Milan J. Math.
https://doi.org/10.1007/s00032-024-00394-9

Milan Journal of Mathematics

On the Transport of Currents

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Abstract. In this work, we consider some evolutionary models for k-currents in $\mathbb{R}^d$. We study a transport-type equation which can be seen as a generalisation of the transport/continuity equation and can be used to model the movement of singular structures in a medium, such as vortex points/lines/sheets in fluids or dislocation loops in crystals. We provide a detailed overview of recent results on this equation obtained mostly in (Bonicatto et al. Transport of currents and geometric Rademacher-type theorems. arXiv:2207.03922, 2022; Bonicatto et al. Existence and uniqueness for the transport of currents by Lipschitz vector fields. arXiv:2303.03218, 2023). We work within the setting of integral (sometimes merely normal) k-currents, covering in particular existence and uniqueness of solutions, structure theorems, rectifiability, and a number of Rademacher-type differentiability results. These differentiability results are sharp and can be formulated in terms of a novel condition we called "Negligible Criticality condition" (NC), which turns out to be related also to Sard's Theorem. We finally provide a new stability result for integral currents satisfying (NC) in a uniform way.

Mathematics Subject Classification. 49Q15, 35Q49.

Keywords. Currents, Transport equation, Continuity equation, Decomposability bundle, Lie derivative.

Contents

Published online: 13 April 2024

1. Introduction

Transport phenomena are widespread in physics and engineering. Given a bounded (time-dependent) vector field $\boldsymbol{b}_t = \boldsymbol{b}(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$, with $t \in [0, 1]$, the *transport equation*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u + \boldsymbol{b}_t \cdot \nabla u = 0 \\ u(0, \cdot) = \overline{u}(\cdot) \end{cases} \tag{TE}$$

describes the transport of *scalar fields* $u \colon [0, 1] \times \mathbb{R}^d \to \mathbb{R}$. On the other hand, the *continuity equation*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \rho + \operatorname{div}(\rho \boldsymbol{b}_t) = 0 \\ \rho(0, \cdot) = \overline{\rho}(\cdot) \end{cases} \tag{CE}$$

characterises the transport of *densities* or, more generally, measures $\mu_t = \mu(t, \cdot)$ representing (possibly singular) mass distributions. The initial data $\overline{u}$ and $\overline{\rho}$ are usually given and the goal is to investigate existence, uniqueness and structure of solutions.

Another instance of a transport phenomenon is the movement of dislocations, serving as the primary mechanism for plastic deformation in crystalline materials, such as metals [1,16]. Dislocations represent topological defects within the crystal lattice, carrying both an orientation and a "topological charge" known as the Burgers vector. When examining fields $\tau_t = \tau(t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ describing continuously-distributed dislocations (with a fixed Burgers vector) being transported by a velocity field $\boldsymbol{b}_t$, the resulting equation is the dislocation-transport equation:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \tau_t + \operatorname{curl}(\boldsymbol{b}_t \times \tau_t) = 0 \\ \tau(0, \cdot) = \overline{\tau}. \end{cases} \tag{DT}$$

The three equations (CE), (TE) and (DT) can be cast into the single, unifying *geometric transport equation*

$$\partial_t T_t + \mathcal{L}_{\boldsymbol{b}_t} T_t = 0 \tag{GTE}$$

where $(T_t)_{t>0}$ is a family of k-*currents* in $\mathbb{R}^d$, $k \in \{0, \dots, d\}$, and $\mathcal{L}_{\boldsymbol{b}_t} T_t$ is the Lie derivative of T_t in the direction of the field $\boldsymbol{b}_t$, defined as

$$\mathcal{L}_{\boldsymbol{b}_t} T_t = -\boldsymbol{b}_t \wedge \partial T_t - \partial(\boldsymbol{b}_t \wedge T_t).$$

This formula is obtained by duality via Cartan's formula for differential forms. We understand (GTE) in a weak sense, meaning that for every $\psi \in \mathrm{C}_c^\infty((0,1))$ and each smooth k-form $\omega \in \mathscr{D}^k(\mathbb{R}^d)$ it holds

$$\int_0^1 \langle T_t, \omega \rangle \psi'(t) - \langle \mathcal{L}_{\boldsymbol{b}_t} T_t, \omega \rangle \psi(t) \, \mathrm{d}t = 0.$$

Written in coordinates for different values of k, the geometric transport equation (GTE) encompasses the classical transport equation ($k = d$), the continuity equation ($k = 0$), as well as the equations for the transport of dislocation lines in crystals ($k = 1$) – and even the movement of membranes in liquids ($k = d-1$). Notice that in all these equations, the case of "singular" objects being transported is just as natural as the case of fields. Apart from dislocations, which were previously discussed, the movement of point masses, lines, or sheets is particularly relevant in fluid mechanics when considering concentrated vorticity. Intermediate-dimensional structures also emerge in the context of Ginzburg-Landau energies, even in static situations (e.g. [2, 13, 17]).

This work offers a broad overview of the state of the art around (GTE). We provide a general theory for the geometric transport equation in the case of transported integral (sometimes only normal) k-currents, including the case of intermediate dimensions ($k \neq 0, d$). We present in particular:

- a comparison between the notions of *weak* solutions and of *space-time* solutions (see below for more details); this includes a detailed analysis of various possible ***definitions of variations***, scattered in the literature and collected and compared here;
- a structure theorem: For a space-time current, this theorem details the structure of its ***disintegration***. We introduce the notion of "critical points" of a space-time current, which turn out to be crucial in the study of transport-type phenomena;
- various ***rectifiability results***: we study under which conditions a time-indexed collection of boundary-less integral k-currents can be seen as the slices of a space-time integral current;
- the ***Advection Theorem***: we show that a boundaryless space-time current satisfies the negligible criticality condition (meaning that critical points are negligible for the mass measure of the current) if and only if its slices are advected by some vector field in the sense of (GTE);
- ***Existence & Uniqueness Theorem*** in the Lipschitz framework: In the case where the driving vector field is assumed Lipschitz, we show the existence and uniqueness of a path of integral currents solving (GTE);
- various ***Rademacher-type Differentiability Theorems***, showing that a time-indexed family of boundaryless integral currents, satisfying suitable Lipschitz assumptions, is a solution to (GTE) for some driving vector field;

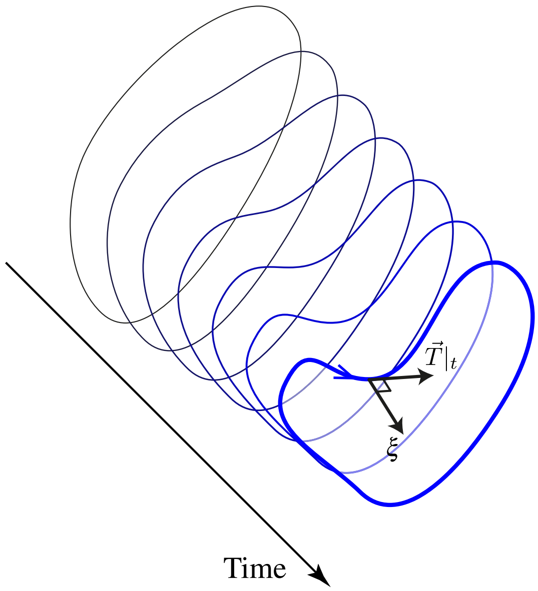

FIGURE 1 Evolution of an integral 1-current described via a space-time integral 2-current

- a new ***stability result***, giving some sufficient conditions for the "equi-integrability" of space-time currents. We show under which assumptions a family of integral space-time currents satisfying the negligible criticality condition in a uniform sense is pre-compact (in the topology of currents) and the limit points also satisfy the criticality condition.

We conclude this short introduction with a few words on one of the pivotal notions of this work, namely space-time solutions to (GTE). This concept builds on the theory introduced in [19] and can be explained, in the case of integral currents, as follows: Let S be a $(k+1)$-integral current in $[0,1]\times\mathbb{R}^d$. Denote by $S|_t$ the slice of S at time t (with respect to the time projection $\mathbf{t}(t,x) := t$) and by $S(t) := \mathbf{p}_*(S|_t)$ its pushforward under the spatial projection $\mathbf{p}(t,x) := x$. Standard theory gives that $S(t)$ is an integral k-current in $\mathbb{R}^d$ and that the orienting map $\vec{S} \in L^\infty(\|S\|; \bigwedge_k(\mathbb{R}\times\mathbb{R}^d))$ (with $|\vec{S}| = 1$ $\|S\|$-a.e.) decomposes orthogonally as

$$\vec{S} = \xi \wedge \vec{S}|_t,$$

where $\vec{S}|_t$ is the orienting map of the slice $S|_t$ (see Fig. 1) and

$$\xi(t,x) := \frac{\nabla^S \mathbf{t}(t,x)}{|\nabla^S \mathbf{t}(t,x)|}.$$

Here, $\nabla^S\mathbf{t}$ is approximate gradient of $\mathbf{t}$ with respect to S, i.e., the projection of $\nabla\mathbf{t}$ onto the approximate tangent space to (the rectifiable carrier of) S. We can now define the geometric derivative of S as the (normal) change of position per time of a point travelling on the current being transported, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t,x) := \frac{\mathbf{p}(\xi(t,x))}{|\mathbf{t}(\xi(t,x))|} = \frac{\mathbf{p}(\xi(t,x))}{|\nabla^S \mathbf{t}(t,x)|},$$

for $\|S\|$-a.e. (t,x). Clearly, this quantity exists only outside the critical set $\mathrm{Crit}(S) := \{(t,x) \in [0,1] \times \mathbb{R}^d : \nabla^S \mathbf{t}(t,x) = 0\}$, which is in turn related to Sard's theorem and play a major role in this work.

We then say that a space-time current S as above is a space-time solution of (GTE) if

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t,x) = \boldsymbol{b}(t,x) \text{ for } \|S\|\text{-a.e. } (t,x). \tag{1.1}$$

One can see without too much effort that space-time solutions give rise to weak solutions: The projected slices $S(t) := \mathbf{p}_*(S|_t)$ of an integral $(k+1)$-current S satisfying (1.1) solve (GTE). The converse question, that is, when a collection of currents $\{T_t\}_t$ solving (GTE) can be realised as the slices of a space-time current is more challenging and will be one of the recurring themes of the paper.

2. Notation and Preliminaries

This section fixes our notation and recalls some basic facts. We refer the reader to [15,18] for notation and the main results we use about differential forms and currents.

2.1. Linear and Multilinear Algebra

Let $d \in \mathbb{N}$ be the ambient dimension. We will often use the projection maps $\mathbf{t} \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $\mathbf{p} \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ from the (Euclidean) space-time $\mathbb{R} \times \mathbb{R}^d$ onto the time and space variables, respectively, which are given as

$$\mathbf{t}(t,x) = t, \qquad \mathbf{p}(t,x) = x.$$

We also define, for every given $t \in \mathbb{R}$, the immersion map $\iota_t \colon \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$ by

$$\iota_t(x) := (t,x), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

If V is a (finite-dimensional, real) vector space, for every $k \in \mathbb{N}$, we let $\bigwedge^k V$ be the space of k-covectors on V, and $\bigwedge_k(V)$ be the space of k-vectors on V. We denote the duality pairing between $v \in \bigwedge_k V$ and $\alpha \in \bigwedge^k V$ by $\langle v, \alpha \rangle$. Referring to [4, Section 5.8], given a k-vector v in V, we denote by $\mathrm{span}(v)$ the smallest linear subspace W of V such that $v \in \bigwedge_k(W)$. A similar definition is given for a k-covector α in V.

Whenever V is an inner product space, we can endow $\bigwedge_k V$ with an inner product (Euclidean) norm $|\cdot|$ by declaring $\mathrm{e}_I := \mathrm{e}_{i_1} \wedge \ldots \wedge \mathrm{e}_{i_k}$, as I varies in the k-multi-indices of $\{1,\ldots,n\}$, as **orthonormal** whenever $\mathrm{e}_1,\ldots,\mathrm{e}_n$ are an orthonormal basis of V. A simple k-vector $\eta \in \bigwedge_k V$ is called a **unit** if there exists an orthonormal family $v_1,\ldots,v_k$ such that $\eta = v_1 \wedge \ldots \wedge v_k$, or equivalently if its Euclidean norm $|\eta|$ equals 1. We define the **comass** of $\alpha \in \bigwedge^k V$ as

$$\|\alpha\| := \sup\bigl\{\langle \eta, \alpha \rangle : \eta \in \textstyle\bigwedge_k V, \text{ simple, unit}\bigr\}$$

and the **mass** of $\eta \in \bigwedge_k V$ as

$$\|\eta\| := \sup\bigl\{\langle \eta, \alpha \rangle : \alpha \in \textstyle\bigwedge^k V, \ \|\alpha\| \le 1\bigr\}.$$

Given a linear map $S\colon V \to W$, we define $\bigwedge^k S\colon \bigwedge_k V \to \bigwedge_k W$ by

$$(\textstyle\bigwedge^k S)(v_1 \wedge \ldots \wedge v_k) = (Sv_1) \wedge \ldots \wedge (Sv_k),$$

on simple vectors and then we extend this definition by linearity to all of $\bigwedge_k V$. If there is no risk of confusion, we will often write simply S instead of $\bigwedge^k S$ to denote the extension of the map S to $\bigwedge^k V$.

We denote by $\mathscr{D}^k(\mathbb{R}^d)$ the space of smooth k-forms on $\mathbb{R}^d$ with compact support. The integer k will also be called the **degree** of $\omega \in \mathscr{D}^k(\mathbb{R}^d)$ and the **comass of a form** $\omega \in \mathscr{D}^k(\mathbb{R}^d)$ is

$$\|\omega\|_\infty := \sup_{x\in\mathbb{R}^d} \|\omega(x)\|.$$

The **pullback of a covector** $\alpha \in \bigwedge^k V$ with respect to a linear map $S : V \to W$ is given by

$$\big\langle v_1 \wedge \ldots \wedge v_k, S^*\alpha\big\rangle := \big\langle (Sv_1) \wedge \ldots \wedge (Sv_k), \alpha\big\rangle$$

on simple k-vectors and then extended by linearity. Therefore,

$$\big\langle \eta, S^*\alpha\big\rangle = \big\langle (\textstyle\bigwedge^k S)\eta, \alpha\big\rangle.$$

If $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is differentiable and proper (meaning that preimages of compact sets are themselves compact) and $\omega \in \mathscr{D}^k(\mathbb{R}^d)$, we define the **pullback** $f^*\omega \in \mathscr{D}^k(\mathbb{R}^d)$ to be the differential form $f^*\omega$ given by

$$\big\langle v, (f^*\omega)(x)\big\rangle := \big\langle Df(x)[v], \omega(f(x))\big\rangle, \qquad v \in \textstyle\bigwedge_k(\mathbb{R}^d).$$

The properness of the pullback map f can be omitted if the pullback form $f^*\omega$ is always integrated against a current of compact support. Here, we usually use $\mathbf{t}$ and $\mathbf{p}$ as pullback maps, which are not proper, but in all instances the compound expressions in which they appear are compactly supported and no issue of well-definedness arises.

2.2. Currents

We refer to [15] for a comprehensive treatment of the theory of currents, summarising here only the main notions that we will need. The space of k-**dimensional currents** $\mathscr{D}_k(\mathbb{R}^d)$ is defined as the dual of $\mathscr{D}^k(\mathbb{R}^d)$, where the latter space is endowed with the locally convex topology induced by local uniform convergence of all derivatives. Then, the notion of **(sequential weak*) convergence** is the following:

$$T_n \overset{*}{\rightharpoonup} T \text{ in the sense of currents } \iff \big\langle T_n, \omega\big\rangle \to \big\langle T, \omega\big\rangle \quad \text{for all } \omega \in \mathscr{D}^k(\mathbb{R}^d).$$

The **boundary** of a current is defined as the adjoint of De Rham's differential: if T is a k-current, then ∂T is the $(k-1)$-current given by

$$\big\langle \partial T, \omega\big\rangle = \big\langle T, d\omega\big\rangle, \qquad \omega \in \mathscr{D}^{k-1}(\mathbb{R}^d).$$

We denote by $\mathrm{M}_k(\mathbb{R}^d)$ the space of k-**currents with finite mass** in $\mathbb{R}^d$, where the **mass** of a current $T \in \mathscr{D}_k(\mathbb{R}^d)$ is defined as

$$\mathbf{M}(T) := \sup\left\{\big\langle T, \omega\big\rangle : \omega \in \mathscr{D}^k(\mathbb{R}^d), \|\omega\|_\infty \le 1\right\}.$$

Let μ be a finite measure on $\mathbb{R}^d$ and let $\tau\colon \mathbb{R}^d \to \bigwedge_k(\mathbb{R}^d)$ be a map in $\mathrm{L}^1(\mu)$. Then we define the current $T := \tau\mu$ as

$$\langle T, \omega \rangle = \int_{\mathbb{R}^d} \langle \tau(x), \omega(x) \rangle \, \mathrm{d}\mu(x).$$

We recall that all currents with finite mass can be represented as $T = \tau\mu$ for a suitable pair τ, μ as above. In the case when $\|\tau\| = 1$ μ-a.e., we denote μ by $\|T\|$ and we call it the **mass measure** of T. As a consequence, we can write $T = \vec{T}\|T\|$, where $\|\vec{T}\| = 1$ $\|T\|$-almost everywhere. One can check that, if $T = \tau\mu$ with $\tau \in \mathrm{L}^1(\mu)$, then $\|T\| = \|\tau\|\mu$, hence

$$\mathbf{M}(T) = \int_{\mathbb{R}^d} \|\tau(x)\| \, \mathrm{d}\mu(x).$$

Given a current $T = \tau\mu \in \mathscr{D}_k(\mathbb{R}^d)$ with finite mass and a vector field $v\colon \mathbb{R}^d \to \mathbb{R}^d$ defined $\|T\|$-a.e., we define the **wedge product**

$$v \wedge T := (v \wedge \tau)\mu \in \mathscr{D}_{k+1}(\mathbb{R}^d).$$

The **pushforward** of T with respect to a proper C^1-map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$\langle f_* T, \omega \rangle = \langle T, f^* \omega \rangle.$$

In the case of measures, we employ instead the standard notation $f_\# \mu$ for the pushforward of μ under a map f, namely, the measure defined by $f_\#\mu(A) = \mu(f^{-1}(A))$.

If T is **simple**, i.e., $\vec{T}$ is a simple k-vector $\|T\|$-almost everywhere, then the same inequality holds with the mass norm $\|\cdot\|$ replaced by the Euclidean norm $|\cdot|$.

Given two currents $T_1 \in \mathscr{D}_{k_1}(\mathbb{R}^{d_1})$ and $T_2 \in \mathscr{D}_{k_2}(\mathbb{R}^{d_2})$, their **product** $T_1 \times T_2$ is a well-defined current in $\mathscr{D}_{k_1+k_2}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$.

A k-current on $\mathbb{R}^d$ is said to be **normal** if both T and ∂T have finite mass. The space of normal k-currents is denoted by $\mathrm{N}_k(\mathbb{R}^d)$. The weak* topology on the space of (normal) currents has good properties of compactness and lower semicontinuity: if $(T_j)_j$ is a sequence of currents with $\mathbf{M}(T_j) + \mathbf{M}(\partial T_j) \le C < +\infty$ for every $j \in \mathbb{N}$, then there exists a normal current T such that, up to a subsequence, $T_j \overset{*}{\rightharpoonup} T$. Furthermore,

$$\mathbf{M}(T) \le \liminf_{j \to +\infty} \mathbf{M}(T_j), \qquad \mathbf{M}(\partial T) \le \liminf_{j \to +\infty} \mathbf{M}(\partial T_j).$$

An **integer-multiplicity rectifiable** k-current T is a k-current of the form

$$T = m \vec{T} \mathscr{H}^k \llcorner R,$$

where:

(1) $R \subset \mathbb{R}^d$ is countably $\mathscr{H}^k$-rectifiable (that is, it can be covered up to a $\mathscr{H}^k$-null set by countably many images of Lipschitz functions from $\mathbb{R}^k$ to $\mathbb{R}^d$) with $\mathscr{H}^k(R \cap K) < \infty$ for all compact sets $K \subset \mathbb{R}^d$;

(2) $\vec{T}\colon R \to \bigwedge_k \mathbb{R}^d$ is $\mathscr{H}^k$-measurable and for $\mathscr{H}^k$-a.e. $x \in R$ the k-vector $\vec{T}(x)$ is simple, unit ($|\vec{T}(x)| = 1$), and its span coincides with the approximate tangent space $\mathrm{Tan}_x R$ to R at x;

(3) $m \in \mathrm{L}^1_{\mathrm{loc}}(\mathscr{H}^k \llcorner R; \mathbb{Z})$;

The map $\vec{T}$ is called the **orientation map** of T and m is the **multiplicity**. Let $T = \vec{T}\|T\|$ be the Radon–Nikodým decomposition of T with the **total variation measure** $\|T\| = |m|\,\mathscr{H}^k \llcorner R \in \mathscr{M}^+_{\mathrm{loc}}(\mathbb{R}^d)$. Then we have

$$\mathbf{M}(T) = \|T\|(\mathbb{R}^d) = \int_R |m(x)| \,\mathrm{d}\mathscr{H}^k(x).$$

We then define the space of **integral** k-currents ($k \in \mathbb{N} \cup \{0\}$):

$$\mathrm{I}_k(\mathbb{R}^d) := \big\{T \text{ integer-multiplicity rectifiable } k\text{-current} : \mathbf{M}(T) + \mathbf{M}(\partial T) < \infty\big\}.$$

For $F \subset \mathbb{R}^d$ closed, the subspaces $\mathrm{I}_k(F)$, $\mathrm{N}_k(F)$ are defined as the spaces of all $T \in \mathrm{I}_k(\mathbb{R}^d)$, or $T \in \mathrm{N}_k(\mathbb{R}^d)$, respectively, with support (in the sense of measures) in F. Since F is closed, these subspaces are weakly* closed.

An important property of integral currents is the *Federer-Fleming compactness theorem* [18, Theorems 7.5.2, 8.2.1]: Let $(T_j)_j \subset \mathrm{I}_k(\mathbb{R}^d)$ with

$$\sup_{j\in\mathbb{N}} \left(\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)\right) < \infty.$$

Then, there exists a (not relabeled) subsequence and a $T \in \mathrm{I}_k(\mathbb{R}^d)$ such that $T_j \overset{*}{\rightharpoonup} T$ in the sense of currents.

2.3. Flat Norms

For $T \in \mathrm{I}_k(\mathbb{R}^d)$, the **(Whitney) flat norm** is given by

$$\mathbf{F}(T) := \inf\{\mathbf{M}(Q) + \mathbf{M}(R) : Q \in \mathrm{N}_{k+1}(\mathbb{R}^d), R \in \mathrm{N}_k(\mathbb{R}^d),\, T = \partial Q + R\} \tag{2.1}$$

and one can also consider the **flat convergence** $\mathbf{F}(T - T_j) \to 0$ as $j \to \infty$. Under the mass bound $\sup_{j\in\mathbb{N}} \big(\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)\big) < \infty$, this flat convergence is equivalent to weak* convergence (see, for instance, [18, Theorem 8.2.1] for a proof). The flat norm admits also a dual representation (see [15, 4.1.12]) as

$$\mathbf{F}(T) = \sup\{\langle T, \omega\rangle \colon\, \omega \in \mathscr{D}^k(\mathbb{R}^d),\, \|\omega\|_\infty \le 1,\, \|d\omega\|_\infty \le 1\}. \tag{2.2}$$

When $\partial T = 0$, one can also consider the **homogeneous flat norm**

$$\mathbb{F}(T) := \inf\{\mathbf{M}(Q) : Q \in \mathrm{N}_{k+1}(\mathbb{R}^d),\, T = \partial Q\}, \tag{2.3}$$

which also admits a dual representation as

$$\mathbb{F}(T) = \sup\{\langle T, \omega\rangle : \omega \in \mathscr{D}^k(\mathbb{R}^d),\, \|d\omega\|_\infty \le 1\}. \tag{2.4}$$

If T is integral, one can also consider the corresponding integral versions of (2.1) and (2.3), called **integral flat norm** and **integral homogeneous flat norm** respectively:

$$\begin{aligned} \mathbf{F}_{\mathrm{I}}(T) &:= \inf\{\mathbf{M}(Q) + M(R) : Q \in \mathrm{I}_{k+1}(\mathbb{R}^d), R \in \mathrm{I}_k(\mathbb{R}^d),\, T = \partial Q + R\}, \\ \mathbb{F}_{\mathrm{I}}(T) &:= \inf\{\mathbf{M}(Q) : Q \in \mathrm{I}_{k+1}(\mathbb{R}^d),\, T = \partial Q\}. \end{aligned}$$

These, however, do not admit a dual representation as in (2.2) and (2.4). Notice that these are not proper norms because $\mathrm{I}_k(\mathbb{R}^d)$ is not a vector space. In the following, we will also consider the homogeneous flat norms $\mathbb{F}, \mathbb{F}_{\mathrm{I}}$ on the whole $\mathrm{N}_k(\mathbb{R}^d)$ or $\mathrm{I}_k(\mathbb{R}^d)$, in which case they are understood to be $+\infty$ on currents that are not boundaryless.

2.4. Slicing and Coarea Formula for Integral Currents

Given a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and $S \in \mathrm{N}_{k+1}(\mathbb{R}^n)$, the **slicing** of S at level t is defined by the following, which will be referred to as the **cylinder formula**:

$$S|_t := \partial(S \llcorner \{f < t\}) - (\partial S) \llcorner \{f < t\}.$$

The slices with respect to f are also characterised by the following property:

$$\int S|_t \psi(t)\, \mathrm{d}t = S \llcorner (\psi \circ f) df \qquad \text{for every } \psi \in \mathrm{C}_c^\infty(\mathbb{R}). \tag{2.5}$$

see, e.g., [9] or also [15, 4.3.2]. For integral currents the following coarea formula holds [19, Section 2.4]: For every non-negative Borel function $g\colon \mathbb{R}^n \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} g(z)|\nabla^S f(z)|\, \mathrm{d}\|S\|(z) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} g(z) \mathrm{d}\|S|_t\|(z) \right) \mathrm{d}t, \tag{2.6}$$

where $\nabla^S f(z)$ denotes the tangential gradient of the map f on the approximate tangent space to S at z, that is, the projection of the vector $\nabla f(z)$ onto the approximate tangent space to S at z (see [7, Theorem 2.90]). The equality (2.6) holds also whenever $g \in \mathrm{L}^1(|\nabla^S f|\|S\|)$.

2.5. Disintegration of Measures

Given the product structure of the space-time $\mathbb{R}\times\mathbb{R}^d$, we will often work with product measures or *generalised product measures* and we will consider the *disintegration* of measures on $\mathbb{R}\times\mathbb{R}^d$ with respect to the map $\mathbf{t}$, for which we follow the approach of [3]. Let $\{\mu_t\} = \{\mu_t : t \in \mathbb{R}\}$ be a family of finite (vector) measures on $\mathbb{R}\times\mathbb{R}^d$. We say that such a family is **Borel** if

$$t \mapsto \int_{\mathbb{R}\times\mathbb{R}^d} \phi \, \mathrm{d}\mu_t$$

is Borel for every test function $\phi \in \mathrm{C}_c(\mathbb{R}\times\mathbb{R}^d)$. Given a measure ν on $\mathbb{R}$ and a family $\{\rho_t : t \in \mathbb{R}\}$ of measures on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}} |\rho_t|(\mathbb{R}^d)\, \mathrm{d}\nu(t) < \infty,$$

we define the generalised product $\nu \otimes \rho_t$ as the measure on $\mathbb{R}\times\mathbb{R}^d$ such that

$$\int_{\mathbb{R}\times\mathbb{R}^d} \phi(t,x)\, \mathrm{d}(\nu\otimes\rho_t)(t,x) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \phi(t,x)\, \mathrm{d}\rho_t(x) \right) \mathrm{d}\nu(t)$$

for every $\phi \in \mathrm{C}_c(\mathbb{R}^d)$.

Let now μ be a (possibly vector-valued) measure in $\mathbb{R}\times\mathbb{R}^d$ and let ν be a measure on $\mathbb{R}$ such that $\mathbf{t}_\#\mu \ll \nu$. Then, there exists a Borel family $\{\mu_t : t \in \mathbb{R}\}$ of (possibly vector-valued) measures on $\mathbb{R}\times\mathbb{R}^d$ such that:

(i) μ_t is supported on $\{t\}\times\mathbb{R}^d$ for ν-a.e. $t \in \mathbb{R}$;
(ii) μ can be decomposed as

$$\mu = \int_{\mathbb{R}} \mu_t \, \mathrm{d}\nu(t),$$

which means

$$\mu(A) = \int_{\mathbb{R}} \mu_t(A) \, \mathrm{d}\nu(t), \tag{2.7}$$

for every Borel set $A \subset \mathbb{R} \times \mathbb{R}^d$.

Any family $\{\mu_t\}$ satisfying the conditions (i) and (ii) above will be called a **disintegration of μ with respect to t and ν**. We remark that, from (2.7) we obtain

$$\int_{\mathbb{R}\times\mathbb{R}^d} \phi \, \mathrm{d}\mu = \int_{\mathbb{R}} \left(\int_{\{t\}\times\mathbb{R}^d} \phi \, \mathrm{d}\mu_t \right) \mathrm{d}\nu(t)$$

for every Borel function $\phi\colon \mathbb{R} \times \mathbb{R}^d \to [0, +\infty]$.

2.6. Decomposability Bundle

We recall from [4] the definition and a few basic facts about the decomposability bundle of a measure.

Given a measure μ on $\mathbb{R}^n$, the *decomposability bundle* is a μ-measurable map $x \mapsto V(\mu, x)$ (defined up to μ-negligible sets) which associates to μ-a.e. x a subspace $V(\mu, x)$ of $\mathbb{R}^n$. The map V satisfies the following property: Every Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at x along the subspace $V(\mu, x)$, for μ-a.e. $x \in \mathbb{R}^n$. Moreover, this map is μ-maximal in a suitable sense, meaning that $V(\mu, x)$ is, for μ-a.e. x, the biggest subspace with this property (see [4, Theorem 1.1]). The directional derivative of f at x in direction $v \in V(\mu, x)$ will be denoted by $Df(x)[v]$. Observe that this is a slight abuse of notation, as the full differential Df might not exist at x, even though the directional derivative exists.

A key fact about the decomposability bundle with regard to the theory of normal currents is the following [4, Theorem 5.10]: Given a normal k-current $T = \vec{T}\|T\|$ in $\mathbb{R}^n$, it holds that

$$\operatorname{span}(\vec{T}) \subseteq V(\|T\|, x) \qquad \text{for } \|T\|\text{-a.e. } x \in \mathbb{R}^n. \tag{2.8}$$

In particular, given any Lipschitz function f, we can define $D_T f$ at $\|T\|$-a.e. point as the restriction of the differential of f to $\operatorname{span}(\vec{T})$. We will usually just write Df instead of $D_T f$ when this differential is evaluated in a direction in $\operatorname{span}(\vec{T})$.

For a normal current $T \in \mathrm{N}_k(\mathbb{R}^d)$, it is possible to define the pushforward f_*T when $f : \mathbb{R}^d \to \mathbb{R}^d$ is merely Lipschitz [15, 4.1.14]. Classically, no explicit formula for this pushforward was available. However, it is shown in [4, Proposition 5.17] that the pushforward formula, in fact, remains true:

Lemma 2.1. *Suppose that $T = \tau\mu$ is a normal k-current in $\mathbb{R}^n$, and $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a proper, injective Lipschitz map. Then, the pushforward current f_*T satisfies*

$$f_*T = \tilde{\tau}\tilde{\mu},$$

where $\tilde{\mu} = f_\#\mu$, and $\tilde{\tau}(y) = Df(x)[\tau(x)] = D_T f(x)[\tau(x)]$ with $y = f(x)$.

3. Space-Time Currents: Variations and Disintegration

3.1. Notions of Variation and AC Integral Currents

Given a (locally compact) metric space (X, d) and a curve $t \mapsto \gamma(t) \in X$, $t \in [a,b]$ ($a < b$), we define the **pointwise variation** of γ as

$$\mathsf{d}\text{-pV}(\gamma;[a,b]) := \sup\left\{\sum_{i=1}^{N} \mathsf{d}(\gamma(t_{i+1}),\gamma(t_i)) : a \le t_1 \le \ldots \le t_N \le b\right\}. \tag{3.1}$$

We further define the **essential variation** of the curve γ as

$$\mathsf{d}\text{-eV}(\gamma;[a,b]) := \inf\left\{\mathsf{d}\text{-pV}(\tilde\gamma;[a,b]) : \gamma(t) = \tilde\gamma(t) \ \text{ for } \mathscr{L}^1\text{-a.e. } t \in [a,b]\right\}.$$

We extend the same definition to curves γ which are only defined for $\mathscr{L}^1$-a.e. $t \in [a,b]$. In this case, the supremum in (3.1) is taken over families of partitions such that γ is defined at t_i for every i. By [5, Remark 2.2], the infimum in the definition of essential variation is achieved and therefore, if $\mathsf{d}\text{-eV}(\gamma : [a,b]) < \infty$, then there exist two **good representatives**, the right-continuous representative γ_+ and the left-continuous representative γ_- such that

$$\mathsf{d}\text{-eV}(\gamma_\pm;[a,b]) = \mathsf{d}\text{-pV}(\tilde\gamma;[a,b]).$$

If $\mathsf{d}\text{-eV}(\gamma;[a,b]) < \infty$, then $\mathsf{d}\text{-eV}(\gamma;\bullet)$ can be extended to a finite measure on the Borel subsets of $[a,b]$.

In this vein, for for $S \in \mathrm{I}_{k+1}([0,1]\times\mathbb{R}^d)$ and $U \in \mathrm{N}_{k+1}([0,1]\times\mathbb{R}^d)$ with $\partial S \llcorner (0,1)\times\mathbb{R}^d = \partial U \llcorner (0,1)\times\mathbb{R}^d = 0$ we set, with a little abuse of notation,

$$\begin{aligned}\mathbb{F}_{\mathrm{I}}\text{-eV}(S;[a,b]) &:= \mathbb{F}_{\mathrm{I}}\text{-eV}(t\mapsto S(t);[a,b]),\\ \mathbb{F}\text{-eV}(U;[a,b]) &:= \mathbb{F}\text{-eV}(t\mapsto U(t);[a,b])\end{aligned}$$

for every closed interval $[a,b] \subset [0,1]$. Recall that we denote by $S(t) := \mathbf{p}_*(S|_t)$ the pushforward of the slice $S|_t$ onto $\mathbb{R}^d$. Likewise we define the slices $U|_t$ and the pushforwards $U(t) := \mathbf{p}_*(U|_t)$ for $\mathscr{L}^1$-a.e. t.

On the other hand, in the work [19] the author introduced the *(space-time) variation* of an integral space-time current. Given a current S of finite mass, i.e., $S \in \mathrm{M}_{k+1}([0,1]\times\mathbb{R}^d)$, we define the **(space-time) variation** of S on the interval $[a,b]$ to be

$$\mathrm{Var}(S;[a,b]) := \int_{[a,b]\times\mathbb{R}^d} \|\mathbf{p}(\vec S)\| \,\mathrm{d}\|S\|.$$

Here and in the following, we will often write $\mathbf{p}(\vec S)$ instead of $\bigwedge^{k+1}\mathbf{p}(\vec S)$ for ease of notation. We remark that, if S is integral, then $\vec S$ is simple and so is $\mathbf{p}(\vec S)$. Therefore, in this case, $\|\mathbf{p}(\vec S)\| = |\mathbf{p}(\vec S)|$. One can further see that $\mathrm{Var}(S;\bullet)$ can be extended to all Borel sets (by the very same formula) to define a non-negative finite measure on $\mathbb{R}$, which will still be denoted by $\mathrm{Var}(S;\bullet)$.

One might wonder which relation exists between these various notions of variation. It is fairly easy to see that the space-time variation bounds from above the pointwise variation; in general, however, the opposite inequality may not hold, due to the possible presence of jumps in the path $t \mapsto S|_t$. Indeed, whenever a jump occurs at a certain time t_0, $\mathrm{Var}(S;\bullet)$ depends on the particular current that connects $S|_{t_0^-}$ and $S|_{t_0^+}$, while $\mathbb{F}_{\mathrm{I}}$-eV always measures the optimal connection. The next theorem entails that jumps are in fact the *only* obstructions to the equality between Var and $\mathbb{F}_{\mathrm{I}}$-eV.

Theorem 3.1. (Equality of variations) *Let* $S \in \mathrm{I}_{k+1}([0,1]\times\mathbb{R}^d)$ *with* $\partial S \llcorner (0,1)\times\mathbb{R}^d = 0$ *and such that* $\mathrm{Var}(S;\bullet)$ *is non-atomic. Then,*

$$\mathrm{Var}(S;\bullet) = \mathbb{F}\text{-eV}(S;\bullet) = \mathbb{F}_{\mathrm{I}}\text{-eV}(S;\bullet).$$

Theorem 3.1 plays a central place in this work and can be seen as a generalisation to any codimension of the following formula, valid for a function $u\colon [0,1]\to\mathbb{R}$ that is continuous and of bounded variation:

$$\mathrm{pV}(u,\mathbb{R}) = \int_{\mathrm{graph}(u)} |\mathbf{p}(\tau)|\,\mathrm{d}\mathscr{H}^1 = \mathrm{Var}(S_u,\mathbb{R}),$$

where $S_u := \tau\mathscr{H}^1\llcorner\mathrm{graph}(u)$ and τ is the forward-pointing unit tangent to $\mathrm{graph}(u)$.

We do not discuss here the proof of Theorem 3.1 and we refer the reader to [10, Theorem 5.3]. We simply remark that it is obtained as a consequence of the following space-time rectifiability result:

Theorem 3.2. (Rectifiability) *Let* $t\mapsto T_t \in \mathrm{I}_k(\mathbb{R}^d)$, $t\in[0,1]$, *with* $\partial T_t = 0$ *for every* $t\in[0,1]$, *and such that*

$$\sup_{t\in[0,1]} \mathbf{M}(T_t) < \infty, \qquad \mathbb{F}_{\mathrm{I}}\text{-eV}(t\mapsto T_t;[0,1]) < \infty.$$

Then, there exists $S\in\mathrm{I}_{k+1}(\mathbb{R}\times\mathbb{R}^d)$ *with* $\partial S\llcorner(0,1)\times\mathbb{R}^d = 0$ *such that:*

(a) $S(t) = \mathbf{p}_*(S|_t) = T_t$ *for* $\mathscr{L}^1$*-a.e.* $t\in[0,1]$*;*
(b) $\mathrm{Var}(S;\bullet) = \mathbb{F}_{\mathrm{I}}\text{-eV}(t\mapsto T_t;\bullet)$ *as measures on* $[0,1]$.

Theorem 3.2 is indeed a space-time *rectifiability* result for general integral currents: Under a BV bound in time, we can "glue" the currents of a path $t\mapsto T_t$ in a space-time *integral* current, whose space-time variation coincides with the pointwise one of the path. These gluing procedures play an important role in our analysis and we will be exploited several times throughout this work.

Remark 3.3. It is possible to show that, if we further assume that the variation is non-atomic, i.e. there are no jumps in the path $t\mapsto T_t$, then the current S in Theorem 3.2 is unique. See, for more details, Corollary [10, Corollary 6.5].

We conclude this section introducing the class of **absolutely continuous** (AC) integral space-time currents:

$$\mathrm{I}^{\mathrm{AC}}_{1+k}([0,1]\times\mathbb{R}^d) := \big\{S\in\mathrm{I}_{1+k}([0,1]\times\mathbb{R}^d) : \mathrm{Var}(S;\bullet)\ll\mathscr{L}^1,\ \mathrm{Var}(\partial S;\bullet)\llcorner(0,1)\ll\mathscr{L}^1\big\}.$$

3.2. Disintegration of Space-Time Integral Currents

We now turn our attention to the disintegration structure of space-time integral currents. Let $S\in\mathrm{I}_{k+1}([0,1]\times\mathbb{R}^d)$ and let $\|S\|$ denote its mass measure. We define the **critical set** of S as

$$\mathrm{Crit}(S) := \big\{(t,x)\in[0,1]\times\mathbb{R}^d : \nabla^S\mathbf{t}(t,x) = 0\big\}. \tag{3.2}$$

Here, $\nabla^S \mathbf{t}$ is approximate gradient of $\mathbf{t}$ with respect to S, i.e., the projection of $\nabla \mathbf{t}$ onto the approximate tangent space to the $\mathscr{H}^{k+1}$-rectifiable carrier of S.

Theorem 3.4. (Disintegration structure) *Let $S \in \mathrm{I}_{k+1}([0,1] \times \mathbb{R}^d)$ and let $\{\mu_t\}$ the disintegration of $\|S\|$ with respect to the map $\mathbf{t}$ and the measure*

$$\lambda := \mathscr{L}^1 + (\mathbf{t}_\# \|S\|)^s,$$

where $(\mathbf{t}_\# \|S\|)^s$ denotes the singular part (with respect to $\mathscr{L}^1$) of $\mathbf{t}_\# \|S\|$, i.e.

$$\|S\| = \int_0^1 \mu_t \, \mathrm{d}\lambda(t) = \int_0^1 \mu_t \, \mathrm{d}t + \int_0^1 \mu_t \, \mathrm{d}\lambda^s(t).$$

Then the following statements hold:

(i) *For λ^s-a.e. $t \in \mathbb{R}$ the measure μ_t is concentrated on* $\mathrm{Crit}(S)$.

(ii) *For $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$ the measure μ_t can be decomposed as*

$$\mu_t = |\nabla^S \mathbf{t}|^{-1} \|S|_t\| + \mu_t^s,$$

where μ_t^s is a measure which is concentrated on $\mathrm{Crit}(S)$ *and is singular with respect to $|\nabla^S \mathbf{t}|^{-1} \|S|_t\|$ and also with respect to $\mathscr{H}^k$.*

The disintegration of the mass measure $\|S\|$ with respect to the map $\mathbf{t}$ can therefore be written as

$$\|S\| = \int_0^1 \left(|\nabla^S \mathbf{t}|^{-1} \|S|_t\| + \mu_t^s \right) \, \mathrm{d}t + \int_0^1 \mu_t \, \mathrm{d}\lambda^s(t). \tag{3.3}$$

Observe that, by the Besicovitch differentiation theorem (see, e.g., [7, Thm. 2.22]), μ_t^s can be identified (for $\mathscr{L}^1$-a.e. t) with the restriction of μ_t to the set

$$\left\{ (t,x) : \limsup_{\rho \to 0} \frac{\mu_t(B_\rho(x))}{\rho^k} = \infty \right\}.$$

This conveys the idea that μ_t^s is *more concentrated* than $\mathscr{H}^k$.

We remark that, in general, all terms in the disintegration (3.3) can be non-zero.

The measure λ^s takes into account singular-in-time behaviour and one can surmise it vanishes if S is (absolutely) continuous in time. The next proposition confirms this intuition and contains a characterisation of AC space-time current via time projections.

Proposition 3.5. *Let $S \in \mathrm{I}_{k+1}([0,1] \times \mathbb{R}^d)$ with $\partial S \llcorner (0,1) \times \mathbb{R}^d = 0$. Then, $S \in \mathrm{I}^{\mathrm{AC}}_{1+k}([0,1] \times \mathbb{R}^d)$ if and only if $\lambda^s = 0$, i.e.,*

$$\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S)) \ll \mathscr{L}^1.$$

On the other hand, the measures μ_t^s in (3.3) account for a completely different type of singularity, measuring a sort of *diffuse concentration* that is "smeared out in time". These aspects will be further investigated in the next section.

3.3. Negligible Criticality Condition and Sard-type Property

The considerations at the end of the previous section inspire the following definitions, which turn out to be central:

Definition 3.6. A space-time current $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ is said to satisfy the **negligible-criticality condition** (with respect to the map $\mathbf{t}$) if

$$\|S\| \llcorner \mathrm{Crit}(S) = 0, \tag{NC}$$

where $\mathrm{Crit}(S)$ is the critical set of S defined in (3.2).

The condition (NC) means that for $\|S\|$-a.e. (t,x), $\mathrm{span}\,(\vec{S}(t,x)) \not\subseteq \{0\} \times \mathbb{R}^d$ (i.e., the approximate tangent space to S has almost always a non-trivial time component).

We also consider the following (in general weaker) condition:

Definition 3.7. A current $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$ is said to have the **Sard property** (with respect to the map $\mathbf{t}$) if

$$\mu_t^s = 0 \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in \mathbb{R}, \tag{S}$$

where μ_t^s is as in Theorem 3.4.

The following lemma clarifies the relationship between (NC) and the Sard property.

Lemma 3.8. *Let $S \in \mathrm{I}_{1+k}([0,1] \times \mathbb{R}^d)$. Then, the following statements are equivalent:*
(i) *S has the Sard property, i.e., $\mu_t^s = 0$ for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$;*
(ii) *$\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S))$ is singular with respect to $\mathscr{L}^1$.*
Furthermore, (NC) implies both of them.

Notice that this is indeed a Sard-type property: if $f : \mathbb{R}^d \to \mathbb{R}$ is a C^1 function and $\Gamma := \{(f(x), x) : x \in \mathbb{R}^d\} \subset \mathbb{R} \times \mathbb{R}^d$ denotes its graph, then f has the classical Sard property (namely, $\mathscr{L}^1(f(\{x : \nabla f(x) = 0\})) = 0$) if and only if the natural integral d-current associated to Γ has the Sard property in the sense of Definition 3.7.

The following structure result offers some equivalent conditions to the negligible criticality property for AC currents.

Proposition 3.9. *Let $S \in \mathrm{I}^{\mathrm{AC}}_{1+k}([0,1] \times \mathbb{R}^d)$. Then, the following are equivalent:*
(i) *S has the (NC) property, i.e., $\|S\| \llcorner \mathrm{Crit}(S) = 0$;*
(ii) *it holds that $\mathbf{t}_\#(\|S\| \llcorner \mathrm{Crit}(S)) \perp \mathscr{L}^1$;*
(iii) *S has the Sard property, i.e., $\mu_t^s = 0$ for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$;*
(iv) *$\|S\| \ll \int_0^1 \|S|_t\|\, \mathrm{d}t$, that is, for every Borel set $A \subset [0,1] \times \mathbb{R}^d$*

$$\int_0^1 \|S|_t\|(A)\, \mathrm{d}t = 0 \implies \|S\|(A) = 0.$$

Furthermore, if any of the above conditions holds, then the disintegration of the mass measure $\|S\|$ with respect to $\mathbf{t}$ and $\mathscr{L}^1$ has the form

$$\|S\| = \int_0^1 |\nabla^S \mathbf{t}|^{-1} \|S|_t\|\, \mathrm{d}t.$$

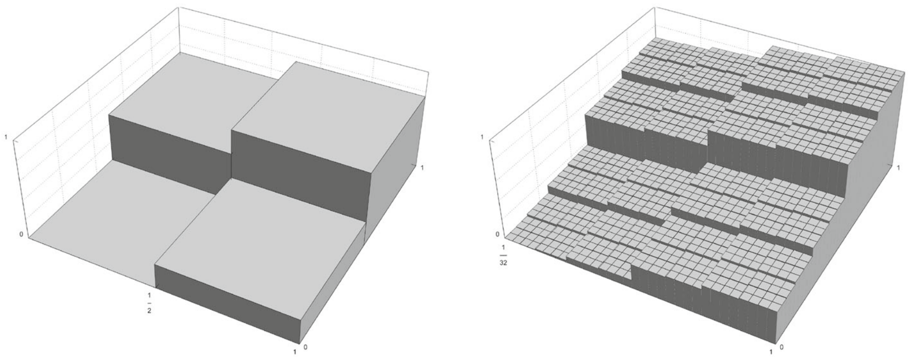

Figure 2 A space-time current without the Negligible Criticality condition

Observe that if $k = 0$, the Sard property is always satisfied, that is, for every $S \in \mathrm{I}_1(\mathbb{R}\times\mathbb{R}^d)$ it holds that $\mathbf{t}_\#(\|S\|\llcorner C) \perp \mathscr{L}^1$, since by the area formula $\mathscr{H}^1(\mathbf{t}(C)) = 0$. So, the effects related to the criticality are not present in all of the classical theory of BV- or AC-maps (which can be recovered as the $k = 0$ endpoint of our theory).

For $k > 1$, instead, the Sard property is not always satisfied and it is possible to construct AC (even Lipschitz) integral currents that do not have the Sard property. We refer the reader to [10, Section 9] for the details of the construction of such an example—see Fig. 2 for a visual depiction of some steps of the construction.

We conclude this section by observing that it is not clear how to extend the theory developed within these paragraphs to *normal* space-time currents. This extension, beside having a purely mathematical interest, would be extremely important in connection with applications, too. Indeed, the "diffuse" setting of normal currents is the natural one to consider, allowing one to deal with *fields* of singular objects, as it is often required in Materials Science.

4. Geometric Derivative and Advection Theorem

In this section, we show one of the main results, more precisely what we call the *advection theorem*. This result entails that for space-time currents satisfying the negligible criticality condition (NC), there exists an advecting vector field – namely their slices satisfy the transport equation (GTE). Furthermore, also the converse holds. These results should be compared with e.g. [8, Theorem 8.3.1], where a similar advection theorem is established within the class of probability measures.

Recall that, if $S \in \mathrm{I}^{\mathrm{AC}}_{k+1}([0,1]\times\mathbb{R}^d)$ then, by Proposition 3.9, the condition (NC) is equivalent to S having the Sard property or also to

$$\|S\| \ll \int_0^1 \|S|_t\| \,\mathrm{d}t.$$

We define the **geometric derivative** of such an S as

$$\boldsymbol{b}(t,x) := \frac{\mathrm{D}}{\mathrm{D}t} S(t,x) := \frac{\mathbf{p}(\xi(t,x))}{|\nabla^S \mathbf{t}(t,x)|}, \qquad (t,x) \in \mathrm{Crit}(S)^c, \tag{4.1}$$

where $\xi = |\nabla^S \mathbf{t}|^{-1}\nabla^S \mathbf{t}$ on $\mathrm{Crit}(S)^c$. Observe that $\boldsymbol{b}$ is well-defined $(\int_0^1 \|S|_t\| \, \mathrm{d}t)$-almost everywhere and under (NC), both ξ and $\boldsymbol{b}$ are well-defined also $\|S\|$-almost everywhere.

The main result reads as follows:

Theorem 4.1. (Advection) *Let $S \in \mathrm{I}^{\mathrm{AC}}_{k+1}([0,1]\times\mathbb{R}^d)$ with $(\partial S) \llcorner (0,1)\times\mathbb{R}^d = 0$ satisfy (NC). Then, the geometric derivative $\boldsymbol{b} := \frac{\mathrm{D}}{\mathrm{D}t}S$ defined in (4.1) belongs to $\mathrm{L}^1(\int_0^1 \|S|_t\| \, \mathrm{d}t)$ and it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t) + \mathcal{L}_{\boldsymbol{b}_t} S(t) = 0. \tag{4.2}$$

Conversely, if there exists a vector field $\boldsymbol{b} \in \mathrm{L}^1(\int_0^1 \|S|_t\| \, \mathrm{d}t)$ such that (4.2) holds, then S satisfies (NC).

Proof. We here sketch only the proof of the sufficiency part, referring the interested reader to the original paper [10] for the complete proof (notice that the proof of the necessity part requires a suitable gluing techique that will be discussed later in this work). Suppose we are given $S \in \mathrm{I}^{\mathrm{AC}}_{k+1}([0,1]\times\mathbb{R}^d)$ with $(\partial S) \llcorner (0,1)\times\mathbb{R}^d = 0$ satisfying (NC). We therefore have the disintegration

$$S \llcorner \mathrm{Crit}(S)^c = \int_0^1 \frac{\xi}{|\nabla^S \mathbf{t}|} \wedge S|_t \, \mathrm{d}t.$$

In particular, since S has (NC), then $S = S \llcorner \mathrm{Crit}(S)^c$ and therefore

$$S = \int_0^1 \frac{\xi}{|\nabla^S \mathbf{t}|} \wedge S|_t \, \mathrm{d}t.$$

It will be convenient to write this conclusion in the following form:

$$\int_0^1 \left(\frac{\xi}{|\nabla^S \mathbf{t}|} \wedge S|_t \right) \psi(t) \, \mathrm{d}t = (\psi \circ \mathbf{t}) S \qquad \text{for every } \psi \in \mathrm{C}^\infty_c((0,1)). \tag{4.3}$$

We now want to prove that for every $\psi \in \mathrm{C}^\infty_c((0,1))$ and for every $\omega \in \mathscr{D}^k(\mathbb{R}^d)$ it holds that

$$\int_0^1 \langle S(t), \omega \rangle \psi'(t) - \langle \mathcal{L}_{\boldsymbol{b}_t} S(t), \omega \rangle \psi(t) \, \mathrm{d}t = 0. \tag{4.4}$$

Notice that

$$
\begin{aligned}
-\int_0^1 \langle \mathcal{L}_{\boldsymbol{b}_t} S(t), \omega\rangle \psi(t)\,\mathrm{d}t &= \int_0^1 \langle b_t \wedge S(t), d\omega\rangle \psi(t)\,\mathrm{d}t \\
&= \left\langle \int_0^1 \boldsymbol{b}_t \wedge S(t)\psi(t)\,\mathrm{d}t, d\omega \right\rangle \\
&= \left\langle \int_0^1 \frac{\mathbf{p}(\xi)}{|\nabla^S \mathbf{t}|} \wedge \mathbf{p}_*(S|_t)\psi(t)\,\mathrm{d}t, d\omega \right\rangle \\
&= \left\langle \int_0^1 \mathbf{p}_*\left(\frac{\xi}{|\nabla^S \mathbf{t}|} \wedge S|_t\right)\psi(t)\,\mathrm{d}t, d\omega \right\rangle \\
&= \left\langle \mathbf{p}_*\left(\int_0^1 \frac{\xi}{|\nabla^S \mathbf{t}|} \wedge S|_t \psi(t)\,\mathrm{d}t\right), d\omega \right\rangle \\
&= \langle \mathbf{p}_*((\psi \circ \mathbf{t})S), d\omega\rangle \\
&= \langle \partial \mathbf{p}_*((\psi \circ \mathbf{t})S), \omega
\end{aligned}
$$

where in the second-to-last equality we used (4.3). Using also the commutativity between boundary and pushforward, we have thus shown

$$
-\int_0^1 \mathcal{L}_{\boldsymbol{b}_t}(S(t))\psi(t)\,dt = \partial \mathbf{p}_*[(\psi \circ \mathbf{t})S] = \mathbf{p}_* \partial[(\psi \circ \mathbf{t})S]. \tag{4.5}
$$

Observe now that for any k-current T, for any $f \in \mathrm{C}_c^\infty(\mathbb{R})$, and any $\omega \in \mathscr{D}^k(\mathbb{R}^d)$, the Leibniz rule holds in the form

$$
\langle \partial(fT), \omega\rangle = \langle f\partial T, \omega\rangle - \langle T \llcorner df, \omega\rangle.
$$

Using this in (4.5) and also taking into account that $d(\psi \circ \mathbf{t}) = (\psi' \circ \mathbf{t})dt$ as well as (2.5),

$$
\begin{aligned}
-\int_0^1 \langle \mathcal{L}_{\boldsymbol{b}_t} S(t), \omega\rangle \psi(t)\,\mathrm{d}t &= \langle \partial[(\psi \circ \mathbf{t})S], \mathbf{p}^*\omega\rangle \\
&= \langle (\psi \circ \mathbf{t})\partial S, \mathbf{p}^*\omega\rangle - \langle S \llcorner d(\psi \circ \mathbf{t}), \mathbf{p}^*\omega\rangle \\
&= \langle (\psi \circ \mathbf{t})\partial S, \mathbf{p}^*\omega\rangle - \langle S \llcorner (\psi' \circ \mathbf{t})dt, \mathbf{p}^*\omega\rangle \\
&= \langle (\psi \circ \mathbf{t})\partial S, \mathbf{p}^*\omega\rangle - \int_0^1 \langle S|_t, \mathbf{p}^*\omega\rangle \psi'(t)\,\mathrm{d}t.
\end{aligned}
$$

Since $\psi \in \mathrm{C}_c^\infty((0,1))$ and $\partial S \llcorner (0,1) \times \mathbb{R}^d = 0$, the term $(\psi \circ \mathbf{t})\partial S$ vanishes. Recalling that, by definition, $S(t) = \mathbf{p}_*(S|_t)$, we finally obtain (4.4). $\square$

5. Well-Posedness for Lipschitz Velocity Fields

Relying on the space-time approach, we now present a well-posedness result for (GTE). We show that a Lipschitz condition on the vector field $\boldsymbol{b}$ is sufficient to ensure existence and uniqueness of solutions to the initial-value problem for the geometric transport equation.

Theorem 5.1. *Let $b\colon \mathbb{R}^d \to \mathbb{R}^d$ be a globally bounded and Lipschitz vector field with flow $\Phi_t = \Phi(t, \cdot)\colon \mathbb{R}^d \to \mathbb{R}^d$ and let $\overline{T} \in \mathrm{N}_k(\mathbb{R}^d)$ be a k-dimensional normal current on $\mathbb{R}^d$. Then, the initial-value problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} T_t + \mathcal{L}_b T_t = 0, & t \in (0,1), \\ T_0 = \overline{T} \end{cases}$$

admits a solution $(T_t)_{t\in(0,1)} \subset \mathrm{N}_k(\mathbb{R}^d)$ of normal k-currents, which is unique in the class of normal k-currents. The solution is given by the pushforward of the initial current under the flow, namely, $T_t = (\Phi_t)_*\overline{T}$. In particular, if $\overline{T}$ is integral, then so is T_t, $t \in (0,1)$.

Notice that, at a technical level, it is not immediately clear why Theorem 5.1 should hold true. We know that solutions to (GTE) can be understood as the transport of the currents T_t along the flow lines of $\boldsymbol{b}$. However, since the flow of $\boldsymbol{b}$ is merely Lipschitz, it may well occur that it is not (fully) differentiable *anywhere* on the support of the transported currents, therefore questioning the possibility of performing pointwise computations.

In order to overcome these difficulties we combine the space-time approach developed in the sections above with a relatively recent tool from Geometric Measure Theory, namely the notion of *decomposability bundle*, introduced by Alberti and Marchese [4]. This tool ensures that, while full differentiability of the flow Φ_t on the support of $\overline{T}$ may fail, the derivative of Φ_t still exists in a sufficiently good sense to define the pushforward $(\Phi_t)_*\overline{T}$ *pointwise* (and not via the homotopy formula) – see Lemma 2.1.

Additionally, we remark that the *existence* part of Theorem 5.1 can be proved by means of a simple approximation argument that does not necessitate the decomposability bundle (but assumes the existence of solutions when the drift is smooth). In this section we will therefore focus on the uniqueness part.

5.1. The Case $k = 0$: Uniqueness for the Continuity Equation

We present first a simple proof of uniqueness for the transport of 0-currents, i.e., signed measures advected via the continuity equation. Our proof differs from the classical one that can be found, e.g., in [8, Proposition 8.3.1] – see also [6,12,14].

In the case of 0-currents the geometric transport equation (GTE) reduces to the continuity equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \operatorname{div}(\boldsymbol{b}\mu_t) = 0 \tag{5.1}$$

where $(\mu_t)_{t\in(0,1)}$, is a family of signed measures. We understand this equation in the usual distributional sense, i.e.

$$\int_0^1 \int_{\mathbb{R}^d} \partial_t\psi(t,x) + \boldsymbol{b}(x)\cdot\nabla\psi(t,x)\, \mathrm{d}\mu_t(x)\, \mathrm{d}t = 0 \tag{5.2}$$

for all $\psi \in \mathrm{C}^1_c((0,1)\times\mathbb{R}^d)$.

We illustrate our proof idea by showing uniqueness under the additional regularity assumption that $\boldsymbol{b}$ is of class C^1. The key is to show directly that the solution is necessarily given by

$$\mu_t = (\Phi_t)_\# \overline{\mu}$$

or, equivalently, that given a solution $(\mu_t)_{t\in(0,1)}$, the map $t \mapsto (\Phi_{-t})_\# \mu_t$ is constant. It is therefore natural to test the weak formulation (5.2) with a function of the form $\psi(t,x) = \alpha(t)\beta(\Phi_{-t}(x))$. The field is C^1, and so is its flow Φ, thus one can differentiate ψ classically. On the one hand we have that

$$\begin{aligned}\partial_t \psi(t,x) &= \alpha'(t)\beta(\Phi_{-t}(x)) + \alpha(t)\nabla\beta(\Phi_{-t}(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{-t}(x) \\ &= \alpha'(t)\beta(\Phi_{-t}(x)) - \alpha(t)\nabla\beta(\Phi_{-t}(x)) \cdot b(\Phi_{-t}(x)),\end{aligned}$$

where we used the defining property of Φ. On the other hand, by an elementary computation on the directional derivative of the flow (see [11, Lemma 2.3]) we have

$$\begin{aligned}\boldsymbol{b}(x) \cdot \nabla_x \psi(t,x) &= \alpha(t)\boldsymbol{b}(x) \cdot \big(\nabla\beta(\Phi_{-t}(x)) D\Phi_{-t}(x)\big) \\ &= \alpha(t)\nabla\beta(\Phi_{-t}(x)) \cdot \big(D\Phi_{-t}(x)[b(x)]\big) \\ &= \alpha(t)\nabla\beta(\Phi_{-t}(x)) \cdot b(\Phi_{-t}(x)).\end{aligned}$$

Plugging the two terms in the weak formulation gives that

$$\int_0^1 \alpha'(t)\left\langle \mu_t, \beta(\Phi_{-t}(x))\right\rangle \mathrm{d}t = 0$$

for all $\alpha \in \mathrm{C}^1((0,1))$ and all $\beta \in \mathrm{C}^1(\mathbb{R}^d)$. From this we deduce that $t \mapsto (\Phi_{-t})_\# \mu_t$ is constant, as required.

5.2. The Lipschitz Case

We now consider the case when $\boldsymbol{b}$ is Lipschitz. Notice that, setting $\mu := \mathscr{L}^1(\mathrm{d}t) \otimes \mu_t(\mathrm{d}x)$, two equivalent ways of formulating the PDE are the following:

$$\int_{(0,1)\times\mathbb{R}^d} (1, \boldsymbol{b}(x)) \cdot \tilde{\nabla}\psi(t,x)\, \mathrm{d}\mu(t,x) = 0 \tag{5.3}$$

for all $\psi \in \mathrm{C}^1_c((0,1)\times\mathbb{R}^d)$, where $\tilde{\nabla}\psi(t,x) := (\partial_t\psi(t,x), \nabla\psi(t,x))$. Equivalently,

$$\int_{(0,1)\times\mathbb{R}^d} D\psi(t,x)[(1,\boldsymbol{b}(x))]\, \mathrm{d}\mu(t,x) = 0$$

for all $\psi \in \mathrm{C}^1_c((0,1)\times\mathbb{R}^d)$.

Having introduced this notation, we can present the following key result.

Lemma 5.2. *In the setting above, $(1, \boldsymbol{b}(x)) \in V(\mu, (t,x))$ for μ-a.e. (t,x). In particular, every Lipschitz function $\psi : (0,1)\times\mathbb{R}^d \to \mathbb{R}$ is differentiable in direction $(1,\boldsymbol{b}(x))$ for μ-a.e. $(t,x) \in (0,1)\times\mathbb{R}^d$.*

Proof. Let U be the 1-current in $(0,1)\times\mathbb{R}^d$ defined by $U := (1,\boldsymbol{b}(x))\mu$. Then (5.3) can be understood as

$$\partial U = 0 \qquad \text{in } (0,1)\times\mathbb{R}^d,$$

hence U is a normal 1-current without boundary in $(0,1)\times\mathbb{R}^d$. Thus, the assertion follows by (2.8). $\square$

One can easily show, as a consequence of Lemma 5.2, that one can use *any* Lipschitz test function in the distributional formulation of the continuity equation (5.1). This crucial ingredient allows us to get the desired uniqueness, as we are about to show.

Proof of the uniqueness statement of Theorem 5.1 in the case $k = 0$. We choose as test function $\psi(t,x) := \alpha(t)\beta(\Phi_{-t}(x))$, where $\alpha \in \mathrm{C}^1_c((0,1))$ and $\beta \in \mathrm{C}^1_c(\mathbb{R}^d)$. Recalling that Φ_{-t} is Lipschitz, also ψ is a Lipschitz function and therefore, by Lemma 5.2,

$$\int_{(0,1)\times\mathbb{R}^d} D\psi(t,x)[(1,\boldsymbol{b}(x))]\,\mathrm{d}\mu(t,x) = 0. \tag{5.4}$$

At this point, one can easily compute (at *every* point) the integrand, showing that

$$D\psi(t,x)[(1,\boldsymbol{b}(x))] = \alpha'(t)\beta(\Phi_{-t}(x)) \qquad \text{for every } (t,x) \in (0,1)\times\mathbb{R}^d. \tag{5.5}$$

Then, plugging (5.5) into (5.4), we have

$$\begin{aligned} 0 &= \int_{(0,1)\times\mathbb{R}^d} \alpha'(t)\beta(\Phi_{-t}(x))\,\mathrm{d}\mu(t,x) \\ &= \int_0^1 \alpha'(t)\,\big\langle \mu_t, \beta\circ\Phi_{-t}\big\rangle\,\mathrm{d}t \\ &= \int_0^1 \alpha'(t)\,\big\langle (\Phi_{-t})_\#\mu_t, \beta\big\rangle\,\mathrm{d}t. \end{aligned}$$

This holds for every $\alpha \in \mathrm{C}^1_c((0,1))$, hence we obtain that the map $t \mapsto \big\langle (\Phi_t)^{-1}_\#\mu_t, \beta\big\rangle$ is constant for every β. Recalling the weak* continuity of $t\mapsto \mu_t$ it follows that

$$\big\langle (\Phi_t)^{-1}_\#\mu_t, \beta\big\rangle = \big\langle \mu_0, \beta\big\rangle \qquad \text{for all } \beta \in \mathrm{C}^1_c(\mathbb{R}^d),$$

and since β is arbitrary we have $(\Phi_t)^{-1}_\#\mu_t = \mu_0$ as measures, i.e. $\mu_t = (\Phi_t)_\#\mu_0$. □

5.3. The Case $k > 0$: Gluing of Transported Currents

The strategy presented above for the continuity equation essentially carries over to the case $k > 0$ and can be adapted with minor modifications. We single out here a single technical difficulty: we need to find a replacement for Lemma 5.2 or, more precisely, for the space-time 1-current U, defined directly as $U = (1,\boldsymbol{b})\mu$ in the proof of Lemma 5.2. In order to do this, we need to resort once again to gluing techniques. The following proposition constitutes another approach (beside the one presented in Theorem 3.2) to turn a path of integral currents into a space-time current. It applies when we know *a priori* that our path of integral currents satisfies the geometric transport equation.

Lemma 5.3. *Let* $(T_t)_{t\in(0,1)} \subset \mathrm{N}_k(\mathbb{R}^d)$ *with* $\partial T_t = 0$ *be a weakly***-continuous solution to* (GTE). *Write* $T_t = \vec{T}_t\|T_t\|$, *with* $\vec{T}_t$ *unit* k*-vectors and let* $\vec{T} : (0,1)\times\mathbb{R}^d \to \bigwedge_k(\mathbb{R}\times\mathbb{R}^d)$ *be the* k*-vector field defined* $\mathscr{L}^1_t \otimes \|T_t\|$*-almost everywhere by*

$$\vec{T}(t,x) := (\iota_t)_*\vec{T}_t(x),$$

where we recall that $\iota_t(x) := (t,x)$. *Define the current*

$$U := [(1,\boldsymbol{b}(x))\wedge\vec{T}(t,x)]\,\mathscr{L}^1(\mathrm{d}t)\otimes\|T_t\|(\mathrm{d}x).$$

Then U is a normal $(k+1)$-current in $(0,1)\times\mathbb{R}^d$, and $\partial U \llcorner (0,1)\times\mathbb{R}^d = 0$.

The fact that U is normal allows one to use the Alberti-Marchese's theory, and in particular we immediately obtain the following result:

Corollary 5.4. *Define the measure $\mu := \mathscr{L}^1(\mathrm{d}t)\otimes\|T_t\|$ in $\mathbb{R}\times\mathbb{R}^d$. Then, we have $\operatorname{span}((1,\boldsymbol{b})\wedge\vec{T}_t(x))\subset V(\mu,(t,x))$ for μ-a.e. (t,x) in $(0,1)\times\mathbb{R}^d$.*

We refer the reader to [11] for the complete proof.

6. Rademacher-type Differentiability Result

We collect in this section two Rademacher-type differentiability theorems for paths of currents. Given a path $t\mapsto T_t\in \mathrm{I}_k(\mathbb{R}^d)$, $\partial T_t = 0$, which is absolutely continuous in time with respect to the homogeneous integral flat norm $\mathbb{F}_{\mathrm{I}}$, we ask when we can find a vector field $\boldsymbol{b}_t\colon\mathbb{R}^d\to\mathbb{R}^d$ that solves the geometric transport equation. The existence of such a vector field implies that the path $t\mapsto T_t$ is differentiable in a geometric sense, hence the designation "Rademacher".

Theorem 6.1. (Weak differentiability) *Let $t\mapsto T_t\in \mathrm{I}_k(\mathbb{R}^d)$, $t\in[0,1]$, with $\partial T_t = 0$ for every $t\in[0,1]$, be a path that is absolutely continuous with respect to the homogeneous integral flat norm $\mathbb{F}$ (which is implied by $\mathbb{F}_{\mathrm{I}}$-absolute continuity), that is,*

$$\mathbb{F}(T_s - T_t)\le\int_s^t g(r)\,\mathrm{d}r$$

for some $g\in \mathrm{L}^1([0,1])$ and all $s<t$. Then, there exists a finite-mass $(k+1)$-current $R_t\in \mathrm{M}_{k+1}(\mathbb{R}^d)$ that solves the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}T_t - \partial R_t = 0.$$

We next ask when the currents R_t are actually of the form $-\boldsymbol{b}_t\wedge T_t$ for some vector field $\boldsymbol{b}_t$, so that we can solve the geometric transport equation in the original formulation (GTE). This question is more subtle, and after the previous sections it is perhaps not surprising that a positive answer is strictly related to the property (NC) or, equivalently, the Sard property (S).

Theorem 6.2. (Strong differentiability) *Let $t\mapsto T_t\in \mathrm{I}_k(\mathbb{R}^d)$, $t\in[0,1]$, be a path that is absolutely continuous with respect to the homogeneous integral flat norm $\mathbb{F}_{\mathrm{I}}$ and such that*

$$\partial T_t = 0,\ t\in[0,1] \qquad \textit{and} \qquad \sup_{t\in[0,1]}\mathbf{M}(T_t)<\infty.$$

Let S be the unique current given by Theorem 3.2 (and Remark 3.3) in this setting. If S satisfies (NC), then there exists $\boldsymbol{b}\in \mathrm{L}^1(\mathscr{L}^1\otimes\|T_t\|)$ such that (GTE) holds, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}T_t + \mathcal{L}_{\boldsymbol{b}_t}T_t = 0.$$

In some special cases – namely when $k \in \{0, d-2, d-1, d\}$ – it is known that $\mathbb{F}_{\mathrm{I}}$ coincides with $\mathbb{F}$, and thus all the results can be stated using the latter norm.

Corollary 6.3. *Let $k \in \{0, d-2, d-1, d\}$ and let $t \mapsto T_t \in \mathrm{I}_k(\mathbb{R}^d)$, $t \in [0,1]$, be a path that is absolutely continuous with respect to the homogeneous (non-integral) flat norm $\mathbb{F}$ and such that*

$$\partial T_t = 0, \ t \in [0,1] \qquad \text{and} \qquad \sup_{t \in [0,1]} \mathbf{M}(T_t) < \infty.$$

Let S be the unique current given by Theorem 3.2 (and Remark 3.3) in this setting. If S satisfies (NC), then there exists $\boldsymbol{b} \in \mathrm{L}^1(\mathscr{L}^1 \otimes \|T_t\|)$ such that (GTE) holds.

For general k it is not known whether, for $T \in \mathrm{I}_k(\mathbb{R}^d)$, the definitions of homogeneous flat norm and homogeneous integral flat norm give rise to equivalent norms.

7. Stability of AC Integral Currents with (NC)

In this final section, we study the stability properties of absolutely continuous space-time integral currents and of the condition (NC).

Proposition 7.1. *Let $(t \mapsto T_t^\varepsilon)_{\varepsilon>0} \subset \mathrm{AC}((0,1); \mathrm{I}_k(\mathbb{R}^d))$ be a family of curves of currents with $\partial T_t^\varepsilon = 0$ for every $\varepsilon > 0$ and $t \in (0,1)$. Assume that*

$$\sup_{\varepsilon>0} \sup_{t \in (0,1)} \mathbf{M}(T_t^\varepsilon) < \infty.$$

For each $\varepsilon > 0$, denote by $S^\varepsilon \in \mathrm{I}^{\mathrm{AC}}_{1+k}(\mathbb{R} \times \mathbb{R}^d)$ the (unique) current given by Theorem 3.2 (and Remark 3.3) in this setting. Then the following statements are true:

(i) *If $\sup_{\varepsilon>0} \mathbb{F}\text{-pV}(t \mapsto T_t^\varepsilon; \mathbb{R}) < \infty$ or, equivalently, $\sup_{\varepsilon>0} \mathrm{Var}(S^\varepsilon; \mathbb{R}) < \infty$, then there exists $S \in \mathrm{I}_{1+k}(\mathbb{R} \times \mathbb{R}^d)$ such that $S^\varepsilon \overset{*}{\rightharpoonup} S$ up to a (non-relabelled) subsequence, in the sense of $(1+k)$-currents on $\mathbb{R} \times \mathbb{R}^d$.*

(ii) *Assume the currents S^ε satisfy the* uniform negligible criticality condition, *i.e. there exist vector fields $\boldsymbol{b}_t^\varepsilon \in \mathrm{L}^1(\mathscr{L}^1 \otimes \|T_t^\varepsilon\|)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t^\varepsilon + \mathcal{L}_{\boldsymbol{b}_t^\varepsilon} T_t^e = 0,$$

and such that the maps $B^\varepsilon \colon (0,1) \to \mathbb{R}$ defined by

$$t \mapsto \int_{\mathbb{R}^d} |\boldsymbol{b}^\varepsilon(t,x)| \, \mathrm{d}\|T_\varepsilon^t\|(x)$$

are uniformly integrable on $(0,1)$. Then there exists $S \in \mathrm{I}^{\mathrm{AC}}_{1+k}(\mathbb{R} \times \mathbb{R}^d)$ such that $S^\varepsilon \overset{*}{\rightharpoonup} S$ up to a (non-relabelled) subsequence, in the sense of $(1+k)$-currents on $\mathbb{R} \times \mathbb{R}^d$.

(iii) *Suppose that the vector fields $\boldsymbol{b}_t^\varepsilon$ defined in Point (ii) converge uniformly, as $\varepsilon \to 0$, to a continuous vector field $\boldsymbol{b}_t$. Then there exists $S \in \mathrm{I}^{\mathrm{AC}}_{1+k}(\mathbb{R} \times \mathbb{R}^d)$ such that $S^\varepsilon \overset{*}{\rightharpoonup} S$ up to a (non-relabelled) subsequence, in the sense of $(1+k)$-currents on $\mathbb{R} \times \mathbb{R}^d$ and the current S has (NC).*

Proof. We split the proof in various parts.

(i) It is enough to apply [19, Theorem 3.7].

(ii) By the same arguments used in the proof of Lemma [10, Lemma 3.5], we have that for every measurable set $I \subset \mathbb{R}$ it holds

$$\mathrm{Var}(S^\varepsilon; I) = \mathbb{F}\text{-pV}(t \mapsto T_t^\varepsilon; I) \le 2 \int_I B^\varepsilon(\tau)\,\mathrm{d}\tau,$$

whence we deduce that the (absolutely continuous) measures $\mathrm{Var}(S^\varepsilon, \bullet)$ have uniform integrable densities. Applying Point (i) and Dunford-Pettis' Theorem we therefore conclude that the family S^ε converges, up to a subsequence, to a current S whose variation is still absolutely continuous.

(iii) It is clear that there exists a limit current $S \in \mathrm{I}^{\mathrm{AC}}_{1+k}(\mathbb{R} \times \mathbb{R}^d)$ because the assumption of Point (ii) is automatically satisfied. We now show that any such limit S has (NC), because its slices solve the geometric transport equation with vector field $\boldsymbol{b}$ (as we know, this is indeed equivalent to (NC) in view of Theorem 4.1). By assumption we have that for every $\varepsilon > 0$, for every $\psi \in \mathrm{C}^\infty_c((0,1))$ and every $\omega \in \mathscr{D}^k(\mathbb{R}^d)$ it holds

$$0 = \int_0^1 \left(\langle T_t^\varepsilon, \omega\rangle \psi'(t) + \langle \boldsymbol{b}_t^\varepsilon \wedge T_t^\varepsilon, d\omega\rangle \psi(t) \right) \mathrm{d}t. \tag{7.1}$$

By the weak convergence of currents we have that $\langle T_t^\varepsilon, \omega\rangle \to \langle T_t, \omega\rangle$, where $T_t := S(t)$ are the projected slices of S and thus, by the Dominated convergence theorem,

$$\int_0^1 \langle T_t^\varepsilon, \omega\rangle \psi'(t)\,\mathrm{d}t \to \int_0^1 \langle T_t, \omega\rangle \psi'(t)\,\mathrm{d}t. \tag{7.2}$$

For the other term, instead, we have

$$\boldsymbol{b}_t^\varepsilon \wedge T_t^\varepsilon - \boldsymbol{b}_t \wedge T_t = (\boldsymbol{b}_t^\varepsilon - \boldsymbol{b}_t) \wedge T_t^\varepsilon + \boldsymbol{b}_t \wedge (T_t^\varepsilon - T_t).$$

For every fixed $t \in (0,1)$, the mass of the first term can be estimated by

$$\mathbf{M}((\boldsymbol{b}_t^\varepsilon - \boldsymbol{b}_t) \wedge T_t^\varepsilon) \le \|\boldsymbol{b}_t^\varepsilon - \boldsymbol{b}_t\|_{C^0_x} \mathbf{M}(T_t^\varepsilon) \to 0, \qquad \text{as } \varepsilon \to 0. \tag{7.3}$$

Furthermore, since all currents T_t^ε and T_t have uniformly bounded mass, we can test them against *continuous* forms, not necessarily smooth and compactly supported. Therefore for every $t \in (0,1)$ we have

$$\langle \boldsymbol{b}_t \wedge (T_t^\varepsilon - T_t), \alpha\rangle = \langle T_t^\varepsilon - T_t, i_{\boldsymbol{b}_t}\alpha\rangle \to 0 \qquad \text{as } \varepsilon \to 0 \tag{7.4}$$

for every $\alpha \in \mathscr{D}^{k+1}(\mathbb{R}^d)$ (it is fairly easy to check that the k-form $i_{\boldsymbol{b}_t}\alpha$ has continuous coefficients, since $\boldsymbol{b}_t$ is continuous). Choosing in particular $\alpha = d\omega$ in (7.4) and combining it with (7.3) and (7.2), we can pass to the limit in (7.1) obtaining

$$0 = \int_0^1 \left(\langle T_t, \omega\rangle \psi'(t) + \langle \boldsymbol{b}_t \wedge T_t, d\omega\rangle \psi(t) \right) \mathrm{d}t,$$

which concludes the proof.

□

Acknowledgements

The content of this note is an expanded version of a seminar the author gave at the VII RISM School, held in Varese within the Riemann Prize Week (10–14 July, 2023). He acknowledges the organisers of this event for their kind invitation. The author is also grateful to Giacomo Del Nin and Filip Rindler for useful discussions.

Funding Information Open access funding provided by Università degli Studi di Trento within the CRUI-CARE Agreement.

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Received: January 8, 2024.
Accepted: March 1, 2024.