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Graded Lie algebras of maximal class in characteristic p, generated by two elements of degree 1 and p

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Introduction

The notion of *maximal class* was first introduced in the context of p-groups by Blackburn [Bla58]: let G be a group of order p^n and c be its nilpotency class, that is, the length of its lower central series. Since $c < n$, when c equals $n-1$ the p-group has maximal nilpotency class and is thus referred to as a group of maximal class. Equivalently, one can define the *coclass* of G as

$$\mathrm{cc}(G) := n - c$$

and say that a p-group of maximal class is a group of minimal coclass, namely $\mathrm{cc}(G) = 1$.

The definition of coclass can be generalized in a natural way to pro-p-groups: if G is a pro-p group, denote by $\gamma_i(G)$ the terms of its lower central series and consider the quotients $G_i := G/\gamma_i(G)$, which are finite p-groups. Then the coclass of G is

$$\mathrm{cc}(G) := \lim_{i\to\infty} \mathrm{cc}(G_i).$$

Leedham-Green and Newman [LGN80] formulated five conjectures regarding the structure of pro-p-groups of given finite coclass, which nowadays have all been proven thanks to the contribution of many authors.

Analogously, one can define the coclass of a finite-dimensional nilpotent Lie algebra L as $\mathrm{cc}(L) := n - c$, where n is the dimension of L and c is its nilpotency class. This can be extended to infinite-dimensional Lie algebras L by defining

$$\mathrm{cc}(L) := \sum_{\substack{i\geqslant 1 \\ L^i\neq 0}} (\dim(L^i/L^{i+1}) - 1),$$

provided L is residually nilpotent, that is, $\bigcap_i L^i = \{0\}$. Clearly, L has finite coclass if and only if all the quotients L^i/L^{i+1} are finite-dimensional and $\dim(L^i/L^{i+1}) \leqslant 1$ for all sufficiently large i. When the coclass is minimal, namely $\mathrm{cc}(L) = 1$, we say that L is of *maximal class*. Equivalently (see [Sha94a]), a Lie algebra of maximal class is a residually nilpotent Lie algebra L such that $\dim(L/L^2) = 2$ and $\dim(L^i/L^{i+1}) \leqslant 1$ for all $i > 1$.

If one considers the family of all Lie algebras of maximal class, it has been shown already by Vergne ([Ver66, Ver70]) that in characteristic zero there are simply too many:

Lie algebras of maximal class form an irreducible component of dimension greater than n^2 in the variety of all nilpotent Lie algebras of fixed dimension n. Shalev and Zelmanov [SZ97] concentrated their attention to Lie algebras of maximal class (and more generally of finite coclass) in characteristic zero. With the assumption that these algebras are $\mathbb{N}$-graded and generated by the first homogeneous component, they were able to develop a coclass theory similar to the one for p-groups established by Leedham-Green [LG94] and Shalev [Sha94b]. In particular, there is only one just infinite algebra, namely

$$M = \langle x, y \ : \ [yx^i y] = 0 \quad \forall i \geqslant 1 \rangle ,$$

which is actually of maximal class and metabelian.

Over a field of positive characteristic p, Riley and Semple [RS94] developed a coclass theory for $\mathbb{N}$-graded *restricted* Lie algebras: they are all finite-dimensional, and the dimension is bounded in terms of p and the coclass. When dealing with non-restricted Lie algebras, this is no longer true even for $\mathbb{N}$-graded algebras of maximal class generated by the first homogeneous component, which have been called *algebras of type* 1 in [CVL00]. In fact, Shalev [Sha94a] proved that there are countably many *insoluble* algebras of type 1 of any given characteristic $p \neq 0$. They are built as positive parts of twisted loop algebras of some finite-dimensional simple algebras constructed by Albert and Frank [AF55], extended by a non-singular derivation. Caranti, Mattarei and Newman [CMN97] showed that starting from those algebras one can get $2^{\aleph_0}$ non-isomorphic algebras of type 1, for any given prime characteristic. If p is odd, these are all the possible algebras of type 1 (see [CN00]). If $p = 2$, there is one additional family of algebras of type 1 (see [Jur05]).

Algebras of type 1 do not exhaust the possibilities of graded Lie algebras of maximal class. For instance, one can consider those graded Lie algebras that are generated by the first and second homogeneous component, with all homogeneous component of dimension at most 1. These are of maximal class, and have been called *algebras of type 2* in [CVL00]. In characteristic zero, Shalev and Zelmanov [SZ97] proved that the only infinite-dimensional algebras of type 2 are

$$\begin{aligned} M &= \langle e_1, e_2 \ : [e_2 e_1^i e_2] = 0 \quad \forall i \geqslant 1 \rangle , \\ M_2 &= \langle e_i \ : [e_i e_1] = e_{i+1} \quad \forall i \geqslant 2, \\ &\qquad\quad [e_i e_2] = e_{i+2} \quad \forall i \geqslant 3, \\ &\qquad\quad [e_i e_j] = 0 \qquad \forall i, j \geqslant 3 \rangle \end{aligned}$$

and the positive part of the Witt algebra, namely

$$W^+ = \langle e_i \ : [e_i e_j] = (i-j) e_{i+j} \quad \forall i, j \geqslant 1 \rangle .$$

These algebras are graded by assigning degree i to each element e_i.

In odd characteristic, Caranti and Vaughan-Lee [CVL00] proved that M and M_2 are still algebras of type 2, but there are several more examples:

- The family of subalgebras of algebras of type 1;
- A family of soluble algebras;
- Another family of soluble algebras in characteristic 3 only.

In characteristic 2, the classification is more *uniform* (see [CVL03] or Chapter 2 of this thesis).

In [Ugo10], the author considers the case of $\mathbb{N}$-graded Lie algebras generated by the first and n-th homogeneous component

$$L := L_1 \oplus \bigoplus_{i=n}^{\infty} L_i,$$

with $\dim(L_1) = \dim(L_i) = 1$ for every $i \geqslant n$. He refers to these as *algebras of type n*. Over fields of positive characteristic greater than $2n$, the author generalizes some of the results of the case $n = 2$.

In this thesis we consider the case of infinite-dimensional algebras of type p over a field of characteristic p, providing a complete description of them. The resulting classification is a generalization of the classification of algebras of type 2 in characteristic 2.

The structure of the thesis is the following: Chapter 1 introduces the reader to the basic definitions and notations. In Chapter 2 we discuss the most important properties of algebras of type p in characteristic p in comparison to those of algebras of type 1, as well as stating the main result of this thesis, namely the classification theorem. The remaining chapters are devoted to proving that theorem: Chapter 3 reduces the possibilities on the *length of the first constituent* (see Chapter 2), Chapter 4 proves the uniqueness of the algebras of type p appearing in the classification theorem, and Chapter 5 provides the existence.

Chapter 1

Preliminaries

This preliminary chapter introduces basic definitions and notations, as well as recalling some well-known identities involving binomial coefficients and their evaluation modulo a prime.

1.1 Graded Lie algebras of maximal class

Let $\mathbb{F}$ be a field of arbitrary characteristic. An algebra L over $\mathbb{F}$ is a Lie algebra if the product satisfies

(i) $x \cdot x = 0$ for any $x \in L$;

(ii) the Jacobi identity: $x \cdot (y \cdot z) + y \cdot (z \cdot x) + z \cdot (x \cdot y)$ for any $x, y, z \in L$.

We use the bracket notation instead of the above one so that, for instance, the Jacobi identity will be written as

$$[x[yz]] + [y[zx]] + [z[xy]] = 0.$$

As a direct consequence of (i) the product is anticommutative, that is, $[xy] = -[yx]$ for any $x, y \in L$.

In what follows we will be dealing with iterated Lie brackets taken in the left-normed notation, namely

$$[xyz] := [[xy]z], \quad \text{and} \quad [yx^n] := [y\underbrace{x \ldots x}_{n}].$$

The following *generalized* Jacobi identity will be useful in computing those iterated brackets:

$$[z[yx^n]] = \sum_{i=0}^{n} (-1)^i \binom{n}{i} [zx^i yx^{n-i}].$$

A Lie algebra L is said to be *G-graded*, where G is an arbitrary abelian group, if the additive group of L is a direct sum $L = \bigoplus_{g \in G} L_g$ such that $[L_g L_h] \subseteq L_{g+h}$ for all $g, h \in G$.

The subspaces L_g are usually referred to as *homogeneous components* of L, regarded of *degree* (or *weight*) g. Moreover, any element $x \in L$ belongs to one - and only one - homogeneous component L_g for some $g \in G$, and we say x is an element of degree g. In this thesis we consider Lie algebras graded over the positive integers, namely of the form

$$L = \bigoplus_{i=1}^{\infty} L_i,$$

meaning that L is actually $\mathbb{Z}$-graded with $L_i = \{0\}$ for every $i \leqslant 0$. From now on, when talking about a graded Lie algebra without further specifications, we implicitly mean that the grading is taken over the positive integers.

A finite-dimensional Lie algebra M is of *maximal class* when the codimension of the Lie powers M^i is precisely i for $2 \leqslant i \leqslant \dim(M)$. More generally, an infinite-dimensional Lie algebra M is of maximal class when the codimension of M^i is precisely i for all $i \geqslant 2$ and M is *residually nilpotent*, namely $\bigcap_i M^i = \{0\}$.

One can grade an algebra of maximal class M with respect to the filtration of the M^i: for all $i \geqslant 2$, let

$$L_i := M^i/M^{i+1}$$

and consider

$$L := \bigoplus_{i=1}^{\infty} L_i.$$

The resulting Lie algebra L is graded and of maximal class, with $\dim(L_1) = 2$ and $\dim(L_i) \leqslant 1$ for all $i \geqslant 2$. Furthermore, L is generated by its first homogeneous component, namely L_1. A graded Lie algebra satisfying these conditions is called *algebra of type* 1 in [CVL00, CVL03].

Viceversa, a graded Lie algebra does not need to be an algebra of type 1 to be of maximal class. For instance, consider a graded Lie algebra

$$L = L_1 \oplus \bigoplus_{i=n}^{\infty} L_i$$

generated by L_1 and L_n for some $n > 1$. If $\dim(L_1) = 1$ and $\dim(L_i) \leqslant 1$ for every $i \geqslant n$, then L is of maximal class. In [CVL00, CVL03] the authors addressed the above kind of graded Lie algebras of maximal class in positive characteristic when $n = 2$, and they called them *algebras of type* 2. As a natural generalization, in [Ugo10] the author calls those algebras with arbitrary n *algebras of type* n. We remark that, restricting the attention to infinite-dimensional algebras of type n, then every homogeneous component has exactly dimension 1 (except the first one, when $n = 1$). Furthermore, these algebras are just-infinite, that is, their proper quotients are all finite-dimensional.

In this thesis we focus on infinite-dimensional algebras of type p over fields of positive characteristic p. Therefore, except when explicitly stated otherwise, $\mathbb{F}$ is a field of positive characteristic p, and every Lie algebra is considered over $\mathbb{F}$ and has infinite dimension. From Chapter 3 onward, we will assume p is odd, since a complete discussion of algebras of type 2 in characteristic 2 can be found in [CVL03].

1.2 Binomial identities

As already mentioned, most of the computations of this thesis involve binomial coefficients arising from the generalized Jacobi identity. The main tool to evaluate binomial coefficients modulo p is due to Lucas ([Luc78]):

Lucas' Theorem. *Let a and b be two non-negative integers with p-adic expansion $a = a_0 + a_1 p + \cdots + a_n p^n$ and $b = b_0 + b_1 p + \cdots + b_n p^n$, where $0 \leqslant a_i, b_i < p$ for every i. Then*

$$\binom{a}{b} \equiv \prod_{i=0}^{n} \binom{a_i}{b_i} \mod p.$$

In particular, for every positive integer h and for every non-negative integers u, v, s, t such that $v, t < p^h$,

$$\binom{up^h + v}{sp^h + t} \equiv \binom{u}{s}\binom{v}{t} \mod p.$$

As an example of application of Lucas' theorem we have that

$$\binom{q-k}{m} \equiv (-1)^m \binom{k+m-1}{m} \pmod p$$

for any non-negative $k, m < p$, where $q \geqslant p$ is a power of p. Indeed,

$$\binom{q-k}{m} \equiv \binom{p-k}{m} \pmod p$$

by Lucas' theorem. By definition of binomial coefficients

$$\begin{aligned} \binom{p-k}{m} &= \frac{(p-k)^{\underline{m}}}{m!} \\ &\equiv \frac{(-k)^{\underline{m}}}{m!} \pmod p, \end{aligned}$$

and

$$\begin{aligned} \frac{(-k)^{\underline{m}}}{m!} &= (-1)^m \frac{k^{\overline{m}}}{m!} \\ &= (-1)^m \frac{(k+m-1)^{\underline{m}}}{m!} \\ &= (-1)^m \binom{k+m-1}{m}. \end{aligned}$$

Here we used $k^{\underline{m}}$ and $k^{\overline{m}}$ to denote respectively the *falling factorial* and the *rising factorial* of k, namely

$$k^{\underline{m}} := k(k-1)\cdots(k-m+1), \quad k^{\overline{m}} := k(k+1)\cdots(k+m-1).$$

Let us finish the section collecting a few elementary binomial identities that will be used in the following:

- For any positive integer n
$$\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0.$$
This is a simple consequence of the evaluation in $x = -1$ of the polynomial identity
$$\sum_{i=0}^{n}\binom{n}{i}x^i = (1+x)^n. \tag{1.1}$$

- More generally, for any positive integer n and any non-negative integer $k \leqslant n$
$$\sum_{i=0}^{k}(-1)^i\binom{n}{i} = (-1)^k\binom{n-1}{k}.$$
We can prove that this is true by induction on k using the well-known identity $\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-1}{k+1}$: indeed, the identity claimed above trivially holds for $k = 0$, and assuming by induction that it holds for a given $k < n$ we have that
$$\begin{aligned}\sum_{i=0}^{k+1}(-1)^i\binom{n}{i} &= \sum_{i=0}^{k}(-1)^i\binom{n}{i} + (-1)^{k+1}\binom{n}{k+1} \\ &= (-1)^k\binom{n-1}{k} + (-1)^{k+1}\binom{n}{k+1} \\ &= (-1)^{k+1}\binom{n-1}{k+1}.\end{aligned}$$

- For any integer $n \geqslant 2$
$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}i = 0.$$
It is enough to take the derivative with respect to x of the polynomial identity (1.1) to get
$$\sum_{i=1}^{n}\binom{n}{i}ix^{i-1} = n(1+x)^{n-1}$$
Substituting $x = -1$, one gets the binomial identity claimed above.

- For any non-negative integers n, m, k

$$\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}.$$

This is also known as Vandermonde's identity, and is a consequence of the polynomial identity

$$(1+x)^n(1+x)^m = (1+x)^{n+m}.$$

Indeed, expansion of the left-hand side of the identity yields

$$\begin{aligned}(1+x)^n(1+x)^m &= \left(\sum_{r=0}^{n}\binom{n}{r}x^r\right)\left(\sum_{s=0}^{m}\binom{m}{s}x^s\right)\\ &= \sum_{k=0}^{n+m}\left(\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}\right)x^k,\end{aligned}$$

while the right-hand side expands to

$$(1+x)^{n+m} = \sum_{k=0}^{n+m}\binom{n+m}{k}x^k.$$

Vandermonde's identity for all integers k with $0 \leqslant k \leqslant m+n$ follows by comparing coefficients of x^k. For larger integers k, both sides of Vandermonde's identity are zero.

Chapter 2

Constituents of graded Lie algebras of maximal class

In this chapter we start investigating the basic properties of algebras of type p and introduce some tools to deal with them. In the third section the reader can find the statement of this thesis' main result, namely the classification theorem for algebras of type p.

2.1 Constituents of algebras of type 1

Suppose $N = \bigoplus_{i\geqslant 1} N_i$ is an *uncovered* algebra of type 1, which means that there is an element $e_1 \in N_1$ such that $[N_i e_1] = N_{i+1}$ for $i \geqslant 1$. Choose $y \in N_1 \setminus \langle e_1\rangle$, and define recursively $e_2 := [ye_1]$, $e_{i+1} := [e_i e_1]$ for $i \geqslant 2$. For every $i \geqslant 2$ we then have that $[e_i y] = \beta_i e_{i+1}$ for some $\beta_i \in \mathbb{F}$. The sequence $(\beta_i)_{i\geqslant 2}$ is called *sequence of two-step centralizers* of N. It completely determines the multiplication table of N, as for any $j, k \geqslant 2$

$$\begin{aligned}
[e_j e_k] &= [e_j [y e_1^{k-1}]] \\
&= \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} [e_{j+i} y e_1^{k-1-i}] \\
&= \left(\sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \beta_{j+i} \right) e_{j+k}.
\end{aligned}$$

Clearly, the above definition of two-step centralizers depends on the choice of generators of N. For instance, consider another generator $y' \in N_1 \setminus \langle e_1\rangle$, and write it as $y' = \lambda y + \delta e_1$ for some $\lambda \in \mathbb{F}^*$ and $\delta \in \mathbb{F}$. We would then have that

$$e'_2 := [y' e_1] = \lambda e_2, \qquad e'_{i+1} := [e'_i e_1] = \lambda e_{i+1} \quad \text{for} \quad i \geqslant 2$$

and

$$\begin{aligned}[e_i'y'] &= \lambda[e_i, \lambda y + \delta e_1] \\ &= (\lambda\beta_i + \delta)\, e_{i+1}',\end{aligned}$$

meaning the the two-step centralizers with respect to the new generator y' are $\beta_i' := \lambda\beta_i + \delta$. Therefore, one can introduce an equivalence relation on sequences of two-step centralizers by saying that two sequences $(\beta_i)_{i\geqslant 2}$ and $(\beta_i')_{i\geqslant 2}$ are equivalent if and only if there exist $\lambda \in \mathbb{F}^*$ and $\delta \in \mathbb{F}$ such that $\beta_i' = \lambda\beta_i + \delta$ for all $i \geqslant 2$, and with this definition we can say that two uncovered algebras of type 1 are isomorphic if and only if their corresponding sequences of two-step centralizers are equivalent. Note that this amounts to *scaling* the sequence by a non-zero factor λ and eventually *translating* the sequence by a factor δ.

Remark 2.1. This definition of two-step centralizers for uncovered algebras of type 1 is equivalent to the one given in [CMN97], where clearly the i-th two-step centralizers in the classical fashion are precisely

$$C_i := C_{N_1}(N_i) = \langle y - \beta_i e_1\rangle.$$

Consider now the sequence of two-step centralizer $(\beta_i)_{i\geqslant 2}$ with respect to a fixed generator $y \in N_1 \setminus \langle e_1\rangle$. Suppose $\beta_2 = \beta_3 = \cdots = \beta_{n-1}$ but $\beta_n \neq \beta_2$ for some n. Then the sequence

$$\beta_2, \beta_3, \ldots, \beta_n$$

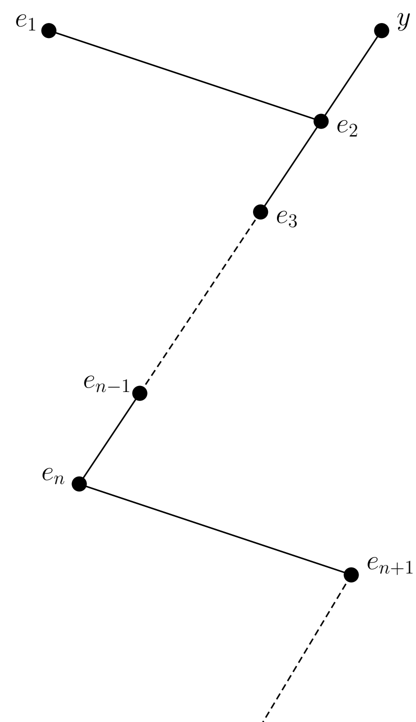
is referred to as the *first constituent* of N, and the length is defined to be n. The other constituents are defined recursively: if $\beta_i, \ldots, \beta_j$ is a constituent already defined, and if $\beta_{j+1} = \cdots = \beta_{j+m-1} = \beta_2$ but $\beta_{j+m} \neq \beta_2$ for some m, then $\beta_{j+1}, \ldots, \beta_{j+m}$ is a constituent, of length m. It turns out (see [CMN97]) that the length of the first constituent equals $2q$ for some $q = p^h$, $h \geqslant 1$. Furthermore, every constituent can only have length of the form

$$2q, \quad \text{or} \quad 2q - p^k \quad \text{for some } 0 \leqslant k \leqslant h.$$

Remark 2.2. Let $N = \bigoplus_{i\geqslant 1} N_i$ be an uncovered algebra of type 1 and suppose, up to scaling and translating, that its first constituent is given by $\beta_2 = \ldots = \beta_{n-1} = 0$, $\beta_n = 1$. Recall that, by definition, this means that

$$[e_i y] = 0 \quad \text{for } 2 \leqslant i < n, \qquad [e_n y] = e_{n+1}.$$

The following graph represents the initial structure of N: it should be looked at from the top to the bottom, in the sense that each line represents the generators of a homogeneous component of N, namely e_1 and y for the first, and e_i for the i-th. Going from a homogeneous component to the following one, we draw an edge between the corresponding generators e_i and e_{i+1} which is pointing to the left if e_{i+1} can be obtained only as a bracket of e_i and e_1, and pointing to the right if $[e_i y] = \gamma e_{i+1}$ for some $\gamma \neq 0$.



Now, let $J := [NN] = \bigoplus_{h \geqslant 2} N_h$. Looking at the picture above we can see that

$$J^2 = \bigoplus_{h \geqslant n+1} N_h,$$

as one can also check explicitly: clearly $e_n \notin J^2$, while

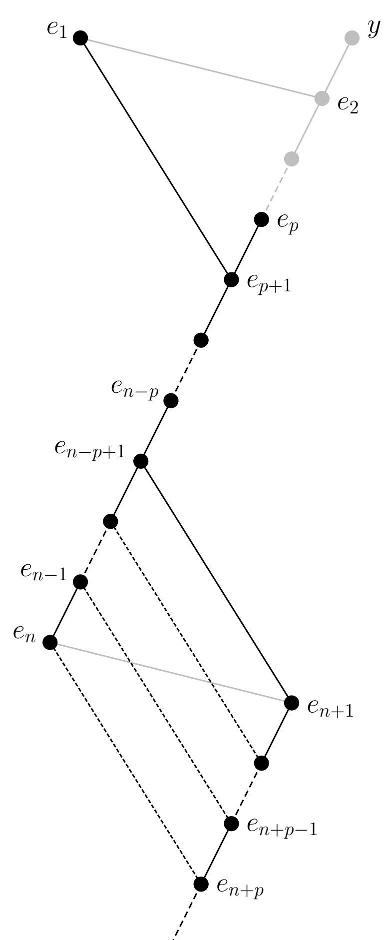
$$e_{n+1} = [e_n y] = -[e_{n-1}[ye_1]] \ \in J^2.$$

Therefore, there is a one-to-one correspondence between the quotient J/J^2 and the first constituent as defined above. It is an easy remark to note that, for any $i \geqslant 1$, the Lie powers J^{i+1} correspond exactly to a change of two-step centralizer, in the sense that if $\gamma_i = \beta_m$ denotes the last two-step centralizer of the i-th constituent, then $J^{i+1} = \bigoplus_{h \geqslant m+1} N_h$. This leads us to the equivalent definition of i-th constituent as the quotient J^i/J^{i+1}, for any $i \geqslant 1$. Moreover, the length of the i-th constituent is equal to the dimension of the quotient J^i/J^{i+1} for every $i > 1$, while the length of the first constituent equals $\dim(J/J^2) + 1$.

Of course, this equivalent definition may be given in more general situations, such as algebras of type p or even non-graded algebras of maximal class. Nonetheless, we believe it is recommendable to slightly refine it when dealing with algebras of type p, as we will soon see.

2.2 Constituents of algebras of type p

Starting fron an uncovered algebra $N = \bigoplus_{i\geqslant 1} N_i$ of type 1, let $L_1 := \langle e_1 \rangle$, $L_i := N_i$ for $i \geqslant p$. Putting $L := L_1 \oplus \bigoplus_{i\geqslant p} L_i$, the resulting graded subalgebra is an algebra of type p with generators e_1 and e_p. Therefore for any uncovered algebra N of type 1 there exist a subalgebra which is of type p. We remark that we may also consider subalgebras of type n for any positive integer n, but the equality between the characteristic of the ground field and the type of the algebra will play an important role later in this thesis. We refer to [Ugo10] for a discussion on the case of algebras of type n in characteristic p, provided n is *small* with respect to p (precisely, $2n < p$). Let us draw an approximative graph of an algebra L of type p that is a subalgebra of an uncovered algebra of type 1:



We may define the constituents of $L \leqslant N$ in the same intrinsic way as the previous section: put $J := [LL]$, and consider the Lie powers J^i. For any $i \geqslant 1$ we say that the i-th constituent of L is the quotient J^i/J^{i+1}, and its length is exactly its dimension as long as $i > 1$, while the length of the first constituent is $n = \dim(J/J^2) + p$. With this definition, every constituent of L other than the first coincides with the corresponding constituent of N, while the first constituent of L is strictly contained in the first constituent of N. Nonetheless, the first constituent length of L equals that of N.

Let us inspect what this definition means in terms of two-step centralizers. As for algebras of type 1, we may look at the adjoint action of the generator e_p: for each $i > p$ we have that $[e_i e_p] = \alpha_i e_{i+p}$ for some $\alpha_i \in \mathbb{F}$, and we refer to the sequence $(\alpha_i)_{i>p}$ as the *sequence of two-step centralizers* of L. Since $L \leqslant N$ and $e_2 = [ye_1]$, $e_{i+1} = [e_i e_1]$, an application of the generalized Jacobi identity gives the relation between the two-step centralizers of L and N:

$$\begin{aligned}\alpha_i e_{i+p} &= [e_i e_p] = [e_i[ye_1^{p-1}]]\\ &= \sum_{j=0}^{p-1}(-1)^j\binom{p-1}{j}[e_i e_1^j y e_1^{p-j-1}]\\ &= \sum_{j=0}^{p-1}[e_{i+j} y e_1^{p-j-1}] = \left(\sum_{j=0}^{p-1}\beta_{i+j}\right) e_{i+p},\end{aligned}$$

hence $\alpha_i = \sum_{j=0}^{p-1}\beta_{i+j}$.

From the theory of constituents for algebras of type 1 (see [CMN97, CN00]) we know that if N is an algebra of type 1 in positive characteristic p, then all elements of a constituent except the last one coincide with β_2, the first two-step centralizer. Furthermore, p is a lower bound for the length of all constituents, and combining these two facts we have that if $L \leqslant N$ is a subalgebra of type p as above, then the expression $\alpha_i = \sum_{j=0}^{p-1}\beta_{i+j}$ actually contains either p constant terms (coinciding with β_2) or $p-1$ constant terms and only one which differs from the others, which is the last term of a constituent of N. Therefore, if we consider a constituent of N

$$\beta_2 = \beta_i = \beta_{i+1} = \cdots = \beta_{i+n-1} \neq \beta_{i+n},$$

then $\alpha_i = \cdots = \alpha_{i+n-p} = 0$ and $\alpha_{i+n-p+1} = \cdots = \alpha_{i+n} = \beta_{i+n} - \beta_2 \neq 0$. Therefore, let us look again at the graph above: up to scaling and translating, we are assuming that $\beta_2 = 0$ and $\beta_n = 1$, which is the last element of the first constituent of N. Hence the sequence of two-step centralizers of L begins with $\alpha_{p+1} = \ldots = \alpha_{n-p} = 0$ and $\alpha_{n-p+1} = \ldots = \alpha_n = 1$, and this corresponds to the first constituent of N. Moreover, from the above considerations we have that the sequence of two-step centralizers of L continues with repetitions of patterns of the form

$$0 \dots 0 \underbrace{\lambda \dots \lambda}_{p}$$

for some $0 \neq \lambda \in \mathbb{F}$. We remark that here the hypothesis that L is of type $p = \operatorname{char} \mathbb{F}$ has already played a role, since the two-step centralizers α_i of an algebra of arbitrary type n that is subalgebra of an algebra of type 1 would be related to the two-step centralizers β_i of the latter by

$$\alpha_i = \sum_{j=0}^{n-1} (-1)^i \binom{n-1}{i} \beta_{i+j},$$

and, without assumptions on n, the lower bound on the length of the constituents of the algebra of type 1 is too weak to get to the same conclusion as above.

Clearly, we can define two-step centralizers and constituents also for arbitrary algebras of type p: if

$$L = L_1 \oplus \bigoplus_{i \geqslant p} L_i$$

is such an algebra, we can choose two generators $e_1 \in L_1 \setminus \{0\}$ and $e_p \in L_p \setminus \{0\}$, and define recursively $e_{i+1} := [e_i e_1]$ for $i \geqslant p$. For each $i > p$ we then have that $[e_i e_p] = \alpha_i e_{i+p}$ for some $\alpha_i \in \mathbb{F}$, and we refer to the sequence $(\alpha_i)_{i>p}$ as the *sequence of two-step centralizers* of L. Similarly to algebras of type 1, by a straightforward application of the generalized Jacobi identity one can deduce that the sequence of two-step centralizers completely determines the multiplication table of L.

Again, this definition depends on the choice of generators of L. For instance, consider another pair of generators $e_1' = \lambda e_1$ of degree 1 and $e_p' = \mu e_p$ of degree p, for some $\lambda, \mu \in \mathbb{F}^*$. We then have that

$$e_i' := [e_{i-1}' e_1'] = \lambda^{i-p} \mu e_i \quad \text{for all } i > p$$

and

$$\begin{aligned} [e_i' e_p'] &= \lambda^{i-p} \mu^2 [e_i, e_p] \\ &= \frac{\mu}{\lambda^p} \alpha_i e_{i+p}', \end{aligned}$$

meaning the the two-step centralizers with respect to the new generators are $\alpha_i' := \frac{\mu}{\lambda^p} \alpha_i$. Therefore, one can introduce an equivalence relation on sequences of two-step centralizers by saying that two sequences $(\alpha_i)_{i>p}$ and $(\alpha_i')_{i>p}$ are equivalent if and only if there exist $\lambda \in \mathbb{F}^*$ such that $\alpha_i' := \lambda \alpha_i$ for all $i > p$, and with this definition two algebras of type p are isomorphic if and only if their corresponding sequences of two-step centralizers are equivalent. Note that this amounts to *scaling* the sequence by a non-zero factor λ, but unlike algebras of type 1 we cannot *translate* the sequence without altering the isomorphism type of the algebra.

Nonetheless, we can translate the sequence of two-step centralizers of L by a factor $\delta \in \mathbb{F}$ getting *another* algebra of type p. This can be done similarly to [CVL03]: let us start from a given algebra L of type p, and regard it as embedded in an associative algebra A. Let e_1 and e_p be generators of L, $[e_i e_1] = e_{i+1}$ for $i \geqslant p$ as customary, and $(\alpha_i)_{i>p}$ the sequence of two-step centralizers. For every $\delta \in \mathbb{F}$ we may consider the Lie subalgebra $L(\delta)$ of A generated by e_1 and $e'_p := e_p + \delta\tilde{e}_p$, where $\tilde{e}_p := e_1^p \in A$. Then

$$\begin{aligned}
[e_i e'_p] &= [e_i, e_p + \delta\tilde{e}_p] \\
&= [e_i e_p] + \delta[e_i \tilde{e}_p] \\
&= \alpha_i e_{i+p} + \delta[e_i e_1^p] \\
&= (\alpha_i + \delta) e_{i+p}.
\end{aligned}$$

Furthermore, $[e'_p e_1] = e_{p+1}$, hence it follows that

$$L(\delta) = L_1 \oplus \langle e'_p \rangle \oplus \bigoplus_{i>p} L_i$$

is an algebra of type p, with sequence of two-step centralizers $\alpha'_i := \alpha_i + \delta$. We have proven that

Lemma 2.3. *Let L be an algebra of type p over a field $\mathbb{F}$ of characteristic p, with sequence of two-step centralizers $(\alpha_i)_{i>p}$. Then for any $\delta \in \mathbb{F}$ there is an algebra of type p with sequence of two-step centralizers $(\alpha_i + \delta)_{i>p}$.*

Remark 2.4. (i) Let M be the (unique) metabelian algebra of type 1, and let L be its subalgebra of type p. Then the sequence of two-step centralizers of L has constant value $\alpha = 0$, and any translated algebra $L(\delta)$ has sequence of two-step centralizers of constant value δ. Note that $L(\delta)$ is not isomorphic to L for any $\delta \neq 0$, since this can happen if and only if there exist $\lambda \in \mathbb{F}^*$ such that $\lambda\alpha = \alpha + \delta$, which would imply $\delta = 0$. On the other hand, different choices of $\delta \neq 0$ lead to isomorphic algebras since in this particular case that amounts to different scalings of the algebra.

(ii) If L is an algebra of type p with non-constant sequence of two-step centralizers (α_i), then $L(\delta)$ is not isomorphic to L unless $\delta = 0$, since this can happen if and only if there exist $\lambda \in \mathbb{F}^*$ such that $\lambda\alpha_i = \alpha_i + \delta$ for any $i > p$, meaning that (α_i) has constant value

$$\alpha_i = \frac{\delta}{\lambda - 1}.$$

Let us now give a definition of constituents for arbitrary algebras of type p that generalizes the one we just gave for subalgebras of algebras of type 1. One may be tempted to adopt the same intrinsic definition as that of algebras of type 1, namely saying that the

i-th constituent of L is the quotient J^i/J^{i+1} for any $i \geqslant 1$, where $J := [LL]$. Equivalently, the two-step centralizers associated to the first constituent would be of the form

$$0, \ldots, 0, \alpha_{n-p+1}, \ldots, \alpha_n$$

with $\alpha_{n-p+1} \neq 0$, and its length would be $n = \dim J/J^2 + p$. Recursively, if $\alpha_i, \ldots, \alpha_j$ is a constituent we have already defined, and if $\alpha_{j+1} = \cdots = \alpha_{j+m-p} = 0$ but $\alpha_{j+m-p+1} \neq 0$, then we would say that $\alpha_{j+1}, \ldots, \alpha_{j+m}$ is a constituent of length m. This is actually how constituents for algebras of type 2 over a field of characteristic 2 were defined, and we will refer to the above as *fake constituents*. The reason for the unpleasant 'fake' label is easily explained: for instance, let us consider the first fake constituent of an algebra of type p, namely

$$0, \ldots, 0, \alpha_{n-p+1}, \ldots, \alpha_n.$$

By definition, the only information on the *tail* of the fake constituent is that $\alpha_{n-p+1} \neq 0$, while the following two-step centralizers might be zero or non-zero. For example, we might be in a situation where $\alpha_n = 0$, namely

$$0, \ldots, 0, \alpha_{n-p+1}, \ldots, \alpha_{n-1}, 0,$$

thus it would be preferable to consider α_n as an element of the second constituent instead, and so on going backwards until we get the last non-zero element of the first fake constituent, say $\alpha_r \neq 0$. Therefore, we refine the definition of constituent for algebras of type p as follows: suppose that $\alpha_{p+1} = \ldots = \alpha_{n-p} = 0$ but $\alpha_{n-p+1} \neq 0$, and let $r \leqslant n$ be maximal with the property that $\alpha_r \neq 0$. Then we refer to the subsequence

$$0, \ldots, 0, \alpha_{n-p+1}, \ldots, \alpha_r$$

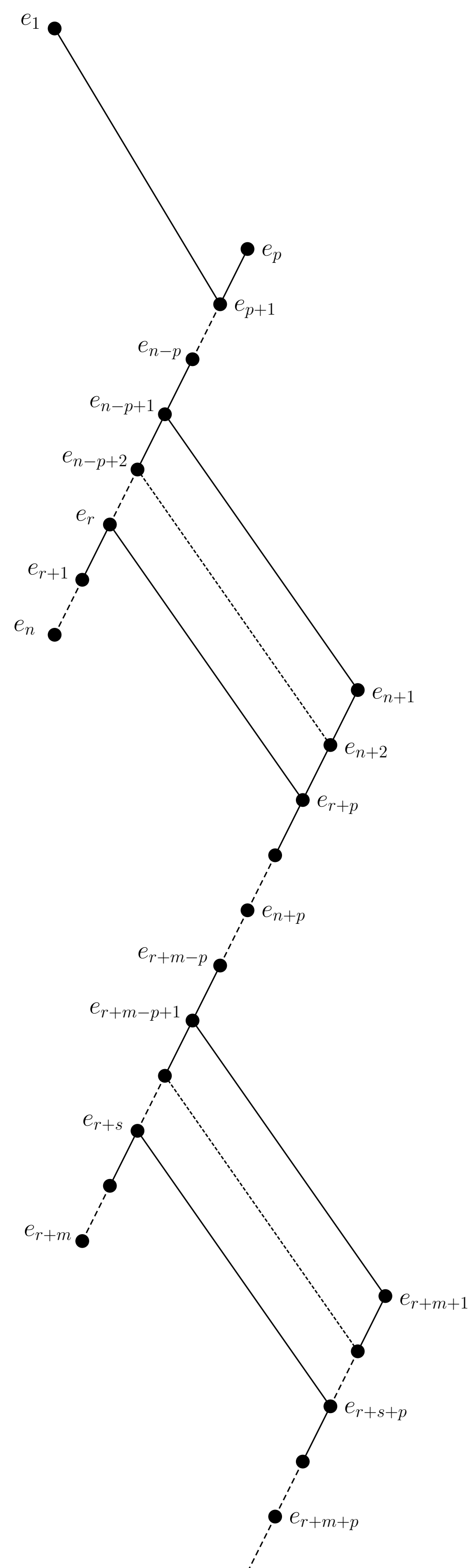
as the first constituent, which we regard of length r and 0-length $n - p$. Recursively, suppose $\alpha_i, \ldots, \alpha_j$ is a constituent we have already defined. If $\alpha_{j+1} = \cdots = \alpha_{j+m-p} = 0$ but $\alpha_{j+m-p+1} \neq 0$ and $s \leqslant m$ is maximal with the property that $\alpha_{j+s} \neq 0$, then we say the subsequence

$$\alpha_{j+1}, \ldots, \alpha_{j+m-p}, \alpha_{j+m-p+1}, \ldots, \alpha_{j+s}$$

is a constituent of length s and 0-length $m - p$. The following graph represents the initial structure of an algebra of type p up to the third constituent, provided that

$$0, \ldots, 0, \alpha_{n-p+1}, \ldots, \alpha_r \quad \text{and} \quad 0, \ldots, 0, \alpha_{r+m-p+1}, \ldots, \alpha_{r+s}$$

are the first and the second constituent, respectively:


e_1
e_p
e_{p+1}
e_{n-p}
e_{n-p+1}
e_{n-p+2}
e_r
e_{r+1}
e_n
e_{n+1}
e_{n+2}
e_{r+p}
e_{n+p}
e_{r+m-p}
$e_{r+m-p+1}$
e_{r+s}
e_{r+m}
e_{r+m+1}
e_{r+s+p}
e_{r+m+p}

Let us also rephrase the above definition of constituents for algebras of type p implicitly, refining the one corresponding to fake constituents. Consider and algebra of type p

$$L = L_1 \oplus \bigoplus_{i \geqslant p} L_i$$

and put $J := [LL] = \bigoplus_{i>p} L_i$. The first fake constituent was defined as J/J^2, while we now define the first constituent to be

$$\mathfrak{C}_1 := {}^{J/J^3}\big/_{Z(J/J^3)}.$$

The length of the first constituent is $\dim(\mathfrak{C}_1) + p$. To have a better insight of the definition, we advise the reader to relate to the last graph we draw: using that notation, we have that

$$J = \bigoplus_{i \geqslant p+1} \langle e_i \rangle, \quad J^2 = \bigoplus_{i \geqslant n+1} \langle e_i \rangle \quad \text{and} \quad J^3 = \bigoplus_{i \geqslant r+m+1} \langle e_i \rangle,$$

hence

$$J/J^3 \approx \bigoplus_{i=p+1}^{r+m} \langle e_i \rangle,$$

where the right-hand side is clearly intended modulo J^3. Furthermore,

$$Z(J/J^3) \approx \bigoplus_{i=r+1}^{r+m} \langle e_i \rangle,$$

hence

$$\mathfrak{C}_1 \approx \bigoplus_{i=p+1}^{r} \langle e_i \rangle$$

as desired. We define the second constituent as

$$\mathfrak{C}_2 := I_2 / C_{I_2}(J/J^4),$$

where $I_2 := Z(J/J^3) \oplus J^3/J^4$, provided we regard $Z(J/J^3)$ as embedded into J/J^4. The length of the second constituent is precisely $\dim(\mathfrak{C}_2)$. To relate to the picture above, let $J^4 = \bigoplus_{i \geqslant k+1} \langle e_i \rangle$ for some $k > r + m + p$. Then

$$I_2 \approx \bigoplus_{i=r+1}^{k} \langle e_i \rangle \quad \text{and} \quad C_{I_2}(J/J^4) \approx \bigoplus_{i=r+s+1}^{k} \langle e_i \rangle,$$

thus

$$\mathfrak{C}_2 \approx \bigoplus_{i=r+1}^{r+s} \langle e_i \rangle.$$

It is now clear how to define recursively the following constituents: for any $k > 2$, the k-th constituent is

$$\mathfrak{C}_k := I_k / C_{I_k}(J/J^{k+2}),$$

where $I_k := C_{I_{k-1}}(J/J^{k+1}) \oplus J^{k+1}/J^{k+2}$. Although this definition of constituent may be suitable when dealing with more general algebras of maximal class, the equivalent explicit definition for algebras of type p in terms of two-step centralizers is the one that will be used through the rest of this thesis.

We remark that if L is the subalgebra of type p of an uncovered algebra of type 1, then all constituents of L coincides with the corresponding fake ones, thus this new definition really generalizes the previous one. Moreover, as we previously observed, every constituent of such an algebra is *ordinary ending in* λ, i.e. is of the form

$$0 \dots 0 \underbrace{\lambda \dots \lambda}_{p}$$

for some $0 \neq \lambda \in \mathbb{F}$. The converse also holds (see [Ugo10]), hence:

Proposition 2.5. *An algebra of type p is a subalgebra of an algebra of type 1 if and only if all its constituents are ordinary.*

On the other hand, there are (infinitely many) examples in which the constituents of an algebra of type p do *not* coincide with its fake constituents: for instance, consider the algebras of Albert-Frank-Shalev $AFS(1, b, n, p)$ with fixed $p > 0$, where the parameters b and n are such that $1 < b < n$, and n may also be infinite (see [Sha94a, CMN97, CN00]). For any choice of b and n put $q := p^b$ and consider the subalgebra L of type p of $AFS(1, b, n, p)$. By Proposition 2.5 L has ordinary constituents, and since $AFS(1, b, n, p)$ has only two distinct two-step centralizers (see [CMN97]), so does L, meaning that every constituent is ordinary ending in $\lambda = -1$, up to scaling. As we previously observed, the length of every constituent of the subalgebra L equals the length of the corresponding constituent of $AFS(1, b, n, p)$, therefore the sequence of constituent lengths of L is (see [CMN97])

$$\left(2p, p^{\frac{q}{p}-2}, 2p-1, \left(p^{\frac{q}{p}-2}, 2p\right)^{\frac{p^n}{q}-2}, p^{\frac{q}{p}-2}\right)^{\infty}.$$

The notation used above is the one used by the authors of [CMN97]: for instance,

$$a_1^m, a_2, (a_3^n, a_4)^{\infty},$$

where a_i are arbitrary elements and m, n are non-negative integers, denotes the sequence

$$\underbrace{a_1, \dots, a_1}_{m}, a_2, \underbrace{a_3, \dots, a_3}_{n}, a_4, \underbrace{a_3, \dots, a_3}_{n}, a_4, \dots.$$

Consider now the translated algebra $\tilde{L} := L(1)$. Clearly, also $\tilde{L}$ has only two different two-step centralizers, namely $\tilde{\alpha}_i = 0$ or 1, and the sequence of two-step centralizers of $\tilde{L}$ is

$$\underbrace{0\dots0}_{q-p}\underbrace{1\dots1}_{p-1}\left(\left(\underbrace{0\dots0}_{q-p}\underbrace{1\dots1}_{p}\right)^{\frac{p^n}{q}-1}\underbrace{0\dots0}_{q-p}\underbrace{1\dots1}_{p-1}\right)^{\infty}.$$

The sequence of constituent lengths of $\tilde{L}$ is thus

$$q+p-1, \left(q^{\frac{p^n}{q}-1}, q-1\right)^{\infty},$$

and every constituent is either ordinary ending in 1 of length q or has length $q-1$ ($q+p-1$ for the first constituent) and is of the form

$$0\dots0\underbrace{1\dots1}_{p-1}.$$

We say that such a constituent is *almost ordinary* ending in 1.

2.3 The classification theorem

The classification of algebras of type 2 in characteristic 2 (see [CVL03]) roughly states that every algebra of type 2 in characteristic 2 is obtainable translating a subalgebra of type 2 of an uncovered algebra of type 1. This is not quite true for algebras of type p in characteristic $p > 2$, since there actually exist a few algebras of type p which are not isomorphic to $L(\delta)$ for any $\delta \in \mathbb{F}$ and any subalgebra L of an uncovered algebra of type 1. For instance, for any $q = p^h > p$ and any positive integer m such that $m < p-1$ there exist an algebra L of type p such that:

(i) the first constituent of L has length $q+m$ and is of the form

$$0, \dots, 0, \alpha_{q-p+m+1}, \dots, \alpha_{q+m},$$

where

$$\alpha_{q-p+h} = \begin{cases} 1+(-1)^{m+1}\binom{h-1}{m}, & \text{if } m+1 \leqslant h \leqslant p; \\ 1, & \text{if } p < h \leqslant p+m; \end{cases}$$

(ii) every other constituent of L is ordinary ending in 1, of length q.

We refer to Chapter 5 for an explicit construction of such algebras.

Nonetheless, let $\mathcal{E}$ be the family of all algebras of type p having constituents as described above, for any $q = p^h > p$ and $1 \leqslant m < p-1$. Then the main result we will prove in this thesis is the following:

Theorem 2.6. *Over a field $\mathbb{F}$ of positive characteristic p, let $\mathcal{F}$ be the family of algebras of type p that are subalgebras of an uncovered algebra of type* 1. *Then every algebra of type p over $\mathbb{F}$ is isomorphic to $L(\delta)$ for some $L \in \mathcal{F} \cup \mathcal{E}$ and some $\delta \in \mathbb{F}$.*

When $p = 2$, this is exactly the classification result found in [CVL03], since $\mathcal{E}$ is empty. Thus, we assume p is odd for the rest of this thesis, unless otherwise stated. Note that, given an algebra L of type p, for the sake of proving Theorem 2.6 we may assume that the length l of its first fake constituent is not *minimal*, i.e. $l > 2p$. Indeed, by definition, $l = 2p$ would mean that the first two-step centralizer is not zero, namely $\alpha_{p+1} = \delta \neq 0$, therefore one can simply replace L with $L(-\delta)$ to obtain an algebra of type p with a null first two-step centralizer.

The proof of Theorem 2.6 can be sketched as follows:

- Assume L has first fake constituent of length $l > 2p$. Then the only possible values for l are either $2q$ or $q+j$, where $q > p$ is a power of p and j is a odd integer such that $1 \leqslant j \leqslant p$. This is Proposition 3.2, and the next chapter is dedicated to its proof.

- If $l = 2q$, then L is isomorphic to a subalgebra of an uncovered algebra of type 1. This is proved in the first section of Chapter 4.

- If $l = q + p$ then every constituent of L is either ordinary ending in λ of length q, or almost ordinary ending in λ of length $q - 1$. Therefore, translating L by $-\lambda$ one gets an algebra with ordinary constituents, which by Proposition 2.5 is isomorphic to a subalgebra of an algebra of type 1.

- If $l = q + j$ for some $j < p$, then $L \in \mathcal{E}$. The proof of both this and the previous point are addressed in the second section of Chapter 4.

One tool that will be useful to prove the classification theorem is *deflation*. Let $L = L_1 \oplus \bigoplus_{i \geqslant p} L_i$ be an algebra of type p, and let $L_i = \langle e_i \rangle$, $[e_i e_1] = e_{i+1}$ for $i \geqslant p$. Regarding L as embedded in an associative algebra A, we may consider the Lie subalgebra N of A generated by $x := e_1^p$ and e_p. It is easy to see that N turns out to be an uncovered algebra of type 1, with $N_1 = \langle x, e_p \rangle$ and $N_i = \langle e_{pi} \rangle$ for $i > 1$. N is said to be the *deflated algebra* of L. Note that, if $(\alpha_i)_{i>p}$ is the sequence of two-step centralizers of L, then

$$[e_{pi} e_p] = \alpha_{pi} e_{p(i+1)} = \alpha_{pi} [e_{pi} x]$$

shows that the sequence of two-step centralizers of N is given by $\beta_i = \alpha_{pi}$.

Finally, let us record a simple fact which will be used in almost every computation we will have to deal with:

Lemma 2.7. *Let L be an algebra of type p over a field of characteristic p with generators and two-step centralizers as above, and let a, b be two non-negative integers. Then the following relations hold:*

$$0 = \sum_{i=0}^{a}(-1)^i \binom{a}{i} \alpha_{b+p+i} + \sum_{i=0}^{b}(-1)^i \binom{b}{i} \alpha_{a+p+i} \tag{2.1}$$

$$\begin{aligned} 0 = \alpha_{a+b+2p} &\sum_{i=0}^{b}(-1)^i \binom{b}{i} \alpha_{a+p+i} + \\ - \alpha_{b+p} &\sum_{i=0}^{b+p}(-1)^i \binom{b+p}{i} \alpha_{a+p+i} + \\ - \alpha_{a+p} &\sum_{i=0}^{b}(-1)^i \binom{b}{i} \alpha_{a+2p+i} \end{aligned} \tag{2.2}$$

Proof. The proof is straightforward, since Relation (2.1) is just the expansion by means of Lucas' theorem of the anticommutativity relation in L, i.e.

$$0 = [e_{b+p} e_{a+p}] + [e_{a+p} e_{b+p}].$$

Similarly, Relation (2.2) is just the expansion of the Jacobi identity

$$0 = [e_{a+p} e_{b+p} e_p] - [e_{a+p}[e_{b+p} e_p]] - [e_{a+p} e_p e_{b+p}].$$

□

We remark that Relations (2.1) and (2.2) are of weight $a + b + 2p$ and $a + b + 3p$ respectively. Furthermore, assuming without loss of generality that $a \leqslant b$, the first one relates two-step centralizers from α_{a+p} to α_{a+b+p}, while the second one relates two-step centralizers from α_{a+p} to α_{a+b+2p}. The general strategy in every computation in this thesis is to use these relations with suitable choices of a and b such that we can obtain non-trivial relations between two-step centralizers of a given (fake) constituent.

Chapter 3

Constituent lengths

This chapter is devoted to discussing the possible fake constituent lengths of an algebra of type p. More precisely, in the first section we give upper and lower bounds on the lengths of every fake constituent other than the first one, and these bounds depend only on the length of the first fake constituent. In the second section we prove that the length of the first fake constituent can assume only certain values.

3.1 Upper and lower bounds for the fake constituent lengths

Let L be an algebra of type p with generators e_1 and e_p and associated sequence of two-step centralizers $(\alpha_i)_{i>p}$. For later convenience, let us introduce a total ordering on two-step centralizers by saying that $\alpha_i \preceq \alpha_j$ if and only if $i \leqslant j$. Suppose the first constituent is

$$0, \dots, 0, \alpha_{l-p+1}, \dots, \alpha_r,$$

with $\alpha_{l-p+1} \neq 0$ and $r \leqslant l$ such that $\alpha_r \neq 0$ and $\alpha_{r+1} = \dots = \alpha_l = 0$. Equivalently, r is the length of the first constituent and l is the length of the first fake one. We claim that the 0-length of every following constituent can be at most equal to the 0-length of the first constituent, namely $l - p$. Before proving that, we remark that this directly implies that the length of every fake constituent after the first one is at most l: this is because clearly the number of zeros at the beginning of a fake constituent is at most equal to the 0-length of the corresponding (real) constituent, and the length of the fake constituent is exactly equal to the number of zeros at its beginning plus p.

To prove the claim, let $\alpha_k \neq 0$ be the last element of a given constituent, and suppose that the following constituent has 0-length greater than $l - p$, namely

$$\alpha_{k+1} = \dots = \alpha_{k+l-p+1} = 0.$$

Consider Relation (2.2) with $a = k - p$ and $b = l - 2p + 1$: we have that $\alpha_{a+b+2p} = \alpha_{k+l-p+1} = 0$, while $\alpha_{a+p} = \alpha_k \neq 0$ and $\alpha_{b+p} = \alpha_{l-p+1} \neq 0$, hence

$$\begin{aligned} 0 &= -\alpha_{l-p+1} \sum_{i=0}^{l-p+1} (-1)^i \binom{l-p+1}{i} \alpha_{k+i} - \alpha_k \sum_{i=0}^{l-2p+1} (-1)^i \binom{l-2p+1}{i} \alpha_{k+p+i} \\ &= -\alpha_{l-p+1}\alpha_k \neq 0, \end{aligned}$$

a contradiction.

The following lemma yields a lower bound on the lengths of the fake constituents following the first one:

Lemma 3.1. *Let L be an algebra of type p with first fake constituent of length $l > 2p$. Then every fake constituent other than the first has length at least $\frac{l}{2}$.*

Proof. First of all, $[e_p e_1^i, e_p e_1^{i+1}] = 0$ for $0 \leqslant i < \frac{l}{2} - p$, since the first fake constituent has length l.

Let us then proceed by induction on the constituents. Regarding the second fake constituent, we have to prove that $\alpha_{l+i} = 0$ for $1 \leqslant i \leqslant \frac{l}{2} - p$, and we do that by secondary induction on i. For $i = 1$ we have that

$$\begin{aligned} 0 &= [e_{l-p}[e_p, e_p e_1]] \\ &= -[e_{l-p}[e_p e_1] e_p] \\ &= \alpha_{l-p+1}\alpha_{l+1} e_{l+p+1}, \end{aligned}$$

hence $\alpha_{l+1} = 0$. Assume by induction that $\alpha_{l+i} = 0$ for $i = 1, \dots r$, where $r < \frac{l}{2} - p$. Then

$$\begin{aligned} 0 &= [e_{l-p-r}[e_p e_1^r, e_p e_1^{r+1}]] \\ &= -[e_{l-p-r}[e_p e_1^{r+1}], e_p e_1^r] \\ &= (-1)^r \alpha_{l-p+1}[e_{l+1}, e_p e_1^r] \\ &= \alpha_{l-p+1}\alpha_{l+r+1} e_{l+p+r+1}, \end{aligned}$$

hence $\alpha_{l+r+1} = 0$.

Let us now consider a general fake constituent other than the first, written in the form

$$\underbrace{0, \dots, 0}_{s}, \alpha_{k+1}, \dots, \alpha_{k+p},$$

hence of length $m = s + p$. By induction hypothesis we assume that this constituent has length greater than $\frac{l}{2}$ (i.e. $s \geqslant \frac{l}{2} - p$), and we aim at proving that the same holds for the following constituent, or equivalently that $\alpha_{k+p+i} = 0$ for $1 \leqslant i \leqslant \frac{l}{2} - p$. We do that again by secondary induction on i and by means of similar computations: for $i = 1$ we have that

$$\begin{aligned} 0 &= [e_k[e_p, e_p e_1]] \\ &= -[e_k[e_p e_1]e_p] \\ &= \alpha_{k+1}\alpha_{k+p+1}e_{k+2p+1}, \end{aligned}$$

hence $\alpha_{k+p+1} = 0$. Assume by induction that $\alpha_{k+p+i} = 0$ for $i = 1, \dots r$, where $r < \frac{l}{2} - p$. Then

$$\begin{aligned} 0 &= [e_{k-r}[e_p e_1^r, e_p e_1^{r+1}]] \\ &= -[e_{k-r}[e_p e_1^{r+1}], e_p e_1^r] \\ &= (-1)^r \alpha_{k+1}[e_{k+p+1}, e_p e_1^r] \\ &= \alpha_{k+1}\alpha_{k+p+r+1}e_{k+2p+r+1} \end{aligned}$$

yields $\alpha_{k+p+r+1} = 0$, which completes the proof. □

3.2 The length of the first fake constituent

As already anticipated, in this section we will prove that the length of the first fake constituent can assume only certain values. More precisely, the main result we will prove is the following:

Proposition 3.2. *Given an algebra of type p over a field of positive characteristic p, the first fake constituent can only have length of the form:*

1. *either $2q$, where $q \geqslant p$ is a power of p,*
2. *or $q + j$, where $q > p$ is a power of p and j is an odd integer such that $1 \leqslant j \leqslant p$.*

Let L be an algebra of type p over a field of characteristic p with generators e_1, e_p and two-step centralizers α_i as costumary. Let l and l_2 denote the lengths of its first and second fake constituent, respectively. The first thing we easily notice is that l has to be even, as a consequence of Relation (2.1) with $a = l - 2p + 1$ and $b = 0$:

$$\begin{aligned} 0 &= \sum_{i=0}^{l-2p+1} (-1)^i \binom{l-2p+1}{i} \alpha_{p+i} + \alpha_{l-p+1} \\ &= \left(1 + (-1)^{l+1}\right) \alpha_{l-p+1}. \end{aligned}$$

Consider now the p-adic expansion of l, i.e.

$$l = a_h p^h + a_{h-1}p^{h-1} + \cdots + a_0, \quad a_h \neq 0.$$

Since for any algebra of type p the fake length is at least $2p$, we have that $h \geqslant 1$.

Let $q := p^h$. In the following, we will prove that $l \leqslant q + p$ or $l = 2q$, thus proving Proposition 3.2. Let us start by showing that $l \leqslant 2q$. Suppose on the contrary that $l > 2q$. Then (see previous section) $l_2 \geqslant \frac{l}{2} > q$, so that in particular

$$\alpha_{l+1} = \ldots = \alpha_{l+q-p+1} = 0. \tag{3.1}$$

In the p-adic expansion of l, let $c := a_{h-1}p^{h-1} + \cdots + a_0$, so that we can write $l = a_h q + c$ with $0 \leqslant c < q$. We now distinguish two cases:

- If $p \leqslant c < q$, consider Relation (2.1) with $a = (a_h + 1)q - p$ and $b = c - p + 1$:

$$\sum_{i=0}^{(a_h+1)q-p} (-1)^i \binom{(a_h+1)q-p}{i} \alpha_{c+1+i} + \sum_{i=0}^{c-p+1} (-1)^i \binom{c-p+1}{i} \alpha_{(a_h+1)q+i} = 0.$$

  The second sum vanishes since all two-step centralizers contained in it are null:

$$\alpha_{(a_h+1)q+i} \succeq \alpha_{a_h q+q} \succ \alpha_{a_h q+c} = \alpha_l$$

  and

$$\alpha_{(a_h+1)q+i} \preceq \alpha_{a_h q+c+q-p+1} = \alpha_{l+q-p+1}.$$

  In the first sum only the terms with $i \equiv 0 \mod p$ have non-vanishing binomial coefficient, hence we get

$$0 = (-1)^{a_h q-p} \binom{(a_h+1)q-p}{a_h q-p} \alpha_{a_h q+c-p+1} = (-1)^{a_h q-p} a_h \alpha_{l-p+1} \neq 0,$$

  a contradiction.

- If $0 \leqslant c < p$, consider Relation (2.1) with $a = a_h q$ and $b = q + c - 2p + 1$ this time:

$$\sum_{i=0}^{a_h q} (-1)^i \binom{a_h q}{i} \alpha_{q+c-p+1+i} + \sum_{i=0}^{q+c-2p+1} (-1)^i \binom{q+c-2p+1}{i} \alpha_{a_h q+p+i} = 0.$$

  The second sum vanishes since all two-step centralizers contained in it are null:

$$\alpha_{a_h q+p+i} \succeq \alpha_{a_h q+p} \succ \alpha_{a_h q+c} = \alpha_l$$

  and

$$\alpha_{a_h q+p+i} \preceq \alpha_{a_h q+c+q-p+1} = \alpha_{l+q-p+1}.$$

  In the first sum only the terms with $i \equiv 0 \mod q$ have non-vanishing binomial coefficient, hence we get

$$0 = (-1)^{(a_h-1)q} \binom{a_h}{a_h-1} \alpha_{a_h q+c-p+1} = (-1)^{a_h} a_h \alpha_{l-p+1} \neq 0,$$

  a contradiction.

We are left with proving that l cannot be in the range $(q+p, 2q)$, so assume by contradiction that $q+p<l<2q$. Note that $q>p$, since l is always greater than or equal to $2p$ by definition. We claim that

$$\alpha_{l-j}=0 \quad \text{for } j=-1,0,\dots,l-q-p.$$

This is trivially true for $j=-1$, hence assume by induction that $\alpha_{l-j}=0$ for all $-1\leqslant j<m$ for a fixed $m\leqslant l-q-p$. Consider Relation (2.1) with $a=l-p+1$ and $b=l-q-p-m$:

$$\sum_{i=0}^{l-p+1}(-1)^i\binom{l-p+1}{i}\alpha_{l-q-m+i} + \sum_{i=0}^{l-q-p-m}(-1)^i\binom{l-q-p-m}{i}\alpha_{l+1+i}=0. \tag{3.2}$$

Let us start by looking at the second sum: it contains two-step centralizers starting from α_{l+1}, which does not lie in the first constituent, until $\alpha_{l+(l-q-m+1)-p}$. The lower bound on the second fake constituent length, i.e. $l_2\geqslant\frac{l}{2}$, implies that

$$\alpha_{l+1}=\ldots=\alpha_{l+\frac{l}{2}-p}=0.$$

Therefore the sum vanishes as long as $l-q-m+1\leqslant\frac{l}{2}$, or equivalently as long as $l\leqslant 2q+2m-2$, and this is true since $l<2q$ (and l is even). Hence, Relation (3.2) is

$$\sum_{i=0}^{l-p+1}(-1)^i\binom{l-p+1}{i}\alpha_{l-q-m+i}=0,$$

and in this sum the only non-vanishing two-step centralizers are the ones at the end of the first constituent, i.e. the ones for $q-p+m+1\leqslant i\leqslant q+m$. But when $q-p+m+1\leqslant i<q$,

$$\binom{l-p+1}{i}=\binom{q+l-q-p+1}{i}=\binom{l-q-p+1}{i}=0,$$

since $1<l-q-p+1<q-p+1\leqslant i$. Also, by induction hypothesis, $\alpha_{l-q-m+i}=0$ when $i>q$, so the only non-zero addend of the sum is the one for $i=q$:

$$-\binom{l-p+1}{q}\alpha_{l-m}=-\alpha_{l-m}=0.$$

Hence $\alpha_{l-j}=0$ for all $j\leqslant l-q-p$ as stated, and this yields a contradiction as long as $l\geqslant q+2p-1$: indeed, if that is the case, then in particular the previous statement holds for $j=p-1$, i.e. $0\neq\alpha_{l-p+1}=0$.

On the other hand, if $q+p<l<q+2p-1$ this is not enough to get a contradiction, and the use of Lemma 2.7, Relation (2.2) will be needed. So let us assume $l=q+p+k$ for some $0<k<p-1$. Since l is even, k must be even too, hence $2\leqslant k\leqslant p-3$. Recall

that by the previous result we know that $\alpha_{q+p} = \alpha_{q+p+1} = \cdots = \alpha_{q+p+k} = 0$, hence the first fake constituent is of the form

$$0, \dots 0, \alpha_{q+k+1}, \dots, \alpha_{q+p-1}, \underbrace{0, \dots 0}_{k+1},$$

where $\alpha_{q+k+1} \neq 0$. We will get a contradiction by proving that $\alpha_{q+p-r} = 0$ for $0 \leqslant r \leqslant p-k-1$, and to do this the following lemma will be needed:

Lemma 3.3. *Let L be an algebra of type p as above, with first fake constituent of length $l = q+p+k$, $0 < k < p-1$. Then $q-2$ is an upper bound for the second fake constituent length.*

Proof. Let l_2 denote the length of the second fake constituent. Put $\gamma_2 := \alpha_{l_2+q+k+1}$ and $\gamma_2' := \alpha_{l_2+q+k+2}$, so that the second fake constituent begins with

$$\alpha_{q+p+k+1} = \underbrace{0, \dots 0}_{l_2-p}, \gamma_2, \gamma_2',$$

where $\gamma_2 \neq 0$. From the computation in the previous section we already know that $l_2 \leqslant q+p-1$, but we can easily refine that upper bound to $l_2 \leqslant q$ by using Relation (2.2) with $0 \leqslant a \leqslant l_2-p-1$ and $b = q-p+k+1$: for such choices of a and b we have that

$$\alpha_{q+p+k+1} \preceq \alpha_{a+b+2p} \preceq \alpha_{l_2+q+k} \prec \gamma_2,$$

hence $\alpha_{a+b+2p} = 0$. Relation (2.2) then yields

$$\alpha_{q+k+1} \sum_{i=0}^{q+k+1} (-1)^i \binom{q+k+1}{i} \alpha_{a+p+i} + \alpha_{a+p} \sum_{i=0}^{q-p+k+1} (-1)^i \binom{q-p+k+1}{i} \alpha_{a+2p+i} = 0,$$

but either $\alpha_{a+p} = 0$ or $\alpha_{a+2p+i} = 0$ for all $i \leqslant q-p+k+1$, hence the second term vanishes. Since $\alpha_{q+k+1} \neq 0$, we have that

$$\sum_{i=0}^{q+k+1} (-1)^i \binom{q+k+1}{i} \alpha_{a+p+i} = 0$$

for all $a \leqslant l_2-p-1$. Now, if we assume that $l_2 > q$, then the previous relation holds in particular for $a = q-p$, that is,

$$0 = \sum_{i=0}^{q+k+1} (-1)^i \binom{q+k+1}{i} \alpha_{q+i} = -\alpha_{q+k+1} \neq 0,$$

a contradiction.

To complete the proof we have to show that $l_2 \neq q, q-1$. Suppose first that $l_2 = q$ and consider Relation (2.2) with $a = q-p+k$ and $b = q-p+1$: for these choices we have that

$$\begin{aligned}\alpha_{a+b+2p} &= \alpha_{2q+k+1} = \gamma_2 \neq 0,\\ \alpha_{a+p} &= \alpha_{b+p} = 0.\end{aligned}$$

Hence we get

$$\begin{aligned}0 &= \gamma_2 \sum_{i=0}^{q-p+1} (-1)^i \binom{q-p+1}{i} \alpha_{q+k+i}\\ &= -\gamma_2 \alpha_{q+k+1} \neq 0.\end{aligned}$$

If on the other hand we assume $l_2 = q-1$, the same relation we just used yields

$$\begin{aligned}0 &= \gamma_2' \sum_{i=0}^{q-p+1} (-1)^i \binom{q-p+1}{i} \alpha_{q+k+i}\\ &= -\gamma_2' \alpha_{q+k+1},\end{aligned}$$

hence $\gamma_2' = 0$. We can then use Relation (2.2) with $a = q-p+k+2$ and $b = q-p-1$ to get

$$\begin{aligned}0 &= \alpha_{q+k+2} \sum_{i=0}^{q-p-1} (-1)^i \binom{q-p-1}{i} \alpha_{q+p+k+2+i}\\ &= -\gamma_2 \alpha_{q+k+2}.\end{aligned}$$

This is a contradiction since $\alpha_{q+k+2} \neq 0$: just use Relation (2.1) with $a = q-p+k+2$ and $b = 0$ to get

$$\begin{aligned}0 &= \sum_{i=0}^{q-p+k+2} (-1)^i \binom{q-p+k+2}{i} \alpha_{p+i} + \alpha_{q+k+2}\\ &= 2\alpha_{q+k+2} - (k+2)\alpha_{q+k+1},\end{aligned}$$

hence $\alpha_{q+k+2} = \frac{k+2}{2}\alpha_{q+k+1} \neq 0$. □

We can now prove that $\alpha_{q+p-r} = 0$ for every r such that $0 \leqslant r \leqslant p-k-1$. This is trivially true when $r = 0$, so fix r such that $0 < r \leqslant p-k-1$ and assume by induction that $\alpha_{q+p-s} = 0$ whenever $0 \leqslant s < r$. Let $\gamma_2 := \alpha_{l_2+q+k+1}$ and $\gamma_2' := \alpha_{l_2+q+k+2}$ like in the proof of the previous lemma, so that γ_2 is the first non-zero two-step centralizer of the second constituent. Recall also that $l_2 < q-1$ as proven there.

If r is even, consider both Relation (2.1) and Relation (2.2) with $a = l_2 - 2p + k + r + 1$ and $b = q - r$. The first relation yields

$$\begin{aligned} 0 &= \sum_{i=0}^{a}(-1)^i\binom{a}{i}\alpha_{q+p-r+i} + \sum_{i=0}^{q-r}(-1)^i\binom{q-r}{i}\alpha_{a+p+i} \\ &= \alpha_{q+p-r} + \sum_{i=0}^{q-r}(-1)^i\binom{q-r}{i}\alpha_{a+p+i}, \end{aligned} \tag{3.3}$$

where we made use of the induction hypothesis and the fact that in the first summation $\alpha_{q+p-r+i} \preceq \alpha_{l_2+q-p+k+1} \prec \gamma_2$. On the other hand, the second relation is

$$0 = \gamma_2 \sum_{i=0}^{q-r}(-1)^i\binom{q-r}{i}\alpha_{a+p+i} - \alpha_{q+p-r}\sum_{i=0}^{q+p-r}(-1)^i\binom{q+p-r}{i}\alpha_{a+p+i}, \tag{3.4}$$

since $\alpha_{a+p} = \alpha_{l_2-p+k+r+1} \prec \alpha_{q-p+k+r} \preceq \alpha_{q+k}$ and hence $\alpha_{a+p} = 0$. Substituting (3.3) in (3.4) yields

$$0 = \gamma_2\alpha_{q+p-r} + \alpha_{q+p-r}\sum_{i=0}^{q+p-r}(-1)^i\binom{q+p-r}{i}\alpha_{a+p+i}, \tag{3.5}$$

and the summation is easy to evaluate: by Lucas' theorem it is equal to

$$\sum_{i=0}^{p-r}(-1)^i\binom{p-r}{i}\alpha_{a+p+i} - \sum_{i=0}^{p-r}(-1)^i\binom{p-r}{i}\alpha_{a+q+p+i},$$

and the first summation is null since $\alpha_{a+p+i} \prec \alpha_{q+k}$ for every $i \leqslant p-r$, while in the second summation only the last term survives, namely $(-1)^{p-r}\gamma_2$. Putting everything together, Equation (3.5) yields

$$\begin{aligned} 0 &= \gamma_2\alpha_{q+p-r} + (-1)^{p-r+1}\gamma_2\alpha_{q+p-r} \\ &= 2\gamma_2\alpha_{q+p-r}, \end{aligned}$$

hence $\alpha_{q+p-r} = 0$.

If r is odd, consider both Relation (2.1) and Relation (2.2) with $a = l_2 - 2p + k + r + 2$ and $b = q - r$ this time. Similarly to the previous case, the first relation yields

$$\begin{aligned} 0 &= \sum_{i=0}^{a}(-1)^i\binom{a}{i}\alpha_{q+p-r+i} + \sum_{i=0}^{q-r}(-1)^i\binom{q-r}{i}\alpha_{a+p+i} \\ &= \alpha_{q+p-r} + \sum_{i=0}^{q-r}(-1)^i\binom{q-r}{i}\alpha_{a+p+i}. \end{aligned}$$

On the other hand, the second relation is

$$0 = \gamma_2' \sum_{i=0}^{q-r} (-1)^i \binom{q-r}{i} \alpha_{a+p+i} - \alpha_{q+p-r} \sum_{i=0}^{q+p-r} (-1)^i \binom{q+p-r}{i} \alpha_{a+p+i},$$

and combining both we get

$$0 = \gamma_2' \alpha_{q+p-r} + \alpha_{q+p-r} \sum_{i=0}^{q+p-r} (-1)^i \binom{q+p-r}{i} \alpha_{a+p+i}. \tag{3.6}$$

Let us evaluate the summation by means of Lucas' theorem like before: it is equal to

$$\sum_{i=0}^{p-r} (-1)^i \binom{p-r}{i} \alpha_{a+p+i} - \sum_{i=0}^{p-r} (-1)^i \binom{p-r}{i} \alpha_{a+q+p+i},$$

and the first summation is null since $\alpha_{a+p+i} \prec \alpha_{q+k+1}$ for every $i \leqslant p-r$, while in the second summation only the last two term survive, namely $(-1)^{p-r-1}(p-r)\gamma_2 + (-1)^{p-r}\gamma_2'$. Putting everything together, Equation (3.6) yields

$$\begin{aligned} 0 &= \gamma_2' \alpha_{q+p-r} + (-1)^{p-r}(p-r)\gamma_2 \alpha_{q+p-r} + (-1)^{p-r+1} \gamma_2' \alpha_{q+p-r} \\ &= (p-r)\gamma_2 \alpha_{q+p-r}, \end{aligned}$$

hence $\alpha_{q+p-r} = 0$ also for every odd $r \leqslant p-k-1$. Therefore $\alpha_{q+p-r} = 0$ for every $r \leqslant p-k-1$, so that in particular $\alpha_{q+k+1} = 0$, a contradiction. This completes the proof of the main result of this section, namely Proposition 3.2.

Chapter 4

Algebras with given first fake constituent length

In this chapter we discuss the uniqueness of algebras of type p with first fake constituent of given length l, or equivalently with first constituent of given 0-length $l-p$. In the first section we consider the case $l = 2q$, and it will turn out that the first fake constituent actually coincides with the first constituent, and such algebras are actually subalgebras of algebras of type 1.

In the second section we address those algebras with first fake constituent length $l = q+j$, where j is an odd integer such that $1 \leqslant j \leqslant p$. It will turn out that, although the cases $j \neq p$ and $j = p$ share some common features regarding the *initial* structure of the algebra, they are substantially different when it comes to the structure of the constituents after the first ones: we will prove that:

- If $j \neq p$, then there are at most two algebras of type p with first fake constituent of length $q+j$. One of them has first fake constituent actually coinciding with the first constituent, while the other has first constituent of length $q+j-1$. In fact, if $j = 1$ only the first case can happen. A posteriori, this can be rephrased as follows: there is at most one algebra of type p with first constituent of length $q+m$ for every m such that $1 \leqslant m < p-1$.

- If $j = p$, then the first constituent is almost ordinary of length $q+p-1$ ending in λ, and every other constituent is either ordinary of length q ending in λ, or almost ordinary of length $q-1$ ending in λ. It is then a straightforward consequence to see that $L(-\lambda)$ has ordinary constituent and is thus isomorphic to a subalgebra of an algebra of type 1.

4.1 First fake constituent length $2q$

Let L be an algebra of type p with first fake constituent of length $2q$, where $q > p$ is a power of p. Equivalently, the first constituent has 0-length $q-p$, and up to scaling it starts with

$$\alpha_{p+1} = \cdots = \alpha_{2q-p} = 0, \quad \alpha_{2q-p+1} = 1.$$

First of all, let us show that $\alpha_{2q} = 1$, meaning in particular that the the first constituent coincides with its fake counterpart. This is a simple consequence of Relation (2.1) with $a = 2q-p+1$ and $b = 0$:

$$\begin{aligned}
0 &= \sum_{i=0}^{2q-p+1} (-1)^i \binom{2q-p+1}{i} \alpha_{p+i} \\
&= \sum_{i=2q-2p+1}^{2q-p} (-1)^i \binom{2q-p+1}{i} \alpha_{p+i} \\
&= -\sum_{i=1}^{p} (-1)^i \binom{q-p+1}{i} \alpha_{2q-p+i} \\
&= \alpha_{2q-p+1} - \alpha_{2q} \\
&= 1 - \alpha_{2q}.
\end{aligned}$$

As a consequence of the lower bound on the length of the second fake constituent, the first constituent is ordinary ending in 1. Indeed $\alpha_{2q-p+1} = 1$ by assumption, hence suppose that $\alpha_{2q-p+h} = 1$ for $h < p-1$ and let us prove the claim by induction. Relation (2.1) with $a = 2q-p+1$ and $b = h$ yields

$$\begin{aligned}
0 &= \sum_{i=0}^{2q-p+1} (-1)^i \binom{2q-p+1}{i} \alpha_{p+h+i} + \sum_{i=0}^{h} (-1)^i \binom{h}{i} \alpha_{2q+i+1} \\
&= \sum_{i=2q-2p-h+1}^{2q-p-h} (-1)^i \binom{2q-p+1}{i} \alpha_{p+h+i} \\
&= (-1)^h \sum_{i=1}^{p} (-1)^i \binom{q-p+1}{q-2p-h+i} \alpha_{2q-p+i} \\
&= \alpha_{2q-p+h} - \alpha_{2q-p+h+1} \\
&= 1 - \alpha_{2q-p+h+1},
\end{aligned}$$

since by Lucas' theorem

$$\binom{q-p+1}{q-2p-h+i} = \begin{cases} -1, & \text{if } i = h, h+1; \\ 0, & \text{otherwise.} \end{cases}$$

Hence the first constituent is ordinary ending in 1 as claimed, i.e.

$$\alpha_{2q-p+1} = \cdots = \alpha_{2q} = 1.$$

We now want to show that also every other constituent is ordinary, thus obtaining that L is a subalgebra of an algebra of type 1 by means of Proposition 2.5.

Proceeding by induction, assume we have already proved this up to a certain constituent, ending as

$$\alpha_{m-p} = \cdots = \alpha_{m-1} = \lambda \in \mathbb{F}^*.$$

Let l be the length of the next fake constituent, and recall that $q \leqslant l \leqslant 2q$. Suppose first that $l = q$, i.e.

$$\alpha_m = \ldots = \alpha_{m+q-p-1} = 0, \quad \alpha_{m+q-p} \neq 0$$

and let

$$\alpha_{m+q-p} = \lambda_0, \quad \alpha_{m+q-p+1} = \lambda_1, \quad \ldots, \quad \alpha_{m+q-1} = \lambda_{p-1}.$$

We remind that the following fake constituent has length at least q, meaning that

$$\alpha_{m+q} = \ldots = \alpha_{m+2q-p-1} = 0.$$

For every $h = 1, \ldots, p-1$, consider Relation (2.2) with $a = m-2p-1$ and $b = 2q-2p+h$: we have that

$$\alpha_{a+b+2p} = \alpha_{m+2q-2p+h-1} \prec \alpha_{m+2q-p-1},$$

thus $\alpha_{a+b+2p} = 0$. Also, $\alpha_{a+p} = 0$ and $\alpha_{b+p} = \alpha_{2q-p+h} = 1$, hence the relation yields

$$\begin{aligned} 0 &= \sum_{i=0}^{2q-p+h} (-1)^i \binom{2q-p+h}{i} \alpha_{m-p-1+i} \\ &= \sum_{i=1}^{p} (-1)^i \binom{q-p+h}{i} \lambda - \sum_{i=1}^{p} (-1)^i \binom{q-p+h}{i} \lambda_{i-1} \\ &= \lambda \sum_{i=1}^{h} (-1)^i \binom{h}{i} + \lambda - \sum_{i=1}^{h} (-1)^i \binom{h}{i} \lambda_{i-1} + \lambda_{p-1} \\ &= -\sum_{i=1}^{h} (-1)^i \binom{h}{i} \lambda_{i-1} + \lambda_{p-1}. \end{aligned}$$

Since this relation holds for every $h = 1, \ldots, p-1$, we obtain that $\lambda_0 = \lambda_1 = \ldots = \lambda_{p-1}$ and hence the constituent is ordinary, as claimed.

Suppose now $l > q$, so that in particular

$$\alpha_m = \cdots = \alpha_{m+q-p} = 0.$$

We first extend this to show that $\alpha_{m+q-p+h}=0$ for $h=0,\dots,p$, so that $l>q+p$. We do that by induction on h: the claim is trivially true for $h=0$, so let us fix $1\leqslant h\leqslant p$ and assume the result holds for every $j<h$. If $h<p$, Relation (2.2) with $a=m-2p$ and $b=q-p+h$ yields

$$\begin{aligned}
0&=\alpha_{m+q-p+h}\sum_{i=0}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{m-p+i}+(-1)^{h+1}\lambda\alpha_{m+q-p+h}\\
&=\alpha_{m+q-p+h}\sum_{i=0}^{h}(-1)^i\binom{h}{i}\lambda+(-1)^{h+1}\lambda\alpha_{m+q-p+h}\\
&=(-1)^{h+1}\lambda\alpha_{m+q-p+h},
\end{aligned}$$

while if $h=p$ the same relation yields $2\lambda\alpha_{m+q}=0$. In any case, $\alpha_{m+q-p+h}=0$ as claimed.

Let us now look at the end of the constituent, and suppose

$$\alpha_{m+l-p}=\lambda_0\neq 0,\quad \alpha_{m+l-p+1}=\lambda_1,\quad \dots,\quad \alpha_{m+l-1}=\lambda_{p-1}.$$

Also, recall that the following fake constituent has length at least q, that is,

$$\alpha_{m+l}=\ldots=\alpha_{m+l+q-p-1}=0.$$

For every $h=1,\dots,p-1$, consider Relation (2.2) with $a=m+l-q-p-h-1$ and $b=2q-2p+h$: we have that $\alpha_{a+b+2p}=\alpha_{m+l+q-p-1}=0$, $\alpha_{b+p}=1$, and $\alpha_{a+p}=\alpha_{m+l-q-h-1}=0$ since

$$m\leqslant m+l-q-h-1\leqslant m+q-2.$$

Therefore, the relation yields

$$\begin{aligned}
0&=\sum_{i=0}^{2q-p+h}(-1)^i\binom{2q-p+h}{i}\alpha_{m+l-q-h-1+i}\\
&=\sum_{i=q-p+h+1}^{q+h}(-1)^i\binom{2q-p+h}{i}\alpha_{m+l-q-h-1+i}\\
&=\sum_{i=1}^{p}(-1)^{i+h}\binom{2q-p+h}{q-p+h+i}\lambda_{i-1}.
\end{aligned}$$

By Lucas' theorem we have

$$\binom{2q-p+h}{q-p+h+i}=\begin{cases}0, & \text{for } i=1,\dots,p-h-1;\\ \binom{h}{p-i}, & \text{for } i=p-h,\dots,p,\end{cases}$$

hence the following relation holds for every $h=1,\dots,p-1$:

$$\sum_{i=0}^{h}(-1)^i\binom{h}{i}\lambda_{p-i-1}=0.$$

This yields $\lambda_0 = \lambda_1 = \ldots = \lambda_{p-1}$, thus proving that also in this case the constituent is ordinary.

4.2 First fake constituent length $q + j$

Now let L be an algebra of type p over a field of characteristic p with first fake constituent of length $l = q + j$, where $q > p$ is a power of p and j is an odd integer such that $1 \leqslant j \leqslant p$. Equivalently, this means that L has first constituent of 0-length $q - p + j$, i.e.

$$\alpha_{p+i} = 0 \quad \text{for } i \leqslant q - p + j, \quad \alpha_{q-p+j+1} \neq 0.$$

First of all, we want to refine the upper and lower bound on the length of the second fake constituent. Thus, consider Relation (2.1) with $a = q - p + j + 1$ and $b = 0$ for $j \neq p$:

$$\begin{aligned} 0 &= \sum_{i=0}^{q-p+j+1} (-1)^i \binom{q-p+j+1}{i} \alpha_{p+i} \\ &= \alpha_{q-p+j+1} + \sum_{i=1}^{j} (-1)^i \binom{j+1}{i} \alpha_{q+i}, \end{aligned}$$

that is,

$$\sum_{i=1}^{j} (-1)^i \binom{j+1}{i} \alpha_{q+i} = -\alpha_{q-p+j+1} \neq 0. \tag{4.1}$$

In particular, this equation ensure the existence of a positive integer $k \leqslant j$ such that $\alpha_{q+k} \neq 0$, and without loss of generality we may assume k is maximal with this property, so that the length of the first constituent of L is exactly $q + k$. Note that, although Equation (4.1) is false for $j = p$, the consequencestill holds, since by definition $\alpha_{q+1} \neq 0$ in that casee. Furthermore, $k < p$ since $\alpha_{q+p} = 0$: the deflated algebra of L has two-step centralizers $\alpha_{2p} = \alpha_{3p} = \ldots = \alpha_q = 0$, hence by the theory of algebras of type 1 a two-step centralizer of the form α_{pi} different from zero cannot occur before α_{2q}.

This simple observation allows us to refine the lower bound on the length of the second fake constituent:

Lemma 4.1. *Let L be an algebra of type p with first fake constituent of length $l = q + j$, where j is an odd integer such that $1 \leqslant j \leqslant p$. Then the second fake constituent of L has length at least $q - 1$.*

Proof. Let l_2 denote the length of the second fake constituent and let $\gamma_2 := \alpha_{l_2+l-p+1} \neq 0$, $\gamma_2' := \alpha_{l_2+l-p+2}$, so that the second fake constituent starts with

$$\alpha_{q+k+1} = 0, \ldots, 0, \gamma_2, \gamma_2'.$$

Assume $l_2 < q-1$, and let k be the integer introduced before. To get a contradiction, we just need to use both Relation (2.1) and (2.2) with $b = q-p+k$ and

$$a = \begin{cases} l_2 + l - q - 2p - k + 1, & \text{if } k \text{ is odd;} \\ l_2 + l - q - 2p - k + 2, & \text{otherwise.} \end{cases}$$

In both cases, the first relation yields

$$\sum_{i=0}^{q-p+k} (-1)^i \binom{q-p+k}{i} \alpha_{a+p+i} = -\alpha_{q+k}. \tag{4.2}$$

Regarding the second relation, first note that

$$\begin{aligned} \alpha_{a+p} &\preceq \alpha_{l_2+l-q-p-k+2} \\ &\prec \alpha_{l-p-k+1} \end{aligned}$$

and thus $\alpha_{a+p} = 0$. Substituting Equation (4.2) in the second relation yields:

- if k is odd,

$$\begin{aligned} 0 &= -\gamma_2 \alpha_{q+k} - \alpha_{q+k} \sum_{i=0}^{q+k} (-1)^i \binom{q+k}{i} \alpha_{a+p+i} \\ &= -\gamma_2 \alpha_{q+k} + (-1)^k \gamma_2 \alpha_{q+k} \\ &= -2\gamma_2 \alpha_{q+k} \neq 0, \end{aligned}$$

  a contradiction.

- if k is even,

$$\begin{aligned} 0 &= -\gamma_2' \alpha_{q+k} - \alpha_{q+k} \sum_{i=0}^{q+k} (-1)^i \binom{q+k}{i} \alpha_{a+p+i} \\ &= -\gamma_2' \alpha_{q+k} + (-1)^k \gamma_2' \alpha_{q+k} + (-1)^{k-1} k \gamma_2 \alpha_{q+k} \\ &= -k\gamma_2 \alpha_{q+k} \neq 0, \end{aligned}$$

  a contradiction.

□

Remark 4.2. From the previous lemma, we have that $\alpha_{q+j+1} = \ldots = \alpha_{2q-p+j-1} = 0$. Consider now Relation (2.1) with $a = 2q-2p+j+1$ and $b = 0$:

$$\begin{aligned} 0 &= \sum_{i=0}^{2q-2p+j} (-1)^i \binom{2q-2p+j+1}{i} \alpha_{p+i} + 2\alpha_{2q-p+j+1} \\ &= -\alpha_{q-p+j+1} - (j+1)\alpha_{2q-p+j} + 2\alpha_{2q-p+j+1}. \end{aligned}$$

Therefore

$$-(j+1)\alpha_{q+l-p} + 2\alpha_{q+l-p+1} = \alpha_{l-p+1} \neq 0,$$

so that in particular at least one between α_{q+l-p} and $\alpha_{q+l-p+1}$ differs from zero. This in turn means that

$$l_2 = q - 1 \qquad \text{or} \qquad l_2 = q.$$

Remark 4.3. Note also that

$$l_2 = q \qquad \Longleftrightarrow \qquad k \text{ is odd},$$

or, equivalently, $l_2 = q$ if and only if L has even length. In particular,

$$l_2 = q - 1 \quad \Rightarrow \quad \alpha_l = 0.$$

Indeed, if k is odd, Relation (2.2) with $a = q - p + k$ and $b = l - 2p - k$ yields

$$\begin{aligned} 0 &= \alpha_{q+k}\alpha_{l+q-p} + (-1)^{k+1}\alpha_{q+k}\alpha_{l+q-p} \\ &= 2\alpha_{q+k}\alpha_{l+q-p}, \end{aligned}$$

hence $\alpha_{l+q-p} = 0$ and $l_2 = q$. Viceversa, if $l_2 = q$ then Relation (2.2) with $a = q - p + k$ and $b = l - 2p - k + 1$ yields

$$\begin{aligned} 0 &= \alpha_{q+k}\alpha_{l+q-p+1} + (-1)^k\alpha_{q+k}(\alpha_{l+q-p+1} - (l-k+1)\alpha_{l+q-p}) \\ &= (1 + (-1)^k)\alpha_{q+k}\alpha_{l+q-p+1}, \end{aligned}$$

therefore k has to be odd.

Unfortunately, the informations we have so far are not yet enough to compute exactly the two-step centralizers contained in the first constituent, which would be the first step to compute all the following constituents by induction. Instead, we need some information regarding the length of the third fake constituent, and we proceed similarly to how we just did for the fake length of the second one.

Lemma 4.4. *Let L be an algebra of type p with first fake constituent length $l = q + j$, where j is an odd integer such that $1 \leqslant j \leqslant p$. Let l_2 and l_3 denote the lengths of the second and third fake constituent, respectively. Then:*

(i) The last two-step centralizer of the second constituent is α_{2q+m} for some m such that $0 \leqslant m \leqslant j$. When $j = p$, the integer m equals at most $p - 1$.

(ii) The third fake constituent length is at least $q - 1$.

(iii) If $l_2 = q - 1$, then $l_3 = q - 1$ or $l_3 = q$.

(iv) If $l_2 = q$, then $l_3 = q$.

Proof. (i) If $j = p$ the statement is trivial, and note that $m < p$ since $\alpha_{2q+p} = 0$ by deflation. Hence, suppose $j < p$ and consider Relation (2.1) with $a = l_2 + l - p + 1$ and $b = 0$:

$$0 = \sum_{i=0}^{l_2+l-p} (-1)^i \binom{l_2 + l - p + 1}{i} \alpha_{p+i}. \tag{4.3}$$

When $l_2 = q - 1$, the previous relation yields

$$\begin{aligned} 0 &= \sum_{i=0}^{2q-p+j-1} (-1)^i \binom{2q - p + j}{i} \alpha_{p+i} \\ &= \sum_{i=0}^{q-p+j} (-1)^i \binom{q - p + j}{i} \alpha_{p+i} + (-1)^{j+1}\gamma_2 - \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} \alpha_{2q+i}, \end{aligned}$$

where γ_2 as usual denotes the first non-zero element of the second constituent, i.e. $\gamma_2 = \alpha_{l_2+l-p+1} = \alpha_{2q-p+j}$. The first sum vanishes, as one can check using Relation (2.1) with $a = q - p + j$ and $b = 0$, hence

$$\sum_{i=0}^{j-1} (-1)^i \binom{j}{i} \alpha_{2q+i} = \gamma_2 \neq 0$$

and this proves that at least one among $\alpha_{2q}, \ldots, \alpha_{2q+j-1}$ is different from zero.

On the other hand, when $l_2 = q$ Relation (4.3) yields

$$\begin{aligned} 0 &= \sum_{i=0}^{2q-p+j} (-1)^i \binom{2q - p + j + 1}{i} \alpha_{p+i} \\ &= \sum_{i=0}^{q-p+j+1} (-1)^i \binom{q - p + j + 1}{i} \alpha_{p+i} + (-1)^j \gamma_2 - \sum_{i=0}^{j} (-1)^i \binom{j+1}{i} \alpha_{2q+i}. \end{aligned}$$

Again, Relation (2.1) with $a = q - p + j + 1$ and $b = 0$ ensures that the first sum vanishes, hence

$$\sum_{i=0}^{j} (-1)^i \binom{j+1}{i} \alpha_{2q+i} = -\gamma_2 \neq 0$$

and this proves that at least one among $\alpha_{2q}, \ldots, \alpha_{2q+j}$ is different from zero.

(ii) Let $\gamma_3 := \alpha_{l_3+l_2+q-p+j+1} \neq 0$, $\gamma_3' := \alpha_{l_3+l_2+q-p+j+2}$, so that the third fake constituent starts with

$$\alpha_{l_2+q+j+1} = 0, \ldots, 0, \gamma_3, \gamma_3'.$$

Assume $l_3 < q-1$, and let m be the integer introduced before. To get a contradiction, we just need to use both Relation (2.1) and (2.2) with $b = 2q - p + m$ and

$$a = \begin{cases} l_3 + l_2 - q - 2p + j - m + 1, & \text{if } m \text{ is even;} \\ l_3 + l_2 - q - 2p + j - m + 2, & \text{otherwise.} \end{cases}$$

In both cases, the first relation yields

$$\sum_{i=0}^{2q-p+m} (-1)^i \binom{2q-p+m}{i} \alpha_{a+p+i} = -\alpha_{2q+m}.$$

Regarding the second relation, first note that

$$\begin{aligned} \alpha_{a+p} &\preceq \alpha_{l_3-p+j-m+2} \\ &\prec \alpha_{q-p+j-m+1} \end{aligned}$$

and thus $\alpha_{a+p} = 0$. Substituting the equation above in the second relation yields:

– if m is even,

$$0 = -\gamma_3\alpha_{2q+m} - \alpha_{2q+m} \sum_{i=0}^{2q+m} (-1)^i \binom{2q+m}{i} \alpha_{a+p+i}, \tag{4.4}$$

and the only non-zero binomial coefficients above are those for $i = rq + s$, where $0 \leqslant r \leqslant 2$ and $0 \leqslant s \leqslant m$. Nonetheless, if $r = 0$ then $\alpha_{a+p+i} \prec \alpha_{q-p+j}$, hence they all equal zero. If $r = 1$,

$$\alpha_{a+p+i} \prec \alpha_{2q-p+j} \preceq \gamma_2$$

and

$$\alpha_{a+p+i} \succ \alpha_{l_3+q-p} \succeq \alpha_{q+j},$$

where the last inequality holds since $l_3 \geqslant \frac{l}{2} = \frac{q+j}{2}$ and thus

$$l_3 - p - j \geqslant \frac{q-2p-j}{2} \geqslant \frac{q-3p}{2} \geqslant 0.$$

Therefore $\alpha_{a+p+i} = 0$ if $r = 1$, as it belongs to the null part of the second constituent. For similar reasons, if $r = 2$ only the very last two-step centralizer survives, namely γ_3, and therefore Equation (4.4) yields

$$\begin{aligned} 0 &= -\gamma_3\alpha_{2q+m} + (-1)^{m+1}\gamma_3\alpha_{2q+m} \\ &= -2\gamma_3\alpha_{2q+m} \neq 0, \end{aligned}$$

a contradiction.

– if m is odd, the computation is almost the same:

$$\begin{aligned}0 &= -\gamma_3'\alpha_{2q+m} - \alpha_{2q+m}\sum_{i=0}^{2q+m}(-1)^i\binom{2q+m}{i}\alpha_{a+p+i}\\ &= -\gamma_3'\alpha_{2q+m} + (-1)^{m+1}\alpha_{2q+m}(\gamma_3' - m\gamma_3)\\ &= -m\gamma_3\alpha_{2q+m}.\end{aligned}$$

Since m is odd, it cannot be zero, hence we have a contradiction.

(iii) Now suppose $l_2 = q-1$, meaning that $\gamma_2 = \alpha_{2q-p+j}$. As we proved in *(ii)*, $l_3 \geqslant q-1$, meaning that $\alpha_{2q+j} = \ldots = \alpha_{3q-p+j-2} = 0$. Consider now Relation (2.1) with $a = 3q-2p+j$ and $b = 0$:

$$\begin{aligned}0 &= \sum_{i=0}^{3q-2p+j-1}(-1)^i\binom{2q-2p+j}{i}\alpha_{p+i} + 2\alpha_{3q-p+j}\\ &= -2\alpha_{2q-p+j} - j\alpha_{3q-p+j-1} + 2\alpha_{3q-p+j}.\end{aligned}$$

Therefore

$$-j\alpha_{3q-p+j-1} + 2\alpha_{3q-p+j} = 2\gamma_2 \neq 0, \tag{4.5}$$

so that in particular at least one between $\alpha_{3q-p+j-1}$ and α_{3q-p+j} differs from zero. This in turn means that

$$l_3 = q-1 \qquad \text{or} \qquad l_3 = q.$$

(iv) Finally, suppose $l_2 = q$, meaning that $\gamma_2 = \alpha_{2q-p+j+1}$. Since $l_3 \geqslant q-1$, we have that $\alpha_{2q+j+1} = \ldots = \alpha_{3q-p+j-1} = 0$. The same relation used in *(iii)* this time yields

$$\alpha_{3q-p+j} = 0,$$

which means that $l_3 \geqslant q$. Assume that $l_3 > q$, and consider Relation (2.2) with $a = l_2 + l - 2p - k + 1$ and $b = q - p + k$: since

$$\begin{aligned}\alpha_{a+b+2p} &= \alpha_{l_2+l+q-p+1}\\ &\prec \alpha_{l_3+l_2+l-p+1} = \gamma_3\end{aligned}$$

and

$$\begin{aligned}\alpha_{a+p} &= \alpha_{l_2+l-p+1-k}\\ &\prec \alpha_{l_2+l-p+1} = \gamma_2,\end{aligned}$$

we have that $\alpha_{a+b+2p} = \alpha_{a+p} = 0$, therefore Relation (2.2) yields

$$\begin{aligned} 0 &= -\alpha_{q+k} \sum_{i=0}^{q+k} (-1)^i \binom{q+k}{i} \alpha_{l_2+l-p-k+1+i} \\ &= -\alpha_{q+k}\alpha_{l_2+l-p+1} \\ &= -\alpha_{q+k}\gamma_2 \neq 0, \end{aligned}$$

a contradiction.

□

In particular, by Lemma 4.4 *(iii)* and *(iv)* we have that there are only three possible cases, which we are going to deal with separately:

(a) $l_2 = l_3 = q-1$;

(b) $l_2 = q-1$, $l_3 = q$;

(c) $l_2 = l_3 = q$.

Remark 4.5. If $j = 1$, L is an algebra with first fake constituent length $q+1$ and one can easily check using Relation (2.1) with $a = q-p+2$ and $b = 0$ that

$$\alpha_{q-p+2} - 2\alpha_{q+1} = 0,$$

hence $\alpha_{q+1} \neq 0$ and the first constituent coincides with its fake counterpart. By Remark 4.3, this means that L has fake second constituent of length q, thus belongs to case (c) above. Nonetheless, for technical reasons we deal with this case separately at the end of the section, thus assume $j \neq 1$ for the time being.

Case (a). Suppose $l_2 = l_3 = q-1$. We claim that this case is actually impossible unless $j = p$, but we will need some work to prove so.

Let us first fix the notation as customary: $\gamma_2 := \alpha_{q+l-p}$ and $\gamma_3 := \alpha_{2q+l-p-1}$ are the first non-zero two-step centralizers of the second and the third constituent, respectively, and $\gamma_2' := \alpha_{q+l-p+1}$, $\gamma_3' := \alpha_{2q+l-p}$. By Remark 4.3, we know that the first fake constituent has at least one null two-step centralizer at the end, namely $\alpha_l = \alpha_{q+j} = 0$. As a consequence, also the second fake constituent ends with a null two-step centralizer: Relation (2.2) with $a = l-p-1$ and $b = 2q-2p$ yields

$$\begin{aligned} 0 &= \gamma_3 \sum_{i=0}^{2q-2p} (-1)^i \binom{2q-2p}{i} \alpha_{l-1+i} - \alpha_{l-1} \sum_{i=0}^{2q-2p} (-1)^i \binom{2q-2p}{i} \alpha_{l+p-1+i} \\ &= \gamma_3(\alpha_{l-1} - \alpha_{q+l-1}) - \gamma_3\alpha_{l-1} \\ &= -\gamma_3\alpha_{q+l-1}, \end{aligned}$$

hence $\alpha_{q+l-1} = 0$. We claim that

$$\alpha_{l-h} = \alpha_{q+l-h-1} \qquad \text{for} \quad 0 \leqslant h < j. \tag{4.6}$$

This is straightforward for $h = 0$ as we just proved that $\alpha_l = 0 = \alpha_{q+l-1}$, thus assume by induction that the claim holds for all non-negative integers smaller than a fixed $h < j$. If h is even, consider Relation (2.2) with $a = q + l - p - h - 1$ and $b = q - 2p + h$:

$$0 = \gamma_3 \sum_{i=0}^{q-2p+h} (-1)^i \binom{q-2p+h}{i} \alpha_{q+l-h-1+i} - \alpha_{q+l-h-1} \sum_{i=0}^{q-2p+h} (-1)^i \binom{q-2p+h}{i} \alpha_{a+2p+i}.$$

The only non-zero two-step centralizer involved in the second sum is the very last one, namely γ_3, hence

$$\begin{aligned} 0 &= \gamma_3 \sum_{i=0}^{q-2p+h} (-1)^i \binom{q-2p+h}{i} \alpha_{q+l-h-1+i} + (-1)^h \gamma_3 \alpha_{q+l-h-1} \\ &= 2\gamma_3 \alpha_{q+l-h-1} + \gamma_3 \sum_{i=1}^{h} (-1)^i \binom{h}{i} \alpha_{q+l-h-1+i}. \end{aligned} \tag{4.7}$$

By induction hypothesis

$$\sum_{i=1}^{h} (-1)^i \binom{h}{i} \alpha_{q+l-h-1+i} = \sum_{i=1}^{h} (-1)^i \binom{h}{i} \alpha_{l-h+i},$$

and we can compute this sum using Relation (2.2) with $a = l - p - h$ and $b = q - 2p + h$:

$$\begin{aligned} 0 &= \gamma_2 \sum_{i=0}^{q-2p+h} (-1)^i \binom{q-2p+h}{i} \alpha_{l-h+i} - \alpha_{l-h} \sum_{i=0}^{q-2p+h} (-1)^i \binom{q-2p+h}{i} \alpha_{a+2p+i} \\ &= \gamma_2 \alpha_{l-h} + \gamma_2 \sum_{i=1}^{h} (-1)^i \binom{h}{i} \alpha_{l-h+i} + (-1)^h \gamma_2 \alpha_{l-h}, \end{aligned}$$

therefore

$$\sum_{i=1}^{h} (-1)^i \binom{h}{i} \alpha_{l-h+i} = -2\alpha_{l-h}.$$

Substitution in (4.7) yields

$$0 = 2\gamma_3 (\alpha_{q+l-h-1} - \alpha_{l-h}),$$

proving the claim for h even. On the other hand, if h is odd consider Relation (2.2) with $a = l - p - h - 1$ and $b = 2q - 2p + h$:

$$\begin{aligned} 0 &= \gamma_3 \sum_{i=0}^{2q-2p+h} (-1)^i \binom{2q-2p+h}{i} \alpha_{l-h-1+i} - \alpha_{l-h-1} \sum_{i=0}^{2q-2p+h} (-1)^i \binom{2q-2p+h}{i} \alpha_{a+2p+i} \\ &= \gamma_3 \sum_{i=0}^{2q-2p+h} (-1)^i \binom{2q-2p+h}{i} \alpha_{l-h-1+i} + (-1)^{h+1} \gamma_3 \alpha_{l-h-1}. \end{aligned}$$

Expanding the above sum by means of Lucas' theorem and using the induction hypothesis we get

$$\begin{aligned}
0 &= 2\alpha_{l-h-1} - h\alpha_{l-h} + \sum_{i=2}^{h}(-1)^i\binom{h}{i}\alpha_{l-h-1+i} - \alpha_{q+l-h-1} - \sum_{i=1}^{h-1}(-1)^i\binom{h}{i}\alpha_{q+l-h-1+i}\\
&= 2\alpha_{l-h-1} - \alpha_{q+l-h-1} - h\alpha_{l-h} + \sum_{i=2}^{h}(-1)^i\binom{h}{i}\alpha_{l-h-1+i} - \sum_{i=1}^{h-1}(-1)^i\binom{h}{i}\alpha_{l-h-1+i}\\
&= \alpha_{l-h-1} - \alpha_{q+l-h-1} + \alpha_{l-h} + \sum_{i=0}^{h}(-1)^i\binom{h+1}{i}\alpha_{l-h-1+i}. \qquad (4.8)
\end{aligned}$$

Relation (2.2) with $a = l-p-h-1$ and $b = q-2p+h+1$ yields

$$0 = \gamma_2\sum_{i=0}^{h}(-1)^i\binom{h+1}{i}\alpha_{l-h-1+i} + \gamma_2\alpha_{l-h-1},$$

which substituted in Equation (4.8) gets the job done also for h odd:

$$0 = \alpha_{l-h} - \alpha_{q+l-h-1}.$$

Before addressing the proof of Equation (4.6) we proved that $\alpha_{2q+j-1} = 0$. Assume that

$$\alpha_{2q+j-2} \neq 0.$$

Hence, without loss of generality, we can suppose that $\alpha_{2q+j-2} = 1$. Relation (2.2) with $a = 2q-p+j-2$ and $b = q-2p+2$ then yields

$$\begin{aligned}
0 &= \gamma_3'\sum_{i=0}^{q-2p+2}(-1)^i\binom{q-2p+2}{i}\alpha_{2q+j-2+i} - \sum_{i=0}^{q-2p+2}(-1)^i\binom{q-2p+2}{i}\alpha_{2q+p+j-2+i}\\
&= \gamma_3' + \gamma_3' - 2\gamma_3,
\end{aligned}$$

therefore $\gamma_3' = \gamma_3$. We have the following

Lemma 4.6. *Under the previous assumptions, the following hold:*

(i) $\alpha_{2q} = \ldots = \alpha_{2q+j-2} = 1$;

(ii) $\alpha_{q+1} = \ldots = \alpha_{q+j-1} = 1$;

(iii) $\alpha_{q-p+h} = 1 - \binom{h-1}{j-1}$ *for* $j < h \leqslant p$;

(iv) $\alpha_{2q-p+h} = 1 - \binom{h}{j-1}$ *for* $j \leqslant h < p$.

Proof. (i) $\alpha_{2q+j-2} = 1$ by hypothesis, hence assume by induction that $\alpha_{2q+j-2} = \ldots = \alpha_{2q+j-h} = 1$ for a fixed $h < j$ and let us prove that $\alpha_{2q+j-h-1} = 1$. If h is even, Relation (2.2) with $a = 2q - p + j - h - 1$ and $b = q - 2p + h$ yields

$$\begin{aligned}
0 &= \gamma_3 \sum_{i=0}^{q-2p+h} (-1)^i \binom{q-2p+h}{i} \alpha_{2q+j-h-1+i} + (-1)^h \gamma_3 \alpha_{2q+j-h-1} \\
&= \gamma_3 \left(2\alpha_{2q+j-h-1} + \sum_{i=1}^{h-1} (-1)^i \binom{h}{i} \right) \\
&= \gamma_3 \left(2\alpha_{2q+j-h-1} + (-1)^{h-1} - 1 \right) \\
&= 2\gamma_3 \left(\alpha_{2q+j-h-1} - 1 \right).
\end{aligned}$$

Similarly, if h is odd, Relation (2.2) with $a = 2q - p + j - h - 1$ and $b = q - 2p + h + 1$ yields

$$\begin{aligned}
0 &= \gamma_3' \sum_{i=0}^{q-2p+h+1} (-1)^i \binom{q-2p+h+1}{i} \alpha_{2q+j-h-1+i} - \alpha_{2q+j-h-1} \left((h+1)\gamma_3 - \gamma_3' \right) \\
&= \gamma_3 \left((1-h)\alpha_{2q+j-h-1} + \sum_{i=1}^{h-1} (-1)^i \binom{h+1}{i} \right) \\
&= \gamma_3 \left((1-h)\alpha_{2q+j-h-1} + (-1)^{h-1} h - 1 \right) \\
&= \gamma_3 \left(\alpha_{2q+j-h-1} - 1 \right).
\end{aligned}$$

(ii) This is a straightforward consequence of (i) thanks to Equation (4.6).

(iii) When $h = p$, the equation claimed is $\alpha_q = 0$, which is true by deflation. When $j < h < p$, consider Relation (2.1) with $a = q - p + h$ and $b = 0$:

$$\begin{aligned}
0 &= \sum_{i=0}^{q-p+h} (-1)^i \binom{q-p+h}{i} \alpha_{p+i} \\
&= \sum_{i=q-2p}^{q-2p+h} (-1)^i \binom{q-p+h}{i} \alpha_{p+i} + \sum_{i=q-p+1}^{q-p+j-1} (-1)^i \binom{q-p+h}{i} \alpha_{p+i} \\
&= \sum_{i=0}^{h} (-1)^i \binom{h}{i} \alpha_{q-p+i} + \sum_{i=1}^{j-1} (-1)^i \binom{h}{i} \\
&= \sum_{i=0}^{h} (-1)^i \binom{h}{i} \alpha_{q-p+i} + \binom{h-1}{j-1} - 1.
\end{aligned}$$

Relation (2.1) with $a = h$ and $b = q - 2p$ lets us compute the above sum:

$$\sum_{i=0}^{h}(-1)^i\binom{h}{i}\alpha_{q-p+i} = -\sum_{i=0}^{q-2p}(-1)^i\binom{q-2p}{i}\alpha_{p+h+i}$$
$$= \alpha_{q-p+h},$$

therefore $\alpha_{q-p+h} = 1 - \binom{h-1}{j-1}$.

(iv) Similarly to (iii), for every h such that $j \leqslant h < p$ consider Relation (2.1) with $a = 2q - p + h$ and $b = 0$:

$$\begin{aligned}0 &= \sum_{i=0}^{2q-p+h}(-1)^i\binom{2q-p+h}{i}\alpha_{p+i}\\ &= \sum_{i=0}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{p+i} - \sum_{i=0}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{q+p+i}\\ &= -\sum_{i=0}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{q+p+i}\end{aligned}$$

since the first sum is null by Relation (2.1) with $a = q - p + h$ and $b = 0$. Thus, we have

$$\begin{aligned}0 &= \sum_{i=0}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{q+p+i}\\ &= \sum_{i=q-2p}^{q-2p+h}(-1)^i\binom{q-p+h}{i}\alpha_{q+p+i} + \sum_{i=q-p}^{q-p+j-2}(-1)^i\binom{q-p+h}{i}\alpha_{q+p+i}\\ &= \sum_{i=0}^{h}(-1)^i\binom{h}{i}\alpha_{2q-p+i} + \sum_{i=0}^{j-2}(-1)^i\binom{h}{i}\\ &= \sum_{i=0}^{h}(-1)^i\binom{h}{i}\alpha_{2q-p+i} - \binom{h-1}{j-2}.\end{aligned}$$

Relation (2.1) with $a = h$ and $b = 2q - 2p$ lets us compute the above sum:

$$\sum_{i=0}^{h}(-1)^i\binom{h}{i}\alpha_{2q-p+i} = -\sum_{i=0}^{2q-2p}(-1)^i\binom{2q-2p}{i}\alpha_{p+h+i}$$
$$= \alpha_{q-p+h} - \alpha_{2q-p+h},$$

therefore

$$\begin{aligned}\alpha_{2q-p+h} &= \alpha_{q-p+h} - \binom{h-1}{j-2}\\ &= 1 - \binom{h-1}{j-1} - \binom{h-1}{j-2}\\ &= 1 - \binom{h}{j-1}.\end{aligned}$$

□

In particular, as a consequence of the previous lemma we have that

$$\alpha_{q+j-1} = 1, \quad \alpha_{2q-p+j} = 1-j, \quad \alpha_{2q-p+j+1} = 1 - \binom{j+1}{2} = 1 - \frac{j(j+1)}{2}$$

but this is actually possible only if $j = p$, as Relation (2.2) with $a = q-p+j-1$ and $b = q-2p+2$ shows:

$$\begin{aligned}0 &= \alpha_{2q-p+j+1}\alpha_{q+j-1} - \alpha_{q+j-1}\left(-\alpha_{2q-p+j+1} + 2\alpha_{2q-p+j}\right)\\ &= 2\alpha_{2q-p+j+1} - 2\alpha_{2q-p+j}\\ &= 2j\left(1-j\right).\end{aligned}$$

Hence, as a consequence of the assumption $\alpha_{2q+j-2} \neq 0$, we got a contradiction as long as $j \neq p$. Note that, if $j = p$ and $\alpha_{2q+j-2} = 1$, Lemma 4.6 shows that the first and second constituent of the algebra are uniquely determined, and are both almost ordinary ending in 1. Furthermore, the second constituent has length $q-1$.

Assume now that

$$\alpha_{2q+j-2} = 0,$$

and recall that by Equation (4.6) this is equivalent to $\alpha_{q+j-1} = 0$. Hence, by Remark 4.3 also $\alpha_{q+j-2} = 0$, while there must be an even integer k smaller than $j-2$ such that $\alpha_{q+k} \neq 0$, and k is maximal. Without loss of generality we may assume that $\alpha_{q+k} = 1$. Relation (2.2) with $a = q-p+k$ and $b = q-2p+j-k+1$ then yields

$$\begin{aligned}0 &= \gamma_2' - \left(-\gamma_2' + (j-k+1)\gamma_2\right)\\ &= 2\gamma_2' - (j-k+1)\gamma_2,\end{aligned}$$

therefore $\gamma_2' = \frac{j-k+1}{2}\gamma_2$. We now claim that

$$\alpha_{q+k-h} = (-1)^h \binom{p - \frac{j-k+1}{2}}{h} \qquad \text{for } 0 \leqslant h < k. \tag{4.9}$$

Note that the left-hand side can also be written as $\binom{h+\frac{j-k-1}{2}}{h}$, since for any $0 \leqslant r, s < p$

$$\binom{p-s}{r} \equiv (-1)^r \binom{s+r-1}{r} \pmod{p}.$$

Of course the claim is true for $h = 0$, hence fix a positive integer $h < k$ and assume by induction that Equation (4.9) holds for indexes $\tilde{h} < h$. If h is odd, Relation (2.2) with $a = q-p+k-h$ and $b = q-2p+j-k+h$ yields

$$\begin{aligned} 0 &= \gamma_2 \sum_{i=0}^{h} (-1)^i \binom{j-k+h}{i} \alpha_{q+k-h+i} + (-1)^{j-k+h} \gamma_2 \alpha_{q+k-h} \\ &= 2\gamma_2 \alpha_{q+k-h} + \gamma_2 \sum_{i=1}^{h} (-1)^i \binom{j-k+h}{i} \alpha_{q+k-(h-i)} \\ &= 2\gamma_2 \alpha_{q+k-h} + (-1)^h \gamma_2 \sum_{i=1}^{h} \binom{j-k+h}{i} \binom{p-\frac{j-k+1}{2}}{h-i}. \end{aligned}$$

We can use the Vandermonde's identity to compute the last sum above, obtaining

$$\begin{aligned} 0 &= 2\gamma_2 \alpha_{q+k-h} - \gamma_2 \left(\binom{p-\frac{j-k+1}{2}+j-k+h}{h} - \binom{p-\frac{j-k+1}{2}}{h} \right) \\ &= 2\gamma_2 \alpha_{q+k-h} - \gamma_2 \left(\binom{h+\frac{j-k-1}{2}}{h} - \binom{p-\frac{j-k+1}{2}}{h} \right) \\ &= 2\gamma_2 \alpha_{q+k-h} - 2\gamma_2 \binom{h+\frac{j-k-1}{2}}{h}, \end{aligned}$$

hence Equation (4.9) holds for odd h. Similarly, if h is even we can use Relation (2.2) with $a = q-p+k-h$ and $b = q-2p+j-k+h+1$ to get

$$\begin{aligned} 0 &= \gamma_2' \sum_{i=0}^{h} (-1)^i \binom{j-k+h+1}{i} \alpha_{q+k-h+i} - \alpha_{q+k-h} \left(-\gamma_2' + (j-k+h+1)\gamma_2 \right) \\ &= \left(2\gamma_2' - (j-k+h-1)\gamma_2 \right) \alpha_{q+k-h} + \gamma_2' \sum_{i=1}^{h} (-1)^i \binom{j-k+h+1}{i} \alpha_{q+k-(h-i)} \\ &= -h\gamma_2 \alpha_{q+k-h} + \frac{j-k+1}{2} \gamma_2 \sum_{i=1}^{h} \binom{j-k+h+1}{i} \binom{p-\frac{j-k+1}{2}}{h-i}. \end{aligned}$$

Again, by the Vandermonde's identity we get

$$\begin{aligned}
0 &= -h\gamma_2\alpha_{q+k-h} + \frac{j-k+1}{2}\gamma_2\left(\binom{p-\frac{j-k+1}{2}+j-k+h+1}{h} - \binom{p-\frac{j-k+1}{2}}{h}\right)\\
&= -h\gamma_2\alpha_{q+k-h} + \frac{j-k+1}{2}\gamma_2\left(\binom{h+\frac{j-k-1}{2}+1}{h} - \binom{h+\frac{j-k-1}{2}}{h}\right)\\
&= -h\gamma_2\alpha_{q+k-h} + \frac{j-k+1}{2}\gamma_2\binom{h+\frac{j-k-1}{2}}{h-1}\\
&= -h\gamma_2\alpha_{q+k-h} + h\gamma_2\binom{h+\frac{j-k-1}{2}}{h}
\end{aligned}$$

hence Equation (4.9) holds also when h is even.

Finally, we can show that this yields a contradiction thanks to Relation (2.1) with $a = q-p+2$ and $b = 0$:

$$\begin{aligned}
0 &= \sum_{i=0}^{q-p+2}(-1)^i\binom{q-p+2}{i}\alpha_{p+i} + \alpha_{q+2}\\
&= 2(\alpha_{q+2} - \alpha_{q+1})\\
&= 2\left(\binom{k-2+\frac{j-k-1}{2}}{k-2} - \binom{k-1+\frac{j-k-1}{2}}{k-1}\right)\\
&= -2\binom{k-2+\frac{j-k-1}{2}}{k-1} \neq 0,
\end{aligned}$$

since

$$k-1 \leqslant k-2+\frac{j-k-1}{2} < p.$$

Summing all up, we have proved the following result:

Lemma 4.7. *Let L be an algebra of type p with first fake constituent length $q+j$.*

(i) If $1 < j < p$, then the second and third fake constituent of L cannot be both of length $q-1$.

(ii) If $j = p$ and both the second and the third fake constituent have length equal to $q-1$, then the first and the second constituent are almost ordinary ending in 1, of lengths $q+p-1$ and $q-1$ respectively. Moreover, the third fake constituent has length $q-1$, and its first non-zero two-step centralizer is equal to 1.

Case (b). Suppose now $l_2 = q-1$ and $l_3 = q$. We claim that if this is the case, then the first two constituents are uniquely determined up to scaling.

First of all, $\gamma_2 = \gamma_3$: this is a direct consequence of Equation (4.5) which was used to prove Lemma 4.4 (iii), recalling that in the case we are now considering

$$\gamma_2 = \alpha_{q+l-p} \quad \text{and} \quad \gamma_3 = \alpha_{2q+l-p}.$$

We claim that

$$\alpha_{l-h} = \alpha_{q+l-h} \qquad \text{for } \ 0 \leqslant h < j. \tag{4.10}$$

This is straightforward for $h = 0$ as $\alpha_l = 0 = \alpha_{q+l}$, thus assume by induction that the claim holds for all non-negative integers smaller than a fixed $h < j$. Relation (2.2) with $a = 2q - 2p + h$ and $b = l - p - h$ yields

$$\begin{aligned}
0 &= \gamma_3 \sum_{i=0}^{l-p-h} (-1)^i \binom{l-p-h}{i} \alpha_{2q-p+h+i} - \alpha_{l-h} \sum_{i=0}^{q+j-h} (-1)^i \binom{q+j-h}{i} \alpha_{2q-p+h+i} \\
&= \gamma_3 \sum_{i=0}^{l-p-h} (-1)^i \binom{l-p-h}{i} \alpha_{2q-p+h+i} - \alpha_{l-h} \left((-1)^{j-h} \gamma_2 + (-1)^{q+j-h} \gamma_3 \right) \\
&= \gamma_3 \sum_{i=0}^{l-p-h} (-1)^i \binom{l-p-h}{i} \alpha_{2q-p+h+i}.
\end{aligned}$$

Taking into account also Relation (2.1) with $a = 2q - 2p + h$ and $b = l - p - h$, we have that

$$\begin{aligned}
0 &= -\gamma_3 \sum_{i=0}^{2q-2p-h} (-1)^i \binom{2q-2p+h}{i} \alpha_{l-h+i} \\
&= -\gamma_3 \left(\sum_{i=0}^{h} (-1)^i \binom{h}{i} \alpha_{l-(h-i)} - \sum_{i=q}^{q+h} (-1)^i \binom{q+h}{i} \alpha_{l-(h-i)} \right) \\
&= -\gamma_3 \left(\alpha_{l-h} + \sum_{i=1}^{h} (-1)^i \binom{h}{i} \alpha_{l-(h-i)} - \alpha_{q+l-h} - \sum_{i=0}^{h} (-1)^i \binom{h}{i} \alpha_{q+l-(h-i)} \right) \\
&= -\gamma_3 \left(\alpha_{l-h} - \alpha_{q+l-h} \right),
\end{aligned}$$

proving the claim. We then have the following

Lemma 4.8. *Let L be an algebra of type p with first fake constituent length $q + j$, where j is an odd integer such that $1 < j \leqslant p$. Assume the second fake constituent has length $q - 1$ and the third fake constituent has length q. Then the first two constituents of L are uniquely determined, up to scaling, by the following:*

(i) $\alpha_{2q} = \ldots = \alpha_{2q+j-1} = 1$;

(ii) $\alpha_{q+1} = \ldots = \alpha_{q+j-1} = 1$;

(iii) $\alpha_{q-p+h} = 1 - \binom{h-1}{j-1}$ *for* $j < h \leqslant p$;

(iv) $\alpha_{2q-p+h} = 1$ *for* $j \leqslant h < p$.

In particular, the first constituent of L *has length* $q+j-1$ *and the second constituent is ordinary ending in* 1 *of length* q. *Moreover, the third fake constituent has length* q, *and its first non-zero two-step centralizer is equal to* 1.

Proof. We know that $\gamma_2 := \alpha_{2q-p+j} \neq 0$, hence without loss of generality one can scale L so that $\gamma_2 = 1$.

(i) For $0 \leqslant h < j$, Relations (2.1) and (2.2) with $a = 2q-p+h$ and $b = l-2p-h$ yield

$$\begin{aligned} 0 &= \gamma_3 \sum_{i=0}^{l-2p-h} (-1)^i \binom{l-2p-h}{i} \alpha_{2q+h+i} + (-1)^{h+1}\gamma_3\alpha_{2q+h} \\ &= -\gamma_3 \sum_{i=0}^{2q-p+h} (-1)^i \binom{2q-p+h}{i} \alpha_{l-p-h+i} + (-1)^{h+1}\gamma_3\alpha_{2q+h}, \end{aligned}$$

and we can compute easily the sum above thanks to Equation (4.10):

$$\begin{aligned} \sum_{i=0}^{2q-p+h} (-1)^i \binom{2q-p+h}{i} \alpha_{l-p-h+i} &= -\sum_{i=0}^{h-1} (-1)^i \binom{h}{i} \alpha_{l-h+i} + (-1)^{h+1}\gamma_2 + \\ &\quad + \sum_{i=0}^{h-1} (-1)^i \binom{h}{i} \alpha_{q+l-h+i} \\ &= (-1)^{h+1}\gamma_2 \\ &= (-1)^{h+1}. \end{aligned}$$

Therefore, $0 = (-1)^h\gamma_3(1-\alpha_{2q+h})$.

(ii) This is a straightforward consequence of (i) thanks to Equation (4.10).

(iii) The proof is exactly the same as that of Lemma 4.6 (iii).

(iv) Similarly to the proof of Lemma 4.6 (iv), consider Relation (2.1) with $a = 2q-p+h$ and $b = 0$ for every h such that $j \leqslant h < p$:

$$\begin{aligned} 0 &= \sum_{i=0}^{2q-p+h} (-1)^i \binom{2q-p+h}{i} \alpha_{p+i} \\ &= \sum_{i=0}^{q-p+h} (-1)^i \binom{q-p+h}{i} \alpha_{p+i} - \sum_{i=0}^{q-p+h} (-1)^i \binom{q-p+h}{i} \alpha_{q+p+i} \\ &= -\sum_{i=0}^{q-p+h} (-1)^i \binom{q-p+h}{i} \alpha_{q+p+i} \end{aligned}$$

since the first sum is null by Relation (2.1) with $a = q - p + h$ and $b = 0$. Thus, we have

$$\begin{aligned}
0 &= \sum_{i=0}^{q-p+h} (-1)^i \binom{q-p+h}{i} \alpha_{q+p+i} \\
&= \sum_{i=q-2p}^{q-2p+h} (-1)^i \binom{q-p+h}{i} \alpha_{q+p+i} + \sum_{i=q-p}^{q-p+j-1} (-1)^i \binom{q-p+h}{i} \alpha_{q+p+i} \\
&= \sum_{i=0}^{h} (-1)^i \binom{h}{i} \alpha_{2q-p+i} + \sum_{i=0}^{j-1} (-1)^i \binom{h}{i} \\
&= \sum_{i=0}^{h} (-1)^i \binom{h}{i} \alpha_{2q-p+i} + \binom{h-1}{j-1}.
\end{aligned}$$

Relation (2.1) with $a = h$ and $b = 2q - 2p$ lets us compute the above sum:

$$\begin{aligned}
\sum_{i=0}^{h} (-1)^i \binom{h}{i} \alpha_{2q-p+i} &= - \sum_{i=0}^{2q-2p} (-1)^i \binom{2q-2p}{i} \alpha_{p+h+i} \\
&= \alpha_{q-p+h} - \alpha_{2q-p+h},
\end{aligned}$$

therefore

$$\begin{aligned}
\alpha_{2q-p+h} &= \alpha_{q-p+h} + \binom{h-1}{j-1} \\
&= 1.
\end{aligned}$$

□

<u>*Remark*</u> 4.9. In particular, when $j = p$ the previous lemma states that the first constituent of L is almost ordinary of length $q + p - 1$ and the second constituent is ordinary of length q, both ending in 1.

Case (c). Suppose now $l_2 = l_3 = q$ and $j \neq 1$. We claim that this can happen only if $j \neq p$, and under that assumption the first two constituents are uniquely determined up to scaling, similarly to case (b).

We start by reminding that there are two integers k and m such that $1 \leqslant k \leqslant j$, $0 \leqslant m \leqslant j$, and they are maximal with the properties

$$\alpha_{q+k} \neq 0, \quad \alpha_{2q+m} \neq 0.$$

In fact, $k > 1$: this is a simple consequence of Relation (2.1) with $a = q - p + 2$ and $b = 0$, which implies that $\alpha_{q+2} = \alpha_{q+1}$. Recall also that by Remark 4.3 the integer k is odd, since

$l_2 = q$. Furthermore, if $j = p$ then actually $1 < k < p$ and $1 \leqslant m < p$: the fact that both k and m cannot equal p has been discussed in the proof of the existence of such integers, while $m \geqslant 1$ is a direct consequence of the hypothesis $l_2 = q$: indeed, when $j = p$, the first non-zero element of the second constituent is $\gamma_2 = \alpha_{2q+1}$.

We claim that $m = k$ necessarily. Assume first that $m > k$, and note that this implies $\alpha_{q+m} = 0$. Relations (2.2) with $a = q-p+m$ and $b = q+l-2p-m+1$ yield a contradiction:

$$\begin{aligned} 0 &= \gamma_3 \sum_{i=0}^{q+l-2p-m+1} (-1)^i \binom{q+l-2p-m+1}{i} \alpha_{q+m+i} \\ &= \gamma_3 \sum_{i=0}^{2q-2p+j-m+1} (-1)^i \binom{2q-2p+j-m+1}{i} \alpha_{q+m+i} \\ &= -\gamma_3 \alpha_{2q+m} \neq 0. \end{aligned}$$

Suppose then $m < k$, and let us show that this is also impossible. To do that, we first claim that under this assumption m has to be zero, namely

$$\alpha_{2q+1} = \ldots = \alpha_{2q+j} = 0. \tag{4.11}$$

Of course $\alpha_{2q+j} = 0$, since $m < k \leqslant j$, hence let us prove by induction that $\alpha_{2q+j-h} = 0$ for $h = 0, \ldots, j-1$. Suppose this is true for indexes smaller than a fixed positive integer $h \leqslant j-1$. If h is odd, Relation (2.2) with $a = 2q-p+j-h$ and $b = q-2p+h+1$ yields

$$\begin{aligned} 0 &= \gamma_3 \sum_{i=0}^{q-2p+h+1} (-1)^i \binom{q-2p+h+1}{i} \alpha_{2q+j-(h-i)} + (-1)^{h+1} \gamma_3 \alpha_{2q+j-h} \\ 0 &= 2\gamma_3 \alpha_{2q+j-h}, \end{aligned}$$

hence the claim holds. If h is even, Relation (2.2) with $a = q-p+j-h$ and $b = 2q-2p+h+1$ yields

$$\begin{aligned} 0 = \gamma_3 & \sum_{i=0}^{2q-2p+h+1} (-1)^i \binom{2q-2p+h+1}{i} \alpha_{q+j-h+i} + \\ & - \alpha_{q+j-h} \sum_{i=0}^{2q-2p+h+1} (-1)^i \binom{2q-2p+h+1}{i} \alpha_{q+p+j-h+i}. \end{aligned} \tag{4.12}$$

The second sum above is easy to evaluate, and equals $\gamma_2 - \gamma_3$. Regarding the first sum, first note that it equals

$$\sum_{i=0}^{q-2p+h+1} (-1)^i \binom{q-2p+h+1}{i} \alpha_{q+j-h+i} - \sum_{i=0}^{q-2p+h+1} (-1)^i \binom{q-2p+h+1}{i} \alpha_{2q+j-h+i},$$

then use Relation (2.2) with $a = q - p + j - h$ and $b = q - 2p + h + 1$ to evaluate the first sum above and induction hypothesis to evaluate the second one, obtaining

$$\alpha_{q+j-h} - \alpha_{2q+j-h}.$$

Putting everything together, Equation (4.12) yields

$$\begin{aligned} 0 &= \gamma_3(\alpha_{q+j-h} - \alpha_{2q+j-h}) - \alpha_{q+j-h}(\gamma_2 - \gamma_3) \\ &= (2\gamma_3 - \gamma_2)\alpha_{q+j-h} - \gamma_3\alpha_{2q+j-h}. \end{aligned}$$

As a consequence of Relation (2.2) with $a = q - p + k$ and $b = q + l - 2p - k + 1$ we have that $2\gamma_3 - \gamma_2 = 0$:

$$\begin{aligned} 0 = \gamma_3 & \sum_{i=0}^{2q-2p+j-k+1} (-1)^i \binom{2q-2p+j-k+1}{i} \alpha_{q+k+i} + \\ - \alpha_{q+k} & \sum_{i=0}^{2q-2p+j-k+1} (-1)^i \binom{2q-2p+j-k+1}{i} \alpha_{q+p+k+i} \\ = \gamma_3\alpha_{q+k} & - \alpha_{q+k}\left((-1)^{k+1}\gamma_2 + (-1)^k\gamma_3\right) \\ = (2\gamma_3 & - \gamma_2)\alpha_{q+k}. \end{aligned}$$

Therefore the claim is true also when h is even and (4.11) holds. Note that this is a contradiction if $j = p$, since in that case $\alpha_{2q+1} = \gamma_2 \neq 0$.

Assume $j < p$. As we have just proved, $m = 0$, that is,

$$\alpha_{2q} \neq 0 \quad \text{and } \alpha_{2q+1} = \ldots 0\alpha_{2q+j} = 0.$$

Without loss of generality, assume $\alpha_{2q} = -1$. For every $h = 1, \ldots, j$, Relation (2.1) with $a = 2q - p + h$ and $b = 0$ yields

$$\sum_{i=1}^{h} (-1)^i \binom{h}{i} \alpha_{q+i} = -1,$$

hence

$$\alpha_{q+1} = \ldots = \alpha_{q+j} = 1$$

as one may easily prove by induction on h. Thus $k = j$, and Equation (4.1) readily yields

$$\alpha_{q-p+j+1} = 2.$$

Furthermore, Relation (2.1) with $a = q - 2p + j + 2$ and $b = 0$ allows us to compute

$$\alpha_{q-p+j+2} = j + 2.$$

Consider now Relation (2.1) with $a = 2q - p + j + 1$ and $b = 0$:

$$\begin{aligned} 0 &= \sum_{i=q}^{2q-p+j+1} (-1)^i \binom{2q-p+j+1}{i} \alpha_{p+i} \\ &= -\gamma_2 - \alpha_{2q} \\ &= -\gamma_2 + 1, \end{aligned}$$

hence $\gamma_2 = 1$. Similarly, Relation (2.1) with $a = 2q - p + j + 2$ and $b = 0$ yields

$$\begin{aligned} 0 &= \sum_{i=q}^{2q-p+j+2} (-1)^i \binom{2q-p+j+2}{i} \alpha_{p+i} \\ &= -(j+2)\gamma_2 + \gamma_2' - \alpha_{2q} \\ &= \gamma_2' - j - 1, \end{aligned}$$

hence $\gamma_2' = j + 1$.

We get a contradiction by considering Relation (2.2) with $a = q - 2p + j$ and $b = q - p + 2$:

$$\begin{aligned} 0 &= \gamma_2' \sum_{i=0}^{q-p+2} (-1)^i \binom{q-p+2}{i} \alpha_{q-p+j+i} - \sum_{i=0}^{q+2} (-1)^i \binom{q+2}{i} \alpha_{q-p+j+i} \\ &= \gamma_2' \left(-2\alpha_{q-p+j+1} + \alpha_{q-p+j+2} + \alpha_{q+j}\right) + 2\alpha_{q-p+j+1} - \alpha_{q-p+j+2} - 2\gamma_2 + \gamma_2' \\ &= (j+1)(j-1) + 1 \\ &= j^2 \neq 0, \end{aligned}$$

since $j < p$. Therefore $m = k$, as claimed.

The next thing we are going to prove is that

$$\alpha_{l-h} = \alpha_{q+l-h} \qquad \text{for} \ \ 0 \leqslant h < j. \tag{4.13}$$

Clearly, this is equivalent to proving that

$$\alpha_{q+k-s} = \alpha_{2q+k-s} \qquad \text{for} \ \ -1 \leqslant s \leqslant k-1, \tag{4.14}$$

since $\alpha_{q+k+1} = \ldots = \alpha_{l+1} = 0$ and $\alpha_{2q+k+1} = \ldots = \alpha_{q+l+1} = 0$. Before doing that, note that $\gamma_2 = \gamma_3$: as a consequence of Relation (2.2) with $a = q - p + k$ and $b = l - 2p - k + 2$,

$$2\gamma_2' - (j - k + 2)\gamma_2 = 0,$$

and similarly from Relation (2.2) with $a = 2q - p + k$ and $b = l - 2p - k + 2$ we deduce

$$2\gamma_3' - (j - k + 2)\gamma_3 = 0.$$

Relation (2.1) with $a = 2q - 2p + j + 2$ and $b = q$ yields

$$2\gamma_2' - (j+2)\gamma_2 + (j+2)\gamma_3 - 2\gamma_3' = 0,$$

which together with the previous equations yields $\gamma_2 = \gamma_3$ as claimed. We are now ready to prove (4.14). This is clearly true for $s = -1$, since $\alpha_{q+k+1} = 0 = \alpha_{2q+k+1}$, hence assume by induction it is true for indexes smaller than a fixed $s \leqslant k-1$ and use Relation (2.2) with $a = q - p + k - s$ and $b = 2q - 2p + j - k + s + 1$ to get

$$\begin{aligned} 0 &= \gamma_3 \left(\alpha_{q+k-s} + \sum_{i=1}^{s} (-1)^i \binom{b}{i} \alpha_{q+k-(s-i)} - \alpha_{2q+k-s} - \sum_{i=1}^{s} (-1)^i \binom{b}{i} \alpha_{2q+k-(s-i)} \right) + \\ &\quad - \alpha_{q+k-s}((-1)^s \gamma_2 + (-1)^{s+1}\gamma_3) \\ &= \gamma_3 \left(\alpha_{q+k-s} - \alpha_{2q+k-s} \right). \end{aligned}$$

We can now state and prove the following

Lemma 4.10. *Let L be an algebra of type p with first fake constituent length $q+j$, where j is an odd integer such that $1 < j < p$. Assume the second fake constituent has length q. Then the third fake constituent has length q as well, and the first two constituents of L are uniquely determined, up to scaling, by the following:*

(i) $\alpha_{2q} = \ldots = \alpha_{2q+j} = 1$;

(ii) $\alpha_{q+1} = \ldots = \alpha_{q+j} = 1$;

(iii) $\alpha_{q-p+h} = 1 + \binom{h-1}{j}$ *for* $j < h \leqslant p$;

(iv) $\alpha_{2q-p+h} = 1$ *for* $j < h < p$.

In particular, the first and second constituents coincides with their fake counterparts, and the second constituent is ordinary ending in 1. Moreover, the third fake constituent has length q, and its first non-zero two-step centralizer is equal to 1.

On the other hand, if $j = p$, then L cannot have second fake constituent length q.

Proof. Without loss of generality one can scale L so that $\gamma_2 = 1$.

(i) For $0 \leqslant h < j$ and $j \neq p$, Relations (2.1) and (2.2) with $a = 2q - p + h + 1$ and $b = l - 2p - h$ yield

$$\begin{aligned} 0 &= \gamma_3 \sum_{i=0}^{l-2p-h} (-1)^i \binom{l-2p-h}{i} \alpha_{2q+h+1+i} + (-1)^{h+1}\gamma_3\alpha_{2q+h+1} \\ &= -\gamma_3 \sum_{i=0}^{2q-p+h+1} (-1)^i \binom{2q-p+h+1}{i} \alpha_{l-p-h+i} + (-1)^{h+1}\gamma_3\alpha_{2q+h+1}, \end{aligned}$$

and we can compute easily the sum above thanks to Equation (4.13):

$$\begin{aligned}\sum_{i=0}^{2q-p+h+1}(-1)^i\binom{2q-p+h+1}{i}\alpha_{l-p-h+i} = {} & (-1)^{h+1}\alpha_{l-p+1}+(-1)^h\gamma_2 + \\ & -\sum_{i=0}^{h}(-1)^i\binom{h+1}{i}\alpha_{l-h+i} + \\ & +\sum_{i=0}^{h}(-1)^i\binom{h+1}{i}\alpha_{q+l-h+i} \\ = {} & (-1)^{h+1}\alpha_{l-p+1}+(-1)^h\gamma_2 \\ = {} & (-1)^{h+1},\end{aligned}$$

where we have used the fact that $\alpha_{l-p+1} = 2\gamma_2 = 2$, which is a consequence of Relation (2.1) with $a = q + l - 2p + 1$ and $b = 0$. Therefore,

$$0 = (-1)^h\gamma_3(1-\alpha_{2q+h+1}),$$

proving the statement for all two-step centralizers but α_{2q}. To prove that also $\alpha_{2q} = 1$, just note that $\alpha_{q+1} = \alpha_{2q+1} = 1$ and use Relation (2.1) with $a = 2q-p+1$ and $b = 0$ to conclude:

$$\begin{aligned}0 &= -\alpha_{q+1}-\alpha_{2q}+2\alpha_{2q+1} \\ &= 1-\alpha_{2q}.\end{aligned}$$

Note that, as long as $h < p-1$, the above computations work for the case $j = p$ as well, meaning that $\alpha_{2q+1} = \ldots = \alpha_{2q+p-1} = 1$ in that case.

(ii) For $j \neq p$, this is a straightforward consequence of (i) thanks to Equation (4.13). On the other hand, if $j = p$ we deduce that $\alpha_{q+1} = \ldots = \alpha_{q+p-1} = 1$ but $\alpha_{q+p} = 0$, meaning that $k = p-1$ is even, a contradiction. For this reason, an algebra of type p with first fake constituent length $q+p$ cannot have second fake constituent length q.

(iii) When $h = p$, the equation claimed is $\alpha_q = 0$, which is true by deflation. When $j < h < p$, consider Relation (2.1) with $a = q-p+h$ and $b = 0$:

$$\begin{aligned}0 &= \sum_{i=0}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{p+i} \\ &= \sum_{i=q-2p}^{q-2p+h}(-1)^i\binom{q-p+h}{i}\alpha_{p+i}+\sum_{i=q-p+1}^{q-p+j}(-1)^i\binom{q-p+h}{i}\alpha_{p+i}.\end{aligned}$$

Use Lucas' theorem on both the above sums, and point (ii) on the second sum's two-step centralizers, to get

$$\begin{aligned} 0 &= \sum_{i=0}^{h}(-1)^i\binom{h}{i}\alpha_{q-p+i} + \sum_{i=1}^{j}(-1)^i\binom{h}{i} \\ &= \sum_{i=0}^{h}(-1)^i\binom{h}{i}\alpha_{q-p+i} - \binom{h-1}{j} - 1. \end{aligned}$$

Relation (2.1) with $a = h$ and $b = q - 2p$ lets us compute the above sum:

$$\begin{aligned} \sum_{i=0}^{h}(-1)^i\binom{h}{i}\alpha_{q-p+i} &= -\sum_{i=0}^{q-2p}(-1)^i\binom{q-2p}{i}\alpha_{p+h+i} \\ &= \alpha_{q-p+h}, \end{aligned}$$

therefore $\alpha_{q-p+h} = 1 + \binom{h-1}{j}$.

(iv) Similarly to (iii), consider Relation (2.1) with $a = 2q - p + h$ and $b = 0$ for every h such that $j < h < p$:

$$\begin{aligned} 0 &= \sum_{i=0}^{2q-p+h}(-1)^i\binom{2q-p+h}{i}\alpha_{p+i} \\ &= \sum_{i=0}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{p+i} - \sum_{i=0}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{q+p+i} \\ &= -\sum_{i=0}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{q+p+i} \end{aligned}$$

since the first sum is null by Relation (2.1) with $a = q - p + h$ and $b = 0$. Thus, we have

$$\begin{aligned} 0 &= \sum_{i=0}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{q+p+i} \\ &= \sum_{i=q-2p}^{q-2p+h}(-1)^i\binom{q-p+h}{i}\alpha_{q+p+i} + \sum_{i=q-p}^{q-p+j}(-1)^i\binom{q-p+h}{i}\alpha_{q+p+i} \\ &= \sum_{i=0}^{h}(-1)^i\binom{h}{i}\alpha_{2q-p+i} + \sum_{i=0}^{j}(-1)^i\binom{h}{i} \\ &= \sum_{i=0}^{h}(-1)^i\binom{h}{i}\alpha_{2q-p+i} - \binom{h-1}{j}. \end{aligned}$$

Relation (2.1) with $a = h$ and $b = 2q - 2p$ lets us compute the above sum:

$$\sum_{i=0}^{h}(-1)^i\binom{h}{i}\alpha_{2q-p+i} = -\sum_{i=0}^{2q-2p}(-1)^i\binom{2q-2p}{i}\alpha_{p+h+i}$$
$$= \alpha_{q-p+h} - \alpha_{2q-p+h},$$

therefore

$$\alpha_{2q-p+h} = \alpha_{q-p+h} - \binom{h-1}{j}$$
$$= 1.$$

□

Case ($j = 1$)**.** Finally, Let L be an algebra of type p with first constituent of length $l = q + 1$. As we anticipated in Remark 4.5, L belongs to case (c) above, meaning that $l_2 = l_3 = q$, since $\alpha_{q+1} \neq 0$. We remind that this was a consequence of the following easy relation in L:

$$2\alpha_{q+1} = \alpha_{q-p+2} \neq 0.$$

Without loss of generality, suppose $\alpha_{q+1} = 1$, hence $\alpha_{q-p+2} = 2$. We claim that Lemma 4.10 holds verbatim also in this case, although the proof is slightly different. Indeed, we can now unravel the structure of the first constituent right away, without even considering the second constituent: we claim that, accordingly to Lemma 4.10,

$$\alpha_{q-p+h} = h \qquad \text{for } \ 1 < h \leqslant p + 1. \tag{4.15}$$

This is clearly true for $h = 2$, and in fact it is true also for $h = p$ and $h = p + 1$: $\alpha_q = 0$ by deflation, and $\alpha_{q+1} = 1$ by assumption. Therefore, we just need to prove Equation (4.15) for $h < p$, and we do that by induction on h: fix $h > 2$ and suppose the equation is true for indexes smaller than h. Relation (2.1) with $a = q - p + h$ and $b = 0$ yields

$$0 = \sum_{i=q-2p+2}^{q-p+h}(-1)^i\binom{q-p+h}{i}\alpha_{p+i}$$
$$= \sum_{i=2}^{p+1}(-1)^{i+1}\binom{q-p+h}{p+h-i}\alpha_{q-p+i}.$$

By Lucas' theorem we have that

$$\binom{q-p+h}{p+h-i} = \begin{cases} (p-1)\binom{h}{i}, & \text{for } i = 2,\dots,h; \\ 0, & \text{for } i = h+1,\dots,p-1; \\ 1, & \text{for } i = p; \\ h, & \text{for } i = p+1. \end{cases}$$

Substitution and induction hypothesis yields

$$\begin{aligned} 0 &= \alpha_q - h\alpha_{q+1} + \sum_{i=2}^{h}(-1)^i\binom{h}{i}\alpha_{q-p+i} \\ &= -h + \sum_{i=2}^{h-1}(-1)^i\binom{h}{i}i + (-1)^h\alpha_{q-p+h} \\ &= \sum_{i=1}^{h-1}(-1)^i\binom{h}{i}i + (-1)^h\alpha_{q-p+h}. \end{aligned}$$

The above sum equals $(-1)^{h-1}h$ by the identity (see Chapter 1)

$$\sum_{i=1}^{h}(-1)^i\binom{h}{i}i = 0,$$

hence $\alpha_{q-p+h} = h$ as claimed.

At this point, one may use the same methods used so far to deduce that $\gamma_3 = \gamma_2 = 1$ and the tail of the second constituent equals the tail of the first one, namely $\alpha_{2q-h} = \alpha_{q-h}$ for $h = -1, \ldots, p+2$. This concludes the analysis of the case $j = 1$, meaning that Lemma 4.10 holds even in this case.

As a consequence of Proposition 3.2 and the results of both this and the previous section, we have the following

Proposition 4.11. *Let L be an algebra of type p over a field of positive characteristic p, with generators e_1 and e_p of degree 1 and p, respectively. Suppose that $[e_pe_1e_p] = 0$ (that is, the first two-step centralizer is null). Then the first constituent can only have length of the form:*

1. *either $2q$, where $q > p$ is a power of p;*
2. *or $q + m$, where $q > p$ is a power of p and m is an integer such that $0 < j < p$.*

Moreover,

(i) If $l = 2q$, then L is a subalgebra of an algebra of type 1.

(ii) If $l = q + m$ with $m \neq p - 1$, then, up to scaling, the algebra is unique. In particular, every constituent other than the first one is ordinary of length q.

(iii) If $l = q + p - 1$, then the first constituent is almost ordinary ending in λ for some $\lambda \neq 0$, while every other constituent can be either ordinary ending in λ of length q, or almost ordinary ending in λ of length $q - 1$.

Proof. The statement regarding the possible lengths of the first constituent is just an easy remark. Furthermore, if $l = 2q$ then everything has already been proved in the first section.

On the other hand, statement (ii) (and similarly for statement (iii)) can be proven by induction on the constituents: as we proved in this section, the second constituent has length q and is ordinary ending in 1, up to scaling. Moreover, we proved that the following one has 0-length $q - p$, and its first non-zero two-step centralizer equals 1. One may now assume that this holds for the $(n-1)$-th constituent ($n > 2$) and prove it for the following one, using the same strategy we used on the second constituent. We sketch the strategy once again:

- Prove that $q - 1$ is a lower bound for the length of the $(n+1)$-th fake constituent, and that in fact it equals q.
- Prove that $\gamma_{n+1} = \gamma_n$, where γ_n and γ_{n+1} are the first non-zero two-step centralizers of the n-th and the $(n+1)$-th constituent, respectively. Recall that $\gamma_n = 1$ by induction.
- Prove that the tail of the n-th constituent equals the tail of the previous one, concluding the proof.

□

Remark 4.12. As a consequence of the previous proposition, algebras of type p with first constituent length $q+m$ with $m \neq p-1$ are soluble: this is a consequence of their periodic structure, as for any $r, s > q$ we have that, writing $s = aq + b$ for some $a > 0$ and $0 \leqslant b < q$,

$$\begin{aligned}
[e_r e_{s+p}] &= [e_r [e_p e_1^s]] \\
&= \sum_{i=0}^{s} (-1)^i \binom{s}{i} \alpha_{r+i} e_{r+s+p} \\
&= \sum_{h=0}^{a} (-1)^h \binom{a}{h} \sum_{j=0}^{b} (-1)^j \binom{b}{j} \alpha_{r+hq+j} e_{r+s+p} \\
&= \sum_{h=0}^{a} (-1)^h \binom{a}{h} \sum_{j=0}^{b} (-1)^j \binom{b}{j} \alpha_{r+j} e_{r+s+p} \\
&= 0.
\end{aligned}$$

Chapter 5

Construction of some Lie algebras

We now give an explicit construction of the class $\mathcal{E}$ consisting of algebras of type p with first constituent length $q+m$, where $q>p$ is a power of p and m is an integer such that $1 \leqslant m < p-1$, thus providing existence of the algebras we dealt with in the previous chapter. As a consequence, the proof of Theorem 2.6 will be complete.

Let $\mathbb{F}$ be a field of characteristic $p>2$, $q>p$ a power of p, and m an integer as above. Let t be an indeterminate over $\mathbb{F}$, and consider the vector space $\mathbb{F}(t)^q$ with standard basis $v_1, \dots, v_q$, written as row vectors. Let us define the following $q \times q$ matrices over $\mathbb{F}(t)$:

$$E := \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ t & & & & \end{bmatrix}, \qquad D := \begin{bmatrix} & D_1 \\ D_2 & \end{bmatrix},$$

where D_1 and D_2 are respectively the $(q-p)\times(q-p)$ and $p \times p$ diagonal matrices

$$D_1 = \begin{bmatrix} \mathrm{Id}_m & \\ & 0_{q-p-m} \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0_m & \\ & t\,\mathrm{Id}_{p-m} \end{bmatrix}.$$

For technical convenience, we may also write the latter matrices as

$$D_1 = diag\left(\lambda_1, \dots \lambda_{q-p}\right), \qquad D_2 = diag\left(\lambda_{q-p+1}t, \dots \lambda_q t\right),$$

with

$$\lambda_j = \mathbb{1}_{[1,m]\cup[q-p+m+1,q]}(j) = \begin{cases} 1, & \text{if } 1 \leqslant j \leqslant m \quad \text{or} \quad q-p+m < j \leqslant q; \\ 0, & \text{if } m < j \leqslant q-p+m. \end{cases}$$

The coefficients λ_j may as well be defined for $j \in \mathbb{Z}/q\mathbb{Z}$ as follows:

$$\lambda_j = \begin{cases} 1, & \text{if } 1 \leqslant \overline{j} \leqslant m \quad \text{or} \quad q-p+m < \overline{j} \leqslant q; \\ 0, & \text{otherwise,} \end{cases}$$

where by $\overline{j}$ we indicate the representative of j such that $1 \leqslant \overline{j} \leqslant q$. This notation will be used in all of the following without further mention.

Lemma 5.1. *For every integer k such that $0 \leqslant k \leqslant q-p$ we have that*

$$[DE^k] = \begin{bmatrix} & A_k \\ B_k & \end{bmatrix},$$

where A_k and B_k are respectively the $(q-p-k)\times(q-p-k)$ and $(p+k)\times(p+k)$ diagonal matrices

$$A_k = diag\left(\mu_1^{(k)}, \ldots \mu_{q-p-k}^{(k)}\right), \qquad B_k = diag\left(\mu_{q-p-k+1}^{(k)} t, \ldots \mu_q^{(k)} t\right)$$

whose entries can be computed by means of

$$\mu_j^{(k)} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \lambda_{j+i} \tag{5.1}$$

for every j ranging from 1 *to q. Indexes in Equation* (5.1) *are to be considered modulo q.*

Proof. The claim is trivially true when $k=0$, as

$$[DE^0] = D = \begin{bmatrix} & D_1 \\ D_2 & \end{bmatrix}$$

and $A_0 = D_1$, $B_0 = D_2$. Hence, let us suppose by induction that the claim is true for a fixed $k < q-p$ and prove it for $k+1$. Since

$$[DE^{k+1}] = [DE^k]E - E[DE^k],$$

we need to compute both the products $[DE^k]E$ and $E[DE^k]$. This can be checked explicitly, but looking at the definition of E we see that it is *almost* a permutation matrix: multiplying by E on the right shifts every column onto the following one, except from the last one which is both multiplied by t and shifted onto the first column. Similarly, multiplying by E on the left shifts every row onto the previous one, except from the first one which is both multiplied by t and shifted onto the last row. Hence

$$[DE^{k+1}] = \begin{bmatrix} & \tilde{A}_{k+1} \\ \tilde{B}_{k+1} & \end{bmatrix},$$

where

$$\tilde{A}_{k+1} = diag\left(\mu_1^{(k)} - \mu_2^{(k)}, \ldots, \mu_{q-p-(k+1)}^{(k)} - \mu_{q-p-k}^{(k)}\right)$$

and

$$\tilde{B}_{k+1} = diag\left((\mu_{q-p-k}^{(k)} - \mu_{q-p-k+1}^{(k)})t, \ldots, (\mu_q^{(k)} - \mu_1^{(k)})t\right).$$

To conclude, we just need to check that $\tilde{A}_{k+1} = A_{k+1}$ and $\tilde{B}_{k+1} = B_{k+1}$. This is true since by induction hypothesis for every j ranging from 1 to q

$$\begin{aligned}
\mu_j^{(k)} - \mu_{j+1}^{(k)} &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \lambda_{j+i} - \sum_{i=0}^{k} (-1)^i \binom{k}{i} \lambda_{j+i+1} \\
&= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \lambda_{j+i} + \sum_{i=1}^{k+1} (-1)^i \binom{k}{i-1} \lambda_{j+i} \\
&= \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} \lambda_{j+i} \\
&= \mu_j^{(k+1)}.
\end{aligned}$$

□

<u>*Remark*</u> 5.2. Note that, by definition of the coefficients λ_i, we have that $\mu_j^{(k)} = 0$ whenever $m < \overline{j} \leqslant q - p - k + m$.

As a consequence of the previous lemma, $[DE^{q-p}] = t\,\mathrm{Id}_q$ is a scalar matrix, hence the Lie algebra spanned by E and D has dimension $q - p + 2$. The following lemma will be useful for the construction of the Lie algebra L we are going to give:

Lemma 5.3. *The matrix D commutes with $[DE^k]$ for every $k \geqslant 1$.*

Proof. The statement is trivially true for every $k \geqslant q - p$, since $[DE^{q-p}]$ is a scalar matrix, hence assume that $k < q - p$.

Let us evaluate first the product $[DE^k]D$. Since k is fixed, in the following we will write μ_j instead of $\mu_j^{(k)}$. We will also consider indexes of λ_i, μ_i, and v_i modulo q. It is convenient to consider $[DE^k]$ as a matrix of row vectors:

$$[DE^k] = \begin{bmatrix} \mu_1 v_{p+k+1} \\ \vdots \\ \mu_{q-p-k} v_q \\ \mu_{q-p-k+1} t v_1 \\ \vdots \\ \mu_q t v_{p+k} \end{bmatrix} = \left(\mu_i t^{\varepsilon_i} v_{p+k+i}\right)_i, \tag{5.2}$$

where

$$\varepsilon_i = \mathbb{1}_{[q-p-k+1,q]}(i) = \begin{cases} 1, & \text{if } q - p - k + 1 \leqslant i \leqslant q; \\ 0, & \text{otherwise.} \end{cases}$$

Now consider D as a matrix of column vectors:

$$\begin{aligned}
D &= \left[\lambda_{q-p+1} t v_{q-p+1}^T, \ldots, \lambda_q t v_q^T, \lambda_1 v_1^T, \ldots, \lambda_{q-p} v_{q-p}^T\right] \\
&= \left(\lambda_{q-p+j} t^{\sigma_j} v_{q-p+j}^T\right)_j,
\end{aligned} \tag{5.3}$$

where

$$\sigma_j = \mathbb{1}_{[1,p]}(j) = \begin{cases} 1, & \text{if } 1 \leqslant j \leqslant p; \\ 0, & \text{otherwise.} \end{cases}$$

We can now compute the (i,j)-entry of $[DE^k]D$, for every $i,j = 1,\dots,q$:

$$\begin{aligned} ([DE^k]D)_{i,j} &= ([DE^k])_i(D)_j \\ &= \mu_i\lambda_{q-p+j}t^{\varepsilon_i+\sigma_j}v_{p+k+i}v^T_{q-p+j} \\ &= \begin{cases} \mu_i\lambda_{q-p+j}t^{\varepsilon_i+\sigma_j}, & \text{if } p+k+i \equiv q-p+j \pmod q; \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \mu_i\lambda_{p+k+i}t^{\varepsilon_i+\sigma_j}, & \text{if } j \equiv 2p+k+i \pmod q; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let us now evaluate the product $D[DE^k]$. It is now convenient to write D and $[DE^k]$ as matrices of respectively row and column vectors:

$$D = \left(\lambda_i t^{\tilde{\varepsilon}_i} v_{p+i}\right)_i, \quad [DE^k] = \left(\mu_{q-p-k+j}t^{\tilde{\sigma}_j}v^T_{q-p-k+j}\right)_j, \tag{5.4}$$

where

$$\tilde{\varepsilon}_i = \mathbb{1}_{[q-p+1,q]}(i) = \begin{cases} 1, & \text{if } q-p+1 \leqslant i \leqslant q; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tilde{\sigma}_j = \mathbb{1}_{[1,p+k]}(j) = \begin{cases} 1, & \text{if } 1 \leqslant j \leqslant p+k; \\ 0, & \text{otherwise.} \end{cases}$$

The (i,j)-entry of $D[DE^k]$ for $i,j = 1,\dots,q$ is:

$$\begin{aligned} (D[DE^k])_{i,j} &= (D)_i([DE^k])_j \\ &= \lambda_i\mu_{q-p-k+j}t^{\tilde{\varepsilon}_i+\tilde{\sigma}_j}v_{p+i}v^T_{q-p-k+j} \\ &= \begin{cases} \lambda_i\mu_{q-p-k+j}t^{\tilde{\varepsilon}_i+\tilde{\sigma}_j}, & \text{if } p+i \equiv q-p-k+j \pmod q; \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \lambda_i\mu_{p+i}t^{\tilde{\varepsilon}_i+\tilde{\sigma}_j}, & \text{if } j \equiv 2p+k+i \pmod q; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the matrix $[DE^kD]$ trivially has zero entries in every position (i,j) with $j \not\equiv 2p+k+i \pmod q$, while when $j \equiv 2p+k+i \pmod q$ its (i,j)-entry is

$$\left([DE^kD]\right)_{i,j} = \mu_i\lambda_{p+k+i}t^{\varepsilon_i+\sigma_j} - \lambda_i\mu_{i+p}t^{\tilde{\varepsilon}_i+\tilde{\sigma}_j}. \tag{5.5}$$

Suppose first that $k \leqslant q-2p$, and let us prove that $\left([DE^kD]\right)_{i,j} = 0$ also when $j \equiv 2p+k+i \pmod q$, for any i such that $1 \leqslant i \leqslant q$.

- If $1 \leqslant i \leqslant m$, we have that

$$m < p + k + i \leqslant q - p + m$$

and

$$m < p + i \leqslant q - p - k + m,$$

thus both λ_{p+k+i} and μ_{p+i} equal zero, the former by definition of λ_{p+k+i} and the latter from Remark 5.2. This implies that $\left([DE^k D]\right)_{i,j} = 0$.

- If $m < i \leqslant q - p - k + m$, we have that both λ_i and μ_i equal zero, therefore $\left([DE^k D]\right)_{i,j} = 0$.

- If $q - p - k + m < i \leqslant q$, we have that

$$q + m < p + k + i \leqslant q + q - p,$$

thus

$$m < \overline{p + k + i} \leqslant q - p$$

and $\lambda_{p+k+i} = 0$. Furthermore, if $q - p - k + m < i \leqslant q - p + m$ we have that $\lambda_i = 0$, while if $q - p + m < i \leqslant q$ then

$$m < \overline{p + i} \leqslant p$$

and $\mu_{p+i} = 0$ by Remark 5.2. In any case, the product $\lambda_i \mu_{p+i}$ equals zero, therefore $\left([DE^k D]\right)_{i,j} = 0$.

This proves the claim for $k \leqslant q - 2p$, since

$$[DE^k D] = 0.$$

Suppose now $q - 2p < k < q - p$, thus let us write $k = q - 2p + h$ for some h such that $1 < h < p$. For any i such that $1 \leqslant i \leqslant q$, let j be an integer such that $1 \leqslant j \leqslant q$, and $j \equiv 2p + k + i \pmod q$, i.e. $j \equiv h + i \pmod q$. We have to prove that $\left([DE^k D]\right)_{i,j} = 0$ for such a choice of j, and we recall that by Equation (5.5)

$$\left([DE^k D]\right)_{i,j} = \mu_i \lambda_{q-p+h+i} t^{\varepsilon_i + \sigma_{\overline{h+i}}} - \lambda_i \mu_{p+i} t^{\tilde{\varepsilon}_i + \tilde{\sigma}_{\overline{h+i}}}.$$

- If $1 \leqslant i \leqslant m - h$, we have that $\lambda_{q-p+h+i} = 0$ since

$$q - p + h + i \leqslant q - p + m$$

and $\mu_{p+i} = 0$ by Remark 5.2, since

$$m < p + i \leqslant p + m - h = q - p - k + m.$$

Therefore $\left([DE^k D]\right)_{i,j} = 0$.

- If $m-h < i \leqslant m$, then $\lambda_i = 1$ trivially, but also $\lambda_{q-p+h+i} = 1$: indeed

$$q-p+m < q-p+h+i < q+m,$$

thus $q-p+m < q-p+h+i \leqslant q$ or $1 < \overline{q-p+h+i} < m$. Therefore

$$\big([DE^kD]\big)_{i,j} = \mu_i t^{\varepsilon_i+\sigma_{h+i}} - \mu_{p+i} t^{\tilde{\varepsilon}_i+\tilde{\sigma}_{h+i}},$$

and we claim that $\mu_i = \mu_{p+i}$ and $\varepsilon_i + \sigma_{h+i} = \tilde{\varepsilon}_i + \tilde{\sigma}_{h+i}$. The first equality is a consequence of the definition of μ_i and μ_{p+i}:

$$\begin{aligned}\mu_i &= \sum_{r=0}^{q-2p+h} (-1)^r \binom{q-2p+h}{r} \lambda_{i+r} \\ &= \sum_{r=0}^{m-i} (-1)^r \binom{h}{r},\end{aligned}$$

where we have used the fact that $\lambda_{i+r} = 1$ only for $r = 0, \ldots, m-i$, while it equals zero for all other values of r. Similarly,

$$\begin{aligned}\mu_{p+i} &= \sum_{r=0}^{q-2p+h} (-1)^r \binom{q-2p+h}{r} \lambda_{p+i+r} \\ &= \sum_{r=q-2p+m-i+1}^{q-2p+h} (-1)^r \binom{q-2p+h}{r} \\ &= -\sum_{r=m-i+1}^{h} (-1)^r \binom{h}{r} \\ &= \sum_{r=0}^{m-i} (-1)^r \binom{h}{r} = \mu_i,\end{aligned}$$

where we have used the fact that $\lambda_{p+i+r} = 1$ only for $r = q-2p+m-i+1, \ldots, q-2p+h$, while it equals zero for all other values of r.

Let us now check the second equality we stated: since $m-h < i \leqslant m$, we have that

$$\tilde{\varepsilon}_i = \mathbb{1}_{[q-p+1,q]}(i) = 0$$

and

$$\tilde{\sigma}_{h+i} = \mathbb{1}_{[1,q-p+h]}(h+i) = 1,$$

while

$$\varepsilon_i = \mathbb{1}_{[p-h+1,q]}(i) = \begin{cases} 1, & \text{if } i > p-h; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma_{h+i} = \mathbb{1}_{[1,p]}(h+i) = \begin{cases} 1, & \text{if } i \leqslant p-h; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\varepsilon_i + \sigma_{h+i} = 1 = \tilde{\varepsilon}_i + \tilde{\sigma}_{h+i},$$

and

$$\left([DE^k D]\right)_{i,j} = \mu_i t - \mu_{p+i} t = 0.$$

- If $m < i \leqslant q-p-k+m = m+p-h$, we have that both λ_i and μ_i equal zero, therefore $\left([DE^k D]\right)_{i,j} = 0$.
- If $m+p-h < i \leqslant q-p+m$, we still have $\lambda_i = 0$, and also $\lambda_{q-p+h+i} = 0$ since

$$m < \overline{q-p+h+i} \leqslant q-2p+m+h < q-p+m.$$

Hence $\left([DE^k D]\right)_{i,j} = 0$.
- If $q-p+m < i \leqslant q+m-h$, then $\lambda_{q-p+h+i} = 0$ since

$$m < \overline{q-p+h+i} \leqslant q-p+m.$$

We also have that $\mu_{p+i} = 0$, since

$$m < \overline{p+i} \leqslant m+p-h = q-p-k+m,$$

therefore $\left([DE^k D]\right)_{i,j} = 0$.
- Finally, if $q+m-h < i \leqslant q$, then $\lambda_i = 1$ trivially, but also $\lambda_{q-p+h+i} = 1$ since

$$q-p+m < \overline{q-p+h+i} \leqslant q-p+h < q.$$

Therefore

$$\left([DE^k D]\right)_{i,j} = \mu_i t^{\varepsilon_i + \sigma_{\overline{h+i}}} - \mu_{p+i} t^{\tilde{\varepsilon}_i + \tilde{\sigma}_{\overline{h+i}}},$$

and we claim that $\mu_i = \mu_{p+i}$ and $\varepsilon_i + \sigma_{\overline{h+i}} = \tilde{\varepsilon}_i + \tilde{\sigma}_{\overline{h+i}}$. The first equality is a consequence of the definition of μ_i and μ_{p+i}:

$$\begin{aligned} \mu_i &= \sum_{r=0}^{q-2p+h} (-1)^r \binom{q-2p+h}{r} \lambda_{i+r} \\ &= \sum_{r=0}^{q+m-i} (-1)^r \binom{h}{r}, \end{aligned}$$

where we have used the fact that $\lambda_{i+r} = 1$ only for $r = 0, \dots, q+m-i$, while it equals zero for all other values of r. Similarly,

$$\begin{aligned}\mu_{p+i} &= \sum_{r=0}^{q-2p+h} (-1)^r \binom{q-2p+h}{r} \lambda_{p+i+r} \\ &= \sum_{r=2q-2p+m-i+1}^{q-2p+h} (-1)^r \binom{q-2p+h}{r} \\ &= -\sum_{r=q+m-i+1}^{h} (-1)^r \binom{h}{r} \\ &= \sum_{r=0}^{q+m-i} (-1)^r \binom{h}{r} = \mu_i,\end{aligned}$$

where we have used the fact that $\lambda_{p+i+r} = 1$ only for $r = 2q-2p+m-i+1, \dots, q-2p+h$, while it equals zero for all other values of r.

Let us now check the second equality we stated: since $q+m-h < i \leqslant q$, we have that

$$\varepsilon_i = \mathbb{1}_{[p-h+1,q]}(i) = 1,$$

$$\sigma_{\overline{h+i}} = \mathbb{1}_{[1,p]}(h+i-q) = 1,$$

$$\tilde{\varepsilon}_i = \mathbb{1}_{[q-p+1,q]}(i) = 1$$

and

$$\tilde{\sigma}_{\overline{h+i}} = \mathbb{1}_{[1,q-p+h]}(h+i-q) = 1.$$

Therefore

$$\varepsilon_i + \sigma_{\overline{h+i}} = 2 = \tilde{\varepsilon}_i + \tilde{\sigma}_{\overline{h+i}},$$

and

$$\left([DE^k D]\right)_{i,j} = \mu_i t^2 - \mu_{p+i} t^2 = 0.$$

This proves the claim also for $q-2p < k < q-p$, hence

$$[DE^k D] = 0$$

for every $k \geqslant 1$. □

Now consider the $(q+1) \times (q+1)$ block matrices

$$e_1 := \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad e_p := \begin{bmatrix} D & 0 \\ v & 0 \end{bmatrix},$$

with $v = \frac{1}{t}v_p$, and let L be the Lie algebra spanned by e_1 and e_p. For every $i > p$ let

$$e_i := [e_p e_1^{i-p}] = \begin{bmatrix} [DE^{i-p}] & 0 \\ vE^{i-p} & 0 \end{bmatrix}$$

and notice that, by definition of E,

$$vE^{i-p} = \frac{1}{t}v_i$$

for $i = p+1, \ldots, q$, while for $i > q$

$$vE^{i-p} = t^{k-1}v_j$$

where $1 \leqslant j \leqslant q$ and $i = kq + j$.

We have that

Proposition 5.4. *The Lie algebra L defined above is an algebra of type p, with generators e_1 and e_p of degree respectively 1 and p. The following relations hold in L:*

(i) $[e_{p+k}e_p] = 0$ *for* $k = 1, \ldots, q - 2p + m$;

(ii) $[e_{q-p+h}e_p] = \left(1 + (-1)^{m+1}\binom{h-1}{m}\right) e_{q+h}$ *for* $h = m+1, \ldots, p$;

(iii) $[e_{kq+h}e_p] = 0$ *for every* $k \geqslant 1$ *and* $m < h \leqslant q - p + m$;

(iv) $[e_{kq+h}e_p] = e_{kq+p+j}$ *for every* $k \geqslant 1$ *and* $1 \leqslant h \leqslant m$ *or* $q - p + m < h \leqslant q$.

In particular, L has first constituent of length $q + m$, while every other constituent has length q and is ordinary ending in 1.

Proof. We only need to prove relations *(i)*-*(iv)*, since these will imply that L is an algebra of type p with constituent lengths as stated above.

(i) Let $1 \leqslant k \leqslant q - 2p + m$. From the definition and Lemma 5.3 we have

$$[e_{p+k}e_p] = \begin{bmatrix} [DE^kD] & 0 \\ vE^kD - v[DE^k] & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ vE^kD - v[DE^k] & 0 \end{bmatrix}.$$

Recalling Equation (5.3) and the notation used in Lemma 5.3, we have

$$\begin{aligned} vE^kD &= \frac{1}{t}v_{p+k}D \\ &= \frac{1}{t}v_{p+k}\left(\lambda_{q-p+j}t^{\sigma_j}v_{q-p+j}^T\right)_{j=1,\ldots,q} \\ &= \frac{1}{t}\lambda_{p+k}t^{\sigma_{2p+k}}v_{2p+k} \\ &= 0, \end{aligned}$$

since $\lambda_{p+k} = \mathbb{1}_{[1,m]\cup[q-p+m+1,q]}(p+k) = 0$. Similarly, by Equation (5.4),

$$\begin{aligned} v[DE^k] &= \frac{1}{t} v_p [DE^k] \\ &= \frac{1}{t} v_p \big(\mu_{q-p-k+j} t^{\tilde{\sigma}_j} v_{q-p-k+j}^T\big)_{j=1,\dots,q} \\ &= \frac{1}{t} \mu_p t^{\tilde{\sigma}_{2p+k}} v_{2p+k} \\ &= 0, \end{aligned}$$

since $\mu_p = 0$. Therefore $[e_{p+k} e_p] = 0$.

(ii) Now let $k = q - 2p + h$ for $m < h \leqslant p$. We have that

$$[e_{q-p+h} e_p] = [e_{p+k} e_p] = \begin{bmatrix} 0 & 0 \\ vE^k D - v[DE^k] & 0 \end{bmatrix},$$

and since

$$e_{q+h} = \begin{bmatrix} 0 & 0 \\ v_h & 0 \end{bmatrix}$$

we want to prove that $vE^k D - v[DE^k] = \left(1 + (-1)^{m+1}\binom{h-1}{m}\right) v_h$: let us start by computing $vE^k D$ in the same way we just did for relations *(i)*:

$$\begin{aligned} vE^k D &= \frac{1}{t} v_{p+k} D \\ &= \frac{1}{t} \lambda_{p+k} t^{\sigma_{\overline{2p+k}}} v_{2p+k} \\ &= v_{2p+k} = v_h, \end{aligned}$$

since $\lambda_{p+k} = 1$ and $\sigma_{\overline{2p+k}} = \sigma_h = 1$. Regarding $v[DE^k]$, we have

$$\begin{aligned} v[DE^k] &= \frac{1}{t} v_p [DE^k] \\ &= \frac{1}{t} \mu_p t^{\tilde{\sigma}_{2p+k}} v_{2p+k} \\ &= \mu_p v_{2p+k} = \mu_p v_h, \end{aligned}$$

and Equation (5.1) yields

$$\begin{aligned} \mu_p &= \sum_{i=0}^{q-2p+h} (-1)^i \binom{q-2p+h}{i} \lambda_{p+i} \\ &= \sum_{i=q-2p+m+1}^{q-2p+h} (-1)^i \binom{q-2p+h}{i} \\ &= -\sum_{i=m+1}^{h} (-1)^i \binom{h}{i} = (-1)^m \binom{h-1}{m}. \end{aligned}$$

Therefore $vE^kD - v[DE^k] = \left(1 + (-1)^{m+1}\binom{h-1}{m}\right) v_h$ as claimed.

(iii) Let $k \geqslant 1$ and $m < h \leqslant q - p + m$. Then

$$\begin{aligned}
[e_{kq+h}e_p] &= \begin{bmatrix} 0 & 0 \\ t^{k-1}v_h & 0 \end{bmatrix} \begin{bmatrix} D & 0 \\ v & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ t^{k-1}v_h D & 0 \end{bmatrix}.
\end{aligned}$$

But

$$\begin{aligned}
v_h D &= v_h \left(\lambda_{q-p+j} t^{\sigma_j} v^T_{q-p+j}\right)_{j=1,\dots,q} \\
&= \lambda_h t^{\sigma_{\overline{p+h}}} v_{p+h} = 0,
\end{aligned}$$

since $\lambda_h = 0$. Therefore $[e_{kq+h}e_p] = 0$.

(iv) Let $k \geqslant 1$, and let us consider separately the case $1 \leqslant h \leqslant m$ and $q-p+m < h \leqslant q$. If the first case occurs,

$$[e_{kq+h}e_p] = \begin{bmatrix} 0 & 0 \\ t^{k-1}v_h D & 0 \end{bmatrix}$$

and

$$\begin{aligned}
v_h D &= v_h \left(\lambda_{q-p+j} t^{\sigma_j} v^T_{q-p+j}\right)_{j=1,\dots,q} \\
&= \lambda_h t^{\sigma_{p+h}} v_{p+h} \\
&= v_{p+h},
\end{aligned}$$

since $\lambda_h = 1$ and $\sigma_{p+h} = 0$. Hence

$$\begin{aligned}
[e_{kq+h}e_p] &= \begin{bmatrix} 0 & 0 \\ t^{k-1}v_{p+h} & 0 \end{bmatrix} \\
&= e_{kq+p+h}.
\end{aligned}$$

Finally, if the second case occurs,

$$[e_{kq+h}e_p] = \begin{bmatrix} 0 & 0 \\ t^{k-1}v_h D & 0 \end{bmatrix},$$

but this time

$$\begin{aligned}
v_h D &= v_h \left(\lambda_{q-p+j} t^{\sigma_j} v^T_{q-p+j}\right)_{j=1,\dots,q} \\
&= \lambda_h t^{\sigma_{p+h-q}} v_{p+h} \\
&= t v_{p+h-q},
\end{aligned}$$

since $\lambda_h = 1$, $\sigma_{p+h-q} = 1$, and $\overline{p+h} = p+h-q$. Hence

$$\begin{aligned}[e_{kq+h}e_p] &= \begin{bmatrix} 0 & 0 \\ t^k v_{p+h-q} & 0 \end{bmatrix} \\ &= e_{kq+p+h}.\end{aligned}$$

□

<u>*Remark*</u> 5.5. (i) The above construction works also for $m = p-1$: the resulting algebra of type p has first constituent of length $q+p-1$ and associated two-step centralizers

$$\alpha_{p+1} = \cdots = \alpha_q = 0, \quad \alpha_{q+1} = \cdots = \alpha_{q+p-1} = 1.$$

The n-th constituent of L, for any $n > 1$, has length q and associated two-step centralizers

$$\alpha_{(n-1)q+p} = \cdots = \alpha_{nq-1} = 0, \quad \alpha_{nq} = \cdots = \alpha_{nq+p-1} = 1.$$

Hence L has only two distinct two-step centralizers, namely 0 and 1. The first constituent is almost ordinary of length $q+p-1$, while all the following constituents are ordinary of length q. The sequence of constituent lengths of L is

$$q+p-1, q^\infty.$$

Therefore, L is a translated algebra of the subalgebra N of type p of $AFS(1, h, \infty, p)$, where h is such that $q = p^h$: indeed $AFS(1, h, \infty, p)$ (and hence N) has only two distinct two-step centralizers and its sequence of constituent lengths is

$$2p, p^{\frac{q}{p}-2}, 2p-1, \left(p^{\frac{q}{p}-2}, 2p\right)^\infty.$$

Up to scaling, the disctinct two-step centralizers of N may be taken as 0 and -1, therefore $N(1) = L$.

(ii) On the other hand, if $1 \leqslant m < p-1$ then the algebras constructed above cannot be translated algebras of any subalgebra of an algebra of type 1. Indeed, suppose L is an algebra with first constituent of length $q+m$ and sequence of two-step centralizers (α_i) given explicitly by the previous proposition, and consider its translation $L(\delta)$ for an arbitrary $\delta \in \mathbb{F}^*$. The sequence of two-step centralizers of $L(\delta)$ starts as

$$\underbrace{\delta, \ldots, \delta}_{q-2p+m}, \tilde{\alpha}_{q-p+m+1}, \ldots, \tilde{\alpha}_q,$$

where

$$\begin{aligned}\tilde{\alpha}_{q-p+h} &= \delta + \alpha_{q-p+h}\\ &= \delta + \left(1 + (-1)^{m+1}\binom{h-1}{m}\right)\end{aligned}$$

for $h = m+1, \ldots, p$. Therefore the first constituent of $L(\delta)$ is ordinary ending in δ of length $2p$, followed by $\frac{q}{p} - 3$ ordinary constituents of length p, but the constituent coming after these is of the form

$$\underbrace{\delta, \ldots, \delta}_{m}, \tilde{\alpha}_{q-p+m+1}, \ldots, \tilde{\alpha}_q,$$

which is clearly not ordinary.

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