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Davide Panizzolo

Maps Between Projective Varieties

Description of the General Fiber of a Fano Mori Contraction

Relatore

Prof. Marco Andreatta

A mia moglie Anna

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CONTENTS

Introduction

Let X and W be projective varieties defined over the field $\mathbb{C}$ and let

$$\varphi : X \to W$$

be a projective and surjective morphism, i.e. a proper surjective morphism which is given by the global sections of a base point free linear system on X.

If the varieties X and W are *normal* and the projective morphism $\varphi : X \to W$ has only *connected fibers*, we say that φ is a *contraction*.

In this thesis we are interested in the study of some special classes of these contractions: in particular we will assume that *the variety X is smooth.*

Notice that this is a crucial assumption since, though some of the results presented in this work hold also if we assume some "mild" singularities, the approach in the smooth case is quite different from the one used in the singular case.

Let $\varphi : X \to W$ be a contraction from a smooth variety onto a normal one and let K_X be the canonical divisor of X. If $-K_X$ is φ-ample, we say that the contraction is *extremal* or *Fano Mori (F-M).*

Fano Mori contractions were introduced at the beginning of the eighties by S. Mori ([Mor82]) in the case of smooth threefolds and were generalized to higher dimensional varieties, admitting also some mild singularities, by Y. Kawamata ([Kaw84b], [Kaw84a]) and V. Shokurov ([Sho85]).

Their results, in the smooth case, can be summarized as follows.

Let X be a smooth variety which contains a curve C with $K_X \cdot C < 0$ (a *negative curve*), let $N_1(X)$ be the $\mathbb{R}$-vector space generated by 1-cycles of X

modulo numerical equivalence, let $NE(X)$ be the cone in $N_1(X)$ generated by the classes of the effective curves and let $\overline{NE}(X)$ be its closure.

- (Cone Theorem) Let $\overline{NE}_{K<0}(X)$ be the part of $\overline{NE}(X)$ formed by the negative curves.
  Then $\overline{NE}_{K<0}(X)$ is locally polyhedral and its rays, i.e. its one dimensional faces, are generated by rational curves with bounded degree with respect to $-K_X$.

Let σ be a face of $\overline{NE}_{K<0}(X)$, then σ is called *extremal face of* X. Given an extremal face σ of X, there exists a nef divisor H on X such that

$$\sigma = \{z \in \overline{NE}(X) \mid H \cdot z = 0\}.$$

The divisor H is called a *supporting divisor of* σ; moreover we can always assume that H is of the form $K_X + \tau L$, with L an ample divisor on X.

- (Contraction Theorem) The extremal faces of X can be contracted.
  This means that, given an extremal face σ with supporting divisor H, the linear system $|mH|$ is base point free for $m \gg 0$ and gives a projective morphism $\varphi : X \to W$ onto a normal projective variety W which is characterized by the following properties:

  1. For any irreducible curve C on X, $\varphi(C)$ is a point in W if and only if $H \cdot C = 0$, i.e. if and only if $[C] \in \sigma$ in $N_1(X)$;
  2. φ has connected fibers;
  3. $H = \varphi^* A$, for some ample Cartier divisor A on W

  The morphism φ is an extremal contraction and is called the *contraction* of the face σ, while H is a *supporting divisor* of φ.

Thus we have that a Fano Mori contraction of a smooth variety is defined by linear systems

$$|m(K_X + \tau L)|,$$

with L an ample divisor on X, $m \gg 0$ and τ a positive integer.

One possible way to study Fano Mori contractions of a smooth n-fold X supported by $K_X + \tau L$ is to classify them according to the values of τ.

From the Cone Theorem one can deduce that

$$\tau \leq n + 1$$

and from the Kobayashi Ochiai theorem $\tau = n+1$ if and only if X is the projective space.
Moreover S. Mori, for the case of smooth threefolds, T. Fujita ([Fuj87]) and P. Ionescu ([Ion86]), in the general case, classified all these contractions for $n-2 \leq \tau \leq n$.
More recently Y. Kawamata ([Kaw89]), M. Andreatta and J. Wiśniewski ([AW98]) dealt with the case $n = 4$ and $\tau = 1$ and gave a complete classification of these contractions.

To go further, i.e. to study the cases $\tau = n-3$ with $n > 4$ and $\tau < n-3$, the first step is to consider the general non trivial (i.e. non 0-dimensional) fibers of the extremal contractions from a smooth n-fold X.

G. Nakamura ([Nak95]) considered the case of F-M contractions supported by $K_X + (n-3)L$, assuming that the "exceptional locus" of the contraction has codimension 0 or 1 in X (i.e. the extremal contraction is *divisorial* or *of fiber type*).
In particular he showed that the general non trivial fibers of these contractions are irreducible and he gave a classification of these general fibers.

In this thesis we classify the general non trivial fibers of some extremal contractions.
In particular, after recalling the classical results, we deal with the case of extremal contractions supported by

$$K_X + (n-3)L$$

to give a different proof of the results in [Nak95], and the case

$$K_X + (n-4)L$$

assuming that the extremal contraction is divisorial or of fiber type.

The main ingredients in this study will be the theory of deformation of curves, developed by S. Mori and others, and some cohomological methods, such as vanishing theorems, due to Y. Kawamata and E. Viehveg, and the classical theory of Fujita Δ-genus.

In particular, in this range of τ, all these ingredients are sufficient to give a description of the general non trivial fiber of the contraction, *if we assume that the fiber is irreducible.*

The most delicate part of this work is the study of the irreducibility of the general non trivial fiber:

Bertini theorems and Generic Smoothness assure the smoothness and so also the irreducibility of the general fiber for a contraction of fiber type from a smooth variety, but they are not sufficient for the singular case or for the birational case.

Indeed the core of our work is to prove that, except in one case, the general non trivial fiber of a F-M contraction, which is divisorial or of fiber type, supported by $K_X + \tau L$ with $\tau = n-3, n-4$, is irreducible. Moreover this "exceptional case" is effective:
in the last Chapter of this thesis we give an example of divisorial elementary F-M contraction from a smooth fivefold supported by $K_X + \tau L$ with $\tau = 5-4 = 1$ whose fibers are all reducible. In order to construct this example we also give an example of birational contraction (not extremal) from a smooth threefold such that all the fibers are reducible.

Going into detais, the contents of the single Chapters are the following.

In Chapter 1, after recalling some results of the intersection theory, we define the cone of curves on a proper scheme and state the Kleiman's Criterion of ampleness. Then we recall the basic facts of Mori Theory for a smooth variety and the definitions and the first properties of Fano varieties and of the Fujita Δ-genus.

In Chapter 2 we develop the theory of deformations of curves on a projective variety to construct the families of deformations of a rational curve which are, is some sense, minimal: in particular we study the *unsplit families*, the *generically unsplit families*, and the *minimal dominating families* of rational curves, to state the Ionescu-Wiśniewski inequality and a very useful property concerning the Picard number of the locus of an unsplit family of rational curves.

In Chapter 3 we introduce the study of elementary F-M contraction (i.e. F-M contraction of a ray of $\overline{NE}(X)$); in particular we deal with the fiber type case to give the classification of the general fiber of an elementary F-M contraction supported by $K_X + \tau L$, with $n-4 \le \tau \le n+1$.

In Chapter 4 we study divisorial F-M contraction supported by $K_X + \tau L$, with $\tau \ge n-4$, using the techniques of T. Fujita and T. Ando, which have been used also by G. Nakamura: in particular we classify the general non trivial fiber of these contractions, under the assumption that this fiber is irreducible.

In Chapter 5 we investigate the irreducibility of the general non trivial fiber of the contraction studied in the previous chapter. The main theorem concerns the case $\tau = n-4$ and it is the following

Theorem. *Let X be a smooth $n-$fold, let $\varphi_X : X \to W$ be a divisorial, elementary F-M contraction supported by $K_X + (n-4)L$ and let E be its exceptional divisor. Then the general fiber G of φ_X is irreducible, except in the following case:*
φ_X is the contraction of a 5-fold associated to an extremal ray of length 1, which maps the irreducible divisor E to a curve and its general non trivial fiber is the union of two $\mathbb{P}^2$-bundles of the form $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$, which meet along a quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

In the Appendix we recall very briefly some results concerning the projectivization of a vector bundle.

CHAPTER 1

Background material

In this chapter we collect the standard material that will be used throughout the thesis; the main references for the topics will be given at the beginning of each section.
We will always work over $\mathbb{C}$, the field of the complex numbers.

1.1 Intersection numbers

The results in this section as well as in the next one are taken from the first chapter of the book [Deb01].
Let X be a proper scheme of dimension n and let $D_1, \ldots, D_n$ be Cartier divisors on X; we want to define the intersection number

$$D_1 \cdots D_n.$$

Theorem 1.1.1. *Let $D_1, \ldots, D_r$ be Cartier divisors on a proper scheme X of dimension n and let $\mathcal{F}$ be a coherent sheaf on X. The function*

$$(m_1, \ldots, m_r) \mapsto \chi(X, \mathcal{F}(m_1 D_1 + \cdots + m_r D_r))$$

takes the same values on $\mathbb{Z}^r$ as a polynomial with rational coefficients of degree at most the dimension of the support of $\mathcal{F}$.

Taking $\mathcal{F} = \mathcal{O}_X$, we get the following definition.

Definition 1.1.2. Let $D_1, \ldots, D_r$ be Cartier divisors on a proper scheme X with $\dim X \leq r$. The intersection number

$$D_1 \cdots D_r$$

is the coefficient of $m_1 \cdots m_r$ in the polynomial

$$\chi(X, m_1 D_1 + \cdots + m_r D_r).$$

If Y is a closed subscheme of X of dimension at most s, we set

$$D_1 \cdots D_s \cdot Y = D_1|_Y \cdots D_s|_Y.$$

Moreover if $\mathcal{F}$ is any coherent sheaf on X whose support has dimension at most r, we can define the intersection number

$$D_1 \cdots D_r \cdot \mathcal{F}$$

as the coefficient of $m_1 \cdots m_r$ in the polynomial

$$\chi(X, \mathcal{F}(m_1 D_1 + \cdots + m_r D_r)).$$

Remark 1.1.3. If $r > \dim X$, then the intersection number $D_1 \cdots D_r = 0$.

Here are the basic general properties of these intersection numbers.

Proposition 1.1.4. *Let $D_1, \ldots, D_n$ be Cartier divisors on a proper scheme X of dimension n.*

1. *The map*
$$(D_1, \ldots, D_n) \mapsto D_1 \cdots D_n$$
is multilinear, symmetric and takes integral values.

2. *If D_n is effective with associated subscheme Y,*
$$D_1 \cdots D_n = D_1 \cdots D_{n-1} \cdot Y.$$

Proposition 1.1.5 (Projection formula). *Let $\pi : X \to Y$ be a surjective morphism between proper varieties and let $D_1, \ldots D_r$ be Cartier divisors on Y, with $\dim X \leq r$. Then*

$$\pi^* D_1 \cdots \pi^* D_r = \deg(\pi)(D_1 \cdots D_r).$$

With these intersection numbers we can generalize the Riemann-Roch theorem to higher dimensional schemes.

Theorem 1.1.6 (Asymptotic Riemann-Roch). *Let D be a Cartier divisor and $\mathcal{F}$ a coherent sheaf on a proper scheme X of dimension n. Then*

$$\chi(X, \mathcal{F}(mD)) = \frac{D^n \cdot \mathcal{F}}{n!} m^n + \textit{terms of lower degree},$$

where D^n is the intersection number

$$D^n = \underbrace{D \cdots D}_{n \ \textit{times}}.$$

1.2 The cone of curves

Let X be a proper scheme of dimension n, D a Cartier divisor on X and let C be a complete curve on X, i.e. an integral and proper subscheme of X of dimension 1. Then we can consider the intersection number

$$D \cdot C,$$

which is, by Definition 1.1.2, the coefficient of the leading term of the polynomial $\chi(C, mD|_C)$.
The Riemann-Roch theorem for curves (see 1.1.6 or also [Har77]) gives

$$\chi(C, mD|_C) = m \deg(\mathcal{O}_C(D)) + \chi(C, \mathcal{O}_C),$$

and so

$$D \cdot C = \deg(\mathcal{O}_C(D)).$$

Using Proposition 1.1.5 we can compute this intersection number in another way, which will be useful in the following chapter.

Proposition 1.2.1. *Let X be a proper scheme of dimension n, D a Cartier divisor on X and let C be a complete curve on X and $\nu : \widetilde{C} \to C$ its normalization. Then*

$$D \cdot C = \deg_{\widetilde{C}} \nu^* D.$$

A $1-$cycle is a formal linear combination of curves $\Gamma = \sum n_i C_i$ with integral coefficients and it is called *effective* if all the coefficients are nonnegative.
We denote with $Z_1(X)$ the free abelian group generated by 1-cycles and with $ZE_1(X)$ the semigroup of effective 1-cycles.
By linearity we can extend the definition of the intersection number $D \cdot \Gamma$ for Cartier divisors and $1-$cycles.

Definition 1.2.2. Let X be a proper scheme.

1. Two Cartier divisors D and D' on X are *numerically equivalent* if

   $$D \cdot C = D' \cdot C$$

   for every curve C in X. In this case we will write $D \equiv D'$.
   The quotient of the group of the Cartier divisors by this equivalence relation is denoted by $N^1(X)_{\mathbb{Z}}$ and we can consider the $\mathbb{R}$-vector space $N^1(X) = N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$.

2. Two $1-$cycles $C, C' \in Z_1(X)$ are *numerically equivalent* if
$$D \cdot C = D \cdot C'$$
for every Cartier divisor D on X. In this case we will write $C \equiv C'$. The quotient group $Z_1(X)/\equiv$ is denoted by $N_1(X)_{\mathbb{Z}}$ and we can consider the $\mathbb{R}$-vector space $N_1(X) = N_1(X)_{\mathbb{Z}} \otimes \mathbb{R}$.

The intersection numbers induce a nondegenerate paring
$$N^1(X) \times N_1(X) \to \mathbb{R}$$
which makes these vector spaces canonically dual.
Moreover by the Neron-Severi theorem we have that these vector spaces are finite dimensional.
The number $\rho(X) = \dim N_1(X) = \dim N^1(X)$ is called the *Picard number of* X.
In $N_1(X)$ we consider

$\overline{NE}(X)$:= the closed convex cone generated by effective $1-$cycles of X (*Mori cone*).

$\overline{NE}_D(X) := \{C \in \overline{NE}(X) \mid D \cdot C \geq 0\}$, with $D \in N^1(X)$.

Definition 1.2.3. Let X be a proper scheme and let D be a Cartier divisor. D is *ample* if some multiple of it is *very ample*, i.e. gives an embedding of X in a projective space. X is *projective* if there exists on it a ample divisor.

We have the following numerical characterization of ampleness for projective variety.

Theorem 1.2.4 (Kleiman's criterion). *Let X be a (projective) variety. A Cartier divisor D on X is ample if and only if*
$$D \cdot \Gamma > 0 \text{ for all } \ \Gamma \in \overline{NE}(X) \setminus \{0\}.$$

The Figure 1.1 gives an intuitive picture of this theorem.
This criterion naturally leads to the following definition.

Definition 1.2.5. Let X be a proper scheme and let D be a Cartier divisor on X. D is *numerically effective (nef)* if and only if
$$D \cdot \Gamma \geq 0 \text{ for all } \ \Gamma \in \overline{NE}(X) \setminus \{0\},$$
or equivalently
$$D \cdot C \geq 0 \ \text{ for all } C \text{ curve in } X.$$

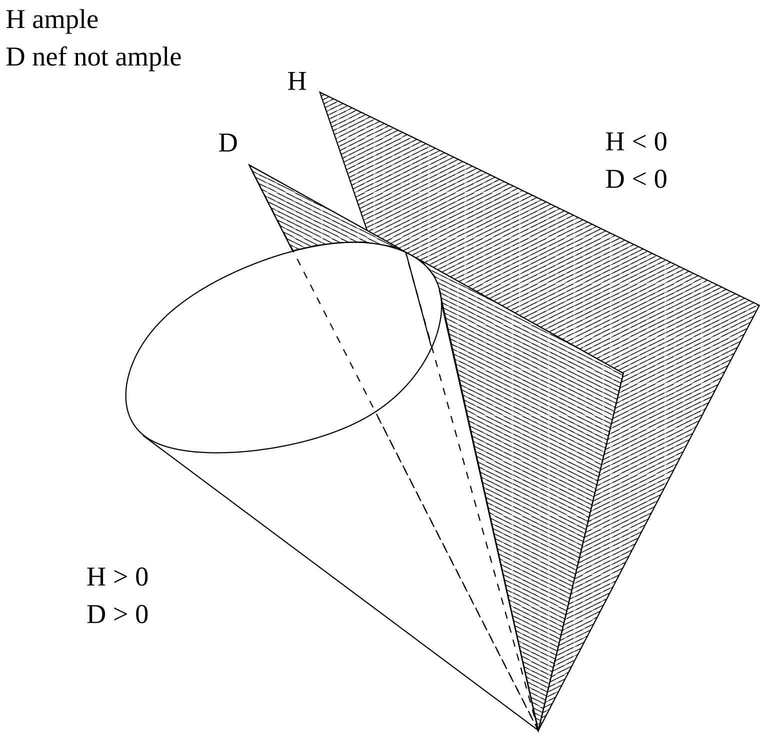


Figure 1.1: The Mori cone with a nef and an ample divisors

Definition 1.2.6. Let X be a proper scheme of dimension n and let D be a nef divisor on X. Then D is *big* if and only if $D^n > 0$.

The following is a technical result due to Kawamata and Viehweg (for the definitions of $\mathbb{Q}$-divisor, round up and fractional part of a $\mathbb{Q}$-divisor and simple normal crossing see [KMM87]).

Theorem 1.2.7 (Kawamata-Viehveg vanishing theorem). *Let X be a smooth complex projective variety and let D be a nef and big $\mathbb{Q}$-divisor on X whose* fractional part *has* simple normal crossing. *Then*

$$H^i(X, K_X + \lceil D \rceil) = 0, \quad \forall\ i > 0,$$

where $\lceil D \rceil$ is the round-up *of the $\mathbb{Q}$-divisor D.*

We conclude this section with a definition concerning 1-cycles on a proper scheme X which will be useful in the next chapters.

Definition 1.2.8. Two 1-cycles $D_1, D_2 \in Z_1(X)$ are *algebraically equivalent* if there exists an effective 1-cycle $E \in ZE_1(X)$ such that $D_1 + E$ and $D_2 + E$

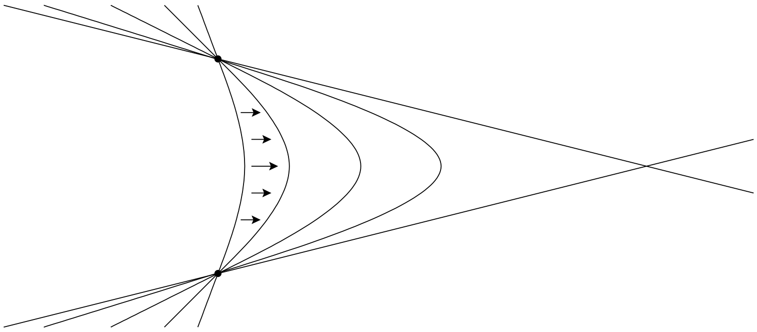


Figure 1.2: Bend and break

are effective and belong to a *flat family* of effective 1-cycles.
We will write $B_1(X)$ for the group of 1-cycles modulo algebraic equivalence.

Definition 1.2.9. Two 1-cycles $D_1, D_2 \in Z_1(X)$ are *rational equivalent* if there exists an effective 1-cycle $E \in ZE_1(X)$ such that $D_1 + E$ and $D_2 + E$ are effective and belong to a *rational flat family* (i.e. a flat family over a rational curve) of effective 1-cycles.
We will write $A_1(X)$ for the group of 1-cycles modulo rational equivalence.

Clearly two 1-cycles on X which are rationally equivalent are also algebraically equivalent and two 1-cycles on X which are algebraically equivalent are also numerically equivalent (in fact $N_1(X)_{\mathbb{Z}}$ is a quotient of $A_1(X)$ and $B_1(X)$).

1.3 Mori theory for smooth varieties

For the results in this section we refer again to the book [Deb01, Chapter 6].

From now on X will be always a smooth projective variety.

Definition 1.3.1. A curve $C \subset X$ is *rational* if and only if it has $\mathbb{P}^1$ as normalization.

Using the existence and the deformations of rational curves on a smooth variety with K_X not nef (see Chapter 2 for details) Mori proved the following facts.

The starting point is the "bend and break" technique (see Figure 1.2); essentially it is divided into two steps.

1. A curve deforming nontrivially in X, while keeping a point fixed, must break into an effective 1-cycle with a rational component which passes through the fixed point.

2. If a rational curve deforms nontrivially while keeping two points fixed, it must break up into an effective 1-cycle with rational components.

These facts leads to the following theorem.

Theorem 1.3.2. *Let X be a smooth projective variety, let H an ample divisor on X and let C be a smooth curve such that $K_X \cdot C < 0$. For every point $x \in C$ there exists a rational curve Γ in X passing through x with*

$$0 < -K_X \cdot \Gamma \leq n+1, \qquad H \cdot \Gamma \leq 2\dim(X)\frac{H \cdot C}{-K_X \cdot C}.$$

As a consequence of this theorem, we have the following

Theorem 1.3.3 (Cone Theorem). *Let X be a smooth projective variety. There exists a countable family $\{\Gamma_i\}_{i\in I}$ of rational curves on X such that*

$$0 < -K_X \cdot \Gamma_i \leq \dim(X) + 1$$

and

$$\overline{NE}(X) = \overline{NE}_{K_X}(X) + \sum_{i\in I} \mathbb{R}^+[\Gamma_i].$$

Moreover the $\mathbb{R}^+[\Gamma_i]$ are locally discrete in the half-space of $N_1(X)$ given by $\{Z \in N_1(X) \mid K_X \cdot Z < 0\}$.

This theorem says that the negative part of the Mori cone $\overline{NE}(X)$ is polyhedral (see Figure 1.3).

Definition 1.3.4. A face in the negative part of the Mori cone is an *extremal face* of X. The $R_i = \mathbb{R}^+[\Gamma_i]$ are the one dimensional extremal faces of $\overline{NE}(X)$ and they are called *extremal rays* of X. Note that, by the Cone Theorem extremal rays are generated by rational curves.

Corollary 1.3.5. *Let X be a smooth projective variety and let σ be an extremal face of X. There exists a nef divisor H on X such that*

1. *$\sigma = \{Z \in \overline{NE}(X) \mid H \cdot Z = 0\}$;*

2. *the divisor $mH - K_X$ is ample for all integers $m \gg 0$.*

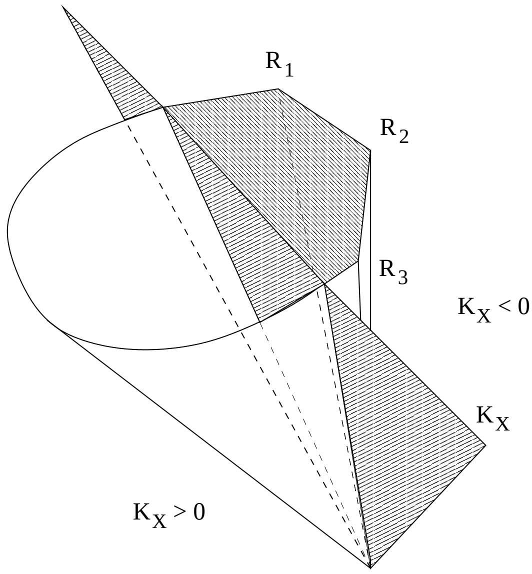


Figure 1.3: The Cone Theorem

Definition 1.3.6. The divisor H is called *supporting divisor* for the extremal face σ.
Notice that, for *2* of Corollary 1.3.5, (a multiple of) the supporting divisor can be written as $mH = K_X + \tau L$ where L is an ample divisor on X and τ is an integer.

The following result is due to Kawamata and Shokurov.

Theorem 1.3.7 (Base-point-free theorem). *Let X be a smooth variety and let H be a nef divisor on X such that $aH - K_X$ is nef and big for some positive number a. Then the linear system $|mH|$ is base-point-free for all integers $m \gg 0$.*

Thus the linear system $|mH|$ of Corollary 1.3.5 for $m \gg 0$ gives a morphism

$$\varphi_{|mH|} : X \to \mathbb{P}(H^0(X, mH)).$$

Consider the *Stein factorization* of this morphism:

$$\begin{tikzcd} X \arrow[rr, "\varphi_{|mH|}"] \arrow[dr, "\varphi"'] & & \mathbb{P}(H^0(X, mH)) \\ & X' \arrow[ur, "p"'] & \end{tikzcd}$$

where φ has connected fibers, X' is a normal variety and p is a finite morphism; it turns out that the connected parts of the morphisms $\varphi_{|mH|}$ and $\varphi_{|(m+1)H|}$ are the same.
As a consequence of this fact, of Corollary 1.3.5 and of Theorem 1.3.7 we have the following theorem, which is a starting point of the Minimal Model Program.

Theorem 1.3.8 (Contraction Theorem). *Let X be a smooth variety and let H be a nef divisor such that $\sigma := H^\perp \cap \overline{NE}(X)$ is entirely containd in $\{Z \in N_1(X) \mid K_X \cdot Z < 0\}$ (i.e. H is a supporting divisor of the extremal face σ of X). Then there exists a projective morphism*

$$\varphi : X \longrightarrow Y,$$

onto a normal projective variety Y, which is characterized by the following properties:

1. *For any curve $C \subset X$, $\varphi(C)$ is a point if and only if $H \cdot C = 0$;*
2. *φ has connected fibers;*
3. *$H = \varphi^* A$ for some ample Cartier divisor A on Y.*

The morphism φ is called the contraction *of the face σ and the divisor H is called also the* supporting divisor *of the contraction φ.*

Remark 1.3.9. A *contraction* (i.e. a proper morphism with connected fibers between normal varieties) always corresponds to a face of the Mori cone (not necessarily in the negative part of it) (see [Deb01][Section 1.3]); this fact is sometimes called *fundamental triviality of the Mori's program.*
The *contraction theorem* gives a class of the faces of the Mori cone of a smooth variety for which the converse holds.

Definition 1.3.10. A *Fano-Mori (F-M) contraction* of a smooth variety X is a contraction such that the anticanonical divisor $-K_X$ is φ-ample.
By the previous remark we have that a F-M contraction of X is a contraction of an extremal face of X.

We conclude this section with a proposition which shows an important property of the contractions.

Proposition 1.3.11. *(See [Deb01][Proposition 1.14]) Let* X, Y *and* Y' *be normal projective varieties and let*

$$\pi : X \longrightarrow Y \qquad \pi' : X \longrightarrow Y'$$

be two contractions, associated to the faces σ_π *and* $\sigma_{\pi'}$ *of the Mori cone of* X.
If $\sigma_\pi \subset \sigma_{\pi'}$*, then there exist a* unique *morphism* $f : Y \to Y'$ *such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "\pi'"] \arrow[dr, "\pi"'] & & Y' \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

1.4 Singularities

The study of the F-M contractions naturally leads to consider some special singularities.

Proposition 1.4.1. *[Har77, Cor. III, 7.7] Let* X *be a projective Cohen-Macaulay scheme of equidimension* n *(e.g.* X *is a local complete intersection). Then there exists a unique coherent sheaf on* X ω_X^0 *which is called* dualizing sheaf *of* X*, such that for each locally free sheaf* $\mathcal{F}$ *on* X *there exist a natural isomorphisms*

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^0)^\vee.$$

Definition 1.4.2. X is called $\mathbb{Q}$-*Gorenstein* if there exists an integer m such that the sheaf $(\omega_X^0)^{\otimes m}$ is a Cartier divisor; X is called *Gorenstein* if ω_X^0 is Cartier and X is Cohen-Macaulay.

If moreover the projective scheme X is normal, we can associate a Weil divisor class K_X to the dualizing sheaf: this class coincides with the class of the closure of the canonical divisor on the smooth locus of X. We will call the divisor K_X the *canonical divisor* of X and the previous definition becomes:

Definition 1.4.3. Let X be a normal projective scheme; X is called $\mathbb{Q}$-*Gorenstein* if there exists an integer m such that the Weil divisor mK_X is also a Cartier divisor; X is called *Gorenstein* if K_X is Cartier and X is Cohen-Macaulay.

In the Minimal Model Program the following classes of singularities are very important.

Definition 1.4.4. A normal projective scheme X is said to have *terminal* (respectively *canonical*) *singularities* if the following conditions are satisfied:

1. X is $\mathbb{Q}$-Gorenstein;
2. There exists a resolution of singularities $f : Y \to X$ such that $K_Y = f^*K_X + \sum a_i E_i$, with $a_i \in \mathbb{Q}$ and $a_i > 0$ (respectively $a_i \geq 0$), where E_i are the exceptional divisors for f.

When X is normal and projective we can give also the following class of singularities:

Definition 1.4.5. X is called $\mathbb{Q}$-*factorial* if for each Weil divisor D on X there exists an integer m such that mD is a Cartier divisor; X is called *factorial* if each Weil divisor is also Cartier.

We conclude this section with a property of a birational morphism whose target is $\mathbb{Q}$-factorial.

Proposition 1.4.6. *[Deb01, Sec. 1.10] Let $\pi : X \to Y$ be a birational morphism and let E be its exceptional locus, i.e.*

$$E = \{x \in X \mid \pi \textit{ is not an isomorphism at } x\}.$$

If Y is normal and $\mathbb{Q}$-factorial then every irreducible component of E has codimension 1 in X and its image has codimension at least 2 in Y.

1.5 Fano varieties

In this section we recall some facts about a very important class of varieties which are called *Fano varieties.*

Definition 1.5.1. A normal projective variety X is called *Fano variety* if

1. X is Gorenstein;
2. the anticanonical divisor $-K_X$ is an ample Cartier divisor.

Let X be a Fano variety, then there is an important invariant of X, which in some cases allows us to classify these varieties.

Definition 1.5.2. The *index of a Fano variety* X is defined as

$$i(X) := \max\{t \in \mathbb{N} \mid -K_X \equiv tH, \text{ for some ample Cartier divisor } H\}$$

We have the following theorem, due to Kobayashi and Ochiai in the smooth case

Theorem 1.5.3. *Let X be a Fano variety of dimension n with rational singularities. Let $i(X)$ be the index and let L be an ample Cartier divisor on X such that $-K_X \equiv i(X)L$. Then*

1. $i(X) \leq n+1$;
2. $i(X) = n+1$ *if and only if* $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$;
3. $i(X) = n$ *if and only if* $(X, L) = (\mathbb{Q}^n, \mathcal{O}(1))$.

Then the n-dimensional Fano varieties with index $i(X) \geq n$ are complitely classified. For the next steps we can give the following definitions

Definition 1.5.4. Let X be a projective variety of dimension n and let L be an ample line bundle on it. the pair (X, L) is a *del Pezzo variety* if

1. X has only Gorenstein singularities;
2. $-K_X = (n-1)L$;
3. $H^q(X, tL) = 0$ for any integers q, t with $0 < q < n$.

Remark 1.5.5. If X is smooth condition 2 of Definition 1.5.4 implies the other two conditions. Moreover this Definition is consistent also when X is non normal (see [Fuj90]).

Definition 1.5.6. Let X be a projective variety of dimension n and let L be an ample line bundle on it. the pair (X, L) is a *Mukai variety* if $i(X) = n-2$.

Remark 1.5.7. If X is Fano with only canonical singularities then $\operatorname{Pic} X$ is torsion free and so $-K_X = i(X)L$.

1.6 Fujita Δ-genus and Hilbert polynomial

The results in this section are contained in [Fuj90].

Definition 1.6.1. Let X be a projective variety of dimension n and let L be an ample line bundle on it. The Δ-*genus* of the pair (X, L) is

$$\Delta(X, L) := n + L^n - h^0(X, L).$$

The number $d = L^n$ is called the *degree* of the pair (X, L).

The Δ-genus of a pair (X, L) is always nonnegative and moreover the following facts, which explain the importance of this invariant, hold.

Theorem 1.6.2. *Let X be a projective variety of dimension n and let L be an ample line bundle on it.*
If $\Delta(X, L) = 0$, the pair (X, L) is one of the following:

1. $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ *if* $d = 1$;
2. $(X, L) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ *if* $d = 2$;
3. (X, L) *is a* $\mathbb{P}^{n-1}$*-bundle on* $\mathbb{P}^1$, $X = \mathbb{P}(\mathcal{E})$, *with* $\mathcal{E}$ *a vector bundle on* $\mathbb{P}^1$ *which is the direct sum of line bundles of positive degree;*
4. $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
5. (X, L) *is a generalized cone on a smooth subvariety* $V \subset X$ *with* $\Delta(V, L_V) = 0$ *(for a definition of generalized cone see [BS95]).*

Proposition 1.6.3. *Let (X, L) be a del Pezzo variety, then $\Delta(X, L) = 1$.*

Proposition 1.6.4. *(see [Fuj90, Sec. I.6]) Let X be a projective variety of dimension n and let L be an ample line bundle on it. Suppose that the following conditions are satisfied:*

1. $\Delta(X, L) = 1$;
2. *X is Gorenstein and Cohen-Macaulay (e.g. X is locally complete intersection (see [Har77, Sec. II.8]));*
3. $-K_X = (n - 1)L$.

Then the pair (X, L) is a del Pezzo variety.

To compute the Δ-genus of a pair (X, L) we will need the following tool, which comes from the intersection numbers defined in Section 1.1.

Definition 1.6.5. Let D be an ample divisor on a projective scheme X of dimension n and let $\mathcal{F}$ be a coherent sheaf on X. The *Hilbert polynomial* of $(X, \mathcal{F})$ is the numerical polynomial

$$P(t) = \chi(X, \mathcal{F}(tD)).$$

If Y is a closed subscheme of X, then the *Hilbert polynomial* of $(Y, \mathcal{O}_Y(D))$ is the numerical polynomial

$$P(t) = \chi(Y, \mathcal{O}_Y(tD)).$$

Remark 1.6.6. If $\dim Y = m$, the coefficient of the leading term of $P(t) = \chi(Y, \mathcal{O}_Y(tD))$ is

$$\frac{(D|_Y)^m}{m!}$$

where $(D|_Y)^m$ is the degree of the pair $(Y, D|_Y)$ (see Theorem 1.1.6).
Moreover if Y is equidimensional the degree of $(Y, D|_Y)$ is the sum of the degrees of the irreducible components: if $Y_1, \ldots, Y_r$ are the irreducible components of Y and $D_i = D|_{Y_i}$ for $i = 1 \ldots r$, we have

$$(D|_Y)^m = \sum D_i^m.$$

CHAPTER 2

Rational curves on projective varieties

In this chapter we collect the results concerning rational curves and their deformations that will be used throughout the study of the Fano-Mori contractions. The main reference for this chapter will be the book [Kol96].

2.1 Parametrizing schemes

Let Y be projective variety and let X be a smooth quasi-projective variety. Then there is a scheme which parametrizes all morphisms $f : Y \to X$; we denote this scheme with $\mathrm{Hom}(Y, X)$ and we write $[f]$ for the point in $\mathrm{Hom}(Y, X)$ corresponding to the morphism f. The scheme $\mathrm{Hom}(Y, X)$ and the evaluation morphism:

$$\mathrm{ev} : Y \times \mathrm{Hom}(Y, X) \longrightarrow X \qquad \text{with} \qquad (y, [f]) \mapsto f(y)$$

have the universal property that for each scheme D and for each morphism $F : Y \times D \to X$ there exists a unique morphism $F' : D \to \mathrm{Hom}(Y, X)$ such that the following diagram commutes:

$$\begin{tikzcd}
Y \times \mathrm{Hom}(Y,X) \arrow[rr, "\mathrm{ev}"] & & X \\
Y \times D \arrow[u, "\mathbb{1}\times F'"] \arrow[urr, "F"'] & &
\end{tikzcd}$$

In general $\mathrm{Hom}(Y, X)$ is not a variety, since usually it has countably many

irreducible components; however each irreducible component of $\mathrm{Hom}(Y, X)$ is a (quasi-projective) variety.
The following theorem gives us very important information about it.

Theorem 2.1.1. *[Kol96, Sec.I.2] Let Y be a projective variety (or a projective scheme without embedded points), X a smooth quasi-projective variety and $f_0 : Y \to X$ a morphism.*

1. *The tangent space of* $\mathrm{Hom}(Y, X)$ *at* $[f_0]$ *is*
$$T_{[f_0]} \mathrm{Hom}(Y, X) \cong H^0(Y, f_0^* T_X),$$
where T_X is the tangent bundle of X;

2. *If* $H^1(Y, f_0^* T_X) = 0$ *then* $\mathrm{Hom}(Y, X)$ *is smooth at the point* $[f_0]$ *and has dimension* $h^0(Y, f_0^* T_X)$*;*

3. *In general* $\dim_{[f_0]} \mathrm{Hom}(Y, X) \geq h^0(Y, f_0^* T_X) - h^1(Y, f_0^* T_X)$.

We can also consider morphisms from Y to X which fix a closed subscheme B of Y. Let $g : B \to X$ a given morphism, then there exists a scheme $\mathrm{Hom}(Y, X, g)$ which parametrizes all the morphisms $f : Y \to X$ such that $f|_B = g$. Clearly $\mathrm{Hom}(Y, X, g)$ is a subscheme of $\mathrm{Hom}(Y, X)$ and has the same properties.
Moreover for the scheme $\mathrm{Hom}(Y, X, g)$ holds an analogous of the previous theorem:

Theorem 2.1.2. *[Mor79, Prop. 3] Let Y be a projective variety (or a projective scheme without embedded points), B a closed subscheme of Y and X a smooth quasi-projective variety. Let moreover $g : B \to X$ a given morphism and let $f_0 : Y \to X$ a morphism such that $f_0|_B = g$.*

1. *The tangent space of* $\mathrm{Hom}(Y, X, g)$ *at* $[f_0]$ *is*
$$T_{[f_0]} \mathrm{Hom}(Y, X, g) \cong H^0(Y, f_0^* T_X \otimes \mathcal{I}_B),$$
where $\mathcal{I}_B$ is the sheaf of the defining ideal of B in Y;

2. *If* $H^1(Y, f_0^* T_X \otimes \mathcal{I}_B) = 0$ *then* $\mathrm{Hom}(Y, X, g)$ *is smooth at the point* $[f_0]$ *and has dimension* $h^0(Y, f_0^* T_X \otimes \mathcal{I}_B)$*;*

3. *In general* $\dim_{[f_0]} \mathrm{Hom}(Y, X, g) \geq h^0(Y, f_0^* T_X \otimes \mathcal{I}_B) - h^1(Y, f_0^* T_X \otimes \mathcal{I}_B)$.

2.1.1 The case of curves

We want to study the scheme $\mathrm{Hom}(C, X)$, where C is a proper curve without embedded points; in this case the previous theorems are simpler:

Theorem 2.1.3. *[Kol96, Theorem II.1.2] Let X be a smooth quasi-projective variety, C a proper curve without embedded points and of genus $g(C)$ and let $f : C \to X$ a morphism.*

1. $T_{[f]} \mathrm{Hom}(C, X) \cong H^0(C, f^*T_X)$;
2. $\dim_{[f]} \mathrm{Hom}(C, X) \geq -f_*[C] \cdot K_X + \dim X(1 - g(C))$.

Proof. Since C is a curve $h^0(C, f^*T_X) - h^1(C, f^*T_X) = \chi(C, f^*T_X)$. Then from Riemann-Roch

$$\chi(C, f^*T_X) = \deg(f^*T_X) + \dim X \cdot \chi(C, \mathcal{O}_C);$$

from the projection formula and linear algebra we get

$$\deg(f^*T_X) + \dim X \cdot \chi(C, \mathcal{O}_C) = -f_*[C] \cdot K_X + \dim X(1 - g(C)).$$

□

The analogous of this theorem concerning the scheme $\mathrm{Hom}(C, X, g)$ which parametrizes morphisms $f : C \to X$ with fixed points is the following

Theorem 2.1.4. *[Kol96, Theorem II.1.7] Let X be a smooth quasi-projective variety, C a proper curve without embedded points and of genus $g(C)$ and let $f : C \to X$ a morphism. Moreover let B be a closed subscheme of C of finite length $l(B)$ and let $g : B \to X$ be a morphism.*

1. $T_{[f]} \mathrm{Hom}(C, X, g) \cong H^0(C, f^*T_X \otimes \mathcal{I}_B)$;
2. $\dim_{[f]} \mathrm{Hom}(C, X, g) \geq -f_*[C] \cdot K_X + \dim X(1 - g(C) - l(B))$.

Remark 2.1.5. [Kol96, Theorem II.1.3] These two theorems holds also when X is a quasi-projective variety with *local complete intersection singularities*, if we assume that the image through f of every irreducible component of the curve C intersects the smooth locus of X.

2.2 Deformations of rational curves

2.2.1 The scheme $\mathrm{Hom}(\mathbb{P}^1, X)$

The aim of this section is to study in more detail the scheme $\mathrm{Hom}(\mathbb{P}^1, X)$.

Let $\mathrm{Hom}_{bir}(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be the open subscheme corresponding to those morphisms which are birational onto their image (indeed this is an open condition in $\mathrm{Hom}(\mathbb{P}^1, X)$). At least set theoretically $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ contains all information about $\mathrm{Hom}(\mathbb{P}^1, X)$ and so we can restrict ourselves to this subscheme.
In fact the scheme $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ is still too large: if $f : \mathbb{P}^1 \to X$ is any morphism and $h \in \mathrm{Aut}(\mathbb{P}^1)$, then $f \circ h$ is counted as a different morphism.
Let $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ be the normalization of $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$: the group $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ and the quotient exists.

Definition 2.2.1. We define the space $\mathrm{RatCurves}^n(X)$ as the quotient of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ by $\mathrm{Aut}(\mathbb{P}^1)$.
We define the space $\mathrm{Univ}(X)$ as the quotient of the product action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1$.

Moreover there is the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \arrow[r, "U"] \arrow[d] \arrow[rr, bend left, "\mathrm{ev}"] & \mathrm{Univ}(X) \arrow[r, "\iota"'] \arrow[d, "\pi"'] & X \\
\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \arrow[r, "u"] & \mathrm{RatCurves}^n(X) &
\end{tikzcd} \tag{2.2.2}$$

where u and U are principal $\mathrm{Aut}(\mathbb{P}^1)$-bundles and π is a universal $\mathbb{P}^1$-bundle.

There exists a "pointed" version of this constructions. Explicitly, let $x \in X$ be a point and let $\mathrm{Hom}_{bir}(\mathbb{P}^1, X, 0 \to x)$ the scheme that parametrizes the morphisms $f : \mathbb{P}^1 \to X$ which sends the point $0 \in \mathbb{P}^1$ to $x \in X$. Let $\mathrm{Aut}(\mathbb{P}^1, 0)$ be the group of the automorphisms of $\mathbb{P}^1$ which fix a point $0 \in \mathbb{P}^1$.
Let $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, 0 \to x)$ be the normalization of $\mathrm{Hom}_{bir}(\mathbb{P}^1, X, 0 \to x)$: the group $\mathrm{Aut}(\mathbb{P}^1, 0)$ acts on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, 0 \to x)$ and the quotient exists.

Definition 2.2.3. We define the space $\mathrm{RatCurves}^n(x, X)$ as the quotient of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, 0 \to x)$ by $\mathrm{Aut}(\mathbb{P}^1, 0)$.
We define the space $\mathrm{Univ}(x, X)$ as the quotient of the product action of $\mathrm{Aut}(\mathbb{P}^1, 0)$ on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, 0 \to x) \times \mathbb{P}^1$.

Moreover the previous diagram becomes

$$\begin{tikzcd}
\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, 0 \to x) \times \mathbb{P}^1 \arrow[r, "U_x"] \arrow[d] \arrow[rr, bend left, "\mathrm{ev}_x"] & \mathrm{Univ}(x, X) \arrow[r, "\iota_x"'] \arrow[d, "\pi_x"'] & X \\
\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, 0 \to x) \arrow[r, "u_x"] & \mathrm{RatCurves}^n(x, X) &
\end{tikzcd} \tag{2.2.4}$$

where u_x and U_x are principal $\mathrm{Aut}(\mathbb{P}^1, 0)$-bundles and π_x is again a universal $\mathbb{P}^1$-bundle.

2.2.2 Families of rational curves

In view of these facts we can give the following definition (which is different from Kollár's one).

Definition 2.2.5. Let X be a quasi-projective variety and let $x \in X$ be a point.

1. A *family of rational curves* in X is an irreducible component V of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ which is closed under the action of $\mathrm{Aut}(\mathbb{P}^1)$.

2. Given a family of rational curves V, we can consider the subfamily of curves of V which pass through a fixed point $x \in X$: $V_x := V \cap \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X, 0 \to x)$.

3. If $C \subset X$ is a rational curve, $f : \mathbb{P}^1 \to C \subset X$ is the normalization and $[f] \in V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$, then we say that V is a *family of the deformations* of the rational curve C.

Given a family of rational curves $V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ and a point $x \in X$, we can consider the restriction of the evaluation map

$$\mathrm{ev} : \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to X$$

to the families V and V_x:

$$\mathrm{ev}_V : V \times \mathbb{P}^1 \to X, \qquad \mathrm{ev}_{V_x} : V_x \times \mathbb{P}^1 \to X.$$

Moreover we can consider the restriction of the diagrams 2.2.2 and 2.2.4 to the families V and V_x (see Figure 2.1).

Definition 2.2.6. With this notation:

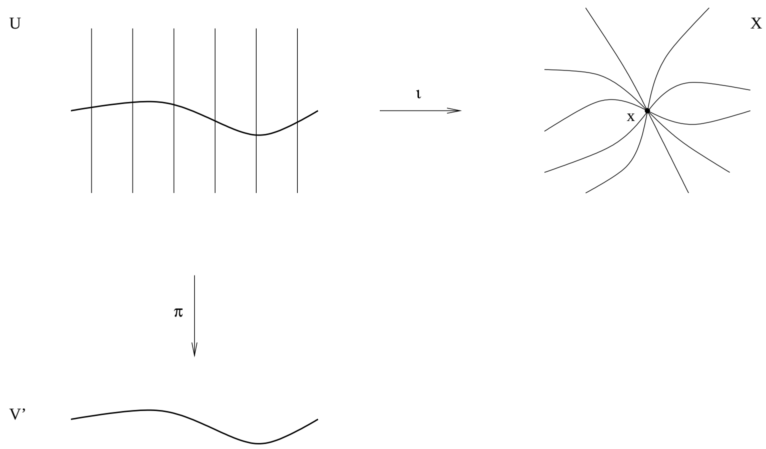


Figure 2.1: The universal family of rational curves

1. $\mathrm{Locus}(V) := \overline{\mathrm{ev}_V(V \times \mathbb{P}^1)} \subset X$ is the *locus of* V ;
2. $\mathrm{Locus}(V_x) := \overline{\mathrm{ev}_{V_x}(V_x \times \mathbb{P}^1)} \subset X$ is the *locus of* V_x.

Definition 2.2.7. Let V be a family of rational curves on X and let $E \subseteq X$ be a closed subset (it can be $E = X$). We say that V *dominates* E or V is a *dominating family* for E if $\mathrm{Locus}(V) = E$.

Let $C \subset X$ be a rational curve and let $f : \mathbb{P}^1 \to C \subset X$ be the normalization. Since each vector bundle on $\mathbb{P}^1$ is decomposable there exist integers $a_1 \geq \cdots \geq a_n$ such that

$$f^*T_X = \mathcal{O}(a_1) + \cdots + \mathcal{O}(a_n);$$

since f is nonconstant, we have $a_1 \geq 2$.
Moreover we can consider the *anticanonical degree* of f:

$$\deg_{-K}(f) := -\deg f^*K_X = \deg f^*T_X = \sum a_i.$$

Definition 2.2.8. We say that $f : \mathbb{P}^1 \to C \subset X$ is *free* if all $a_i \geq 0$, i.e. f^*T_X is nef.

Let $V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ be the family of the deformations of the rational curve C; then the numbers a_i are common to the general curves in the family. Then we can give the following definition.

Definition 2.2.9. With the previous notations, the *splitting type* of the tangent bundle of X on the family V is the n-tuple $(a_1, \ldots, a_n)$ defined

before for the general morphism f.
The *anticanonical degree* of the family V is

$$\deg_{-K}(V) := \deg_{-K}(f) = \sum a_i.$$

Remark 2.2.10. Given an ample line bundle H on X, it is well defined the degree of the family V with respect to H:

$$\deg_H(V) := \deg_H(f) := \deg f^*H.$$

In some cases the splitting type of the family V gives some information about the locus of the family V.

Proposition 2.2.11. *[Kol96, Prop. IV.2.7, Cor. IV.2.9] Let V be a family of rational curves on a variety X with splitting type $(a_1, \dots, a_n)$ and let x be a general point in* $\operatorname{Locus}(V)$*:*

1. *if $a_i \geq -1\ \forall\ i$, then* $\dim \operatorname{Locus}(V) = \#\{i \mid a_i \geq 0\}$*;*
2. *if $a_i \geq 0\ \forall\ i$ (i.e. f is free for general $[f] \in V$), then*

$$\dim \operatorname{Locus}(V_x) = \#\{i \mid a_i \geq 1\}.$$

2.2.3 Minimizing families of rational curves

Definition 2.2.12. Let X be a quasi-projective variety and consider a family of rational curves $V \subset \operatorname{Hom}^n_{bir}(\mathbb{P}^1, X)$.

1. V is *unsplit* if $V/\operatorname{Aut}(\mathbb{P}^1)$ is proper.
2. Let
$$\Pi : V \longrightarrow X \times X, \qquad \text{with} \qquad [f] \mapsto (f(0), f(\infty));$$
we say that V is *generically unsplit* if the general non trivial fiber of Π (i.e. the fiber over the generic point of $\operatorname{Im}\Pi$) has dimension at most zero.

2.2.13 (Explanation). Let us explain these definitions.

1. $V/\operatorname{Aut}(\mathbb{P}^1)$ is an irreducible component of $\operatorname{RatCurves}^n(X)$ and this has a natural inclusion in $\operatorname{Chow}(X)$, the scheme that parametrizes the effective cycles of X.
Let W be the image of $V/\operatorname{Aut}(\mathbb{P}^1)$ in $\operatorname{Chow}(X)$: $V/\operatorname{Aut}(\mathbb{P}^1)$ is proper if and only if W is closed in $\operatorname{Chow}(X)$. A point $w \in \overline{W} \setminus W$ corresponds to a 1-cycle $\sum a_i[C_i]$, with C_i rational curves and $\sum a_i \geq 2$. Thus the general rational curve in $V/\operatorname{Aut}(\mathbb{P}^1)$ degenerates into a reducible cycle.

2. V is generically unsplit if for two general points of $\mathrm{Locus}(V)$ pass only a finite number of rational curves of V. Moreover, it can be shown that if V is unsplit then it is also generically unsplit: using Mori bend and break it can be shown that every nontrivial fiber of the map $\Pi : V \to X \times X$ is zero dimensional if the family V is unsplit.

Generically unsplit families of rational curves are important, thanks to the following result.

Theorem 2.2.14. *Let X be a projective scheme and let $W \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be a family of rational curves. Then there exists a family of rational curves $V \subset \mathrm{Hom}(\mathbb{P}^1, X)$ such that*

1. *V is generically unsplit;*
2. *for every 1-cycle $z \in V/\mathrm{Aut}(\mathbb{P}^1)$ there exists an effective 1-cycle $E \in ZE_1(X)$ such that $z + E$ is effective and algebraically equivalent to a 1-cycle $z' \in W/\mathrm{Aut}(\mathbb{P}^1)$;*
3. *$\mathrm{Locus}(W) = \mathrm{Locus}(V)$.*

Definition 2.2.15. A free morphism $f : \mathbb{P}^1 \to C \subset X$ is called *minimal* if

$$f^*T_X = \mathcal{O}(2) + \underbrace{\mathcal{O}(1) + \cdots + \mathcal{O}(1)}_{(d-2)\ times} + \mathcal{O} + \cdots + \mathcal{O},$$

where $d = \deg_{-K}(f)$.

Using Proposition 2.2.11 it can be shown the following result.

Proposition 2.2.16. *Let V be a family of rational curves on X such that f is free for the general $[f] \in V$. Then:*

1. *V dominates X, i.e. $\mathrm{Locus}(V) = X$;*
2. *V is generically unsplit if and only if f is minimal for general $[f] \in V$; in this case, if x is a general point of $\mathrm{Locus}(V)$*

$$\dim \mathrm{Locus}(V_x) = \deg_{-K}(V) - 1.$$

There is also a kind of converse of the first statement of this proposition (for which is necessary that the characteristic is 0):

Proposition 2.2.17. *Let X be a smooth quasi-projective variety and let V be a family of rational curves which dominates X. Then for the general $[f] \in V$, f is free.*

The following definition is due to Jan Wierzba and Jaroslaw A. Wiśniewski (see [WW02]).

Definition 2.2.18. Let X be a projective variety with an ample divisor H, let $E \subseteq X$ a closed subset and let V be a family of rational curves on X. We say that V is a *minimal dominating family* for E if V dominates E and $\deg_H(V)$ is minimal among all families of rational curves which dominate E.

Proposition 2.2.19. *With the above notation if V is a minimal dominating family, then it is generically unsplit. Moreover if x is a general point of $E = \mathrm{Locus}(V)$, then the family V_x is unsplit (in particular this fact shows that a generically unsplit family is not, in general, a minimal dominating family).*

2.2.4 Ionescu-Wiśniewski inequality

In this section we will prove a very important inequality concerning the dimension of the loci of families of rational curves which are generically unsplit.

Proposition 2.2.20. *Let X be a projective variety and let V be a family of rational curves on X.*
If V is generically unsplit *and x is a* general *point in* $\mathrm{Locus}(V)$ *we have*

$$\dim V = \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1.$$

Moreover if we assume that V is unsplit*, then for* every *point $x \in \mathrm{Locus}(V)$ we have*

$$\dim V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1.$$

Proof. Let x be a point in $\mathrm{Locus}(V)$ and consider the diagram

$$\begin{tikzcd} V \times \mathbb{P}^1 \arrow[r, "\mathrm{ev}_V"] \arrow[d, "p"] & \mathrm{Locus}(V) \ni x \\ V & \end{tikzcd}$$

Note that $V_x = \{[f] \in V \mid f(0) = x\} = p(\mathrm{ev}_V^{-1}(x))$; by upper semi continuity of the fiber dimension we have

$$\dim p(\mathrm{ev}_V^{-1}(x)) = \dim \mathrm{ev}_V^{-1}(x) - 1 \geq \dim V - \dim \mathrm{Locus}(V)$$

Similarly, if $y \in \mathrm{Locus}(V_x)$ we have

$$\begin{tikzcd} V_x \times \mathbb{P}^1 \arrow[r, "\mathrm{ev}_{V_x}"] \arrow[d, "p_x"] & \mathrm{Locus}(V_x) \ni y \\ V_x & \end{tikzcd}$$

In this case $\{[f] \in V \mid f(0) = x, f(\infty) = y\} = p_x(\mathrm{ev}_{V_x}^{-1}(y))$ and so

$$\begin{aligned} \dim p_x(\mathrm{ev}_{V_x}^{-1}(y)) &\geq \dim V_x - \dim \mathrm{Locus}(V_x) \geq \\ &\geq \dim V - \dim \mathrm{Locus}(V) - \dim \mathrm{Locus}(V_x) \end{aligned}$$

Moreover these two inequalities are equalities for general x and y.
Let $\Pi : V \longrightarrow X \times X$ as in Definition 2.2.12. By construction

$$\Pi^{-1}(x, y) = p_x(\mathrm{ev}_{V_x}^{-1}(y))$$

is not empty and, since by definition V is closed under $\mathrm{Aut}(\mathbb{P}^1)$, we have $\dim \Pi^{-1}(x, y) \geq 1$.
Now, if V is generically unsplit we have $\dim \Pi^{-1}(x, y) = 1$ for general x and y in $\mathrm{Locus}(V)$, while if V is unsplit $\dim \Pi^{-1}(x, y) = 1$ for every x and y in $\mathrm{Locus}(V)$.
Thus, if V is generically unsplit and x is general, we have:

$$1 = \dim V - \dim \mathrm{Locus}(V) - \dim \mathrm{Locus}(V_x),$$

while, if V is unsplit and x is any point in $\mathrm{Locus}(V)$, we get:

$$1 \geq \dim V - \dim \mathrm{Locus}(V) - \dim \mathrm{Locus}(V_x).$$

□

Using the results in Subsection 2.1.1 we get the following inequality, due to Ionescu and Wiśniewski. We state it allowing some singularities of the variety X.

Corollary 2.2.21. *Let X be a projective variety with only local complete intersection singularities and let V be a family of rational curves such that* $\mathrm{Locus}(V)$ *meets the smooth locus of X.*
Assume that V is generically unsplit and let x be a general point in $\mathrm{Locus}(V)$*, or that V is unsplit and x is any point of* $\mathrm{Locus}(V)$*. Then*

$$\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1 \geq \dim X + \deg_{-K}(V).$$

Remark 2.2.22. In the previous notations, let C be a rational curve in X such that, if $f : \mathbb{P}^1 \to C \subset X$ is its normalization, $[f] \in V$. Then the Ionescu Wiśniewski inequality is

$$\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1 \geq \dim X - K_X \cdot C.$$

2.2.5 Unsplit families and Picard number

In Chapter 1 we have defined for a proper scheme X the groups $B_1(X)$ and $A_1(X)$ consisting of 1-cycles modulo algebraic (resp. rational) equivalence.

Proposition 2.2.23. *Let $f : X \to Y$ be a morphism between proper schemes. Assume that every fiber of f is a connected curve with (geometrically) rational components. Let $i : Z \hookrightarrow X$ be a closed subscheme such that $f \circ i : Z \to Y$ is surjective.*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[dr, bend right, two heads] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Then

$$A_1(X)_{\mathbb{Q}} = \langle [\textit{irred. components of fibers of } f], i_*A_1(Z)_{\mathbb{Q}} \rangle.$$

Therefore also

$$N_1(X) = \langle [\textit{irred. components of fibers of } f], i_*N_1(Z) \rangle.$$

Proof. See [Kol96, Prop. II.4.19]. $\square$

Corollary 2.2.24. *Let $f : X \to Y$ and $i : Z \to X$ as before. Let $p : X \to W$ be a proper and dominant morphism such that $p \circ i(Z) = point$.*

$$\begin{tikzcd} Z \arrow[r, "i"'] \arrow[dr, bend right, two heads] \arrow[rr, bend left] & X \arrow[d, "f"] \arrow[r, "p"] & W \ni pt \\ & Y & \end{tikzcd}$$

Then

$$A_1(W)_{\mathbb{Q}} = \langle p_*[\textit{irred. components of fibers of } f] \rangle.$$

Therefore also

$$N_1(W) = \langle p_*[\textit{irred. components of fibers of } f] \rangle.$$

Proof. Since p is dominant we have $A_1(W)_{\mathbb{Q}} = p_*A_1(X_{\mathbb{Q}})$; then for the previous proposition

$$A_1(W)_{\mathbb{Q}} = p_*\langle [\text{irred. components of fibers of } f] \rangle + (p \circ i)_*A_1(Z)_{\mathbb{Q}};$$

since $p \circ i(Z) = point$ we get $(p \circ i)_*A_1(Z)_{\mathbb{Q}} = 0$ and so we are done. $\square$

Corollary 2.2.25. *Let X be a projective variety, let x be a point of X and V be a family of rational curves such that V_x is unsplit. Then*

$$\dim B_1(\mathrm{Locus}(V_x))_{\mathbb{Q}} = \rho(\mathrm{Locus}(V_x)) = 1.$$

Proof. Considering the restriction of the diagram 2.2.4 to the family V_x, we obtain the universal family (see Figure 2.1):

$$\begin{tikzcd} U_x \arrow[r, "\iota_x"] \arrow[d, "\pi_x"'] & X \\ V_x & \end{tikzcd}$$

The point x gives a (multi-)section of π_x, say $i : V_x \to U_x$. We can apply the previous corollary with $f = \pi_x : U_x \to V_x$ and $p = \iota_x : U_x \to \mathrm{Locus}(V_x) = W$:

$$\begin{tikzcd} V_x \arrow[r, "i"'] \arrow[rr, bend left] \arrow[dr, bend right] & U_x \arrow[r, "p"] \arrow[d, "f"] & \mathrm{Locus}(V_x) \ni pt \\ & V_x & \end{tikzcd}$$

to get

$$B_1(\mathrm{Locus}(V_x))_{\mathbb{Q}} = p_*\langle[\text{fibers of } f]\rangle.$$

Since V_x is connected, all the fibers of f are algebraically equivalent, and so we are done. □

As a consequence of the results stated in this subsection, we have the following proposition.

Proposition 2.2.26. *Let X be a projective variety (not necessarily normal) and let x be a point of X. Suppose that there exists a family of rational curves $V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ such that the subfamily V_x is unsplit and it is also dominant for X. Then the Picard number $\rho(X) = 1$.*

CHAPTER 3

Fano Mori contractions

In this chapter we will develop the study of Fano Mori contractions, introduced in the first chapter. Our general reference is [KMM87].

3.1 General facts

Let X be a smooth n-fold and let $\varphi : X \to W$ be a F-M contraction of an extremal face σ of the Mori cone.
There are different types of F-M contraction, according to the dimension of the "exceptional locus" E of the contraction φ defined as

$$E := \{x \in X \mid \varphi \text{ is not an isomorphism at } x\},$$

which is a closed subscheme of X (it can be $E = X$); of course $\dim E$ denotes the maximum of the dimensions of the components of E.

Definition 3.1.1. If $E = X$, i.e. $\dim X > \dim W$, then we will say that φ is of *fiber type*, otherwise φ is *birational*.
In the latter case if $\dim E = n - 1$ we will say that φ is *divisorial*, otherwise φ is *small*.
If $\dim \sigma = 1$, i.e. σ is an extremal ray, we will say that φ is an *elementary contraction*.

The following proposition gives information about the Picard groups of the target and the source of a F-M contraction.

Proposition 3.1.2. *Let $\varphi : X \to W$ be a F-M contraction of an extremal face σ of the Mori cone and let $\varphi^* : \mathrm{Pic}(W) \to \mathrm{Pic}(X)$ be the induced map*

between the Picard groups. Then the image of the map φ^* coincides with the set $\{D \in \mathrm{Pic}(X) \mid D \cdot z = 0,\ \forall\, z \in \sigma\}$.
This gives also that the following dual sequences are exact

$$0 \longrightarrow N_1(X/W) \longrightarrow N_1(X) \longrightarrow N_1(W) \longrightarrow 0,$$

$$0 \longrightarrow N^1(W) \longrightarrow N^1(X) \longrightarrow N^1(X/W) \longrightarrow 0,$$

where $N_1(X/W)$ is the vector space generated by the curves in X which are contracted by φ and $N^1(X/W)$ is its dual via the intersection number with $N_1(X/W)$.

Corollary 3.1.3. *Let F be a general fiber of a F-M contraction $\varphi : X \to W$ supported by $K_X + \tau L$. Then $(K_X + \tau L)|_F = \mathcal{O}_F$.*

Proof. From the previous proposition we have $(K_X + \tau L)|_F \equiv \mathcal{O}_F$ and from the Base Point Free Theorem we have that there exists an integer m_0 such that $m(K_X + \tau L)$ is base point free for $m > m_0$. Thus we have that $m(K_X + \tau L)|_F = (m+1)(K_X + \tau L)|_F = \mathcal{O}_F$ and so $(K_X + \tau L)|_F = \mathcal{O}_F$. □

Lemma 3.1.4. *If two curves C_1, C_2 in a fiber F of φ are such that $[C_1] = [C_2]$ in $N_1(F)$, then $[C_1] = [C_2]$ also in $N_1(X)$.*

Proof. Let C_1 and C_2 be two curves in F and let $D \in \mathrm{Pic}(X)$. Since these two curves are numerically equivalent in F, we have

$$C_1 \cdot D = C_1 \cdot D|_F = C_2 \cdot D|_F = C_2 \cdot D,$$

and so C_1 and C_2 are numerically equivalent in X. □

The contraction theorem asserts that the exceptional locus of a F-M contraction of an extremal face σ is the union of the curves whose numerical classes are in σ; we can say more:

Proposition 3.1.5. *(cf. [Deb01, Prop. 6.10]) Let $\varphi : X \to W$ be a F-M contraction of an extremal face σ of the Mori cone. Then the exceptional locus E of φ is covered by a family of rational curves whose class is in σ.*

Proof. First we will show that through each point of E passes a rational curve of σ of bounded degree.
Any point x of E lies in a curve C whose class is in σ by definition of F-M contraction. The face σ is generated by a finite number of extremal rays $R_1, \ldots, R_k$: if $\Gamma_1, \ldots, \Gamma_k$ are the rational curves that generate these rays we have that

$$C \equiv \sum a_i \Gamma_i, \quad a_i \in \mathbb{R}.$$

Let H be the supporting divisor of the contraction φ, let L be an ample divisor and let

$$a = \max\{L \cdot \Gamma_i \mid i = 1, \dots, k\} > 0,$$

$$b = \min\{-K_X \cdot \Gamma_i \mid i = 1, \dots, k\} > 0.$$

For each curve C in σ we have

$$\frac{L \cdot C}{-K_X \cdot C} = \frac{L \cdot \sum a_i \Gamma_i}{-K_X \cdot \sum a_i \Gamma_i} < \frac{a}{b}.$$

Thus for each C in σ there exists an integer m, which depends on σ and not on C, such that

$$m > 2\dim(X)\frac{L \cdot C}{-K_X \cdot C}.$$

The Cartier divisor $mH + L$ is ample; thus by Theorem 1.3.2 there exists in X a rational curve Γ through the point x such that

$$0 < (mH + L) \cdot \Gamma \leq 2\dim(X)\frac{(mH + L) \cdot C}{-K_X \cdot C} = 2\dim(X)\frac{L \cdot C}{-K_X \cdot C} < m$$

from which it follows that $H \cdot \Gamma = 0$ and $L \cdot \Gamma < m$: the class of Γ is in σ and so Γ is contained in the exceptional locus of φ.
The exceptional locus of φ is therefore the union of all the rational curves of L-degree at most m, whose classes are in σ.
The scheme $\mathrm{Hom}(\mathbb{P}^1, X)$ has at most countably many components and we have shown that it dominates E; thus there exist at least one irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X)$ which dominates E, i.e. E is covered by a family of rational curves whose class is in σ. □

3.1.1 Local setup

Let $\varphi : X \to W$ be a F-M contraction from a smooth variety. If we want to study the fibers of φ it is useful to shrink the target and to consider a local contraction:
choose a fiber F of φ and an open affine subset $Z \subset W$ such that $\varphi(F) \in Z$ and let $Y = \varphi^{-1}(Z)$; we will call $\varphi : Y \to Z$ a *local F-M contraction* around F. If L is a φ-ample divisor on Y and τ is a rational number such that $K_Y + \tau L$ is trivial on the fiber of φ, we say that the local contraction φ is supported by the divisor $K_Y + \tau L$.
Using this construction Andreatta and Wiśniewski proved the following theorem

Theorem 3.1.6. *[AW93] Let X be a smooth variety and let $\varphi : X \to W$ be a local F-M contraction supported by $K_X + \tau L$, with L φ-ample divisor on X. Let F be a fiber of φ and assume that*

$\dim F < \tau + 1$ *if φ is of fiber type*

$\dim F \leq \tau + 1$ *if φ is birational*

then $|L|$ is base-point-free.

From this theorem they get the following results

Lemma 3.1.7 (Horizontal slicing). *[AW93] Suppose that $\varphi : X \to W$ is a local contraction supported by $K_X + \tau L$, and let X' be a general divisor in the linear system $|L|$. Then, outside of the base locus of $|L|$, the singularities of X' are not worse than those of X and any section of L on X' extends to X.*
Moreover, if we set $\varphi' := \varphi_{|X'}$ and $L' = L_{X'}$, then $K_{X'} + (\tau - 1)L'$ is φ'-trivial. If $\tau \geq 1 + \varepsilon(\dim X - \dim Z)$ then φ' is a contraction, i.e. has connected fibers.

Lemma 3.1.8 (Vertical slicing). *[AW93] Assume that $\varphi : X \to W$ is a local contraction supported by $K_X + \tau L$ and let $X'' \subset X$ be a non trivial divisor defined by a global function $h \in H^0(X, K_X + \tau L) = H^0(X, \mathcal{O}_X)$; then, for a general choice of h, X'' has singularities not worse than those of X and any section of L on X'' extends to X.*

3.1.2 Elementary contractions

From now on we want to study in more detail elementary F-M contraction from a smooth n-fold.
Let $\varphi : X \to W$ be a contraction of an extremal ray R. From the Cone Theorem there exists at least one rational curve in R; one of the most important invariant for the ray R and thus for the contraction associated to it is the following.

Definition 3.1.9. Let R be an extremal ray; then

$$l(R) = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } R\}$$

is the *length* of the extremal ray R. A rational curve in R that realizes the length of the ray is called *minimal extremal curve.*

Remark 3.1.10. From Theorem 1.3.2 we have $1 \leq l(R) \leq \dim(X) + 1$. On the other hand we have that if R is supported by a divisor of the form $K_X + \tau L$, then $l(R) \geq \tau$.

Lemma 3.1.11. *Let $V \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be the family of the deformations of a minimal extremal rational curve; then V is unsplit.*

Proof. Let C be a curve in V and let R be the extremal ray to which the curve belongs.
Assume by contradiction that $[C] = a_1[C_1] + a_2[C_2]$ in $\overline{NE}(X)$, with a_1 and a_2 integers. Since C is extremal we have that C_1 and C_2 are rational curves which belong to R; but this is impossible, since C is a curve which has minimal intersection number with K_X among the curves of R. □

From Proposition 3.1.5 we have that the exceptional locus E of an elementary F-M contraction is covered by a family of rational curves whose class is in the extremal ray associated to the contraction. Thus we can give the following definition.

Definition 3.1.12. Let R be an extremal ray of a smooth projective variety. Then

$$\mathrm{Locus}(R) := \bigcup \{\text{curves of } R\} = E,$$

where E is the exceptional locus of the contraction associated to R.

Moreover from Theorem 2.2.14 we have that E is covered by a generically unsplit family of rational curves whose class is in the extremal ray. In general it is not true that E can be covered by an unsplit family of rational curves, as shown in the following example.

Example 3.1.13. (see [Mor82]) Let $\varphi : X \to W$ an elementary F-M contraction of fiber type from a smooth 3-fold to a smooth surface which is a conic bundle (i.e. the fibers of this map are conics).
The general fiber of φ is a smooth conic, while there exists a possibly empty curve in W such that the fibers over this curve are singular, hence reducible, conics; the generically unsplit family which covers X is the family of the smooth fibers which are of degree 2, while there exist rational curves of degree 1 (i.e. the irreducible components of the singular fibers) which are contracted by φ: these lines belongs to the extremal ray associated to φ, they are the minimal rational curves and do not cover X.

In view of these facts, we can apply the theory of deformation of rational curves developed in the previous chapter to the case of generically unsplit family coming from extremal rays: this provides important informations about the fibers of an elementary F-M contraction.

Lemma 3.1.14. *Let X be a smooth variety and let $\varphi : X \to W$ be the elementary contraction of the extremal ray R; let C be a (extremal) rational* *curve in R, let $V \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be the family of deformations of C and let x be a point of $\mathrm{Locus}(V) \subset X$. Then $\mathrm{Locus}(V_x)$ is entirely contained in the fiber of φ which contains C (i.e. the fiber over $\varphi(x)$).*

Proof. Every curve in V and so in V_x is numerically equivalent to C and so is in R; then every curve in V_x is in a fiber of φ and passes through x and so is in the fiber of φ over x. □

Thanks to this fact, we can state the Ionescu-Wiśniewski inequality in the case of extremal contraction.

Proposition 3.1.15. *Let X be a smooth variety and let $\varphi : X \to W$ be the elementary contraction of the extremal ray R, supported by $K_X + \tau L$, with L ample divisor on X. Let C be a rational curve in R whose family of deformations $V \subset \mathrm{Hom}(\mathbb{P}^1, X)$ is generically unsplit and dominates the exceptional locus E of φ; if we denote with F an irreducible component of any fiber of this map and with $l(R)$ the length of the ray R and we let x be a general point of X, we have*

$$\begin{aligned} &\dim E + \dim F + 1 \geq \\ &\quad \geq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1 \geq \dim X + \tau L \cdot C, \end{aligned} \tag{3.1.16}$$

From this it follows that

$\dim F \geq l(R) - 1 \geq \tau - 1$ *if φ is of fiber type,*

$\dim F \geq l(R) \geq \tau$ *if φ is birational.*

We conclude this section with a lemma concerning varieties with at least two extremal rays.

Lemma 3.1.17. *Let X be a smooth variety which has at least two extremal rays. Then the fibers of two different elementary contractions cannot have a common curve.*

Proof. Each curve in a fiber of an elementary contraction belongs to the extremal ray associated to this contraction: if a curve in X belongs to two fibers of two different contractions, it should belong to two different extremal rays; thus this two extremal rays should meet in a point of the Mori cone different from the vertex of the cone and this is not possible. □

3.2 Fibers of a contraction of fiber type

The aim of this section is to give a classification of the general fibers of F-M contractions of extremal rays with high length, which are of fiber type. More precisely we want to study the elementary contractions of fiber type from a smooth n-fold X, $n \geq 3$, supported by $K_X + \tau L$, with $n - 4 \leq \tau \leq n + 1$.

These results come from classical Mori theory and adjunction theory (see for example [Bel87] and [BS95]).

3.2.1 (Set-up). Let X be a smooth n-fold and let

$$\varphi : X \to W$$

be an elementary F-M contraction of an extremal ray R of X, which is of fiber type and is supported by $K_X + \tau L$, with L an ample line bundle on X; with G we will denote the general fiber of the map φ.
We will always assume that the ample line bundle L is *numerically reduced on X with respect to R* (i.e. it does not exist an ample line bundle L' on X such that $[L] = m[L']$, with $m > 1$ in $N^1(X/W) \cong \mathbb{R}$ or, equivalently, $L \cdot C = mL' \cdot C$ for a curve C in R).

Remark 3.2.2. In this set-up we have that *τ is an integer.*

Proof. Suppose, by contradiction, that

$$\tau = a + \frac{b}{c},$$

where a, b, c are integers with $a \geq 0$ the integral part of τ and $0 < b < c$.
Then, in $N^1(X/W)$, we have that

$$[-K_X - aL] = \frac{b}{c}[L]$$

and so the line bundle $-K_X - aL$ is φ-ample.
Thus, for a suitable choice of an ample divisor A on W, the line bundle L' on X defined as

$$L' := -K_X - aL + \varphi^* A$$

is ample on X and in $N^1(X/W)$ we have

$$[L] = \frac{c}{b}[L'], \quad \text{with } \frac{c}{b} > 1.$$

But this contradicts the fact that L is numerically reduced on X with respect to R, and so τ must be an integer. $\square$

Remark 3.2.3. The assumption "L numerically reduced on X with respect to R" is only needed to make the classification simpler. We will classify the general fibers of F-M contraction according to the value of τ and if we have a contraction supported by $K_X + \tau L$ with $[L] = m[L']$ in $N^1(X/W)$, then this contraction is supported also by $K_X + \tau m L'$. Thus it is enough to study the case L numerically reduced.

Remark 3.2.4. The assumption "L numerically reduced on X with respect to R" does not imply that the line bundle L_G is numerically reduced in G (see Example 3.2.9).

The following proposition gives us some information about the "geometry" of the general fiber of a contraction of fiber type.

Proposition 3.2.5 (Generic smoothness). *(see [Har77, Cor. III.10.7]) Let $\varphi : X \to W$ be a morphism from a smooth variety X. Then there exists a nonempty subset V of W such that, if we call $U = \varphi^{-1}(V)$, the morphism $\varphi|_U : U \to V$ is smooth.*
In particular we have that all the general fibers of φ are smooth, of the same dimension and have the same Hilbert polynomial.

The main tool in the classification will be the theory of the deformations of rational curve developed in the second chapter and the Proposition 3.1.15.

Remark 3.2.6. With the notations as in Proposition 3.1.15, in this case $E = \text{Locus}(V) = X$. Thus, from Proposition 2.2.17, the rational curve C is free and so, by Propositions 2.2.16 and 3.1.15

$$n \geq \dim G \geq \dim \text{Locus}(V_x) = \tau L \cdot C - 1. \tag{3.2.7}$$

Hence we get $\tau \leq n + 1$.

Proposition 3.2.8. *Let X be a smooth n-fold and let $\varphi : X \longrightarrow W$ be an elementary F-M contraction of an extremal ray R which is of fiber type and is supported by $K_X + \tau L$, with L numerically reduced. Then the general fiber G of φ is a smooth Fano variety with index $i(G)$ which is a multiple of τ.*

Proof. Since φ is of fiber type, by adjunction we have that $K_G = K_X|_G$ and from generic smoothness we have that G is smooth.
Thus from Corollary 3.1.3 we have that $-K_G = -K_X|_G = \tau L_G$, which is ample and so G is Fano with index $i(G)$ a multiple of τ. □

The following table summarizes the classification of the pairs (G, L_G), according to the values of τ.

Classification of the pairs (G, L_G)

	$\dim G$	$\dim W$	G	n
$\boldsymbol{\tau = n+1}$	n	0	$\mathbb{P}^n$	any
$\boldsymbol{\tau = n}$	n	0	$\mathbb{Q}^n$	any
	$n-1$	1	$\mathbb{P}^{n-1}$	
$\boldsymbol{\tau = n-1}$	n	0	del Pezzo	any
	$n-1$	1	$\mathbb{Q}^{n-1}$	
	$n-2$	2	$\mathbb{P}^{n-2}$	
$\boldsymbol{\tau = n-2}$	n	0	Mukai	≥ 6
	$n-1$	1	del Pezzo	
	$n-2$	2	$\mathbb{Q}^{n-2}$	
	$n-3$	3	$\mathbb{P}^{n-3}$	
$\boldsymbol{\tau = n-3}$	n	0	$i(G) = n-3$	≥ 8
	$n-1$	1	Mukai	
	$n-2$	2	del Pezzo	
	$n-3$	3	$\mathbb{Q}^{n-3}$	
	$n-4$	4	$\mathbb{P}^{n-4}$	
$\boldsymbol{\tau = n-4}$	n	0	$i(G) = n-4$	≥ 10
	$n-1$	1	$i(G) = n-3$	
	$n-2$	2	Mukai	
	$n-3$	3	del Pezzo	
	$n-4$	4	$\mathbb{Q}^{n-4}$	
	$n-5$	5	$\mathbb{P}^{n-4}$	

In all the cases we have $L_G \cong \mathcal{O}_G(1)$

We give a sketch of the proof for the first values of τ. For the other values the proofs go in the same way.

$\tau = n + 1$

In this case inequality 3.1.16 gives that

$$\dim G + 1 \geq n + 1, \quad \text{i.e. } \dim G = n,$$

which implies that $G = X$ and $\dim W = 0$. On the other hand $i(G) = i(X) \geq n + 1$ and so Theorem 1.5.3 gives that $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

$\tau = n$

From inequality 3.2.7 we have that

$$L \cdot C = 1 \quad \text{and so} \quad \dim G \geq \dim \mathrm{Locus}(V_x) = n - 1.$$

Then we have two different cases:

$\dim G = n$ and so $G = X$, $\dim W = 0$ and $i(G) = i(X) = n$; Theorem 1.5.3 gives that $(X, L) = (\mathbb{Q}^n, \mathcal{O}(1))$.

$\dim G = n-1$ and so $\dim W = 1$ and $i(G) = n = \dim G+1$; again Theorem 1.5.3 gives $(G, L) = (\mathbb{P}^{n-1}, \mathcal{O}(1))$.
Moreover we have that W is a smooth curve (since it is normal) and so $\varphi : X \to W$ is a flat morphism whose general fiber is isomorphic to $\mathbb{P}^{n-1}$ and has degree one with respect to L. Thus any fiber is isomorphic to $\mathbb{P}^{n-1}$ and has degree one with respect to L and so $\varphi : X \to W$ is a $\mathbb{P}$-bundle.

$\tau = n - 1$

Inequality 3.2.7 gives the following possibilities:

$$L \cdot C = \begin{cases} 2 & \text{with } n = 3 \\ 1 & \text{with } n \geq 3 \end{cases}$$

moreover the first case is possible only if $\dim G = \dim \mathrm{Locus}(V_x) = 3$: thus $X = G$ is a del Pezzo threefold of degree at least 8 and this implies that $(X, L) = (\mathbb{P}^3, \mathcal{O}(2))$ (see [Fuj90]), and so L is not numerically reduced.
Consider now the case $L \cdot C = 1$: inequality 3.2.7 becomes

$$\dim G \geq \dim \mathrm{Locus}(V_x) = n - 2,$$

and Proposition 3.2.8 implies that G is a Fano manifold of index $i(G) = n-1$; thus we have the following possibilities:

$\dim G = n$ and so $G = X$, $\dim W = 0$ and $i(X) = i(G) = n-1$: thus X is a del Pezzo manifold.

$\dim G = n-1$ and so $\dim W = 1$. Thus G is a smooth Fano variety with index $i(G) = \dim G$: Theorem 1.5.3 gives that $(G, L_G) = (\mathbb{Q}^{n-1}, \mathcal{O}(1))$. Moreover we have that W is a smooth curve (since it is normal) and so $\varphi : X \to W$ is a flat morphism; this implies that φ is a quadric fibration.

$\dim G = n-2$ and so W is a surface and $i(G) = \dim G + 1$. Theorem 1.5.3 gives $(G, L) = (\mathbb{P}^{n-2}, \mathcal{O}(1))$.
Moreover it can be shown that W is smooth and that $\varphi : X \to W$ is a scroll (see [Fuj87]).

The other cases

For the cases $\tau = n-2, n-3, n-4$ we gave only a rough classification; in particular the list of the general fibers holds for n big enough. Indeed from inequality 3.2.7 we have that, in the considered range of τ, if n is big enough the line bundle L_G has intersection number 1 with the minimal extremal curve and so $L_G = \mathcal{O}(1)$.
The situation becomes more complicated for variety of smaller dimension: in particular we will show with an example in dimension four that the general fiber of these contractions can be out of the list given before.
The techniques used to produce the tables of the fibers and also the "exceptional cases" in low dimension are similar to the ones used for the previous cases.

This example was inspired by [ABW92, Example 2.3].

Example 3.2.9. Let S be a smooth projective surface whose Brauer group is non-trivial (for example an abelian surface, see [Ele82]): suppose in particular that there exists a smooth fibration, from a smooth fourfold X,

$$\varphi : X \to S$$

whose fibers F are $\mathbb{P}^2$, which is not a projective bundle over S.
First of all we have to show that φ is a F-M contraction: since we know that it has connected fibers, it is enough to see that $-K_X$ is relatively ample, i.e.

it is ample on the fibers F of φ.
By adjunction we have

$$-K_X|_F = -K_F = \mathcal{O}_{\mathbb{P}^2}(3),$$

and so $\varphi : X \to S$ is a F-M contraction of fiber type, which is also elementary (the curves in each fiber are numerically proportional since the fibers are projective spaces and the curves in two different fiber are proportional since φ is a flat morphism).
Let $L = -K_X + \varphi^*(A)$, where A is a ample divisor on S: we can assume that L is ample on X and so we have that $K_X + L = \varphi^*(A)$ is a supporting divisor for φ whose fibers are $(F, L_F) = (\mathbb{P}^2, \mathcal{O}(3))$, which are not contained in the previous list.
We claim that L is numerically reduced. If not, there should exist a line bundle L' on X such that $L'_F = \mathcal{O}(1)$: but in this case [Fuj87, Lemma 2.12] gives that $\varphi : X \to S$ is a projective bundle and this contradicts the hypothesis.

3.3 Small contractions

Only a few words about the small contractions. We recall that a F-M contraction $\varphi : X \to W$ is *small* if it is birational and the codimension in X of its exceptional locus is ≥ 2.
From inequality 3.1.16 it is easy to see that, if φ is supported by $K_X + \tau L$, the contraction can be a small contraction only if $\tau \leq n-3$ (with, of course, $n \geq 4$).
Small contractions supported by $K_X + (n-3)L$ where studied by Kawamata ([Kaw89]) for $n = 4$ and by Andreatta, Ballico and Wiśniewski ([ABW93]) for $n \geq 5$, while the ones supported by $K_X + (n-4)L$ where studied by Occhetta ([Occ]) and Zhang ([Zha95]).

Remark 3.3.1. Unlike the fiber type case, in the study of birational F-M contraction $\varphi : X \to W$, it is important to study the normal bundle $N_{E/X}$ of the exceptional locus E of the contraction in the variety X or, at least, its restriction to the fibers of φ.

Here we will recall briefly the results concerning small contractions supported by $K_X + (n-3)L$.

Definition 3.3.2. Let $\varphi : X \to W$ be a small elementary contraction; the *flip* of φ is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "tr_\varphi"] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & W & \end{tikzcd}$$

where $\varphi^+ : X^+ \to W$ is a birational morphism from a normal projective variety with only terminal singularities such that the canonical divisor of X^+ is φ^+-ample and X^+ is isomorphic to X in codimension 1 via tr_φ.

Theorem 3.3.3. *[ABW93, Theorem A] Let X be a smooth variety and let $\varphi : X \to W$ be an elementary small contraction whose supporting divisor is $K_X + (n-3)L$.*
Then the exceptional locus E of φ is a disjoint union of its irreducible components E_i $(i = 1 \dots s)$, such that $E_i = \mathbb{P}^{n-2}$ and $N_{E_i/X} = \mathcal{O}_{\mathbb{P}^{n-2}}(-1)^{\oplus 2}$.
Moreover there exists the flip and the variety X^+ is non-singular and projective.

3.3.4 (Description of the flip). (see Figure 3.1) Let $\beta : Z \to X$ be the blow-up of X with center E: its exceptional locus E' is a disjoint union of $\mathbb{P}^{n-2} \times \mathbb{P}^1$'s with normal bundles $\mathcal{O}(-1,-1)$.

$$\begin{tikzcd} & Z \arrow[dl, "\beta"'] \arrow[dd, "(\varphi\circ\beta)"] \\ X \arrow[dr, "\varphi"'] & \\ & W \end{tikzcd}$$

The map $(\varphi \circ \beta)$**.** Clearly this map has connected fibers (each $\mathbb{P}^{n-2} \times \mathbb{P}^1$ is a fiber of this composition) and moreover $-K_Z$ is $(\varphi \circ \beta)$-ample and so this composition is a F-M contraction. To prove this we have to show that $-K_Z$ is ample on each $\mathbb{P}^{n-2} \times \mathbb{P}^1$: it suffices to show that $-K_Z$ has positive intersection number on the generators of $N_1(\mathbb{P}^{n-2} \times \mathbb{P}^1)$ which are a line ℓ in $\mathbb{P}^{n-2}$ and Γ, one of the $\mathbb{P}^1$'s.
Since Γ is a fiber of the blow-up β, we have that $-K_Z \cdot \Gamma = 1$; moreover, from the projection formula and since $-K_Z = -\beta^* K_X - E'$, we have that

$$-K_Z \cdot \ell = (-\beta^* K_X - E') \cdot \ell = -K_X \cdot l - E' \cdot \ell,$$

where $l = \beta_* \ell$ is a line in one of the irreducible components of E; thus

$$-K_Z \cdot \ell = 1 + 1 = 2.$$

Construction of X^+. Since φ and β are elementary contractions, we have that $N_1(Z/W)$ is two-dimensional, i.e. there exists on Z an extremal ray different from the one contracted by β which is contracted by $(\varphi \circ \beta)$: by Nakano contractibility criterion the contraction of this ray is the contraction of E' "in the other direction": E' is the exceptional locus of a smooth blow-down whose non trivial fibers are $\mathbb{P}^{n-2}$.

$$\begin{tikzcd} & Z \arrow[dl, "\beta"'] \arrow[dr, "\psi"] & \\ X \arrow[rr, dashed, "tr_\varphi"] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & W & \end{tikzcd}$$

In this way we obtain the blow-down $\psi : Z \to X^+$, where X^+ is a smooth projective variety, with an induced morphism $\varphi^+ : X^+ \to W$ (see Proposition 1.3.11). Moreover $\psi(E') = C$ is a smooth curve in X^+ and K_{X^+} is φ^+-ample.

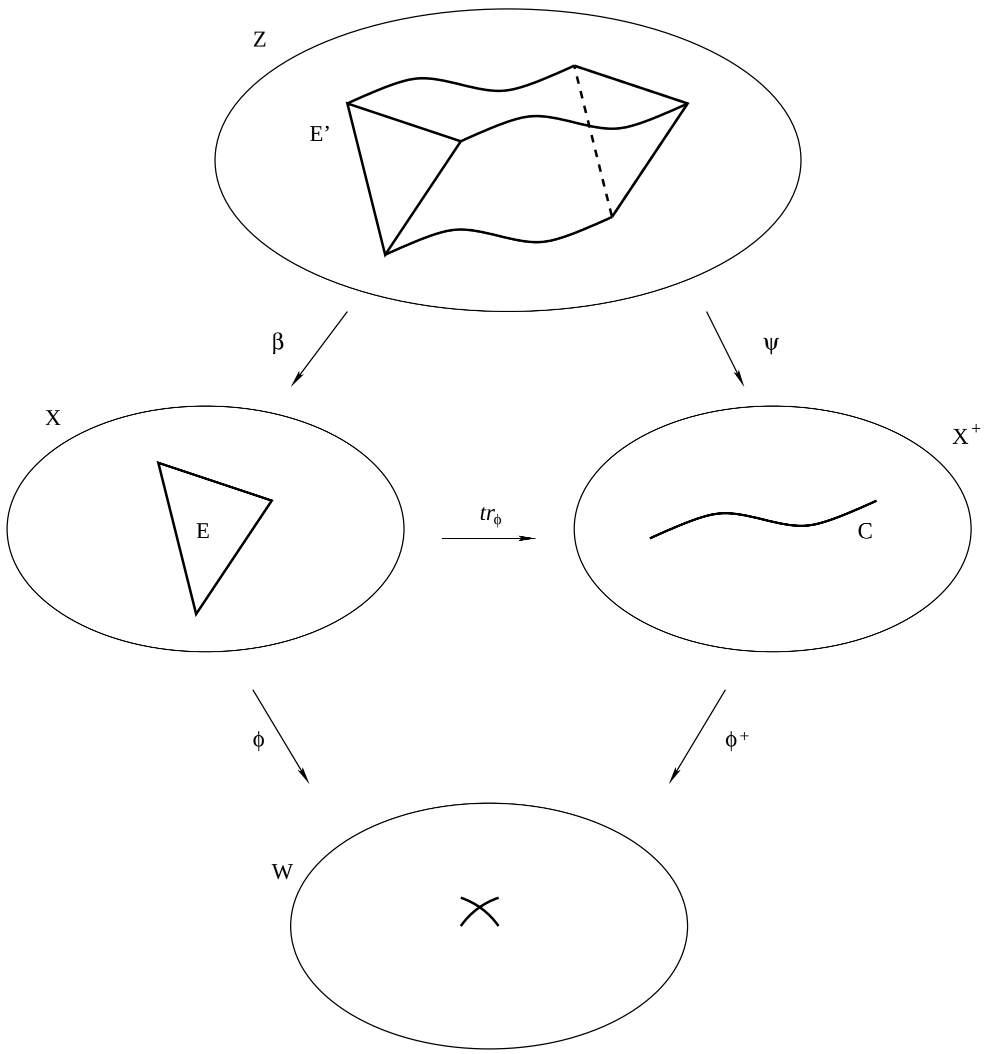


Figure 3.1: The flip of Kawamata

CHAPTER 4

Fibers of divisorial contractions

In this chapter we will study divisorial Fano Mori contractions; in particular we will classify the general fibers of elementary divisorial contractions from a smooth n-fold X supported by $K_X + \tau L$ with $n-4 \leq \tau \leq n-1$.

4.1 First properties

4.1.1 (Setup). Let X be a smooth n-fold and let

$$\varphi : X \to W$$

be an elementary F-M contraction of an extremal ray R of X, which is supported by $K_X+\tau L$, with L an ample line bundle on X and whose exceptional locus $E \subset X$ is a divisor.
With F we will denote *any* non trivial (i.e. non 0-dimensional) fiber of φ, while G will be a *general* non trivial fiber of this map. Moreover we will consider these fibers as subschemes of X *with the reduced structure.*
We will always assume that the ample line bundle L is *numerically reduced on X with respect to R* (i.e. it does not exist an ample line bundle L' on X such that $[L] = m[L']$, with $m > 1$ in $N^1(X/W) \cong \mathbb{R}$ or, equivalently, $L \cdot C = mL' \cdot C$ for a curve C in R). (See also the Remarks that follow (3.2.1)).

Remark 4.1.2. Since we consider E with the reduced structure, from a theorem of Bertini (see [Jou83, Theoreme 6.3]), to consider F and G with the reduced structure, is equivalent to consider F and G as any and the general fiber of $\varphi|_E : E \to \varphi(E)$.

Unlike the fiber type case, the geometry of the general non trivial fibers of φ is not so simple. In particular it is not true that the fiber G is smooth: it can be even not irreducible.
We have the following general result.

Theorem 4.1.3. *(see [Deb01, Proposition 6.10 (b)]) Let $\varphi : X \to W$ be an elementary divisorial contraction with exceptional divisor E. Then E is irreducible.*

Proof. Let H be a supporting divisor for φ; since $H = \varphi^*(A)$, where A is an ample Cartier divisor on W and φ is birational, we have that H is nef and big.
Then from Theorems 1.2.7 and 1.1.6 we have

$$H^0(X, K_X + mH) = \chi(X, K_X + mH) \sim \frac{H^n}{n!} m^n + \cdots > 0$$

for $m \gg 0$; thus the linear system $|K_X + mH| \neq \emptyset$ for $m \gg 0$.
Let

$$D = \sum d_i D_i \in |K_X + mH|$$

be an element of this linear system (of course we have $d_i > 0$) and let C be a curve of the extremal ray contracted by φ; we have

$$\begin{cases} H \cdot C = 0 & \text{since } H \text{ is a supporting divisor,} \\ K_X \cdot C < 0 & \text{since } K_X \text{ is } \varphi\text{-ample,} \end{cases}$$

and so $D \cdot C < 0$.
Then there exists an irreducible component D_i of D, which is an effective divisor on X such that $D_i \cdot C < 0$.
This implies that all the curves in the extremal ray contracted by φ have negative intersection with D_i, hence they are contained in D_i.
Then we have that $E \subseteq D_i$: thus, since E is a divisor, we have that $E = D_i$ and so E is irreducible. □

Corollary 4.1.4. *(See [KMM87, Proposition 5.1.7]) Let $\varphi : X \to W$ be an elementary divisorial contraction. Then the variety W is $\mathbb{Q}$-factorial.*

Proof. In the proof of the previous theorem we have shown that $E \cdot C < 0$, where C is an extremal curve. Then for any Weil divisor A' of W, if we call A its strict transform by φ, there exists a rational number $q \in \mathbb{Q}$ such that $(A + qE) \cdot C = 0$. Thus from Proposition 3.1.2 we have that there exists a Cartier divisor A_0 on W and an integer r such that $r(A + qE) = \varphi^* A_0$: thus $rA' = A_0$ and so A' is $\mathbb{Q}$-Cartier. □

Our aim, in this chapter, is to give a list of all the possible terns:

$$(G, L_G, (N_{E/X})|_G)$$

where G is a general fiber of an elementary divisorial F-M contraction $\varphi : X \to W$, supported by $K_X + \tau L$ and $N_{E/X}$ is the normal bundle of the exceptional locus E of φ in X.

Lemma 4.1.5. *(See [And85]) Let $\varphi : X \to W$ be an elementary divisorial F-M contraction supported by $K_X + \tau L$, let E be the exceptional divisor of φ and let F be any fiber of $\varphi|_E : E \to \varphi(E)$.*
Then the image of the restriction map $\operatorname{Pic} X \to \operatorname{Pic} F$ *is of rank one, generated by $L_F = L|_F$.*
From Corollary 3.1.3 we have that $K_X|_F = -\tau L_F$; moreover there exists an integer $q \geq 1$ such that

$$E|_F = (N_{E/X})|_F = -qL_F.$$

Proof. Consider the inclusion

$$N_1(F) \to N_1(X);$$

since the contraction φ is elementary, all the curves in F are numerically proportional in X and so $\operatorname{Im}(N_1(F) \to N_1(X))$ is of rank one.
The dual map is the restriction $N^1(X) \to N^1(F)$, thus we have that the subspace $\operatorname{Im}(N^1(X) \to N^1(F))$ is of dimension 1.
Moreover from Proposition 3.1.2 and Corollary 3.1.3 we have that if we have $A \in \operatorname{Pic} X$ with $A|_F \equiv 0$ then $A|_F \sim 0$: this implies that since $\operatorname{Im}(N^1(X) \to N^1(F))$ is of rank one, then also $\operatorname{Im}(\operatorname{Pic} X \to \operatorname{Pic} F)$ is of rank one.
Since L is ample and it is numerically reduced, L_F is the positive generator of this image.
Notice that, since $E \cdot C < 0$ for each curve C contracted by φ, $-E$ is φ-ample i.e. $-E|_F$ is ample of F and so it is a *positive* multiple of L_F. $\square$

4.1.1 Vertical slicing technique

Following Ando [And85], to continue the study of the general fibers of the contraction $\varphi : X \to W$, we have to use the vertical slicing technique:

4.1.6 (Construction). (see Figure 4.1) Let $\varphi : X \to W$ be an elementary divisorial F-M contraction with exceptional divisor E and let $r = \dim \varphi(E)$; if $r = 0$ then the contraction φ maps E to a point and so there is a unique

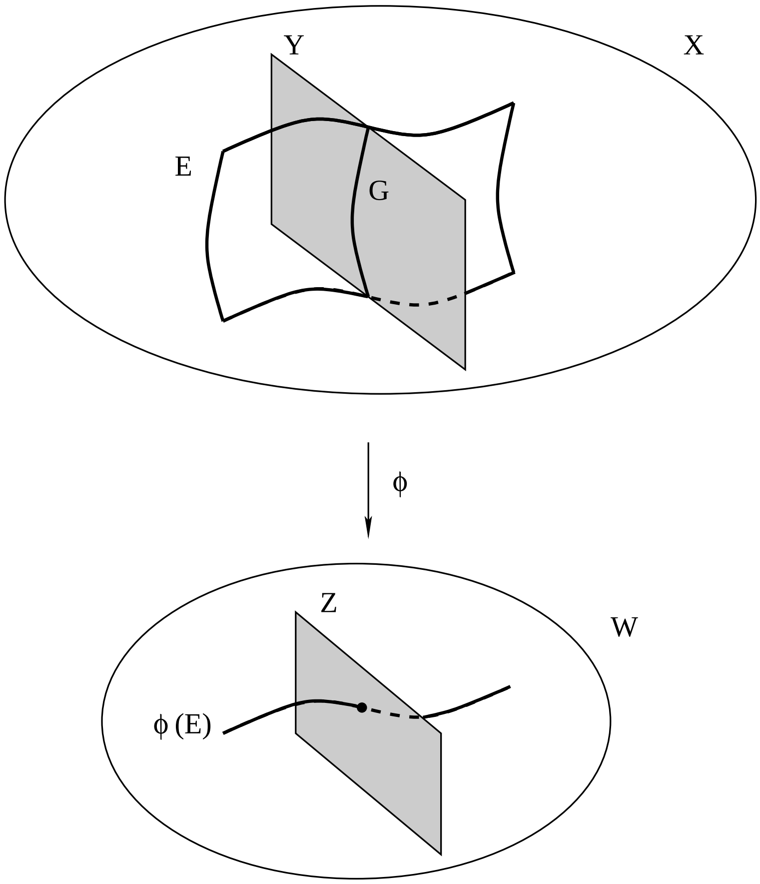


Figure 4.1: Vertical slicing

fiber E. If $r > 0$, we can take r general very ample divisors $Z_1, \dots, Z_r$ on W; if we call $Y_i = \varphi^*(Z_i)$, we can consider:

$$Y := \bigcap_i Y_i, \qquad Z := \bigcap_i Z_i.$$

From Bertini theorem we have that Y is smooth and Z is normal and $\mathbb{Q}$-factorial. Moreover the general fiber G of φ is a connected component of $E \cap Y$.

Lemma 4.1.7. *The map $\varphi_Y : Y \to Z$, which is the restriction of φ to Y, is a F-M contraction supported by $K_Y + \tau L_Y$ which maps G to a point in Z.*

Proof. First of all notice that φ_Y has connected fibers: this map contracts only the fibers of φ over the points of $Z \cap \varphi(E)$.

We will show that $(K_Y + \tau L_Y)|_G = \mathcal{O}_G$ and so we are done.
Since Y is the complete intersection $Y = \cap_i \varphi^*(Z_i)$, we have

$$\det N_{Y/X} = \varphi^*(Z_1 + \cdots + Z_r)|_Y.$$

By adjunction

$$K_Y = (K_X + \det N_{Y/X})|_Y$$

and so we have

$$K_Y + \tau L_Y = (K_X + \tau L)|_Y + \varphi^*(Z_1 + \cdots + Z_r)|_Y.$$

The right side term is trivial on G, since G is a fiber of the contraction φ, and so also the left side term is trivial and we are done. $\square$

Remark 4.1.8. In general φ_Y is not an elementary contraction and so G can be not irreducible.

Since we have that G is a divisor on Y and Y is a smooth variety, we have that G is a local complete intersection and so it is well defined its normal bundle in Y.
Another connection between the starting contraction φ and φ_Y concerns the normal bundle of the exceptional locus. Precisely the following holds

Lemma 4.1.9. *In the previous notations* $N_{G/Y} = (N_{E/X}|_G)$.

Proof. From the sequences of inclusions

$$G \to Y \to X, \qquad G \to E \to X,$$

we have the exact sequences of normal bundles

$$0 \to N_{G/Y} \to N_{G/X} \to (N_{Y/X})|_G \to 0,$$

$$0 \to N_{G/E} \to N_{G/X} \to (N_{E/X})|_G \to 0.$$

Taking the determinants of these sequences and observing that $N_{G/Y}$ and $(N_{E/X})|_G$ are line bundles and so they coincide with their determinants, we obtain

$$\det N_{G/X} = N_{G/Y} \otimes \det(N_{Y/X})|_G = \det N_{G/E} \otimes (N_{E/X})|_G$$

Since $Y = \cap_i \varphi^*(Z_i)$, as in the proof of the previous Lemma we have that $\det(N_{Y/X})|_G = \mathcal{O}_G$; moreover, since G is a fiber of the fibration $\varphi_E : E \to \varphi(E)$, we have also that $\det N_{G/E} = \mathcal{O}_G$. Then we have

$$N_{G/Y} = \det N_{G/X} = (N_{E/X})|_G.$$

$\square$

Since G is a local complete intersection, using adjunction formula, we can compute its canonical bundle: from Lemma 4.1.5 we have

$$K_X|_G = -\tau L_G, \qquad G|_G = N_{G/Y} = (N_{E/X}|_G) = -qL_G,$$

and so we obtain the following equality:

$$K_G = K_X|_G + G|_G = -(\tau + q)L_G.$$

4.2 The Hilbert polynomial

The aim of this section is to compute the Hilbert polynomial of (G, L_G), a general fiber of the contraction $\varphi : X \to W$ supported by $K_X + \tau L$, as in 4.1.1; we will follow the ideas in [Nak95].
Before all, we need some vanishing for the higher cohomology groups of the fibers of φ_E, due to Ando and Andreatta and Wiśniewski, which comes from the Kawamata-Viehveg vanishing theorem.

Lemma 4.2.1. *([And85, Lemma 2.2],[AW98, Lemma 1.2.1]) Let things be as in 4.1.1 and let P be a Cartier divisor on X such that $-K_X + P$ is relatively nef and big; then $H^i(G, \mathcal{O}_G(P)) = 0$ for $i > 0$.*
Moreover if F' is a subscheme of X with $F'_{red} \subset F$, then $H^s(F', \mathcal{O}_{F'}(P)) = 0$, where $s = \dim F$.

Corollary 4.2.2. *In the previous notations we have that, if $t \geq -\tau$,*

$$H^i(G, tL_G) = 0, \quad i > 0.$$

Proof. We have only to show that $-K_X + tL$ is relatively nef and big for $t \geq -\tau$. From Proposition 3.1.3 we have that, for each non trivial fiber F of the contraction φ, holds the equality $-K_X|_F = \tau L_F$ and so

$$(-K_X + tL)|_F = (\tau + t)L_F,$$

which is ample on F if $t > -\tau$ and it is $\mathcal{O}_F$ if $t = -\tau$, which is nef and big, and we are done. □

From this fact we can deduce the following

Proposition 4.2.3. *The Hilbert polynomial $P(t) = \chi(G, tL_G)$ is such that*

$$P(t) = \begin{cases} 1 & \text{if } t = 0 \\ h^0(L_G) & \text{if } t = 1 \\ 0 & \text{if } t = -1, \dots, -\tau \end{cases} \tag{4.2.4}$$

From Serre duality, if as usual $r = \dim \varphi(E)$ *and* q *is the integer defined by the equality* $E|_F = -qL_F$ *(see Lemma 4.1.5), we have also that*

$$P(t) = (-1)^{n-r-1} P(-\tau - q - t). \tag{4.2.5}$$

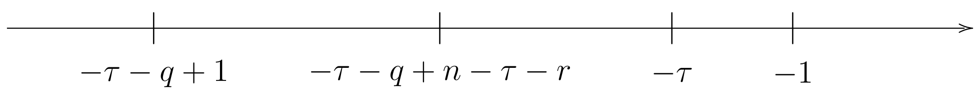
Moreover it holds that $1 \leq q \leq n - \tau - r$.

Proof. From the previous Corollary we have that, for $t \geq -\tau$, the Hilbert polynomial of (G, L_G) is $P(t) = h^0(G, tL_G)$. Since L_G is ample, it is clear that $h^0(G, tL_G) = 0$ if $t < 0$, while the other two equalities of 4.2.4 are straightforward.
Moreover, since $K_G = -(\tau + q)L_G$, the equality 4.2.5 is a direct cosequence of Serre duality.

The last thing to prove is that $q \leq n - \tau - r$.
$P(t)$ is the Hilbert polynomial of the general fiber of the morphism $\varphi_E : E \to \varphi(E)$ from a variety of dimension $n - 1$ to a variety of dimension r and so it is a polynomial of degree $n - r - 1$, which is the dimension of the general fiber of φ_E.
Assume now, by contradiction, that $q \geq n - \tau - r + 1$. Then the integers

$$-\tau - q + 1, \; -\tau - q + 2, \; \ldots, \; -\tau - q + n - \tau - r$$

are smaller than $-\tau$:



and thus they are distinct from

$$-\tau, -\tau + 1, \ldots, -1.$$

Hence 4.2.4 gives τ roots of $P(t)$ and 4.2.5 gives $(n - \tau - r)$ other roots which are distinct from the previous ones: so we get $\tau + (n - \tau - r) = (n - r)$ distinct roots of $P(t)$, which is a non zero polynomial of degree $n - r - 1$; but this is impossible and so we are done. □

So we have an upper bound for q:

$$q \leq n - \tau - r;$$

moreover, using these properties and these techniques, we can compute explicitly the Hilbert polynomial of the pairs (G, L_G) when q is big enough with respect to the upper bound $n-\tau-r$. In particular we will get explicitly this Hilbert polynomial when

$$q = \begin{cases} n-\tau-r \\ n-\tau-r-1 \end{cases}$$

and we can obtain some information also when

$$q = \begin{cases} n-\tau-r-2 \\ n-\tau-r-3 \end{cases}$$

if these cases are effective, i.e.

$$\textit{under the condition that } q \geq 1. \tag{4.2.6}$$

Using this description, *in the hypothesis that G is irreducible*, we will be able to compute the Δ-genus of the pair (G, L_G):

$$\Delta(G, L_G) = \dim G + L_G^{\dim G} - h^0(L_G) = n-r-1 + L_G^{n-r-1} - P(1)$$

to get the classification of the pairs (G, L_G), according to the values of q.

a) $q = n - \tau - r$

In this case 4.2.4 and 4.2.5 gives exactly

$$\tau + (n-\tau-r-1) = n-r-1 = \deg P(t)$$

distinct roots of $P(t)$ and so

$$P(t) = \frac{d}{(n-r-1)!}(t+1)\dots(t+(n-r-1)),$$

where $d = L_G^{n-r-1}$.
Since 4.2.4 gives that $P(0) = 1$, we have that $d = 1$ and so, recalling Remark 1.6.6, we have that G is irreducible.
Thus we can compute the Δ-genus of (G, L_G); since in this case $P(1) = n-r$ we have

$$\Delta(G, L_G) = n-r-1+d-P(1) = 0$$

and so from Theorem 1.6.2

$$(G, L_G, N_{E/X}|_G) = (\mathbb{P}^{n-r-1}, \mathcal{O}(1), \mathcal{O}(-(n-\tau-r)))$$

.

b) $q = n - \tau - r - 1$

In this case 4.2.4 and 4.2.5 gives exactly

$$\tau + (n - \tau - r - 2) = n - r - 2 = \deg P(t) - 1$$

distinct roots of $P(t)$ and so

$$P(t) = \frac{d}{(n-r-1)!}(t+1)\dots(t+(n-r-2))(t-\alpha),$$

where, as before, $d = L_G^{n-r-1}$ and $\alpha \in \mathbb{C}$ is the other root.
Thanks to 4.2.5, we have two different expressions of the roots of $P(t)$:

$$\begin{aligned} &\{-1, \dots, -(n-r-2), \alpha\} = \\ &\qquad = \{-(n-r-1)+1, \dots, -1, -(n-r-1)-\alpha\}. \end{aligned}$$

Of course the sum of the elements of one set is equal to the sum of the elements of the other:

$$\begin{aligned} &-\frac{(n-r-2)(n-r-1)}{2} + \alpha = \\ &\qquad = -\frac{(n-r-2)(n-r-1)}{2} - (n-r-1) - \alpha \end{aligned}$$

and so we find

$$\alpha = -\frac{n-r-1}{2};$$

from the condition $P(0) = 1$ we have also

$$d\alpha = -(n-r-1) \quad \text{and so} \quad d = 2.$$

Moreover it follows that, *if G is irreducible*, we can compute the Δ-genus of (G, L_G); since in this case $P(1) = n - r + 1$ we have

$$\Delta(G, L_G) = n - r - 1 + d - P(1) = 0$$

and so from Theorem 1.6.2, *under the hypothesis that G is irreducible*, we get

$$(G, L_G, N_{E/X}|_G) = (\mathbb{Q}^{n-r-1}, \mathcal{O}(1), \mathcal{O}(-(n-\tau-r-1))).$$

c) $q = n - \tau - r - 2$

In this case, following the same argument as before, we have

$$P(t) = \frac{d}{(n-r-1)!}(t+1)\dots(t+(n-r-3))(t-\alpha)(t-\beta),$$

where $d = L_G^{n-r-1}$ and $\alpha, \beta \in \mathbb{C}$ are the other roots.
Again, thanks to 4.2.5, we have two different expressions of the roots of $P(t)$:

$$\{-1, \dots, -(n-r-3), \alpha, \beta\} = \\ = \{-(n-r-2)+1, \dots, -1, -(n-r-2)-\alpha, -(n-r-2)-\beta\}.$$

Taking the sum we have

$$\alpha + \beta = -(n-r-2)$$

and from the condition $P(0) = 1$ we get

$$\alpha\beta = \frac{(n-r-2)(n-r-1)}{d} \quad \text{and so} \quad h^0(L_G) = P(1) = d + n - r - 2.$$

As before, *if G is irreducible*, we can compute the Δ-genus of (G, L_G):

$$\Delta(G, L_G) = n - r - 1 + d - P(1) = 1.$$

In this case we have also that

$$K_G = -(\tau + q)L_G = (n-r-2)L_G = (\dim G - 1)L_G$$

and so, since G is locally complete intersection, Proposition 1.6.4 applies to get that, *if G is irreducible*, the pair (G, L_G) is a del Pezzo variety.

d) $q = n - \tau - r - 3$

In this case we have

$$P(t) = \frac{d}{(n-r-1)!}(t+1)\dots(t+(n-r-4))(t-\alpha)(t-\beta)(t-\gamma),$$

where $d = L_G^{n-r-1}$ and $\alpha, \beta, \gamma \in \mathbb{C}$ are the other roots.
Similar arguments as before show that, *if G is irreducible*, we can compute the Δ-genus of (G, L_G):

$$\Delta(G, L_G) = \frac{d}{2}.$$

Moreover we have also that

$$K_G = -(\tau + q)L_G = (n-r-3)L_G = (\dim G - 2)L_G,$$

and so, *if G is irreducible*, the pair (G, L_G) is a Mukai variety.

4.3 Classification of the general fibers

Using the results obtained in the previous section, we will give the list of all the possible general fibers of an elementary divisorial contraction supported by $K_X + \tau L$, according to the values of τ, with $n-4 \leq \tau \leq n-1$, *in the hypothesis that these fibers are irreducible.*
The irreducibility of these fibers is studied in great detail in the next chapter.

Remark 4.3.1. Since $\varphi : X \to W$ is divisorial, Proposition 3.1.15 gives

$$n-1 \geq \dim F \geq \dim \operatorname{Locus}(V_x) \geq \tau L \cdot C, \tag{4.3.2}$$

where F is any non trivial fiber of φ, C is a rational curve in the extremal ray R contracted by φ, whose family of deformations $V \subset \operatorname{Hom}(\mathbb{P}^1, X)$ is generically unsplit and dominates the exceptional divisor E of φ, and x is a general point of $\operatorname{Locus}(V)$ (see Proposition 3.1.15 and Subsection 3.1.2 for details).
Moreover we will assume that V is a minimal dominating family for E, with respect to K_X (see Definition 2.2.18).

4.3.1 $\tau = n-1$

From inequality 4.3.2 we have that

$$L \cdot C = 1 \qquad \text{and} \qquad \dim \operatorname{Locus}(V_x) = \dim F = n-1.$$

Thus we have that $F = E$, i.e. E is mapped by φ to a point in W. Using condition 4.2.6, we have that only the case a) in Section 4.2 is effective, with $q = 1$, and so we have that G is irreducible and

$$(G, L_G, N_{E/X}|_G) = (\mathbb{P}^{n-1}, \mathcal{O}(1), \mathcal{O}(-1)).$$

Moreover, following [AO], we can apply Castelnuovo theorem ([Har77, Theorem V.5.7]) to our case to see that $\varphi : X \to W$ is a smooth blow-down which maps E to a smooth point of W.

4.3.2 $\tau = n-2$

Inequality (4.3.2) gives the following possibilities:

$$L \cdot C = \begin{cases} 2 & \text{with } n = 3 \\ 1 & \text{with } n \geq 3 \end{cases}$$

The first case is not possible. From inequality 4.3.2 we have that

$$\dim G = \dim \mathrm{Locus}(V_x) = 2$$

and so $E = G$ is a del Pezzo surface and $r = \dim \varphi(E) = 0$.
Thus, from condition 4.2.6, we have that only the cases a) and b) in Section 4.2 are effective.
Moreover, since the family V is a minimal dominating family for E, from Proposition 2.2.26 we have that $\operatorname{Pic} E = \mathbb{Z}$ and so E can not be a (2-dimensional) quadric.
Thus we are in case a) of Section 4.2, i.e. $(E, L_E) = (\mathbb{P}^2, \mathcal{O}(1))$ and, since $L \cdot C = 2$, we have that C is a conic in $\mathbb{P}^2$: but the family of conics in $\mathbb{P}^2$ is not a minimal dominating family, since the minimal dominating family is the family of the lines, and so also this case is not possible.
The case $L \cdot C = 1$. From inequality 4.3.2 we have two distinct cases:

$\dim G = n - 1$ and so $r = \dim \varphi(E) = 0$. In this case we have that $G = E$ and so *it is irreducible*. Moreover we have that only the cases a) and b) of Section 4.2 are effective and so we have two possibilities for E, according to the possible values of q:

$$(E, L_E, N_{E/X}) = \begin{cases} (\mathbb{P}^{n-1}, \mathcal{O}(1), \mathcal{O}(-2)), & \text{if } q = 2 \\ (\mathbb{Q}^{n-1}, \mathcal{O}(1), \mathcal{O}(-1)), & \text{if } q = 1 \end{cases}$$

$\dim G = n - 2$ and so $r = \dim \varphi(E) = 1$. In this case only case a) of Section 4.2 is effective, with $q = 1$, and so we have that G *is irreducible* and

$$(G, L_G, N_{E/X}|_G) = (\mathbb{P}^{n-2}, \mathcal{O}(1), \mathcal{O}(-1)).$$

The results are summarized in the following table.

$\dim \varphi(E)$	G	L_G	$N_{E/X}\|_G = -qL_G$
1	$\mathbb{P}^{n-2}$	$\mathcal{O}(1)$	$\mathcal{O}(-1)$
0	$\mathbb{P}^{n-1}$	$\mathcal{O}(1)$	$\mathcal{O}(-2)$
	$\mathbb{Q}^{n-1}$	$\mathcal{O}(1)$	$\mathcal{O}(-1)$

4.3.3 $\tau = n - 3$

Inequality 4.3.2 gives the following possibilities:

$\dim G = n-3$ and so $r = \dim \varphi(E) = 2$. In this case only case a) of Section 4.2 is effective, with $q = 1$. Thus *we have that G is irreducible* and

$$(G, L_G, N_{E/X}|_G) = (\mathbb{P}^{n-3}, \mathcal{O}(1), \mathcal{O}(-1)).$$

$\dim G = n-2$ and so $r = \dim \varphi(E) = 1$. We have that only the cases a) and b) of Section 4.2 are effective, with $q = 2, 1$ respectively.
Notice that if $q = 1$, i.e. *if we are in case b) of Section 4.2, the irreducibility of G is not automatic;* we will prove it in the next chapter. Once proved the irreducibility of G for $q = 1$, we have two possibilities for G, according to the possible values of q:

$$(G, L_G, N_{E/X}|_G) = \begin{cases} (\mathbb{P}^{n-2}, \mathcal{O}(1), \mathcal{O}(-2)), & \text{if } q = 2 \\ (\mathbb{Q}^{n-2}, \mathcal{O}(1), \mathcal{O}(-1)), & \text{if } q = 1 \end{cases}$$

$\dim G = n-1$ and so $r = \dim \varphi(E) = 0$. In this case $G = E$ and so it is irreducible. Moreover only the cases a), b) and c) of Section 4.2 are effective, with $q = 3, 2$ and 1 respectively; thus we have

$$(E, L_E, N_{E/X}) = \begin{cases} (\mathbb{P}^{n-1}, \mathcal{O}(1), \mathcal{O}(-3)), & \text{if } q = 3 \\ (\mathbb{Q}^{n-1}, \mathcal{O}(1), \mathcal{O}(-2)), & \text{if } q = 2 \\ (\text{del Pezzo}, \mathcal{O}(1), \mathcal{O}(-1)), & \text{if } q = 1. \end{cases}$$

The results are summarized in the following table.

$\dim \varphi(E)$	G	L_G	$N_{E/X}\|_G = -qL_G$
2	$\mathbb{P}^{n-3}$	$\mathcal{O}(1)$	$\mathcal{O}(-1)$
1	$\mathbb{P}^{n-2}$	$\mathcal{O}(1)$	$\mathcal{O}(-2)$
	$\mathbb{Q}^{n-2}$	$\mathcal{O}(1)$	$\mathcal{O}(-1)$
0	$\mathbb{P}^{n-1}$	$\mathcal{O}(1)$	$\mathcal{O}(-3)$
	$\mathbb{Q}^{n-1}$	$\mathcal{O}(1)$	$\mathcal{O}(-2)$
	del Pezzo	$\mathcal{O}(1)$	$\mathcal{O}(-1)$

4.3.4 $\tau = n - 4$

In this case the irreducibility of G is not automatic except the cases $\dim G = n-4$ and $\dim G = n-1$ (see below for details). In the next chapter we will study this problem and we will show that G is irreducible, except in one case and we will construct an example that shows that this "exceptional case" is effective.
Inequality 4.3.2 gives the following possibilities:

$\dim G = n-4$ and so $r = \dim \varphi(E) = 3$. In this case only case a) of Section 4.2 is effective, with $q = 1$. Thus *we have that G is irreducible* and

$$(G, L_G, N_{E/X}|_G) = (\mathbb{P}^{n-4}, \mathcal{O}(1), \mathcal{O}(-1)).$$

$\dim G = n-3$ and so $r = \dim \varphi(E) = 2$. We have that only the cases a) and b) of Section 4.2 are effective, with $q = 2, 1$ respectively. *Assuming that G is irreducible*, there are two possibilities for G, according to the possible values of q:

$$(G, L_G, N_{E/X}|_G) = \begin{cases} (\mathbb{P}^{n-3}, \mathcal{O}(1), \mathcal{O}(-2)), & \text{if } q = 2 \\ (\mathbb{Q}^{n-3}, \mathcal{O}(1), \mathcal{O}(-1)), & \text{if } q = 1 \end{cases}$$

$\dim G = n-2$ and so $r = \dim \varphi(E) = 1$. Moreover only the cases a), b) and c) of Section 4.2 are effective, with $q = 3, 2$ and 1 respectively; *assuming that G is irreducible*, we have

$$(G, L_G, N_{E/X}|_G) = \begin{cases} (\mathbb{P}^{n-2}, \mathcal{O}(1), \mathcal{O}(-3)), & \text{if } q = 3 \\ (\mathbb{Q}^{n-2}, \mathcal{O}(1), \mathcal{O}(-2)), & \text{if } q = 2 \\ (\text{del Pezzo}, \mathcal{O}(1), \mathcal{O}(-1)), & \text{if } q = 1. \end{cases}$$

$\dim G = n-1$ and so $r = \dim \varphi(E) = 0$. In this case $G = E$ and so *it is irreducible*. Moreover the cases a), b), c) and d) of Section 4.2 are effective, with $q = 4, 3, 2$ and 1 respectively; thus we have

$$(E, L_E, N_{E/X}) = \begin{cases} (\mathbb{P}^{n-1}, \mathcal{O}(1), \mathcal{O}(-4)), & \text{if } q = 4 \\ (\mathbb{Q}^{n-1}, \mathcal{O}(1), \mathcal{O}(-3)), & \text{if } q = 3 \\ (\text{del Pezzo}, \mathcal{O}(1), \mathcal{O}(-2)) & \text{if } q = 2 \\ (\text{Mukai}, \mathcal{O}(1), \mathcal{O}(-1)) & \text{if } q = 1. \end{cases}$$

The results, in the hypothesis that G is irreducible, are summarized in the following table.

$\dim \varphi(E)$	G	L_G	$N_{E/X}\|_G = -qL_G$
3	$\mathbb{P}^{n-4}$	$\mathcal{O}(1)$	$\mathcal{O}(-1)$
2	$\mathbb{P}^{n-3}$	$\mathcal{O}(1)$	$\mathcal{O}(-2)$
	$\mathbb{Q}^{n-3}$	$\mathcal{O}(1)$	$\mathcal{O}(-1)$
1	$\mathbb{P}^{n-2}$	$\mathcal{O}(1)$	$\mathcal{O}(-3)$
	$\mathbb{Q}^{n-2}$	$\mathcal{O}(1)$	$\mathcal{O}(-2)$
	del Pezzo	$\mathcal{O}(1)$	$\mathcal{O}(-1)$
0	$\mathbb{P}^{n-1}$	$\mathcal{O}(1)$	$\mathcal{O}(-4)$
	$\mathbb{Q}^{n-1}$	$\mathcal{O}(1)$	$\mathcal{O}(-3)$
	del Pezzo	$\mathcal{O}(1)$	$\mathcal{O}(-2)$
	Mukai	$\mathcal{O}(1)$	$\mathcal{O}(-1)$

CHAPTER 5

Irreducibility of the general fiber

In this chapter we will study the irreducibility of the general fiber of an elementary divisorial F-M contraction $\varphi : X \to W$, from a smooth n-fold X, supported by $K_X + \tau L$, with $\tau = n-3, n-4$.
In particular we will show that the general fiber of the contractions supported by $K_X + (n-3)L$ is irreducible, while there is a (unique) case of elementary divisorial contraction supported by $K_X + (n-4)L$ in which the general fiber is reducible.
At the end of this chapter we will give an example, suggested to me by Jaroslaw A. Wiśniewski, which shows that such an "exceptional case" exists.

5.1 The case $\tau = n-3$

Proposition 5.1.1. *Let $\varphi : X \to W$ be an elementary divisorial F-M contraction from a smooth n-fold X, supported by $K_X + (n-3)L$ (of course $n \geq 4$). Then the general fiber G of this contraction is irreducible.*

Proof. From Subsection 4.3.3, if E is the exceptional divisor of φ and q is the integer defined by the equality $N_{E/X}|_G = -qL_G$, we have that G is irreducible except the case $\dim \varphi(E) = 1$, $q = 1$ and so, for b) of Section 4.2 we have also that $d = L_G^{\dim G} = 2$.
The rest of the proof is by contradiction.

Suppose G reducible. From Remark 1.6.6 we have that $G = G_1 + G_2$ and that the pair (G_i, L_{G_i}) (with $i = 1, 2$) is a polarized variety with

$$L_{G_i}^{\dim G} = 1.$$

On the other hand, from [AW97, Theorem 1.10], we have that every irreducible component S of each fiber of φ is normal and $\Delta(S, L_S) = 0$. Then $\Delta(G_i, L_{G_i}) = 0$ and $L_{G_i}^{\dim G} = 1$ and so, from Theorem 1.6.2 we have that

$$(G_i, L_{G_i}) = (\mathbb{P}^{n-2}, \mathcal{O}(1)).$$

Vertical slicing. From Subsection 4.1.1 we can construct a F-M contraction, from a smooth variety Y of dimension $n-1$, $\varphi_Y : Y \to Z$, which is divisorial and maps G to a point. Since G is reducible, we have that φ_Y is not elementary (see Proposition 4.1.3) and so φ_Y is the contraction of a face σ of the Mori cone spanned by at least two different extremal rays: R_1 and R_2.

The space $N_1(Y/Z)$. From Lemma 3.1.4, since $N_1(G_i) = \mathbb{R}$, we have that all the curves which are contained in one G_i are numerically proportional in Y: then if C_1 is a curve of R_1 which is contained in G_1, then each curve in G_1 belongs to R_1. Since two extremal rays cannot have a common curve, the curves of R_2 must be in G_2 and so as before each curve in G_2 belongs to R_2.

Conclusion. Since G is connected, from Serre inequality (see [Har77, Theorem I.7.2]) we have that

$$\dim(G_1 \cap G_2) \geq \dim G_1 + \dim G_2 - \dim Y = n - 3 \geq 1;$$

then we can find a curve in $G_1 \cap G_2$ and this curve belongs to R_1 and to R_2. From Lemma 3.1.17 this is impossible, and so we are done. $\square$

In this way, we have that the general fibers of an elementary divisorial contraction from a smooth n-fold supported by $K_X + (n-3)L$ are the ones listed in the previous chapter.

5.2 The case $\tau = n - 4$

This section is devoted to the proof of the following Theorem.

Theorem 5.2.1. *Let X be a smooth $n-$fold, let $\varphi_X : X \to W$ be a divisorial, elementary F-M contraction supported by $K_X + (n-4)L$ and let E be its exceptional divisor. Then the general fiber G of φ_X is irreducible, except in the following case.*
φ_X is the contraction of a 5-fold associated to an extremal ray of length 1, which maps the irreducible divisor E to a curve and its general non trivial fiber is the union of two $\mathbb{P}^2$-bundles of the form $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$, which meet along a quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark 5.2.2. From Inequality 4.3.2, we have that $\dim \varphi_X(E) \leq 3$; Subsection 4.3.4 gives that if $\dim \varphi_X(E) = 0$ or $\dim \varphi_X(E) = 3$, the general fiber of φ_X is irreducible. Moreover if $\dim \varphi_X(E) = 2$ the argument used in the case $\tau = n - 3$ gives that G is irreducible.

Thus in the rest of the Section we will deal with the case $\dim \varphi_X(E) = 1$.

5.2.3 (Setup). From Subsection 4.1.1 with a vertical slice we can produce

$$\varphi : Y \to Z$$

F-M contraction from a smooth m-fold (with $m = n - 1$), supported by

$$K_Y + (n-4)L = K_Y + (m-3)L$$

which maps a divisor G to a point. Moreover

$K_Y|_G = -(m-3)L_G,$

$G|_G = N_{G/Y} = -qL_G$ with $q = 1, 2, 3$ (see Subsection 4.3.4),

and so, by adjunction, $K_G = K_Y|_G + N_{G/Y} = -(m - 3 + q)L_G$.

Strategy. We will study in great detail the contraction $\varphi : Y \to Z$ to show that, apart from the case cited in Theorem 5.2.1, this map is an elementary contraction and thus, thanks to Theorem 4.1.3, G is irreducible.

5.2.4 (Assumption). Suppose that G is reducible, say

$$G = G_1 + \cdots + G_s \quad \text{with } s > 1,$$

and so φ is not elementary; let

$$R_1, \ldots, R_k \quad (k > 1)$$

be the extremal rays that generate the face σ contracted by φ and let

$$\varphi_i : Y \to Z_i$$

the elementary contraction of the ray R_i, for $i = 1, \ldots, k$. From Proposition 1.3.11 the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, "\varphi"] \arrow[dr, "\varphi_i"'] & & Z \\ & Z_i \arrow[ur, "\psi_i"'] & \end{tikzcd}$$

and so the contraction φ_i is birational and it is supported by $K_Y + (m-3)L'$ with L' ample divisor on Y of the form $L' = L + a\varphi_i^*(A_i)$, where $a \gg 0$ and A_i is a ample divisor on Z_i.

5.2.5 (Notations). For each $i = 1, \ldots, s$, we can consider the decomposition of G

$$G = G_i + \widetilde{G}_i,$$

where $\widetilde{G}_i$ is the sum of all the components of G which are different from G_i. Moreover we can consider

$$D_i := \widetilde{G}_i \cap G_i.$$

Since we have assumed that G is reducible, $\widetilde{G}_i$ is effective and since G is a general fiber of a F-M contraction we have that it is connected; this implies that D_i is non empty and thus we have that D_i is an *effective* divisor on G (see Figure 5.1).

In the rest of this section we will show that the divisor D_i, apart from the case cited in Theorem 5.2.1, cannot be effective: thus G must be irreducible.

5.2.1 Deformation of rational curves

One of the main argument to get to the conclusion is a refined study of the families of deformation of rational curves.
With the notations 5.2.5, recalling also the Setup 5.2.3, for each $i = 1, \ldots, s$ we can compute the normal bundle of G_i in Y

$$N_{G_i/Y} = G_i|_{G_i} = G|_{G_i} - \widetilde{G}_i|_{G_i} = -qL_{G_i} - D_i \tag{5.2.6}$$

and so, by adjunction, we have

$$K_{G_i} = -(m - 3 + q)L_{G_i} - D_i.$$

Deformations in Y. Let C be a rational curve in G, let $V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, Y)$ be the family of deformations of C in Y and let x be a general point of $\mathrm{Locus}(V)$.
If V is generically unsplit, since from Proposition 3.1.3 we have

$$-K_Y|_G = (m - 3)L_G,$$

we can state the Ionescu-Wiśniewski inequality (see Corollary 2.2.21):

$$\begin{aligned} &\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1 \geq \\ &\qquad \geq \dim Y - K_Y \cdot C = m + (m - 3)L \cdot C. \end{aligned} \tag{5.2.7}$$

Deformations in G_i. Suppose that the curve C is contained in the irreducible component G_i of G; thus we can consider the deformations of

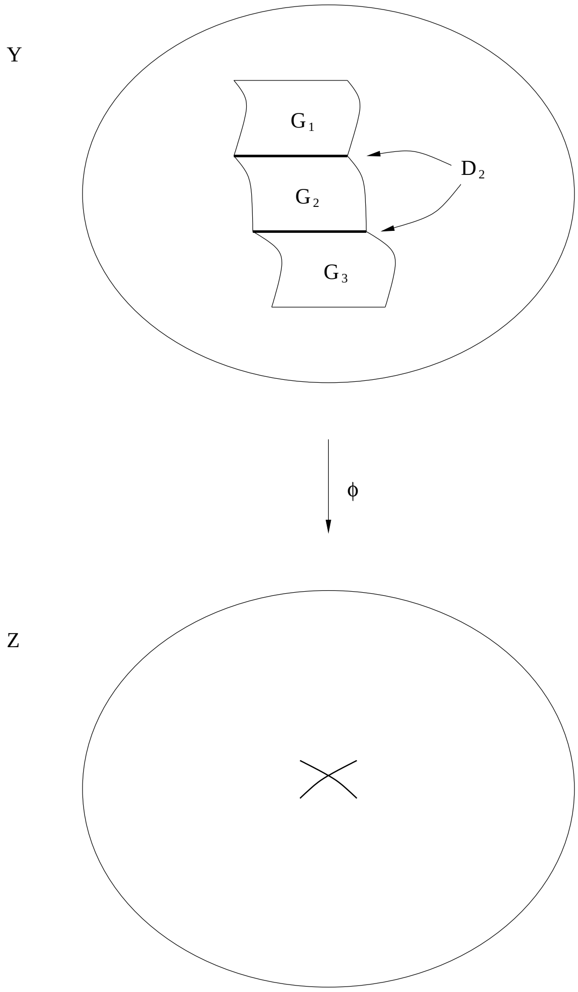


Figure 5.1: The divisor D_i

C in G_i in the following way: every element of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, G_i)$ is an element of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, Y)$ and so we can take an irreducible component T_i of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, G_i)$ which is contained in V; this, under the assumption that V is generically unsplit, together with Subsection 2.2.4, implies that

$$\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1 = \dim V \geq \dim T_i.$$

On the other hand, since G_i is a locally complete intersection subvariety of Y and *if moreover we assume that C meets the smooth locus of G_i*, we can apply the results in the Subsection 2.1.1 to have

$$\dim T_i \geq \dim G_i - K_{G_i} \cdot C$$

and so we obtain the following inequality

$$\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1 \geq \\ \geq m - 1 + (m - 3 + q)L \cdot C + D_i \cdot C \quad (5.2.8)$$

Remark 5.2.9. Since D_i is an effective divisor on G_i, we have the following facts:

1. If C is a curve in G_i which passes through a point x of D_i and $D_i \cdot C = 0$, then the curve C is entirely contained in D_i.

2. If moreover the family V of deformations of C is unsplit and $D_i \cdot C = 0$, we have that $\mathrm{Locus}(V_x) \subset D_i$.

Lemma 5.2.10. *Let C_i be a minimal extremal curve of the ray R_i; then $L \cdot C_i = 1$.*

Proof. Since L is ample, we have that $L \cdot C_i \geq 1$. Suppose, by contradiction, that $L \cdot C_i \geq 2$.
From inequality 5.2.7 and since $m \geq 4$, if we call V the *unsplit* family of deformations of C_i, we have that

$$\dim \mathrm{Locus}(V) = m - 1$$

Let G_i be the irreducible component of G which contains C_i and $\mathrm{Locus}(V)$, then we have that $\mathrm{Locus}(V) = G_i$ (indeed $\mathrm{Locus}(V)$ is an irreducible divisor which is contained in G_i and so coincides with it).
Thus if we apply inequality 5.2.8 we get

$$\dim \mathrm{Locus}(V_x) + 1 \geq 2(m - 2) + D_i \cdot C_i.$$

Since D_i is an effective divisor on G_i and since there exists a curve in V which is not entirely contained in D_i, we have that $D_i \cdot C_i \geq 0$. Then

$$m - 1 \geq \dim \mathrm{Locus}(V_x) \geq 2m - 5 = m - 1 + (m - 4);$$

since $m \geq 4$, the right side term is strictly greater then the left side term, unless $m = 4$.
Thus, from inequality 5.2.8, the unique possible case is

$$\begin{cases} m = 4, \\ q = 1, \\ D_i \cdot C_i = 0, \\ \mathrm{Locus}(V) = \mathrm{Locus}(V_x) = G_i. \end{cases}$$

From the last condition, Proposition 2.2.26 gives that $\rho(G_i) = 1$; thus D_i, which is an effective divisor on G_i, is also ample. This contradicts the condition $D_i \cdot C_i = 0$ and so we are done. □

Remark 5.2.11. Let W be a minimal dominating family for G_i (see Definition 2.2.18), let x be a general point in G_i and let Γ be a curve in W which passes through x. Since W_x is unsplit, the same argument used in the proof of the previous Lemma shows that $L \cdot \Gamma = 1$ and so W is unsplit.

Thus we have the following

Proposition 5.2.12. *For each i, the component G_i is covered by a family of rational curves of degree one with respect to L and so this family is unsplit.*

We conclude this section with a general fact concerning unsplit families of rational curves; for the notations we refer to Chapter 2.

Definition 5.2.13. Let X be a projective variety, let V be an unsplit family of rational curves and let Y be a subset of X.
$\mathrm{Locus}(V)_Y$ is the set of points $x \in X$ such that there exists a curve C of V such that

$$C \cap Y \neq \emptyset, \qquad \text{and} \qquad x \in C,$$

i.e. $\mathrm{Locus}(V)_Y$ is the set of points of X that can be joined to Y by a curve of V.

Lemma 5.2.14. *Let X be a projective variety let Y be a closed subset of X and let V be an unsplit family of rational curves. Then $\mathrm{Locus}(V)_Y$ is closed in X and every curve in $\mathrm{Locus}(V)_Y$ is numerically equivalent to a linear combination with rational coefficients of curves contained in Y and curves of V.*

Proof. Let $H = V/\mathrm{Aut}(\mathbb{P}^1)$ be the image of V in $\mathrm{RatCurves}^n(X)$ and consider the universal family

$$\begin{array}{ccc} U & \xrightarrow{\iota} & X \\ {\scriptstyle\pi}\downarrow & & \\ H & & \end{array}$$

Since V is unsplit, we have that the scheme H and the morphism ι are proper. Let $H(Y) = \pi(\iota^{-1}(Y \cap \mathrm{Locus}(V)))$ be the subset of H parametrizing curves of V meeting Y; $\mathrm{Locus}(V)_Y$ is just $\iota(\pi^{-1}(H(Y)))$; since Y and $\mathrm{Locus}(V)$ are closed, we have that $\mathrm{Locus}(V)_Y$ is closed in X.

Let C be a curve contained in $\mathrm{Locus}(V)_Y$; if $C \subset Y$ or C is a curve parametrized by V we have nothing to prove, so we can suppose that this is not the case.
In particular we have that $\iota^{-1}(C)$ contains an irreducible curve C' which is not contained in a fiber of π and dominates C via ι; let S' be the surface $S' = \pi^{-1}(\pi(C'))$, let B' be the curve $\pi(C') \subset H(Y)$ and let $\nu : B \to B'$ be the normalization of B'. By base change we obtain the following diagram:

$$\begin{tikzcd} S_B \arrow[r, "\bar{\nu}"] \arrow[d] & U \arrow[r, "\iota"] \arrow[d, "\pi"] & X \\ B \arrow[r, "\nu"] & H & \end{tikzcd}$$

Let now $\mu : S \to S_B$ be the normalization of S_B: by standard arguments (see [Wiś89]) it can be shown that S is a ruled surface over the curve B.
Consider now the following diagram, where $j = \mu \circ \bar{\nu} \circ \iota$ and p is the projection:

$$\begin{tikzcd} S \arrow[r, "j"] \arrow[d, "p"'] & X \\ B & \end{tikzcd}$$

Let f be a fiber of p and let C_Y be a curve in S which dominates B and whose image via j is contained in Y; such a curve exists since the image via j of every fiber of p meets Y. Since S is a ruled surface, every curve in S is algebraically equivalent to a linear combination with rational coefficients of C_Y and f.
Therefore every curve in $j(S)$ is algebraically, hence numerically equivalent in X to a linear combination with rational coefficients of $j_*(C_Y)$ and $j_*(f)$; in particular

$$C \equiv \alpha j_*(C_Y) + \beta j_*(f),$$

where $j_*(C_Y)$ is a curve in Y or is the zero cycle, and $j_*(f)$ is a curve of the family V. $\square$

Remark 5.2.15. The proof of the above lemma actually yields that a curve C in $\mathrm{Locus}(V)_Y$ is numerically equivalent to

$$\alpha j(C_Y) + \beta j(f),$$

with $\alpha \geq 0$; in fact, let C_S be an irreducible curve in S which dominates C via j: in S we have that $C_S \equiv \alpha C_Y + \mu f$ and , intersecting with f, we have $\alpha \geq 0$.

5.2.2 Unsplit dominating families

From Proposition 5.2.12, each component of G is covered by an unsplit family of rational curves of degree one with respect to L. For $i = 1, \dots, s$ let W^i be the family which covers G_i and let Γ_i be a rational curve of W^i; we will call this family unsplit dominating family for G_i. Inequality 5.2.8 gives

$$m - 1 \geq \dim \operatorname{Locus}(W^i_x) \geq m - 4 + q + D_i \cdot \Gamma_i \tag{5.2.16}$$

Remark 5.2.17. Since W^i covers G_i we can find a curve Γ_i of W^i which is not contained in the effective divisor D_i and so $D_i \cdot \Gamma_i \geq 0$.

Lemma 5.2.18. *Let G_1 and G_2 be two irreducible components of G with non empty intersection. Then through each point of $G_1 \cap G_2$ passes a curve of W^1 and a curve of W^2 which are contained in $G_1 \cap G_2$.*

Proof. If we show that for each i the divisor D_i is "ruled" by curves of W^i we are done: the common components of D_1 and D_2 are the components of $G_1 \cap G_2$ and so they are "ruled" by curves of W^1 and W^2.
From inequality 5.2.16, we have three cases, according to the dimension of $\operatorname{Locus}(W^i_x)$.

If $\dim \operatorname{Locus}(W^i_x) = m - 3$, inequality 5.2.16 gives that $D_i \cdot \Gamma_i = 0$ for any $x \in G_i$, and so also for $x \in D_i$; thanks to Remark 5.2.9 we get $\operatorname{Locus}(W^i_x) \subset D_i$ and so we are done.

If $\dim \operatorname{Locus}(W^i_x) = m - 2$ inequality 5.2.16 gives two possibilities: $D_i \cdot \Gamma_i = 0, 1$.
If $D_i \cdot \Gamma_i = 0$, Remark 5.2.9 gives that $\operatorname{Locus}(W^i_x) \subset D_i$ and so we are done;
If $D_i \cdot \Gamma_i = 1$, pick a point $y \in D_i$; from the definition of D_i, there exists G_j such that $y \in G_j$, and so

$$\operatorname{Locus}(W^i_y) \cap G_j \subset \operatorname{Locus}(W^i_y) \cap D_i.$$

From Serre inequality, this implies that $\operatorname{Locus}(W^i_y) \cap D_i$ contains a curve; thus there exists a curve Γ in W^i_y which passes through a point of $\operatorname{Locus}(W^i_y) \cap D_i$ different from y: since we have $D_i \cdot \Gamma = 1$, then Γ must be contained in D_i and so we are done.

The case $\dim \operatorname{Locus}(W^i_x) = m - 1$ is not possible; indeed, Proposition 2.2.26 gives that $\rho(G_i) = 1$ and so each curve in G_i is numerically proportional to the curves of W^i, thus the curves in D_i are all multiple up to numerical equivalence; but in the previous steps we have shown that in D_i are contained curves which belongs to unsplit families different from W^i and so we have a contradiction.

□

Corollary 5.2.19. *If W is an unsplit dominating family for a component of G, then* $\dim \operatorname{Locus}(W_x) \leq m-2$.

Proof. The last part of the proof of the previous Lemma gives the assertion. □

Corollary 5.2.20. *The integer q defined by the equality $N_{G/Y} = -qL$ is $q = 1$.*

Proof. Suppose, by contradiction, that $q > 1$. Then, with the notations of the previous lemma, inequality 5.2.16 gives only one possibility:

$$\dim \operatorname{Locus}(W_x^i) = m-2, \text{ and } D_i \cdot \Gamma_i = 0.$$

Thus $\operatorname{Locus}(W_x^i)$ is an irreducible component of D_i with Picard number 1; from the lemma we have that each component of D_i contains curves of two different unsplit families and so its Picard number cannot be 1, contradiction. □

5.2.3 The small rays

Suppose that there is a small ray among the extremal rays R_i's contracted by $\varphi : Y \to Z$.
In this section we will prove each irreducible component of the locus of a small ray meets only one component of G (the one in which it is included).

5.2.21 (Assumption). Suppose that there exists a ray R_1 which is small; since its contraction $\varphi_1 : Y \to Z_1$ is supported by $K_X + (m-3)L'$, Section 3.3 gives the description of this contraction. In particular we have that each connected component of the exceptional locus is isomorphic to $\mathbb{P}^{m-2}$ and its normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Fix one of these components and call it T: of course $T \subset G$ and so there exists a component G_1 such that $T \subset G_1$.

Suppose, by contradiction, that T meets an irreducible component $G_2 \neq G_1$ of G.

Lemma 5.2.22. *The divisor $D_1 \subset G_1$ contains a curve ℓ of T.*

Proof. From the definition of D_1, we have that

$$\emptyset \neq T \cap G_2 \subset G_1 \cap G_2 \subset D_1$$

and, from Serre inequality, $\dim T \cap G_2 \geq m-3 \geq 1$. □

Let $\overline{D}$ be an irreducible component of $G_1 \cap G_2$ which contains the curve ℓ. From Subsection 5.2.2, we have that through each point of $\overline{D}$ passes a curve Γ_1 of W^1 and a curve Γ_2 of W^2 which are contained in $\overline{D}$.

Claim. For each curve C in $\overline{D}$ it holds that $C \equiv a\Gamma_1 + b\Gamma_2$ (in Y), with $a, b \in \mathbb{Q}$ and $a, b \geq 0$.

Proof of the Claim. Consider first $\mathrm{Locus}(W^2)_{\Gamma_1} \subseteq G_2$ (see Definition 5.2.13): this is a closed subset of X and it holds that

$$\mathrm{Locus}(W^2)_{\Gamma_1} = \bigcup_{x \in \Gamma_1} \mathrm{Locus}(W^2_x)$$

Thus we have two possibilities, according to the dimension of $\mathrm{Locus}(W^2_x)$:

If $\dim \mathrm{Locus}(W^2_x) = m - 2$, we have that $\dim \mathrm{Locus}(W^2)_{\Gamma_1} = m - 1$, and so $\mathrm{Locus}(W^2)_{\Gamma_1} = G_2$.

If $\dim \mathrm{Locus}(W^2_x) = m - 3$, we have that $\dim \mathrm{Locus}(W^2)_{\Gamma_1} = m - 2$ and moreover we have that $\mathrm{Locus}(W^2)_{\Gamma_1}$ is contained in $\overline{D}$ (since in this case $\mathrm{Locus}(W^2_x) \subset \overline{D}$ for every $x \in \overline{D}$, see Proof of Lemma 5.2.18) and so must coincide with it.

In both cases, we have that $\overline{D} \subseteq \mathrm{Locus}(W^2)_{\Gamma_1}$.
Then, thanks to Lemma 5.2.14 and Remark 5.2.15, there exist $a_1, b_1 \in \mathbb{Q}$, with $a_1 \geq 0$, such that

$$C \equiv a_1\Gamma_1 + b_1\Gamma_2.$$

We can repeat this argument with $\mathrm{Locus}(W^1)_{\Gamma_2} \subseteq G_1$ to show that there exist $a_2, b_2 \in \mathbb{Q}$, with $b_2 \geq 0$, such that

$$C \equiv a_2\Gamma_1 + b_2\Gamma_2.$$

Since $[\Gamma_1], [\Gamma_2] \in N_1(Y)$ are linearly independent, the decomposition of $[C]$ is unique, and so

$$a = a_1 = a_2 \geq 0, \qquad b = b_1 = b_2 \geq 0$$

$\square$

Hence, since ℓ is a curve in $\overline{D}$, there exist $\alpha, \beta \in \mathbb{Q}$, with $\alpha, \beta \geq 0$ such that

$$\ell \equiv \alpha\Gamma_1 + \beta\Gamma_2.$$

Since ℓ is extremal in Y we have that at least one of Γ_1 and Γ_2 belongs to the extremal ray R_1, and this is not possible.

Thus we have shown that each irreducible component of the locus of a small ray meets only one component of G (the one in which it is included).

Corollary 5.2.23. *Let C_i be a minimal extremal curve whose ray R_i is small and such that* $\mathrm{Locus}(R_i) \subset G_i$*; we have*

$$\begin{cases} G_j \cdot C_i = 0 & i \neq j \\ G_i \cdot C_i = -1 \end{cases}$$

Proof. Since $L \cdot C_i = 1$, we have

$$G \cdot C_i = -L \cdot C_i = -1.$$

On the other hand we have that

$$G \cdot C_i = G_1 \cdot C_i + \cdots + G_i \cdot C_i + \ldots$$

and C_i meets only G_i. □

5.2.4 The divisorial rays

Fix one component of G, say G_1.

Remark 5.2.24. Since G is connected, we can find a curve $\Gamma \subset G$ which has positive intersection with G_1 and is not contained in it; since Γ belongs to the face σ, it is a positive linear combination of extremal rational curve and so we can find a minimal extremal curve, say C_2, which has positive intersection with G_1.

Lemma 5.2.25. *The extremal ray R_2, generated by C_2, is divisorial and if we call $G_2 =$* $\mathrm{Locus}(R_2)$*, we have that G_2 meets G_1 and $G_2 \neq G_1$.*

Proof. Since C_2 has *positive* intersection with one component of G, from the previous section we have that R_2 is divisorial; moreover it is clear that $\mathrm{Locus}(R_2)$ meets G_1.
Thus we have to show that $\mathrm{Locus}(R_2) \neq G_1$. Since C_2 is an extremal curve, we have that $G \cdot C_2 = -L \cdot C_2 = -1$ and so there exists one component of G which has *negative* intersection number with C_2; hence this irreducible divisor is different from G_1 and it contains $\mathrm{Locus}(R_2)$ and so coincides with it. □

Summarizing, we have the following

5.2.26 (Setup). G_1 is a component of G, R_2 is an extremal ray, generated by the minimal rational curve C_2, whose locus is G_2, which has positive intersection with G_1. Thus we have that $G_1 \cap G_2 \neq \emptyset$.

Remark 5.2.27. If R_i is an extremal ray which is divisorial and if we let $G_i = \mathrm{Locus}(R_i)$, we can apply inequality 5.2.8; moreover, since the divisor D_i is effective on G_i, it holds that $D_i \cdot C_i \geq 0$.

Thus, for the extremal curve C_2 and the component G_2, inequality 5.2.8 gives

$$m - 1 \geq \dim \mathrm{Locus}(V_x) \geq m - 3 + D_2 \cdot C_2 \geq m - 3. \tag{5.2.28}$$

Lemma 5.2.29. *The contraction φ_2 of the ray R_2 does not map its exceptional divisor G_2 to a point.*

Proof. Suppose, by contradiction that

$$\dim \varphi_2(G_2) = 0.$$

Then the curves in G_2 and so also the ones in D_2 are all numerically proportional; but this is not possible, since D_2 contains curves of the unsplit dominating family of G_1. $\square$

Proposition 5.2.30. *G_2 is a $\mathbb{P}^{m-2}$-bundle over a curve (i.e. $\dim \varphi_2(G_2) = 1$) and the divisor G_2 meets only G_1.*

Proof. Thanks to Lemma 5.2.29, if V^2 is the unsplit family of deformations of C_2 and x is a point of G_2, we have $\dim \mathrm{Locus}(V^2_x) \leq m - 2$ and so inequality 5.2.28, gives

$$0 \leq D_2 \cdot C_2 \leq 1 :$$

recalling the definition of D_2, we have

$$0 \leq D_2 \cdot C_2 = G_1 \cdot C_2 + G_3 \cdot C_2 + \cdots \leq 1$$

and all the terms are non negative. This means that the curve C_2 has positive intersection with at most one component of G which does not include it.
Moreover we have chosen the curve C_2 such that $G_1 \cdot C_2 > 0$. Thus we have

$$0 < G_1 \cdot C_2 = G_1|_{G_2} \cdot C_2 \leq D_2 \cdot C_2 \leq 1 \tag{5.2.31}$$

and so $D_2 \cdot C_2 = 1$.
Hence, from inequality 5.2.28, we have that

$$\dim \mathrm{Locus}(V^2_x) = m - 2.$$

This implies, from Section 4.3.3, that the morphism

$$\varphi_2 : Y \to Z_2$$

makes G_2 into a $\mathbb{P}$-bundle on a smooth curve with $(m-2)$-dimensional fibers f_2.
Moreover, since $G_1 \cdot C_2 = 1$, we have that the fibers f_2 meet G_2 transversally and so $\dim(f_2 \cap G_1) = m - 3$.
Thus the first part is done.

Claim: G_2 meets only G_1. From inequality 5.2.31, we have that if there is another component G_3 of G which meets G_2, we have that

$$G_3 \cdot C_2 = G_3|_{G_2} \cdot C_2 = 0$$

and so, from Remark 5.2.9 and since the fibers f_2 are $m-2$-dimensional, we have that $G_3 \cap G_2$ is the finite union of the fibers f_2 which meet it and so each curve in this intersection belongs to R_2.
But, from Section 5.2.2, through each point of $G_3 \cap G_2$ passes a curve of the unsplit dominating family W^3 and these curve can not belong to R_2 and so G_3 can not meet G_2. □

Remark 5.2.32. Since G is connected, there is an extremal curve C_1 in G which has positive intersection with G_2. Since G_2 meets only G_1, we have that C_1 is in G_1, its ray is divisorial and so $\mathrm{Locus}(R_1) = G_1$.

Repeating the argument used before we can prove the following

Lemma 5.2.33. *G_1 meets only G_2 and it is a $\mathbb{P}$-bundle over a curve.*

Claim 5.2.34. This case is possible only when $m = \dim Y = 4$.

Proof. Thus $G = G_1 + G_2$ and for $i = 1, 2$

$$\varphi_i : Y \to Z_i$$

makes G_i into a $\mathbb{P}$-bundle on a smooth curve with $(m-2)$-dimensional fibers f_i. Of course, there exist fibers f_1 and f_2 which meet and so, using Serre inequality, we have

$$\dim(f_1 \cap f_2) \geq m - 4.$$

Thus if $m \geq 5$ there exists a curve which belongs to two extremal rays, and this is impossible. □

Description of G

Now we can give a precise description of G.
From what we said before, we have that $G = G_1 + G_2$ with G_1 and G_2 which are $\mathbb{P}^2$-bundles over smooth curves; moreover, for $i = 1, 2$, each G_i is the

locus of a divisorial ray R_i, generated by a minimal extremal curve C_i such that, if $j \neq i$,

$$\begin{cases} G_j \cdot C_i = 1 \\ G_i \cdot C_i = -2 \end{cases}$$

Moreover, if as usual we denote $D = G_1 \cap G_2$, we have

$$D \cdot C_i = 1$$

either in G_1 or in G_2 (see the proof of Proposition 5.2.30).

Remark 5.2.35. Since the G_1 and G_2 are interchangeable, we will deal on with only one component of G, say G_1: everything can be repeated with G_2.

For a brief overview of the general theory of $\mathbb{P}$-bundle as well as for the notations, we refer to the Appendix.

G_1 is a $\mathbb{P}^2$-bundle over a rational curve. It suffices to prove that this $\mathbb{P}^2$-bundle has a (multi)section which is rational. We have already shown that in G_1 there are curves which are minimal extremal curves of the ray R_2: these curves cannot be included in the fibers of the $\mathbb{P}^2$-bundle structure and so they are "transverse"; this means that they are rational (multi)sections, and so we are done.

The vector bundle $\mathcal{E}$. Since G_1 is a $\mathbb{P}^2$-bundle over a smooth curve, there exists $\mathcal{E}$ a rank 3 vector bundle over $\mathbb{P}^1$ such that G_1 coincides with its projectivization $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$.
Moreover $\mathcal{E}$ is decomposable and we can assume that it is of the form:

$$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(a) \oplus \mathcal{O}(b)$$

with $b \geq a \geq 0$.

C_2 is a section. To show this fact we have to look at the Mori cone of G_1 and at the extremal face σ of Y contracted by φ.
The cone of effective 1-cycles of G_1, $\overline{NE}(G_1)$, is 2-dimensional and generated by C_1, which is a line in the fiber, and C_0, which is a section with minimal intersection number with the tautological bundle $\xi_{\mathcal{E}}$ among the sections of G_1 (see Figure 5.2).
The face σ is generated by two extremal rational curves that are C_1 and C_2 (see Figure 5.3).

Claim 5.2.36. The curve C_2 is the section with minimal intersection number with the tautological bundle $\xi_{\mathcal{E}}$ among the sections of G_1, i.e. $C_2 = C_0$ in the notations used in the Appendix.

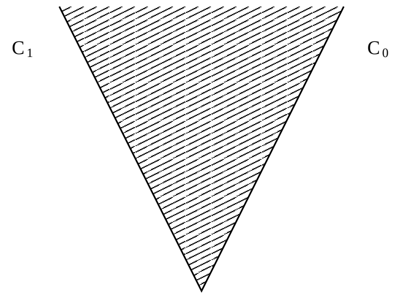


Figure 5.2: The cone $\overline{NE}(G_1)$

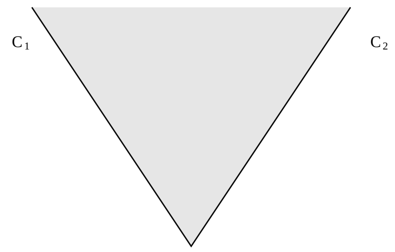


Figure 5.3: The extremal face σ of Y

Proof. Consider the image of $\overline{NE}(G_1)$ in $\overline{NE}(Y)$: since G_1 is contracted by φ, this image is contained in σ (Figure 5.4).
On the other hand, we have shown that C_2 is a (multi)section of G_1; thus we have that C_2 is included in the image of $\overline{NE}(G_1)$ in $\overline{NE}(Y)$ and so also σ is included in it (Figure 5.5).
Thus we have that the two subcones $\overline{NE}(G_1)$ and σ of $\overline{NE}(Y)$ must coincide and so we have that $C_2 \equiv C_0$ is a section of G_1 with minimal intersection number with the tautological bundle $\xi_{\mathcal{E}}$. □

Remark 5.2.37. Thanks to the choice of the normalization of $\mathcal{E}$, we have $\xi_{\mathcal{E}} \cdot C_2 = 0$ in G_1.

Proposition 5.2.38. *The vector bundle $\mathcal{E}$ is of the form*

$$\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2).$$

Proof. First of all, notice that there is a one parameter family of sections of G_1 with minimal intersection number with $\xi_{\mathcal{E}}$:
each fiber f_2 of the $\mathbb{P}^2$-bundle G_2 meets G_1 and the intersection is a curve which is algebraically (and so numerically) equivalent to C_2; since there is

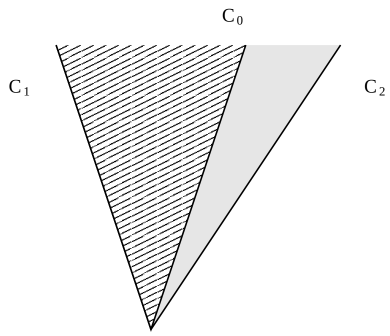


Figure 5.4: The image of $\overline{NE}(G_1)$ is in σ

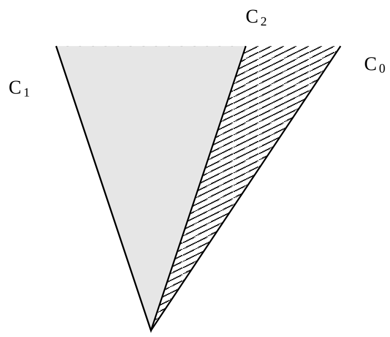


Figure 5.5: σ is in the image of $\overline{NE}(G_1)$

a one parameter family of fibers f_2, there is also a one parameter family of sections C_2.
Hence from a result in the Appendix we have that

$$\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(b).$$

Each divisor Z on $\mathbb{P}(\mathcal{E})$ can be written as

$$Z = a_Z \xi_{\mathcal{E}} + b_Z f,$$

where f is a fiber of the $\mathbb{P}^2$-bundle structure of G_1. Using the intersection numbers with the curves C_1 and C_2 we can compute the following decompositions:

$$\begin{cases} G_1|_{G_1} = a_1 \xi_{\mathcal{E}} + b_1 f \\ G_2|_{G_1} = a_2 \xi_{\mathcal{E}} + b_2 f \end{cases}$$

Recalling that C_1 is a line in the fiber and C_2 is a section of G_1, we have

$$\begin{cases} \xi_{\mathcal{E}} \cdot C_1 = 1, & f \cdot C_1 = 0 \\ \xi_{\mathcal{E}} \cdot C_2 = 0, & f \cdot C_2 = 1 \end{cases}$$

Thus, using C_1, we have

$$\begin{cases} -2 & = G_1|_{G_1} \cdot C_1 = a_1 \xi_{\mathcal{E}} \cdot C_1 + b_1 f \cdot C_1 = a_1 + 0 = a_1 \\ 1 & = G_2|_{G_1} \cdot C_1 = a_2 \xi_{\mathcal{E}} \cdot C_1 + b_2 f \cdot C_1 = 0 + b_2 = b_2 \end{cases}$$

and using C_2 we have

$$\begin{cases} 1 & = G_1|_{G_1} \cdot C_2 = a_1 \xi_{\mathcal{E}} \cdot C_2 + b_1 f \cdot C_2 = 0 + b_1 = b_1 \\ 1 & = G_2|_{G_1} \cdot C_2 = a_2 \xi_{\mathcal{E}} \cdot C_2 + b_2 f \cdot C_2 = a_2 + 0 = a_2 \end{cases}$$

and so we have the following expressions:

$$\begin{cases} G_1|_{G_1} = -2\xi_{\mathcal{E}} + f \\ G_2|_{G_1} = \xi_{\mathcal{E}} + f \end{cases}$$

Recalling the formula of the canonical bundle of a projective bundle, we have

$$K_{G_1} + 3\xi_{\mathcal{E}} = \varphi_1^*(K_{\mathbb{P}^1} + \det \mathcal{E})$$

and so, since in our case $\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(b)$,

$$K_{G_1} = -3\xi_{\mathcal{E}} + (b-2)f.$$

On the other hand, since $-K_Y = L$, we have that $-K_Y \cdot C_i = 1$ and so, arguing as before, $K_Y|_{G_1} = -\xi_{\mathcal{E}} - f$; thus using adjunction formula we have

$$K_{G_1} = K_Y|_{G_1} + G_1|_{G_1} = -\xi_{\mathcal{E}} - f - 2\xi_{\mathcal{E}} + f = -3\xi_{\mathcal{E}},$$

and so $b = 2$; hence we get

$$\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2).$$

□

Proposition 5.2.39. *$D = G_1 \cap G_2$ is a quadric.*

Proof. Recall that D can be seen in G_1 as the locus of the one parameter family of sections C_2.
Consider the surjections of vector bundles over $\mathbb{P}^1$:

$$\mathcal{E} \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O} \to 0,$$

which corresponds to the injections of projective bundles over $\mathbb{P}^1$:

$$0 \hookrightarrow \mathbb{P}(\mathcal{O}) \hookrightarrow \mathbb{P}(\mathcal{O} \oplus \mathcal{O}) \hookrightarrow \mathbb{P}(\mathcal{E}).$$

This shows that in $\mathbb{P}(\mathcal{E})$ there is a quadric (which is $\mathbb{P}(\mathcal{O}\oplus\mathcal{O})$) as a subbundle. Moreover we have that the inclusions of the sections C_2 in $\mathbb{P}(\mathcal{E})$, factor through the inclusion of this quadric. In particular we have that the sections of the quadric are the curves C_2 and so this quadric is the locus of the one parameter family of this section and so it coincides with D. □

This concludes the description of G and the proof of the Theorem 5.2.1. In the following section we will construct an example of elementary contraction with reducible fibers to show that our "exceptional case" is effective.

5.3 Examples

These examples were kindly suggested to me by Jaroslaw A. Wiśniewski.

We begin with an example in dimension 3 of a crepant elementary contraction which contracts an irreducible divisor to a curve and such that any positive dimensional fiber consists of two rational curves. This will give the idea of the construction of our real example. We will use toric geometry to make it explicit.

5.3.1 Crepant resolution with a $\mathbb{Z}_2$-action and Moebius strip construction

We construct the example in two steps: first of all we consider a non elementary birational crepant contraction of surfaces whose exceptional locus consists of two rational curves and then from this we will construct the desired crepant contraction.

Let $\phi : S_2 \to A_2$ be a resolution of an A_2 surface singularity $x^2 + y^2 + z^3 = 0$. The morphism ϕ is crepant (i.e. K_{S_2} is ϕ-trivial) and its exceptional locus consists of two (-2)-curves. Changing x with y gives a $\mathbb{Z}_2$-action which interchanges the (-2)-curves in S_2.

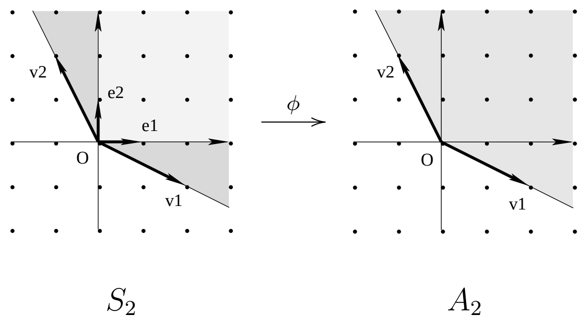


Figure 5.6: Toric representation of the resolution of the A_2 singularity

Now we give a description of this resolution by means of toric geometry.
Let e_1, e_2 be the generators of a two dimensional lattice and let $v_1 = 2e_1 - e_2$ and $v_2 = 2e_2 - e_1$.
The surface singularity A_2 is given by the cone $\sigma = \langle v_1, v_2 \rangle$ and its resolution $\phi : S_2 \to A_2$ is obtained subdividing the cone σ with the vectors e_1 and e_2, i.e. S_2 is given by the fan Δ containing the two dimensional cones $\langle v_2, e_2 \rangle$, $\langle e_2, e_1 \rangle$, $\langle e_1, v_1 \rangle$ and their rays (see Figure 5.6). This subdivision corresponds to the introduction of the two (-2)-curves in the resolution (which are the orbits of the torus action of the rays e_1 and e_2).
Moreover the $\mathbb{Z}_2$-action is the interchanging of the generators of the lattice e_1, e_2 and hence also v_1, v_2.
The fact that the morphism ϕ is crepant can be seen easily from the picture: the boundary of the convex hull of the origin O and the four vectors e_1, e_2, v_1, v_2 that generate Δ is flat in a neighborhood of e_1 and e_2 and this implies that the canonical divisor of S_2 is trivial along the two (-2)-curves.

Now we are ready to construct the elementary contraction of threefolds such that any positive dimensional fiber consists of two rational curves; in fact the vertical slicing of this contraction is the resolution ϕ constructed before.
Let C' be a smooth curve with a free $\mathbb{Z}_2$-action, so that the action induces an étale covering $C' \to C$ of degree 2. Take the product actions $\pi : S_2 \times C' \to X_3$ and $\pi' : A_2 \times C' \to Y_3$, where of course X_3 and Y_3 are the quotients; these product actions are free and so X_3 is smooth.

Remark 5.3.1 (Universal property of quotients of a group action). Let A be a variety with a group action $(G, \cdot)$ and let $\psi : A \to B$ be the quotient map. Then every morphism $\rho : A \to D$ to another variety D factors through ψ iff $\rho(p) = \rho(g \cdot p)$, for every $p \in A$ and every $g \in G$.

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\psi"'] & D \\ B \arrow[ur] & \end{tikzcd}$$

In our case, if $\rho = \pi' \circ (\phi \times \mathbb{1}) : S_2 \times C' \to Y_3$, then the universal property gives a morphism $\varphi : X_3 \to Y_3$ such that the following diagram commutes:

$$\begin{tikzcd} S_2 \times C' \arrow[r, "\phi\times\mathbb{1}"] \arrow[d, "\pi"'] & A_2 \times C' \arrow[d, "\pi'"] \\ X_3 \arrow[r, "\varphi"] & Y_3 \end{tikzcd} \tag{5.3.2}$$

The exceptional locus of φ is as follows. Let E' be the union of the two (-2)-curves in S_2; the map φ is clearly birational and it maps its exceptional locus, which is $E = \pi(E' \times C')$, to the curve $C = \pi'(\{O\} \times C') \subset Y_3$.

$$\begin{tikzcd} E' \times C' \arrow[r, "\phi\times\mathbb{1}"] \arrow[d, "\pi"'] & \{O\} \times C' \arrow[d, "\pi'"] \\ E \arrow[r, "\varphi"] & C \end{tikzcd}$$

Claim 5.3.3. The map $\varphi : X_3 \to Y_3$ is a crepant elementary contraction contracting the divisor E to the curve C and any positive dimensional fiber consists of two rational curves.

Proof. We have already shown that φ is a birational morphism from a smooth threefold which maps E to a curve; since π is finite we have that $\dim E = \dim(E' \times C') = 2$, and so E is a divisor on X_3.

To study the positive dimensional fibers we consider a "vertical slicing" of the diagram 5.3.2.

Let $c \in C$ be a point and let $\{O\} \times \{c'_1, c'_2\} \in \{O\} \times C'$ be inverse image of c under π'. Take a very ample divisor A on Y_3 which meets C in c; after possibly shrinking Y_3 to an affine open subset we can suppose that c is the unique point of the intersection. Call $S = \varphi^*A$ and consider the restriction of the diagram 5.3.2 to these "hyperplane sections". The exceptional locus of $\varphi_{|S}$ is the fiber of φ over $c \in C$; moreover the vertical arrows are trivial étale covering and so the diagram is as follows:

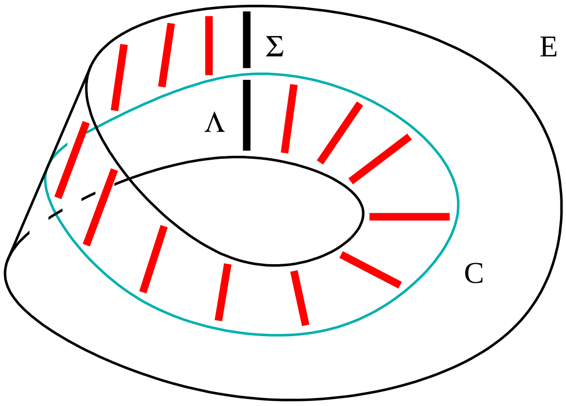


Figure 5.7: The curve Σ is algebraically equivalent to the curve Λ

$$\begin{array}{ccc} S \times \{c'_1, c'_2\} & \xrightarrow{\phi \times \mathbb{1}} & A \times \{c'_1, c'_2\} \\ \downarrow{\scriptstyle \pi_{|S\times\{c'_1,c'_2\}}} & & \downarrow{\scriptstyle \pi'_{|A\times\{c'_1,c'_2\}}} \\ S & \xrightarrow{\varphi_{|S}} & A \end{array}$$

Thus the map $\varphi_{|S}$ is the map ϕ and so the contraction $\varphi : X_3 \to Y_3$ is a crepant contraction such that any positive dimensional fiber consists of two rational curves.

The last thing to prove is the fact that the contraction is elementary, i.e. that all the curves that are contracted are numerically proportional.

Let Γ' be one of the two rational curves contracted by ϕ and let $\Gamma = \pi(\Gamma' \times c') \subset X_3$; note that Γ and Γ' are isomorphic.

We will show that the flat family of rational curves $\Gamma' \times C'$ parametrizes the curves that are contacted by φ (Figure 5.7 gives an intuitive idea of this fact). The product $\Gamma' \times C'$ is sent onto E by π; moreover each curve in E which is contracted by φ is the image of (Γ', c') for some $c' \in C'$: then all the curves contracted by φ are algebraically and so numerically equivalent.

□

5.3.2 Fano Mori contraction with a $\mathbb{Z}_2$-action and Moebius strip construction

We will construct an example of an elementary Fano-Mori contraction of a smooth fivefold contracting a divisor to a curve and such that any positive

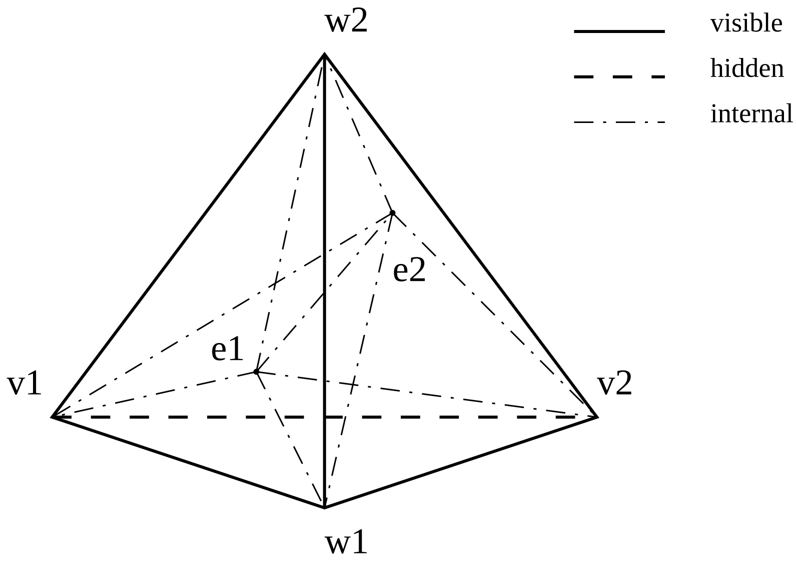


Figure 5.8: Three dimensional model of Δ

dimensional fiber is reducible (and it is as we have predicted in the previous section).
Again the construction will be in two steps: first we will construct the general fiber of the contraction and then we will fit it in a suitable five dimensional manifold.
To construct the fibers we will use toric geometry.
Let e_1, e_2, v_1, v_2 be a basis of a four dimensional lattice and let $w_1 = 2e_1 - e_2 - v_1$ and $w_2 = -e_1 + 2e_2 - v_2$. Let Δ be the fan generated by these six vectors and containing the following maximal cones:

$$\begin{array}{llll} \langle e_1, v_1, w_1, v_2 \rangle & \langle e_1, v_1, w_1, w_2 \rangle & \langle e_1, e_2, v_1, w_2 \rangle & \langle e_1, e_2, v_1, v_2 \rangle \\ \langle e_2, v_2, w_2, v_1 \rangle & \langle e_2, v_2, w_2, w_1 \rangle & \langle e_1, e_2, v_2, w_1 \rangle & \langle e_1, e_2, w_1, w_2 \rangle \end{array}$$

Let X_4 be the variety associated to this fan and let Y_4 be the affine toric variety associated to the cone $\sigma = \langle v_1, v_2, w_1, w_2 \rangle$. The fan Δ is a subdivision of σ, obtained introducing the vectors e_1 and e_2, and let $\phi : X_4 \to Y_4$ be the proper birational morphism associated to this subdivision (see Figures 5.8,5.9).

Claim 5.3.4. The map $\phi : X_4 \to Y_4$ is a non elementary divisorial F-M contraction from a smooth fourfold whose exceptional locus consists of two divisors, which are $\mathbb{P}^2$-bundles over $\mathbb{P}^1$ of the form $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O})$, and which intersect along a quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

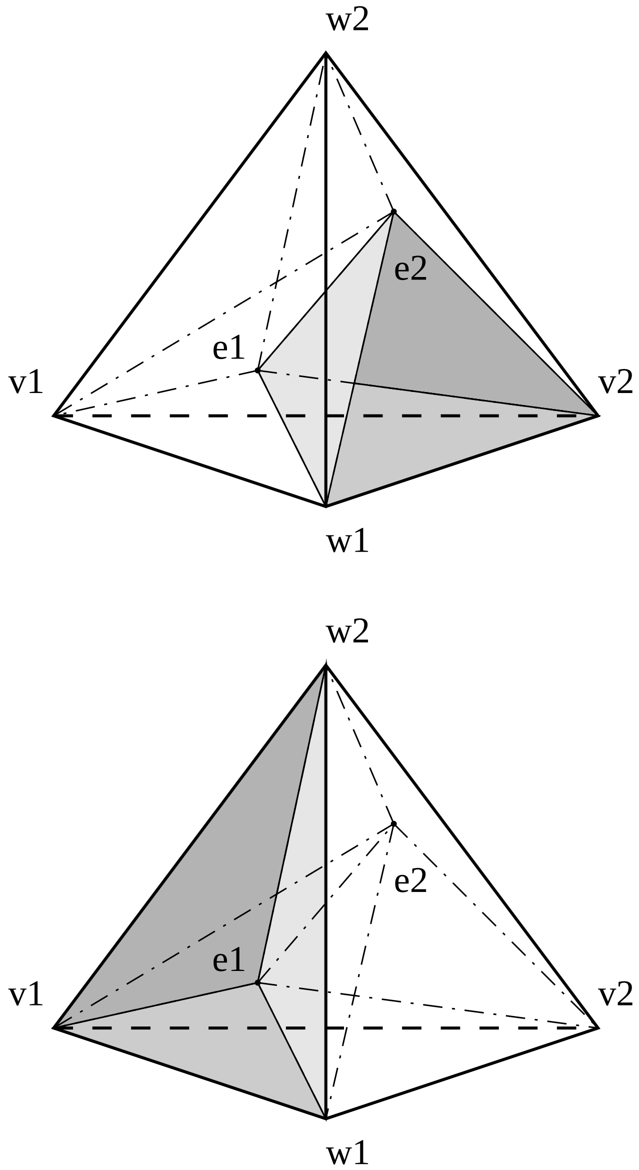


Figure 5.9: Typical elements of the subdivision of Δ

Moreover there is a $\mathbb{Z}_2$-action which interchanges e_1 with e_2, v_1 with v_2 and these two divisors.

Proof. First, notice that there is a $\mathbb{Z}_2$-action in this situation: everything is symmetric with respect to the interchanging of the indices 1 and 2 and so if we prove something for e_1 it is true also for e_2.
We will divide the proof in several steps.

X_4 **is smooth.** To see that the fourfold X_4 is smooth, we have to show that each cone in Δ is generated by a subset of a basis of the lattice and this is a straightforward computation. Notice that it suffices to prove this for the maximal cones in Δ, which are exactly the 8 cones mentioned before.

Exceptional locus of ϕ. The morphism ϕ is proper and birational, since it corresponds to a subdivision of a cone; let E be its exceptional locus.
This consists of two divisors which are the closure of the orbits of the torus action of the two rays ρ_1 and ρ_2 generated by e_1 and e_2 respectively; call them E_1 and E_2. Moreover these divisors meet along a subvariety which is the closure of the orbit of the two dimensional subcone which is generated by e_1 and e_2.
Thanks to the $\mathbb{Z}_2$-action it suffices to study E_1, the closure of the orbit of ρ_1. This divisor corresponds to the star of ρ_1; recall that the star of a cone τ in a fan Δ of cones in a lattice N is the toric variety associated to the fan of cones of Δ which contains τ as a face, seen in the lattice $N(\tau) = N/(\tau \cap N)$.
Let us consider the three dimensional lattice generated by e'_2, v'_1, v'_2. Then E_1 is the toric variety associated to the fan Δ_1 which is generated by the vectors $e'_2, v'_1, v'_2, w'_1, w'_2$, where $w'_1 = -e'_2 - v'_1$ and $w'_2 = 2e'_2 - v'_2$, and contains these maximal cones:

$$\begin{array}{ccc} \langle v'_1, w'_1, v'_2 \rangle & \langle v'_1, w'_1, w'_2 \rangle & \langle e'_2, v'_1, w'_2 \rangle \\ \langle e'_2, v'_1, v'_2 \rangle & \langle e'_2, v'_2, w'_1 \rangle & \langle e'_2, w'_1, w'_2 \rangle \end{array}$$

Since these five vectors satisfy the relations

$$e'_2 + v'_1 + w'_1 = 0, \quad v'_2 + w'_2 = 2e'_2,$$

we have that E_1 (and so E_2) is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ of the form $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O})$ (see [Ful93], Chapter 2, note 12). The intersection D of E_1 and E_2 corresponds to the star of the two dimensional cone generated by e_1 and e_2: in a two dimensional lattice generated by v''_1, v''_2 consider the fan generated by the vectors $v''_1, v''_2, w''_1, w''_2$, where $w''_1 = -v_1$ and $w''_2 = -v_2$, which contains the maximal cones:

$$\langle v''_1, v''_2 \rangle \quad \langle v''_1, w''_2 \rangle \quad \langle v''_2, w''_1 \rangle \quad \langle w''_1, w''_2 \rangle$$

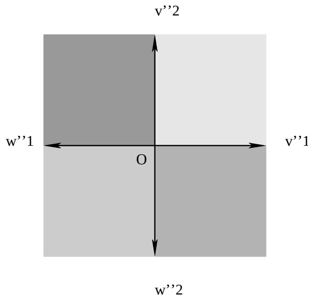


Figure 5.10: Fan associated to a quadric

and this is the fan associated to the quadric (see Figure 5.10).
Notice that, from this toric description of the map ϕ, it is clear that its exceptional locus is mapped to a point (the orbit of the maximal cone σ).

ϕ is a F-M contraction. We have shown that the map ϕ is proper, has connected fibers and maps the reducible divisor described before to a point. To show that ϕ is a Fano Mori contraction it suffices to prove that $-K_{X_4}$ is ample on E, i.e. that for every curve $C \subset E$, $-K_{X_4} \cdot C > 0$.
The cone $\overline{NE(X_4/Y_4)} \subset N_1(X_4)$ is generated by the orbits of the three dimensional cones ω in Δ which have e_1 or e_2 among their one dimensional faces, and these numerical classes of curves correspond to the linear relations between the generators of the four dimensional cones in Δ which are separated by ω. Moreover the coefficient of each generator in these relations is the intersection number of the curve and the divisor that corresponds to the generator.
In our case we have only two relations between these generators:

$$\begin{array}{ll} e_2 + v_1 + w_1 - 2e_1 = 0 & \text{which corresponds to } [C_1] \\ e_1 + v_2 + w_2 - 2e_2 = 0 & \text{which corresponds to } [C_2] \end{array}$$

This implies that, if we call E_i, V_i, W_i the divisors associated to the vectors e_i, v_i, w_i,

$$\begin{array}{ll} C_1 \cdot E_2 = C_1 \cdot V_1 = C_1 \cdot W_1 = 1, & C_1 \cdot E_1 = -2, \\ C_2 \cdot E_1 = C_2 \cdot V_2 = C_2 \cdot W_2 = 1, & C_2 \cdot E_2 = -2, \end{array}$$

while all the other intersection numbers are zero.
The canonical divisor of X_4 is

$$K_{X_4} = -\sum_{i=1}^{2} (E_i + V_i + W_i),$$

and so

$$-K_{X_4} \cdot C_1 = -K_{X_4} \cdot C_2 = 1.$$

This implies that $-K_{X_4}$ is ϕ-ample and so ϕ is a Fano Mori contraction. Moreover $R_1 = \mathbb{R}_+[C_1]$ and $R_2 = \mathbb{R}_+[C_2]$ are the two extremal rays contracted by ϕ (since $E_i \cdot C_i < 0$, we have that $\mathrm{Locus}(R_i) = E_i$). □

Remark 5.3.5. $N_1(E)$ is generated by two classes of rational curves $[C_1]$ and $[C_2]$: in E_1, $[C_1]$ is the class of curves in the fibers and $[C_2]$ is the class of the section of this $\mathbb{P}^2$-bundle corresponding to the surjection $\mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}$, while in E_2 these classes are interchanged; moreover the $\mathbb{Z}_2$ action sends E_1 to E_2 and so interchanges $[C_1]$ and $[C_2]$.

Now exactly the same construction of the crepant contraction of threefold gives the elementary Fano Mori contraction of a fivefold contracting a divisor to a curve and such that any positive dimensional fiber is reducible; in fact the vertical slicing of this contraction is the map $\phi : X_4 \to Y_4$. Let C' be a smooth curve with a free $\mathbb{Z}_2$-action and take the product action $\pi : X_4 \times C' \to X$ and $\pi' : Y_4 \times C' \to Y$, where X, Y are the quotients; these product actions are free and so X is smooth.
Thanks to the universal property of the quotients of the group action (5.3.1), we have the following diagram:

$$\begin{tikzcd} X_4 \times C' \arrow[r, "\phi \times \mathbb{1}"] \arrow[d, "\pi"'] & Y_4 \times C' \arrow[d, "\pi'"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd} \tag{5.3.6}$$

Claim 5.3.7. The map $\varphi : X \to Y$ is the desired F-M contraction, whose exceptional locus is $G = \pi(E \times C')$ (where E is the exceptional locus of ϕ), which is mapped to $C \subset Y$.

Proof. Everything is as in the proof of Claim 5.3.3 except the fact that φ is elementary, and so we will prove only this fact.
Consider the product $C_1 \times C'$: it is a flat family of rational curves and it is mapped by π into the exceptional locus of φ. From the construction, it is clear that for each fiber of φ, whose exceptional locus is $\widetilde{E}$, there are two curves in this family that corresponds to the two generators of $N_1(\widetilde{E})$; thus these two generators are numerically equivalent in X and we are done. □

APPENDIX A

Projectivization of vector bundles

Let $\mathcal{E}$ be a vector bundle of rank $r \geq 2$ over a smooth projective variety M:

$$\pi : \mathcal{E} \to M.$$

Thus we can consider the smooth variety $\mathbb{P}_M(\mathcal{E})$, which is the projectivization of $\mathcal{E}$ over M (in particular it is a fiber bundle over M whose fiber over a point $m \in M$ is the projective space $\mathbb{P}(\mathcal{E}_m^\vee) = \{\text{hyperplanes of } \mathcal{E}_m\}$).
There exists a *tautological* line bundle on $\mathbb{P}(\mathcal{E})$, denoted with $\xi_{\mathcal{E}}$, uniquely determined by the conditions

$$\xi_{\mathcal{E}}|_f = \mathcal{O}_f(1), \qquad\qquad p_*\xi_{\mathcal{E}} = \mathcal{E},$$

where $p : \mathbb{P}(\mathcal{E}) \to M$ is the projection morphism and f is the fiber of p (of course we have that $f = \mathbb{P}^{r-1}$ and $p : \mathbb{P}(\mathcal{E}) \to M$ is a $\mathbb{P}^{r-1}$-bundle).
Explicitly $\xi_{\mathcal{E}}$ is given in the following way. Consider the diagram

$$\begin{tikzcd} p^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "\pi"] \\ \mathbb{P}(\mathcal{E}) \arrow[r, "p"] & M \end{tikzcd}$$

Let $\mathcal{S}$ be the universal subbundle of $p^*\mathcal{E}$ (i.e. the subbundle whose fiber over a point (p, Λ) in $\mathbb{P}(\mathcal{E})$ is $S_{(p,\Lambda)} = \Lambda$), then the tautological bundle is defined by the following exact sequence of vector bundles

$$0 \to \mathcal{S} \to p^*\mathcal{E} \to \xi_{\mathcal{E}} \to 0.$$

The Picard group of $\mathbb{P}(\mathcal{E})$ can be expressed as follows

$$\operatorname{Pic}\mathbb{P}(\mathcal{E}) = \mathbb{Z} \cdot \xi_{\mathcal{E}} \oplus p^*(\operatorname{Pic} M),$$

and so a similar expression holds for $N_1(\mathbb{P}(\mathcal{E}))$.
Moreover we have the following relation between the canonical bundles of $\mathbb{P}(\mathcal{E})$ and of M:

$$K_{\mathbb{P}(\mathcal{E})} + r\xi_{\mathcal{E}} = p^*(K_M + \det \mathcal{E}).$$

If $\mathcal{L}$ is a line bundle on M, replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{L}$, its twist with $\mathcal{L}$, does not affect the projectivization and

$$\xi_{\mathcal{E}\otimes\mathcal{L}} = \xi_{\mathcal{E}} + p^*\mathcal{L}.$$

Claim. There is a natural 1-1 correspondence between *sections* of p, i.e. morphisms

$$s : M \to \mathbb{P}(\mathcal{E})$$

such that $p \circ s = \mathbb{1}_M$, and *surjections* of the vector bundle $\mathcal{E}$ of the form

$$\mathcal{E} \to \mathcal{L} \to 0,$$

where $\mathcal{L}$ is a line bundle on M (or dually *injections* in the dual vector bundle $\mathcal{E}^\vee$ of the form $0 \to \mathcal{L}^\vee \to \mathcal{E}^\vee$).

Explicitly, given a section $s : M \to \mathbb{P}(\mathcal{E})$, we have the following diagram

$$\begin{tikzcd}
s^*p^*\mathcal{E} = \mathcal{E} \arrow[r] \arrow[d] & p^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "\pi"] \\
M \arrow[r, "s"] & \mathbb{P}(\mathcal{E}) \arrow[r, "p"] & M
\end{tikzcd}$$

From the definition of the tautological bundle, we have a line bundle $\mathcal{L} = s^*\xi_{\mathcal{E}}$ on M with the exact sequence

$$0 \to s^*\mathcal{S} \to s^*p^*\mathcal{E} = \mathcal{E} \to s^*\xi_{\mathcal{E}} \to 0$$

and so we have the desired surjection.
Conversely, given a line bundle $\mathcal{L}$ on M and a surjective morphism

$$v : \mathcal{E} \to \mathcal{L},$$

in each fiber $\mathcal{E}_m$ of $\mathcal{E}$ over a point $m \in M$ we can consider the hyperplane

$$\mathrm{Ker}(v_m) \subset \mathcal{E}_m :$$

thus we can define the section s of $\mathbb{P}(\mathcal{E})$:

$$\begin{aligned} M &\xrightarrow{s} \mathbb{P}(\mathcal{E}) \\ m &\longmapsto (m, \mathrm{Ker}(v_m)) \end{aligned}$$

These constructions are inverse to each other; in particular the section $s : M \to \mathbb{P}(\mathcal{E})$ has the property that $s^*\xi_{\mathcal{E}} = \mathcal{L}$.

Suppose now that $M = \mathbb{P}^1$. Then $\mathcal{E}$ is decomposable in the direct sum of line bundles over $\mathbb{P}^1$:

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r),$$

where $a_1 \leq a_2 \leq \cdots \leq a_r$ are integers.
Since there is a surjection of vector bundles over $\mathbb{P}^1$

$$\mathcal{E} \to \mathcal{O}(a_1) \to 0,$$

there is a section $s : \mathbb{P}^1 \to \mathbb{P}(\mathcal{E})$ with $s(\mathbb{P}^1) = C_0 \subset \mathbb{P}(\mathcal{E})$ with the property $s^*\xi_{\mathcal{E}} = \mathcal{O}_{\mathbb{P}^1}(a_1)$.
Thus we have

$$\xi_{\mathcal{E}} \cdot C_0 = \deg \xi_{\mathcal{E}}|_{C_0} = \deg s^*(\xi_{\mathcal{E}}) = a_1$$

and so the term $\mathcal{O}(a_1)$, which is the term of $\mathcal{E}$ with minimal degree, corresponds to a section of $\mathbb{P}(\mathcal{E})$ with the minimal intersection number with the tautological bundle.
Since we can twist the vector bundle $\mathcal{E}$ with a line bundle on M without changing the projective bundle $\mathbb{P}(\mathcal{E})$, we can suppose that the vector bundle $\mathcal{E}$ is of the form

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r),$$

with $a_r \geq \cdots \geq a_2 \geq 0$, and so we have that the section C_0 which has minimal intersection with the tautological bundle is such that

$$\xi_{\mathcal{E}} \cdot C_0 = 0.$$

Moreover this section corresponds to the surjection $\mathcal{E} \to \mathcal{O} \to 0$, or equivalently to the injection $0 \to \mathcal{O} \to \mathcal{E}^\vee$ and so these sections are parameterized by

$$H^0(\mathcal{E}^\vee) = H^0(\mathcal{O} \oplus \mathcal{O}(-a_2) \oplus \cdots \oplus \mathcal{O}(-a_r)).$$

Hence, if on $\mathbb{P}(\mathcal{E})$ there is a one parameter family of sections with minimal intersection with $\xi_{\mathcal{E}}$, we have that $a_2 = 0$.

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