangle$. We may see $\langle \nu(Y') \rangle$ as the target of μ, i.e., we may see μ as a submersion $\mu : \langle \nu(Y) \rangle \setminus \langle \nu(Y'') \rangle \to \langle \nu(Y') \rangle$ with all fibers isomorphic to $\mathbb{A}^1$. Since q is concise, $q \notin \langle \nu(Y'') \rangle$ and hence $\mu(q) \in \langle \nu(Y') \rangle$ is well-defined. Since q is concise for Y, $\mu(q)$ is concise for Y'. We say that a multiprojective space W is obtained from Y by *a finite sequence of linear projections* if there is a finite sequence of linear projections from a point of one of the factors; at different steps we allow to change the factor. Note that $\dim Y - \dim W$ is the number of linear projections from one point used to get W from Y. If q is concise, we get a unique $q' \in \langle W \rangle$ concise for W iterating the definition of μ.

3. Existence Results for Type I and Type II Tensors

Proposition 2. *Write $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k \geq 3$, such that $n_1 \geq \cdots \geq n_k > 0$.*

(i) If Y has a type II concise rank 4 triple (q, A, B), then

$$n_1 \leq 3 \text{ and } \sum_{i=1}^{k} (n_i - 1) \leq 4 \tag{1}$$

(ii) Assume (1). Y has a Type II concise rank 4 triple (q, A, B) if either $n_1 = n_2 = 3$ or $n_2 = 2$ and $k \geq 4$ or $k \geq 5$.

Proof. Take q_1, q_2, $A = A_1 \sqcup A_2$, $B = B_1 \sqcup B_2$ as in Definition 2. Since $\#A_i = \#B_i \geq 2$, we get that q_1 and q_2 have rank 2. Let Y' (resp. Y'') be the minimal multiprojective space containing q_1 (resp. q_2). We have $Y' \cong Y'' \cong \mathbb{P}^1 \times \mathbb{P}^1$ ([14] (Proposition 3.2)). Thus, $n_1 \le 3$ and $\sum_{i=1}^{k}(n_i - 1) \le 4$ are the conditions required to be the minimal multiprojective space containing 2 different multiprojective subspaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, concluding the proof of (i). Now, we prove part (ii).

(a) Assume $n_1 = 3$. Since the case $n_1 = n_2 = 3$ is done in Proposition 6, we assume $n_2 \le 2$. Fix lines $L, R \subset \mathbb{P}^3$ such that $L \cap R = \emptyset$, i.e., $\langle L \cup R \rangle = \mathbb{P}^3$. Fix lines $D \subseteq \mathbb{P}^{n_2}$ and $D' \subseteq \mathbb{P}^{n_3}$. Fix $o_i \in \mathbb{P}^{n_i}$, $3 \le i \le k$ and u_j, $j = 2$ and $j \ge 4$. For $3 < i \le k$ assume $o_i \ne u_i$. If $n_2 = 2$, assume $u_2 \notin R$. If $n_3 = 2$ assume $o_3 \notin D'$. Set $Y' := L \times D \times \{o_3\} \times \cdots \times \{o_k\}$ and $Y'' := R \times \{u_2\} \times D' \times \cdots \times \{u_k\}$. Fix a general $q_1 \in \langle \nu(Y') \rangle$, a general $q_2 \in \langle \nu(Y'') \rangle$ and a general $(A_i, B_i) \in \mathcal{S}(q_i) \times \mathcal{S}(q_i)$, $i = 1, 2$. Set $A := A_1 \cup A_2$ and $B = B_1 \cup B_2$. Take a general $q \in \langle \{q_1, q_2\} \rangle$. Since $n_1 = 3$, to prove that (q, A, B) has Type II and that q is concise it is sufficient to prove that q is concise. Note that Y is the minimal multiprojective space containing A. Assume that q is not concise. Thus, there is $i \in \{1, \dots, k\}$ such that $q \in \langle \nu(H) \rangle$ for some $H \in |\mathcal{O}_Y(\varepsilon_i)|$. Since $A \nsubseteq H$, Ref. [14] (Lemma 5.1) gives $h^1(\mathcal{I}_{A \setminus A \cap H}(\hat{\varepsilon}_i)) > 0$. Thus $h^1(\mathcal{I}_A(\hat{\varepsilon}_i)) > 0$. First assume $i > 1$. Note that $\langle \pi_1(A) \rangle = \langle L \cup R \rangle = \mathbb{P}^3$ (Lemma 4). Thus, $h^1(\mathcal{I}_A(\varepsilon_1)) = 0$. Hence, $h^1(\mathcal{I}_A(\hat{\varepsilon}_i)) = 0$, a contradiction. Now assume $i = 1$. Take $E \in \mathcal{S}(q)$. By concision ([1] (Proposition 3.1.3.1)) $E \subset H$. Thus, Ref. [14] (Lemma 5.1) gives $h^1(\mathcal{I}_{A \setminus A \cap H}(\hat{\varepsilon}_1)) > 0$. Thus, $h^1(\mathcal{I}_A(\hat{\varepsilon}_1)) > 0$. Since $k \ge 3$, we get $h^1(\mathcal{I}_A(0, 1, 1, 0, \dots)) > 0$, contradicting Lemma 4.

(b) Assume $n_1 = 2$ and $k \ge 4$.

(b1) Assume $n_4 = 1$. We take as L, R two general lines of $\mathbb{P}^{n_1}$, D as a line of $\mathbb{P}^{n_2}$ and D' as a line of $\mathbb{P}^{n_3}$. We also take points $o_i \in \mathbb{P}^{n_i}$ for all $i \ge 3$ and points $u_i \in \mathbb{P}^{n_i}$ for $i = 2$ and $i \ge 4$ such that $u_i \ne o_i$ for all $i \ge 4$. If $n_2 = 2$ assume $u_2 \notin D$. If $n_3 = 2$ assume $o_3 \notin D'$. Set $Y' := L \times D \times \{o_3\} \times \cdots \times \{o_k\}$ and $Y'' := R \times \{u_2\} \times D' \times \{u_4\} \cdots \times \{u_k\}$. Note that Y is the minimal multiprojective space containing $Y' \cup Y''$ and hence (Lemma 4) the minimal multiprojective space containing A (or containing B). Thus, to prove that (q, A, B) is a concise triple of Type II it is sufficient to prove that q has rank 4. Assume that q has rank ≤ 3 and take $E \in \mathcal{S}(q)$. Since $\#E \le 3$, there is $H_1 \in |\mathcal{O}_Y(\varepsilon_1)|$ and $H_4 \in |\mathcal{O}_Y(\varepsilon_4)|$ such that $E \subset H_1 \cup H_4$. Since $o_4 \ne u_4$, $A \nsubseteq H_1 \cup H_4$. Thus, $h^1(\mathcal{I}_{A \setminus (A \cap (H_1 \cup H_4))}(0, 1, 1, 0, \dots)) > 0$ ([14] (Lemma 5.1)). Hence, $h^1(\mathcal{I}_A(0, 1, 1, 0, \dots)) > 0$, contradicting Lemma 4.

(b2) Assume $n_4 = 2$. This case is easier, but we need different Y' and Y'' to get a concise q. We take as L, R, D, D' lines of $\mathbb{P}^2$ and points $o_i \in \mathbb{P}^{n_i}$, $i \ge 3$, $u_i \in \mathbb{P}^{n_i}$, $i = 1, 2$ and $i \ge 4$. We assume $o_i \ne u_i$ for $i > 4$, $u_1 \notin L$, $u_2 \notin D$, $o_3 \notin R$ and $o_4 \notin D'$. We take $Y' = L \times D \times \{o_3\} \times \cdots \times \{o_k\}$ and $Y' := \{u_1\} \times \{u_2\} \times R \times D' \times \cdots$. Then we continue as in step (b1).

(c) Assume $k \ge 5$. By steps (a) and (b) we may assume $n_1 = 1$. Fix $o_i \in \mathbb{P}^1, 3 \le i \le k$ and $u_i \in \mathbb{P}^1$, $i = 1, 2$ and $5 \le i \le k$. Assume $o_i \ne u_i$ for all $i \ge 5$. We take $Y' = \mathbb{P}^1 \times \mathbb{P}^1 \times \{o_3\} \times \cdots \times \{o_k\}$ and $Y'' = \{u_1\} \times \{u_2\} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \{u_5\} \times \cdots \{u_k\}$. We first check that q is concise. Assume that q is not concise. Thus, there is $i \in \{1, \dots, k\}$ such that $q \in \langle \nu(H_i) \rangle$ for some $H_i \in |\mathcal{O}_Y(\varepsilon_i)|$. Since $A \nsubseteq H_i$, [14] (Lemma 5.1) gives $h^1(\mathcal{I}_{A \setminus A \cap H}(\hat{\varepsilon}_i)) > 0$. Thus $h^1(\mathcal{I}_A(\hat{\varepsilon}_i)) > 0$. This is false because $h^1(\mathcal{I}_{A_1}(\varepsilon_j)) = 0$ for $j = 1, 2$ and $h^1(\mathcal{I}_{A_2}(\varepsilon_h)) = 0$ for $h = 3, 4$ (Lemma 4). Thus, it is sufficient to prove that q has rank 4. Assume that q has rank ≤ 3 and take $E \in \mathcal{S}(q)$. Take $a_i \in \mathbb{P}^1$, $i = 2, 4, 5$ such that $E \subset H_2 \cup H_4 \cup H_5$, where $H_i := \pi_i^{-1}(a_i)$. Since $\pi_2(A_1)$ spans $\mathbb{P}^1$ and $\pi_3(A_2)$ spans $\mathbb{P}^1$ (Lemma 4), we have $h^1(\mathcal{I}_A(1, 0, 1, 0, 0)) = 0$, contradicting [14] (Lemma 5.1). Now assume $A \subset H_2 \cup H_4 \cup H_5$. By Lemma 4 $\cup_{B_1 \in \mathcal{S}(q_1)} B_1$ is a non-empty open subset of Y' and that $\cup_{B_2 \in \mathcal{S}(q_2)} B_2$ is an open subset of Y''. Thus, taking another E_1, E_2, $E_i \in \mathcal{S}(q_i)$ instead of A_1, A_2 we get $Y' \cup Y'' \subseteq H_2 \cup H_3 \cup H_5$ contradicting the assumption $o_5 \ne u_5$. □

Proposition 3. *Write $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k \ge 3$, with $n_1 \ge \cdots \ge n_k > 0$. Assume $n_1 \le 3$, $n_2 \le 3$ and $k \ge 5$. Then Y has a concise rank 2 tensor q of Type I_2.*

Proof. Take lines $L \subseteq \mathbb{P}^{n_1}$, $R \subseteq \mathbb{P}^{n_2}$ and $o_i \in \mathbb{P}^{n_i}$, $3 \le i \le k$, and set $Y' := L \times R \times \{o_3\} \times \cdots \times \{o_k\} \subset Y$. Fix a general $q' \in \langle \nu(Y') \rangle$ and take $A' \in \mathcal{S}(q')$. Note that $\#A' = 2$. Fix

2 generals $u, v \in Y$ and set $A := \{u, v\} \cup A'$. Note that Y is the minimal multiprojective space containing A. Fix a general $q \in \langle \{q', \nu(u), \nu(v)\} \rangle$. To complete the proof it is sufficient to prove that q is concise and that it has rank 4. Fix $E \in \mathcal{S}(q)$.

(a) Assume that q is not concise, i.e., assume the existence of $i \in \{1, \dots, k\}$ and $H \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $q \in \langle \nu(H) \rangle$. Concision ([1] (Proposition 3.1.3.1)) gives $E \subset H$. We have $h^1(\mathcal{I}_{A \setminus A \cap H}(\hat{\varepsilon}_i)) > 0$ ([14] (Lemma 5.1)) and hence $h^1(\mathcal{I}_A(\hat{\varepsilon}_i)) > 0$. Lemma 4 gives $h^1(\mathcal{I}_{A'}(\varepsilon_h)) = 0$ for $h = 1, 2$ and hence for some $h \neq i$. Since we took (u, v) general in $Y \times Y$ after fixing A', we get $h^1(\mathcal{I}_A(\hat{\varepsilon}_i)) = 0$, a contradiction.

(b) Now, we prove that q has rank 4, i.e., $\#E = 4$. Assume $\#E \leq 3$. Set $S := A \cup E$. Since $q \notin \langle \nu(A_1) \rangle$ for any $A_1 \subsetneq A$, $h^1(\mathcal{I}_S(1, \dots, 1)) > 0$. Take a general $M \in |\mathcal{O}_Y(\varepsilon_k)|$ such that $o_k \in \pi_k(M)$. By [14] (Lemma 5.1) $h^1(\mathcal{I}_{S \setminus S \cap M)}(\hat{\varepsilon}_k)) > 0$. Recall that u, v are general in Y and that $A' \subset S \cap M$. Hence, $\#(S \setminus S \cap M) \leq 5$. Since E evinces a rank, $\eta_{k|E}$ is injective (Lemma 1). Thus $\#(\eta_k(S \setminus S \cap M)) = \#(S \setminus S \cap M)$ and $h^1(Y_k, \mathcal{I}_{\eta_k(S \setminus S \cap M)}(1, \dots, 1)) > 0$. The generality of $(u, v) \in Y \times Y$ means that the minimal multiprojective space containing $\{\eta_k(u), \eta_k(v)\}$ is isomorphic to $(\mathbb{P}^1)^{k-1}$. Since $\#E \leq 3$ and $k \geq 5$, Ref. [13] (Theorem 1.1 and Proposition 6.2) provide a contradiction. □

Definition 3. *Take a concise q of rank 4. We say that (q, A, B) has Type $I_2(e)$ if it has Type I_2 and $e = h^1(\mathcal{I}_{A \cup B}(1, \dots, 1))$.*

Proposition 4. *Assume $k \geq 3$. Take a concise q of rank 4 such that (q, A, B) has Type $I_2(e)$. Set $A' := A \setminus A \cap B$ and $B' := B \setminus A \cap B$. Let $Y' \subset Y$ be the minimal multiprojective space containing A'.*

(a) $Y' \cong \mathbb{P}^1 \times \mathbb{P}^1$ and there is a smooth curve $C \subset Y'$ of bidegree $(1, 1)$ containing $A' \cup B'$.

(b) We have $1 \leq e \leq 2$ and $e = 2$ if and only if $C \cap (A \cap B) \neq \emptyset$.

(c) If (q, A, B) has Type $I_2(2)$, then $\#(C \cap (A \cap B)) = 1$, $n_i \leq 2$ for the 2 factors of Y containing Y' and $n_h = 1$ for all h such that $\pi_h(Y')$ is a point.

Proof. Since $A \neq B$, $h^1(\mathcal{I}_{A \cup B}(1, \dots, 1)) > 0$. Take q' in the definition of Type I_2. Concision says that q' is concise for Y' and that Y' is the minimal multiprojective space containing B'. Since $A', B' \in \mathcal{S}(q')$, Ref. [12] (Proposition 3.2) gives $Y' \cong \mathbb{P}^1 \times \mathbb{P}^1$. Note that $h^1(\mathcal{I}_{A' \cup B'}(1, 1)) = 1$, i.e., q' is in a unique hyperplane section $\nu(C)$ of $\nu(Y')$. Up to the identification of Y' with $\mathbb{P}^1 \times \mathbb{P}^1$ we have $C \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)|$. Since Y' is the minimal multiprojective space containing either A' or B', C is smooth.

Since Y is the minimal multiprojective space containing A, $\#(C \cap (A \cap B)) \leq 1$ and $h^1(\mathcal{I}_{A \cup B}(1, \dots, 1)) = h^1(\mathcal{I}_{(A \cup B) \cap C}(1, \dots, 1))$. □

4. $Y = \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^1$

We start with a result true for all multiprojective spaces $Y = \mathbb{P}^m \times \mathbb{P}^m \times \mathbb{P}^1$, $m \geq 2$, although we only need the case $m = 3$. The following result is an easy consequence of [10] (Theorem 2). The strength of [10] (Theorem 2) shows the usefulness of [10].

Proposition 5. *Write $Y = \mathbb{P}^m \times \mathbb{P}^m \times \mathbb{P}^1$ for some $m \geq 2$. Fix $A \subset Y$ such that $\#(A) = \#(\pi_3(A)) = m + 1$ and $\langle \pi_i(A) \rangle = \mathbb{P}^m$ for $i = 1, 2$. Take any $q \in \langle \nu(A) \rangle$ such that $q \notin \langle \nu(A') \rangle$ for any $A' \subsetneq A$. Then q is concise and $\mathcal{S}(q) = \{A\}$.*

Proof. Since Y is the minimal multiprojective space containing A, it is sufficient to prove that $\{A\} = \mathcal{S}(q)$. If not, by [10] (Theorem 2)). there is a set $E \subseteq A$ such that $\#E \geq 2$ and

$$2(\#E) \geq 2 + \sum_{i=1}^{3} \dim \langle \pi_i(E) \rangle \tag{2}$$

Since $\langle \pi_i(A) \rangle = \mathbb{P}^m$ for $i = 1, 2$, we have $\dim \langle \pi_i(E) \rangle = \#E - 1$ for $i = 1, 2$. Since $\pi_{3|A}$ is injective and $\#E \geq 2$, $\langle \pi_3(E) \rangle = \mathbb{P}^1$. Thus, (2) fails. □

Remark 5. *Note that Proposition 5 gives the uniqueness of the tensor decomposition for a general tensor of format $(m+1)\times(m+1)\times 2$.*

Theorem 4. *Take $Y = \mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^1$ and q concise for Y, with rank 4 and with $\#\mathcal{S}(q) > 1$. Take $A, B \in \mathcal{S}(q)$ such that $A \neq B$. Then either (q, A, B) is of Type II or it is of type I_x with $x \in \{1,2\}$.*

Proof. Set $S{:} = A\cup B$. Since Y is the minimal multiprojective space containing A, $\#\pi_3(A) \geq 2$. Proposition 5 gives $2 \leq \#\pi_3(A) \leq 3$.

Claim 1. *For each $u \in \mathbb{P}^1$ we have $\#(A\cap\pi_3^{-1}(u)) = \#(B\cap\pi_3^{-1}(u))$.*

Proof. Set $H{:} = \pi_3^{-1}(u) \in |\mathcal{O}_Y(\varepsilon_3)|$. Assume that Claim 1 fails for u. Exchanging if necessary the role of A and B we may assume that $e{:} = \#(H\cap A) > f{:} = \#(H\cap B)$. We have $\#(A\setminus A\cap H) = 4-e$ and $\#(B\setminus B\cap H) = 4-f > 4-e$. Concision gives $e \leq 3$. Let M be a general element of $|\mathcal{I}_{A\setminus A\cap H}(\varepsilon_2)|$. Since $\langle\pi_2(A)\rangle = \langle\pi_2(B)\rangle = \mathbb{P}^3$, $\#(B\cap M) \leq 4-e$. Since $A \subset H\cup M$, $B \nsubseteq H\cup M$ and $h^1(\mathcal{I}_B(1,0,0)) = 0$, Ref. [14] (Lemma 5.1) gives a contradiction and proves Claim 1. □

Observation 1. *Claim 1 gives $\pi_3(A) = \pi_3(B)$.*

Since $\#\pi_3(A) \in \{2,3\}$, there is $o \in \mathbb{P}^1$ such that $g{:} = \#(H\cap A) > 1$, where $H{:} = \pi_3^{-1}(o)$. Set $A = A'\sqcup A''$ and $B = B'\sqcup B''$ with $A' = H\cap A$ and $B' = H\cap B$.

Claim 2. *$\eta_{3|S}$ is injective.*

Proof. Assume the existence of $u, v \in S$ such that $\eta_3(u) = \eta_3(v)$ and $u \neq v$. Lemma 1 gives $\#(\{u,v\}\cap A) = 1$, say $u \in A\setminus A\cap B$ and $v \in B\setminus A\cap B$. Take $H_2 \in |\mathcal{O}_Y(\varepsilon_2)|$ such that $(A\setminus\{u\}) \subset H_2$. Since Y is the minimal multiprojective space containing A, $u \notin H_2$. Claim 1 gives $\#(H_2\cap B) = 3$. Since $\pi_2(u) = \pi_2(v)$, $v \notin H_2$. Since $u \neq v$, Ref. [14] (Lemma 5.1) gives $h^1(\mathcal{I}_{\{u,v\}}(1,0,1)) > 0$, contradicting the assumption $\pi_3(u) \neq \pi_3(v)$. □

By Claim 1 and Observation 1 there is $o \in \mathbb{P}^1$ such that $e{:} = \#(A\cap\pi_3^{-1}(o)) = \#(B\cap\pi_3^{-1}(o)) \geq 2$. Set $H{:} = \pi_3^{-1}(o) \in |\mathcal{O}_Y(\varepsilon_2)|$. Write $A = A_1\sqcup A_2$ and $B = B_1\sqcup B_2$ with $A_1 = A\cap H$ and $B_1 = B\cap H$.

(a) Assume $A\cap B = \emptyset$. First assume $e = 3$. Thus, $S\setminus S\cap H = \{u,v\}$ with $u \in A$ and $v \in B$. Since $A\cap B = \emptyset$, $u \neq v$. Thus, Ref. [14] (Lemma 5.1) gives $h^1(\mathcal{I}_{\{u,v\}}(1,1,0)) > 0$. Since $\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^3}(1,1)$ is very ample, we get $\eta_3(u) = \eta_3(v)$, contradicting Claim 2. Assume $e = 2$. Since $A_2\cap B_2 = \emptyset$, Ref. [14] (Lemma 5.1) gives $h^1(\mathcal{I}_{A_2\cup B_2}(1,1,0)) > 0$. Take a general $M \in |\mathcal{I}_{A_2}(0,1,0)|$. Since $A \subset H\cup M$ and $h^1(\mathcal{I}_{B_2}(1,0,0)) = 0$, we get $B_2 \subset M$. In the same way we see that A_2 is contained in the general $M' \in |\mathcal{I}_{B_2}(0,1,0)|$. Take a general $D \in |\mathcal{I}_{A_1}(0,1,0)|$. Claim 2 gives $\#(B\cap D) = 2$. Since $\langle\pi_2(A)\rangle = \langle\pi_2(B)\rangle = \mathbb{P}^3$, we get $D\cap B = B_1$. Similarly a general $D' \in |\mathcal{I}_{A_1}(1,0,0)|$ satisfies $D'\cap B = B_1$. Thus, there are lines $L_1, R_1 \subset \mathbb{P}^3$ such that $A_1\cup B_1 \subset L_1\times R_1\times\{o\}$.

Claim 3. *We have $\#\pi_3(A_2) = 1$.*

Proof. Assume $\#\pi_3(A_2) = 2$, say $\{o_1, o_2\} = \pi_3(A_2)$. We get $\#(\pi_3^{-1}(o_i)\cap A) = 1$ and Claim 1 gives $\#(\pi_3^{-1}(o_i)\cap B) = 1$. Set $\{u\}{:} = A_2\cap\pi_3^{-1}(o_2)$ and $\{v\}{:} = B_2\cap\pi_3^{-1}(o_2)$. Take a general $H_2 \in |\mathcal{I}_{A_1}(\varepsilon_2)|$. Note that $A\setminus(H_2\cup\pi_3^{-1}(o_1))\cap A = \{u\}$ and $B\setminus(H_2\cup\pi_3^{-1}(o_1))\cap B = \{v\}$. We have $h^1(\mathcal{I}_{\{u,v\}}(1,0,0)) > 0$ by [14] (Lemma 5.1). Thus, $\pi_1(u) = \pi_1(v)$. Using a general $H_1 \in |\mathcal{I}_{A_1}(\varepsilon_1)|$ instead of H_2 we get $\pi_2(u) = \pi_2(v)$. Thus, $\eta_3(u) = \eta_3(v)$, contradicting Claim 2. □

Since $\#\pi_3(A_2) = 1$ (Claim 3), we get in the same way the existence of lines L_2, R_2 of $\mathbb{P}^3$ and $o' \in \mathbb{P}^1\setminus\{o\}$ such that $A_2\cup B_2 \subset L_2\times R_2\times\{o'\}$. Moreover $\langle L_1\cup L_2\rangle = \langle R_1\cup R_2\rangle = \mathbb{P}^3$.

To conclude that q is of Type II it is sufficient to prove that $\langle\nu(A)\rangle\cap\langle\nu(B)\rangle=\langle((\langle\nu(A_1)\rangle\cap\langle\nu(B_1)\rangle)\cup(\langle\nu(A_2)\rangle\cap\langle\nu(B_2)\rangle))\rangle$. Since $A\cap B=\emptyset$, we have $\dim(\langle\nu(A)\rangle\cap\langle\nu(B)\rangle)=h^1(\mathcal{I}_{A\cup B}(1,1,1))-1$, $\dim(\langle\nu(A_1)\rangle\cap\langle\nu(B_1)\rangle)=h^1(\mathcal{I}_{A_1\cup B_1}(1,1,1))-1$, and $\dim(\langle\nu(A_2)\rangle\cap\langle\nu(B_2)\rangle)=h^1(\mathcal{I}_{A_2\cup B_2}(1,1,1))-1$ (here, we use that $\langle\nu(A_1)\rangle\cap\langle\nu(B_1)\rangle\neq\emptyset$ and $\langle\nu(A_1)\rangle\cap\langle\nu(B_1)\rangle\neq\emptyset$). The residual exact sequence of H gives $h^1(\mathcal{I}_{A\cup B}(1,1,1))\le h^1(\mathcal{I}_{A_1\cup B_1}(1,1,1))$ $+h^1(\mathcal{I}_{A_2\cup B_2}(1,1,1))$. Let Y' be the minimal multiprojective space containing A_1 and let Y'' be the minimal multiprojective space containing A_2. Since Y is the minimal multiprojective space containing A, it is the minimal multiprojective space $Y'\cup Y''$, $\langle\nu(Y')\rangle\cap\langle\nu(Y'')\rangle=\emptyset$ and $\langle\nu(Y)\rangle=\langle\nu(Y')\cup\nu(Y'')\rangle$. Thus, (q,A,B) has Type II.

(b) Assume $\#(A\cap B)=1$. Set $A_1:=A\setminus A\cap B$ and $B_1:=B\setminus A\cap B$. Set $\{M\}:=|\mathcal{I}_{A_1}(1,0,0)|$. Since $\langle\pi_1(A)\rangle=\mathbb{P}^3$, $M\cap A=A_1$. Claim 1 gives $M\cap B=B_1$. Since $\langle\pi_1(A)\rangle=\mathbb{P}^3$ and $\eta_{3|S}$ is injective (Claim 2), $h^1(\mathcal{I}_{A\cap B}(1,0,0))=0$. Hence, the residual exact sequence of M gives $h^1(M,\mathcal{I}_{A_1\cup B_1}(1,1,1))>0$. Take a general $q_1\in(\langle\nu(A_1)\rangle\cap\langle\nu(B_1)\rangle)$. Since $\langle\pi_2(A)\rangle=\mathbb{P}^3$, $h^1(\mathcal{I}_{A\cap B}(0,1,1))=0$. Since $\dim A\cap B=0$, we have $h^2(\mathcal{I}_{A\cap B}(0,1,1))=h^2(\mathcal{O}_Y(0,1,1))=0$. Thus, the residual exact sequence of M gives an isomorphism

$$\tau: H^1(\mathcal{I}_{A\cup B}(1,1,1))\to H^1(M,\mathcal{I}_{A_1\cup B_1}(1,1,1))\cong H^1(\mathcal{I}_{A_1\cup B_1}(1,1,1)).$$

Let Y' be the minimal multiprojective space containing A_1. Since A_1 and B_1 evince the rank of q_1, $B_1\subset Y'$ and $\nu(Y')$ is the minimal Segre whose linear span contains q_1. Recall that Y is the minimal multiprojective space containing A_1 and the point $A\cap B$. Thus, $\nu(A\cap B)\notin\langle\nu(Y')\rangle$. Thus, for any linear subspace W of $\langle\nu(Y')\rangle$ we have $\dim\langle\nu(A\cap B)\cup W\rangle=\dim W+1$. We get $\dim(\langle\nu(A)\rangle\cap\langle\nu(B)\rangle)=1+\dim(\langle\nu(A_1)\rangle\cap\langle\nu(B_1)\rangle)$. Hence, (q,A,B) is of Type I_1.

(c) Assume $\#(A\cap B)=2$. Set $A_1:=A\setminus A\cap B$ and $B_1:=B\setminus A\cap B$. Take a general $M\in|\mathcal{I}_{A_1}(1,0,0)|$. Since $\langle\pi_1(A)\rangle=\mathbb{P}^3$, $M\cap A=A_1$. Claim 1 gives $M\cap B=B_1$. Since $\langle\pi_1(A)\rangle=\mathbb{P}^3$ and $\eta_{3|S}$ is injective (Claim 2), $h^1(\mathcal{I}_{A\cap B}(1,0,0))=0$. Hence, the residual exact sequence of M gives $h^1(M,\mathcal{I}_{A_1\cup B_1}(1,1,1))>0$. Since $\langle\pi_2(A)\rangle=\mathbb{P}^3$, $h^1(\mathcal{I}_tA\cap B(0,1,1))=0$. Since $\dim A\cap B=0$, we have $h^2(\mathcal{I}_{A\cap B}(0,1,1))=h^2(\mathcal{O}_Y(0,1,1))=0$. Thus, the residual exact sequence of M gives an isomorphism $H^1(\mathcal{I}_{A\cup B}(1,1,1))\to H^1(M,\mathcal{I}_{A_1\cup B_1}(1,1,1))\cong H^1(\mathcal{I}_{A_1\cup B_1}(1,1,1))$. Thus we have $\langle\nu(A_1)\rangle\cap\langle\nu(B_1)\rangle\neq\emptyset$ and any $q_1\in\langle\nu(A_1)\rangle\cap\langle\nu(B_1)\rangle$ is associated to a tensor equivalent to a 2×2 matrix. To conclude it would be sufficient to prove that $q\in\langle((\langle\nu(A_1)\rangle\cap\langle\nu(B_1)\rangle)\cup\nu(A\cap B))\rangle$. Write $A\cap B=\{u,v\}$. Let Y' be the minimal multiprojective space containing A_1 and Y'' the minimal multiprojective space containing $A_1\cup\{u\}$. Since $\langle\pi_1(A)\rangle=\mathbb{P}^3$, we have $Y'\subsetneq Y''\subsetneq Y$. Apply twice the last part of step (b). □

Proposition 6. *Take $Y=\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^1$ and q concise for Y and of rank 4.*

(a) Assume that (q,A,B) has Type II for some $A,B\in\mathcal{S}(q)$ such that $A\neq B$. Then (q,E,F) has Type II for all $E,F\in\mathcal{S}(q)$ such that $E\neq F$. Moreover the rank 2 tensors $\{q_1,q_2\}$ associated to partitions $A=A_1\sqcup A_2$ and $B=B_1\sqcup B_2$ are uniquely determined by q. Moreover $\mathcal{S}(q)\cong U\times U$ with U the complement of a smooth conic in a projective plane.

(b) Assume that (q,A,B) has Type I_x. Then (q,E,F) has Type I_x for all $E,F\in\mathcal{S}(q)$ such that $E\neq F$. Moreover the rank $4-x$ tensor q' such that $(A\setminus A\cap B)\in\mathcal{S}(q')$ and $(B\setminus A\cap B)\in\mathcal{S}(q')$ does not depend on the choice of A and B.

Proof. Fix $E\in\mathcal{S}(q)$ such that $E\neq A$ and $E\neq B$. It is sufficient to prove that (q,A,B) and (q,A,E) have the same type.

(a) Assume that (q,A,B) has Type II, i.e., assume $A\cap B=\emptyset$. In step (a) of the proof of Theorem 4 we proved that $\#\pi_3(A)=2$, say $\pi_3(A)=\{o_1,o_2\}$, that $\pi_3(A)=\pi_3(B)$ and that the partitions $A=A_1\sqcup A_2$ and $B=B_1\sqcup B_2$ with associated rank 2 tensors q_1 and q_2 is given by $A_i=A\cap\pi_3^{-1}(o_i)$ and $B_i=B\cap\pi_3^{-1}(o_i)$. Since $\nu(A)$ irredundantly spans q, q_i is the unique element of $\langle\nu(A_i)\rangle$ such that $q\in\langle\nu(A_i)\rangle$. First assume $E\cap A=\emptyset$.

Theorem 4 gives that (q, A, E) has Type II, say with respect to q_1' and q_2'. Step (a) of the proof of Theorem 4 gives $\pi_3(F) = \{o_1, o_2\}$, that $F = F_1 \sqcup F_2$ with $F_i := F \cap \pi_3^{-1}(o_i)$ and that the decomposition of A is the same as the decomposition of (q, A, B). Thus, $q_i' = q_i$, $i = 1, 2$. Thus, $E_i \in \mathcal{S}(q_i)$.

Now, assume $E \cap A \neq \emptyset$. Fix a finite set $K \subset Y$. Note that $\cap_{U \in \mathcal{S}(q_i)} U = \emptyset$. Thus, there if $F \in \mathcal{S}(q)$ such that (q, A, F) has Type II with q_1 and q_2 and $K \cap F = \emptyset$. Taking $K := E$ we get that (q, F, E) has Type II with q_1 and q_2 as rank 2 tensors. Recall that $\mathcal{S}(q_i)$ is isomorphic to the complement of a smooth conic in a projective plane (Lemma 4). We proved that $\mathcal{S}(q) \cong \mathcal{S}(q_1) \times \mathcal{S}(q_2)$.

(b) Assume that (q, A, B) has Type I_x, $x = 1, 2$, with $x = \#(A \cap B)$. Set $A_1 := A \setminus A \cap B$, $B_1 := B \setminus A \cap B$ and call $q' \in \langle \nu(A_1) \rangle \cap \langle \nu(B_1) \rangle$ such that $q \in \langle \{q'\} \cup \nu(A \cap B) \rangle$. Take $E \in \mathcal{S}(q)$ such that $E \neq A$ and $E \neq B$. It is sufficient to prove that $A \cap E = A \cap B$. Since $A \cap B \neq \emptyset$, part (a) proved in step (a) gives $A \cap E \neq \emptyset$ and $B \cap E \neq \emptyset$. By Proposition 4 (q, A, E) and (q, B, E) have Type I with $q_1 \in \langle \nu(E \setminus A \cap E) \rangle$ and $q_1 \in \langle \nu(E \setminus B \cap E) \rangle$. Since q_1 has rank $4 - x$, we get $\#(B \cap E) = \#(A \cap E) = x$. Fix a finite set K. In all cases listed in [12] there is $G_1 \in \mathcal{S}(q_1)$ such that $G_1 \cap K = \emptyset$ and $G_1 \cap A \cap B = \emptyset$. Thus, (q, A, G) has Type I with $A \cap B = A \cap G$. Taking $K := E$ we get $G \cap E \subseteq A \cap B$. Thus, $E \cap A = A \cap B$ and (q, A, E) has Type I_x with $q_1 = q'$. □

5. Examples

For any Segre variety $\nu(Y) \subset \mathbb{P}^r$ let $\tau(\nu(Y))$ denote its tangential variety. The following result shows that the tangential variety produces a large family of tensors q for which uniqueness fails and for which $\dim \mathcal{S}(Y, q)$ is very large.

Proposition 7. *Fix a concise $q \in \tau(\nu(Y)) \setminus \nu(Y)$ and call k the rank of q. Then $Y = (\mathbb{P}^1)^k$ and* $\dim \mathcal{S}(Y, q) \geq 2k - 2$.

Proof. Since q is concise, $Y = (\mathbb{P}^1)^k$ ([15–17]). Since $q \notin \nu(Y)$, $k \geq 2$. If $k = 2$, then $\nu(Y)$ is a smooth quadric surface and $\mathcal{S}(q)$ is the complement of a smooth conic in a projective plane. Assume $k \geq 3$. In this case, there is a unique degree 2 connected zero-dimensional scheme $v \subset Y$ such that $q \in \langle \nu(v) \rangle$. Set $\{o\} := v_{\mathrm{red}}$ with $o = (o_1, \dots, o_k)$. Let $\mathcal{C}$ be the set of all smooth and connected curve of bidegree $(1, \dots, 1)$. Fix $C, D \in \mathcal{C}$. Then $C \cong \mathbb{P}^1$ and there is $f \in (\mathrm{Aut}(\mathbb{P}^1)^k$ such that $f(C) = D$. We have $\dim \mathcal{C} = 3k - 3$, the set of all $C \in \mathcal{C}$ containing o has dimension $2k - 2$ and the set $\mathcal{V}$ of all C containing v has dimension $k - 1$. Fix $C \in \mathcal{V}$. The curve $\nu(C)$ is a degree k rational normal curve in its linear span. Since $v \subset C$, $q \in \langle \nu(C) \rangle$. A theorem of Sylvester says that q has $\nu(C)$-rank k. Hence, $E \in \mathcal{S}(q)$ for any $E \subset C$ such that E evinces the $\nu(C)$-rank of q. The set of all such sets $E \subset C$ has dimension $k - 1$. To prove that, varying $C \in \mathcal{V}$, we get a family of dimension $2k - 2$ contained in $\mathcal{S}(q)$ it would be sufficient to prove that for all $C, D \in \mathcal{V}$, $C \neq D$, $\#((C \setminus v) \cap (D \setminus v)) < k$. We claim that $\#(C \cap D) \leq 2$ for all $C, D \in \mathcal{C}$ such that $C \neq D$. Fix $C, D \in \mathcal{C}$ such that $C \neq D$ and assume the existence of 3 distinct points $u, v, w \in C \cap D$. Fix 3 distinct points, 0, 1 and ∞, of $\mathbb{P}^1$. There are unique isomorphisms $f : \mathbb{P}^1 \to C$ and $g : \mathbb{P}^1 \to D$ such that $f(0) = g(0) = u$, $f(1) = g(1) = v$ and $f(\infty) = g(\infty) = w$. The embedding f and g of $\mathbb{P}^1$ into Y as a curve of multidegree $(1, \dots, 1)$ are uniquely determined by their components $\pi_i \circ f : \mathbb{P}^1 \to \mathbb{P}^1$ and $\pi_i \circ g : \mathbb{P}^1 \to \mathbb{P}^1$. These isomorphisms $\pi_i \circ f$ and $\pi_i \circ g$ are uniquely determined by the images of 0, 1 and ∞, i.e., the points $\pi_i(u)$, $\pi_i(v)$, $\pi_i(w)$. Thus, $f = g$ and hence $C = f(\mathbb{P}^1) = g(\mathbb{P}^1) = D$, a contradiction. □

Example 1. *Take $Y = (\mathbb{P}^1)^4$ and a concise q in the tangential variety $\tau(\nu(Y))$ of $\nu(Y)$. Then q has rank 4 and $\mathcal{S}(q)$ is positive-dimensional ([15,17]). A general $q \in \tau(\nu(Y))$ is concise. Proposition 7 gives* $\dim \mathcal{S}(q) \geq 6$. *We do not know any case with rank 4 and at least 3 factors with larger* $\dim \mathcal{S}(q)$.

In the next 2 examples we use that the algebraically closed base field has either characteristic 0 or characteristic ≥ 7 for the quotation of a theorem of Sylvester ([18] (p. 22)). See Remark 6 for the general case.

Example 2. *Fix integers $n_1 \geq \cdots \geq n_k > 0$ such that $k \geq 3$ and $n_1 + \cdots + n_k = 6$. Take $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $C \subset Y$ be an integral, smooth and rational curve of multidegree $(n_1, \ldots, n_k)$ such that $\pi_{i|C}$ is an isomorphism if $n_i = 1$ and an embedding if $n_i \geq 2$. If $n_i \geq 2$, then $\pi_i(C)$ is a rational normal curve. Thus, $\nu(C)$ is a rational normal curve of degree 6 in its linear span. Take $A \subset C$ such that $\#A = 4$ and take $q \in \langle \nu(A) \rangle$ such that $q \notin \langle \nu(A') \rangle$ for any $A' \subsetneq A$. By a theorem of Sylvester there are ∞^1 sets $B \subset C$ such that $\#B = 4$ and $q \in \langle \nu(B) \rangle$. Assume for the moment that q has tensor rank 4. Note that $\pi_{i|A}$ is injective for all i and hence q is neither of Type I nor of Type II. Now, assume that q has tensor rank $e \leq 3$ and take $E \in \mathcal{S}(q)$. Fix any integers i, j such that $1 \leq i < j \leq k$ and call $\pi_{i,j} : Y \to \mathbb{P}^{n_i} \times \mathbb{P}^{n_j}$ the projection onto these factors of Y. Since $\pi_{i,j}(C)$ is a smooth curve of bidegree $(1, 1)$, there is $M \in |\mathcal{O}_Y(\varepsilon_i + \varepsilon_j)|$ containing A. By [14] (Lemma 5.1) we have $h^1(\mathcal{I}_{E \setminus E \cap M}(\hat{\varepsilon}_i - \varepsilon_j)) > 0$. Since $e \leq 3$ we get that E depends on at most 3 coordinates, 2 of them being i and j. Thus there is $h \in \{1, \ldots, k\}$ and $H \in |\mathcal{I}_E(\varepsilon_h)|$. Note that $\#(H \cap C) = 1$. By [14] (Lemma 5.1) we get $h^1(\mathcal{I}_{A \setminus A \cap H}(\hat{\varepsilon}_h)) > 0$. This is false, because $\deg(\mathcal{O}_C(\hat{\varepsilon}_h)) = 5$ and C is a rational normal curve in its linear span.*

Example 3. *Fix integers $n_1 \geq \cdots \geq n_k > 0$ such that $k \geq 3$ and $n_1 + \cdots + n_k = 5$. Take $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $C \subset Y$ be an integral, smooth and rational curve of multidegree $(n_1, \ldots, n_k)$ such that $\pi_{i|C}$ is an isomorphism if $n_i = 1$ and an embedding if $n_i \geq 2$. Thus, $\nu(C)$ is a rational normal curve of degree 5 in its linear span. Take a connected zero-dimensional scheme $Z \subset C$ with $\deg(Z) = 3$ and take $q \in \langle \nu(Z) \rangle$ such that $q \notin \langle \nu(Z') \rangle$ for any $Z' \subset Z$. By a theorem of Sylvester there are ∞^1 sets $A \subset C$ such that $\#A = 4$ and A evinces the $\nu(C)$-rank of q. Assume for the moment that q has tensor rank 4. Note that $\pi_{i|A}$ and $\pi_{i|B}$ are injective for all i and hence (q, A, B) is neither of Type I nor of Type II. Now, assume that q has tensor rank $e \leq 3$ and take $E \in \mathcal{S}(q)$. Fix any integers i, j such that $1 \leq i < j \leq k$ and call $\pi_{i,j} : Y \to \mathbb{P}^{n_i} \times \mathbb{P}^{n_j}$ the projection onto these factors of Y. Since $\pi_{i,j}(C)$ is a smooth curve of bidegree $(1, 1)$, there is $M \in |\mathcal{O}_Y(\varepsilon_i + \varepsilon_j)|$ containing A. By [14] (Lemma 5.1) we have $h^1(\mathcal{I}_{E \setminus E \cap M}(\hat{\varepsilon}_i, -\varepsilon_j)) > 0$. Since $e \leq 3$ we get that E depends on at most 3 coordinates, 2 of them being i and j. Thus there are $h \in \{1, \ldots, k\}$ and $H \in |\mathcal{I}_E(\varepsilon_k)|$. Note that $\#(H \cap C) = 1$. By [14] (Lemma 5.1) we get $h^1(\mathcal{I}_{A \setminus A \cap H}(\hat{\varepsilon}_h)) > 0$. This is false, because $\deg(\mathcal{O}_C(\hat{\varepsilon}_h)) = 4$ and C is a rational normal curve in its linear span.*

6. End of the Proofs

Proof of Theorem 1. Use Example 3. □

Proof of Theorem 2. Use Example 2. □

Proof of Theorem 3. Each case is obtained from a sequence of linear projections from the case $Y = \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^1$, which is true by Theorem 4 and Proposition 6. We need to check that at each step the tensor $\mu(q)$ in the definition of a linear projection from a point with respect to one of the factors is not only concise, but it also has rank 4 and not lower rank. Concision was proved in Section 2.1. In our case, with $n_1 = 3$, every concise tensor has rank ≥ 4. □

Proof of Proposition 1. Fix a plane $M \subseteq \mathbb{P}^{n_1}$, lines $L \subseteq \mathbb{P}^{n_2}$, $R \subseteq \mathbb{P}^{n_3}$ and points $o_i \in \mathbb{P}^{n_i}$, $4 \leq i \leq k$. Set $Y' := M \times L \times R \times \{o_4\} \times \cdots \times \{o_k\} \subset Y$. Take $q' \in \langle \nu(Y') \rangle$ of rank 3 as in [12] (Example 3.6). Hence, q' is concise for Y' and $\dim \mathcal{S}(q') = 4$ with 2 irreducible components of dimension 4. Moreover, there are $A'', B'' \in \mathcal{S}(q')$ such that $\#(A'' \cap B'') = 1$, say $A'' \cap B'' = \{u\}$. Set $A' := A'' \setminus \{u\}$ and $A' := A'' \setminus \{u\}$. Since (q', A'', B'') has Type I_1, there is $q'' \in \langle \nu(A') \rangle \cap \langle \nu(B') \rangle$ such that $q' \in \langle \{q'', \nu(u)\} \rangle$. Let Y'' be the minimal multiprojective space containing A'. We have $Y'' \cong \mathbb{P}^1 \times \mathbb{P}^1$ and q'' is concise for Y'' ([17], [12] (Proposition 3.2)). Since Y' is the minimal multiprojective space containing $Y'' \cup \{u\}$, the first positive dimensional factor of Y'' is a line $L_1 \subset M$. The second positive dimensional factor of Y'' is either L or R and both cases occur for certain (A'', B''). We take (A'', B'') such that $Y = L_1 \times L_2 \times \{o_3\} \times \cdots \times \{o_k\}$. Fix a general $p \in Y$ and set $A: = A'' \cup \{p\}$ and $B: = B'' \cup \{p\}$ and take a general $q \in \langle\{q', \nu(p)\}\rangle$. Since p is general Y is the minimal multiprojective space containing A. If q has rank 4, then q is concise for Y, (q, A, B) has Type I_2 and $\dim \mathcal{S}(q) \geq 4$. Assume that Y has rank $e \leq 3$ and take $E \in \mathcal{S}(q)$. Set $S: = E \cup A$. Since $n_1 = 3$, there is $H \in |\mathcal{I}_E(\varepsilon_1)|$. Since Y is the minimalmultiprojective space containing A, $A \nsubseteq H$. By [14] (Lemma 5.1) we have $h^1(\mathcal{I}_{A\setminus A\cap H}(\hat{\varepsilon}_1)) > 0$. Thus, $h^1(\mathcal{I}_A(\hat{\varepsilon}_1)) > 0$. We have $h^1(\mathcal{I}_{A'}(\varepsilon_2)) = 0$. For a general $u \in Y'$ we have $h^1(\mathcal{I}_{A'\cup\{u\}}(0, 1, 1, 0 \ldots, 0)) > 0$. For a general $p \in Y \setminus Y'$ we have $h^1(\mathcal{I}_A(\hat{\varepsilon}_1)) = 0$, a contradiction. $\square$

Remark 6. *Everything works over an arbitrary algebraically closed field $\mathbb{K}$, except at 3 places. In Examples 2 and 3 (used to prove Theorems 1 and 2) we quoted a theorem of Sylvester ([18] (p. 22)). Let $\nu(C)$ be a rational normal curve of degree 5 (as in Example 3. Fix any zero-dimensional scheme $W \subset C$ such that $\deg(W) = 3$ and any set $A \subset C$ such that $\#A = 5$ and $A \cap W = \emptyset$. Since $C \cong \mathbb{P}^1$ and $\nu(C)$ is a rational normal curve, we get $\langle \nu(A \cup W)\rangle = \langle \nu(C)\rangle$. We used Sylvester's theorem in the proof of Proposition 7 with respect to a rational normal curve $\nu(C)$ of degree k. The proof in [15] that q has tensor rank k is characteristic free. We need to check that q has rank k with respect to $\nu(C)$. Obviously the $\nu(C)$-rank of q is at least k. Since $\dim\langle \nu(C)\rangle$ and $\nu(C)$ is smooth, q has at most rank k by [19].*

Now take an infinite field K and let $\mathbb{K}$ be its algebraic closure. If $q \in \mathbb{P}^r(K)$ has (q, A, B) of Type II (or I) over $\mathbb{K}$ and with $A \subset Y(K)$, $B \subset Y(K)$, then it has Type II (or I) over K, because all points of A and B are defined over K. The examples may be constructed only using K. The examples may be constructed over any field with enough elements.

7. Effectiveness and Further Questions

Remark 7. *Fix a multiprojective space $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $a, b \in Y$ such that $a \neq b$. Write $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$. The minimal multiprojective space containing $\{a, b\}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if there are $1 \leq i_1 < i_2 \leq k$ such that $a_j = b_j$ for all $j \in \{1, \ldots, k\} \setminus \{i_1, i_2\}$, $a_{i_1} \neq b_{i_1}$ and $a_{i_2} \neq b_{i_2}$.*

Remark 8. *Take a rank 4 tensor $q \in \mathbb{P}^r = \langle \nu(Y)\rangle$ and $A \in \mathcal{S}(q)$. It is very easy to check if there is $B \in \mathcal{S}(q)$ such that (q, A, B) has Type II. Indeed, B exists if and only if there is a partition $A = A_1 \sqcup A_2$ such that $\#A_1 = \#A_2 = 2$ and the minimal multiprojective spaces $Y(i)$, $i = 1, 2$, are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This is effective by Remark 7. Now, we take q with $\mathrm{rank}(q) = 5$ and $A \in \mathcal{S}(q)$. There is $B \in \mathcal{S}(q)$ such that (q, A, B) has Type II if and only if there is a partition $A = A_1 \sqcup A_2$ such that $\#A_1 = 2$, $\#A_2 = 3$, the minimal multiprojective space Y' containing A_2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (easy to test by Remark 7) and, calling Y'' the minimal multiprojective space containing A_3, the pair (Y'', A_2) is in the list of [12] (Theorem 7.1).*

Remark 9. *Take a multiprojective space Y, a concise $q \in \langle \nu(Y)\rangle$ and $A \in \mathcal{S}(q)$. If $\mathrm{rank}(q) \leq 5$ there is a differential test ($h^1(\mathcal{I}_{2A}(1, \ldots, 1)) > 0$ with 2A as defined in [20]) which is necessary to be either of Type I or of Type II.*

We discuss here the differential criterion hinted in Remark 9 (see the references in [20] for proofs). Take a multiprojective space Y and a finite set $A \subset Y$, $A \neq \emptyset$. Set $t: = \#A$. The set $\nu(A) \subset \langle \nu(Y)\rangle$ is associated to an additive decomposition of many tensors q, all tensors in the linear span of $\nu(A)$, but not in the linear span of a proper subset of $\nu(A)$. Fix any such q and call $\mathcal{S}(Y, q, t)$ the set of all $B \subset Y$ such that $\#B = t$, $q \in \langle \nu(B)\rangle$ and q is . We have $\mathcal{S}(Y, q, t) = \emptyset$ if $t < \mathrm{rank}(q)$ and $\mathcal{S}(Y, q, \mathrm{rank}(q)) = \mathcal{S}(q)$. The set $\mathcal{S}(Y, q, t)$ has an algebraic structure and there is a differentiable map α such that at each $A \in \mathcal{S}(Y, q, t)$ the integer $h^1(\mathcal{I}_{2A}(1, \ldots, 1))$ is the dimension of the kernel of the differential of α at A. Thus, if $h^1(\mathcal{I}_{2A}(1, \ldots, 1)) = 0$, then A is an isolated point of $\mathcal{S}(Y, q, t)$, i.e., no $\tilde{A} \subset Y$ " near " A, but

$\tilde{A} \neq A$, is an element of $\mathcal{S}(Y, q, t)$. Moreover this criterion is stable for small modifications of q and A. This powerful criterion shows that some additive decompositions are stable under small modifications. This criterion is not an " if and only if " criterion (the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is injective but its differential vanishes at 0). In [20], the authors classify the pairs (Y, A), where Y is a multiprojective space, A is a finite set with $\#A \le 3$, Y is the minimal multiprojective space containing A, $h^1(\mathcal{I}_A(1,\dots,1)) > 0$ and $h^0(\mathcal{I}_A(1,\dots,1)) > 0$.

Open Problem 1. *Extend [20] to the case $\#A = 4$.*

Definition 4. *Take a multiprojective space Y and a concise tensor q for Y. Assume the existence of $A, B \in \mathcal{S}(q)$ such that $A \cap B = \emptyset$. We say that (q, A, B) has Type III if there are partitions $A = A_0 \sqcup A_1 \sqcup A_2$, $B = B_0 \sqcup B_1 \sqcup B_2$ and tensors $q_1, q_2 \in \mathbb{P}^r$ such that $q \in \langle \{q_1, q_2\}$, $A_0 \cup A_i \in \mathcal{S}(q_i)$ and $B_0 \cup B_i \in \mathcal{S}(q_i)$ for $i = 1, 2$. We say that q has Type III if there are $A, B \in \mathcal{S}(q)$ such that (q, A, B) has Type III.*

In Definition 4 we do not assume that q_1 and q_2 are concise for Y. We do not have examples of Type III tensors, but we expect that they exist.

Open Problem 2. *Construct examples of Type III tensors.*

8. Methods and Conclusions

We provide full proofs of the results we stated, but we leave open a conjecture in the introduction (Conjecture 1) with an explanation (Remark 1) of our main technical difficulty. Using linear projections to prove the conjecture it would be sufficient to prove for all $n_1 \ge \cdots \ge n_k > 0$ such that $k \ge 3$, $n_1 \le 3$ and $n_1 + \cdots + n_k = 7$. We proved the case $k = 3$ with $n_1 = n_2 = 3$ and $n_3 = 1$. In the last section, we discuss the effectiveness of our results and and two open problems, one on tensors of rank 4 and one on tensor of higher rank.

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