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Γ-CONVERGENCE FOR FUNCTIONALS DEPENDING ON VECTOR FIELDS. II. CONVERGENCE OF MINIMIZERS*

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Abstract. Given a family of locally Lipschitz vector fields $X(x) = (X_1(x), \dots, X_m(x))$ on $\mathbb{R}^n$, $m \le n$, we study integral functionals depending on X. Using the results in [A. Maione, A. Pinamonti, and F. Serra Cassano, *J. Math. Pures Appl.* (9), 139 (2020), pp. 109–142], we study the convergence of minima, minimizers, and momenta of those functionals. Moreover, we apply these results to the periodic homogenization in Carnot groups and prove an H-compactness theorem for linear differential operators of the second order depending on X.

Key words. homogenization, Carnot groups, H-convergence, Γ-convergence, vector fields

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1. Introduction.

In this paper we deal with the asymptotic behavior of minima, minimizers, and momenta, as $h \to \infty$, of the following sequence of minimization problems:

$$\inf \left\{ F_h(u) + G(u) : \, u \in W^{1,p}_X(\Omega), \, u - \varphi \in W^{1,p}_{X,0}(\Omega) \right\}. \tag{1}$$

Here F_h, $G : L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$ denote the functionals

$$F_h(u) := \begin{cases} \int_\Omega f_h(x, Xu(x))dx & \text{if } u \in W^{1,p}_X(\Omega), \\ \infty & \text{otherwise} \end{cases} \tag{2}$$

and

$$G(u) := \int_\Omega g(x, u(x))\, dx \,, \tag{3}$$

with $f_h : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $h \in \mathbb{N}$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ Carathéodory functions and $X(x) := (X_1(x), \dots, X_m(x))$ a given family of first order linear differential operators with Lipschitz coefficients on a bounded open set $\Omega \subset \mathbb{R}^n$, that is,

$$X_j(x) = \sum_{i=1}^{n} c_{ji}(x)\partial_i, \quad j = 1, \dots, m,$$

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with $c_{ji}(x) \in \mathrm{Lip}(\Omega)$ for $j = 1, \dots, m$, $i = 1, \dots, n$.

In the following, we will refer to X and f_h as *X-gradient* and *integrand function*, respectively. The environment spaces $W^{1,p}_X(\Omega)$ and $W^{1,p}_{X,0}(\Omega)$ are the Sobolev spaces associated through the X-gradient in a classical way, according to Folland and Stein [FS] (see Definitions 2.3 and 2.6), and $\varphi \in W^{1,p}_X(\Omega)$ is a given function that plays the role of boundary datum in (1). As usual, we identify each X_j with the vector field

$$(c_{j1}(x), \dots, c_{jn}(x)) \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$$

and we call

$$C(x) = [c_{ji}(x)]_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \tag{4}$$

the *coefficient matrix of the X-gradient*.

Throughout this paper, we will assume some structural conditions on the class of integrand functions and the X-gradient. An integrand function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ will typically satisfy the following conditions:

(I_1) $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is Borel measurable on Ω;

(I_2) for a.e. $x \in \Omega$, the function $f(x, \cdot) : \mathbb{R}^m \to \mathbb{R}$ is convex;

(I_3) there exist two positive constants $c_0 \le c_1$ and two nonnegative functions $a_0, a_1 \in L^1(\Omega)$ such that

$$c_0 \, |\eta|^p - a_0(x) \le f(x, \eta) \le c_1 \, |\eta|^p + a_1(x) \tag{5}$$

for a.e. $x \in \Omega$ and for each $\eta \in \mathbb{R}^m$.

We will denote by $I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ the class of such integrand functions and by $I_{m,p}(\Omega, c_0, c_1)$ if $a_0 = a_1 \equiv 0$. Similarly, function g in (3) will satisfy a suitable growth condition (see (10) and (11)).

As far as the structural conditions on the X-gradient are concerned, we will need two assumptions: the former is an algebraic condition and the latter is a metric condition.

DEFINITION 1.1. *We say that a family of vector fields $X(x) = (X_1(x), \dots, X_m(x))$ satisfies the linear independence condition (LIC) on an open set $\Omega \subset \mathbb{R}^n$ if there exists a set $\mathcal{N}_X \subset \Omega$, closed in the topology of Ω, such that $|\mathcal{N}_X| = 0$ and $X_1(x), \dots, X_m(x)$ are linearly independent as vectors of $\mathbb{R}^n$ for each $x \in \Omega_X := \Omega \setminus \mathcal{N}_X$. Here $|A|$ denotes the n-dimensional Lebesgue measure of a measurable subset $A \subset \mathbb{R}^n$.*

Notice that, if X satisfies the LIC on Ω, then $m \le n$. In some results, we will also assume that X is defined and Lipschitz continuous on an open neighborhood Ω_0 of $\overline{\Omega}$ and that the following conditions hold:

(H1) Let $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$ be the so-called Carnot–Carathéodory distance function induced by X (see, for instance, [FSSC2, section 2]). Then, $d(x, y) < \infty$ for any $x, y \in \Omega_0$, so that d is a standard distance in Ω_0, and d is continuous with respect to the usual topology of $\mathbb{R}^n$.

(H2) For any compact set $K \subset \Omega_0$ there exist a radius r_K and a positive constant C_K, depending on K, such that

$$|B_d(x, 2r)| \le C_K |B_d(x, r)|$$

for any $x \in K$ and $r < r_K$. $B_d(x, r)$ denotes the (open) metric ball with respect to d, that is, $B_d(x, r) := \{y \in \Omega_0 \mid d(x, y) < r\}$.

(H3) There exist geometric constants $c, C > 0$ such that $\forall B = B_d(\overline{x}, r)$ with $cB := B_d(\overline{x}, cr) \subseteq \Omega_0$, $\forall u \in \mathrm{Lip}(c\overline{B})$, and $\forall x \in \overline{B}$

$$\left| u(x) - \frac{1}{|B|}\int_B u(y)dy \right| \le C \int_{cB} |Xu(y)| \frac{d(x,y)}{|B_d(x, d(x,y))|}\, dy\,.$$

Let us point out that the LIC embraces relevant and wide families of vector fields studied in the literature (see [MPSC1, Example 2.2]), as well as important families of vector fields satisfying conditions (H1), (H2), and (H3) (see Remark 2.8).

Each functional (2) always admits an integral representation with respect to the Euclidean gradient. Indeed, for instance, functional (2) can be represented as follows:

$$F_h(u) = \int_\Omega f_{h,e}(x, Du(x))\, dx \quad \text{for each } u \in \mathbf{C}^1(\Omega)\,,$$

where $f_{h,e} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ now denotes the *Euclidean integrand*, defined as

$$f_{h,e}(x,\xi) := f_h(x, C(x)\xi) \quad \text{ for a.e. } x \in \Omega \text{ for each } \xi \in \mathbb{R}^n. \tag{6}$$

Notice also that we cannot reverse this representation (see [MPSC1, Counterexample 3.14]) and the representation with respect to the Euclidean gradient could yield a loss of coercivity (see [MPSC1]). Nonetheless, we will show that by replacing the Euclidean gradient with the X-gradient, we can get rid of this drawback.

In this paper, we will exploit as main tools for studying minimization problems (1) some results of Γ-convergence for functionals depending on vector fields. In particular, let us recall a Γ-compactness theorem for the sequence $(F_h)_h$ defined in (2) (see Theorem 3.5), obtained in [MPSC1], to which we will refer for all the relevant definitions. More precisely, if the X-gradient satisfies the LIC and the sequence of integrand functions $(f_h)_h \subset I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, then, up to a subsequence not relabeled, we can assume the existence of a functional $F : L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$ and $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ such that

$$F = \Gamma(L^p(\Omega))\text{-}\lim_{h\to\infty} F_h \tag{7}$$

and F admits the following representation:

$$F(u) := \begin{cases} \int_\Omega f(x, Xu(x))dx & \text{if } u \in W^{1,p}_X(\Omega), \\ \infty & \text{otherwise.} \end{cases} \tag{8}$$

Let us now describe the main results of the present paper and some of their applications. First, recall the following Poincaré inequality on $W^{1,p}_{X,0}(\Omega)$, $1 \le p < \infty$, which holds provided that Ω is a bounded domain of $\mathbb{R}^n$ and X satisfies conditions (H1), (H2), and (H3) (see Proposition 2.16). More precisely, there exists a positive constant $c_{p,\Omega}$, depending only on p and Ω, such that

$$c_{p,\Omega} \int_\Omega |u|^p\, dx \le \int_\Omega |Xu|^p\, dx \quad \text{for each } u \in W^{1,p}_{X,0}(\Omega)\,. \tag{9}$$

We will also assume that $c_{p,\Omega}$ is the best constant in the Poincaré inequality (9), that is, it is the largest constant for which (9) holds.

Let us begin with a result concerning the convergence of minima and minimizers for minimization problems (1). For any fixed $\varphi \in W^{1,p}_X(\Omega)$, let $\mathbb{1}_\varphi : L^p(\Omega) \to \{0; \infty\}$ denote the indicator function of the affine subspace of $W^{1,p}_X(\Omega)$

$$W^{1,p}_{X,\varphi}(\Omega) := \varphi + W^{1,p}_{X,0}(\Omega)$$

and assume that the Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ in (3) satisfies the following growth condition: there exist two constants d_0, d_1 and two nonnegative functions $b_0, b_1 \in L^1(\Omega)$ such that

$$d_0|s|^p - b_0(x) \le g(x, s) \le d_1|s|^p + b_1(x) \tag{10}$$

for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$, with

$$d_1 > 0 \quad \text{and} \quad - c_0\, c_{p,\Omega} < d_0 \le d_1\,. \tag{11}$$

Theorem 1.2. *Let Ω be a bounded and connected open set, $1 < p < \infty$, and let X satisfy the LIC and conditions* (H1), (H2), *and* (H3). *Let $f_h, f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, let g satisfy* (10) *and* (11), *and let F_h, G, F be the functionals in* (2), (3), *and* (8), *respectively. For any fixed $\varphi \in W^{1,p}_X(\Omega)$, let $\Xi^\varphi_h, \Xi^\varphi : L^p(\Omega) \to \mathbb{R}\cup\{\infty\}$ be, respectively, defined as*

$$\Xi^\varphi_h := F_h + G + \mathbb{1}_\varphi \quad \textit{and} \quad \Xi^\varphi := F + G + \mathbb{1}_\varphi\,. \tag{12}$$

If $(F_h)_h$ Γ-converges to F in the strong topology of $L^p(\Omega)$, then

(i) *for each $h \in \mathbb{N}$, both Ξ^φ_h and Ξ^φ attain their minima in $L^p(\Omega)$ and*

$$\min_{u\in L^p(\Omega)} \Xi^\varphi(u) = \lim_{h\to\infty} \min_{u\in L^p(\Omega)} \Xi^\varphi_h(u)\,; \tag{13}$$

(ii) *if $(u_h)_h$ is a sequence of minimizers of $(\Xi^\varphi_h)_h$, that is,*

$$\Xi^\varphi_h(u_h) = \min_{u\in L^p(\Omega)} \Xi^\varphi_h(u) \quad \textit{for any } h \in \mathbb{N}\,,$$

then there exists $\bar u \in W^{1,p}_{X,\varphi}(\Omega)$ such that, up to subsequences,

$$u_h \to \bar u \textit{ weakly in } W^{1,p}_X(\Omega) \textit{ and strongly in } L^p(\Omega) \tag{14}$$

and

$$\Xi^\varphi(\bar u) = \min_{u\in L^p(\Omega)} \Xi^\varphi(u)\,. \tag{15}$$

The second main result deals with the convergence of the momenta associated with the sequence of functionals $(F_h)_h$ satisfying (7). The result is inspired by [ADMZ2] and it is a partial extension of those results to integral functionals depending on vector fields.

Theorem 1.3. *Let Ω be a bounded open set, $1 < p < \infty$, and let X satisfy the LIC. Let $f_h, f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, let F_h, F be, respectively, the functionals in* (2) *and* (8), *satisfying* (7), *and define $\mathcal{F}_h, \mathcal{F} : L^p(\Omega)^m \to \mathbb{R}$ as*

$$\mathcal{F}_h(\Phi) := \int_\Omega f_h(x, \Phi(x))\, dx \tag{16}$$

and

$$\mathcal{F}(\Phi) := \int_\Omega f(x, \Phi(x))\, dx \tag{17}$$

for any $\Phi \in L^p(\Omega)^m$ and for any $h \in \mathbb{N}$. Assume the following:

(i) *fixed* $0 \le \alpha \le \min\{1, p-1\}$*, there exist a positive constant* $\overline{c}$ *and a nonnegative function* $b \in L^p(\Omega)$ *such that*

$$|\nabla_\eta f_h(x,\eta_1) - \nabla_\eta f_h(x,\eta_2)| \le \overline{c}\,|\eta_1 - \eta_2|^\alpha \left(|\eta_1| + |\eta_2| + b(x)\right)^{p-1-\alpha}$$

for a.e. $x \in \Omega$*, for any* η_1, $\eta_2 \in \mathbb{R}^m$*, and for any* $h \in \mathbb{N}$*;*

(ii) *the map* $\mathbb{R}^m \ni \eta \mapsto f_h(x,\eta)$ *belongs to* $\mathbf{C}^1(\mathbb{R}^m)$ *for a.e.* $x \in \Omega$ *and for any* $h \in \mathbb{N}$*;*

(iii) *the map* $\mathbb{R}^m \ni \eta \mapsto f(x,\eta)$ *belongs to* $\mathbf{C}^1(\mathbb{R}^m)$ *for a.e.* $x \in \Omega$*;*

(iv) *there exist* $u_h, u \in W^{1,p}_X(\Omega)$ *such that*

$$u_h \to u \text{ in } L^p(\Omega) \quad \text{and} \quad \mathcal{F}_h(Xu_h) \to \mathcal{F}(Xu) \quad \text{as } h \to \infty\,.$$

Then, the convergence of momenta associated with $(F_h)_h$ *holds, that is,*

$$\partial_\Phi \mathcal{F}_h(Xu_h) = \nabla_\eta f_h(\cdot, Xu_h) \to \nabla_\eta f(\cdot, Xu) = \partial_\Phi \mathcal{F}(Xu) \tag{18}$$

weakly in $L^{p'}(\Omega)^m$*, where* $\partial_\Phi \mathcal{F}_h$ *and* $\partial_\Phi \mathcal{F}$ *denote, respectively, the Gateaux derivatives of functionals* $\mathcal{F}_h$ *and* $\mathcal{F}$ *(see* (83)*).*

As a consequence of Theorems 1.2 and 1.3, we can infer the convergence of both minimizers and momenta associated with minimization problems (1).

COROLLARY 1.4. *Let* Ω *be open, bounded, and connected, let* $1 < p < \infty$*, and let* X *satisfy the LIC and conditions* (H1), (H2)*, and* (H3)*. Let* $f_h, f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$*, let* g *satisfy* (10) *and* (11)*, let* G *be the functional* (3)*, and let* $F_h, \mathcal{F}_h, F, \mathcal{F}$ *satisfy the hypotheses of Theorem* 1.3*. For any fixed* $\varphi \in W^{1,p}_X(\Omega)$*, consider functionals* $\Xi^\varphi_h, \Xi^\varphi$ *defined in* (12)*. If* $(u_h)_h$ *is a sequence of minimizers of* $(\Xi^\varphi_h)_h$*, then, up to subsequences, there exists a minimizer* u *of* Ξ^φ *such that*

$$u_h \to u \text{ weakly in } W^{1,p}_X(\Omega) \text{ and strongly in } L^p(\Omega)\,.$$

Moreover, (18) *holds.*

We will also provide two interesting applications of the previous results to the periodic homogenization of functionals in Carnot groups and the H-convergence for linear differential operators of the second order depending on X.

Let us recall that Γ-convergence for functionals in (2) has been studied in the case in which the integrand f depends also on u [EPV, EV], and in the framework of Dirichlet forms [Fu, MR], but for special integrand functions f and X-gradient satisfying the Hörmander condition (see, for instance, [BPT2, BT, Mo] and references therein). We also point out that Γ-convergence for functionals defined in Cheeger–Sobolev metric measure spaces has been also studied (see [AHM] and references therein).

Homogenization in Carnot groups has been intensively studied so far (see, for instance, [BMT, BPT1, BPT2, FGVN, FT, MV]). Here we are interested in the recent paper [DDMM], where a Γ-convergence result for the periodic homogenization of functionals in Heisenberg groups has been proved (see Theorem 5.3). By using this result, we prove the convergence of minimizers for minimization problems (1) for each boundary datum $\varphi \in W^{1,p}_X(\Omega)$, as well as the convergence of the associated momenta (see Corollary 5.5).

The H-convergence for subelliptic PDEs has been also studied in the setting of Carnot groups (see [BFT, BFTT, FTT, Ma2, MPV]). In the previous papers, the main tool for showing a compactness result for H-convergence is the nontrivial extension to Carnot groups of the so-called *compensated compactness* [BFTT], originally introduced in the classical Euclidean setting by Murat and Tartar [Mu]. Here we get a compactness result for H-convergence in a broader setting than a Carnot group (see Theorem 6.2), by using Theorem 1.3, without applying the compensated compactness, since it is not clear whether it still holds in our framework.

The plan of this paper is as follows: in section 2, we introduce and study the Sobolev spaces associated with the X-gradient. In section 3, we prove a criterion for the Γ-convergence with respect to the weak convergence in spaces $W^{1,p}_X(\Omega)$ and $W^{1,p}_{X,0}(\Omega)$ (see Theorem 3.1). We also recall two key results of Γ-compactness for integral functions depending on X, with respect to $L^p(\Omega)$-topology: the former is a light extension of a result already shown in [MPSC1] (see Theorem 3.5) and the latter is a Γ-compactness result including Dirichlet boundary conditions (see Theorem 3.6). In section 4, we prove Theorems 1.2 and 1.3 and Corollary 1.4. Finally, in sections 5 and 6, we apply Theorems 1.2 and 1.3 and Corollary 1.4 to the case of periodic homogenization in Heisenberg groups (see Corollary 5.5) and to the H-convergence for linear differential operators of the second order depending on X (see Theorem 6.2), respectively.

2. Functional setting.

Throughout this paper, $\Omega \subset \mathbb{R}^n$ is a fixed open set and $\overline{\mathbb{R}} = [-\infty, \infty]$. If $v, w \in \mathbb{R}^n$, we denote by $|v|$ and $\langle v, w\rangle$ the Euclidean norm and the scalar product, respectively. If Ω and Ω' are subsets of $\mathbb{R}^n$, then $\Omega' \Subset \Omega$ means that Ω' is compactly contained in Ω. Moreover, $B(x, r)$ is the open Euclidean ball of radius r centered at x. If $A \subset \mathbb{R}^n$, χ_A and $\mathbb{1}_A$ are, respectively, the characteristic and the indicator function of A, $|A|$ is its n-dimensional Lebesgue measure $\mathcal{L}^n$, and, by notation a.e. $x \in A$, we will simply mean $\mathcal{L}^n$-a.e. $x \in A$. In what follows, we denote by $\mathbf{C}^k(\Omega)$ the space of $\mathbb{R}$-valued functions k times continuously differentiable and by $\mathbf{C}^k_c(\Omega)$ the subspace of $\mathbf{C}^k(\Omega)$ whose functions have support compactly contained in Ω.

Definition 2.1. *For any $u \in L^1(\Omega)$ we define Xu as an element of $\mathcal{D}'(\Omega; \mathbb{R}^m)$ as follows:*

$$\begin{aligned} Xu(\psi) : &= (X_1u(\psi_1), \dots, X_mu(\psi_m)) \\ &= -\int_\Omega u \left(\sum_{i=1}^n \partial_i(c_{1i}\, \psi_1), \dots, \sum_{i=1}^n \partial_i(c_{mi}\, \psi_m) \right) dx \end{aligned}$$

for any $\psi = (\psi_1, \dots, \psi_m) \in \mathbf{C}^\infty_c(\Omega; \mathbb{R}^m)$.

If we set $X^T\psi := (X_1^T\psi_1, \dots, X_m^T\psi_m)$ with

$$X_j^T \varphi := -\sum_{i=1}^n \partial_i(c_{ji}\, \varphi) = -(\mathrm{div}(X_j) + X_j)\, \varphi \tag{19}$$

for any $\varphi \in \mathbf{C}^\infty_c(\Omega)$ and $j = 1, \dots, m$, then the aspect of the previous definition is even more familiar,

$$Xu(\psi) = \int_\Omega u\, X^T\psi\, dx \quad \text{for any } \psi \in \mathbf{C}^\infty_c(\Omega; \mathbb{R}^m)\,.$$

Remark 2.2. *By the well-known extension result for Lipschitz functions, without loss of generality, we can assume that vector fields' coefficients $c_{ji} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $j = 1, \dots, m$ and $i = 1, \dots, n$.*

DEFINITION 2.3. *For $1 \le p \le \infty$ we set*

$$W^{1,p}_X(\Omega) := \{u \in L^p(\Omega) : X_j u \in L^p(\Omega) \text{ for } j = 1, \dots, m\},$$
$$W^{1,p}_{X;loc}(\Omega) := \left\{u : u|_{\Omega'} \in W^{1,p}_X(\Omega') \textit{ for every open set } \Omega' \Subset \Omega\right\}.$$

REMARK 2.4. *Since vector fields X_j have locally Lipschitz continuous coefficients, $\partial_i c_{ji} \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ for any $j = 1, \dots, m$ and $i = 1, \dots, n$. Then, by definition,*

$$W^{1,p}(\Omega) \subset W^{1,p}_X(\Omega) \quad \forall\, p \in [1,\infty] \tag{20}$$

for any open bounded set $\Omega \subset \mathbb{R}^n$. Moreover, for any $u \in W^{1,p}(\Omega)$

$$Xu(x) = C(x)\, Du(x) \quad \textit{ for a.e. } x \in \Omega\,,$$

where $W^{1,p}(\Omega)$ denotes the classical Sobolev space, or, equivalently, the space $W^{1,p}_X(\Omega)$ associated to

$$X = D := (\partial_1, \dots, \partial_n)\,.$$

It is easy to see that inclusion (20) *can be strict and turns out to be continuous. As well, there is the inclusion*

$$W^{1,p}_{\text{loc}}(\Omega) \subset W^{1,p}_{X;\text{loc}}(\Omega) \quad \forall\, p \in [1,\infty]\,.$$

The following proposition is proved in [FS] and [Ma, Lemma 2.3.29].

PROPOSITION 2.5. *$W^{1,p}_X(\Omega)$ endowed with the norm*

$$\|u\|_{W^{1,p}_X(\Omega)} := \left(\int_\Omega |u|^p\, dx + \int_\Omega |Xu|^p\, dx\right)^{\frac{1}{p}}$$

is a reflexive Banach space if $1 < p < \infty$.

Moreover, if $p > 1$, then functional $\|\cdot\|^p_{W^{1,p}_X(\Omega)} : W^{1,p}_X(\Omega) \to [0,\infty)$ is lower semicontinuous and sequentially coercive in the weak topology of $W^{1,p}_X(\Omega)$.

DEFINITION 2.6. *For $1 \le p \le \infty$ we set*

$$H^{1,p}_X(\Omega) \text{ the closure of } \mathbf{C}^1(\Omega) \cap W^{1,p}_X(\Omega) \text{ in } W^{1,p}_X(\Omega)\,,$$

$$W^{1,p}_{X,0}(\Omega) \text{ the closure of } \mathbf{C}^1_c(\Omega) \cap W^{1,p}_X(\Omega) \text{ in } W^{1,p}_X(\Omega)\,.$$

It is proved in [FS] that the normed spaces $(H^{1,p}_X(\Omega), \|\cdot\|_{W^{1,p}_X(\Omega)})$ and $(W^{1,p}_{X,0}(\Omega), \|\cdot\|_{W^{1,p}_X(\Omega)})$ are Banach spaces for any $1 \le p \le \infty$.

As for the usual Sobolev spaces, it holds that $H^{1,p}_X(\Omega) \subset W^{1,p}_X(\Omega)$. The classical result $H = W$, of Meyers and Serrin [MS] still holds true for these anisotropic Sobolev spaces as proved, independently, in [FSSC1] and [GN1]. Analogous results, under some additional assumptions, are proved in [FSSC2], for the weighted case, and in [APS], where a generalization to metric measure spaces is given.

THEOREM 2.7. *Let Ω be an open subset of $\mathbb{R}^n$ and $1 \le p < \infty$. Then,*

$$H^{1,p}_X(\Omega) = W^{1,p}_X(\Omega)\,.$$

We conclude this section recalling that when Ω is bounded and the family X satisfies properties (H1), (H2), and (H3), then $W^{1,p}_{X,0}(\Omega)$ can be compactly embedded in $L^p(\Omega)$ for any $1 \leq p < \infty$. Moreover, we prove that if, in addition, Ω is connected, then

$$\|u\|_{W^{1,p}_{X,0}(\Omega)} := \left(\int_\Omega |Xu|^p\, dx\right)^{\frac{1}{p}}$$

defines an equivalent norm on $W^{1,p}_{X,0}(\Omega)$ for any $1 \leq p < \infty$.

Let us point out some classes of relevant vector fields satisfying properties (H1), (H2), and (H3).

REMARK 2.8. (i) *(Hörmander vector fields) If the vector fields are smooth and the rank of the Lie algebra generated by $X_1, \ldots, X_m$ equals n at any point of Ω_0 (the so-called Hörmander condition), then* (H1)*,* (H2)*, and* (H3) *hold (see* [NSW] *for* (H1) *and* (H2) *and* [FLW] *for* (H3)*).*

(ii) *(Grushin vector fields) If the vector fields are as in* [F2]*,* [F1]*, and* [FL]*, then conditions* (H1)*,* (H2)*, and* (H3) *still hold (see* [F2, F1, FL] *for* (H1) *and* (H2) *and* [FSSC2, Remark 2.8] *for* (H3)*).*

The following results are proved in [FSSC2, Theorems 2.11 and 3.4].

THEOREM 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 \leq p < \infty$, and let X satisfy conditions* (H1)*,* (H2)*, and* (H3)*. Then, for each metric ball $B = B_d(x, r) \subset \Omega$ and for every $u \in W^{1,p}_X(\Omega)$, there exist a constant c, depending on B and u, and a constant C, not depending on u, such that*

$$\int_B |u(x) - c|^p\, dx \leq C\, r^p \int_B |Xu|^p\, dx\,.$$

THEOREM 2.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 \leq p < \infty$, and let X satisfy conditions* (H1)*,* (H2)*, and* (H3)*. Then, $W^{1,p}_{X,0}(\Omega)$ is compactly embedded in $L^p(\Omega)$.*

An interesting consequence of Theorem 2.10 is the following result, which can be proved exactly as in the Euclidean case. For this reason we omit the proof. For the reader's convenience, we remind that a proof can be found in [Ma, Proposition 1.2.17].

PROPOSITION 2.11. *Under the assumptions of Theorem* 2.10*, $W^{1,p}_X(\Omega)$ can be compactly embedded in $L^p_{loc}(\Omega)$.*

Let us point out that we can get a global compact embedding in Proposition 2.11, by requiring further regularity on Ω, in the sense of the following definition.

DEFINITION 2.12. *Let (M, d) be a metric space. A bounded set $\Omega \subset M$ is said to be a uniform domain if there exists $\varepsilon > 0$ such that for each $x, y \in \Omega$ there exists a continuous rectifiable curve $\gamma : [0, 1] \to \Omega$, with*

$$\gamma(0) = x, \quad \gamma(1) = y\,,$$

$$\text{length}(\gamma) \leq \frac{1}{\varepsilon}\, d(x, y)\,,$$

and, for each $t \in [0, 1]$,

$$\text{dist}(\gamma(t), \partial\Omega) \geq \varepsilon\, \min\{\text{length}(\gamma|_{[0,t]}), \text{length}(\gamma|_{[t,1]})\}\,.$$

REMARK 2.13. *The characterization of uniform domains in metric spaces is a difficult task. Few examples of such domains are known and also in the framework of the Carnot–Carathéodory distance. A comprehensive account of uniform domains, with respect to the Carnot–Carathéodory distance, can be found in* [Mon] *(see also* [FPS] *for an interesting example).*

If Ω is a (bounded) uniform domain in the metric space (Ω_0, d), then, by using an extension result for functions in $W^{1,p}_X(\Omega)$ (see [GN2]), by applying Theorem 2.10 and by a localization argument in a neighborhood of Ω, we get the following result.

THEOREM 2.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 \le p < \infty$, and let X satisfy conditions* (H1), (H2), *and* (H3). *Moreover, assume that Ω is a uniform domain in the metric space (Ω_0, d). Then, $W^{1,p}_X(\Omega)$ can be compactly embedded in $L^p(\Omega)$.*

REMARK 2.15. *Actually, Theorem* 2.14 *still holds for an even more general class of metric regular sets, namely the so-called PS-domains (see* [GN1, Theorem 1.28]*). In particular, the metric balls with respect to the Carnot–Carathéodory distance are PS domains (see* [FGW, GN1]*).*

As a consequence of Theorems 2.9 and 2.10, a global Poincaré inequality holds in $W^{1,p}_{X,0}(\Omega)$. For the reader's convenience, we remind that a proof can be found in [Ma, Proposition 1.2.18].

PROPOSITION 2.16. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 \le p < \infty$, and let X satisfy conditions* (H1), (H2), *and* (H3). *Moreover, assume that Ω is connected. Then, there exists a positive constant $c_{p,\Omega}$, depending on p and Ω, such that*

$$c_{p,\Omega} \int_\Omega |u|^p\,dx \le \int_\Omega |Xu|^p\,dx \quad \textit{for any } u \in W^{1,p}_{X,0}(\Omega)\,.$$

COROLLARY 2.17. *Let Ω, p, and X be as in Proposition* 2.16. *Then,*

$$\|u\|_{W^{1,p}_{X,0}(\Omega)} := \left(\int_\Omega |Xu|^p\,dx\right)^{\frac{1}{p}}$$

is a norm in $W^{1,p}_{X,0}(\Omega)$ equivalent to $\|\cdot\|_{W^{1,p}_X(\Omega)}$.

We conclude this section recalling the following estimate that will be useful to prove the coercivity of functionals Ξ^φ_h, defined in (12). It can be proved as in [DM, Lemma 2.7].

LEMMA 2.18. *Let $c_{p,\Omega}$ be the Poincaré constant in* (9)*, let $\varphi \in W^{1,p}_X(\Omega)$, and let $c < c_{p,\Omega}$. Then, there exist a positive constant k_1, depending only on c and $c_{p,\Omega}$, and a nonnegative constant k_2, depending on c, $c_{p,\Omega}$, and $\|\varphi\|_{W^{1,p}_X(\Omega)}$, such that*

$$\int_\Omega |Xu|^p\,dx - c\int_\Omega |u|^p\,dx \ge k_1\left(\int_\Omega |Xu|^p\,dx + \int_\Omega |u|^p\,dx\right) - k_2$$

for every $u \in W^{1,p}_{X,\varphi}(\Omega)$.

3. Γ-convergence results for integral functionals depending on vector fields.

In this section, we study Γ-convergence results for classes of integral functionals depending on vector fields, with respect to the weak topology of $W^{1,p}_{X,0}(\Omega)$ and $W^{1,p}_X(\Omega)$, namely Theorem 3.1, and the strong topology of $L^p(\Omega)$; see Theorems 3.5, 3.6, and 3.7.

3.1. Γ-convergence in the weak topology of $W^{1,p}_{X,0}(\Omega)$ and $W^{1,p}_X(\Omega)$. First, we show that, if X satisfies conditions (H1), (H2), and (H3), then the pointwise convergence of the sequence $(f_h(\cdot,\eta))_h$ a.e. in Ω for any $\eta\in\mathbb{R}^m$ implies the Γ-convergence of the corresponding integral functionals in the weak topology of $W^{1,p}_{X,0}(\Omega)$ and $W^{1,p}_X(\Omega)$.

THEOREM 3.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, $1<p<\infty$, and let X satisfy the LIC and conditions* (H1), (H2), *and* (H3). *Let $f_h, f\in I_{m,p}(\Omega,c_0,c_1,a_0,a_1)$, with $a_i\in L^\infty(\Omega)$ $(i=0,1)$, and let F_h, $F: W^{1,p}_{X,0}(\Omega)\to\mathbb{R}$ be the corresponding integral functionals, defined as*

$$F_h(u):=\int_\Omega f_h(x,Xu(x))\,dx\,,\quad F(u):=\int_\Omega f(x,Xu(x))\,dx \tag{21}$$

for any $u\in W^{1,p}_{X,0}(\Omega)$ and for any $h\in\mathbb{N}$. Assume that

$$f_h(\cdot,\eta)\to f(\cdot,\eta)\quad \textit{a.e. in } \Omega \textit{ and for any } \eta\in\mathbb{R}^m.$$

Then, $(F_h)_h$ Γ-converges to F in the weak topology of $W^{1,p}_{X,0}(\Omega)$.

Moreover, if $W^{1,p}_X(\Omega)$ is compactly embedded in $L^p(\Omega)$, then $(F_h+G)_h$ Γ-converges to $F+G$ in the weak topology of $W^{1,p}_X(\Omega)$. In this case, F_h, $F: W^{1,p}_X(\Omega)\to\mathbb{R}$ are defined as in (21), *while $G: W^{1,p}_X(\Omega)\to\mathbb{R}$ is the functional in* (3) *such that g satisfies* (10), *with $0<d_0\le d_1$.*

See, e.g., Theorem 2.14 for an example of a set Ω where there exists a compact embedding of $W^{1,p}_X(\Omega)$ in $L^p(\Omega)$. Before proving Theorem 3.1 we need two technical lemmas.

LEMMA 3.2. *Let $f\in I_{m,p}(\Omega,c_0,c_1,0,a)$, with $a\in L^\infty(\Omega)$, and let $r>0$. There exist $R=R(r)>r$ and a Borel measurable function $g_r:\Omega\times\mathbb{R}^m\to\mathbb{R}$ such that $g_r(x,\cdot)$ is convex for a.e. $x\in\Omega$ and*

$$0\le g_r(x,\eta)\le f(x,\eta)\quad \textit{for a.e. } x\in\Omega \textit{ and every } \eta\in\mathbb{R}^m\,; \tag{22}$$

$$g_r(x,\eta)=f(x,\eta)\quad \textit{for a.e. } x\in\Omega \textit{ and every } \eta\in\overline{B_r(0)}\,; \tag{23}$$

$$g_r(x,\eta)\le c_0|\eta|^p\quad \textit{for a.e. } x\in\Omega \textit{ and every } \eta\in\mathbb{R}^m\setminus\overline{B_R(0)}\,. \tag{24}$$

Proof. Observe that, without loss of generality, we can assume that

$$f(x,\cdot):\mathbb{R}^m\to[0,\infty) \text{ is convex for each } x\in\Omega\,. \tag{25}$$

Indeed, by (I_2) and since the n-dimensional Lebesgue measure $\mathcal{L}^n$ is Borel regular, there exists a negligible Borel set $N\subset\Omega$ such that $f(x,\cdot):\mathbb{R}^m\to[0,\infty)$ is convex for each $x\in\Omega\setminus N$. By redefining $f(x,\eta):=0$ for each $x\in N$ and $\eta\in\mathbb{R}^m$, we get the desired conclusion.

By (25) and [Ro, Theorem 10.4], $f(x,\cdot):\mathbb{R}^m\to[0,\infty)$ is locally Lipschitz for any $x\in\Omega$. In particular, in fixed $x\in\Omega$, $\eta_0\in\mathbb{R}^m$, and $r>0$, there exists a positive constant L, depending on f, x, η_0, and r, such that

$$|f(x,\eta_1)-f(x,\eta_2)|\le L\,|\eta_1-\eta_2| \tag{26}$$

for any η_1, $\eta_2\in\overline{B_r(\eta_0)}$, and

$$L:=\frac{1}{r}\sup_{B_{2r}(\eta_0)} f(x,\cdot)\,.$$

Fix $x \in \Omega$ and $\eta_0 \in \mathbb{R}^m$. By [Cla, Proposition 2.22] and (26), for any $v \in \mathbb{R}^m$ there exists the directional derivative

$$f'_x(\eta_0, v) := \lim_{t\to 0} \frac{f(x, \eta_0 + tv) - f(x, \eta_0)}{t} \in \mathbb{R}\,.$$

It is also clear that

$$f'_x(\eta_0, v) = \lim_{h\to\infty} h\left(f\left(x, \eta_0 + \frac{1}{h}v\right) - f(x, \eta_0)\right).$$

By (26) and [Cla, Corollary 4.26],

$$f'_x(\eta_0, v) = \max_{\xi\in\partial f_x(\eta_0)} \langle \xi, v\rangle \quad \text{for any } v \in \mathbb{R}^m, \tag{27}$$

where $\partial f_x(\eta_0)$ denotes the subdifferential of $f(x, \cdot)$ at η_0.

By (27), the map $\mathbb{R}^m \ni v \mapsto f'_x(\eta, v)$ is positively homogeneous of degree one, convex, and so subadditive, continuous, and finite. Moreover, since $f : \Omega \times \mathbb{R}^m \to [0, \infty)$ is Borel measurable, then the map $\Omega \times \mathbb{R}^m \ni (x, v) \mapsto f'_x(\eta, v)$ is Borel measurable for any $\eta \in \mathbb{R}^m$.

Fix $r > 0$ and $\eta_0, \eta \in \mathbb{R}^m$. We define

$$G_{\eta_0}(x, \eta) := f(x, \eta_0) + f'_x(\eta_0, \eta - \eta_0)$$

and

$$g_r(x, \eta) := \sup_{\eta_0\in\mathbb{Q}^m\cap\overline{B_r(0)}} G_{\eta_0}(x, \eta)\,.$$

We first claim that $g_r(x, \eta) < \infty$ for a.e. $x \in \Omega$ for every $\eta \in \mathbb{R}^m$, $g_r : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is Borel measurable and that $g_r(x, \cdot) : \mathbb{R}^m \to \mathbb{R}$ is convex for a.e. $x \in \Omega$.

Let $x \in \Omega$ and define $\overline{\xi}$ the element of $\partial f_x(\eta_0)$ such that

$$f'_x(\eta_0, \eta - \eta_0) = \langle\overline{\xi}, \eta - \eta_0\rangle\,. \tag{28}$$

Since $f \in I_{m,p}(\Omega, c_0, c_1, 0, a)$, then

$$|G_{\eta_0}(x, \eta)| \le c_1|\eta_0|^p + |\overline{\xi}||\eta - \eta_0| + a(x) \tag{29}$$

a.e. $x \in \Omega$, for any $\eta_0, \eta \in \mathbb{R}^m$, and

$$c_0\,|\eta|^p \le\, f(x, \eta) \le\, c_1\,|\eta|^p + a(x) \le\, c_1 2^p\, r^p + \|a\|_{L^\infty(\Omega)} \tag{30}$$

for a.e. $x \in \Omega$ and for any $\eta \in \overline{B_{2r}(0)}$. Moreover, by (26) and (30), and, arguing as in [Cla, Proposition 4.14], there exists a positive constant M, depending only on c_1, $\|a\|_{L^\infty(\Omega)}$, and r, such that

$$|\overline{\xi}| \le M \quad \text{for any } \eta_0 \in \overline{B_r(0)}$$

which, together with (29), gives

$$g_r(x, \eta) \le c_1 r^p + M(|\eta| + r) + \|a\|_{L^\infty(\Omega)} < +\infty \tag{31}$$

for a.e. $x \in \Omega$ and $\forall\eta \in \mathbb{R}^m$.

Since g_r is a pointwise supremum of the countable family of Borel measurable functions $\Omega \times \mathbb{R}^m \ni (x,\eta) \mapsto G_{\eta_0}(x,\eta)$, with $\eta_0 \in \mathbb{Q}^m \cap \overline{B_r(0)}$, then it is Borel measurable. As well, since $g_r(x,\cdot) : \mathbb{R}^m \to \mathbb{R}$ is a pointwise supremum of the countable family of convex functions $\mathbb{R}^m \ni \eta \mapsto G_{\eta_0}(x,\cdot)$ a.e. $x \in \Omega$, with $\eta_0 \in \mathbb{Q}^m \cap \overline{B_r(0)}$, then, by [Cla, Proposition 2.20], it is a convex function.

Let us now prove that g_r satisfies (22) and (23). Let $x \in \Omega$ be such that $f(x,\cdot)$ is convex. Then, fixed $\eta_0 \in \mathbb{R}^m$,

$$f(x,\eta) \geq f(x,\eta_0) + \langle \xi, \eta - \eta_0 \rangle \tag{32}$$

for any $\eta \in \mathbb{R}^m$ and $\xi \in \partial f_x(\eta_0)$. Let $\overline{\xi} \in \partial f_x(\eta_0)$ satisfy (28). By (32),

$$f(x,\eta) \geq G_{\eta_0}(x,\eta)$$

and, passing to the supremum, we get

$$g_r(x,\eta) \leq \ f(x,\eta) \quad \text{ for a.e. } x \in \Omega \text{ and every } \eta \in \mathbb{R}^m. \tag{33}$$

On the other hand, if $\eta \in \overline{B_r(0)}$ and $(\eta_h)_{h\in\mathbb{N}} \subset \mathbb{Q}^m \cap \overline{B_r(0)}$ are such that $\eta_h \to \eta$ as $n \to \infty$, then

$$g_r(x,\eta) \geq f(x,\eta_h) + f'_x(\eta_h, \eta - \eta_h) \quad \forall h \in \mathbb{N}. \tag{34}$$

Moreover, since

$$|f'_x(\eta_n, \eta - \eta_h)| \leq M|\eta - \eta_h| \quad \forall h \in \mathbb{N}\,,$$

we conclude that

$$\lim_{h\to\infty} f'_x(\eta_h, \eta - \eta_h) = 0\,. \tag{35}$$

Therefore, by (33), (34), (35), and by the continuity of $f(x,\cdot)$ in $\mathbb{R}^m$, we obtain (23).

Now we fix $x \in \Omega$ such that both $f(x,\cdot)$ and $g_r(x,\cdot)$ are convex in $\mathbb{R}^m$. By (23) and the Weierstrass theorem, there exists $\eta_1 \in \overline{B_r(0)}$ such that

$$f(x,\eta_1) = \ g_r(x,\eta_1) = \min_{\eta\in\overline{B_r(0)}} g_r(x,\eta) \tag{36}$$

and, since $f \in I_{m,p}(\Omega, c_0, c_1, 0, a)$, then

$$g_r(x,\eta) \geq 0 \quad \text{for any } \eta \in \overline{B_r(0)}\,. \tag{37}$$

Assume, by contradiction, the existence of $\eta_2 \in \mathbb{R}^m \setminus \overline{B_r(0)}$ such that

$$g_r(x,\eta_2) < 0\,. \tag{38}$$

Then, there exist $\eta_3 \in \overline{B_r(0)}$ and $\bar{t} \in (0,1)$ such that $\eta_3 = \bar{t}\eta_1 + (1-\bar{t})\eta_2$ and, since $g_r(x,\cdot)$ is convex in $\mathbb{R}^m$, (36) and (38) give

$$g_r(x,\eta_1) \leq g_r(x,\eta_3) \leq \bar{t}g_r(x,\eta_1) + (1-\bar{t})g_r(x,\eta_2) < g_r(x,\eta_1)\,,$$

which yields a contradiction. Then, by (37), we get (22).

Finally, since $p > 1$ and $c_0, M > 0$, we have

$$\lim_{|\eta|\to\infty} \frac{c_0|\eta|^p}{c_1 r^p + M(|\eta| + r) + \|a\|_{L^\infty(\Omega)}} = +\infty$$

and, by (31), (24) also follows. ∎

LEMMA 3.3. *Let* $f_h \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, *with* $a_0, a_1 \in L^\infty(\Omega)$, *and assume that*

$$f_h(\cdot, \eta) \to f(\cdot, \eta) \quad \text{a.e. in } \Omega, \text{ for each } \eta \in \mathbb{R}^m.$$

Then we have the following:

(i) $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$;

(ii) *if* $(\Phi_h)_h$ *weakly converges to* Φ *in* $L^p(\Omega)^m$, *then functionals* $\mathcal{F}_h, \mathcal{F} : L^p(\Omega)^m \to \mathbb{R}$, *defined in* (16) *and* (17), *satisfy*

$$\mathcal{F}(\Phi) \le \liminf_{h\to\infty} \mathcal{F}_h(\Phi_h)\,, \text{ i.e.,}$$

$$\int_\Omega f(x, \Phi(x))\, dx \le \liminf_{h\to\infty} \int_\Omega f_h(x, \Phi_h(x))\, dx\,.$$

Proof. (i) The proof is immediate.

(ii) Let $(\Phi_h)_h \subset L^p(\Omega)^m$ be weakly convergent to Φ in $L^p(\Omega)^m$. Then, there exists a positive constant M such that

$$\int_\Omega |\Phi_h|^p\, dx \le M \quad \text{for any } h \in \mathbb{N}\,.$$

Moreover, since $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, then $f(\cdot, \Phi(\cdot)) \in L^1(\Omega)$. Therefore, by the absolute continuity of the Lebesgue's integral, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\int_A |f(x, \Phi(x))|\, dx < \varepsilon \tag{39}$$

for any measurable subset A of Ω such that $|A| < \delta$.

Let us fix $R > 0$ and let us consider $\overline{B_R(0)} \subset \mathbb{R}^m$. Then, for any $\varrho > 0$, there exist $\eta_1, .., \eta_k \in B_R(0)$ such that

$$\overline{B_R(0)} \subset \cup_{i=1}^k B_\varrho(\eta_i)\,. \tag{40}$$

Since $f_h, f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, then, by [Ro, Theorem 10.4], there exists a positive constant L_R such that

$$\begin{aligned} &|f_h(x, \eta) - f_h(x, \eta_i)| \le L_R|\eta - \eta_i| \le L_R\varrho\,, \\ &|f(x, \eta) - f(x, \eta_i)| \le L_R|\eta - \eta_i| \le L_R\varrho \end{aligned} \tag{41}$$

for any $h \in \mathbb{N}$, $i = 1, \ldots, k$, and $\eta \in B_\varrho(\eta_i) \cap \overline{B_R(0)}$.

If $x \in \Omega$ and $\eta \in \overline{B_R(0)}$ then, by (40), there exists $i \in \{1, \ldots, k\}$ such that $\eta \in B_\varrho(\eta_i)$ and, by (41),

$$|f_h(x, \eta) - f(x, \eta)| \le 2L_R\varrho + |f_h(x, \eta_i) - f(x, \eta_i)|\,. \tag{42}$$

Since $f_h(x, \eta_i) \to f(x, \eta_i)$ for a.e. $x \in \Omega$ and for any $i \in \{1, \ldots, k\}$, then, by the Severini–Egoroff theorem, there exist $A_1, \ldots, A_k$, measurable subsets of Ω, such that $|A_i| < \frac{\delta}{2k}$ and such that

$$\lim_{h\to\infty} \left[\sup_{x\in\Omega\setminus A_i} |f_h(x, \eta_i) - f(x, \eta_i)| \right] = 0\,.$$

Let $A^\delta := \cup_{i=1}^k A_i$. Thus, $|A^\delta| < \frac{\delta}{2}$ and

$$\lim_{h\to\infty} z_h := \lim_{h\to\infty}\left[\max_{i\in\{1,\dots,k\}} \sup_{x\in\Omega\setminus A^\delta} |f_h(x,\eta_i) - f(x,\eta_i)|\right] = 0\,. \tag{43}$$

Therefore, for any $x \in \Omega \setminus A^\delta$ and for any $\eta \in \overline{B_R(0)}$, by (42),

$$|f_h(x,\eta) - f(x,\eta)| \le 2L_R\varrho + z_h\,. \tag{44}$$

Fix $r > 0$ and define $\varphi_h := f_h + a_0$ for any $h \in \mathbb{N}$ and $\varphi := f + a_0$. Then, trivially, $\varphi_h, \varphi \in I_{m,p}(\Omega, c_0, c_1, 0, a_0 + a_1)$ and, by Lemma 3.2, there exist $R(r) > r$ and $g_r : \Omega \times \mathbb{R}^m \to [0,\infty)$ such that

$$g_r(x,\eta) \le \varphi(x,\eta) \quad \text{for a.e. } x \in \Omega \text{ and every } \eta \in \mathbb{R}^m, \tag{45}$$

$$g_r(x,\eta) = \varphi(x,\eta) \quad \text{for a.e. } x \in \Omega \text{ and every } \eta \in \overline{B_r(0)}, \tag{46}$$

$$g_r(x,\eta) \le c_0|\eta|^p \quad \text{for a.e. } x \in \Omega \text{ and every } \eta \in \mathbb{R}^m \setminus \overline{B_R(0)}\,. \tag{47}$$

Notice that if $x \in \Omega$ and $\eta \in \mathbb{R}^m \setminus \overline{B_R(0)}$, then, by (47)

$$\varphi_h(x,\eta) \ge c_0|\eta|^p \ge g_r(x,\eta) \tag{48}$$

while, if $x \in \Omega \setminus A^\delta$ and $\eta \in \overline{B_R(0)}$, then by (44) and (45)

$$\varphi_h(x,\eta) \ge \varphi(x,\eta) - 2L_R\varrho - z_h \ge g_r(x,\eta) - 2L_R\varrho - z_h. \tag{49}$$

Moreover, since $(\Phi_h)_h$ weakly converges to Φ in $L^p(\Omega)^m$, then

$$\liminf_{h\to\infty} \int_{\Omega\setminus A^\delta} g_r(x,\Phi_h)\,dx \ge \int_{\Omega\setminus A^\delta} g_r(x,\Phi)\,dx\,. \tag{50}$$

Therefore, by (43), (46), (48), (49), and (50), and by Fatou's lemma

$$\begin{aligned}
\liminf_{h\to\infty} \int_\Omega \varphi_h(x,\Phi_h)\,dx &\ge \liminf_{h\to\infty} \int_{\Omega\setminus A^\delta} \varphi_h(x,\Phi_h)\,dx \\
&\ge \liminf_{h\to\infty} \left[\int_{\Omega\setminus A^\delta} g_r(x,\Phi_h)\,dx - (2L_R\varrho + z_h)|\Omega|\right] \\
&\ge \int_{\Omega\setminus A^\delta} g_r(x,\Phi)\,dx - 2L_R\varrho|\Omega| \\
&\ge \int_{\Omega\setminus(A^\delta\cup\{|\Phi|>r\})} g_r(x,\Phi)\,dx - 2L_R\varrho|\Omega| \\
&= \int_{\Omega\setminus(A^\delta\cup\{|\Phi|>r\})} \varphi(x,\Phi)\,dx - 2L_R\varrho|\Omega|\,.
\end{aligned}$$

Moreover, by Chebyshev's inequality,

$$|\{|\Phi| > r\}| \le \frac{1}{r^p}\int_\Omega |\Phi(x)|^p\,dx\,.$$

Let us choose r such that $|\{|\Phi| > r\}| < \frac{\delta}{2}$. Thus, by (39)

$$\liminf_{h\to\infty} \int_\Omega \varphi_h(x,\Phi_h)\,dx \ge \int_\Omega \varphi(x,\Phi)\,dx - \varepsilon - 2L_R\varrho|\Omega|\,,$$

that is,

$$\liminf_{h\to\infty}\int_\Omega f_h(x,\Phi_h)\,dx \geq \int_\Omega f(x,\Phi)\,dx - \varepsilon - 2L_R\varrho|\Omega|,$$

and, as ε and ϱ go to zero, we get the thesis. ☐

Proof of Theorem 3.1. By (5), there exists $\Psi_1 : W^{1,p}_{X,0}(\Omega) \to \mathbb{R}$ such that $\Psi_1 \leq F_h$ for any $h \in \mathbb{N}$ and

$$\lim_{\|u\|_{W^{1,p}_{X,0}(\Omega)}\to\infty} \Psi_1(u) = +\infty\,.$$

Since, by Theorem 2.10, $W^{1,p}_{X,0}(\Omega)$ is compactly embedded in $L^p(\Omega)$, then by [DM, Proposition 8.10], we can characterize the Γ-limit of $(F_h)_h$ in terms of sequences, that is, fixed $u \in W^{1,p}_{X,0}(\Omega)$, it suffices to show the following:

(a) for any $(u_h)_h$ weakly convergent to u in $W^{1,p}_{X,0}(\Omega)$, then

$$F(u) \leq \liminf_{h\to\infty} F_h(u_h)\,;$$

(b) there exists $(v_h)_h$ weakly convergent to u in $W^{1,p}_{X,0}(\Omega)$ such that

$$F(u) = \lim_{h\to\infty} F_h(v_h)\,.$$

Let $(u_h)_h$ be weakly convergent to u in $W^{1,p}_{X,0}(\Omega)$. Then, $(Xu_h)_h$ weakly converges to Xu in $L^p(\Omega)^m$ and (a) follows, by Lemma 3.3.

Let $v_h := u$ for any $h \in \mathbb{N}$. Since $(f_h(\cdot, Xu))_h$ converges to $f(\cdot, Xu)$ a.e. in Ω by hypothesis, then, by the dominated convergence theorem, the sequence $(F_h(u))_h$ converges pointwise to $F(u)$ and (b) also follows.

Similarly, by (5) and (10), there exists $\Psi_2 : W^{1,p}_X(\Omega) \to \mathbb{R}$ such that $\Psi_2 \leq F_h + G$ in $W^{1,p}_X(\Omega)$ for any $h \in \mathbb{N}$ and

$$\lim_{\|u\|_{W^{1,p}_{X}(\Omega)}\to\infty} \Psi_2(u) = +\infty\,.$$

Then, since $W^{1,p}_X(\Omega)$ is compactly embedded in $L^p(\Omega)$ by hypothesis, we can characterize the Γ-limit of $(F_h + G)_h$ in terms of sequences, in virtue of [DM, Proposition 8.10] and, since G is sequentially continuous in the weak topology of $W^{1,p}_X(\Omega)$, then $(F_h + G)_h$ Γ-converges to $F + G$ in the weak topology of $W^{1,p}_X(\Omega)$ by the first part of the proof, and the thesis follows. ☐

An analogous result in the strong topology of $W^{1,p}_X(\Omega)$ still holds true and a proof can be found in [Ma, Proposition 2.3.24].

THEOREM 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, and let X satisfy the LIC. Let $f_h, f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, and let F_h, $F : W^{1,p}_X(\Omega) \to \mathbb{R}$ be the corresponding integral functionals, defined as*

$$F_h(u) := \int_\Omega f_h(x, Xu(x))\,dx\,, \quad F(u) := \int_\Omega f(x, Xu(x))\,dx$$

for any $u \in W^{1,p}_X(\Omega)$ and for any $h \in \mathbb{N}$. Then, $(F_h)_h$ converges pointwise to F in $W^{1,p}_X(\Omega)$ if and only if $(F_h)_h$ Γ-converges to F in the strong topology of $W^{1,p}_X(\Omega)$.

3.2. Γ-convergence in the strong topology of $L^p(\Omega)$. The first result of this section in an extension of [MPSC1, Theorem 4.11] to our class of integrands and a proof can be found in [Ma, Theorem 2.3.12].

THEOREM 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\mathcal{A}$ be the class of all open subsets of Ω, $1 < p < \infty$, and let X satisfy the LIC. Let $f_h \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, and let $F_h : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ be the local functional defined as*

$$F_h(u, A) := \begin{cases} \int_A f_h(x, Xu(x))dx & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A), \\ \infty & \text{otherwise.} \end{cases} \tag{51}$$

Then, there exist local functionals $F : L^p(\Omega)\times\mathcal{A} \to \mathbb{R}\cup\{\infty\}$ and $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ such that, up to subsequences,

$$F(\cdot, A) = \Gamma(L^p(\Omega))\text{-}\lim_{h\to\infty} F_h(\cdot, A) \text{ for each } A \in \mathcal{A} \tag{52}$$

and F admits the following representation:

$$F(u, A) := \begin{cases} \int_A f(x, Xu(x))dx & \text{if } A \in \mathcal{A}, u \in W^{1,p}_X(A), \\ \infty & \text{otherwise.} \end{cases} \tag{53}$$

Following [DM, Theorem 21.1], using [MPSC1, Theorem 4.16] instead of [DM, Theorem 19.6], we get the Γ-convergence for functionals with boundary data. A proof can be found in [Ma, Theorem 2.3.23].

THEOREM 3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\mathcal{A}$ be the class of all open subsets of Ω, $1 < p < \infty$, and let X satisfy the LIC. Let $f_h \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, let $F_h : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ be the functional in (51), and, with a little abuse of notation, denote*

$$F_h(u) := F_h(u, \Omega) \quad \text{for any } u \in L^p(\Omega)\,.$$

Fix $\varphi \in W^{1,p}_X(\Omega)$ and assume that $(F_h)_h$ Γ-converges in the strong topology of $L^p(\Omega)$ to F satisfying (53), with $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$. Then, $(F_h + \mathbb{1}_\varphi)_h$ Γ-converges to $F + \mathbb{1}_\varphi$ in the strong topology of $L^p(\Omega)$.

We conclude this section by providing a Γ-convergence result of perturbed functionals in the strong topology of $L^p(\Omega)$.

THEOREM 3.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\mathcal{A}$ be the class of all open subsets of Ω, let $1 < p < \infty$, and let X satisfy the LIC. Let $f_h, f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ and F_h, F be the functionals in (51) and (53) satisfying (52). For any fixed $\Phi \in L^p(\Omega)^m$, let $G^\Phi_h : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ be the local functional defined as*

$$G^\Phi_h(u, A) := \begin{cases} \int_A f_h(x, Xu(x) + \Phi(x))\, dx & \text{if } A \in \mathcal{A}, u \in W^{1,p}_X(A), \\ \infty & \text{otherwise.} \end{cases}$$

Then, there exists $G^\Phi : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ such that, up to subsequences,

$$G^\Phi(\cdot, A) = \Gamma(L^p(\Omega))\text{-}\lim_{h\to\infty} G^\Phi_h(\cdot, A) \text{ for each } A \in \mathcal{A}, \tag{54}$$

and G^Φ admits the following representation:

$$G^\Phi(u, A) := \begin{cases} \int_A f(x, Xu(x) + \Phi(x))\, dx & \text{if } A \in \mathcal{A}, u \in W^{1,p}_X(A), \\ \infty & \text{otherwise.} \end{cases}$$

Before giving the proof of Theorem 3.7, let us recall the following classical result. For the reader's convenience, we remind that a proof can be found in [Ma2, Lemma 2.3.32] and follows the technique of [Da, Proposition 2.32].

LEMMA 3.8. *Let $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$. Then, there exists a nonnegative constant c_2, depending only on p and c_1, such that*

$$|f(x,\eta_1) - f(x,\eta_2)| \leq c_2\, |\eta_1 - \eta_2| \left(|\eta_1| + |\eta_2| + a(x)^{1/p}\right)^{p-1}, \tag{55}$$

where $a(x) := a_0(x) + a_1(x)$, for a.e. $x \in \Omega$, for each $\eta_1, \eta_2 \in \mathbb{R}^m$.

Proof of Theorem 3.7. Fix $\Phi \in L^p(\Omega)^m$ and, for each $h \in \mathbb{N}$, define $g_h^\Phi(x,\eta) := f_h(x, \eta + \Phi(x))$ a.e. $x \in \Omega$ and for any $\eta \in \mathbb{R}^m$. Then,

$$g_h^\Phi \in I_{m,p}(\Omega, c_0, c_3, \tilde{a}_0, \tilde{a}_1)\,, \tag{56}$$

with $\tilde{a}_0(x) := a_0(x) - c_0|\Phi(x)|^p$ and $\tilde{a}_1(x) := a_1(x) + c_3|\Phi(x)|^p$, for a suitable positive constant c_3 (depending only on p and c_1).

By Theorem 3.5, there exist $G^\Phi : L^p(\Omega) \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ and

$$g^\Phi \in I_{m,p}(\Omega, c_0, c_3, \tilde{a}_0, \tilde{a}_1) \tag{57}$$

such that, up to subsequences, (54) holds and G^Φ can be represented as

$$G^\Phi(u,A) := \begin{cases} \int_A g^\Phi(x, Xu(x))\, dx & \text{if } A \in \mathcal{A},\ u \in W_X^{1,p}(A), \\ \infty & \text{otherwise.} \end{cases}$$

To conclude, we show that

$$G^\Phi(u,A) = \int_A f(x, Xu(x) + \Phi(x))\, dx \tag{58}$$

for each $A \in \mathcal{A}$ and $u \in W_X^{1,p}(A)$. We divide the proof of (58) into three steps.

Step 1. Let us first prove the existence of a positive constant c_4, depending only on c_0, c_1, c_2, a_0, a_1, and p, such that

$$\begin{aligned} &|G^{\Phi_1}(u,A) - G^{\Phi_2}(u,A)| \\ &\qquad \leq c_4\, \|\Phi_1 - \Phi_2\|_{L^p} \left(\|Xu\|_{L^p} + \|\Phi_1\|_{L^p} + \|\Phi_2\|_{L^p} + 1\right)^{p-1} \end{aligned} \tag{59}$$

for any Φ_1, $\Phi_2 \in L^p(\Omega)^m$, $A \in \mathcal{A}$, and $u \in W_X^{1,p}(A)$, where all norms above refer to A.

Fix Φ_1, $\Phi_2 \in L^p(\Omega)^m$, $A \in \mathcal{A}$, and $u \in W_X^{1,p}(A)$. By (54) and [DM, Proposition 8.1], there exists a sequence $(u_h)_h \subset L^p(\Omega) \cap W_X^{1,p}(A)$, strongly convergent to u in $L^p(\Omega)$, such that

$$G^{\Phi_2}(u,A) = \lim_{h\to\infty} G_h^{\Phi_2}(u_h, A) \tag{60}$$

and

$$G^{\Phi_1}(u,A) \leq \liminf_{h\to\infty} G_h^{\Phi_1}(u_h, A)\,. \tag{61}$$

Then, by (55), (56), and Hölder's inequality, there exist positive constants α_1, α_2, depending only c_0, c_1, c_2, a_0, a_1, and p, such that

$$
\begin{aligned}
&G_h^{\Phi_1}(u_h, A) - G_h^{\Phi_2}(u_h, A) \\
&\qquad \le \int_A |f_h(x, Xu_h + \Phi_1) - f_h(x, Xu_h + \Phi_2)| \, dx \\
&\qquad \le \alpha_1 \|\Phi_1 - \Phi_2\|_{L^p} \left(\|Xu_h\|_{L^p} + \|\Phi_1\|_{L^p} + \|\Phi_2\|_{L^p} + 1\right)^{p-1} \\
&\qquad \le \alpha_2 \|\Phi_1 - \Phi_2\|_{L^p} \left(G_h^{\Phi_2}(u_h, A)^{1/p} + \|\Phi_1\|_{L^p} + \|\Phi_2\|_{L^p} + 1\right)^{p-1},
\end{aligned} \tag{62}
$$

and by (57), (60), and (61), there exists a positive constant c_4, depending only on c_0, c_1, c_2, a_0, a_1, and p, such that

$$
\begin{aligned}
&G^{\Phi_1}(u, A) - G^{\Phi_2}(u, A) \\
&\qquad \le \alpha_2 \|\Phi_1 - \Phi_2\|_{L^p} \left(G^{\Phi_2}(u, A)^{1/p} + \|\Phi_1\|_{L^p} + \|\Phi_2\|_{L^p} + 1\right)^{p-1} \\
&\qquad \le c_4 \|\Phi_1 - \Phi_2\|_{L^p} \left(\|Xu\|_{L^p} + \|\Phi_1\|_{L^p} + \|\Phi_2\|_{L^p} + 1\right)^{p-1}.
\end{aligned}
$$

By exchanging the roles of Φ_1 and Φ_2, we obtain (59).

Step 2. Let us prove (58), when Φ has the form

$$
\Phi(x) = C(x)\, \tilde{\Phi}(x) \quad \text{a.e. } x \in \Omega \tag{63}
$$

for some $\tilde{\Phi} \in L^p(\Omega)^n$, where $C(x)$ denotes the coefficient matrix of the X-gradient (4). Let us divide this step into three cases.

Case 1. Suppose $\tilde{\Phi}(x) = \xi \in \mathbb{R}^n$ constant and denote $\Phi_\xi(x) := C(x)\, \xi$. If $u_\xi(x) := \langle \xi, x\rangle$ for any $x \in \mathbb{R}^n$, then

$$
G_h^{\Phi_\xi}(u, A) = F_h(u + u_\xi, A),
$$

and by (52), (53), and (54), we get

$$
G^{\Phi_\xi}(u, A) = F(u + u_\xi, A) = \int_A f(x, Xu(x) + \Phi_\xi(x))\, dx \tag{64}
$$

for any $A \in \mathcal{A}$ and $u \in W^{1,p}_X(A)$.

Case 2. Suppose $\tilde{\Phi}$ *piecewise constant*, i.e., there exist $\xi^1, \dots, \xi^N \in \mathbb{R}^n$ and $A_1, \dots, A_N$ pairwise disjoint open sets such that $|\Omega \setminus \cup_{i=1}^N A_i| = 0$ and

$$
\tilde{\Phi}(x) := \sum_{i=1}^N \chi_{A_i}(x)\, \xi^i.
$$

Fix $A \in \mathcal{A}$ and $u \in W^{1,p}_X(A)$ and denote $\Phi_{\xi^i}(x) := C(x)\, \xi^i$. Since $G^\Phi(u, \cdot)$ is a measure, then, by additivity on pairwise disjoint open sets and locality, it holds that

$$
G^\Phi(u, A) = \sum_{i=1}^N G^\Phi(u, A \cap A_i) = \sum_{i=1}^N G^{\Phi_{\xi^i}}(u, A \cap A_i). \tag{65}
$$

Let $\tilde{u}(x) := \langle \tilde{\Phi}(x), x\rangle$ for any $x \in \mathbb{R}^n$. Then,

$$
\tilde{u}(x) = u_{\xi^i}(x) = \langle \xi^i, x\rangle \quad \text{a.e. } x \in A \cap A_i
$$

for any $i = 1, \dots, N$ and, by locality of F,

$$F(u + \tilde{u}, A \cap A_i) = F(u + u_{\xi^i}, A \cap A_i)\,. \tag{66}$$

Therefore, by (53), (64), (65), (66), and additivity of F on pairwise disjoint open sets, we get

$$\begin{aligned} G^{\Phi}(u, A) &= \sum_{i=1}^{N} G^{\Phi_{\xi^i}}(u, A \cap A_i) = \sum_{i=1}^{N} F(u + u_{\xi^i}, A \cap A_i) \\ &= \sum_{i=1}^{N} F(u + \tilde{u}, A \cap A_i) = F(u + \tilde{u}, A) \\ &= \int_A f(x, Xu(x) + \Phi(x))\, dx \end{aligned} \tag{67}$$

for any $A \in \mathcal{A}$ and $u \in W^{1,p}_X(A)$.

Case 3. Let Φ have the form (63), let $(\tilde{\Phi}_j)_j$ be a sequence of piecewise constant functions converging to $\tilde{\Phi}$ strongly in $L^p(\Omega)^n$, as $j \to \infty$, and define $\Phi_j(x) := C(x)\,\tilde{\Phi}_j(x)$ a.e. $x \in \Omega$. Since

$$C \in L^{\infty}(\Omega)^{mn}, \tag{68}$$

then

$$(\Phi_j)_j \text{ strongly converges to } \Phi \text{ in } L^p(\Omega)^m. \tag{69}$$

If $A \in \mathcal{A}$ and $u \in W^{1,p}_X(A)$, then by (59),

$$G^{\Phi_j}(u, A) \to G^{\Phi}(u, A) \quad \text{as } j \to \infty. \tag{70}$$

and by (67) and Hölder's inequality, it holds that

$$\begin{aligned} &\left| G^{\Phi_j}(u, A) - \int_A f(x, Xu + \Phi)\, dx \right| \\ &\qquad \le \int_A |f(x, Xu + \Phi_j) - f(x, Xu + \Phi)|\, dx \\ &\qquad \le \alpha_1 \, \|\Phi_j - \Phi\|_{L^p} \, \left(\|Xu\|_{L^p} + \|\Phi_j\|_{L^p} + \|\Phi\|_{L^p} + 1\right)^{p-1}, \end{aligned} \tag{71}$$

where α_1 is the positive constant given in (62). Therefore, (58) follows from (69), (70), and (71).

Step 3. Let us finally prove (58) in the general case.

Fix $\Phi \in L^p(\Omega)^m$ and $x \in \Omega_X$ (see Definition (1.1)). Then, in virtue of [MPSC1, Lemma 3.3], there exists $\tilde{\Phi}(x) \in \mathbb{R}^n$ such that

$$C(x)\,\tilde{\Phi}(x) = \Phi(x)$$

and $\tilde{\Phi}$ can be represented as

$$\tilde{\Phi}(x) = C(x)^T\, B(x)^{-1}\, \Phi(x)\,, \tag{72}$$

where $B(x)$ is the $m \times m$ symmetric invertible matrix defined by

$$B(x) := C(x)\, C(x)^T.$$

Since $B(x)$ is positive semidefinite for any $x \in \Omega$ and it is positive definite if and only if $x \in \Omega_X$, it holds that

$$|\Omega \setminus \Omega_X| := |\mathcal{N}_X| = 0 \tag{73}$$

and

$$\Omega_X = \{x \in \Omega :\ \mathrm{det} B(x) > 0\}\,, \quad \mathcal{N}_X = \{x \in \Omega :\ \mathrm{det} B(x) = 0\}\,.$$

For any $\varepsilon > 0$, define

$$\Omega_\varepsilon := \{x \in \Omega : \mathrm{det} B(x) > \varepsilon\}\,.$$

Since $B \in L^\infty(\Omega)^{m^2}$, then by Cramer's rule and (68) and (72),

$$B^{-1} \in L^\infty(\Omega_\varepsilon)^{m^2} \quad \text{and} \quad \tilde{\Phi} \in L^p(\Omega_\varepsilon)^n. \tag{74}$$

Let $\tilde{\Phi}_\varepsilon : \Omega \to \mathbb{R}^n$ and $\Phi_\varepsilon : \Omega \to \mathbb{R}^m$ be, respectively, defined by

$$\tilde{\Phi}_\varepsilon(x) := \begin{cases} \tilde{\Phi}(x) & \text{if } x \in \Omega_\varepsilon, \\ 0 & \text{if } x \in \Omega \setminus \Omega_\varepsilon \end{cases}$$

and

$$\Phi_\varepsilon(x) := C(x)\tilde{\Phi}_\varepsilon(x) = \begin{cases} \Phi(x) & \text{if } x \in \Omega_\varepsilon, \\ 0 & \text{if } x \in \Omega \setminus \Omega_\varepsilon. \end{cases}$$

By (74),

$$\tilde{\Phi}_\varepsilon \in L^p(\Omega)^n, \tag{75}$$

and by (73) and Hölder's inequality, it follows that

$$\Phi_\varepsilon \to \Phi \quad \text{strongly in } L^p(\Omega)^m \text{ as } \varepsilon \to 0\,. \tag{76}$$

If $A \in \mathcal{A}$ and $u \in W^{1,p}_X(A)$, then, by (55), (59), (75), by Hölder's inequality and the second step of the proof, there exists a positive constant c_5, depending only on c_0, c_1, c_2, a_0, a_1, and p, such that

$$\begin{aligned} &\left| G^\Phi(u,A) - \int_A f(x, Xu + \Phi)\, dx \right| \\ &\qquad \le \left| G^\Phi(u,A) - G^{\Phi_\varepsilon}(u,A) \right| + \left| G^{\Phi_\varepsilon}(u,A) - \int_A f(x, Xu + \Phi)\, dx \right| \\ &\qquad \le c_5 \left\| \Phi_\varepsilon - \Phi \right\|_{L^p(A)^m} \left(\|Xu\|_{L^p(A)^m} + \|\Phi_\varepsilon\|_{L^p(A)^m} + \|\Phi\|_{L^p(A)^m} + 1 \right)^{p-1} \end{aligned}$$

and (58) follows by (76), as $\varepsilon \to 0$. ∎

4. Convergence of minima, minimizers, and momenta.

4.1. Convergence of minima and minimizers. By Theorem 3.5, there exist $F: L^p(\Omega)\times\mathcal{A} \to \mathbb{R}\cup\{\infty\}$ and $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$ such that, up to subsequences,

$$F(\cdot, A) = \Gamma(L^p(\Omega))\text{-}\lim_{h\to\infty} F_h(\cdot, A) \quad \text{for each } A \in \mathcal{A}$$

and F admits the representation (53). If, in addition, the sequence $(F_h(\cdot, A))_h$ is equicoercive in $L^p(\Omega)$ (see [DM, Definition 7.6]), then by [DM, Theorem 7.8], $F(\cdot, A)$ attains its minimum in $L^p(\Omega)$ and

$$\min_{u\in L^p(\Omega)} F(u, A) = \lim_{h\to\infty} \inf_{u\in L^p(\Omega)} F_h(u, A) \quad \text{for each } A \in \mathcal{A}\,.$$

Let us now study the asymptotic behavior of minima and minimizers of problems (1). We first need two preliminary results, namely Theorem 4.1 and Lemma 4.3. The first result can be proved as [DM, Theorem 2.6] and [DM, Theorem 2.8], and a proof can be found in [Ma, Theorem 2.3.27]. The second one can be instead proved following [DM, Proposition 2.10] and [DM, Proposition 2.11].

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $1 < p < \infty$, and let X satisfy the LIC and conditions* (H1), (H2), *and* (H3). *Let $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, let g satisfy* (10), *and let $F, G : W^{1,p}_X(\Omega) \to \mathbb{R}$ be the functionals defined, respectively, by*

$$F(u) := \int_\Omega f(x, Xu(x))\,dx \quad \textit{and} \quad G(u) := \int_\Omega g(x, u(x))\,dx\,.$$

For any fixed $\varphi \in W^{1,p}_X(\Omega)$, let $\Xi, \Xi^\varphi : W^{1,p}_X(\Omega) \to \mathbb{R}$ be, respectively, defined by

$$\Xi := F + G \quad \textit{and} \quad \Xi^\varphi := F + G + \mathbb{1}_\varphi\,.$$

Then, the minimum problems

$$\min_{u\in W^{1,p}_X(\Omega)} \Xi(u) \tag{77}$$

and

$$\min_{u\in W^{1,p}_{X,\varphi}(\Omega)} \Xi^\varphi(u) \tag{78}$$

have at least a solution, provided that

$$0 < d_0 \le d_1, \tag{79}$$

and (11) *holds, respectively, where d_0 and d_1 are the constants in* (10). *If, in addition,*

(i) *$g(x, \cdot)$ is strictly convex on $\mathbb{R}$ for a.e. $x \in \Omega$, then both solutions in* (77) *and* (78) *are unique;*

(ii) *$f(x, \cdot)$ is strictly convex on $\mathbb{R}^m$ and $g(x, \cdot)$ is convex on $\mathbb{R}$ for a.e. $x \in \Omega$; then the solution in* (78) *is unique.*

Moreover,

$$\min_{u\in W^{1,p}_X(\Omega)} \Xi(u) = \inf_{u\in \mathbf{C}^1(\Omega)\cap W^{1,p}_X(\Omega)} \Xi(u)\,. \tag{80}$$

Remark 4.2. *Observe that, arguing as in* [DM, Corollary 2.9], *also a linear functional $G: L^p(\Omega) \to \mathbb{R}$, $G(u) := \int_\Omega g(x)\,u(x)\,dx$, for a given function $g \in L^{p'}(\Omega)$, is allowed in minimization problem* (78).

LEMMA 4.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$, let $1 < p < \infty$, and let X satisfy the LIC and conditions* (H1), (H2), *and* (H3). *For any fixed $\varphi \in W^{1,p}_X(\Omega)$, let $\Psi^\varphi : L^p(\Omega) \to [0,\infty]$ be defined by*

$$\Psi^\varphi(u) := \begin{cases} \int_\Omega |Xu(x)|^p\,dx & \text{if } u \in W^{1,p}_{X,\varphi}(\Omega),\\ \infty & \text{otherwise.}\end{cases} \tag{81}$$

Then, Ψ^φ is coercive and lower semicontinuous in the strong topology of $L^p(\Omega)$.

We are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. (i) By (78), both functionals Ξ^φ_h and Ξ^φ attain their minima in $L^p(\Omega)$ and, by Theorem 3.6,

$$(F_h + \mathbb{1}_\varphi)_h \quad \Gamma\text{-converges to} \quad F + \mathbb{1}_\varphi$$

in the strong topology of $L^p(\Omega)$. Moreover, since G is continuous in the strong topology of $L^p(\Omega)$ (it is readily seen, proceeding exactly as in the proof of Theorem 4.1), then

$$(\Xi^\varphi_h)_h \quad \Gamma\text{-converges to} \quad \Xi^\varphi$$

in the strong topology of $L^p(\Omega)$, in virtue of [DM, Proposition 6.21].

Let Ψ^φ be the functional defined in (81). By (5), (10), and Lemma 2.18, there exist a positive constant k_3, depending on c_0, d_0, k_1, and a positive constant k_4, depending on a_0, b_0, k_2, such that

$$\Xi^\varphi_h(u) \ge k_3 \Psi^\varphi(u) - k_4 \quad \text{for any } u \in L^p(\Omega) \text{ for any } h \in \mathbb{N}\,, \tag{82}$$

where k_1, k_2 are the constants in Lemma 2.18. Then, by [DM, Proposition 7.7], $(\Xi^\varphi_h)_h$ is equicoercive in the strong topology of $L^p(\Omega)$, and, by [DM, Theorem 7.8], Ξ^φ is also coercive and (13) follows.

(ii) Let $(u_h)_h$ be a sequence of minimizers of $(\Xi^\varphi_h)_h$. Without loss of generality, we may assume $(u_h)_h \subset W^{1,p}_{X,\varphi}(\Omega)$. As in (82),

$$\infty > \Xi^\varphi_h(u_h) \ge k_3 \|u_h\|^p_{W^{1,p}_X(\Omega)} - k_4 \quad \text{for any } h \in \mathbb{N}\,,$$

i.e., $(u_h)_h$ in bounded in $W^{1,p}_{X,\varphi}(\Omega)$ and, arguing as in the proof of Lemma 4.3, there exists $\bar{u} \in W^{1,p}_{X,\varphi}(\Omega)$ such that, up to subsequences, (14) holds. Finally, by [DM, Corollary 7.20], we get (15). ∎

4.2. Convergence of momenta. We now deal with the convergence of the momenta associated with functionals $(F_h)_h$. The results contained in this section are inspired by [ADMZ2] and they partially extend those results to integral functionals depending on vector fields.

Let $f \in I_{m,p}(\Omega, c_0, c_1, a_0, a_1)$, let $f(x,\cdot) : \mathbb{R}^m \to \mathbb{R}$ be of class $\mathbf{C}^1(\mathbb{R}^m)$ for a.e. $x \in \Omega$, and denote by $\nabla_\eta f(x,\eta)$ its gradient at $\eta \in \mathbb{R}^m$.

By Lemma 3.8, there exists a nonnegative constant c_2, depending only on p and c_1, such that

$$|\nabla_\eta f(x,\eta)| \le\, c_2\,(2\,|\eta| + (a_0(x) + a_1(x))^{1/p})^{p-1} \quad \text{for a.e. } x \in \Omega\ \forall \eta \in \mathbb{R}^m.$$

Then, functional $\mathcal{F}$ in (17) is of class $\mathbf{C}^1(L^p(\Omega)^m)$ and its Gateaux derivative $\partial_\Phi \mathcal{F} : L^p(\Omega)^m \to L^{p'}(\Omega)^m$ is given by

$$\partial_\Phi \mathcal{F}(\Phi)(x) = \nabla_\eta f(x, \Phi(x)) \quad \text{for a.e. } x \in \Omega\,. \tag{83}$$

The main results of this section are Theorem 1.3 and Corollary 1.4. Since the proof of Theorem 1.3 is an adaptation of a technique introduced in [DMFT, Lemma 4.11] and [ADMZ2, Theorem 4.5], we omit it. For an explicit proof, see, e.g., [Ma2, Theorem 3.1.6].

REMARK 4.4. *We are aware that assumption* (iii) *in Theorem* 1.3*, on integrand function $f(x,\cdot)$ to be $\mathbf{C}^1(\mathbb{R}^m)$, is quite strong. Since the techniques exploited in the Euclidean setting do not seem to work in our framework, we do not know whether assumption* (ii) *actually implies it (see* [GP, Proposition 3.5] *and* [ADMZ2, Theorem 2.8]*). However, we will be able to prove this result in two relevant cases: the periodic homogenization in Carnot groups and the case of quadratic forms (see sections* 5 *and* 6 *below).*

Proof of Corollary 1.4. Let $(u_h)_h$ be a sequence of minimizers of $(\Xi_h^\varphi)_h$. By Theorem 1.2, there exists a minimizer u of Ξ^φ such that, up to subsequences,

$$u_h \to u \text{ weakly in } W^{1,p}_X(\Omega) \text{ and strongly in } L^p(\Omega)$$

and

$$\Xi_h^\varphi(u_h) \to \Xi^\varphi(u)\,. \tag{84}$$

Since G is continuous in the strong topology of $L^p(\Omega)$, then

$$G(u_h) \to G(u)$$

and, by (84),

$$\mathcal{F}_h(Xu_h) = F_h(u_h) \to F(u) = \mathcal{F}(Xu)\,. \tag{85}$$

Then, the thesis follows by Theorem 1.3. ∎

5. Convergence of minimizers and momenta in the case of homogenization in Carnot groups. We are going to study the convergence of momenta for a Γ-convergent sequence of functionals in the case of the periodic *homogenization* in Carnot groups. The asymptotic behavior for the periodic homogenization of sequences of functionals and differential operators in Carnot groups has been the object of an intensive study in the last two decades. Here, for the sake of simplicity, we will restrict to the case of periodic homogenization in the setting of Heisenberg groups, which turn out to be the simplest Carnot groups.

Let us recall that the s-dimensional Heisenberg group $\mathbb{H}^s := (\mathbb{R}^n, \cdot)$, with $n = 2s+1 = m+1$ and $s \geq 1$ integer, is a Lie group with respect to the group law

$$x \cdot y := (x_1 + y_1, \ldots, x_m + y_m, x_{m+1} + y_{m+1} + \omega(x,y))\,,$$

where

$$\omega(x,y) := \frac{1}{2}\sum_{i=1}^{s}(x_i\, y_{s+i} - y_i\, x_{s+i})$$

if $x = (x_1, \ldots, x_{m+1})$ and $y = (y_1, \ldots, y_{m+1})$ are in $\mathbb{R}^n$. Moreover, $\mathbb{H}^s$ can be equipped with a one-parameter family of *intrinsic dilations* $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ $(\lambda > 0)$, defined as

$$\delta_\lambda(x) := (\lambda\, x_1, \ldots, \lambda\, x_m, \lambda^2\, x_{m+1}) \text{ if } x = (x_1, \ldots, x_m, x_{m+1})\,,$$

which are also automorphisms of the group. A standard basis of the Lie algebra associated to $\mathbb{H}^s$ is given by the following family of n left-invariant vector fields:

$$X_j := \begin{cases} \partial_j - \frac{x_{s+j}}{2}\partial_n & \text{if } 1 \le j \le s, \\ \partial_j + \frac{x_{j-s}}{2}\partial_n & \text{if } s+1 \le j \le m = 2s, \\ \partial_n & \text{if } j = n = m+1. \end{cases}$$

Notice that the only nontrivial commutations among vector fields X_j's are given by

$$[X_j, X_{s+j}] = X_n \text{ for each } j = 1, \dots, s\,. \tag{86}$$

The family of vector fields

$$X := (X_1, \dots, X_m) \tag{87}$$

is called a *horizontal gradient* of the Heisenberg group $\mathbb{H}^s$. It is immediate to see that X satisfies the LIC and, by (86), the Hörmander condition. Thus, by Remark 2.8 (i), the horizontal gradient X satisfies assumptions (H1), (H2), and (H3).

Since $\mathbb{H}^s$ is a homogeneous Lie group, it is possible to introduce two natural notions of intrinsic periodicity and linearity, with respect to its algebraic structure (see [DDMM, Definition 2.3]).

Definition 5.1. *A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be H-periodic whenever*

$$g(2k \cdot x) = g(x) \quad \textit{for any } x \in \mathbb{R}^n \textit{ for any } k \in \mathbb{Z}^n. \tag{88}$$

Moreover, a function $l : \mathbb{R}^n \to \mathbb{R}$ is said to be H-linear if

$$l(x \cdot y) = l(x) + l(y) \quad \textit{and} \quad l(\delta_\lambda(x)) = \lambda\, l(x)$$

for each x, $y \in \mathbb{R}^n$ and $\lambda > 0$.

It is well known that each H-linear function $l : \mathbb{R}^n \to \mathbb{R}$ can be represented by a unique $\eta \in \mathbb{R}^m$ in such a way that

$$l(x) = l_\eta(x) := \langle \eta, \pi_m(x) \rangle, \tag{89}$$

where $\langle \cdot, \cdot \rangle$ and $\pi_m : \mathbb{R}^n \to \mathbb{R}^m$ denote, respectively, the scalar product on $\mathbb{R}^m$ and the projection map

$$\pi_m(x) := (x_1, \dots, x_m) \quad \text{if } x = (x_1, \dots, x_{m+1}) \in \mathbb{R}^n.$$

Definition 5.2. *A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be H-affine if there exist $\eta \in \mathbb{R}^m$ and $a \in \mathbb{R}$ such that*

$$u(x) = l_\eta(x) + a \quad \textit{for each } x \in \mathbb{R}^n.$$

Let $X = (X_1, \dots, X_m)$ denote the horizontal gradient (87) and fix $\eta \in \mathbb{R}^m$. One can prove that, for any H-affine function u,

$$Xu(x) = \eta \quad \text{if and only if} \quad u(x) = l_\eta(x) + a \tag{90}$$

for some $a \in \mathbb{R}$ and for any $x \in \mathbb{R}^n$ (see, e.g., [DDMM, Lemma 3.1]).

Let us recall a Γ-convergence result for the periodic homogenization of a sequence of integral functionals in $\mathbb{H}^s$.

THEOREM 5.3 (see [DDMM, Theorem 5.2]). *Let $\mathcal{A}_0$ be the class of all open bounded subsets of $\mathbb{R}^n$, $1 < p < \infty$, and $f \in I_{m,p}(\mathbb{R}^n, c_0, c_1, 0, 1)$, which satisfy*

$$f(\cdot, \eta) : \mathbb{R}^n \to [0, \infty) \text{ is } H\text{-periodic for every } \eta \in \mathbb{R}^m. \tag{91}$$

Let $F_\varepsilon : L^p_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A}_0 \to [0, \infty]$ be the local functional defined as

$$F_\varepsilon(u, A) := \begin{cases} \int_A f(\delta_{1/\varepsilon}(x), Xu(x))\, dx & \text{if } A \in \mathcal{A}_0,\ u \in W^{1,p}_X(A), \\ \infty & \text{otherwise.} \end{cases} \tag{92}$$

Then, there exist a local functional $F_0 : L^p_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A}_0 \to [0, \infty]$ and a convex function $f_0 : \mathbb{R}^m \to [0, \infty)$, not depending on x and satisfying

$$c_0|\eta|^p \le f_0(\eta) \le c_1|\eta|^p + 1 \quad \text{for any } \eta \in \mathbb{R}^m,$$

such that, up to subsequences,

$$F_0(\cdot, A) = \Gamma(L^p_{\mathrm{loc}}(\mathbb{R}^n))\text{-}\lim_{h\to\infty} F_{\varepsilon_h}(\cdot, A) \text{ for each } A \in \mathcal{A}_0 \tag{93}$$

for any infinitesimal sequence $(\varepsilon_h)_h$, and F_0 can be represented as

$$F_0(u, A) := \begin{cases} \int_A f_0(Xu(x))\, dx & \text{if } A \in \mathcal{A}_0,\ u \in W^{1,p}_X(A), \\ \infty & \text{otherwise.} \end{cases} \tag{94}$$

We are going to prove the following regularity result for the integrand function f_0 which represents the Γ-limit. It is an extension to functionals depending on vector fields of [GP, Proposition 3.5], when $X = D$, and of [ADMZ2, Theorem 2.8], which applies to a more general setting.

PROPOSITION 5.4. *Under the hypotheses of Theorem 5.3, suppose that*

$$\mathbb{R}^m \ni \eta \mapsto f(x, \eta) \text{ belongs to } \mathbf{C}^1(\mathbb{R}^m) \text{ for a.e. } x \in \Omega \tag{95}$$

and, fixed $0 \le \alpha \le \min\{1, p-1\}$, that there exists a positive constant $\overline{c}$ such that

$$|\nabla_\eta f(x, \eta_1) - \nabla_\eta f(x, \eta_2)| \le \overline{c}\, |\eta_1 - \eta_2|^\alpha\ (|\eta_1| + |\eta_2| + 1)^{p-1-\alpha} \tag{96}$$

a.e. $x \in \Omega$, for any η_1, $\eta_2 \in \mathbb{R}^m$. Then, $f_0 \in \mathbf{C}^1(\mathbb{R}^m)$.

Proof. We follow the same strategy of [GP, Proposition 3.5].

Let $f_h : \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty)$ be the function

$$f_h(y, \eta) := f(\delta_{1/\varepsilon_h}(y), \eta) \quad \text{for any } y \in \mathbb{R}^n, \eta \in \mathbb{R}^m, h \in \mathbb{N} \tag{97}$$

and denote B_1 the unit ball in $\mathbb{R}^n$ centered at 0. For any fixed $\eta \in \mathbb{R}^m$, let $u : \mathbb{R}^n \to \mathbb{R}$ be the H-affine function (89), that is,

$$u(y) = l_\eta(y) \quad \text{for any } y \in \mathbb{R}^n.$$

If $(u_h)_h$ is a recovery sequence for u, then, by (93) and (94),

$$u_h \to l_\eta \quad \text{strongly in } L^p(B_1) \tag{98}$$

and

$$F_0(l_\eta, B_1) = f_0(\eta)\,|B_1| = \lim_{h\to\infty} F_{\varepsilon_h}(u_h, B_1)\,. \tag{99}$$

By (99) and (I_3), $(Xu_h)_h$ is bounded in $L^p(B_1)^m$ and, by (55),

$$|\nabla_\eta f_h(y, Xu_h(y))| \le c_2\,(2|Xu_h(y)|+1)^{p-1}$$

for each $h \in \mathbb{N}$ and for a.e. $y \in B_1$. Then, $(\nabla_\eta f_h(\cdot, Xu_h))_h$ is bounded in $L^{p'}(B_1)^m$ and there exists $\psi \in \mathbb{R}^m$ such that, up to subsequences,

$$\frac{1}{|B_1|}\int_{B_1} \nabla_\eta f_h(y, Xu_h(y))\,dy \to \psi\,. \tag{100}$$

Let $(t_j)_j$ be a decreasing infinitesimal sequence. By the convexity of f_h in the second variable,

$$\begin{aligned}\int_{B_1} f_h(y, Xu_h(y)+t_j\zeta) - f_h(y, Xu_h(y))\,dy \\ \le t_j \int_{B_1} \langle \nabla_\eta f_h\left(y, Xu_h(y)+t_j\zeta\right), \zeta\rangle\,dy\end{aligned} \tag{101}$$

for any $j \in \mathbb{N}$ and $\zeta \in \mathbb{R}^m$.

By the Γ-$\liminf$ inequality and (99), we can find $h_j \in \mathbb{N}$ such that

$$\frac{f_0\left(\eta+t_j\zeta\right)-f_0\left(\eta\right)}{t_j} - \frac{1}{j} \le \frac{1}{|B_1|}\int_{B_1} \langle \nabla_\eta f_{h_j}\left(y, Xu_{h_j}(y)+t_j\zeta\right), \zeta\rangle\,dy$$

and so

$$\begin{aligned}\limsup_{j\to\infty} \frac{f_0\left(\eta+t_j\zeta\right)-f_0\left(\eta\right)}{t_j} \\ \le \frac{1}{|B_1|}\limsup_{j\to\infty}\int_{B_1} \langle \nabla_\eta f_{h_j}\left(y, Xu_{h_j}(y)+t_j\zeta\right), \zeta\rangle\,dy\,.\end{aligned} \tag{102}$$

By (96), (100), and [ADMZ2, Lemma 4.4], with $H_j := \nabla_\eta f_{h_j}$, $\Phi_j := Xu_{h_j}$, $\Psi_j := t_j\,\zeta$, and $\Phi \equiv 1$, we can infer that

$$\begin{aligned}&\lim_{j\to\infty}\int_{B_1} \langle \nabla_\eta f_{h_j}\left(y, Xu_{h_j}(y)+t_j\zeta\right), \zeta\rangle\,dy \\ &= \lim_{j\to\infty}\int_{B_1} \langle \nabla_\eta f_{h_j}\left(y, Xu_{h_j}(y)\right), \zeta\rangle\,dy = \langle \psi, \zeta\rangle\,|B_1|\,.\end{aligned}$$

By (102), for every subgradient v of the convex function f_0 at η,

$$\langle v, \zeta\rangle \le \limsup_{j\to\infty}\frac{f_0\left(\eta+t_j\zeta\right)-f_0\left(\eta\right)}{t_j} \le \langle \psi, \zeta\rangle \quad \text{for each } \zeta \in \mathbb{R}^m\,.$$

Then, $v = \psi$ and there is a unique subgradient of f_0 at η. Therefore, f_0 is differentiable at η for each $\eta \in \mathbb{R}^m$, by [Ro, Theorem 25.1]. On the other hand, by [Ro, Corollary 25.5.1], any finite convex differentiable function on an open convex set must be of class $\mathbf{C}^1$. ∎

COROLLARY 5.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and connected open set, let $1 < p < \infty$, and let X be the horizontal gradient defined in* (87)*. Let $f \in I_{m,p}(\Omega, c_0, c_1, 0, 1)$ satisfy* (91), (95), *and* (96), *let f_h be as in* (97), *let g satisfy* (10) *and* (11), *let G be the functional in* (3), *and with a little abuse of notation, denote*

$$F_{\varepsilon_h}(u) := F_{\varepsilon_h}(u, \Omega) \quad \text{and} \quad F_0(u) := F_0(u, \Omega) \quad \text{for any } u \in L^p(\Omega)\,,$$

where F_{ε_h} and F_0 are, respectively, as in (92) *and* (94) *such that* (93) *holds for each infinitesimal sequence $(\varepsilon_h)_h$. For any fixed $\varphi \in W^{1,p}_X(\Omega)$, consider functionals $\Xi^\varphi_h, \Xi^\varphi_0 : L^p(\Omega) \to \mathbb{R} \cup \{\infty\}$, defined by*

$$\Xi^\varphi_h := F_{\varepsilon_h} + G + \mathbb{1}_\varphi \quad \text{and} \quad \Xi^\varphi_0 := F_0 + G + \mathbb{1}_\varphi\,.$$

If $(u_h)_h \subset L^p(\Omega)$ is a sequence of minimizers of $(\Xi^\varphi_h)_h$, then there exists a minimum u of Ξ^φ such that, up to subsequences,

$$u_h \to u \text{ weakly in } W^{1,p}_X(\Omega) \text{ and strongly in } L^p(\Omega)\,.$$

Moreover, the convergence of momenta also holds, that is,

$$\nabla_\eta f_h(\cdot, Xu_h(\cdot)) \to \nabla_\eta f_0(Xu(\cdot)) \tag{103}$$

weakly in $L^{p'}(\Omega)^m$ as $h \to \infty$.

Proof. In virtue of Remark 2.8 (i), by choosing $\Omega_0 = \mathbb{R}^n$, the horizontal gradient X satisfies (H1), (H2), and (H3) as well as the LIC. By (93), $(F_{\varepsilon_h})_h$ Γ-converges to F_0 in the strong topology of $L^p(\Omega)$. Therefore, by Proposition 5.4, we can apply Corollary 1.4 and we get the desired conclusions. ∎

6. *H*-convergence for linear operators in *X*-divergence form. Throughout this section $X = (X_1, \dots, X_m)$ denotes a family of Lipschitz continuous vector fields on an open neighborhood Ω_0 of Ω, open subset of $\mathbb{R}^n$. Moreover, we denote by $H^1_{X,0}(\Omega)$ the space $W^{1,2}_{X,0}(\Omega)$ and by $H^{-1}_X(\Omega)$ its dual space. Since $H^1_{X,0}(\Omega)$ turns out to be a Hilbert space, in a standard way, then we can construct the triplet

$$H^1_{X,0}(\Omega) \subset L^2(\Omega) \subset H^{-1}_X(\Omega)\,,$$

with the space $L^2(\Omega)$ as the *pivot space.*

Let $a(x) := [a_{ij}(x)]$ be an $m \times m$ symmetric matrix, with $a_{ij} \in L^\infty(\Omega)$ for every $i, j \in \{1, \dots, m\}$, and assume the existence of $c_0 \le c_1$, positive constants, such that

$$c_0|\eta|^2 \le \langle a(x)\eta, \eta\rangle \le c_1|\eta|^2 \quad \text{a.e. } x \in \Omega \text{ and for any } \eta \in \mathbb{R}^m. \tag{104}$$

As a consequence of Corollary 1.4, we can infer a *H-compactness* result for the class of linear partial differential operators in X-divergence form,

$$\mathcal{L} = \operatorname{div}_X(a(x)X) := \sum_{j,i=1}^m X_j^T(a_{ij}(x)X_i)\,, \tag{105}$$

whose domain $D(\mathcal{L})$ is the set of functions $u \in W^{1,2}_X(\Omega)$ such that the distribution defined by the right-hand side belongs to $L^2(\Omega)$. Here, $X_j^T := -(\operatorname{div}(X_j) + X_j)$ denotes the (formal) adjoint of X_j in $L^2(\Omega)$, as in (19). In accordance with [DM, Chapter 13], we denote this class of operators as $\mathcal{E}(\Omega) := \mathcal{E}(\Omega, c_0, c_1)$.

REMARK 6.1. *For each $\mathcal{L} \in \mathcal{E}(\Omega)$ and for each $\mu \geq 0$, the linear operator*

$$\mu \operatorname{Id} + \mathcal{L} : \, H^1_{X,0}(\Omega) \to H^{-1}_X(\Omega)$$

is coercive and then it is an isomorphism. Moreover, it is well known that, fixed $g \in L^2(\Omega)$, $u = (\mu \operatorname{Id} + \mathcal{L})^{-1} g$ is the (unique) solution to

$$\begin{cases} \mu v + \mathcal{L}(v) = g \text{ in } \Omega, \\ v \in H^1_{X,0}(\Omega), \end{cases}$$

and the (unique) minimizer of $F + G : \, H^1_{X,0}(\Omega) \to \mathbb{R}$, where

$$F(v) = \frac{1}{2} \int_\Omega \langle a(x) X v(x), X v(x) \rangle \, dx \,, \quad G(v) := \int_\Omega \left(\frac{\mu}{2} v^2 - g v \right) dx$$

if $v \in H^1_{X,0}(\Omega)$.

We are going to prove the following H-compactness result for operators belonging to $\mathcal{E}(\Omega)$.

THEOREM 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let X satisfy the LIC and conditions* (H1), (H2), *and* (H3). *Let $\mathcal{L}_h \in \mathcal{E}(\Omega)$, and let $a^h(x) = [a^h_{ij}(x)]$ be the associated matrix, in accordance with* (105). *Then, there exist a symmetric matrix $a = [a_{ij}(x)]$, satisfying* (104), *and an operator $\mathcal{L}_\infty := \operatorname{div}_X(a(x)X) \in \mathcal{E}(\Omega)$ such that, for any $g \in L^2(\Omega)$, $\mu \geq 0$ and $h \in \mathbb{N}$, if u_h and u_∞ denote, respectively, the (unique) solutions to*

$$\begin{cases} \mu u + \mathcal{L}_h(u) = g \text{ in } \Omega, \\ u \in H^1_{X,0}(\Omega) \end{cases} \qquad \text{and} \qquad \begin{cases} \mu u + \mathcal{L}_\infty(u) = g \text{ in } \Omega, \\ u \in H^1_{X,0}(\Omega). \end{cases}$$

Then, up to subsequences, the following convergences hold:

$$u_h \to u_\infty \text{ strongly in } L^2(\Omega) \quad \text{(convergence of solutions)} \tag{106}$$

and

$$a^h X u_h \to a X u_\infty \text{ weakly in } L^2(\Omega)^m \quad \text{(convergence of momenta).}$$

Proof. Step 1. Let us prove that, up to a subsequence, there exists an operator $\mathcal{L} = \operatorname{div}_X(a(x)X) \in \mathcal{E}(\Omega)$ for which (106) holds.

Let $(a^h)_h$ be the sequence of matrices associated to $(\mathcal{L}_h)_h$, and let $F_h : L^2(\Omega) \to [0, \infty]$ be the quadratic functional defined by

$$F_h(u) := \begin{cases} \frac{1}{2} \int_\Omega \langle a^h(x) X u(x), X u(x) \rangle \, dx & \text{if } u \in W^{1,2}_X(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

By [MPSC1, Theorem 4.20], there exist $F : L^2(\Omega) \to [0, \infty]$ and a symmetric matrix $a = [a_{ij}(x)]$, satisfying (104), such that, up to subsequences, $(F_h)_h$ Γ-converges in the strong topology of $L^2(\Omega)$ to F and F can be represented as

$$F(u) := \begin{cases} \frac{1}{2} \int_\Omega \langle a(x) X u(x), X u(x) \rangle \, dx & \text{if } u \in W^{1,2}_X(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}_\infty$ be the elliptic operator associated with a on $L^2(\Omega)$, as in (105). It is easy to see that $\mathcal{L}_\infty$ is the operator associated, in the sense of [DM, Definition 12.8], to functional $F^0 : L^2(\Omega) \to [0,\infty]$ defined by

$$F^0(u) = \begin{cases} \frac{1}{2}\int_\Omega \langle a(x)Xu(x), Xu(x)\rangle\, dx & \text{if } u \in H^1_{X,0}(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Let us consider $F^0_h : L^2(\Omega) \to [0,\infty]$ defined by

$$F^0_h(u) = \begin{cases} \frac{1}{2}\int_\Omega \langle a^h(x)Xu(x), Xu(x)\rangle\, dx & \text{if } u \in H^1_{X,0}(\Omega), \\ \infty & \text{otherwise,} \end{cases}$$

whose associated operators are the $(\mathcal{L}_h)_h$. By Theorem 3.6, with $\varphi = 0$ and $A = \Omega$, it holds that

$$(F^0_h)_h \ \Gamma\text{-converges to } F^0 \text{ in the strong topology of } L^2(\Omega)\,. \tag{107}$$

Fix $\mu \geq 0$ and $g \in L^2(\Omega)$, and denote by $G : L^2(\Omega) \to \mathbb{R}$ the functional

$$G(u) := \int_\Omega \left(\frac{\mu}{2}u^2 - gu\right) dx\,.$$

Since G is (strongly) continuous in $L^2(\Omega)$, then, by (107) and in virtue of [DM, Proposition 6.21],

$$(F^0_h + G)_h \ \Gamma\text{-converges to } F^0 + G \text{ in the strong topology of } L^2(\Omega)\,. \tag{108}$$

By Remark 6.1, u_h and u turn out to the unique minimizers of $F_h + G$ and $F + G$, respectively. Therefore, by Theorem 1.2, we get (106) and

$$\begin{aligned} \lim_{h\to\infty} \left(F^0_h(u_h) + G(u_h)\right) &= \lim_{h\to\infty} \min_{u\in H^1_{X,0}(\Omega)} \left(F^0_h(u) + G(u)\right) \\ &= \min_{u\in H^1_{X,0}(\Omega)} \left(F^0(u) + G(u)\right) = F^0(u_\infty) + G(u_\infty)\,. \end{aligned}$$

Step 2. For a.e. $x \in \Omega$ for any $\eta \in \mathbb{R}^m$ and for any $h \in \mathbb{N}$, let

$$f_h(x,\eta) := \langle a^h(x)\eta, \eta\rangle \quad \text{and} \quad f(x,\eta) := \langle a(x)\eta, \eta\rangle\,.$$

It is easy to see that f_h and f satisfy assumptions (i), (ii), and (iii) of Theorem 1.3. Moreover,

$$\partial_\Phi \mathcal{F}_h(Xu_h) = a_h\, Xu_h \quad \text{and} \quad \partial_\Phi \mathcal{F}(Xu) = a\, Xu\,.$$

Therefore, by the first step of the proof and by Corollary 1.4, we get the convergence of momenta. ∎

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