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# Diffusion and Age in Population Dynamics

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# Introduction

Empirical evidence suggests that both the spatial diffusion of individuals and the internal heterogeneity of a population have a deep influence on the dynamics of the population itself. On the one hand these considerations led several authors to be interested in models for the spatial spreading of the individuals of the population, and, on the other hand, to deal with the modeling of structured population, giving a great deal of attention especially to age-structured populations. However, the study of models combining both these important features, diffusion and age, is still quite incomplete.

In the last section of the book "Diffusion and Ecological Problems" ([46]), which represents a reliable evidence of the importance of diffusion models in ecology and especially in population dynamics, Okubo and Levin stress the importance of taking into account both age and spatial diffusion in order to obtain more realistic models and present the state of art of the study of models for age-dependent dispersal, providing an almost complete bibliography concerning this kind of problems. Taking cue from this starting point, the aim of this thesis is to provide a contribution to the study of age-structured diffusion models.

Before entering into the details of the discussion, let us introduce the reader into the problems of modeling a spreading age-structured population by giving a brief description of the arising mathematical problem and by recalling some known results.

In its more general form, the problem describing the dynamics of an age-structured population that diffuses in an open bounded region $\Omega \subset \mathbb{R}^n$ (obviously the biological meaningful cases are $n = 1, 2, 3$) consists in a reaction-diffusion equation of the form

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\nu_j p) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\kappa_j \frac{\partial p}{\partial x_j}\right) - \mu(a; S_1(t,x), \ldots, S_m(t,x))p, \tag{0.0.1}$$

where $p(t, a, x)$ denote the density, per unit volume and age, at time $t$ of the population (here $t \geq 0$ is the time variable, $a \in [0, a_\dagger]$ denotes age and $x = (x_1, \ldots, x_n) \in \Omega$ is the space variable), $\nu_j$ and $\kappa_j$ are, respectively, the advection velocity and the diffusivity, and $\mu(a; S_1(t,x), \ldots, S_m(t,x))$ represents the mortality, which depends on age and on a set of $m$ variables (sizes) representing different ways of weighing the population. The initial distribution of the population is supposed to be given:

$$p(0, a, x) = p_0(a, x), \tag{0.0.2}$$

while the presence of the age-structured leads to the following boundary condition at age 0

$$p(t, 0, x) = \int_0^{a_\dagger} \beta(a; S_1(t,x), \ldots, S_m(t,x)) p(t, a, x) da, \tag{0.0.3}$$

where $\beta(a; S_1(t,x), \ldots, S_m(t,x))$ denotes the age-specific fertility. Obviously boundary conditions on the boundary $\partial\Omega$ of $\Omega$ have to be added to the problem, according to the specific features of the population and the environment, for example we shall take homogeneous Dirichlet boundary conditions if we aim to model a hostile habitat at the boundary of the environment, or zero-flux boundary conditions if we need to assume that individuals do not go out of the region $\Omega$.

Without expecting to be exhaustive, we now recall some of the most important results on this kind of model, a more complete discussion of some of them is given in Chapter 2.
First results concerning linear models for age-dependent diffusion have been given in 1976-1977 by Gopalsamy ([21], [22]), who used Fourier transforms in order to obtain the solution to the linear one-dimensional problem with constant diffusion.
In 1978-1979 Di Blasio and Lamberti ([15], [16]), considered a nonlinear model with homogeneous Neumann boundary conditions and constant diffusion and proved that the problem has a unique positive solution which continuously depends on the initial data, providing also some estimates of the growth rate of the solution.
In 1979 Marcati and Serafini ([40]) considered a model with space-dependent diffusion and used an abstract formulation of the problem in order to prove an existence and uniqueness result.
At the beginning of the eighties Gurtin and MacCamy ([39], [25], [26]) studied age-structured models taking into account two different diffusion processes, namely random dispersal and movement to avoid crowding (directed diffusion).
In the same years also Langlais studied age-dependent population diffusion models ([32], [33], [34]), focusing his attention on the asymptotic behaviour of the solution to nonlinear models (see e.g. [34]).
In 1983 Busenberg and Iannelli provided a method useful to treat a broad class of nonlinear age-structured population problems that involve spatial diffusion (either linear or nonlinear) (see [8]). The method introduced by Busenberg and Iannelli consists in separating the age-dependent part of the problem from the diffusion mechanism.
In the nineties some results, concerning linear models, has been provided by Chan and Guo ([9]) and Huyer ([27]), who analyzed models with, respectively, homogeneous Dirichlet boundary conditions and homogeneous Neumann boundary conditions. More recent results are due to Ainseba and Aniţa ([1], [2]), who also took into account a control problem for a liner model.

The goal of this work is to provide some new results in the subject of modeling spreading age-structured population. This thesis is organized as follows.
Chapter 1 is devoted to a historical introduction to both diffusion equations and models for age-structured population dynamics. In particular, in Section 1.1, we give a brief overview on diffusion models, starting from the simplest diffusion equation (derived both from a simple random walk and via flux considerations), passing through the advection-diffusion equation and the telegraph equation, and ending up with reaction-diffusion problems and models for chemotaxis. In Section 1.2 we focus our attention on well known problems describing the dynamics of age-structured populations without spatial spreading of individuals, namely the linear Lotka-McKenrick model and the nonlinear Gurtin-MacCamy model. Concerning the linear problem existence and uniqueness results are recalled, in particular two possible approaches are analyzed: the first one is based on the equivalence of the Lotka-McKendrick problem with a Volterra integral equation of the second kind, the so-called renewal equation, and then makes use of known

results of the theory of integral equations, while the second one consists in giving an abstract formulation to the model and using semigroup theory. The asymptotic behaviour of the solution to the linear problem is also discussed.

Whereas in Chapter 1 the diffusion of individuals and the age-structure of the populations are separately taken into account, in Chapter 2 we consider reaction-diffusion models describing the spatial spreading of an age-structured population. In particular, the aim of Section 2.1 is to give an overview on previous results on age-structured diffusion models, taking into account both linear and nonlinear models. In Section 2.2 a model describing an age-structured population which diffuses with a constant velocity in a bounded open region of $\mathbb{R}^2$ is analyzed, existence and uniqueness of the solution to this problem are proved, following the abstract approach given by Marcati and Serafini in [40]. Finally, at the end of Section 2.2, we study the asymptotic behaviour of the solution to a one-dimensional age-structured diffusion model with homogeneous Neumann boundary conditions.

After this introductory review, we pass to the analysis of some original works obtained in the last four years. In particular Chapter 3 contains some results obtained in collaboration with Gabriela Marinoschi and Mimmo Iannelli ([11], [12], [13]). In Section 3.1 we consider a nonlinear model for the spatial diffusion of an age-structured population in an open bounded region of $\mathbb{R}^n$ and prove an existence and uniqueness theorem for this problem by using the theory of m-accretive operators.
In Section 3.2 we deal with the same nonlinear model in a one-dimensional environment, namely an interval $[0, L]$, and prove that the solution to this problem can be approximated by using a multi-layer model. In particular it is proved that the solution of the problem set up in a $n$-layer environment converges to the solution of the continuous one (the problem in the interval $[0, L]$) as $n$ goes to infinity.
Finally, the goal of Section 3.3 is to find an analytical expression for the solution to the linear two-layer model in terms of a new variable representing the flux at the interface between the two layers. It is shown that the new variable must satisfy an integral equation containing both a Volterra and a Fredholm term and uniqueness and existence of a solution to this equation are proved by using a fixed point argument.

Finally, in Chapter 4 we deal with another kind of approach to diffusion models for age-structured populations, namely we focus our attention on the effect of diffusion on the stability of the steady states of a reaction-diffusion system, looking for the conditions for diffusion driven instability. First of all, in Section 4.1, we give a brief introduction on the mechanism leading to the evolution of a stable pattern (Turing mechanism) and some examples of pattern formation in systems describing the interaction of two species. In a second moment, namely in Section 4.2, in view of future applications to age-structured models, we consider the case of a population divided into two classes, depending on age, namely juveniles and adults. We show that, under certain conditions on the reaction terms, spatially homogeneous steady state, which are linearly stable in the absence of diffusion, evolve to a stable inhomogenous spatial pattern. Moreover, we introduce an age-structured model for juveniles-adults interaction, which will be the subject of future studies.

# Chapter 1

# Preliminaries

In this chapter we first give an overview on the models describing the diffusion of a population, without taking into account the internal structure of the population itself, concentrating our attention on the derivation of such models (from the simplest diffusion equation to the reaction-diffusion equation). In the second part of chapter, we discuss the problems modeling age-structured population, where spatial spreading is disregarded. In particular we recall some known result concerning linear and nonlinear models.

## 1.1 Diffusion models

In this section we give a brief review of some models describing the diffusion of a population.

In Sections 1.1.1 and 1.1.2 we derive the diffusion equation in two different ways, the first one is based on the formulation of a simple random walk process (Lagrangian approach), while the second one comes from physics and uses an argument based on fluxes (Eulerian approach).

In Section 1.1.3 we consider a population moving according to a random walk which is not simple, i.e. there is a preferred direction in movements of individuals, and we derive the advection-diffusion equation. Section 1.1.4 is devoted to the so-called telegraph equation, which is derived from a random walk with correlated steps.

The models described in Sections 1.1.1-1.1.4 are derived considering only individuals' movements, without taking into account the dynamics of the population (births and deaths, interactions between individuals, etc.). In Section 1.1.5 we consider more realistic models in which also the population dynamics is taken into account.

Finally, in Section 1.1.6 we present the problem of modeling chemotaxis, although models for chemotaxis are not the subject of this thesis.

### 1.1.1 From a simple random walk to the diffusion equation

In describing the derivation of a model for the diffusion of a population we analyze only the one-dimensional case, the procedures for higher-dimensional cases being the same but with more complicated algebra. We suppose that the individuals of the population move along a line in discrete time steps $\Delta t$ by jumping one spatial step $\Delta x$ to the right or left without a preferred direction, i.e. the probability of either move is equal to $1/2$. Such a random walk is called a *simple*, or *isotropic*, random walk. Moreover we suppose that the choice of moving to the right or to the left does not depend on the directions of previous moves. Thus the probability $f(x,t)$ that an individual is at location $x$ at time $t$ turns out to be

$$f(x,t) = \frac{1}{2} f(x+\Delta x, t-\Delta t) + \frac{1}{2} f(x-\Delta x, t-\Delta t). \tag{1.1.1}$$

Subtracting $f(x,t-\Delta t)$ from both sides of (1.1.1) and dividing the equation by $\Delta t$, we obtain the following identity

$$\frac{f(x,t) - f(x,t-\Delta t)}{\Delta t} =$$

$$= \frac{(\Delta x)^2}{2\Delta t} \frac{f(x+\Delta x, t-\Delta t) - 2f(x,t-\Delta t) + f(x-\Delta x, t-\Delta t)}{(\Delta x)^2}.$$

Notice that the left-hand side is nothing but the discrete version of the first derivative of $f$ with respect to time, while the right-hand side is the discrete version of the second derivative of $f$ with respect to space multiplied by the factor $(\Delta x)^2/2\Delta t$. Thus, taking the limit as $\Delta x$ and $\Delta t$ go to zero and assuming that $\Delta x$ and $\Delta t$ shrink to zero in a manner such that the limit

$$D = \lim_{\Delta x, \Delta t \to 0} \frac{(\Delta x)^2}{2\Delta t} \tag{1.1.2}$$

exists, we obtain the one-dimensional equation for diffusion

$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}. \tag{1.1.3}$$

If we multiply $f$ by the total number of individuals, we obtain the individuals' concentration $p$, for which the following equation, obtained from (1.1.3), holds

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2}. \tag{1.1.4}$$

The situation described above is the simplest possible, the model can be generalized in several ways, for example allowing individuals to move to the right or to the left with different probabilities; before analyzing this case, we present another procedure that can be used in order to get the diffusion equation (1.1.4).

### 1.1.2 The diffusion equation derived using flux considerations

The aim of this Section is to derive the diffusion equation using an argument based on fluxes. Also in this case, for simplicity, we deal with the one-dimensional case, we can imagine, for example, a population moving in a thin long tube.

We consider how the concentration of individuals changes in a section $(x, x+\Delta x)$ of the tube. We denote by $\mathbf{J}(x,t)$ the flux of particles at $(x,t)$, which is given by the rate at which individuals cross $x$ in the positive direction (from left to right) minus the rate at which individuals cross $x$ in the negative direction. Moreover we define $p(x,t)$ as the concentration of individuals (i.e. the number of individuals per unit length). The net change of individuals' number in the section $(x, x+\Delta x)$ is given by the net change (entry minus departure) at $x$ plus the net change at $x+\Delta x$, thus, in terms of the functions defined above, we have the following relation

$$\frac{\partial}{\partial t}[p(x,t)\Delta x] = \mathbf{J}(x,t) - \mathbf{J}(x+\Delta x,t). \tag{1.1.5}$$

Dividing both sides of (1.1.5) by $\Delta x$, we get

$$\frac{\partial p}{\partial t} = -\frac{\mathbf{J}(x+\Delta x,t) - \mathbf{J}(x,t)}{\Delta x},$$

and then, taking the limit as $\Delta x \to 0$, we end up with the following balance equation

$$\frac{\partial p}{\partial t} = -\frac{\partial \mathbf{J}(x,t)}{\partial x}. \tag{1.1.6}$$

The next step consist in relating the flux $\mathbf{J}(x,t)$ with the spatial change of individuals' concentration. In order to do this we recall that the flux due to random motion of individuals is proportional to the gradient (in our case the derivative with respect to $x$) of the individuals' concentration (*Fick's law*), i.e.

$$\mathbf{J}(x,t) = -D\frac{\partial p}{\partial x},$$

where $D$ is supposed to be a constant, called *diffusivity*.

Thus, substituting this expression for $\mathbf{J}(x,t)$ in (1.1.6), we obtain again the diffusion equation

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}.$$

### 1.1.3 From a random walk to the advection-diffusion equation

As we have seen in Section 1.1.1, the diffusion equation can be derived from a random walk process in which the individuals move to the right or to the left with the same probability.

We now want to analyze the situation in which, at each time step $\Delta t$, an individual moves one spatial step $\Delta x$ to the right with probability $\alpha$ or one spatial step $\Delta x$ to the left with probability $\beta = 1-\alpha$. Again, as in Section 1.1.1, we suppose that the probability of moving a given direction at a certain time step is not correlated with the direction moved in the previous time step.

With these assumptions the probability $f(x,t)$ that an individual is at location $x$ at time $t$ turns out to be

$$f(x,t) = \alpha f(x-\Delta x, t-\Delta t) + \beta f(x+\Delta x, t-\Delta t). \tag{1.1.7}$$

We suppose that $\Delta x$ and $\Delta t$ are small compared to, respectively, $x$ and $t$, and that each term on the right-hand side of (1.1.7) can be expanded in a Taylor series in $x$ and $t$,

$$\begin{aligned} f(x-\Delta x, t-\Delta t) &= f(x,t) - \Delta x\frac{\partial f}{\partial x} - \Delta t\frac{\partial f}{\partial t} + \frac{(\Delta x)^2}{2}\frac{\partial^2 f}{\partial x^2} + \\ &\quad + \Delta x\Delta t\frac{\partial^2 f}{\partial x \partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 f}{\partial t^2} + \cdots, \end{aligned} \tag{1.1.8}$$

$$\begin{aligned} f(x+\Delta x, t-\Delta t) &= f(x,t) + \Delta x\frac{\partial f}{\partial x} - \Delta t\frac{\partial f}{\partial t} + \frac{(\Delta x)^2}{2}\frac{\partial^2 f}{\partial x^2} + \\ & -\Delta x\Delta t\frac{\partial^2 f}{\partial x\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 f}{\partial t^2} + \cdots, \end{aligned} \tag{1.1.9}$$

where all the right-hand side derivatives are evaluated at $(x,t)$.
Substituting (1.1.8) and (1.1.9) in (1.1.7), using the fact that $\alpha+\beta=1$ and defining $\varepsilon=\alpha-\beta$, we get

$$\frac{\partial f}{\partial t} = -\frac{\varepsilon\Delta x}{\Delta t}\frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 f}{\partial x^2} + \varepsilon\Delta x\frac{\partial^2 f}{\partial x\partial t} + \frac{\Delta t}{2}\frac{\partial^2 f}{\partial t^2} + \cdots, \tag{1.1.10}$$

where the parameters $\Delta x$, $\Delta t$ and $\varepsilon$ are assumed to be constant.

The idea is now to consider the limit as these parameters go to zero, in order to do this we add the following assumptions:

$$\lim_{\Delta x,\Delta t,\varepsilon\to 0}\frac{\varepsilon\Delta x}{\Delta t} = u, \qquad \lim_{\Delta x,\Delta t\to 0}\frac{(\Delta x)^2}{2\Delta t} = D.$$

Thus, taking the limit as $\Delta x$, $\Delta t$ and $\varepsilon$ go to zero in (1.1.10), we obtain the following equation

$$\frac{\partial f}{\partial t} = -u\frac{\partial f}{\partial x} + D\frac{\partial^2 f}{\partial x^2}.$$

As we did in section 1.1.1, in order to get an equation for the concentration of individuals, we multiply $f$ by the number of individuals, obtaining the equation

$$\frac{\partial p}{\partial t} = -u\frac{\partial p}{\partial x} + D\frac{\partial^2 p}{\partial x^2}. \tag{1.1.11}$$

Notice that equation (1.1.4) corresponds to (1.1.11) when advection $u$ is absent ($\alpha=\beta=1/2$).

In performing the above calculation we always considered constant diffusion and advection, but in concrete examples $D$ and $u$ may be functions of time, space and population density (see [48], Section A.2.4, for the discussion of some of models involving variable coefficients).

### 1.1.4 The telegraph equation

In Section 1.1.1 and in Section 1.1.3 we supposed that there is no correlation between two consecutive moves, but we have to consider the fact that, in concrete examples, an individual has the tendency to proceed in the same direction, at least in a small time period.

In [20], Goldstein developed a random walk model with correlated steps, obtaining the following equation for the probability $f(x,t)$ for an individual to be in $x$ at time $t$

$$\frac{\partial^2 f}{\partial t^2} + \frac{1-\gamma}{\Delta t}\frac{\partial f}{\partial t} = \frac{(\Delta x)^2}{(\Delta t)^2}\frac{\partial^2 f}{\partial x^2},$$

where $\gamma$ represents the correlation between two successive steps and the vanishing terms (when $\Delta x$ and $\Delta t$ go to zero) have been neglected. We pass to the limit as $\Delta x, \Delta t \to 0$, using the following assumptions

$$\lim_{\Delta x,\Delta t\to 0}\frac{(\Delta x)^2}{(\Delta t)^2} = \nu^2, \qquad \lim_{\gamma\to 1,\Delta t\to 0}\frac{1-\gamma}{\Delta t} = \frac{2}{T}, \tag{1.1.12}$$

and we multiply $f$ by the number of individuals, then we end up with the so-called *telegraph equation* for the concentration of individuals $p$

$$\frac{\partial^2 p}{\partial t^2} + \frac{2}{T}\frac{\partial p}{\partial t} = \nu^2 \frac{\partial^2 p}{\partial x^2}, \tag{1.1.13}$$

where T is a characteristic time of step correlation.

In the telegraph equation (1.1.13) we find elements of both diffusion and wave motion: the concentration $p$ is transmitted with a wave speed $\nu$ while being dispersed.

Let us analyze the difference between the diffusion equation (1.1.4) and the telegraph equation (1.1.13). In the case of the diffusion equation (see Section 1.1.1), we took the limits in such a way that (see (1.1.2)) $\frac{(\Delta x)^2}{\Delta t}$ is finite, this implies that the walk speed has to be infinite, in fact

$$\lim_{\Delta x, \Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta x, \Delta t \to 0} \frac{(\Delta x)^2}{\Delta x \Delta t} = D \lim_{\Delta x \to 0} \frac{1}{\Delta x} = \infty.$$

On the other hand, in deriving the telegraph equation, limits were taken in such a way that (see (1.1.12)) $\frac{\Delta x}{\Delta t} = \nu$ is finite, and then a finite walk speed is implied.

Thus, since no "real" organism can diffuse with a infinite speed, the telegraph equation (1.1.13) seems to be more realistic than the diffusion equation (1.1.4).

However, when $t \gg T$, i.e. if we consider times $t$ much longer than the duration of step correlation $T$, the difference between the two equations is very small, in fact in this case (1.1.13) can be approximated as

$$\frac{\partial p}{\partial t} = \frac{\nu^2 T}{2}\frac{\partial^2 p}{\partial x^2},$$

i.e. a diffusion equation in which the diffusivity is expressed by $\frac{\nu^2 T}{2}$.

On the other hand, for $t \ll T$, the telegraph equation (1.1.13) takes the form of the wave equation

$$\frac{\partial^2 p}{\partial t^2} = \nu^2 \frac{\partial^2 p}{\partial x^2},$$

and then we conclude that individuals, starting to move from an origin, first diffuse as a wave in all directions with velocity $\nu$.

### 1.1.5 Some examples of reaction-diffusion models

In the previous sections we disregarded the dynamics of the population, namely we did not take into account the fact that the number of individuals is affected by births, deaths, individuals' interaction, etc. When we consider also these factors we have to deal with reaction-diffusion equations of the kind

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + F(p), \tag{1.1.14}$$

where the reaction term $F$ describes the internal dynamics.

This problem was first analyzed by Fisher (see [19]), who considered a logistic growth, namely he studied the problem

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + \alpha p\left(1 - \frac{p}{K}\right), \tag{1.1.15}$$

where $K$ is a constant representing the carrying capacity of the environment. We shall come back to Fisher model later, in Chapter 4.

In [47], Skellam considered a spreading population with exponential growth, namely he took $F(p) = \alpha p$, with $\alpha$ constant.

A case of ecological importance for which the logistic-like growth conditions are not satisfied is the so-called Allee effect, which occurs when an increase of the number of individuals causes an improvement of life conditions; in other words, the population growth is negative at low density. This kind of model has been studied by Fife and McLeod in [18] and by Lewis and Kareiva in [37]. In particular, Lewis and Kareiva considered in (1.1.14) the following reaction term

$$F(p) = kp(p-a)(1-p),$$

here $p$ is the population density expressed in units of carrying capacity (i.e. $K = 1$), and $a$ is the fraction of the carrying capacity below which population growth is negative. They observed that this choice for the function $F$ describes an Allee effect when $0 < a < 1$ and investigated how it influences the spatio-temporal dynamics of ecological invasions.

### 1.1.6 Models for chemotaxis

Chemotaxis is the phenomenon in which individuals direct their movements according to certain chemical substances present in their environment. A lot of examples of chemotactic effect can be found in studying bacteria (e.g. *Escherichia coli*, *Salmonella*), amoebae (e.g. the slime mold *Dictyostelium discoideum*) or animal cells, but, as Murray pointed out in [42], also animals move responding to chemical stimuli.

The first model describing chemotaxis is known as *Keller-Segel model* after Keller and Segel, who, in [29], [30] and [31], gave a mathematical formulation to the chemotactic phenomenon and analyzed some consequences of chemotactic interaction between cells (e.g. aggregation in many species of cellular slime mold, see [29]).

Here we present a model for chemotaxis as derived in [42]. Let us consider a population of cells whose motion is influenced by the molecules of a *critical substrate* composed of a chemical attractant (usually produced by the cells themselves). The general conservation law for the number of cells, $p(x,t)$, is

$$\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{J} = f(p), \tag{1.1.16}$$

where $\mathbf{J}$ is the flux and $f(p)$ represents the growth term for the cells. The flux of cells $\mathbf{J}$ is composed of two components:

$$\mathbf{J} = \mathbf{J}_d + \mathbf{J}_c,$$

the first one ($\mathbf{J}_d$) comes from "usual" diffusion and is of the form

$$\mathbf{J}_d = -D\nabla p,$$

where $D$ is the diffusion coefficient of the cells, while the second one ($\mathbf{J}_c$) is the flux due to chemotaxis. Let us analyze $\mathbf{J}_c$: the presence of a gradient in the attractant $a(x,t)$ causes a movement of the cells up the gradient, thus the flux of cells increases with the number of cells:

$$\mathbf{J}_c = p\chi(a)\nabla a.$$

Substituting $\mathbf{J}$ in (1.1.16), we obtain the following *reaction-diffusion-chemotaxis equation*

$$\frac{\partial p}{\partial t} = f(p) - \nabla \cdot p\chi(a)\nabla a + \nabla \cdot D\nabla p.$$

Now we have to add another equation describing the spreading of the attractant, for which we have the following diffusion equation

$$\frac{\partial a}{\partial t} = g(a,p) + \nabla \cdot D_a \nabla a,$$

where $D_a$ is the diffusion coefficient of the attractant and usually it is $D_a > D$. Notice that the source term $g(a,p)$ may depend also on the number of cells $p$, because, as we pointed out before, usually the attractant is produced by the cells themselves. In particular, in [31], Keller and Segel took $g(a,p) = hp - ka$, with $h$ and $k$ positive constants.

## 1.2 Age-structured models

Let us now disregard the spatial diffusion of individuals and consider models which take into account the internal structure of the population, namely we concentrate on age-structure, the age of individuals being one of their most important features. In this section we give a review of the classical models in the theory of age-structured population dynamics.

In Sections 1.2.1-1.2.5 we deal with linear models for age-structured populations. In particular, in Section 1.2.1 we present the Lotka-McKendrick model, in Section 1.2.2 we recall an existence and uniqueness result for this model and in Section 1.2.3 we analyze the asymptotic behaviour of the solution using classical Laplace transform techniques.

In Section 1.2.4 we present a general nonlinear model and we analyze the stability of the equilibria of this problem.

Finally, in Section 1.2.5, we consider another approach to the Lotka-McKendrick model, based on semigroup theory.

### 1.2.1 The Lotka-McKendrick model

Let us consider a population living in an isolated and invariant environment and suppose that all of its individuals are equal but for their age, in other words for the moment we disregard differences between individuals (e.g. sex, size) and intra specific dynamics (e.g. prey-predator interactions, Allee effect).

The evolution of such a population is described by a function of time $t$ and age $a$ , denoted by $p(t,a)$, which represents the age density of the population. We assume that $a \in [0, a_\dagger]$, where

$a_\dagger < +\infty$ represents the maximum age that an individual can reach. With this definition we have that the integral

$$P(t) = \int_0^{a_\dagger} p(t,a)\ da$$

gives the total population at time $t$, while the integral

$$\int_{a_1}^{a_2} p(t,a)\ da, \quad 0 < a_1 < a_2 < a_\dagger$$

gives the number of the individuals that, at time $t$, have age between $a_1$ and $a_2$.

Concerning the vital rates (fertility and mortality), since we are assuming that individuals are influenced only by their age, they are functions of age only. Fertility is represented by the age specific fertility $\beta(a)$, which gives the number of offsprings coming from an individual of age in the infinitesimal interval $[a, a+da]$ per unit time. Thus the total birth rate (i.e. the number of newborns per time unit) turns out to be

$$B(t) = \int_0^{a_\dagger} \beta(a)p(t,a)\ da. \tag{1.2.1}$$

Concerning mortality we denote by $\mu(a)$ the age-specific mortality and then we have that the probability for an individual to survive at age $a \in [0, a_\dagger]$ (survival probability) is given by

$$\Pi(a) = e^{-\int_0^a \mu(\sigma)d\sigma}.$$

Since an individual can not exceed age $a_\dagger$, we have to assume that $\Pi(a_\dagger) = 0$ and then $\mu$ must satisfy

$$\int_0^{a_\dagger} \mu(\sigma)\ d\sigma = +\infty. \tag{1.2.2}$$

The dynamics of the population is then described by the classical *Lotka-McKendrick model*

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu(a)p = 0, \quad \text{in } [0,+\infty) \times [0, a_\dagger], \tag{1.2.3}$$

$$p(0,a) = p_0(a), \quad \text{for } a \in [0, a_\dagger], \tag{1.2.4}$$

$$p(t,0) = \int_0^{a_\dagger} \beta(a)p(t,a)\ da = B(t), \quad \text{for } t \in [0,+\infty), \tag{1.2.5}$$

where $p_0$ is the initial age-distribution which is supposed to be given.

In the following sections we present some well-known results concerning the linear problem (1.2.3)-(1.2.5).

### 1.2.2 Existence and uniqueness for the linear age-structured model

In order to prove that problem (1.2.3)-(1.2.5) has an unique solution, we need some mathematical assumptions, namely we assume that

$$\beta \in L^\infty([0, a_\dagger]), \ \beta \geq 0, \tag{1.2.6}$$

$$\mu \in L^1_{loc}([0, a_\dagger)), \ \mu \geq 0, \tag{1.2.7}$$

$$p_0 \in L^1([0, a_\dagger]), \ p_0 \geq 0 \ a.e. \ in \ [0, a_\dagger]. \tag{1.2.8}$$

The existence and uniqueness of the solution of (1.2.3)-(1.2.5) are proved using an equivalent formulation (see e.g. [28]). This equivalent formulation is obtained by integrating equation (1.2.3) along characteristics and using conditions (1.2.4) and (1.2.5), ending up with the following expression for the solution $p$:

$$p(t,a) = \begin{cases} p_0(a-t)\dfrac{\Pi(a)}{\Pi(a-t)} & \text{if } a \geq t, \\ \\ B(t-a)\Pi(a) & \text{if } a < t. \end{cases} \tag{1.2.9}$$

Notice that formula (1.2.9) does not give an explicit solution of the problem (1.2.3)-(1.2.5), in fact it involves the total birth rate $B(t)$ which in turn depends on $p$ (see (1.2.1)). But (1.2.9) allows us to get the following Volterra integral equation of the second kind, known as *renewal equation* or *Lotka equation*, for $B(t)$,

$$B(t) = F(t) + \int_0^t K(t-s)B(s)\, ds, \tag{1.2.10}$$

where

$$\begin{aligned} F(t) = & \int_t^\infty \beta(a)\frac{\Pi(a)}{\Pi(a-t)}p_0(a-t)\, da = \\ = & \int_0^\infty \beta(a+t)\frac{\Pi(a+t)}{\Pi(a)}p_0(a)\, da, \end{aligned} \tag{1.2.11}$$

and $K$ is the so-called maternity function:

$$K(t) = \beta(t)\Pi(t). \tag{1.2.12}$$

Notice that (1.2.10) makes sense if the functions $\beta$, $\Pi$, $p_0$ are extended by zero outside the interval $[0, a_\dagger]$.

Under the hypothesis (1.2.2) and (1.2.6)-(1.2.8), one can prove, using well-known results in the theory of Volterra integral equations, that (1.2.10) with (1.2.11) and (1.2.12) has a unique solution $B \in \mathcal{C}([0, +\infty))$, such that $B(t) \geq 0$ for all $t$.

Once shown the existence of a solution $B$ to the renewal equation, under suitable assumptions (see [28] for the detailed hypothesis), it is possible to prove that the function $p$ defined in (1.2.9) satisfies

$$p \in \mathcal{C}([0,+\infty) \times [0,a_\dagger]), \quad p(t,a) \geq 0, \quad \mu(\cdot)p(\cdot,t) \in L^1(0,a_\dagger) \ \forall t > 0, \tag{1.2.13}$$

$$\frac{\partial p}{\partial t}(t,a), \ \frac{\partial p}{\partial a}(t,a) \ \textit{exist a.e. in} \ [0,+\infty) \times [0,a_\dagger] \tag{1.2.14}$$

and it is the only solution of problem (1.2.3)-(1.2.5).

### 1.2.3 The asymptotic behaviour for the linear model

The aim of this section is to investigate the asymptotic behaviour of the solution to problem (1.2.3)-(1.2.5). As we have seen, it is enough to study the behaviour of the total birth rate $B(t)$, every result for $B(t)$ being transfered to $p(t,a)$ via formula (1.2.9).

It is well known that the asymptotic behaviour of a function $f$ can be related to the singularities of its Laplace transform $\widehat{f}$. Thus, being $B(t)$ absolutely Laplace transformable (as observed in [28]), we are led to the problem of investigating the singularities of its Laplace transform $\widehat{B}(\lambda)$.

From the renewal equation (1.2.10) we get

$$\widehat{B}(\lambda) = \frac{\widehat{F}(\lambda)}{1-\widehat{K}(\lambda)} = \widehat{F}(\lambda) + \frac{\widehat{F}(\lambda)\widehat{K}(\lambda)}{1-\widehat{K}(\lambda)}, \tag{1.2.15}$$

where $\widehat{F}$ and $\widehat{K}$ denote the Laplace transforms of $F$ and $K$.
Since $F(t)$ and $K(t)$ vanish for $t > a_\dagger$, their transforms $\widehat{F}(\lambda)$ and $\widehat{K}(\lambda)$ are entire analytical functions of $\lambda$, thus, from equation (1.2.15), we have that the poles of $\widehat{B}(\lambda)$ are the roots of the *Lotka characteristic equation*

$$\widehat{K}(\lambda) = 1. \tag{1.2.16}$$

Concerning equation (1.2.16) we have the following result

**Theorem 1.2.1** *Equation* (1.2.16) *has one and only one real solution* $\lambda^*$, *called the Malthusian parameter.* $\lambda^*$ *is a simple root and it is negative if and only if* $\int_0^\infty K(t)dt < 1$. *Moreover any other solution* $\lambda$ *of* (1.2.16) *is such that* $\mathcal{R}e\lambda < \lambda^*$.

As a consequence of Theorem 1.2.1 and of some classical results stating the relations between the asymptotic behaviour of a function and the singularities of its Laplace transform, we end up with

**Theorem 1.2.2** *Let* $p_0 \in L^1([0,a_\dagger])$, $p_0 \geq 0$ *a.e. in* $[0,a_\dagger]$ *and let* $\lambda^*$ *be the unique real solution of the characteristic equation* (1.2.16), *then*

$$B(t) = b_0 \exp(\lambda^* t)(1+\Omega(t)), \tag{1.2.17}$$

*where* $b_0 \geq 0$ *is a constant and* $\Omega(t)$ *is such that* $\lim_{t\to+\infty} \Omega(t) = 0$.

Theorem 1.2.1 and Theorem 1.2.2 allow us to determine the growth of the population, in fact we can state the following

**Theorem 1.2.3** *Let* $R = \int_0^{a_\dagger} \beta(a)\Pi(a)\ da$, *namely* $R$ *gives the number of newborns that an individual is expected to produce during his life and it is called net reproduction rate. The solution* $p(t,a)$ *of the problem* (1.2.3)-(1.2.5) *is increasing (as function of time) when* $R > 1$, *decreasing when* $R < 1$, *stable if* $R = 1$.

The proof of this result follows immediately from Theorem 1.2.2 by observing that $R = \widehat{K}(0)$ and then, from Theorem 1.2.1,

$$R > 1 \quad \textit{if and only if} \quad \lambda^* > 0,$$

$$R = 1 \quad \textit{if and only if} \quad \lambda^* = 0,$$

$$R < 1 \quad \textit{if and only if} \quad \lambda^* < 0.$$

### 1.2.4 Nonlinear age-structured models

In the previous sections we have considered a population in which the vital rates depend only on the age of individuals and we have seen that this assumption leads to linear models. Now we take into account the fact that the population itself causes modifications on its own conditions of life, and consequently on fertility and mortality.

In [24] Gurtin and MacCamy allowed the vital rates to be density dependent, in particular being

$$P(t) = \int_0^{a_\dagger} p(t,a)\ da, \tag{1.2.18}$$

the total population at time $t$ (see (1.2.1)), they supposed $\beta = \beta(a, P(t))$ and $\mu = \mu(a, P(t))$, obtaining the so called *Gurtin-MacCamy model*

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu(a, P(t))p = 0, \qquad \text{in } [0, +\infty) \times [0, a_\dagger], \tag{1.2.19}$$

$$p(0,a) = p_0(a), \qquad \text{for } a \in [0, a_\dagger], \tag{1.2.20}$$

$$p(t,0) = \int_0^{a_\dagger} \beta(a, P(t))p(t,a)\ da = B(t), \qquad \text{for } t \in [0, +\infty), \tag{1.2.21}$$

The method employed by Gurtin and MacCamy in analyzing this model was analogous to that of the classical linear case, that was to apply the method of characteristics to convert the problem to a system of Volterra integral equation involving the birth rate $B(t)$. We refer to [50] for a complete analysis of this method in the case of the Gurtin-MacCamy model.

The formulation of a more general nonlinear model comes from the assumption that the vital rates depend upon a set of $n$ significant variables (called sizes) which represent different ways of weighing the age distribution. As a consequence we have now to deal with the following nonlinear problem

$$\frac{\partial p}{\partial t}(t,a)+\frac{\partial p}{\partial a}(t,a)+\mu(a;S_1(t),\dots,S_n(t))p(t,a)=0, \quad \text{in } [0,+\infty)\times[0,a_\dagger], \tag{1.2.22}$$

$$p(0,a)=p_0(a), \quad \text{for } a\in[0,a_\dagger], \tag{1.2.23}$$

$$p(t,0)=\int_0^{a_\dagger}\beta(a;S_1(t),\dots,S_n(t))p(t,a)\ da, \quad \text{for } t\in[0,+\infty), \tag{1.2.24}$$

$$S_i(t)=\int_0^{a_\dagger}\gamma_i(a)p(t,a)\ da, \quad \text{for } t\in[0,+\infty), \quad i=1,\dots,n. \tag{1.2.25}$$

We add to the problem the following assumptions

$$\beta(\cdot;x_1,\dots,x_n)\in L^1(0,a_\dagger), \quad \forall(x_1,\dots,x_n)\in\mathbb{R}^n,$$

$$0\le\beta(a;x_1,\dots,x_n)\le\beta_+ \text{ a.e. in } [0,a_\dagger], \quad \forall(x_1,\dots,x_n)\in\mathbb{R}^n,$$

$$\mu(\cdot;x_1,\dots,x_n)\in L^1_{loc}([0,a_\dagger)), \quad \forall(x_1,\dots,x_n)\in\mathbb{R}^n,$$

$$\mu(a;x_1,\dots,x_n)\ge 0 \text{ a.e. in } [0,a_\dagger], \quad \forall(x_1,\dots,x_n)\in\mathbb{R}^n,$$

$$\int_0^{a_\dagger}\mu(a;x_1,\dots,x_n)\ da=+\infty,$$

$$\gamma_i\in L^\infty(0,a_\dagger), \quad \gamma_i\ge 0, \quad \text{a.e. in } [0,a_\dagger], \quad i=1,\dots,n.$$

5 Moreover we suppose that $\beta$ and $\mu$ are locally Lipschitz continuous on $\mathbb{R}^n$ uniformly with respect to age, namely we suppose that there exists a constant $L(R)>0$ such that, if $\|(x_1,\dots,x_n)\|_{\mathbb{R}^n},\|(\overline{x}_1,\dots,\overline{x}_n)\|_{\mathbb{R}^n}\le R$, then

$$|\beta(a;x_1,\dots,x_n)-\beta(a;\overline{x}_1,\dots,\overline{x}_n)|\le L(R)\|(x_1-\overline{x}_1,\dots,x_n-\overline{x}_n)\|_{\mathbb{R}^n},$$

$$|\mu(a;x_1,\dots,x_n)-\mu(a;\overline{x}_1,\dots,\overline{x}_n)|\le L(R)\|(x_1-\overline{x}_1,\dots,x_n-\overline{x}_n)\|_{\mathbb{R}^n}.$$

Concerning existence and uniqueness of a solution to problem (1.2.22)-(1.2.25), we refer to [50] and to [28], while the aim of this section is to investigate the stability of equilibria of the problem (1.2.22)-(1.2.25).

Let $p^*$ be a stationary solution of the problem (1.2.22)-(1.2.25), i.e. the solution of the problem

$$\frac{\partial p^*}{\partial a}(a)+\mu(a;S_1^*,\dots,S_n^*)p^*(a)=0, \quad \text{for } a\in[0,a_\dagger], \tag{1.2.26}$$

$$p^*(0) = \int_0^{a_\dagger} \beta(a; S_1^*, \dots, S_n^*) p^*(a)\ da,$$

$$S_i^* = \int_0^{a_\dagger} \gamma_i(a) p^*(a)\ da, \quad i = 1, \dots, n,$$

then, by integrating equation (1.2.26) we have

$$p^*(a) = p^*(0)\Pi(a; S_1^*, \dots, S_n^*), \tag{1.2.27}$$

where

$$\Pi(a; S_1, \dots, S_n) = \exp\left(-\int_0^a \mu(\sigma; S_1, \dots, S_n) d\sigma\right),$$

represents the survival probability when the sizes have fixed values $S_1, \dots, S_n$.

We linearize the problem (1.2.22)-(1.2.25) at $p^*(a)$, obtaining the following problem for the function $u(t,a) = p(t,a) - p^*(a)$,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu(a; S^*)u + D_1\mu(a; S^*)p^*(a)U_1(t) + \cdots + D_n\mu(a; S^*)p^*(a)U_n(t) = 0, \tag{1.2.28}$$

$$\text{for } (t,a) \in [0, +\infty) \times [0, a_\dagger],$$

$$u(0,a) = u_0(a) = p_0(a) - p^*(a), \qquad \text{for } a \in [0, a_\dagger], \tag{1.2.29}$$

$$\begin{aligned} u(t,0) &= \int_0^{a_\dagger} \beta(a; S^*)u(t,a)\ da + \\ &+ U_1(t)\int_0^{a_\dagger} D_1\beta(a; S^*)p^*(a)\ da + \cdots + \\ &+ U_n(t)\int_0^{a_\dagger} D_n\beta(a; S^*)p^*(a)\ da, \qquad \text{for } t \geq 0, \end{aligned} \tag{1.2.30}$$

$$U_i(t) = \int_0^{a_\dagger} \gamma_i(a)u(t,a)\ da, \quad i = 1, \dots, n,$$

where we denoted $S^* = (S_1^*, \dots, S_n^*)$ and $D_j f(a; x_1, \dots, x_n) = \dfrac{\partial f}{\partial x_j}$. By integrating equation (1.2.28) along characteristics and using conditions (1.2.29) and (1.2.30), we obtain that $u(t,a)$ has the for

$$u(t,a) = \begin{cases} \dfrac{\Pi(a; S^*)}{\Pi(a-t; S^*)} u_0(a-t) + \\ \qquad - \displaystyle\int_0^t p^*(0)\Pi(a; S^*) \sum_{j=1}^n D_j\mu(a-\sigma; S^*)U_j(t-\sigma)\ d\sigma, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } a \geq t, \\ b(t-a)\Pi(a; S^*) + \\ \qquad - \displaystyle\int_0^a p^*(0)\Pi(a; S^*) \sum_{j=1}^n D_j\mu(a-\sigma; S^*)U_j(t-\sigma)\ d\sigma, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } a < t, \end{cases} \tag{1.2.31}$$

where $b(s) = u(s,0)$.

Now we are interested in finding equations for $b(t), U_1(t), \dots, U_n(t)$. Using the definition of $b(t)$ and (1.2.27), we obtain

$$\begin{aligned} b(t) &= \int_0^{a_\dagger} \beta(a;S^*)u(t,a)\ da+ \\ &\quad +p^*(0)\sum_{j=1}^{n} U_j(t)\left(\int_0^{a_\dagger} D_j\beta(a;S^*)\Pi(a;S^*)\ da\right) \\ &= \int_0^{a_\dagger} \beta(a;S^*)u(t,a)\ da + \sum_{j=1}^{n} a_{0j}U_j(t), \end{aligned} \tag{1.2.32}$$

where we have used the notation

$$a_{0j} = p^*(0)\int_0^{a_\dagger} D_j\beta(a;S^*)\Pi(a;S^*)\ da, \quad j=1,\dots,n.$$

Notice that it is possible to consider $a_\dagger = +\infty$, extending the involved functions to 0 on $(a_\dagger, +\infty)$.

Now, using (1.2.31) in the first term of the expression for $b(t)$, we have

$$\begin{aligned} &\int_0^{\infty} \beta(a;S^*)u(t,a)\ da = \int_0^{t} \beta(a;S^*)u(t,a)\ da + \int_t^{\infty} \beta(a;S^*)u(t,a)\ da = \\ &= \int_0^{t} \beta(a;S^*)b(t-a)\Pi(a;S^*)\ da+ \\ &- \int_0^{t} \beta(a;S^*)\int_0^{a} p^*(0)\Pi(a;S^*)\sum_{j=1}^{n} D_j\mu(a-\sigma;S^*)U_j(t-\sigma)\ d\sigma\ da+ \\ &+ \int_t^{\infty} \beta(a;S^*)u_0(a-t)\frac{\Pi(a;S^*)}{\Pi(a-t;S^*)}\ da+ \\ &- \int_t^{\infty} \beta(a;S^*)\int_0^{t} p^*(0)\Pi(a;S^*)\sum_{j=1}^{n} D_j\mu(a-\sigma;S^*)U_j(t-\sigma)\ d\sigma\ da. \end{aligned} \tag{1.2.33}$$

Observe that the third term is independent of $b(t), U_1(t), \dots, U_n(t)$, i.e. it is known, and we denote it by

$$F_0(t) = \int_t^{\infty} \beta(a;S^*)u_0(a-t)\frac{\Pi(a;S^*)}{\Pi(a-t;S^*)}\ da.$$

Moreover we define

$$A_{00}(a) = \beta(a;S^*)\Pi(a;S^*)\chi_{[0,a_\dagger]}(a),$$

where $\chi_{[0,a_\dagger]}$ is the characteristic function of the interval $[0, a_\dagger]$. Thus, from (1.2.33), we have

$$\int_0^\infty \beta(a;S^*)u(t,a)\;da = F_0(t) + (A_{00} * b)(t) +$$

$$-p^*(0)\sum_{j=1}^n \left[\int_0^t \beta(a;S^*)\Pi(a;S^*)\int_0^a D_j\mu(a-\sigma;S^*)U_j(t-\sigma)\;d\sigma\;da + \right. \tag{1.2.34}$$

$$\left. + \int_t^\infty \beta(a;S^*)\Pi(a;S^*)\int_0^t D_j\mu(a-\sigma;S^*)U_j(t-\sigma)\;d\sigma\;da\right],$$

where the symbol $*$ denotes the convolution product, i.e.

$$(f*g)(t) = \int_0^t f(t-s)g(s)\;ds.$$

Now,

$$\int_0^t \beta(a;S^*)\Pi(a;S^*)\int_0^a D_j\mu(a-\sigma;S^*)U_j(t-\sigma)\;d\sigma\;da +$$

$$+ \int_t^\infty \beta(a;S^*)\Pi(a;S^*)\int_0^t D_j\mu(a-\sigma;S^*)U_j(t-\sigma)\;d\sigma\;da =$$

$$= \int_0^t U_j(t-\sigma)\int_\sigma^t A_{00}(a)D_j\mu(a-\sigma;S^*)\;da\;d\sigma + \tag{1.2.35}$$

$$+ \int_0^t U_j(t-\sigma)\int_t^\infty A_{00}(a)D_j\mu(a-\sigma;S^*)\;da\;d\sigma =$$

$$= \int_0^t U_j(t-\sigma)\int_0^\infty A_{00}(s+\sigma)D_j\mu(s;S^*)\;ds\;d\sigma.$$

Let us define

$$A_{0j}(a) = -p^*(0)\int_0^\infty D_j\mu(s;S^*)A_{00}(s+a)\;ds,$$

then, from (1.2.32), using (1.2.34) and (1.2.35), we have

$$b(t) = F_0(t) + \sum_{j=1}^n a_{0j}U_j(t) + (A_{00}*b)(t) + \sum_{j=1}^n (A_{0j}*U_j)(t).$$

The calculations for finding equations for $U_i,\ i = 1,\dots,n$, are similar, with $\gamma_i(a)$ instead of $\beta(a)$, thus, defining the following functions:

$$F_i(t) = \int_t^\infty \gamma_i(a)u_0(a-t)\frac{\Pi(a;S^*)}{\Pi(a-t;S^*)}\;da,$$

$$A_{i0}(a) = \gamma_i(a)\Pi(a;S^*)\chi_{[0,a_\dagger]}(a),$$

$$A_{ij}(a) = -p^*(0) \int_0^{a_\dagger} D_j\mu(s; S^*) A_{i0}(s+a)\, ds,$$

we conclude that $b(t), U_1(t), \ldots, U_n(t)$ solve the following system of Volterra integral equations

$$\begin{cases} b(t) = F_0(t) + \sum_{j=1}^{n} a_{0j} U_j(t) + (A_{00} * b)(t) + \sum_{j=1}^{n} (A_{0j} * U_j)(t), \\ U_i(t) = F_i(t) + (A_{i0} * b)(t) + \sum_{j=1}^{n} (A_{ij} * U_j)(t), \qquad i = 1, \ldots, n. \end{cases}$$

Thus we conclude that the stability of the equilibrium $(p^*, S_1^*, \ldots, S_n^*)$ is related to the roots of the following characteristic equation

$$\det\left(\delta_{ij} - a_{ij} - \widehat{A}_{ij}(\lambda)\right) = 0,$$

where we took $a_{00} = a_{ij} = 0$ for $i \neq 0$ and where $\widehat{f}$ denotes the Laplace transform of a function $f$.

### 1.2.5 Abstract formulation of the linear age-structured model

In this section we analyze another approach to the problem (1.2.3)-(1.2.5), consisting in formulate it as an abstract Cauchy problem in the Banach space $X = L^1(0, a_\dagger)$.

First of all observe that, being $p(t, \cdot)$, solution of (1.2.3)-(1.2.5), the density of a population, it has to be non-negative, thus we shall work in the positive cone $X_+$ of $X$.

Let us consider the linear operator $A : D(A) \subseteq X \to X$ defined by

$$(Af)(a) = -f'(a) - \mu(a) f(a), \tag{1.2.36}$$

with domain

$$D(A) = \left\{ f \in AC[0, a_\dagger] : \mu f \in L^1(0, a_\dagger), f(0) = \int_0^{a_\dagger} \beta(a) f(a)\, da \right\}. \tag{1.2.37}$$

Thus the problem (1.2.3)-(1.2.5) can be rewritten as the following abstract Cauchy problem

$$p'(t) = (Ap)(t), \quad \text{for } t \geq 0, \tag{1.2.38}$$

$$p(0) = p_0. \tag{1.2.39}$$

Concerning the operator $A$, one can prove the following

**Lemma 1.2.1** *The operator $A$ defined in* (1.2.36), *with domain $D(A)$ defined in* (1.2.37), *is closed and $D(A)$ is dense in $L^1(0, a_\dagger)$.*

Moreover

**Lemma 1.2.2** *The spectrum $\sigma(A)$ of the operator $A$ defined in* (1.2.36)-(1.2.37) *consists of eigenvalues, whose are the solutions of the Lotka equation*

$$\widehat{K}(\lambda) = 1,$$

*which is actually* (1.2.16).

Finally

**Lemma 1.2.3** *The resolvent* $R(\lambda, A) = (\lambda I - A)^{-1}$ *of the operator* $A$ *satisfies the following estimate*

$$\|R(\lambda, A)f\|_{L^1(0,a_\dagger)} \leq \frac{1}{\mathcal{R}e\lambda - \|\beta\|_{L^\infty(0,a_\dagger)}} \|f\|_{L^1(0,a_\dagger)}, \qquad \textit{for } \lambda \in \mathbb{C} \setminus \sigma(A).$$

Thus we can conclude (see e.g. [17], Corollary II.3.6) that the operator $A$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. As a consequence, by Theorem II.6.7 in [17], the Cauchy problem (1.2.38)-(1.2.39) is well-posed. Thus we conclude, using the fact that problem (1.2.38)-(1.2.39) is equivalent to problem (1.2.3)-(1.2.5), that the semigroup $(T(t))_{t\geq 0}$ is given by

$$(T(t)p_0)(a) = \begin{cases} p_0(a-t)\dfrac{\Pi(a)}{\Pi(a-t)} & \text{if } a \geq t, \\ \\ B(t-a)\Pi(a) & \text{if } a < t, \end{cases}$$

for $p_0 \in D(A)$.

# Chapter 2

# Age-structure and diffusion

Whereas in Chapter 1 we separately took into account the diffusion of individuals and the age-structure of the population, this chapter is devoted to the introduction of models combining both these features. We derive the reaction-diffusion problem describing an age-structured population spreading in a bounded region of $\mathbb{R}^n$ and recall some known results. Finally we concentrate on the linear diffusion age-structured model in the case in which the population spreads with a constant diffusivity, proving existence and uniqueness of the solution to this problem following a procedure introduced by Marcati and Serafini in [40].

## 2.1 Age-structured diffusion models

In this section, we give the derivation of the equation describing the diffusion of an age-structured population (Section 2.1.1) and a list of previous results on this kind of problems (Sections 2.1.2 and 2.1.3).

### 2.1.1 Deriving the diffusion model for an age-structured population

We consider the diffusion of an age-structured population in a bounded region $\Omega \subset \mathbb{R}^n$ (obviously, in biologically meaningful cases it will be $n = 1, 2, 3$). Suppose that the vital rates are functions of age only: $\beta = \beta(a)$ and $\mu = \mu(a)$. Let us denote by $p(t, a, \mathbf{x})$ the density, per unit volume and age, at time $t$ of the population, where $t \geq 0$, $a \in [0, a_\dagger]$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega$. Thus the integral

$$P(t, \mathbf{x}) = \int_0^{a_\dagger} p(t, a, \mathbf{x}) \; da$$

represents the total population density at time $t$ and position $\mathbf{x}$, while the integral

$$\mathcal{P}(t) = \int_\Omega P(t, \mathbf{x}) \; dV = \int_\Omega \int_0^{a_\dagger} p(t, a, \mathbf{x}) \; da \; dV$$

gives the total population in $\Omega$ at time $t$.

Moreover, if we denote by $\mathbf{J}(t, a, \mathbf{x})$ the population vector flux density and by $n(\mathbf{x})$ the inward unit normal to the boundary $\partial\Omega'$ of a subregion $\Omega' \subset \Omega$, we have that the integral

$$\int_{\partial\Omega'} \mathbf{J}(t, a, \mathbf{x}) \cdot n(\mathbf{x}) \; d\sigma$$

gives the number of individuals of age $a$ who cross $\partial\Omega'$ per time unit.

With these definitions, the continuity equation for $p(t,a,\mathbf{x})$ reads

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\int_{\Omega'} p(t,a,\mathbf{x})\ dV=\int_{\partial\Omega'}\mathbf{J}(t,a,\mathbf{x})\cdot n(\mathbf{x})\ d\sigma-\int_{\Omega'}\mu(a)p(t,a,\mathbf{x})\ dV, \tag{2.1.1}$$

for every measurable subset $\Omega'\subset\Omega$. Thus, using divergence theorem, we get

$$\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}=-\mathrm{div}\mathbf{J}-\mu(a)p.$$

As we have seen in Section 1.1, the flux is proportional to the gradient of $p$, i.e.

$$\mathbf{J}(t,a,\mathbf{x})=-D\nabla p(t,a,\mathbf{x}),$$

where $\nabla$ represents the gradient taken with respect to the space variable $\mathbf{x}$ and $D>0$ represents the diffusivity of individuals. In general, $D$ is a function of time and age, but, for the moment, we suppose it to be constant, actually, in [23], Gurtin suggested that the flux should be proportional to the gradient of the total population density $P(t,\mathbf{x})$).
We end up with the following equation for $p$

$$\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}=D\Delta p-\mu(a)p, \qquad \text{in } (0,+\infty)\times(0,a_\dagger)\times\Omega, \tag{2.1.2}$$

to which we add the initial condition

$$p(0,a,\mathbf{x})=p_0(a,\mathbf{x}), \qquad \text{for } (a,\mathbf{x})\in(0,a_\dagger)\times\Omega, \tag{2.1.3}$$

where $p_0$ is supposed to be given, and the usual condition at age 0

$$p(t,0,\mathbf{x})=\int_0^{a_\dagger}\beta(a)p(t,a,\mathbf{x})\ da, \qquad \text{for } t>0,\ \mathbf{x}\in\Omega. \tag{2.1.4}$$

Concerning the boundary conditions we can take either homogeneous Dirichlet boundary conditions

$$p(t,a,\mathbf{x})=0, \qquad \text{for } t>0,\ a\in(0,a_\dagger),\ \mathbf{x}\in\partial\Omega, \tag{2.1.5}$$

or homogeneous Neumann boundary conditions

$$\frac{\partial p}{\partial n}(t,a,\mathbf{x})=0, \qquad \text{for } t>0,\ a\in(0,a_\dagger),\ \mathbf{x}\in\partial\Omega. \tag{2.1.6}$$

More complicated models can be obtained by taking a more general form for the flux $\mathbf{J}(t,a,\mathbf{x})$, namely, if $J_j$ denotes the flux per unit age in the $j$-th direction across a unit area at $\mathbf{x}$ in a unit time ($j=\ \dots,n$), then $J_j$ has the general form

$$J_j(t,a,\mathbf{x})=\nu_j(t,a,\mathbf{x})p(t,a,\mathbf{x})-\kappa_j(t,a,\mathbf{x})\frac{\partial p}{\partial x_j},$$

where $\nu_j$ and $\kappa_j$ are, respectively, the advection velocity and diffusivity for the age density function. In this case we end up with the following equation for $p$

$$\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}=-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(\nu_j p)+\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\kappa_j\frac{\partial p}{\partial x_j}\right)-\mu(a)p.$$

### 2.1.2 Previous results on linear models

The model (2.1.2)-(2.1.4) with either Dirichlet condition (2.1.5) or Neumann condition (2.1.6) has been investigate by several authors; in this section we recall some known results.

Some investigations concerning the one-dimensional case are due to Gopalsamy (see e.g. [21] and [22]), who obtained the solution of problem (2.1.2)-(2.1.5) via Fourier transforms in the particular case in which age-specific fertility $\beta$ and age-specific mortality $\mu$ are positive constants and $a_\dagger = +\infty$, namely he analyzed the problem

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = D\frac{\partial^2 p}{\partial x^2} - \mu p, \qquad \text{in } (0,+\infty)\times(0,+\infty)\times\mathbb{R}$$

$$p(0,a,x) = p_0(a,x), \qquad \text{for } a \in (0,+\infty),\ x \in \mathbb{R},$$

$$p(t,0,x) = \beta\int_0^{a_\dagger} p(t,a,x)\,da, \qquad \text{for } t \in (0,+\infty),\ x \in \mathbb{R},$$

$$p(t,a,\pm\infty) = 0, \qquad \text{for } t \in (0,+\infty),\ a \in (0,+\infty).$$

A higher-dimensional problem was investigated by Chan and Guo in [9], namely they analyzed the problem (2.1.2)-(2.1.4) with the homogeneous Dirichlet boundary condition (2.1.5) using a semigroup approach in $X = L^2((0,a_\dagger)\times\Omega)$, in particular they considered the operator $\mathbb{A} : D(\mathbb{A}) \subset X \to X$ defined as

$$(\mathbb{A}\phi)(a,\mathbf{x}) = -\frac{\partial\phi}{\partial a}(a,\mathbf{x}) - \mu(a)\phi(a,\mathbf{x}) + D\Delta\phi(a,\mathbf{x}),$$

with domain

$$D(\mathbb{A}) = \left\{\phi \in X : \phi\mid_{\partial\Omega} = 0,\ \phi(0,\mathbf{x}) = \int_0^{a_\dagger}\beta(a)\phi(a,\mathbf{x})da,\ \mathbb{A}\phi \in X\right\}.$$

Using the expansion with respect to the eigenfunctions of the Laplace operator in $L^2$ they showed that $\mathbb{A}$ generates a strongly continuous semigroup and investigated its spectral properties in order to study the asymptotic behaviour of the solution of the problem.

In [27] Huyer investigated a linear model with age-dependent diffusion in a bounded domain $\Omega \subset \mathbb{R}^n$ with homogeneous Neumann boundary conditions, namely

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = D(a)\Delta p - \mu(a)p, \qquad \text{in } (0,+\infty)\times(0,+\infty)\times\Omega, \tag{2.1.7}$$

$$p(0,a,\mathbf{x}) = p_0(a,\mathbf{x}), \qquad \text{for } a \in (0,+\infty),\ \mathbf{x} \in \Omega, \tag{2.1.8}$$

$$p(t,0,\mathbf{x}) = \int_0^{\infty}\beta(a)p(t,a,\mathbf{x})\,da, \qquad \text{for } t \in (0,+\infty),\ \mathbf{x} \in \Omega, \tag{2.1.9}$$

$$\frac{\partial p}{\partial \nu}(t,a,\mathbf{x}) = 0, \qquad \text{for } t \in (0,+\infty),\ a \in (0,+\infty),\ \mathbf{x} \in \partial\Omega, \tag{2.1.10}$$

where $\partial/\partial\nu$ denotes the exterior normal at the boundary $\partial\Omega$ of $\Omega$.
In that paper the well-posedness of the problem (2.1.7)-(2.1.10) is proved by using semigroup theory in the Hilbert space $H = L^2((0,+\infty)\times\Omega;\mathbb{R})$. In particular Huyer defined the following operators on $H$: the population operator $A : D(A) \in H \to H$

$$(A\phi)(a,\mathbf{x}) = -\frac{\partial\phi}{\partial a}(a,\mathbf{x}) - \mu(a)\phi(a,\mathbf{x}),$$

with domain

$$\begin{aligned} D(A) &= \Big\{\phi \in H : \phi(\cdot,\mathbf{x}) \text{ is locally absolutely continuous on } (0,+\infty), \\ &\qquad \phi(0,\mathbf{x}) = \int_0^\infty \beta(a)\phi(a,\mathbf{x})da,\ A\phi \in H\Big\}, \end{aligned}$$

the diffusion operator $B : D(B) \in H \to H$

$$(B\phi)(a,\mathbf{x}) = D(a)\Delta\phi(a,\mathbf{x}),$$

with domain

$$\begin{aligned} D(B) &= \Big\{\phi \in H : \phi(a,\cdot) \in H^2(\Omega,\mathbb{R}), \\ &\qquad \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \partial\Omega \text{ for almost all } a \geq 0, \Delta\phi \in H\Big\}, \end{aligned}$$

and proved that the operator $\mathbb{A} = A + B$ with domain $D(\mathbb{A}) = D(A)\cap D(B)$ is the infinitesimal generator of a strongly continuous semigroup on $H$.

### 2.1.3 Some nonlinear models with age and diffusion

In the previous sections we considered only linear problems, i.e. we assumed that the vital rates $\beta$ and $\mu$ are function of age only, now we recall some known results about nonlinear models.

A first step in building up nonlinear models is to allow fertility and mortality to depend on the total population density at time $t$ and position $\mathbf{x}$

$$P(t,\mathbf{x}) = \int_0^{a_\dagger} p(t,a,\mathbf{x})\ da, \quad t \geq 0,\ \mathbf{x} \in \Omega.$$

In [15] Di Blasio considered the model (2.1.2)-(2.1.4) with the homogeneous Neumann boundary condition (2.1.6) with $a_\dagger = +\infty$, constant diffusion and with $\beta = \beta(a, P(t,\mathbf{x}))$, $\mu = \mu(a, P(t,\mathbf{x}))$, proved that the problem admits a unique non-negative solution, provided that the initial distribution $p_0$ is non-negative, and showed that this solution is continuously dependent on the initial data. Moreover she established some estimates of the growth rate of $p$ and $P$.

The same problem has been investigated by Langlais in [34], in particular in that paper it has been proved that when $t$ goes to infinity either the solution of the problem goes to zero or stabilizes to a nontrivial stationary solution.

In [8] Busenberg and Iannelli considered a general class of nonlinear models for the diffusion of an age-dependent population, in that paper they dealt with the following problem

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = b\left(t, x, P, \frac{\partial P}{\partial x}\right)\frac{\partial p}{\partial x} + c\left(t, x, P, \frac{\partial P}{\partial x}, \frac{\partial^2 P}{\partial x^2}\right)p, \qquad \text{in } (0,+\infty)\times(0,+\infty)\times(0,L), \tag{2.1.11}$$

$$p(0,a,x) = p_0(a,x), \qquad \text{for } a \in (0,+\infty),\ x \in (0,L), \tag{2.1.12}$$

$$p(t,0,x) = \int_0^\infty \beta(t,a,x)p(t,a,x)\ da, \qquad \text{for } t \in (0,+\infty),\ x \in (0,L). \tag{2.1.13}$$

Here the diffusion is represented by the term $b\dfrac{\partial p}{\partial x} + cp$, which include operators of the form

$$\frac{\partial}{\partial x}\left(pk(P)\frac{\partial p}{\partial x}\right). \tag{2.1.14}$$

In particular this model include the case in which, in (2.1.14), $k$ is supposed to be constant; this particular case has been investigated also by MacCamy in [39].
Busenberg and Iannelli studied the existence, the uniqueness and the asymptotic behaviour of the solution of the problem (2.1.11)-(2.1.13). The method described by Busenberg and Iannelli consists in separating the age-dependent part of the problem from the diffusion mechanism, and it is useful to treat a broad class of nonlinear (density-dependent) age-structured population problems that involve spatial diffusion (linear and nonlinear). In particular it is useful notice that, although the model has been set up in a one-dimensional environment, the interval $[0,L]$, the same results obtained by Busenberg and Iannelli can be stated also for higher-dimensional cases.

## 2.2 A linear age-structured model with spatial diffusion

Up to now we did not enter into details of the analysis of age-structured diffusion models, in this section we analyze in depth a problem modeling a population spreading with constant diffusion in a bounded open subset of $\mathbb{R}^2$. In particular, after a brief presentation of the model (Section 2.2.1), in Section 2.2.2 we discuss an existence and uniqueness result for a solution to this model due to Marcati and Serafini ([40]).

Finally, in Section 2.2.3 we consider a model describing the diffusion of a population in a one-dimensional environment and we analyze the asymptotic behaviour of the solution to this problem.

### 2.2.1 Statement of the problem

The model we are going to deal with can be derived with the calculations described in Section 2.1.1, ending up with the problem

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = D\Delta p - \mu(a)p, \qquad \text{in } (0,+\infty)\times(0,a_\dagger)\times\Omega, \tag{2.2.1}$$

$$p(0,a,x) = p_0(a,x), \qquad \text{for } a \in (0,a_\dagger),\ x \in \Omega, \tag{2.2.2}$$

$$p(t,0,x) = \int_0^{a_\dagger} \beta(a)p(t,a,x)\ da, \qquad \text{for } t > 0,\ x \in \Omega, \tag{2.2.3}$$

with either homogeneous Dirichlet boundary conditions

$$p(t,a,x) = 0, \qquad \text{for } t > 0,\ a \in (0,a_\dagger),\ x \in \partial\Omega, \tag{2.2.4}$$

or homogeneous Neumann boundary conditions

$$\frac{\partial p}{\partial n}(t,a,x) = 0, \qquad \text{for } t > 0,\ a \in (0,a_\dagger),\ x \in \partial\Omega. \tag{2.2.5}$$

We have used the following notations: $\beta(a)$ and $\mu(a)$ are the age-specific fertility and mortality respectively, $D$ represent the diffusivity and $p_0$ is the initial distribution, which is supposed to be given.

### 2.2.2 Existence and uniqueness

In this section we prove an existence and uniqueness result for the problem (2.2.1)-(2.2.4) (or (2.2.1)-(2.2.3) with (2.2.5)). We give an abstract formulation to the problem following the method used by Marcati and Serafini in dealing with a more general problem (in [40] they considered a diffusion dependent on space, namely they took $\mathbf{J}(t,a,x) = -c(x)\nabla p(t,a,x)$).

Let $X = L^2(\Omega)$, we consider the linear operator $A : D(A) \subset X \to X$ defined by

$$Af = D\Delta f.$$

The choice of the domain $D(A)$ of the operator $A$ depends on the boundary conditions we are dealing with, in particular we take

$$D(A) = \{f \in H_0^1(\Omega) :\ Af \in X\}$$

if we have homogeneous the Dirichlet boundary condition (2.2.4), and

$$D(A) = \left\{f \in H^1(\Omega) :\ Af \in X, \frac{\partial f}{\partial n}\Big|_{\partial\Omega} = 0\right\}$$

if we consider the Neumann boundary condition (2.2.5).

It is well known from semigroup theory that the operator $A$ generates a strongly continuous semigroup $(T(t))_{t\geq 0} = (e^{tA})_{t\geq 0}$ of bounded linear operators on $X$ and that there exists $\omega \in \mathbb{R}$ such that the following estimate holds

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq e^{\omega t}, \quad \forall t \geq 0, \tag{2.2.6}$$

where $\mathcal{L}(X)$ is the Banach space of bounded linear operators from $X$ to $X$ and $\|\cdot\|_{\mathcal{L}(X)}$ denotes the usual norm on $\mathcal{L}(X)$.

First of all, since the solution $p$ has to be non-negative, we restrict our attention to the positive convex cone $X_+ = \{f \in X : f \geq 0\}$ and we observe that $e^{tA}X_+ \subset X_+$ for every $t > 0$.

In order to work with $a \in (0, +\infty)$, we suppose $\beta \equiv 0$ and $\mu \equiv 0$ on $(a_\dagger, +\infty)$. Moreover we assume that

$$\beta \in L^\infty(0, a_\dagger), \quad \beta \geq 0, \quad \beta(0) = 0,$$

$$\mu \in L^1_{loc}(0, a_\dagger), \quad \int_0^a \mu(s)\ ds < +\infty \ \forall a < a_\dagger,$$

$$\int_0^{a_\dagger} \mu(a)\ da = +\infty. \tag{2.2.7}$$

In particular condition (2.2.7) implies that the survival probability,

$$\Pi(a) = e^{-\int_0^a \mu(a)\ da},$$

vanishes at the maximum age $a_\dagger$.

With these premises we can rewrite the problem as the following abstract one

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = Ap - \mu(a)p, \qquad \text{in } (0, +\infty) \times (0, +\infty), \tag{2.2.8}$$

$$p(0, a) = p_0(a), \qquad \text{for } a > 0, \tag{2.2.9}$$

$$p(t, 0) = \int_0^\infty \beta(a)p(t, a)\ da = B(t), \qquad \text{for } t > 0. \tag{2.2.10}$$

Concerning this problem we give the following

**Definition 2.2.1** *We say that $p$ is a solution of the problem* (2.2.8)-(2.2.10) *if and only if* $p \in \mathcal{C}^0((0, +\infty) \times (0, +\infty); X_+ \cap D(A)) \cap \mathcal{C}^1((0, +\infty) \times (0, +\infty); X)$ *and it verifies* (2.2.8)-(2.2.10).

By integrating along characteristics the equation (2.2.8) we have that the solution $p$ to the problem (2.2.8)-(2.2.10) can be written as

$$p(t, a) = \begin{cases} p_0(a - t)e^{tA}\dfrac{\Pi(a)}{\Pi(a - t)} & \text{if } a \geq t, \\ \\ B(t - a)e^{aA}\Pi(a) & \text{if } a < t, \end{cases} \tag{2.2.11}$$

where $B(t)$ is defined in (2.2.10) and it is the solution of the renewal equation

$$B(t) = f(t) + \int_0^t k(t - s)B(s)\ ds, \tag{2.2.12}$$

where

$$f(t) = \int_t^\infty \beta(a)\frac{\Pi(a)}{\Pi(a - t)}p_0(a - t)e^{tA}\ da, \tag{2.2.13}$$

and

$$k(t) = e^{tA}\beta(t)\Pi(t) = K(t)e^{tA}, \tag{2.2.14}$$

where $K$ is the maternity function $K(t) = \beta(t)\Pi(t)$.

**Definition 2.2.2** *We say that a solution $B$ of the renewal equation* (2.2.12) *is a regular solution if and only if* $B \in \mathcal{C}^0(0,+\infty; X_+ \cap D(A)) \cap \mathcal{C}^1(0,+\infty; X)$.

In order to get an existence and uniqueness result for the solution $p$ to the problem (2.2.8)-(2.2.10), it is enough to prove that the renewal equation (2.2.12) has one and only one solution $B$, the solutions $p$ and $B$ being related via formula (2.2.11).

**Theorem 2.2.1** *Assume that* $p_0 \in \mathcal{C}^0_c(0,+\infty; X_+ \cap D(A)) \cap \mathcal{C}^1(0,+\infty; X)$ *satisfies*

$$p_0(a) = 0 \ \forall a \geq a_\dagger,$$

$$p_0(0) = \int_0^{a_\dagger} \beta(a) p_0(a) \, da,$$

*then there exists one and only one regular solution of the equation* (2.2.12) *with* (2.2.13) *and* (2.2.14).

**Proof.**
We first prove that the renewal equation (2.2.12) has a solution on $[0,\tau]$ for every fixed $\tau > 0$ by reducing it to the following fixed point problem

$$B(t) = (\mathcal{S}B)(t), \qquad \text{for } t \in (0,\tau),$$

with $\mathcal{S}$ defined on $\mathcal{C}^0(0,\tau; X_+)$ as

$$(\mathcal{S}v)(t) = f(t) + \int_0^t k(t-s) v(s) \, ds, \quad t \in (0,\tau), \quad v \in \mathcal{C}^0(0,\tau; X_+).$$

First of all notice that $\mathcal{S}$ maps $\mathcal{C}^0(0,\tau; X_+)$ into itself, moreover, if we denote by $\mathcal{S}^{(n)}$ the $n$-th iterate of $\mathcal{S}$, we have that

$$\|\mathcal{S}^{(n)} v - \mathcal{S}^{(n)} z\|_{\mathcal{C}^0(0,\tau;X)} \leq \kappa^n \frac{\tau^n}{n!} \|v - z\|_{\mathcal{C}^0(0,\tau;X)},$$

for every $v, z \in \mathcal{C}^0(0,\tau; X_+)$, with $\kappa > 0$ constant. Thus we conclude that there exists a unique solution $B \in \mathcal{C}^0(0,\tau; X_+)$ to the renewal equation.

It remains to show that the solution $B$ is actually a regular solution, according to Definition 2.2.2. First of all let us prove that $B(t) \in D(A)$ for every $t \geq 0$.

For every $n > \omega$ let as consider

$$B_n(t) = J_n B(t), \qquad f_n(t) = J_n f(t),$$

where $J_n$ is the Yosida approximation, i.e. $J_n = nR(n,A)$, where we denote by $R(n,A) = (nI - A)^{-1}$ the resolvent operator of $A$. Then we know that $(B_n), (f_n) \subset \mathcal{C}^0(0,+\infty; X_+ \cap D(A))$ and

$$B_n \xrightarrow[n\to\infty]{} B \quad \text{in } \mathcal{C}^0(0,+\infty; X),$$

while, since $f \in \mathcal{C}_c^0(0,+\infty;X_+ \cap D(A)) \cap \mathcal{C}^1(0,+\infty;X)$,

$$f_n \xrightarrow[n\to\infty]{} f \quad \text{in } \mathcal{C}^0(0,+\infty;X_+ \cap D(A)).$$

In order to prove that $B(t) \in D(A)$ it is enough to prove that $(AB_n)$ converges in $\mathcal{C}^0(0,+\infty;X)$ and then use the closedness of the operator $A$, in particular we shall show that $(AB_n)$ is a Cauchy sequence in $\mathcal{C}^0(0,+\infty;X)$.

Let $n,m > \omega$ and $t \geq 0$ and define $\beta_+ = \sup_{a\geq 0}\beta(a)$, $\mu_- = \inf_{a\geq 0}\mu(a)$ and $\alpha = \beta_+ + \mu_-$, then we have

$$\begin{aligned} \|AB_n(t) - AB_m(t)\|_X &\leq \|Af_n(t) - Af_m(t)\|_X + \\ &+\beta_+ e^{\alpha t}\int_0^t e^{-\alpha s}\|AB_n(s) - AB_m(s)\|_X \; ds, \end{aligned} \tag{2.2.15}$$

where we have used the fact that $B(t)$ solves the renewal equation (2.2.12) and the estimate (2.2.6).

From (2.2.15) we have

$$\begin{aligned} e^{-\alpha t}\|AB_n(t) - AB_m(t)\|_X &\leq \sup_{t\geq 0}\|Af_n(t) - Af_m(t)\|_X + \\ &+\beta_+ \int_0^t e^{-\alpha s}\|AB_n(s) - AB_m(s)\|_X \; ds, \end{aligned}$$

and then, using Gronwall inequality,

$$\|AB_n(t) - AB_m(t)\|_X \leq e^{(\beta_+ + \alpha)t}\sup_{t\geq 0}\|Af_n(t) - Af_m(t)\|_X .$$

Thus, since $(Af_n)$ is a Cauchy sequence in $\mathcal{C}^0(0,+\infty;X)$, so it is $(AB_n)$, as claimed.

Finally we have that $B \in \mathcal{C}^1(0,+\infty;X)$, indeed

$$\frac{d}{dt}B(t) = \frac{d}{dt}f(t) + \int_0^t k'(t-s)B(s)\; ds,$$

where

$$k'(t)v = \beta'(t)\Pi(t)e^{tA}v + \beta(t)\Pi(t)e^{tA}(A - \mu(t)I)v, \quad \forall v \in X.$$

□

We refer to [40] for the analysis of the asymptotic behaviour of the solution of the age-structured model with diffusion in a bounded region of $\mathbb{R}^2$, while in the next section we study the behaviour of the solution in the one-dimensional case.

### 2.2.3 Asymptotic behaviour in the one-dimensional case

We now consider an age-structured population diffusing in a closed interval $[0, L]$. As before, we denote by $p(t, a, x)$, with $t \geq 0$, $a \in [0, a_\dagger]$ and $x \in [0, L]$, the density per unit surface and age at time $t$ of the population.

The equation for $p$ now reads

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu(a)p(t,a,x) = D\frac{\partial^2 p}{\partial x^2} \quad \text{in } (0,+\infty) \times (0, a_\dagger) \times (0, L), \tag{2.2.16}$$

where $D$ represents the diffusivity and it is supposed to be constant.

Moreover we have the initial condition

$$p(0,a,x) = p_0(a,x), \qquad \text{for } a \in (0, a_\dagger),\ x \in (0, L), \tag{2.2.17}$$

and the condition at age 0

$$p(t,0,x) = \int_0^{a_\dagger} \beta(a)p(t,a,x)\ da, \qquad \text{for } t > 0,\ x \in (0, L). \tag{2.2.18}$$

Concerning the boundary conditions, we focus on homogeneous Neumann boundary conditions, namely

$$\frac{\partial p}{\partial x}(t,a,0) = \frac{\partial p}{\partial x}(t,a,L) = 0, \qquad \text{for } t > 0,\ a \in (0, a_\dagger), \tag{2.2.19}$$

i.e. we assume that the individuals do not cross the boundary of the environment.

We look for the solution of problem (2.2.16)-(2.2.19) using separation of variables, namely we look for $p(t, a, x)$ of the form

$$p(t,a,x) = P(t,a)X(x),$$

thus equation (2.2.16) becomes

$$\frac{P_t}{P} + \frac{P_a}{P} + \mu(a) = D\frac{X_{xx}}{X}, \tag{2.2.20}$$

where the subscripts denote partial derivatives, i.e. $P_t = \dfrac{\partial P}{\partial t}$, $P_a = \dfrac{\partial P}{\partial a}$ and $X_{xx} = \dfrac{\partial^2 X}{\partial x^2}$.

Since the left hand side of (2.2.20) is constant with respect to $x$, we are led to the following eigenvalues problem

$$X_{xx} = \lambda X, \tag{2.2.21}$$

with

$$X_x(0) = X_x(L) = 0. \tag{2.2.22}$$

It is well known that, for every $n \geq 0$ we have that

$$X_n(x) = c_n \cos\left(\frac{n\pi}{L}x\right), \tag{2.2.23}$$

with $c_n$ constant with respect to $x$, solves (2.2.21)-(2.2.22) with $\lambda = \lambda_n = -\dfrac{n^2\pi^2}{L^2}$.

We conclude that the solution $p$ to problem (2.2.16)- (2.2.19) can be written as

$$p(t,a,x) = \sum_{n=0}^{\infty} c_n P_n(t,a) \cos\left(\frac{n\pi}{L}x\right),$$

where, for every $n \geq 0$, $P_n$ solves

$$\frac{\partial P_n}{\partial t} + \frac{\partial P_n}{\partial a} + \mu(a)P_n - D\lambda_n P_n = 0, \quad \text{in } (0,+\infty)\times(0,a_\dagger), \tag{2.2.24}$$

with the initial condition

$$P_n(0,a) = c_n \int_0^L p_0(a,x)\cos\left(\frac{n\pi}{L}x\right)\, dx = P_n^0(a), \quad \text{for } a \in (0,a_\dagger), \tag{2.2.25}$$

and the additional condition at age 0

$$P_n(t,0) = \int_0^{a_\dagger} \beta(a)P_n(t,a)\, da, \quad \text{for } t \in (0,+\infty). \tag{2.2.26}$$

Notice that the problem (2.2.24)-(2.2.26) is actually the classical linear Lotka-McKendrick model (see Section 1.2.1) with age-specific mortality $\mu_n(a) = \mu(a) - D\lambda_n$, namely equation (2.2.24) can be written as

$$\frac{\partial P_n}{\partial t} + \frac{\partial P_n}{\partial a} + \mu_n(a)P_n = 0, \quad \text{in } (0,+\infty)\times(0,a_\dagger). \tag{2.2.27}$$

We know, from Section 1.2.2, that the solution to equation (2.2.27) with conditions (2.2.25) and (2.2.26) is given by

$$P_n(t,a) = \begin{cases} P_n^0(a-t)\dfrac{\Pi_n(a)}{\Pi_n(a-t)} & \text{if } a \geq t, \\[2ex] B_n(t-a)\Pi_n(a) & \text{if } a < t, \end{cases}$$

where $\Pi_n(a) = e^{-\int_0^a \mu_n(s)\, ds}$ is the survival probability corresponding to the mortality $\mu_n(a)$, and $B_n(t) = \int_0^{a_\dagger} \beta(a)P_n(t,a)da$ and it is the solution of the renewal equation

$$B_n(t) = F_n(t) + \int_0^t K_n(t-s)B_n(s)\, ds,$$

where

$$F_n(t) = \int_0^{\infty} \beta(a+t)\frac{\Pi_n(a+t)}{\Pi_n(a)}P_n^0(a)\, da,$$

and

$$K_n(t) = \beta(t)\Pi_n(t).$$

Thus, for $t$ large, the solution $p$ to (2.2.16)-(2.2.19) can be written as the sum of infinite modes

$$p(t,a,x) = \sum_{n=0}^{\infty} c_n B_n(t-a)\Pi_n(a)\cos\left(\frac{n\pi}{L}x\right). \tag{2.2.28}$$

In order to study the asymptotic behaviour of $p$ it is enough to analyze the behaviour of the modes.

For every fixed $n \geq 0$, the asymptotic behaviour of the $n$-th mode is related to the behaviour of $B_n$ and then, as we have seen in section 1.2.3, to the roots of the characteristic equation

$$\widehat{K}_n(\alpha_n) = 1. \tag{2.2.29}$$

From Theorem 1.2.1 we have that the equation (2.2.29) has a unique real solution $\alpha_n^*$ and, applying Theorem 1.2.2, $B_n(t)$ can be written as

$$B_n(t) = b_n \exp(\alpha_n^* t)(1 + \Omega_n(t)), \tag{2.2.30}$$

where $b_n \geq 0$ is constant with respect to $t$ and $\lim_{t \to +\infty} \Omega_n(t) = 0$.

Now we want to relate the roots of the characteristic equation (2.2.29) to the solutions of the characteristic equation

$$\widehat{K}(\alpha) = 1, \tag{2.2.31}$$

where $K$ is the maternity function $K(t) = \beta(t)\Pi(t)$, with $\Pi(t) = e^{-\int_0^t \mu(s)ds}$.
Observe that

$$K_n(t) = \beta(t)\Pi(t)e^{D\lambda_n t} = e^{D\lambda_n t} K(t),$$

and then

$$\widehat{K}_n(\alpha_n) = \int_0^{a_\dagger} e^{-\alpha_n t} K_n(t)\ dt = \int_0^{a_\dagger} e^{-(\alpha_n - D\lambda_n)t} K(t)\ dt = \widehat{K}(\alpha_n - D\lambda_n).$$

Thus every solution $\alpha_n$ of (2.2.29) can be written as

$$\alpha_n = \alpha + D\lambda_n,$$

where $\alpha$ is a root of the characteristic equation (2.2.31).

In particular, the unique real solution $\alpha_n^*$ can be written as

$$\alpha_n^* = \alpha^* + D\lambda_n, \tag{2.2.32}$$

where $\alpha^*$ is the unique real solution of equation (2.2.31), which exists by Theorem 1.2.1.

Thus, from (2.2.28), using (2.2.30) and (2.2.32), we conclude that, for $t$ large, the solution $p$ to problem (2.2.16)-(2.2.19) has the form

$$\begin{aligned} p(t,a,x) &= \sum_{n=0}^{\infty} c_n b_n \exp(\alpha_n^*(t-a))(1+\Omega_n(t-a))\Pi_n(a)\cos\left(\frac{n\pi}{L}x\right) = \\ &= \Pi(a)\exp(-\alpha^* a)\sum_{n=0}^{\infty} c_n b_n \exp\left((\alpha^* + D\lambda_n)t\right)\left(1+\Omega_n(t-a)\right)\cos\left(\frac{n\pi}{L}x\right). \end{aligned}$$

# Chapter 3

# Age-structure, diffusion and multi-layers

In this chapter we prove an existence and uniqueness result for a nonlinear age-structure model, by reducing it to an abstract Cauchy problem and by using the theory of m-accretive operators. Moreover we prove how this problem, in the one-dimensional case, can be approximated by a diffusion age-structured problem set up in a multi-layer environment. Finally we consider the two-layer linear problem and we find an analytical expression of its solution.
These results are contained in three papers written in collaboration with Mimmo Iannelli and Gabriela Marinoschi, [11], [12], [13].

## 3.1 A nonlinear age-structured model with spatial diffusion

In this section we consider a nonlinear model for the diffusion of an age-structured population spreading in a region $\Omega \subset \mathbb{R}^n$ (obviously, the biologically meaningful cases are $n = 1, 2, 3$). In Section 3.1.1 we present the model and the main hypotheses on the vital rates and on the diffusion coefficient. In Section 3.1.2 an abstract formulation of the problem is given, according to the procedure presented in [13]. In particular, we shall prove that the solution in the sense of distributions of the age-diffusion model is actually the solution of an abstract Cauchy problem associated to a nonlinear operator $A$. The proof of an existence and uniqueness result (Section 3.1.4) is based on the quasi m-accretiveness of the operator $A$, which is proved in Section 3.1.3.

### 3.1.1 Model formulation

We consider an age-structured population diffusing in a domain $\Omega \subset \mathbb{R}^n$ $(n = 1, 2, 3)$, and we denote by $p(t, a, x)$ the density per unit space and age of the population at time $t$. We suppose that the age-specific fertility $\beta(a, x, S(x, t))$ and the age-specific mortality $\mu(a, x, S(x, t))$ depend upon a size $S(t, x)$, defined as

$$S(t, x) = \int_0^{a_\dagger} \int_\Omega \gamma(a, x, z) p(t, a, z)\ dz da.$$

Moreover we suppose that the population flux density is given by

$$\mathbf{J} = K(a,x)\nabla p,$$

where $\nabla p$ denotes the gradient of $p$ with respect to the spatial variable $x$.
Thus the problem we deal with is the following

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu(a,x,S(t,x))p - \sum_{i=0}^{n} \frac{\partial}{\partial x_i}\left(K(a,x)\frac{\partial p}{\partial x_i}\right) = f, \quad \text{in } (0,T)\times(0,a_\dagger)\times\Omega, \quad (3.1.1)$$

$$p(0,a,x) = p_0(a,x), \quad \text{in } (0,a_\dagger)\times\Omega, \quad (3.1.2)$$

$$p(t,0,x) = \int_0^{a_\dagger} \beta(a,x,S(t,x))p(t,a,x)\, da, \quad \text{in } (0,T)\times\Omega, \quad (3.1.3)$$

$$K(a,x)\frac{\partial p}{\partial \nu} = 0, \quad \text{on } (0,T)\times(0,a_\dagger)\times\partial\Omega. \quad (3.1.4)$$

Here $\partial/\partial\nu$ denotes the exterior normal at the boundary $\partial\Omega$ of $\Omega$, in particular the homogeneous Neumann boundary condition (3.1.4) amounts to assume that the population does not go through the boundary. Moreover $f$ is a source term representing population supply and it is supposed to be given.

Concerning the mathematical assumptions, we suppose that fertility, mortality and the weight function are nonnegative and that the fertility and the weight function are essentially bounded, in particular we assume that

$$0 \le \beta(a,x,s) \le \beta_+, \quad (3.1.5)$$

$$\mu(a,x,s) \ge 0, \text{ with } \mu(a,x,0) = 0, \quad (3.1.6)$$

$$\gamma \in L^\infty((0,a_\dagger)\times\Omega\times\Omega), \quad 0 \le \gamma(a,x,z) \le \gamma_+. \quad (3.1.7)$$

Moreover we suppose that the functions $\beta(a,x,s)$ and $\mu(a,x,s)$ are measurable and locally Lipschitz continuous on $\mathbb{R}$ in the variable $s$, uniformly with respect to $a$ and $x$, i.e. for every $R > 0$ there exist $L_\beta(R) > 0$ and $L_\mu(R) > 0$ such that, if $|s| < R$ and $|\overline{s}| < R$, then

$$|\beta(a,x,s) - \beta(a,x,\overline{s})| \le L_\beta(R)|s - \overline{s}|, \quad (3.1.8)$$

$$|\mu(a,x,s) - \mu(a,x,\overline{s})| \le L_\mu(R)|s - \overline{s}|. \quad (3.1.9)$$

Concerning diffusion, we impose the standard conditions

$$K \in L^\infty((0,a_\dagger)\times\Omega), \quad 0 < K_0 \le K(a,x) \le K_+ = \|K\|_{L^\infty}. \quad (3.1.10)$$

We have to specify that, in general, the mortality includes an intrinsic term $\mu_0(a)$, namely: $\mu(a,x,S(t,x)) = \mu_0(a) + \mu_1(a,x,S(x))$. This intrinsic mortality has the property that

$$\int_0^{a_\dagger} \mu_0(a)da = +\infty,$$

that amounts to assume that the probability $\pi_0(a) = e^{-\int \mu_0(\sigma)d\sigma}$ of survive at age 0 in the absence of the extra-mortality $\mu_1$ vanishes at the maximum age $a_\dagger$. In this case the problem can be reduced to (3.1.1)-(3.1.4) by performing a change of variables, namely the following functions are defined

$$p(t,a,x) = \widetilde{p}(t,a,x)\Pi_0(a),$$

$$\widetilde{\gamma}(a,x,z) = \gamma(a,x,z)\Pi_0(a),$$

$$\widetilde{S}(t,x) = \int_0^{a_\dagger}\int_\Omega \widetilde{\gamma}(a,x,z)\widetilde{p}(t,a,x)\ dzda,$$

$$\widetilde{\beta}(a,\widetilde{S}(t,x)) = \beta(a,\widetilde{S}(t,x))\Pi_0(a),$$

$$\widetilde{p}_0(t,a,x) = \frac{p_0(t,a,x)}{\Pi_0(a)},$$

$$\widetilde{f}(t,a,x) = \frac{f(t,a,x)}{\Pi_0(a)},$$

$$\widetilde{\mu}_1(a,x,\widetilde{S}(t,x)) = \mu_1(a,x,\widetilde{S}(t,x)).$$

With these definitions the equation for $\widetilde{p}$ reads

$$\frac{\partial \widetilde{p}}{\partial t} + \frac{\partial \widetilde{p}}{\partial a} + \widetilde{\mu}_1(a,x,\widetilde{S}(x))p - \sum_{i=0}^{n}\frac{\partial}{\partial x_i}\left(K(a,x)\frac{\partial \widetilde{p}}{\partial x_i}\right) = \widetilde{f},$$

while the initial and boundary conditions read exactly like (3.1.2)-(3.1.4).

Finally, we have to observe that here we suppose that the vital rates $\beta$ and $\mu$ are functions of $a$, $x$ and of one size only, but the results obtained in the following section can be proved also in the case in which $\beta = \beta(a,x;S_1(t,x),\ldots,S_m(t,x))$ and $\mu = \mu(a,x;S_1(t,x),\ldots,S_m(t,x))$, where $S_1(t,x),\ldots,S_m(t,x)$ are $m$ sizes defined using different weights $\gamma_1(a,x,z),\ldots,\gamma_m(a,x,z)$.

### 3.1.2 Functional setting of the problem

We consider the following variational triplet

$$V = H^1(\Omega) \subset H = L^2(\Omega) \subset V' = H^{-1}(\Omega),$$

we denote by $Q = (0,a_\dagger)\times\Omega$ and define $H_Q = L^2(Q) \equiv L^2(0,a_\dagger;H)$. Consequently we can consider the following triplet

$$L^2(0,a_\dagger;V) \subset H_Q \subset L^2(0,a_\dagger;V').$$

We first define an operator $A_0 : D(A_0) \subset L^2(0,a_\dagger,;V) \to L^2(0,a_\dagger;V')$ by

$$\ll A_0u,\psi \gg = \ll u_a,\psi \gg + \int_Q \left[\mu(a,x,S(x))u\psi + K(a,x)\nabla u\cdot\nabla\psi\right]\ dxda, \quad \forall\psi\in L^2(0,a_\dagger;V), \tag{3.1.11}$$

where $\ll \cdot,\cdot \gg$ indicates the pairing between the space $L^2(0,a_\dagger;V)$ and its dual $L^2(0,a_\dagger;V')$, i.e.

$$\ll \Phi, \psi \gg = \int_0^{a_\dagger} < \Phi(a), \psi(a) >_{V',V} \, da,$$

(where $< \cdot,\cdot >_{V',V}$ is the pairing between $V$ and its dual $V'$) and $S(x)$ is given by

$$S(x) = \int_Q \gamma(a,x,z)u(a,z) \; dzda. \tag{3.1.12}$$

The domain $D(A_0)$ of the operator $A_0$ is

$$\begin{aligned} D(A_0) \quad = \quad \Big\{ & u \in L^2(0,a_\dagger;V), \; u_a \in L^2(0,a_\dagger;V'), \\ & u(0,x) = \int_0^{a_\dagger} \beta(a,x,S(x))u(a,x) \; da \Big\}. \end{aligned} \tag{3.1.13}$$

In the above definition we have used the subscript to indicate the partial derivative, i.e. $u_a = \dfrac{\partial u}{\partial a}$ and we have denoted by $x \cdot y$ the scalar product in $\mathbb{R}^n$.
Then we define the operator $A : D(A) \subset H_Q \to H_Q$ by setting

$$Au = A_0 u, \quad \forall u \in D(A) = \{u \in D(A_0) : \; A_0 u \in H_Q\}. \tag{3.1.14}$$

Thus we are led to the following abstract Cauchy problem

$$\frac{dp}{dt} + Ap = f, \qquad \text{a.e. } t \in (0,T), \tag{3.1.15}$$

$$p(0) = p_0, \tag{3.1.16}$$

for every fixed $T > 0$. At this point of the discussion we do not need to make particular assumptions on the regularity of the function $f$, which will be specified later.
The problem is set up in the space $W^{1,2}(0,T;H_Q) = \Big\{ u \in L^2(0,T;H_Q) \; : \; \dfrac{du}{dt} \in L^2(0,T;H_Q) \Big\}$.
Recall (see e.g. [3]) that any $u \in W^{1,2}(0,T;H_Q)$ is absolutely continuous on $[0,T]$ and $\dfrac{du}{dt}$ exists a.e. on $(0,T)$. Moreover $u$ at $a = 0$ makes sense, since $u \in L^2(0,a_\dagger;V)$ and $u_a \in L^2(0,a_\dagger;V')$ imply that $u \in \mathcal{C}([0,a_\dagger];H)$.

Concerning this abstract Cauchy problem we recall the following definition (see e.g. [3]),

**Definition 3.1.1** *A function $p \in \mathcal{C}^0(0,T;D(A)) \cap \mathcal{C}^1(0,T;H_Q)$ which satisfies* (3.1.15)-(3.1.16) *for all $t \in [0,T]$ is called a strong solution of* (3.1.15)-(3.1.16).

It is well known that, in order to prove existence and uniqueness of a solution to the problem (3.1.1)-(3.1.4), it is enough to show that there exists a unique strong solution of (3.1.15)-(3.1.16).

First of all we consider the functions $F, E : H_Q \to H_Q$ defined by

$$(F(u))(a,x) = \beta(a,x,S(x))u(a,x), \tag{3.1.17}$$

$$(E(u))(a,x) = \mu(a,x,S(x))u(a,x), \tag{3.1.18}$$

and we observe that, under our hypotheses, the functions $F, E$ are locally Lipschitz continuous, namely we have

**Lemma 3.1.1** *Assume that* (3.1.5)-(3.1.9) *are satisfied, then for any* $R > 0$ *there exist* $B(R) > 0$ *and* $M(R) > 0$*, such that, if* $\|u\|_{H_Q} \le R$ *and* $\|\widetilde{u}\|_{H_Q} \le R$*, then*

$$\|F(u) - F(\widetilde{u})\|_{H_Q} \le B(R)\|u - \widetilde{u}\|_{H_Q}, \tag{3.1.19}$$

$$\|E(u) - E(\widetilde{u})\|_{H_Q} \le M(R)\|u - \widetilde{u}\|_{H_Q}. \tag{3.1.20}$$

**Proof.**
Let $u, \widetilde{u} \in H_Q$ be such that $\|u\|_{H_Q} \le R$ and $\|\widetilde{u}\|_{H_Q} \le R$. Let $S(x) = \displaystyle\int_Q \gamma(a,x,z)u(a,z)dzda$ and $\overline{S}(x) = \displaystyle\int_Q \gamma(a,x,z)\widetilde{u}(a,z)dzda$. First of all observe that

$$\begin{aligned} |S(x) - \overline{S}(x)| &= \left| \int_Q \gamma(a,x,z)[u(a,z) - \widetilde{u}(a,z)]\ dzda \right| \le \\ &\le \gamma_+ \int_Q |u(a,z) - \widetilde{u}(a,z)|\ dzda \le \\ &\le \gamma_+ \sqrt{a_\dagger}\sqrt{|\Omega|}\|u - \widetilde{u}\|_{H_Q}, \end{aligned} \tag{3.1.21}$$

where $\gamma_+$ has been defined in (3.1.7) and $|\Omega|$ denotes the measure of $\Omega$. With the same argument in can be proved that

$$|S(x)| \le \gamma_+\sqrt{a_\dagger}\sqrt{|\Omega|}\|u\|_{H_Q} \le \gamma_+\sqrt{a_\dagger}\sqrt{|\Omega|}R,$$

and

$$|\overline{S}(x)| \le \gamma_+\sqrt{a_\dagger}\sqrt{|\Omega|}\|\widetilde{u}\|_{H_Q} \le \gamma_+\sqrt{a_\dagger}\sqrt{|\Omega|}R.$$

Thus, being $\beta$ locally Lipschitz continuous, we have

$$\begin{aligned} |\beta(a,x,S(x)) - \beta(a,x,\overline{S}(x))| &\le L_\beta(\gamma_+\sqrt{a_\dagger}\sqrt{|\Omega|}R)|S(x) - \overline{S}(x)| \\ &\le \gamma_+\sqrt{a_\dagger}\sqrt{|\Omega|}L_\beta(\gamma_+\sqrt{a_\dagger}\sqrt{|\Omega|}R)\|u - \widetilde{u}\|_{H_Q} = \\ &= C_\beta(R)\|u - \widetilde{u}\|_{H_Q}, \end{aligned} \tag{3.1.22}$$

where we have used (3.1.8) and (3.1.21). Thus we have

$$|\beta(a,x,S(x))u(a,x) - \beta(a,x,\overline{S}(x))\widetilde{u}(a,x)| \le$$

$$\le |\beta(a,x,S(x))||u(a,x) - \widetilde{u}(a,x)| + |\beta(a,x,S(x)) - \beta(a,x,\overline{S}(x))||\widetilde{u}(a,x)| \le$$

$$\le \beta_+|u(a,x) - \widetilde{u}(a,x)| + C_\beta(R)\|u - \widetilde{u}\|_{H_Q}|\widetilde{u}(a,x)|.$$

Thus we conclude that

$$\begin{aligned}\|F(u)-F(\widetilde{u})\|_{H_Q}^2 &= \int_Q |\beta(a,x,S(x))u(a,x)-\beta(a,x,\overline{S}(x))\widetilde{u}(a,x)|^2\ dxda \leq \\ &\leq \int_Q \left(\beta_+|u(a,x)-\widetilde{u}(a,x)|+C_\beta(R)\|u-\widetilde{u}\|_{H_Q}|\widetilde{u}(a,x)|\right)^2\ dxda \leq \\ &\leq (\beta_+ + RC_\beta(R))^2\,\|u-\widetilde{u}\|_{H_Q}^2.\end{aligned}$$

Thus we have obtained (3.1.19) with $B(R)=\beta_+ + RC_\beta(R)$.

With calculations similar to those used in order to get (3.1.22) we have that

$$|\mu(a,x,S(x))-\mu(a,x,\overline{S}(x))| \leq C_\mu(R)\|u-\widetilde{u}\|_{H_Q}, \tag{3.1.23}$$

with $C_\mu(R)=\gamma_+\sqrt{a_\dagger}\sqrt{|\Omega|}L_\mu(\gamma_+\sqrt{a_\dagger}\sqrt{|\Omega|}R)$. Moreover, being $\mu(a,x,0)=0$, from (3.1.23) we have also

$$|\mu(a,x,S(x))| \leq C_\mu(R)\|u\|_{H_Q}.$$

Thus we get

$$\begin{aligned}&|\mu(a,x,S(x))u(a,x)-\mu(a,x,\overline{S}(x))\widetilde{u}(a,x)| \leq \\ &\leq |\mu(a,x,S(x))||u(a,x)-\widetilde{u}(a,x)| + |\mu(a,x,S(x))-\mu(a,x,\overline{S}(x))||\widetilde{u}(a,x)| \leq \\ &\leq C_\mu(R)\|u\|_{H_Q}|u(a,x)-\widetilde{u}(a,x)| + C_\mu(R)\|u-\widetilde{u}\|_{H_Q}|\widetilde{u}(a,x)|,\end{aligned}$$

and then

$$\begin{aligned}\|E(u)-E(\widetilde{u})\|_{H_Q}^2 &= \int_Q |\mu(a,x,S(x))u(a,x)-\mu(a,x,\overline{S}(x))\widetilde{u}(a,x)|^2\ dxda \leq \\ &\leq 4R^2(C_\mu(R))^2\|u-\widetilde{u}\|_{H_Q},\end{aligned}$$

thus we have obtained (3.1.20) with $M(R)=2RC_\mu(R)$. □

### 3.1.3 The m-accretiveness of the operator $A$

In order to prove existence and uniqueness of a solution to problem (3.1.15)-(3.1.16), we are going to show the quasi m-accretiveness of the operator $A$. For a first analysis we shall work under a stronger hypothesis on the functions $F$ and $E$ defined in (3.1.17) and (3.1.18), namely throughout this section we will assume that $F$ and $E$ are Lipschitz continuous functions. After then, in Theorem 3.1.6, the problem will be reduced to this previous case according to a method presented in [4]. Thus now we assume that there exist constants $B>0$ and $M>0$ such that for $u,\widetilde{u}\in H_Q$ we have

$$\|F(u)-F(\widetilde{u})\|_{H_Q} \leq B\|u-\widetilde{u}\|_{H_Q}, \tag{3.1.24}$$

$$\|E(u) - E(\widetilde{u})\|_{H_Q} \leq M\|u - \widetilde{u}\|_{H_Q}. \tag{3.1.25}$$

We first recall some definitions. Let $H$ be a Hilbert space with scalar product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$ and let $A : D(A) \subset H \to H$ an operator, then we recall the following definitions:

**Definition 3.1.2** *The operator $A$ is called accretive if*

$$(Au - A\widetilde{u}, u - \widetilde{u})_H \geq 0, \quad \forall u, \widetilde{u} \in D(A).$$

**Definition 3.1.3** *The operator $A$ is called m-accretive if it is accretive and*

$$\mathcal{R}(I + A) = H$$

*where $I$ is the unit operator in $H$ and $\mathcal{R}(I + A)$ denotes the range of the operator $I + A$.*

**Definition 3.1.4** *The operator $A$ is called quasi accretive if for every $u, \widetilde{u} \in D(A)$ and some $\omega \in \mathbb{R}$, $\omega > 0$, the operator $\omega I + A$ is accretive.*

**Definition 3.1.5** *The operator $A$ is called quasi m-accretive if for all $u, \widetilde{u} \in D(A)$ and some $\omega \in \mathbb{R}$, $\omega > 0$, $\omega I + A$ is accretive (i.e. $A$ is quasi-accretive), and*

$$\mathcal{R}(\lambda I + A) = H, \quad \textit{for } \lambda > \omega.$$

We will need also the following definitions, concerning an operator $B : V \to V'$, where $V$ is a Banach space, $V'$ is its dual and $< \cdot, \cdot >_{V',V}$ denotes the pairing between $V'$ and $V$.

**Definition 3.1.6** *The operator $B : V \to V'$ is called coercive if there exist $\alpha > 0$ and $c \geq 0$ such that*

$$< Bv, v >_{V',V} \geq \alpha\|v\|_V^2 + c, \forall v \in V.$$

**Definition 3.1.7** *The operator $B : V \to V'$ is called monotone if*

$$< Bv - B\overline{v}, v - \overline{v} >_{V',V} \geq 0, \quad \forall v, \overline{v} \in V.$$

**Definition 3.1.8** *The operator $B : V \to V'$ is called hemicontinuous if*

$$B(v + \lambda w) \xrightarrow[\lambda \to 0]{w} Bv, \quad \forall v, w \in V,$$

*where $\xrightarrow{w}$ indicates the weak convergence in $V$.*

Finally we recall the following result due to Lions (see [38])

**Theorem 3.1.1** *Let $V \subset H \subset V'$ a variational triplet and let $B : V \to V'$ be a monotone and hemicontinuous operator that satisfies*

$$< Bu, u >_{V',V} \geq \alpha\|v\|_V^2 + c_1, \quad \forall u \in V, \quad \textit{with } \alpha > 0,$$

$$\|Bu\|_{V'} \leq c_2(\|u\|_V + 1), \quad \forall u \in V.$$

*Suppose that* $u_0 \in H$ *and* $f \in L^2(0,T;V')$. *Then there exists a unique solution to the problem*

$$\frac{du}{dt}(t) + Bu(t) = f(t), \qquad a.e. \ t \in (0,T),$$

$$u(0) = u_0,$$

*that is* $V'$*-valued, absolutely continuous on* $[0,T]$ *and satisfies*

$$u \in \mathcal{C}([0,T];H) \cap L^2(0,T;V)$$

$$\frac{du}{dt} \in L^2(0,T;V').$$

A proof of this theorem is given in [38], but also in [3] or in [41].

We are now ready to start to prove the quasi accretiveness of $A$.

**Lemma 3.1.2** *Assume* (3.1.5)-(3.1.7), (3.1.10) *and the additional conditions* (3.1.24) *and* (3.1.25), *then the operator* $A$ *is quasi accretive.*

**Proof.** According to Definition 3.1.4, we have to prove that for $\omega > 0$ sufficiently large we have

$$((\omega I + A)u - (\omega I + A)\widetilde{u}, u - \widetilde{u})_{HQ} \geq 0, \tag{3.1.26}$$

for $u, \widetilde{u} \in D(A)$. We have

$$\begin{aligned}((\omega I + A)u - (\omega I + A)\widetilde{u}, u - \widetilde{u})_{HQ} &= \omega \|u - \widetilde{u}\|^2_{HQ} + \\ + \int_Q [(u_a - \widetilde{u}_a)(u - \widetilde{u}) + (E(u) - E(\widetilde{u}))(u - \widetilde{u})] \ dxda &+ \\ + \int_Q K(a,x)|\nabla(u - \widetilde{u})|^2 \ dxda &\geq \\ \geq \omega \|u - \widetilde{u}\|^2_{HQ} + \frac{1}{2}\int_\Omega (u - \widetilde{u})^2 \ |_{a=a_\dagger} \ dx - \frac{1}{2}\int_\Omega (u - \widetilde{u})^2 \ |_{a=0} \ dx &+ \\ - \left| \int_Q (E(u) - E(\widetilde{u}))(u - \widetilde{u}) \ dxda \right| + K_0 \||\nabla(u - \widetilde{u})|\|^2_{HQ}&.\end{aligned} \tag{3.1.27}$$

Now, using the fact that $u, \widetilde{u} \in D(A)$ and the Lipschitz continuity of $F$ (namely (3.1.24)),

$$\begin{aligned}\int_\Omega (u - \widetilde{u})^2 \ |_{a=0} \ dx &= \int_\Omega \left| \int_0^{a_\dagger} (F(u) - F(\widetilde{u})) \ da \right|^2 dx \leq \\ &\leq a_\dagger \|F(u) - F(\widetilde{u})\|^2_{HQ} \leq a_\dagger B^2 \|u - \widetilde{u}\|^2_{HQ}.\end{aligned}$$

Moreover, using the fact that $E$ is Lipschitz continuous (namely (3.1.25)),

$$\left| \int_Q (E(u) - E(\widetilde{u}))(u - \widetilde{u}) \ dxda \right| \leq \|E(u) - E(\widetilde{u})\|_{HQ} \|u - \widetilde{u}\|_{HQ} \leq M \|u - \widetilde{u}\|^2_{HQ}.$$

Thus, from (3.1.27) we get

$$((\omega I + A)u - (\omega I + A)\widetilde{u}, u - \widetilde{u})_{H_Q} \geq \omega\|u - \widetilde{u}\|^2_{H_Q} - \frac{a_\dagger B^2}{2}\|u - \widetilde{u}\|^2_{H_Q} + -M\|u - \widetilde{u}\|^2_{H_Q} + K_0\|\|\nabla(u - \widetilde{u})\|\|^2_{H_Q} \geq 0,$$

for $\omega$ large enough, $\omega \geq \dfrac{a_\dagger B^2}{2} + M$. □

The following lemma states the quasi m-accretiveness of $A$.

**Lemma 3.1.3** *Assume the same conditions as in Lemma 3.1.2, then the operator $A$ is quasi m-accretive.*

**Proof.** According to Definition 3.1.5 it remains to show that

$$\mathcal{R}(\lambda I + A) = H_Q,$$

for $\lambda > 0$ large enough, in particular, according to the proof of Lemma 3.1.2, it has to be $\lambda > \dfrac{a_\dagger B^2}{2} + M$.

In other words, we have to prove that for any $f \in H_Q$ there exists $u \in D(A)$ such that

$$\lambda u + Au = f, \tag{3.1.28}$$

To come to this end we define the operator $A_V(a) : V \to V'$ as

$$< A_V(a)v, \psi >_{V',V} = \int_\Omega K(a,x)\nabla v \cdot \nabla\psi \; dx, \quad \forall \psi \in V,$$

thus solving (3.1.28) means that we should find $u \in L^2(0, a_\dagger; V)$ with $u_a \in L^2(0, a_\dagger; V')$ solution of the following Cauchy problem

$$u_a + (\lambda I + A_V(a))u = f - E(u), \tag{3.1.29}$$

$$u(0,x) = \int_0^{a_\dagger} F(u)(a,x) \; da. \tag{3.1.30}$$

In order to prove existence and uniqueness of a solution of (3.1.29)-(3.1.30), we use a fixed point argument, namely, for a fixed $\omega \in H_Q$ we consider the problem

$$u_a + (\lambda I + A_V(a))u = f - E(\omega), \tag{3.1.31}$$

$$u(0,x) = \int_0^{a_\dagger} F(\omega)(a,x) \; da, \tag{3.1.32}$$

then, once proved the existence of a solution to (3.1.31)-(3.1.32), it will be enough to show that the application $\mathcal{P} : H_Q \to H_Q$ that associates to $\omega \in H_Q$ the corresponding solution $u$ to problem (3.1.31)-(3.1.32) is a contraction on $H_Q$.

First of all observe that the linear operator $\lambda I + A_V(a)$ is coercive, indeed

$$\begin{aligned} <(\lambda I + A_V(a))v, v>_{V',V} &\geq \lambda\|v\|_H^2 + K_0\||\nabla v|\|_H^2 = \\ &= (\lambda - K_0)\|v\|_H^2 + K_0(\|v\|_H^2 + \||\nabla v|\|_H^2) = \\ &= (\lambda - K_0)\|v\|_H^2 + K_0\|v\|_V^2, \end{aligned}$$

and bounded, since

$$\|A_V(a)v\|_{V'} = \sup_{\|\psi\|_V \leq 1} | < A_V(a)v, \psi >_{V',V} | \leq K_+\|v\|_V.$$

Moreover, since $\omega$ is fixed in $H_Q$, we have that $u(0,\cdot) \in H$, indeed

$$\int_\Omega |u(0,x)|^2 dx = \int_\Omega \left| \int_0^{a_\dagger} F(\omega)(a,x)\ da \right|^2 dx \leq$$

$$\leq \beta_+^2 a_\dagger \int_\Omega \int_0^{a_\dagger} |\omega(a,x)|^2\ dadx = \beta_+^2 a_\dagger \int_Q |\omega(a,x)|^2 dxda < \infty.$$

Furthermore,

$$f - E(\omega) \in H_Q \equiv L^2(0, a_\dagger; H).$$

Thus the hypotheses of Theorem 3.1.1 are verified and we can conclude that the Cauchy problem (3.1.31)-(3.1.32) has a unique solution

$$u \in L^2(0, a_\dagger; V), \quad u_a \in L^2(0, a_\dagger; V').$$

Now it remains to show that the application $\mathcal{P} : H_Q \to H_Q$, $\mathcal{P}\omega = u$, is a contraction on $H_Q$.
We consider $\omega, \overline{\omega} \in H_Q$ and $u = \mathcal{P}\omega$, $\widetilde{u} = \mathcal{P}\overline{\omega}$, and multiply the equation

$$(u - \widetilde{u})_a + \lambda(u - \widetilde{u}) + A_V(a)(u - \widetilde{u}) + E(\omega) - E(\overline{\omega}) = 0,$$

by $(u - \widetilde{u})$ and integrate over $Q$. We obtain

$$\frac{1}{2}\int_\Omega (u(a_\dagger,x) - \widetilde{u}(a_\dagger,x))^2\ dx - \frac{1}{2}\int_\Omega (u(0,x) - \widetilde{u}(0,x))^2\ dx + \lambda\|u - \widetilde{u}\|_{H_Q}^2 +$$

$$+ \int_Q K(a,x)|\nabla u - \nabla\widetilde{u}|^2\ dxda + \int_Q (E(\omega) - E(\overline{\omega}))(u - \widetilde{u})\ dxda = 0,$$

and then

$$\begin{aligned} \lambda\|u - \widetilde{u}\|^2 &\leq \frac{1}{2}\int_\Omega \left( \int_0^{a_\dagger} F(\omega)(a,x)da - \int_0^{a_\dagger} F(\overline{\omega})(a,x)da \right)^2 dx + \\ &\quad + \|E(\omega) - E(\overline{\omega})\|_{H_Q}\|u - \widetilde{u}\|_{H_Q} \leq \\ &\leq \frac{a_\dagger}{2}\|F(\omega) - F(\overline{\omega})\|_{H_Q}^2 + \|E(\omega) - E(\overline{\omega})\|_{H_Q}\|u - \widetilde{u}\|_{H_Q}. \end{aligned}$$

By using (3.1.24), (3.1.25) and applying Young inequality to the second term, we obtain

$$\lambda\|u-\widetilde{u}\|^2_{H_Q} \le \frac{a_\dagger B^2}{2}\|\omega-\overline{\omega}\|^2_{H_Q} + \frac{M^2}{2\lambda}\|\omega-\overline{\omega}\|^2_{H_Q} + \frac{\lambda}{2}\|u-\widetilde{u}\|^2_{H_Q},$$

from which we get

$$\lambda\|u-\widetilde{u}\|^2_{H_Q} \le \left(a_\dagger B^2 + \frac{M^2}{\lambda}\right)\|\omega-\overline{\omega}\|^2_{H_Q},$$

i.e.

$$\|\mathcal{P}\omega-\mathcal{P}\overline{\omega}\|^2_{H_Q} \le \frac{1}{\lambda}\left(a_\dagger B^2 + \frac{M^2}{\lambda}\right)\|\omega-\overline{\omega}\|^2_{H_Q},$$

and then $\mathcal{P}$ is a contraction for $\lambda$ sufficiently large, namely

$$\lambda > \lambda_c = \frac{a_\dagger B^2 + \sqrt{(a_\dagger B^2)^2 + 4M^2}}{2}. \tag{3.1.33}$$

Thus we conclude that the equation $\mathcal{P}u = u$ has a solution, which is actually the solution of (3.1.29)-(3.1.30), moreover $u \in D(A)$, in fact $u \in L^2(0,a_\dagger;V)$, $u_a \in L^2(0,a_\dagger;V')$ and, since $f \in H_Q$, we have $Au = f - \lambda u \in H_Q$. Thus it follows that $A$ is quasi m-accretive in $H_Q$. □

### 3.1.4 Existence and properties of the solution

We are now ready to state an existence result for the solution to problem (3.1.15)-(3.1.16).

**Theorem 3.1.2** *Assume* (3.1.5)-(3.1.7), (3.1.10) *and* (3.1.24)-(3.1.25) *and let*

$$f \in W^{1,1}(0,T;H_Q), \quad p_0 \in D(A). \tag{3.1.34}$$

*Then the problem* (3.1.15)-(3.1.16) *has an unique strong solution* $p \in \mathcal{C}^0(0,T;D(A))$ *such that*

$$p \in L^\infty(0,T;D(A)) \cap W^{1,\infty}(0,T;H_Q).$$

The proof of Theorem 3.1.2 follows immediately from the fundamental results concerning the existence for evolution equations with m-accretive operators in Hilbert spaces (see e.g. [41], Theorem 2.2.6).

The following result states some properties of the solution $p$:

**Theorem 3.1.3** *Assume the same hypotheses of Theorem 3.1.2, then the solution p to problem* (3.1.15)-(3.1.16) *satisfies the following estimate*

$$\begin{aligned} &\|p(t)\|^2_{H_Q} + \int_0^t \|p(\tau)(a_\dagger,\cdot)\|^2_H \, d\tau + \int_0^t\int_0^{a_\dagger} \|p(\tau)(a,\cdot)\|^2_V \, d\tau da \le \\ &\le \frac{1}{K_{min}}\left(\|p_0\|^2_{H_Q} + \int_0^T \|f(\tau)\|^2_{H_Q}\, d\tau\right)\exp[(\beta_+^2 a_\dagger + 2)t] < +\infty, \end{aligned} \tag{3.1.35}$$

*for every* $t \in [0,T]$, *where* $K_{min} = \min(1, K_0)$.

*Moreover if* $p$ *and* $\overline{p}$ *are two solutions of* (3.1.15)-(3.1.16) *corresponding to the initial data* $p_0, f$ *and* $\overline{p}_0, \overline{f}$ *respectively, then*

$$\|p(t)-\overline{p}(t)\|^2_{H_Q} \leq \left(\|p_0-\overline{p}_0\|^2_{H_Q} + \int_0^T \|f(\tau)-\overline{f}(\tau)\|^2_{H_Q}\, d\tau\right)\exp[\alpha_0 t], \tag{3.1.36}$$

*for every* $t \in [0,T]$, *where* $\alpha_0 = (a_\dagger B^2 + 2M + 1)$.

**Proof.**
In order to obtain the estimate (3.1.35), we multiply (3.1.15) by $p$ and integrate over $(0,t)$, after some calculations we get

$$\|p(t)\|^2_{H_Q} + 2\int_0^t (Ap(\tau),p(\tau))_{H_Q}\, d\tau = \|p_0\|^2_{H_Q} + 2\int_0^t (f(\tau),p(\tau))_{H_Q}\, d\tau. \tag{3.1.37}$$

Observe that, by (3.1.5), (3.1.6) and (3.1.10), the operator $A$ satisfies, for all $u \in D(A)$,

$$\begin{aligned} 2(Au,u)_{H_Q} &= \int_\Omega |u(a_\dagger,x)|^2\, dx - \int_\Omega |u(0,x)|^2\, dx + \\ &+2\int_Q \mu(a,x,S(x))u^2(a,x)\, dxda + 2\int_Q K(a,x)|\nabla u(a,x)|^2\, dxda \geq \\ &\geq \|u(a_\dagger,\cdot)\|^2_H - \beta_+^2 a_\dagger \|u\|^2_{H_Q} + 2K_0\int_0^{a_\dagger} |||\nabla u(a,\cdot)|||^2_H\, da. \end{aligned} \tag{3.1.38}$$

Using estimate (3.1.38) in (3.1.37), we obtain

$$\begin{aligned} &\|p(t)\|^2_{H_Q} + \int_0^t \|p(\tau)(a_\dagger,\cdot)\|^2_H\, d\tau - \beta_+^2 a_\dagger \int_0^t \|p(\tau)\|^2_{H_Q}\, d\tau + \\ &+K_0\int_0^t\int_0^{a_\dagger} |||\nabla p(\tau)(a,\cdot)|||^2_H\, dad\tau \leq \|p_0\|^2_{H_Q} + 2\int_0^t (f(\tau),p(\tau))_{H_Q}\, d\tau. \end{aligned}$$

By adding to both sides $\int_0^t \|p(\tau)\|^2_{H_Q} d\tau$ and using the estimate $2(f(\tau),p(\tau))_{H_Q} \leq \|p(\tau)\|^2_{H_Q} + \|f(\tau)\|^2_{H_Q}$, we end up with

$$\begin{aligned} &\|p(t)\|^2_{H_Q} + \int_0^t \|p(\tau)(a_\dagger,\cdot)\|^2_H\, d\tau + K_0\int_0^t |||\nabla p(\tau)(a,\cdot)|||^2_H\, dad\tau + \int_0^t \|p(\tau)\|^2_{H_Q}\, d\tau \leq \\ &\leq \|p_0\|^2_{H_Q} + \int_0^t \|f(\tau)\|^2_{H_Q}\, d\tau + (\beta_+^2 a_\dagger + 2)\int_0^t \|p(\tau)\|^2_{H_Q}\, d\tau, \end{aligned}$$

in particular we have the following inequalities:

$$\|p(t)\|^2_{H_Q} \leq \|p_0\|^2_{H_Q} + \int_0^t \|f(\tau)\|^2_{H_Q}\, d\tau + (\beta_+^2 a_\dagger + 2)\int_0^t \|p(\tau)\|^2_{H_Q}\, d\tau \tag{3.1.39}$$

and

$$K_{min}\left[\|p(t)\|_{H_Q}^2+\int_0^t\|p(\tau)(a_\dagger,\cdot)\|_H^2\,d\tau+\int_0^t\int_0^{a_\dagger}\|p(\tau)(a,\cdot)\|_V^2\,dad\tau\right]\leq$$
$$\leq\|p_0\|_{H_Q}^2+\int_0^t\|f(\tau)\|_{H_Q}^2\,d\tau+(\beta_+^2a_\dagger+2)\int_0^t\|p(\tau)\|_{H_Q}^2\,d\tau, \tag{3.1.40}$$

where $K_{min}=\min(1,K_0)$.
Applying Gronwall lemma to (3.1.39), we have

$$\|p(t)\|_{H_Q}^2\leq\left(\|p_0\|_{H_Q}^2+\int_0^t\|f(\tau)\|_{H_Q}^2\,d\tau\right)\exp[(\beta_+^2a_\dagger+2)t],$$

and then, using this in (3.1.40),

$$K_{min}\left[\|p(t)\|_{H_Q}^2+\int_0^t\|p(\tau)(a_\dagger,\cdot)\|_H^2\,d\tau+\int_0^t\int_0^{a_\dagger}\|p(\tau)(a,\cdot)\|_V^2\,dad\tau\right]\leq$$
$$\leq\|p_0\|_{H_Q}^2+\int_0^t\|f(\tau)\|_{H_Q}^2\,d\tau+(\beta_+^2a_\dagger+2)\left(\|p_0\|_{H_Q}^2+\int_0^t\|f(\tau)\|_{H_Q}^2\,d\tau\right)\int_0^t e^{(\beta_+^2a_\dagger+2)\tau}\,d\tau=$$
$$=\left(\|p_0\|_{H_Q}^2+\int_0^t\|f(\tau)\|_{H_Q}^2\,d\tau\right)\exp((\beta_+^2a_\dagger+2)t),$$

and this implies (3.1.35) as claimed.

Now we consider two solutions $p$ and $\overline{p}$ of (3.1.15)-(3.1.16), corresponding to the initial data $p_0,f$ and $\overline{p}_0,\overline{f}$ respectively and we multiply the equation

$$\frac{d}{dt}(p-\overline{p})+Ap-A\overline{p}=f-\overline{f}$$

by $(p-\overline{p})$ to obtain

$$\frac{d}{dt}\|p(t)-\overline{p}(t)\|_{H_Q}^2=-2(Ap(t)-A\overline{p}(t),p(t)-\overline{p}(t))_{H_Q}+2(f(t)-\overline{f}(t),p(t)-\overline{p}(t))_{H_Q},$$

now, using the quasi accretiveness of $A$, namely (3.1.26) with $\omega=\dfrac{a_\dagger B^2}{2}+M$, we get

$$\frac{d}{dt}\|p(t)-\overline{p}(t)\|_{H_Q}^2\leq\alpha_0\|p(t)-\overline{p}(t)\|_{H_Q}^2+\|f(t)-\overline{f}(t)\|_{H_Q}^2.$$

Thus, integrating over $(0,t)$ with $t\in[0,T]$,

$$\|p(t)-\overline{p}(t)\|_{H_Q}^2\leq\|p_0-\overline{p}_0\|_{H_Q}^2+\int_0^t\|f(\tau)-\overline{f}(\tau)\|_{H_Q}^2\,d\tau+\alpha_0\int_0^t\|p(\tau)-\overline{p}(\tau)\|_{H_Q}^2\,d\tau,$$

and, applying Gronwall lemma, we obtain (3.1.36). □

A further result concerns non-negativeness of the solution, according with the biological meaning of the problem. In order to prove that the solution $p$ to problem (3.1.15)-(3.1.16) is non-negative, we are going to show that its negative part $p^-(t)$ is zero a.e. in $Q$ for each $t \in [0,T]$. Concerning the properties of the positive and negative parts of a function we recall the following results:

**Theorem 3.1.4** (Stampacchia lemma) *Let $u \in H^1(D)$, $D \subset \mathbb{R}^n$, then its positive part $u^+ = \max(u,0)$ belongs to $H^1(D)$ and*

$$\left(\frac{\partial}{\partial x_i}u^+\right)(x) = \begin{cases} \dfrac{\partial u}{\partial x_i}(x) & a.e. \ in \ \{x \in D; u(x) > 0\}, \\ 0 & a.e. \ in \ \{x \in D; u(x) \le 0\}, \end{cases}$$

*for all $i = 1, 2, \ldots, n$.*

**Corollary 3.1.1** *Let $u \in H^1(D)$, $D \subset \mathbb{R}^n$, then its negative part $u^- = -\min(u,0)$ belongs to $H^1(D)$ and*

$$\left(\frac{\partial}{\partial x_i}u^-\right)(x) = \begin{cases} -\dfrac{\partial u}{\partial x_i}(x) & a.e. \ in \ \{x \in D; u(x) < 0\}, \\ 0 & a.e. \ in \ \{x \in D; u(x) \ge 0\}, \end{cases}$$

*for all $i = 1, 2, \ldots, n$.*

Now we are ready to prove the following non-negativeness result.

**Theorem 3.1.5** *Assume the conditions of Theorem 3.1.2 and let*

$$f \ge 0 \ a.e. \ in \ (0,T) \times Q, \tag{3.1.41}$$

$$p_0 \ge 0 \ a.e. \ in \ Q. \tag{3.1.42}$$

*Then the solution $p$ to problem (3.1.15)-(3.1.16) satisfies*

$$p(t) \ge 0 \ a.e. \ in \ Q \ for \ each \ t \in [0,T].$$

**Proof.** As already said, we are going to show that the negative part $p^-(t)$ of $p(t)$ is zero a.e. in $Q$ for each $t \in [0,T]$.

We multiply equation (3.1.15) by $p^-$ and integrate over $(0,t) \times Q$, for any $t \in [0,T]$, we obtain

$$\int_0^t \int_Q \frac{\partial p}{\partial \tau} p^- \, dx da d\tau + \int_0^t \int_Q \frac{\partial p}{\partial a} \, dx da d\tau + \int_0^t \int_Q \mu p p^- \, dx da d\tau +$$

$$+ \int_0^t \int_Q K(a,x) \nabla p \cdot \nabla p^- \, dx da d\tau = \int_0^t \int_Q f p^- dx da d\tau.$$

Using Corollary 3.1.1 we have

$$-\frac{1}{2}\int_0^t\int_Q \frac{\partial(p^-)^2}{\partial\tau} - \frac{1}{2}\int_0^t\int_\Omega (p^-)^2(\tau)(a_\dagger,x)\ dxd\tau - \int_0^t\int_Q K(a,x)|\nabla p^-|^2\ dxdad\tau =$$

$$= -\frac{1}{2}\int_0^t\int_\Omega (p^-)^2(\tau)(0,x)\ dxd\tau + \int_0^t\int_Q \mu(p^-)^2\ dxdad\tau + \int_0^t\int_Q fp^-\ dxdad\tau,$$

and then

$$\begin{aligned}
&\frac{1}{2}\|p^-(t)\|^2_{H_Q} - \frac{1}{2}\|p^-(0)\|^2_{H_Q} \le \\
&\le \frac{1}{2}\|p^-(t)\|^2_{H_Q} - \frac{1}{2}\|p^-(0)\|^2_{H_Q} + \int_0^t\int_\Omega (p^-)^2(\tau)(a_\dagger,x)\ dxdad\tau = \\
&= \frac{1}{2}\int_0^t\int_\Omega (p^-)^2(\tau)(0,x)\ dxdad\tau - \int_0^t\int_Q \mu(p^-)^2\ dxdad\tau - \int_0^t\int_Q fp^-\ dxdad\tau \le \\
&\le \frac{1}{2}\int_0^t\int_\Omega (p^-)^2(\tau)(0,x)\ dxdad\tau,
\end{aligned} \tag{3.1.43}$$

in the last estimate we have used (3.1.6) and (3.1.41).

Recall that if $\phi(t) = \phi_1(t) - \phi_2(t)$, with $\phi_1(t) \ge 0$ and $\phi_2(t) \ge 0$, then it follows that $\phi^-(t) \le \phi_2(t)$, in fact

$$\phi^-(t) = \begin{cases} 0 & \text{if } \phi_1(t) \ge \phi_2(t), \\ \phi_2(t) - \phi_1(t) \le \phi_2(t) & \text{if } \phi_1(t) \le \phi_2(t). \end{cases}$$

Now, we apply this to

$$p(\tau)(0,x) = \int_0^{a_\dagger} \beta(a,x,S(\tau,x))p^+(\tau)(a,x)\ da - \int_0^{a_\dagger} \beta(a,x,S(\tau,x))p^-(\tau)(a,x)\ da,$$

and then, from (3.1.43), we have

$$\begin{aligned}
\frac{1}{2}\|p^-(t)\|^2_{H_Q} - \frac{1}{2}\|p^-(0)\|^2_{H_Q} &\le \int_0^t\int_\Omega \left(\int_0^{a_\dagger} \beta(a,x,S(\tau,x))p^-(\tau)(a,x)\ da\right)^2\ dxd\tau \le \\
&\le \frac{\beta_+^2 a_\dagger}{2}\int_0^t \|p^-(\tau)\|^2_{H_Q}\ d\tau,
\end{aligned}$$

from which we get

$$\|p^-(t)\|^2_{H_Q} \le \|p^-(0)\|^2_{H_Q} + \frac{\beta_+^2 a_\dagger}{2}\int_0^t \|p^-(\tau)\|^2_{H_Q}\ d\tau.$$

Applying Gronwall lemma and using (3.1.42) we finally conclude that $\|p^-(t)\|^2_{H_Q} = 0$ for all $t \in [0,T]$, hence $p(t) \geq 0$ a.e. on $Q$ for each $t \in [0,T]$. □

Now we are ready to pass to the proof of the existence result under the main assumptions of locally Lipschitz continuity of the functions $\beta$ and $\mu$. Namely, under the assumptions (3.1.8) and (3.1.9), the following Theorem holds

**Theorem 3.1.6** *Assume the conditions* (3.1.5)-(3.1.10) *and* (3.1.34). *Then the problem* (3.1.15)-(3.1.16) *has a unique strong solution* $p \in \mathcal{C}([0,T];H_Q)$ *such that*

$$p \in W^{1,\infty}(0,T;H_Q) \cap L^\infty(0,T;D(A)),$$

*and*

$$\|p(t)\|^2_{H_Q} \leq \frac{1}{K_{min}} \left( \|p_0\|^2_{H_Q} + \int_0^t \|f(\tau)\|^2_{H_Q}\, d\tau \right) \exp[(\beta_+^2 a_\dagger + 2)t], \tag{3.1.44}$$

*for any* $t \in [0,T]$.

**Proof.**
We have already proved, in Lemma 3.1.1, that, under the assumptions (3.1.8) and (3.1.9), the functions $F$ and $E$ defined in (3.1.17) and (3.1.18) are locally Lipschitz continuous from $H_Q$ to $H_Q$. We now reduce the problem to the previous case for which $F$ and $E$ are globally Lipschitz continuous, using a technique described in [4]. Namely, fixed $N \geq 1$, we approximate $F$ and $E$ setting

$$F_N(u) = \begin{cases} F(u) & \text{for } \|u\|_{H_Q} \leq N, \\ F\left(\dfrac{Nu}{\|u\|_{H_Q}}\right) & \text{for } \|u\|_{H_Q} > N, \end{cases}$$

and

$$E_N(u) = \begin{cases} E(u) & \text{for } \|u\|_{H_Q} \leq N, \\ E\left(\dfrac{Nu}{\|u\|_{H_Q}}\right) & \text{for } \|u\|_{H_Q} > N. \end{cases}$$

Actually these truncated functions are Lipschitz continuous on $H_Q$ (for each fixed $N$). Therefore, we consider the approximating problem

$$\frac{dp_N}{dt} + A_N p_N = f, \tag{3.1.45}$$

$$p_N(0) = p_0, \tag{3.1.46}$$

where the operator $A_N$ is defined in (3.1.14) in which the functions $F$ and $E$ are replaced by $F_N$ and $E_N$ respectively. Since for each $N$, the assumptions of Theorem 3.1.2 are fulfilled, we find that for $f \in W^{1,1}(0,T;H_Q)$ and $p_0 \in H_Q$, for each $N$, the problem (3.1.45)-(3.1.46) has a solution

$$p_N \in \mathcal{C}([0,T];H_Q) \cap L^\infty(0,T;V),\ \ (p_N)_a \in L^\infty(0,T;V'),$$

$$A_N p_N \in H_Q,\ p_N(t,0,x) = \int_0^{a_\dagger} (F_N(p_N)(t))(a,x)\ da.$$

In particular, this solution satisfies (3.1.35) which is independent of the Lipschitz constants and of $N$ because (see the proof of Theorem 3.1.3)

$$2(A_N u, u)_{H_Q} \geq \|u(a_\dagger)\|_H^2 - \beta_+^2 a_\dagger \|u\|_{H_Q}^2 + 2K_0 \int_0^{a_\dagger} \||\nabla u(a)|\|_H^2\ da,$$

where we have used

$$(F_N(u))(a,x) \leq \beta_+ u(a,x).$$

Namely, for each $T$ we have

$$\|p_N(t)\|_{H_Q}^2 \leq R^2 \quad \text{for } t \in [0,T],$$

with

$$R = \frac{1}{K_{min}} \left( \|p_0\|_{H_Q}^2 + \int_0^T \|f(\tau)\|_{H_Q}^2\ d\tau \right) \exp[(\beta_+^2 a_\dagger + 2)T] < \infty, \tag{3.1.47}$$

and $R$ is independent of $N$.

In conclusion, for $N$ sufficiently large, namely $N > R$, we get that

$$A_N p_N(t) = A p_N(t),$$

$$p_N(t,0,x) = \int_0^{a_\dagger} F(p_N(t))(a,y)\ da,$$

so that $p_N(t)$ is actually a solution to the problem (3.1.15)-(3.1.16).

To prove the uniqueness we suppose that there exist two solutions $p$ and $\overline{p}$ corresponding to the same data $f$ and $p_0$. Then, by the previous proof, we have that if

$$N > \sup_{t\in[0,T]} \|p(t)\|_{H_Q} + \sup_{t\in[0,T]} \|\overline{p}(t)\|_{H_Q},$$

then $p(t) = p_N(t)$ and $\overline{p}(t) = p_N(t)$, where $p_N$ is the solution to (3.1.45)-(3.1.46). This prove that the solution is unique.

Finally, estimate (3.1.44) is a direct consequence of (3.1.47).

□

## 3.2 Approximation of the one-dimensional problem via a multi-layer model

Here we consider a particular case of the model analyzed in the previous section, namely the one-dimensional problem, and we show how this problem can be approximated using a model set up in a multi-layer habitat.

In particular, we shall consider the problem (3.1.1)-(3.1.4) in a one-dimensional environment $\Omega = (x_0, x_L)$ and the corresponding problem in the same interval divided into $n$ sub-intervals (layers), we shall refer to these problems as $(P)$ and $(P^n)$ respectively. In Section 3.2.2, following the procedure described in Section 3.1, we will write $(P)$ and $(P^n)$ as abstract Cauchy problems and, in Section 3.2.3, we will give the mathematical assumptions that guarantee existence and uniqueness of the solutions to $(P)$ and $(P^n)$. The convergence of the solution to problem $(P^n)$ to the solution to problem $(P)$ is proved using a result given in [7] by Brezis and Pazy (see Theorem 3.2.1 in Section 3.2.3). In order to apply this Theorem some previous results are necessary, these preliminaries are given in Section 3.2.4. Finally, in Sections 3.2.5 and 3.2.6 the convergence of the multi-layer model to the continuous one is proved.

### 3.2.1 Statement of the problem

Let us consider the problem (3.1.1)-(3.1.4) in a one-dimensional environment $\Omega = (x_0, x_L)$. We introduce the following notation

$$Q = (0, a_\dagger) \times (x_0, x_L),$$

$$\Gamma_0 = \{(0,x) : \ x \in (x_0, x_L)\}, \quad \Gamma_{a_\dagger} = \{(a_\dagger, x) : \ x \in (x_0, x_L)\},$$
$$\Gamma_{x_0} = \{(a, x_0) : \ a \in (0, a_\dagger)\}, \quad \Gamma_{x_L} = \{(a, x_L) : \ a \in (0, a_\dagger)\}.$$

The problem in one dimension reads

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu(a, x, S(t,x))p - \frac{\partial}{\partial x}\left(K(a,x)\frac{\partial p}{\partial x}\right) = f, \qquad \text{in } (0,T) \times Q,$$

$$p(0,a,x) = p_0(a,x), \qquad \text{in } Q,$$

$$p(t,0,x) = \int_0^{a_\dagger} \beta(a,x,S(t,x))p(a,t,x)\ da, \qquad \text{in } (0,T) \times (x_0, x_L),$$

$$S(t,x) = \int_Q \gamma(a,x,z)p(t,a,z)\ dzda,$$

$$K(a,x)\frac{\partial p}{\partial x} = 0, \qquad \text{on } (0,T) \times \Gamma_{x_0},$$

$$K(a,x)\frac{\partial p}{\partial x} = 0, \qquad \text{on } (0,T) \times \Gamma_{x_L}.$$

From now on we shall refer to this problem as problem $(P)$.

If we suppose that the habitat is rapidly changing, we can think to approximate this problem with a model set up in a $n$-layer environment, in such a way that in each layer the population has its own growth and diffusion parameters, so that in each layer the dynamics is not subject to environmental variations, while changes occur from one layer to another, according to different conditions. In order to introduce the $n$-layer structure, we consider $x_0 \leq x_1 \leq \cdots \leq x_n = x_L$, we denote

$$Q_j = (0, a_\dagger) \times (x_{j-1}, x_j), \quad j = 1, \dots, n,$$
$$\Gamma_{x_j} = \{(a, x_j) : a \in (0, a_\dagger)\}, \quad j = 1, \dots, n,$$

and consider the following problems for $j = 1, \dots, n$, on the variables $p_j^n(t,a,x)$ defined on $[0,T] \times Q_j$

$$\frac{\partial p_j^n}{\partial t} + \frac{\partial p_j^n}{\partial a} + \mu_j^n(a, S_j^n(t))p_j^n - K_j^n(a)\frac{\partial^2 p_j^n}{\partial x^2} = f_j^n, \quad \text{in } (0,T) \times Q_j, \tag{3.2.1}$$

$$p_j^n(0,a,x) = p_{0,j}^n(a,x), \quad \text{in } Q_j, \tag{3.2.2}$$

$$p_j^n(t,0,x) = \int_0^{a_\dagger} \beta_j^n(a, S_j^n(t))p_j^n(t,a,x)\, da, \quad \text{in } (0,T) \times (x_{j-1}, x_j), \tag{3.2.3}$$

$$S_j^n(t) = \sum_{k=1}^{n} \int_0^{a_\dagger} \int_{x_{k-1}}^{x_k} \gamma_j^n(a,z)p_k^n(t,a,z)\, dzda, \tag{3.2.4}$$

endowed with the boundary conditions

$$K_1^n(a)\frac{\partial p_1^n}{\partial x} = 0, \quad \text{on } (0,T) \times \Gamma_{x_0}, \tag{3.2.5}$$

$$K_n^n(a)\frac{\partial p_n^n}{\partial x} = 0, \quad \text{on } (0,T) \times \Gamma_{x_L}, \tag{3.2.6}$$

and with the additional conditions on each interface between two layers

$$p_j^n = p_{j+1}^n, \quad \text{on } (0,T) \times \Gamma_{x_j}, \quad j = 1, \ldots, n-1, \tag{3.2.7}$$

$$K_j^n(a)\frac{\partial p_j^n}{\partial x} = K_{j+1}^n(a)\frac{\partial p_{j+1}^n}{\partial x}, \quad \text{on } (0,T) \times \Gamma_{x_j}, \quad j = 1, \ldots, n-1. \tag{3.2.8}$$

Here we used the superscript "$n$", since this problem is associated to the partition of the space interval $(x_0, x_L)$ in $n$ layers; we shall refer to it as problem $(P^n)$. All the functions $K_j^n$, $\mu_j^n$, $\beta_j^n$, $f_j^n$, $\gamma_j^n$ are approximations of the respective functions in problem $(P)$ and all of them, but $\gamma_j^n$, are constants within each layer, with respect to the space variable $x$.

### 3.2.2 Functional framework

In this section we give the formulation of a functional framework to treat problems $(P)$ and $(P^n)$, following the procedure used in Section 3.1.2. Namely, we shall formulate these problems as abstract Cauchy problems in the space $H_Q = L^2(Q)$. As we have seen in Section 3.1.2, we can rewrite problem $(P)$ as the following abstract Cauchy problem,

$$\frac{dp}{dt} + Ap = f, \quad \text{a.e. } t \in (0,T), \tag{3.2.9}$$

$$p(0) = p_0, \tag{3.2.10}$$

where $A$ is the operator defined in Section 3.1.2

**Remark 3.2.1** *We know that every strong solution to problem* (3.2.9)-(3.2.10) *is actually a solution to problem* $(P)$ *in the sense of distributions.*

Now we aim to give the same abstract formulation to problem $(P^n)$, to come to this end we first need to build new functions on $(0,T)\times Q$ by a stepwise definition (see [13]), namely

$$p^n(t,a,x)=\begin{cases} p_1^n(t,a,x), & x\in(x_0,x_1),\\ \cdots \\ p_n^n(t,a,x), & x\in(x_{n-1},x_L),\end{cases} \tag{3.2.11}$$

$$S^n(t,x)=\begin{cases} S_1^n(t), & x\in(x_0,x_1),\\ \cdots \\ S_n^n(t), & x\in(x_{n-1},x_L),\end{cases} \tag{3.2.12}$$

$$\beta^n(a,x,s)=\begin{cases} \beta_1^n(a,s), & x\in(x_0,x_1),\\ \cdots \\ \beta_n^n(a,s), & x\in(x_{n-1},x_L),\end{cases} \tag{3.2.13}$$

$$\mu^n(a,x,s)=\begin{cases} \mu_1^n(a,s), & x\in(x_0,x_1),\\ \cdots \\ \mu_n^n(a,s), & x\in(x_{n-1},x_L),\end{cases} \tag{3.2.14}$$

$$\gamma^n(a,x,z)=\begin{cases} \gamma_1^n(a,z), & x\in(x_0,x_1),\\ \cdots \\ \gamma_n^n(a,z), & x\in(x_{n-1},x_L),\end{cases} \tag{3.2.15}$$

$$K^n(a,x)=\begin{cases} K_1^n(a), & x\in(x_0,x_1),\\ \cdots \\ K_n^n(a), & x\in(x_{n-1},x_L),\end{cases} \tag{3.2.16}$$

$$p_0^n(a,x)=\begin{cases} p_{0,1}^n(a,x), & x\in(x_0,x_1),\\ \cdots \\ p_{0,n}^n(a,x), & x\in(x_{n-1},x_L),\end{cases} \tag{3.2.17}$$

$$f^n(a,x)=\begin{cases} f_1^n(a,x), & x\in(x_0,x_1),\\ \cdots \\ f_n^n(a,x), & x\in(x_{n-1},x_L).\end{cases} \tag{3.2.18}$$

Notice that, after these definitions, $S^n(t,x)$ can be rewritten as

$$S^n(t,x)=\int_Q \gamma^n(a,x,z)p(t,a,z)\ dzda.$$

Now we can proceed as before in order to rewrite the problem $(P^n)$ as an abstract Cauchy problem. Namely we first define, for every $n\in\mathbb{N}$, the operator

$$A_0^n : D(A_0^n)\subset L^2(0,a_\dagger;V)\to L^2(0,a_\dagger;V'),$$

as

$$\ll A_0^n u,\psi \gg = \ll u_a,\psi \gg + \int_Q \left[\mu^n(a,x,S^n(x))u\psi + K^n(a,x)u_x\psi_x\right]\ dxda,$$

for all $\psi \in L^2(0, a_\dagger; V)$, on the domain

$$D(A_0^n) = \Bigg\{ u \in L^2(0, a_\dagger; V),\ u_a \in L^2(0, a_\dagger; V'),$$

$$u(0,x) = \int_0^{a_\dagger} \beta^n(a, x, S^n(x)) u(a, x)\ da \Bigg\},$$

with

$$S^n(x) = \int_Q \gamma^n(a, x, z) u(a, z)\ dz da.$$

We have to specify that $S^n(x)$ is in fact the functional $S^n : H_Q \to H$ defined by

$$(S^n u)(x) = \int_Q \gamma^n(a, x, z) u(a, z)\ dz,$$

but, for the writing simplicity we skip this notation and write simply $S^n(x)$.
Finally we define the operator

$$A^n : D(A^n) \subset H_Q \to H_Q,$$

by setting

$$D(A^n) = \{u \in D(A_0^n),\ A_0^n u \in H_Q\},$$

and

$$A^n u = A_0^n u,\ \forall u \in D(A^n).$$

Thus we are led to the abstract Cauchy problem corresponding to $(P^n)$

$$\frac{dp^n}{dt} + A^n p^n = f^n \quad \text{a.e. } t \in (0, T), \tag{3.2.19}$$

$$p^n(0) = p_0^n. \tag{3.2.20}$$

**Remark 3.2.2** *Also in this case, every strong solution to problem* (3.2.19)-(3.2.20) *is actually a solution to problem* $(P^n)$ *in the sense of distributions.*

We notice that problems $(P)$ and $(P^n)$ are the same as mathematical form, but the equations have different coefficients. Theorem 3.1.2 provide an existence and uniqueness result for the solution to problem $(P)$, in a similar way it can be proved that, for every $n \in \mathbb{N}$, the operator $A^n$ is quasi m-accretive and then that the problem $(P^n)$ has a unique solution.

The next step consists in showing that, under suitable convergence assumptions, the solution to problem $(P^n)$ converges, as $n \to \infty$, to the solution to problem $(P)$. Before, in the next section, we recall the hypotheses that guarantee existence and uniqueness of the solutions to problems $(P)$ and $(P^n)$ and introduce the convergence assumptions.

### 3.2.3 Problem hypotheses and preliminaries

First of all, in order to have existence and uniqueness of the solution to problem $(P^n)$ (see Section 3.1.1), we assume that, for each $n \in \mathbb{N}$, all the functions $\beta^n(a,x,s)$, $\mu^n(a,x,s)$, $\gamma^n(a,x,z)$ and $K^n(a,x)$ are measurable and we suppose that $\beta^n(a,x,s)$ and $\mu^n(a,x,s)$ are locally Lipschitz continuous, uniformly with respect to $a$ and $x$, namely for each $R>0$ there exist $L_\beta(R)>0$ and $L_\mu(R)>0$ such that for any $s,\overline{s}\in\mathbb{R}$ with $|s|\le R$ and $|\overline{s}|\le R$,

$$|\beta^n(a,x,s)-\beta^n(a,x,\overline{s})| \le L_\beta(R)|s-\overline{s}|, \tag{3.2.21}$$

$$|\mu^n(a,x,s)-\mu^n(a,x,\overline{s})| \le L_\mu(R)|s-\overline{s}|. \tag{3.2.22}$$

Moreover we assume that, for each $n \in \mathbb{N}$,

$$0 \le \beta^n(a,x,s) \le \beta_+, \tag{3.2.23}$$

$$\mu^n(a,x,s) \ge 0, \ \text{with } \mu^n(a,x,0)=0, \tag{3.2.24}$$

$$0 \le \gamma^n(a,x,s) \le \gamma_+, \tag{3.2.25}$$

$$0 < K_0 \le K^n(a,x) \le K_+. \tag{3.2.26}$$

Note that these assumptions directly concern the functions defined on $Q$ and built in (3.2.11)-(3.2.18). We should have assumed the same hypotheses for the functions in each layer, but in that case the same properties hold for the functions defined on $Q$ (see [13]). Moreover we stress the fact that these assumptions introduce uniformity with respect to the index $n$.

We now introduce the following set of convergence hypotheses to the functions involved in problem $(P)$:

$$\beta^n(a,x,s) \xrightarrow[n\to\infty]{} \beta(a,x,s), \ \text{uniformly with respect to } a, \ x \text{ and } s, \tag{3.2.27}$$

$$\mu^n(a,x,s) \xrightarrow[n\to\infty]{} \mu(a,x,s), \ \text{uniformly with respect to } a, \ x \text{ and } s, \tag{3.2.28}$$

$$\gamma^n(a,x,z) \xrightarrow[n\to\infty]{} \gamma(a,x,z), \ \text{uniformly with respect to } a, \ x \text{ and } z, \tag{3.2.29}$$

$$K^n(a,x) \xrightarrow[n\to\infty]{} K(a,x), \ \text{uniformly with respect to } a \text{ and } x, \tag{3.2.30}$$

Using these convergence assumptions we have that $\beta(a,x,s)$, $\mu(a,x,s)$, $\gamma(a,x,z)$ and $K(a,x)$ satisfy (3.2.21)-(3.2.26) with the same constants $L_\beta(R)$, $L_\mu(R)$, $\beta_+$, $\gamma_\infty$, $K_0$ and $K_+$.

Before proving the convergence result, we recall that, in order to prove existence and uniqueness of the solution of problems $(P)$ and $(P^n)$, it is enough to replace the locally Lipschitz continuity of $\beta(a,x,s)$, $\mu(a,x,s)$, $\beta^n(a,x,s)$ and $\mu^n(a,x,s)$ with conditions of global Lipschitz continuity of the functions $F: H_Q \to H_Q$, $E: H_Q \to H_Q$, $F^n: H_Q \to H_Q$ and $E^n: H_Q \to H_Q$ defined as (see (3.1.17)-(3.1.18))

$$(F(u))(a,x) = \beta(a,x,S(x))u(a,x), \tag{3.2.31}$$

$$(E(u))(a,x) = \mu(a,x,S(x))u(a,x).$$

$$(F^n(u))(a,x) = \beta^n(a,x,S(x))u(a,x),$$

$$(E^n(u))(a,x) = \mu^n(a,x,S(x))u(a,x).$$

Actually we stress the fact that, under our hypotheses, these functions are only locally Lipschitz continuous, but we can truncate them by introducing the approximated functions (see the proof of Theorem 3.1.6)

$$F_N(u) = \begin{cases} F(u) & \text{for } \|u\|_{H_Q} \le N, \\ F\left(\dfrac{Nu}{\|u\|_{H_Q}}\right) & \text{for } \|u\|_{H_Q} > N, \end{cases} \tag{3.2.32}$$

$$E_N(u) = \begin{cases} E(u) & \text{for } \|u\|_{H_Q} \le N, \\ E\left(\dfrac{Nu}{\|u\|_{H_Q}}\right) & \text{for } \|u\|_{H_Q} > N, \end{cases} \tag{3.2.33}$$

$$F_N^n(u) = \begin{cases} F^n(u) & \text{for } \|u\|_{H_Q} \le N, \\ F^n\left(\dfrac{Nu}{\|u\|_{H_Q}}\right) & \text{for } \|u\|_{H_Q} > N, \end{cases} \tag{3.2.34}$$

$$E_N^n(u) = \begin{cases} E^n(u) & \text{for } \|u\|_{H_Q} \le N, \\ E^n\left(\dfrac{Nu}{\|u\|_{H_Q}}\right) & \text{for } \|u\|_{H_Q} > N, \end{cases} \tag{3.2.35}$$

whose are globally Lipschitz continuous on $H_Q$, for each fixed $N$, and we can deal with the corresponding operators $A_N$ and $A_N^n$ whose coincide with $A$ and $A^n$ respectively on the ball of radius $N$.

We conclude that it is enough to develop the further theory for the truncated problems because, as we have seen in the proof of Theorem 3.1.6, the solutions to the original problems are in fact the solutions to these ones. Thus from now on we will deal with the truncated solutions $p_N(t)$ and $p_N^n(t)$ and with the operators $A_N$ and $A_N^n$ associated to them. For the writing simplicity, we shall omit the subscript $N$, however the original functions coincide with the truncated ones in the ball of radius $N$, with a fixed $N$ large enough.

In order to prove the convergence of the solution of $(P^n)$ to the solution of $(P)$ we will use the following result, given in [7], concerning a sequence of quasi m-accretive operators in a Banach space $X$, $\mathcal{A}^n : D(\mathcal{A}^n) \subset X \to X$ and their "limit" $\mathcal{A} : D(\mathcal{A}) \subset X \to X$, stated in terms of their resolvents:

**Theorem 3.2.1** (Brezis-Pazy) *Let $\mathcal{A}^n$ and $\mathcal{A}$ be quasi m-accretive operators and let $\mathcal{U}^n(t)$ and $\mathcal{U}(t)$ be the semigroups generated by $-\mathcal{A}^n$ and $-\mathcal{A}$ respectively. If*

$$\lim_{n\to\infty} (\lambda I + \mathcal{A}^n)^{-1} g = (\lambda I + \mathcal{A})^{-1} g,$$

*for every* $g \in \overline{D}$ *and* $\lambda > \lambda_0$*, where* $\overline{D} = \bigcap_{n \geq 1} \overline{D(\mathcal{A}^n)} \cap \overline{D(\mathcal{A})}$ *and* $\lambda_0$ *is independent of* $n$*, then*

$$\lim_{n \to \infty} \mathcal{U}^n(t)g = \mathcal{U}(t)g,$$

*for every* $g \in \overline{D}$ *and the limit is uniform on bounded intervals for* $t$*.*

Thus, in order to apply Theorem 3.2.1, it remains to prove the convergence of the sequence of the resolvents. To come to this end, we first need some intermediate results concerning the closure of $D(A^n)$ and $D(A)$, some convergence results and estimates for the solutions to the resolvent problems. Next section is devoted to these preliminaries.

### 3.2.4 Study of the resolvents

First of all we study the density of the domains of the quasi m-accretive operators $A^n$ and $A$. Let us mention once more that actually these operators are the truncations $A_N^n$ and $A_N$ defined through the functions $F_N(\cdot)$, $E_N(\cdot)$, $F_N^n(\cdot)$ and $E_N^n(\cdot)$ introduced in (3.2.32)-(3.2.35).

**Proposition 3.2.1** *The domains of the operators* $A$ *and* $A^n$ *satisfy*

$$\overline{D(A)} = \overline{D(A^n)} = H_Q.$$

**Proof.**
We restrict our attention to the case of the operator $A$, the proof of the density of the domain of $A^n$ being identical.

Let us define a linear operator $A_\Delta(a) : V \to V'$ as

$$< A_\Delta(a)u, \psi >_{V',V} = \int_{x_0}^{x_L} K(a,x) u_x \psi_x \, dx, \quad \forall\, u \in V,\ \psi \in V,$$

and consider the associated Cauchy problem

$$\frac{du}{da} + A_\Delta(a)u = 0,$$

$$u(0) = u_0.$$

It is known that, if $u_0 \in H$, this problem has a unique solution that we denote by $u(a,x;u_0)$, such that $u \in L^2(0,a_\dagger;V)$, $u_a \in L^2(0,a_\dagger;V')$ and

$$\|u(a,\cdot;u_0)\|_H \leq \|u_0\|_H. \tag{3.2.36}$$

Let now $f \in H_Q = L^2(Q)$, then there exist a sequence $(f_n) \in \mathcal{C}^\infty([0,a_\dagger] \times [x_0,x_L])$ such that $f_n \to f$ in $H_Q$.
For every $n \in \mathbb{N}$, we introduce the following mapping $T_n : L^2(0,a_\dagger;V) \to H_Q$

$$(T_n g)(a,x) = \rho(na) f_n(a,x) + (1 - \rho(na)) u(a,x;(\phi g)(x)),$$

where $\phi : H_Q \to H$ is defined by

$$(\phi g)(x) = \int_0^{a_\dagger} F(g)(\sigma, x)\ d\sigma,$$

and $\rho \in \mathcal{C}^\infty(\mathbb{R}_+)$ is such that

$$0 \le \rho(z) \le 1, \quad \rho(z) = 0 \text{ for } 0 \le z \le \frac{1}{2}, \quad \rho(z) = 1 \text{ for } z \ge 1.$$

We note that, according to the definition of the function $F$ given in (3.2.31) and to the fact that $F$ is globally Lipschitz continuous with Lipschitz constant $B(N)$ (recall that actually we are working with the truncated function $F_N$), we have that

$$\|\phi g - \phi h\|_H \le B(N)\sqrt{a_\dagger}\|g - h\|_{H_Q}. \tag{3.2.37}$$

Moreover, for every $f, g \in H_Q$, we have

$$\begin{aligned}
\|T_n g - T_n h\|^2_{H_Q} &= \int_0^{1/n}\int_{x_0}^{x_L} |(1-\rho(na))u\Big(a, x; (\phi g)(x) - (\phi h)(x)\Big)|^2\ dxda \le \\
&\le \int_0^{1/n}\int_{x_0}^{x_L} |u\Big(a, x; (\phi g)(x) - (\phi h)(x)\Big)|^2\ dxda \le \\
&\le \int_0^{1/n}\int_{x_0}^{x_L} |(\phi g)(x) - (\phi h)(x)|^2\ dxda = \\
&= \frac{1}{n}\|\phi g - \phi h\|^2_H \le \frac{(B(N))^2 a_\dagger}{n}\|g - h\|^2_{H_Q}.
\end{aligned} \tag{3.2.38}$$

Here we have used the fact that for $na \ge 1$ we have $1 - \rho(na) = 0$ and that for $na \le 1$ we have $1 - \rho(na) \le 1$, consequently the integral on $[1/n, a_\dagger]$ vanishes. Moreover we have used (3.2.36) and (3.2.37).

Thus, from (3.2.38), we conclude that $T_n$ is a contraction on $H_Q$ and then the fixed-point equation $T_n g_n = g_n$ has a unique solution

$$g_n(a, x) = \rho(na) f_n(a, x) + (1 - \rho(na)) u\left(a, x; \int_0^{a_\dagger} F(g_n)(\sigma, x) d\sigma\right). \tag{3.2.39}$$

Observe that

$$\begin{aligned}
g_n(0, x) &= \rho(0) f_n(0, x) + (1 - \rho(0)) u\left(0, x; \int_0^{a_\dagger} F(g_n)(\sigma, x) d\sigma\right) = \\
&= u\left(0, x; \int_0^{a_\dagger} F(g_n)(\sigma, x) d\sigma\right) = \int_0^{a_\dagger} F(g_n)(\sigma, x) d\sigma = \\
&= \int_0^{a_\dagger} \beta(\sigma, x, S(x)) g_n(\sigma, x) d\sigma,
\end{aligned}$$

moreover it is not difficult to prove that $A_0 g_n \in H_Q$ and then, by definition, $g_n \in D(A)$. Furthermore, from (3.2.39), we have

$$\begin{aligned}
\frac{1}{2}\|g_n\|^2_{H_Q} &\leq \|f_n\|^2_{H_Q} + \int_0^{1/n}\int_{x_0}^{x_L} |u(a,x;(\phi g_n)(x))|^2 \, dxda \leq \\
&\leq \|f_n\|^2_{H_Q} + \int_0^{1/n} \|\phi g_n\|^2_H da \leq \\
&\leq \|f_n\|^2_{H_Q} + \frac{1}{n}\|\phi g_n\|^2_H \leq \\
&\leq \|f_n\|^2_{H_Q} + \frac{(B(N))^2 a_\dagger}{n}\|g_n\|^2_{H_Q},
\end{aligned}$$

so that, for $n$ sufficiently large, $n \geq 4(B(N))^2 a_\dagger$, the sequence $(g_n)$ is bounded, indeed

$$\|g_n\|^2_{H_Q} \leq 4\|f\|^2_{H_Q}. \tag{3.2.40}$$

On the other hand,

$$\begin{aligned}
f(a,x) - g_n(a,x) &= f(a,x) - \rho(na) f_n(a,x) - (1-\rho(na))u\Big(a,x;(\phi g_n)(x)\Big) = \\
&= \rho(na)(f(a,x) - f_n(a,x)) + \\
&\quad + (1-\rho(na))\Big[f(a,x) - u\Big(a,x;(\phi g_n)(x)\Big)\Big],
\end{aligned}$$

thus

$$\begin{aligned}
\|f - g_n\|_{H_Q} &\leq \|f - f_n\|_{H_Q} + \left(\int_0^{1/n}\int_{x_0}^{x_L} |f(a,x)|^2 \, dxda\right)^{1/2} + \\
&\quad + \frac{B(N)\sqrt{a_\dagger}}{\sqrt{n}}\|g_n\|_{H_Q},
\end{aligned}$$

and, consequently, using (3.2.40), we have that

$$f = \lim_{n\to\infty} g_n, \qquad \text{in } H_Q,$$

proving that $\overline{D(A)} = H_Q$. As already observed, the same argument shows that also $\overline{D(A^n)} = H_Q$, completing the proof.

□

Moreover, the operator $A$, being quasi m-accretive, generates a nonlinear semigroup of quasi contraction on $\overline{D(A)} = H_Q$, and then the following result holds

**Proposition 3.2.2** *Let $f \in L^1(0,T;H_Q)$ and $p_0 \in \overline{D(A)} = H_Q$, then there exists a weak (mild) solution to the Cauchy problem*

$$\frac{dp}{dt} + Ap = f, \qquad a.e.\ t \in (0,T),$$

$$p(0) = p_0,$$

*satisfying*

$$\frac{1}{2}\|p(t) - \xi\|^2_{H_Q} \leq \frac{1}{2}\|p(s) - \xi\|^2_{H_Q} - \int_s^t (f - A\xi, p(\tau) - \xi)\ d\tau,$$

*for all $\xi \in H_Q$ and for all $s$ such that $0 \leq s \leq t \leq T$.*

Let us recall that, being the operator $A$ quasi m-accretive, in particular, we have

$$\mathcal{R}(\lambda I + A) = H_Q$$

for $\lambda$ large enough, namely

$$\lambda > \lambda_c = \frac{(B(N))^2 + \sqrt{a_\dagger{}^2(B(N))^2 + 4(M(N))^2}}{2},$$

as showed in the proof of Lemma 3.1.3. We now need some estimates for the solution to the resolvent problem associated to the problem $(P)$, namely we shall prove the following result

**Proposition 3.2.3** *Let $\lambda > \lambda_c$ and $g \in H_Q$ and let $u \in L^2(0,a_\dagger;V)$ with $u_a \in L^2(0,a_\dagger;V')$ be the solution of*

$$(\lambda I + A)u = g, \tag{3.2.41}$$

*then $u$ satisfies the following estimates*

$$\lambda\|u\|^2_{H_Q} + \|u(a_\dagger,\cdot)\|^2_H + K_0\|u\|^2_{L^2(0,a_\dagger;V)} \leq \|g\|^2_{H_Q}, \tag{3.2.42}$$

*for $\lambda \geq \lambda_0 = \max(1 + a_\dagger\beta_+^2 + K_0, \lambda_c)$, and*

$$\|u_a\|_{L^2(0,a_\dagger;V')} \leq \left(\sqrt{\lambda} + \frac{K_+}{K_0} + \frac{M(N)}{\sqrt{\lambda}} + 1\right)\|g\|_{H_Q}. \tag{3.2.43}$$

*Moreover, if $u$ and $\widetilde{u}$ are the solutions of the resolvent problem (3.2.41) with $g$ and $\overline{g}$ respectively, then*

$$\begin{aligned}[2\lambda - (1 + a_\dagger B(N)^2 + M(N) + K_0)]\|u - \widetilde{u}\|^2_{H_Q} + \|u(a_\dagger,\cdot) - \widetilde{u}(a_\dagger,\cdot)\|^2_H + \\ + K_0\|u - \widetilde{u}\|_{L^2(0,a_\dagger;V)} \leq \|g - \overline{g}\|_{H_Q},\end{aligned} \tag{3.2.44}$$

*for every $\lambda > \lambda_c$.*

**Proof.**
In order to prove estimate (3.2.42), we multiply equation (3.2.41) by $u$, then we obtain

$$\lambda\|u\|_{H_Q}^2 + \frac{1}{2}\|u(a_\dagger,\cdot)\|_H^2 + \frac{1}{2}\|u(0,\cdot)\|_H^2 + \int_Q K(a,x)u_x^2 dx \leq \|g\|_{H_Q}\|u\|_{H_Q}.$$

Now,

$$\begin{aligned}\|u(0,\cdot)\|_H^2 &= \int_{x_0}^{x_L} |u(0,x)|^2 dx = \int_{x_0}^{x_L} \left|\int_0^{a_\dagger} \beta(a,x,S(s))u(a,x)da\right|^2 dx \leq \\ &\leq a_\dagger\beta_+^2 \int_{x_0}^{x_L}\int_0^{a_\dagger} |u(a,x)|^2 da dx = a_\dagger\beta_+^2\|u\|_{H_Q}^2,\end{aligned}$$

and then

$$[2\lambda - (1 + a_\dagger\beta_+^2 + K_0)]\|u\|_{H_Q}^2 + \|u(a_\dagger,\cdot)\|_H^2 + K_0\|u\|_{L^2(0,a_\dagger;V)}^2 \leq \|g\|_{H_Q}.$$

Thus we conclude that, for

$$\lambda \geq \lambda_0 = \max(1 + a_\dagger\beta_+^2 + K_0, \lambda_c),$$

we obtain (3.2.42) as claimed.

Now, for estimating the norm of $u_a$, we use the following definition for the norm in $L^2(0,a_\dagger;V')$:

$$\|u_a\|_{L^2(0,a_\dagger;V')} = \sup_{\substack{\psi \in L^2(0,a_\dagger;V)\\ \|\psi\| \leq 1}} |\ll u_a, \psi \gg|,$$

where, as specified before, $\ll \cdot,\cdot \gg$ denotes the pairing between $L^2(0,a_\dagger;V)$ and $L^2(0,a_\dagger;V')$, i.e., by definition

$$\ll \Phi, \psi \gg = \int_0^{a_\dagger} < \Phi(a), \psi(a) >_{V',V} \; da,$$

for all $\psi \in L^2(0,a_\dagger;V)$ and $\Phi \in L^2(0,a_\dagger;V')$.
By using the definition of the operator $A$, we have

$$\begin{aligned}|\ll u_a, \psi \gg| &= \left|\int_0^{a_\dagger} < u_a(a,\cdot)\psi(a,\cdot) >_{V',V} \; da\right| = \\ &= \left|\int_Q u_a(a,x)\psi(a,x)dxda\right| = \\ &= \left|\int_Q (-\lambda u\psi - Ku_x\psi_x - \mu(a,x,S(x))u\psi + g\psi)dxda\right| \leq \\ &\leq \Big(\lambda\|u\|_{H_Q} + K_+\|u\|_{L^2(0,a_\dagger;V)} + M(N)\|u\|_{H_Q} + \|g\|_{H_Q}\Big)\|\psi\|_{L^2(0,a_\dagger;V)} \leq \\ &\leq \left(\sqrt{\lambda} + \frac{K_+}{\sqrt{K_0}} + \frac{M(N)}{\sqrt{\lambda}} + 1\right)\|g\|_{H_Q}\|\psi\|_{L^2(0,a_\dagger;V)},\end{aligned}$$

and then (3.2.43) is proved for every $\lambda$ large enough.

Now it remains to prove estimate (3.2.44), we consider the following equation

$$\lambda(u - \widetilde{u}) + (Au - A\widetilde{u}) = g - \overline{g},$$

and we multiply it by $(u - \widetilde{u})$. Thus, developing the calculations as did in proving (3.2.42), and using the fact that

$$\|u(0,\cdot) - \widetilde{u}(0,\cdot)\|_H^2 = \int_{x_0}^{x_L} |u(0,x) - \widetilde{u}(0,x)|^2 dx \leq a_\dagger (B(N))^2 \|u - \widetilde{u}\|_{H_Q}^2,$$

we obtain (3.2.44), completing the proof.

□

We observe that the results of Proposition 3.2.2 and Proposition 3.2.3 also hold for the sequence of operators $A^n$ and all the estimates are uniform with respect to $n$. In particular we have the following

**Proposition 3.2.4** *Let $\lambda > \lambda_c$ and $g \in H_Q$ and let $u^n \in L^2(0, a_\dagger; V)$ with $u_a^n \in L^2(0, a_\dagger; V')$ be the solution of*

$$(\lambda I + A^n)u^n = g, \tag{3.2.45}$$

*then $u^n$ satisfies the following estimates*

$$\lambda \|u^n\|_{H_Q}^2 + \|u^n(a_\dagger, \cdot)\|_H^2 + K_0 \|u^n\|_{L^2(0,a_\dagger;V)}^2 \leq \|g\|_{H_Q}^2,$$

*for $\lambda \geq \lambda_0 = \max(1 + a_\dagger \beta_+^2 + K_0, \lambda_c)$, and*

$$\|u_a^n\|_{L^2(0,a_\dagger;V')} \leq \left( \sqrt{\lambda} + \frac{K_+}{K_0} + \frac{M(N)}{\sqrt{\lambda}} + 1 \right) \|g\|_{H_Q}.$$

*Moreover, if $u^n$ and $\overline{u^n}$ are the solutions of the resolvent problem (3.2.45) with $g$ and $\overline{g}$ respectively, then*

$$[2\lambda - (1 + a_\dagger B(N)^2 + M(N) + K_0)] \|u^n - \overline{u^n}\|_{H_Q}^2 + \|u^n(a_\dagger, \cdot) - \overline{u^n}(a_\dagger, \cdot)\|_H^2 +$$

$$+ K_0 \|u^n - \overline{u^n}\|_{L^2(0,a_\dagger;V)} \leq \|g - \overline{g}\|_{H_Q},$$

*for every $\lambda > \lambda_c$.*

### 3.2.5 Convergence of the semigroup

In this section we are going to apply Theorem 3.2.1 in order to prove that the sequence of the semigroups generated by $-A^n$ converges to the semigroup generated by $-A$. To come to this end, we first prove an intermediate convergence result

**Lemma 3.2.1** *Assume that the hypotheses* (3.2.23)-(3.2.26) *are satisfied and that the convergence assumptions* (3.2.27)-(3.2.30) *hold. Moreover suppose that* $u^n \to u$ *strongly in* $H_Q$ *and weakly in* $L^2(0, a_\dagger; V)$. *Then*

$$F^n(u^n) \to F(u) \quad \textit{strongly in } H_Q, \tag{3.2.46}$$

$$E^n(u^n) \to E(u) \quad \textit{strongly in } H_Q, \tag{3.2.47}$$

$$K^n u_x^n \to K u_x \quad \textit{weakly in } H_Q. \tag{3.2.48}$$

**Proof.**
For proving (3.2.46) we first note that, for any fixed $\varepsilon > 0$, and for $n$ sufficiently large, we have

$$\begin{aligned}
|(F^n(u))(a,x) - (F(u))(a,x)| &= |\beta^n(a,x,S^n(x))u(a,x) - \beta(a,x,S(x))u(a,x)| \leq \\
&\leq |\beta^n(a,x,S^n(x)) - \beta(a,x,S^n(x))||u(a,x)| + \\
&\quad +|\beta(a,x,S^n(x)) - \beta(a,x,S(x))||u(a,x)| \leq \\
&\leq \varepsilon|u(a,x)| + L_\beta(N)|S^n(x) - S(x)||u(a,x)| \leq \\
&\leq \varepsilon(1 + L_\beta(N))|u(a,x)|,
\end{aligned}$$

where we have used (3.2.27) and (3.2.29). Thus we have

$$\begin{aligned}
\|F^n(u^n) - F(u)\|_{H_Q} &\leq \|F^n(u^n) - F^n(u)\|_{H_Q} + \|F^n(u) - F(u)\|_{H_Q} \leq \\
&\leq B(N)\|u^n - u\|_{H_Q} + \varepsilon(1 + L_\beta(N))\|u\|_{H_Q}.
\end{aligned}$$

The convergence (3.2.47) can be proved in the same way, by using the Lipschitz continuity of the function $E$ and the convergence hypotheses.

Concerning (3.2.48), we observe that, for any $\psi \in H_Q$ and $\varepsilon > 0$, and for $n$ sufficiently large, we have

$$\begin{aligned}
\left|\int_Q (K^n u_x^n - K u_x)\psi dx da\right| &\leq \left|\int_Q (K^n - K)u_x^n \psi dx da\right| + \left|\int_Q K\psi(u_x^n - u_x) dx da\right| \leq \\
&\leq \varepsilon \left|\int_Q u_x^n \psi dx da\right| + \left|\int_Q K\psi(u_x^n - u_x) dx da\right| \leq \\
&\leq \varepsilon \|u_x^n\|_{H_Q} \|\psi\|_{H_Q} + \left|\int_Q K\psi(u_x^n - u_x) dx da\right|.
\end{aligned}$$

Thus, since $\|u_x^n\|_{H_Q}$ is bounded and $K\psi \in H_Q$, we get (3.2.48). □

Now, in view of Theorem 3.2.1, we prove the convergence of the sequence of the resolvents $(\lambda I + A^n)^{-1}$ to $(\lambda I + A)^{-1}$.

**Theorem 3.2.2** *Assume the properties* (3.2.23)-(3.2.26) *and the convergence hypotheses* (3.2.27)-(3.2.30). *Then, for every* $g \in H_Q$

$$\lim_{n\to\infty} (\lambda I + A^n)^{-1} g = (\lambda I + A)^{-1} g \ \text{in}\ H_Q.$$

**Proof.**
For $g \in H_Q$ we denote by $u^n$ the solution to the resolvent problem associated to $A^n$, namely $u^n = (\lambda I + A^n)^{-1} g$. We have to prove that $u^n$ converges in $H_Q$ and the limit is exactly $(\lambda I + A)^{-1} g$.

First of all, observe that, from Proposition 3.2.4, we have that the sequence $u^n$ is bounded in $H_Q \cap L^2(0, a_\dagger; V)$, the sequence $u^n_a$ is bounded in $L^2(0, a_\dagger; V')$ and $u^n(a_\dagger, \cdot)$ lies in a bounded subset of $H$.

Thus, extracting a subsequence, we have that

$$u^n \to u \text{ weakly in } L^2(0, a_\dagger; V), \tag{3.2.49}$$

$$u^n_a \to u_a \text{ weakly in } L^2(0, a_\dagger; V'), \tag{3.2.50}$$

$$u^n(a_\dagger, \cdot) \to \zeta \text{ weakly in } V'.$$

It remains to show that $u = (\lambda I + A)^{-1} g$ and $\zeta = u(a_\dagger, 0)$.
From (3.2.49) and (3.2.50) and since $V$ is compact in $H$, we have that

$$u^n \to u \text{ strongly in } L^2(0, a_\dagger; H) \equiv H_Q.$$

Now, let us consider the following equality

$$\int_Q \Big(\lambda u^n \psi + u^n_a \psi + K^n(a,x) u^n_x \psi + \mu^n(a, x, S^n(x)) u^n \psi - g\psi\Big) dx da = 0,$$

for every $\psi \in L^2(0, a_\dagger; V)$, and pass to the limit, then, using (3.2.49),(3.2.50) and Lemma 3.2.1, we obtain

$$\int_Q \Big(\lambda u \psi + u_a \psi + K(a,x) u_x \psi + \mu(a, x, S(x)) u \psi - g\psi\Big) dx da = 0.$$

Then it is obvious that $u$ is the solution to the resolvent problem $(\lambda I + A)u = g$.

Moreover

$$u^n(0, \cdot) \to u(0, \cdot) \text{ strongly in } H,$$

in fact

$$\begin{aligned} \|u^n(0,\cdot) - u(0,\cdot)\|_H^2 &= \int_{x_0}^{x_L} |u^n(0,x) - u(0,x)|^2 dx = \\ &= \int_{x_0}^{x_L} \left| \int_0^{a_\dagger} (F^n(u^n)(a,x) - F(u)(a,x)) da \right|^2 dx \le \\ &\le a_\dagger \|F^n(u^n) - F(u)\|_{H_Q}^2 \le a_\dagger (B(N))^2 \|u^n - u\|_{H_Q}^2. \end{aligned}$$

Finally we also have that

$$u^n(a_\dagger,\cdot) \to u(a_\dagger,\cdot) \text{ weakly in } V',$$

in fact we can write

$$u^n(a_\dagger,x) = u^n(0,x) + \int_0^{a_\dagger} u_a^n(a,x)da,$$

since we have already proved that $u^n(0,x) \to u(0,x)$ strongly in $H$, it remains to show the convergence of the integral. Actually , by using (3.2.50), for every $\phi \in V$, we have

$$\int_{x_0}^{x_L} \phi(x) \int_0^{a_\dagger} u_a^n(a,x)dadx = \int_{x_0}^{x_L}\int_0^{a_\dagger} \phi(x)u_a^n(a,x)dadx \to$$

$$\to \int_{x_0}^{x_L}\int_0^{a_\dagger} \phi(x)u_a(a,x)dadx = \int_{x_0}^{x_L} \phi(x)\int_0^{a_\dagger} u_a(a,x)dadx.$$

□

### 3.2.6 Convergence for the Cauchy problems

Before proving the convergence of the solution to problem $(P^n)$ to the solution to problem $(P)$, we first consider the Cauchy problems associated to the sequence of operators $A^n$ with the same data $f \in L^1(0,T;H_Q)$ and $p_0 \in H_Q$ as in problem $(P)$, namely we consider the problem

$$\frac{d\widetilde{p}^n}{dt} + A^n\widetilde{p}^n = f,$$

$$\widetilde{p}^n(0) = p_0.$$

We shall refer to this problem as problem $(\widetilde{P^n})$. The following result holds

**Theorem 3.2.3** *Let* $f \in L^2(0,T;H_Q)$, $p_0 \in H_Q$ *and assume* (3.2.23)-(3.2.26) *and* (3.2.27)-(3.2.30). *Then the solution to problem* $(\widetilde{P^n})$ *tends to the solution to problem* $(P)$, *i.e.*

$$\widetilde{p}^n(t) \to p(t) \text{ uniformly for } t \in [0,T].$$

**Proof.**
Let us first take $f \in W^{1,1}(0,\infty;H_Q)$. According to a procedure proposed in [14], we consider the space $\mathcal{X} = H_Q \times L^1(0,\infty;H_Q)$ endowed with the norm

$$\|(u,\gamma)\|_{\mathcal{X}} = \|u\|_{H_Q} + \int_0^\infty \|\gamma(t)\|_{H_Q}dt.$$

We now define the following operator in the space $\mathcal{X}$

$$\mathcal{A}_n : D(\mathcal{A}_n) = D(A) \times W^{1,1}(0,\infty;H_Q) \subset \mathcal{X} \to \mathcal{X},$$

by

$$\mathcal{A}_n(u,\gamma) = \begin{pmatrix} A^n u - \gamma(0) \\ -\gamma' \end{pmatrix},$$

for all $(u,\gamma) \in D(A) \times W^{1,1}(0,\infty; H_Q)$.
If we denote

$$\mathcal{P}^n(t) = \begin{pmatrix} \widetilde{p}^n(t) \\ \Gamma(t) \end{pmatrix},$$

where $\Gamma \in W^{1,\infty}(0,T; W^{1,1}(0,\infty; H_Q))$ is defined by

$$\Gamma(t)(s) = f(t+s), \ \forall s \in (0,+\infty),$$

then it follows that the problem

$$\frac{d\mathcal{P}^n}{dt}(t) + \mathcal{A}^n \mathcal{P}^n(t) = 0, \ \text{a.e. } t \in (0,T), \tag{3.2.51}$$

$$\mathcal{P}^n(0) = \begin{pmatrix} p_0 \\ f(s) \end{pmatrix}, \tag{3.2.52}$$

is equivalent to problem $(\widetilde{P^n})$. Indeed, the second component of problem (3.2.51)-(3.2.52) reads

$$\frac{\partial\gamma(t,s)}{\partial t} - \frac{\partial\gamma(t,s)}{\partial s} = 0,$$

$$\Gamma(0,s) = f(s),$$

that are necessarily satisfied by $\Gamma(t,s) = f(t+s)$.
On the other hand, the first component of problem (3.2.51)-(3.2.52) gives

$$\frac{d\widetilde{p}^n}{dt}(t) + A^n \widetilde{p}^n(t) - \Gamma(t)(0) = 0, \ \text{a.e. } t \in (0,T),$$

and then, since $\Gamma(t)(0) = f(t)$, we obtain exactly problem $(\widetilde{P^n})$.

Of course the same construction performed above can be repeated for problem $(P)$, in particular we define an operator $\mathcal{A} : D(\mathcal{A}) = D(A) \times W^{1,1}(0,\infty; H_Q) \subset \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{A}(u,\gamma) = \begin{pmatrix} Au - \gamma(0) \\ -\gamma' \end{pmatrix}.$$

First of all, in order to apply Theorem 3.2.1, we observe that, according to the results given in [10] and [14], the operators $\mathcal{A}^n$ and $\mathcal{A}$ are quasi m-accretive on $\mathcal{X}$. Moreover, from Proposition 3.2.1, we deduce that $\overline{D(\mathcal{A}^n)} = \overline{D(\mathcal{A})} = \mathcal{X}$.

Finally we prove that the sequence of the resolvents of the operators $\mathcal{A}^n$ converge to the resolvent of $\mathcal{A}$. To do this we fix $\begin{pmatrix} g \\ \phi \end{pmatrix} \in \mathcal{X}$, then, using the m-accretiveness of the operator $\mathcal{A}^n$, for $\lambda$ sufficiently large we may consider $\begin{pmatrix} u^n \\ \gamma^n \end{pmatrix} \in D(A^n) \times W^{1,1}(0,\infty; H_Q)$, given by

$$\begin{pmatrix} u^n \\ \gamma^n \end{pmatrix} = (\lambda I + \mathcal{A}^n)^{-1} \begin{pmatrix} g \\ \phi \end{pmatrix}.$$

In other words we have

$$\lambda u^n + A^n u^n - \gamma^n(0) = g, \tag{3.2.53}$$

$$\lambda \gamma^n - (\gamma^n)' = \phi. \tag{3.2.54}$$

By solving equation (3.2.54) we have that

$$\gamma^n(s) = e^{\lambda s}\left(C - \int_0^s e^{-\lambda\tau}\phi(\tau)d\tau\right),$$

where the constant $C$ should be determined such that $\gamma^n \in W^{1,1}(0,\infty; H_Q)$. Hence we impose that $\lim_{s\to\infty} \gamma^n(s) = 0$, it follows that

$$C = \int_0^\infty e^{-\lambda\tau}\phi(\tau)d\tau,$$

and then

$$\gamma^n(s) = e^{\lambda s}\int_s^\infty e^{-\lambda\tau}\phi(\tau)d\tau.$$

Notice that this solution does not depend on $n$, i.e. $\gamma^n(s) = \gamma(s)$. Now we introduce it in equation (3.2.53), we obtain

$$\lambda u^n + A^n u^n = g + \int_0^\infty e^{-\lambda\tau}\phi(\tau)d\tau.$$

Since $\int_0^\infty e^{-\lambda\tau}\phi(\tau)d\tau \in H_Q$, we have, by Theorem 3.2.2, that

$$\begin{pmatrix} u^n \\ \gamma^n \end{pmatrix} = \begin{pmatrix} (\lambda I + A^n)^{-1}\left(g + \int_0^\infty e^{-\lambda\tau}\phi(\tau)d\tau\right) \\ e^{\lambda s}\int_s^\infty e^{-\lambda\tau}\phi(\tau)d\tau \end{pmatrix} \xrightarrow[n\to\infty]{}$$

$$\xrightarrow[n\to\infty]{} \begin{pmatrix} (\lambda I + A)^{-1}\left(g + \int_0^\infty e^{-\lambda\tau}\phi(\tau)d\tau\right) \\ e^{\lambda s}\int_s^\infty e^{-\lambda\tau}\phi(\tau)d\tau \end{pmatrix} = \begin{pmatrix} u \\ \gamma \end{pmatrix}.$$

It is obvious that

$$\begin{pmatrix} u \\ \gamma \end{pmatrix} = (\lambda I + \mathcal{A})^{-1}\begin{pmatrix} g \\ \phi \end{pmatrix}.$$

Thus the assumptions of Theorem 3.2.1 are satified, and then we conclude that the sequence of the semigroups generated by the operators $-\mathcal{A}^n$ corresponding to problem $(\widetilde{P^n})$ converges to the semigroup generated by the operator $-\mathcal{A}$, i.e.

$$\mathcal{P}^n(t) = \begin{pmatrix} \widetilde{p}^n(t) \\ \Gamma(t) \end{pmatrix} \xrightarrow[n\to\infty]{} \begin{pmatrix} p(t) \\ \Gamma(t) \end{pmatrix} = \mathcal{P}(t),$$

uniformly for $t \in [0, T]$.

Now, if $f \in L^2(0,T;H_Q)$, it is sufficient to extend it by 0 on $(T, \infty)$ and consider a sequence $f_m \in W^{1,1}(0,\infty;H_Q)$, such that $f_m \xrightarrow[m\to\infty]{} f$ in $L^2(0,\infty)$; then the resul is preserved, on the basis of estimate (3.1.36) rewritten for problem $(P^n)$.

□

The following theorem gathers all the results obtained up to now and concludes the proof of the convergence of the multi-layer model to the continuous one.

**Theorem 3.2.4** *Let* $f^n \in L^2(0,T;H_Q)$, $p_0 \in H_Q$ *and assume* (3.2.23)-(3.2.26), (3.2.27)-(3.2.30) *and the convergence hypotheses*

$$f^n \to f \text{ strongly in } L^2(0,T;H_Q), \tag{3.2.55}$$

$$p_0^n \to p_0 \text{ strongly in } L^2(0,T;H_Q). \tag{3.2.56}$$

*Then, if we denote by* $p^n$ *and* $p$ *the solutions to problem* $(P^n)$ *and* $(P)$ *respectively, we have*

$$\lim_{n\to\infty} p^n(t) = p(t) \text{ uniformly for any } t \in [0,T].$$

**Proof.**

Let us consider problems $(\widetilde{P^n})$ and $(P^n)$, they are in fact the same problem with different data: $f$, $p_0$ and $f^n$ , $p_0^n$ respectively. We extend $f^n$ to $(0,\infty)$ by setting $f^n(t) = 0$ for $t \in [T,\infty)$.
By (3.1.36), (3.2.55) and (3.2.56) we have that

$$\|p^n(t) - \widetilde{p}^n(t)\|^2_{H_Q} \xrightarrow[n\to\infty]{} 0,$$

so that, using the result of Theorem 3.2.3, we conclude that the solution $p^n(t)$ of problem $(P^n)$ tends to the solution $p(t)$ of problem $(P)$ uniformly for $t \in [0,T]$.

□

## 3.3 Analytical solution for the linear two-layer model

In this section we consider a particular case of the multi-layer model, namely the linear case in a two-layer environment, and look for an analytical expression of solution of the problem.
In particular, after a brief description of the model (Section 3.3.1), in Sections 3.3.2 and 3.3.3 we study the problem in a single layer, giving an analytical expression of the solution in terms of a new variable $q$ representing the flux at the interface between the two layers.
In Section 3.3.4 we impose the continuity of the solution at the interface between the layers, ending up with an integral equation for $q$, containing both a Volterra type and a Fredholm type term. Finally, in Section 3.3.5, we analyze this integral equation and prove that it has a unique solution, by using a fixed point argument.

### 3.3.1 Description of the linear model

As already specified, here we deal with a particular case of the multi-layer model, namely we consider a one-dimensional spatial domain $[x_0, x_2]$ composed of two layers of the same thickness $h$, with $x_0 = 0$ and then $x_1 = h$ and $x_2 = 2h$. Moreover we suppose that the age-specific fertility $\beta(a)$ and the age-specific mortality $\mu(a)$ are the same in the two layers. We assume that the diffusion coefficients $K_1$ and $K_2$, in the first and in the second layer respectively, do not depend on age. Finally we suppose that there is not an individual source, namely $f_1 = f_2 = 0$. Thus now we deal with the following linear problem $(j = 1, 2)$

$$\frac{\partial p_j}{\partial t} + \frac{\partial p_j}{\partial a} + \mu(a)p_j - K_j\frac{\partial^2 p_j}{\partial x^2} = 0, \qquad \text{in } (0,T) \times Q_j \tag{3.3.1}$$

$$p_j(0,a,x) = p_{j0}(a,x), \qquad \text{for } (a,x) \in Q_j, \tag{3.3.2}$$

$$p_j(t,0,x) = \int_0^{a_\dagger} \beta(a)p_j(t,a,x)da \qquad \text{for } (t,x) \in (0,T) \times (x_{j-1}, x_j), \tag{3.3.3}$$

$$\frac{\partial p_1}{\partial x}(t,a,0) = \frac{\partial p_2}{\partial x}(t,a,2h) = 0, \tag{3.3.4}$$

$$p_1(t,a,h^-) = p_2(t,a,h^+), \tag{3.3.5}$$

$$K_1\frac{\partial p_1}{\partial x}(t,a,h^-) = K_2\frac{\partial p_2}{\partial x}(t,a,h^+), \tag{3.3.6}$$

Here we denote by $p_j(t,a,x)$ ($t \in (0,T)$, $a \in (0,a_\dagger)$, $x \in (x_{j-1}, x_j)$) the age-specific density of the population at time $t$ in the layer $j$. Notice that now we do not use the same notation as in problem (3.2.1)-(3.2.8), namely we omit the superscript $n = 2$.
Let us suppose that all the hypotheses that guarantee existence and uniqueness of the solution to problem (3.3.1)-(3.3.6) are satisfied, in particular we assume conditions (3.2.23)-(3.2.26).

In order to simplify the problem, following a standard procedure, we perform the change of variables

$$p_j(t,a,x) = \Pi(a)\widetilde{p_j}(t,a,x),$$

$$p_{j0}(a,x) = \Pi(a)\widetilde{p}_{j0}(a,x),$$

where $\Pi(a)$ is the survival probaibility defined as

$$\Pi(a) = e^{-\int_0^a \mu(\sigma)d\sigma}.$$

This change of variables transforms the problem into the next one for the unknowns $\widetilde{p_j}$. However, for the writing simplicity, we shall no longer indicate the $\sim$ symbol. Thus the problem we deal with reads $(j = 1, 2)$

$$\frac{\partial p_j}{\partial t} + \frac{\partial p_j}{\partial a} - K_j\frac{\partial^2 p_j}{\partial x^2} = 0, \qquad \text{in } (0,T) \times Q_j \tag{3.3.7}$$

$$p_j(0,a,x) = p_{j0}(a,x), \qquad \text{for } (a,x) \in Q_j, \tag{3.3.8}$$

$$p_j(t,0,x) = \int_0^{a_\dagger} m(a)p_j(t,a,x)da \qquad \text{for } (t,x) \in (0,T) \times (x_{j-1}, x_j), \tag{3.3.9}$$

$$\frac{\partial p_1}{\partial x}(t,a,0) = \frac{\partial p_2}{\partial x}(t,a,2h) = 0, \tag{3.3.10}$$

$$p_1(t,a,h^-) = p_2(t,a,h^+), \tag{3.3.11}$$

$$K_1\frac{\partial p_1}{\partial x}(t,a,h^-) = K_2\frac{\partial p_2}{\partial x}(t,a,h^+), \tag{3.3.12}$$

where $m(a) = \beta(a)\Pi(a)$ is the maternity function.

From Section 3.2.2, we know that the problem (3.3.7)-(3.3.12) is equivalent to the abstract Cauchy problem

$$\frac{dp}{dt} + \mathcal{A}p = 0,$$

$$p(0) = p_0,$$

here the operator $\mathcal{A}$ is actually the operator $\mathcal{A} = A^n$ defined in Section 3.2.2 with $n = 2$. We recall that the operators $A^n$ defined in Section 3.2.2, have the same properties as the operator $A$ defined in Section 3.1.2. In particular we know that the operator $-\mathcal{A} = -A^2$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup $e^{-t\mathcal{A}}$ on the space $H_Q = L^2(Q)$, satisfying the exponential bound (see (3.1.35))

$$\|e^{-t\mathcal{A}}\|_{H_Q} \le e^{(\beta_+^2 a_\dagger + 2)t}.$$

Now we aim to a constructive approach to the problem, providing an analytical representation of $e^{-t\mathcal{A}}$ and we start focusing on the resolvent $(\lambda I + \mathcal{A})^{-1}$ of the operator $\mathcal{A}$. Namely, for any fixed $f \in H_Q$, we consider the following problem

$$\lambda u_j(\lambda,a,x) + \frac{\partial u_j}{\partial a}(\lambda,a,x) - K_j\frac{\partial^2 u_j}{\partial x^2}(\lambda,a,x) = f_j(a,x), \quad \text{in } Q_j, \tag{3.3.13}$$

$$u_j(\lambda,0,x) = \int_0^{a_\dagger} m(a)u_j(\lambda,a,x)da, \quad \text{for } x \in (x_0,x_L), \tag{3.3.14}$$

$$\frac{\partial u_1}{\partial x}(\lambda,a,0) = \frac{\partial u_2}{\partial x}(\lambda,a,2h) = 0, \quad \text{for } a \in (0,a_\dagger), \tag{3.3.15}$$

$$K_1\frac{\partial u_1}{\partial x}(\lambda,a,h^-) = K_2\frac{\partial u_2}{\partial x}(\lambda,a,h^+), \quad \text{for } a \in (0,a_\dagger), \tag{3.3.16}$$

$$u_1(\lambda,a,h^-) = u_2(\lambda,a,h^+), \quad \text{for } a \in (0,a_\dagger), \tag{3.3.17}$$

where $f_j$ is the restriction of $f$ to the domain $Q_j$, i.e.

$$f(a,x) = \begin{cases} f_1(a,x), & \text{if } x \in (0,h), \\ f_2(a,x), & \text{if } x \in (h,2h). \end{cases}$$

Notice that $u_j$ are actually the Laplace transforms with respect to $t$ of the functions $p_j$.

Wishing to give an analytical expression to the solution of this problem, we will first consider the problem in each layer and then we will impose the conditions at the interface between the two layers to build the whole solution. Next section is devoted to the first step.

### 3.3.2 The solution in a single layer

As explained at the end of the previous section, we focus on a single layer and, in order to take into account the boundary conditions, we introduce a new variable $q(\lambda, a)$, representing the flux at the interface between the layers, namely we take (see condition (3.3.16))

$$q(\lambda, a) = K_1 \frac{\partial u_1}{\partial x}(t, a, h^-) = K_2 \frac{\partial u_2}{\partial x}(t, a, h^+).$$

Giving this variable as known, we consider the following problems in the first and in the second layer respectively

$$\lambda u_1(\lambda, a, x) + \frac{\partial u_1}{\partial a}(\lambda, a, x) - K_1 \frac{\partial^2 u_1}{\partial x^2}(\lambda, a, x) = f_1(a, x), \tag{3.3.18}$$

$$u_1(\lambda, 0, x) = \int_0^{a_\dagger} m(a) u_1(\lambda, a, x) \, da, \tag{3.3.19}$$

$$K_1 \frac{\partial u_1}{\partial x}(\lambda, a, h^-) = q(\lambda, a), \tag{3.3.20}$$

and

$$\lambda u_2(\lambda, a, x) + \frac{\partial u_2}{\partial a}(\lambda, a, x) - K_2 \frac{\partial^2 u_2}{\partial x^2}(\lambda, a, x) = f_2(a, x), \tag{3.3.21}$$

$$u_2(\lambda, 0, x) = \int_0^{a_\dagger} m(a) u_2(\lambda, a, x) \, da, \tag{3.3.22}$$

$$K_2 \frac{\partial u_2}{\partial x}(\lambda, a, h^+) = q(\lambda, a). \tag{3.3.23}$$

The continuity of the solution at the interface between the two layers will be guaranteed by imposing the following condition

$$u_1(\lambda, a, h^-) = u_2(\lambda, a, h^+). \tag{3.3.24}$$

Now, in order to deal with non-homogeneous boundary conditions, we look for $u_1$ and $u_2$, solutions to problems (3.3.18)-(3.3.20) and (3.3.21)-(3.3.23) respectively, of the form

$$u_j(\lambda, a, x) = U_j(\lambda, a, x) + \phi_j(x) \frac{q(\lambda, a)}{K_j}, \quad j = 1, 2, \tag{3.3.25}$$

where $\phi_1(x) = \dfrac{x^2}{2h}$ and $\phi_2(x) = -\dfrac{(x-2h)^2}{2h}$ and $U_1$ and $U_2$ are the solutions of the problem in the first and in the second layer with homogeneous boundary conditions. In particular $U_1$ solves

$$\lambda U_1 + \frac{\partial U_1}{\partial a} - g_1 = K_1 \frac{\partial^2 U_1}{\partial x^2}, \tag{3.3.26}$$

where

$$g_1 = g_1(\lambda, a, x) = f_1(a, x) + \frac{1}{h} q(\lambda, a) - \frac{\phi_1(x)}{K_1} w(\lambda, a), \tag{3.3.27}$$

with

$$w(\lambda, a) = \lambda q(\lambda, a) + \frac{\partial q}{\partial a}(\lambda, a), \tag{3.3.28}$$

together with the following conditions

$$U_1(\lambda, 0, x) = \int_0^{a_\dagger} m(a) U_1(\lambda, a, x) da, \tag{3.3.29}$$

and

$$\frac{\partial U_1}{\partial x}(\lambda, a, 0) = \frac{\partial U_1}{\partial x}(\lambda, a, h) = 0. \tag{3.3.30}$$

While $U_2$ solves

$$\lambda U_2 + \frac{\partial U_2}{\partial a} - g_2 = K_2 \frac{\partial^2 U_2}{\partial x^2}, \tag{3.3.31}$$

where

$$g_2 = g_2(\lambda, a, x) = f_2(a, x) - \frac{1}{h} q(\lambda, a) - \frac{\phi_2(x)}{K_2} w(\lambda, a), \tag{3.3.32}$$

with the conditions

$$U_2(\lambda, 0, x) = \int_0^{a_\dagger} m(a) U_2(\lambda, a, x) da, \tag{3.3.33}$$

and

$$\frac{\partial U_2}{\partial x}(\lambda, a, h) = \frac{\partial U_2}{\partial x}(\lambda, a, 2h) = 0. \tag{3.3.34}$$

Having reduced the problem to the problems (3.3.26)-(3.3.30) and (3.3.31)-(3.3.34), we now concentrate our attention on their solutions by Fourier expansion.
Let us denote by $\lambda_n$ the eigenvalues and by $X_j^{(n)}$ the respective normalized eigenfunctions of the Laplace operator in the $j$-th layer with homogeneous Neumann boundary conditions, namely we consider the eigenvalue problems ($j = 1, 2$)

$$\frac{\partial^2 X_j^{(n)}}{\partial x^2} + \lambda_n X_j^{(n)} = 0, \qquad \text{in } (x_{j-1}, x_j),$$

$$\frac{\partial X_j^{(n)}}{\partial x}(x_{j-1}) = \frac{\partial X_j^{(n)}}{\partial x}(x_j) = 0,$$

and then we have $\lambda_n = \dfrac{n^2 \pi^2}{h^2}$ and $X_j^{(n)}(x) = \sqrt{\dfrac{2}{h}} \cos\left[\dfrac{n\pi}{h}(x - x_{j-1})\right]$.
Now we can write the functions $U_j$, $f_j$ and $g_j$ respectively as

$$U_j(\lambda, a, x) = \sum_{n=0}^{\infty} U_j^{(n)}(\lambda, a) X_j^{(n)}(x), \tag{3.3.35}$$

$$f_j(a, x) = \sum_{n=0}^{\infty} f_j^{(n)}(a) X_j^{(n)}(x),$$

$$g_j(\lambda, a, x) = \sum_{n=0}^{\infty} g_j^{(n)}(\lambda, a) X_j^{(n)}(x).$$

In particular we have

$$g_1^{(0)}(\lambda,a) = f_1^{(0)}(a) + \frac{1}{h}q(\lambda,a) - \frac{\phi_1^{(0)}}{K_1}w(\lambda,a),$$

$$g_1^{(n)}(\lambda,a) = f_1^{(n)}(a) - \frac{\phi_1^{(n)}}{K_1}w(\lambda,a),\ n \geq 1,$$

where $\phi_1^{(n)}$ are the Fourier coefficients of $\phi_1(x)$, i.e.

$$\phi_1^{(0)} = \sqrt{\frac{2}{h}}\frac{h^2}{6}, \quad \phi_1^{(n)} = \sqrt{\frac{2}{h}}\frac{(-1)^n}{\lambda_n},\ n \geq 1,$$

and

$$g_2^{(0)}(\lambda,a) = f_2^{(0)}(a) - \frac{1}{h}q(\lambda,a) - \frac{\phi_2^{(0)}}{K_2}w(\lambda,a),$$

$$g_2^{(n)}(\lambda,a) = f_2^{(n)}(a) - \frac{\phi_2^{(n)}}{K_2}w(\lambda,a),\ n \geq 1,$$

where $\phi_2^{(n)}$ are the Fourier coefficients of $\phi_2(x)$, i.e.

$$\phi_2^{(0)} = -\sqrt{\frac{2}{h}}\frac{h^2}{6}, \quad \phi_2^{(n)} = -\sqrt{\frac{2}{h}}\frac{1}{\lambda_n},\ n \geq 1.$$

We are now interested in finding an expression for $U_1(\lambda,a,x)$ and $U_2(\lambda,a,x)$ in terms of $q(\lambda,a)$ and $w(\lambda,a)$, we shall develop the calculations only for $U_1$, the ones for $U_2$ being very similar.

By substituting (3.3.35) in (3.3.26) and in (3.3.30), we have, for every fixed $n$, the following problem for $U_1^{(n)}$

$$\frac{\partial U_1^{(n)}}{\partial a} + (\lambda + \lambda_n K_1)\, U_1^{(n)} = g_1^{(n)}, \tag{3.3.36}$$

$$U_1^{(n)}(\lambda,0) = \int_0^{a_\dagger} m(a)U_1^{(n)}(\lambda,a)da. \tag{3.3.37}$$

By integrating equation (3.3.36), we have

$$\begin{aligned} U_1^{(n)}(\lambda,a) = {} & U_1^{(n)}(\lambda,0)e^{-(\lambda+\lambda_n K_1)a} + \\ & + \int_0^a e^{-(\lambda+\lambda_n K_1)(a-\tau)} g_1^{(n)}(\lambda,\tau)d\tau, \end{aligned}$$

and, imposing the initial condition (3.3.37), we end up with

$$\begin{aligned} U_1^{(n)}(\lambda,a) = {} & e^{-(\lambda+\lambda_n K_1)a}\frac{I_1^{(n)}(\lambda)}{1-\widehat{m}(\lambda+\lambda_n K_1)} \\ & + \int_0^a e^{-(\lambda+\lambda_n K_1)(a-\tau)} g_1^{(n)}(\lambda,\tau)d\tau, \end{aligned} \tag{3.3.38}$$

where $\widehat{m}$ denotes the Laplace transform of the maternity function $m$ and

$$I_1^{(n)}(\lambda) = \int_0^{a_\dagger} m(a) \int_0^a e^{-(\lambda+\lambda_n K_1)(a-\tau)} g_1^{(n)}(\lambda,\tau) d\tau da. \tag{3.3.39}$$

Thus, by substituting (3.3.38)in (3.3.35), we get

$$\begin{aligned} U_1(\lambda,a,x) &= e^{-\lambda a} \sum_{n=0}^{\infty} \left( \int_0^a e^{\lambda\tau} e^{-\lambda_n K_1(a-\tau)} g_1^{(n)}(\lambda,\tau) d\tau \right) X_1^{(n)}(x) + \\ & +e^{-\lambda a} \sum_{n=0}^{\infty} e^{-\lambda_n K_1 a} \frac{I_1^{(n)}(\lambda)}{1-\widehat{m}(\lambda+\lambda_n K_1)} X_1^{(n)}(x) = \\ &= \mathbf{A} + \mathbf{B}. \end{aligned} \tag{3.3.40}$$

Now,

$$\begin{aligned} \mathbf{A} &= e^{-\lambda a} \int_0^a e^{\lambda\tau} \left( \sum_{n=0}^{\infty} e^{-\lambda_n K_1(a-\tau)} g_1^{(n)}(\lambda,\tau) X_1^{(n)}(x) \right) d\tau = \\ &= e^{-\lambda a} \int_0^a e^{\lambda\tau} \left( \sum_{n=0}^{\infty} e^{-\lambda_n K_1(a-\tau)} X_1^{(n)}(x) \int_0^h g_1^{(n)}(\lambda,\tau,\sigma) X_1^{(n)}(\sigma) d\sigma \right) d\tau = \\ &= e^{-\lambda a} \int_0^a \int_0^h e^{\lambda\tau} g_1(\lambda,\tau,\sigma) \left( \sum_{n=0}^{\infty} e^{-\lambda_n K_1(a-\tau)} X_1^{(n)}(x) X_1^{(n)}(\sigma) \right) d\sigma d\tau = \\ &= \int_0^a \int_0^h g_1(\lambda,\tau,\sigma) N_1(a,\tau,x,\sigma) d\sigma d\tau, \end{aligned}$$

where we have denoted

$$N_1(a,\tau,\sigma) = e^{-\lambda(a-\tau)} \sum_{n=0}^{\infty} e^{-\lambda_n K_1(a-\tau)} X_1^{(n)}(x) X_1^{(n)}(\sigma).$$

Concerning the second term in (3.3.40), we first observe that, from (3.3.39), we have

$$\begin{aligned} I_1^{(n)} &= \int_0^{a_\dagger} \int_\tau^{a_\dagger} m(a) e^{-(\lambda+\lambda_n K_1)(a-\tau)} g_1^{(n)}(\lambda,\tau) da d\tau = \\ &= \int_0^{a_\dagger} \int_0^{a_\dagger-\tau} m(\rho+\tau) e^{-(\lambda+\lambda_n K_1)\rho} \int_0^h g_1(\lambda,\tau,\sigma) X_1^{(n)}(\sigma) d\sigma d\rho d\tau, \end{aligned}$$

and then

$$\mathbf{B} = e^{-\lambda a}\int_0^{a_\dagger}\int_0^h g_1(\lambda,\tau,\sigma)\int_0^{a_\dagger-\tau} m(\rho+\tau)e^{-\lambda\rho}\times$$

$$\times\left[\sum_{n=0}^{\infty}\frac{e^{-\lambda_n K_1\rho}e^{-\lambda_n K_1 a}}{1-\widehat{m}(\lambda+\lambda_n K_1)}X_1^{(n)}(x)X_1^{(n)}(\sigma)\right]d\rho d\sigma d\tau =$$

$$= \int_0^{a_\dagger}\int_0^h g_1(\lambda,\tau,\sigma)M_1(\lambda,a,\tau,x,\sigma)d\sigma d\tau,$$

with

$$M_1(\lambda,a,\tau,x,\sigma) = \sum_{n=0}^{\infty}\frac{e^{-(\lambda-\lambda_n K_1)a}X_1^{(n)}(x)X_1^{(n)}(\sigma)}{1-\widehat{m}(\lambda+\lambda_n K_1)}\int_0^{a_\dagger-\tau} m(\rho+\tau)e^{-(\lambda+\lambda_n K_1)\rho}d\rho.$$

Concerning $M_1$, it should be specified that the integral can be calculated either on $(0, a_\dagger-\tau)$ or on $(0,\infty)$, by assuming $m(a)=0$ for $a>a_\dagger$, thus we can rewrite the expression for $M_1$ as

$$M_1(\lambda,a,\tau,x,\sigma) = \sum_{n=0}^{\infty}\gamma_1^{(n)}(\lambda,\tau)e^{-(\lambda+\lambda_n K_1)a}X_1^{(n)}(x)X_1^{(n)}(\sigma),$$

with

$$\gamma_1^{(n)}(\lambda,\tau) = \frac{\widehat{\eta}(\lambda+\lambda_n K_1,\tau)}{1-\widehat{m}(\lambda+\lambda_n K_1)},\ n\geq 0, \tag{3.3.41}$$

where $\widehat{\eta}$ denotes the Laplace transform of the function $\eta(\rho,\tau)=m(\rho+\tau)$.
Thus we have obtained the following expression for $U_1$

$$\begin{aligned} U_1(\lambda,a,x) &= \int_0^a\int_0^h g_1(\lambda,\tau,\sigma)N_1(a,\tau,x,\sigma)d\sigma d\tau + \\ &+ \int_0^{a_\dagger}\int_0^h g_1(\lambda,\tau,\sigma)M_1(\lambda,a,\tau,x,\sigma)d\sigma d\tau. \end{aligned} \tag{3.3.42}$$

By developing similar calculations for $U_2$ we get

$$\begin{aligned} U_2(\lambda,a,x) &= \int_0^a\int_0^h g_2(\lambda,\tau,\sigma)N_2(a,\tau,x,\sigma)d\sigma d\tau + \\ &+ \int_0^{a_\dagger}\int_0^h g_2(\lambda,\tau,\sigma)M_2(\lambda,a,\tau,x,\sigma)d\sigma d\tau, \end{aligned} \tag{3.3.43}$$

where

$$N_2(a,\tau,\sigma) = e^{-\lambda(a-\tau)}\sum_{n=0}^{\infty}e^{-\lambda_n K_2(a-\tau)}X_2^{(n)}(x)X_2^{(n)}(\sigma),$$

and

$$M_2(\lambda,a,\tau,x,\sigma) = \sum_{n=0}^{\infty}\gamma_2^{(n)}(\lambda,\tau)e^{-(\lambda+\lambda_n K_2)a}X_2^{(n)}(x)X_2^{(n)}(\sigma),$$

with

$$\gamma_2^{(n)}(\lambda,\tau) = \frac{\widehat{\eta}(\lambda+\lambda_n K_2,\tau)}{1-\widehat{m}(\lambda+\lambda_n K_2)},\ n \geq 0. \tag{3.3.44}$$

In the next section we shall obtain, by substituting the expression obtained for $U_1$ and $U_2$ in (3.3.25), the expressions for the solutions $u_1$ and $u_2$ to problems (3.3.18)-(3.3.20) and (3.3.21)-(3.3.23).

### 3.3.3 More about $u_1$ and $u_2$

As already specified at the end of the previous section, we now aim to obtain the solutions $u_1$ and $u_2$. As before, we shall develop the calculations for $u_1$ only. In particular we substitute (3.3.42) in (3.3.25) and, by using the definition (3.3.27) of $g_1$, we obtain

$$\begin{aligned} u_1(\lambda,a,x) \;=\; & \int_0^a \int_0^h f_1(\tau,\sigma) N_1(a,\tau,x,\sigma)\,d\sigma d\tau + \\ & + \int_0^{a_\dagger} \int_0^h f_1(\tau,\sigma) M_1(\lambda,a,\tau,x,\sigma)\,d\sigma d\tau + \\ & + \frac{1}{h}\int_0^a \int_0^h q(\lambda,\tau) N_1(a,\tau,x,\sigma)\,d\sigma d\tau + \\ & + \frac{1}{h}\int_0^{a_\dagger} \int_0^h q(\lambda,\tau) M_1(\lambda,a,\tau,x,\sigma)\,d\sigma d\tau + \\ & - \frac{1}{K_1}\int_0^a \int_0^h \phi_1(\sigma) w(\lambda,\tau) N_1(a,\tau,x,\sigma)\,d\sigma d\tau + \\ & - \frac{1}{K_1}\int_0^{a_\dagger} \int_0^h \phi_1(\sigma) w(\lambda,\tau) M_1(\lambda,a,\tau,x,\sigma)\,d\sigma d\tau + \\ & + \phi_1(x)\frac{q(\lambda,a)}{K_1}. \end{aligned}$$

Observe that the third term vanishes, indeed $q(\lambda,\tau)$ is independent of $\sigma$ and

$$\int_0^h N_1(a,\tau,x,\sigma)\,d\sigma = e^{-\lambda(a-\tau)} \sum_{n=0}^{\infty} e^{-\lambda K_1(a-\tau)} X_1^{(n)}(x) \int_0^h X_1^{(n)}(\sigma)\,d\sigma = 0,$$

in the same way it can be proved that also the fourth term is zero. On the other hand

$$\int_0^a \int_0^h \phi_1(\sigma) w(\lambda,\tau) N_1(a,\tau,x,\sigma) d\sigma d\tau =$$

$$= \int_0^a w(\lambda,\tau) \left( \sum_{n=0}^{\infty} e^{-(\lambda+\lambda_n K_1)(a-\tau)} X_1^{(n)}(x) \left( \int_0^h \phi_1(\sigma) X_1^{(n)}(\sigma) d\sigma \right) \right) d\tau =$$

$$= \int_0^a w(\lambda,\tau) \left( \sum_{n=0}^{\infty} e^{-(\lambda+\lambda_n K_1)(a-\tau)} X_1^{(n)}(x) \phi_1^{(n)} \right) d\tau,$$

and, similarly,

$$\int_0^{a_\dagger} \int_0^h \phi_1(\sigma) w(\lambda,\tau) M_1(\lambda,a,\tau,x,\sigma) d\sigma d\tau =$$

$$= \int_0^{a_\dagger} w(\lambda,\tau) \left( \sum_{n=0}^{\infty} e^{-(\lambda+\lambda_n K_1) a \gamma_1^{(n)}(\lambda,\tau)} X_1^{(n)}(x) \phi_1^{(n)} \right) d\tau.$$

Thus we conclude that

$$\begin{aligned} u_1(\lambda,a,x) &= H_{10}(\lambda,a,x) - \int_0^a w(\lambda,\tau) v_1(a,\tau,x) d\tau + \\ &\quad - \int_0^{a_\dagger} w(\lambda,\tau) f_1(\lambda,a,\tau,x) d\tau + \phi_1(x) \frac{q(\lambda,a)}{K_1}, \end{aligned} \tag{3.3.45}$$

where

$$\begin{aligned} H_{10}(\lambda,a,x) &= \int_0^a \int_0^h f_1(\tau,\sigma) N_1(a,\tau,x,\sigma) d\sigma d\tau + \\ &\quad + \int_0^{a_\dagger} \int_0^h f_1(\tau,\sigma) M_1(\lambda,a,\tau,x,\sigma) d\sigma d\tau, \end{aligned} \tag{3.3.46}$$

$$v_1(a,\tau,x) = \frac{1}{K_1} \sum_{n=0}^{\infty} e^{-(\lambda+\lambda_n K_1)(a-\tau)} X_1^{(n)}(x) \phi_1^{(n)},$$

and

$$f_1(\lambda,a,\tau,x) = \frac{1}{K_1} \sum_{n=0}^{\infty} e^{-(\lambda+\lambda_n K_1) a} \gamma_1^{(n)}(\lambda,\tau) X_1^{(n)}(x) \phi_1^{(n)}.$$

Concerning $u_2$, similar calculations lead to

$$\begin{aligned} u_2(\lambda,a,x) &= H_{20}(\lambda,a,x) - \int_0^a w(\lambda,\tau) v_2(a,\tau,x) d\tau + \\ &\quad - \int_0^{a_\dagger} w(\lambda,\tau) f_2(\lambda,a,\tau,x) d\tau + \phi_2(x) \frac{q(\lambda,a)}{K_2}, \end{aligned}$$

where

$$\begin{aligned} H_{20}(\lambda, a, x) &= \int_0^a \int_h^2 hf_2(\tau,\sigma)N_2(a,\tau,x,\sigma)d\sigma d\tau + \\ &+ \int_0^{a_\dagger}\int_h^2 hf_2(\tau,\sigma)M_2(\lambda,a,\tau,x,\sigma)d\sigma d\tau, \end{aligned} \tag{3.3.47}$$

$$v_2(a,\tau,x) = \frac{1}{K_2}\sum_{n=0}^{\infty} e^{-(\lambda+\lambda_n K_2)(a-\tau)} X_2^{(n)}(x)\phi_2^{(n)},$$

and

$$f_2(\lambda,a,\tau,x) = \frac{1}{K_2}\sum_{n=0}^{\infty} e^{-(\lambda+\lambda_n K_2)a}\gamma_2^{(n)}(\lambda,\tau)X_2^{(n)}(x)\phi_2^{(n)}.$$

The next step consists in writing $u_1$ and $u_2$ in terms of $q$ only, to come to this end we first notice that, using the definition of $w$ (see (3.3.28)) and integrating by parts, we have

$$\begin{aligned} \int_0^a w(\lambda,\tau)v_1(a,\tau,x)d\tau = \frac{1}{K_1}\left[\frac{h}{3}+\sum_{n=1}^{\infty}\phi_1^{(n)}X_1^{(n)}(x)\right]\left[q(\lambda,a)-e^{-\lambda a}q(\lambda,0)\right] + \\ -\int_0^a\left(\sum_{n=1}^{\infty}\lambda_n\phi_1^{(n)}X_1^{(n)}(x)e^{-(\lambda+\lambda_n K_1)(a-\tau)}\right)q(\lambda,\tau)d\tau, \end{aligned} \tag{3.3.48}$$

and, with similar calculations,

$$\begin{aligned} \int_0^{a_\dagger} w(\lambda,\tau)f_1(\lambda,a,\tau,x)d\tau = \frac{1}{K_1}e^{-\lambda a}q(\lambda,0)\left[\frac{h}{3}+\sum_{n=1}^{\infty}\phi_1^{(n)}X_1^{(n)}(x)\right] + \\ -\int_0^{a_\dagger}\left(\sum_{n=1}^{\infty}\lambda_n\phi_1^{(n)}X_1^{(n)}(x)e^{-(\lambda+\lambda_n K_1)a}\gamma_1^{(n)}(\lambda,\tau)\right)q(\lambda,\tau)d\tau. \end{aligned} \tag{3.3.49}$$

We have to specify that in the above calculations we have used

$$\frac{\partial\gamma_1^{(n)}}{\partial\tau}(\lambda,\tau) = (\lambda+\lambda_n K_1)\gamma_1^{(n)}(\lambda,\tau) - \frac{m(\tau)}{1-\widehat{m}(\lambda+\lambda_n K_1)},$$

and the fact that we can identify $a_\dagger = \infty$, by extending all the functions involved to 0 after $a_\dagger$, thus, in particular, in the integration by parts we can take $q(\lambda,\infty) = 0$.
Substituting (3.3.48) and (3.3.49) in (3.3.45), we end up with the following expression for $u_1$

$$\begin{aligned} u_1(\lambda,a,x) = H_{10}(\lambda,a,x) - \frac{1}{K_1}\left[\frac{h}{3}+\sum_{n=1}^{\infty}\phi_1^{(n)}X_1^{(n)}(x)-\phi_1(x)\right]q(\lambda,a) + \\ +\int_0^a\left(\sum_{n=1}^{\infty}\lambda_n\phi_1^{(n)}X_1^{(n)}(x)e^{-(\lambda+\lambda_n K_1)(a-\tau)}\right)q(\lambda,\tau)d\tau + \\ +\int_0^{a_\dagger}\left(\sum_{n=1}^{\infty}\lambda_n\phi_1^{(n)}X_1^{(n)}(x)e^{-(\lambda+\lambda_n K_1)a}\gamma_1^{(n)}(\lambda,\tau)\right)q(\lambda,\tau)d\tau. \end{aligned} \tag{3.3.50}$$

Finally, with similar calculations for $u_2$, we obtain

$$u_2(\lambda,a,x) = H_{20}(\lambda,a,x) - \frac{1}{K_2}\left[-\frac{h}{3} + \sum_{n=1}^{\infty}\phi_2^{(n)}X_2^{(n)}(x) - \phi_2(x)\right]q(\lambda,a)+$$
$$+\int_0^a\left(\sum_{n=1}^{\infty}\lambda_n\phi_2^{(n)}X_2^{(n)}(x)e^{-(\lambda+\lambda_nK_2)(a-\tau)}\right)q(\lambda,\tau)d\tau+$$
$$+\int_0^{a_\dagger}\left(\sum_{n=1}^{\infty}\lambda_n\phi_2^{(n)}X_2^{(n)}(x)e^{-(\lambda+\lambda_nK_2)a}\gamma_2^{(n)}(\lambda,\tau)\right)q(\lambda,\tau)d\tau.$$

### 3.3.4 Imposing the continuity condition

Now we have to impose the continuity of the solution at the interface between the two layers, namely $u_1$ and $u_2$ must satisfy condition (3.3.24).
We first calculate $u_1(\lambda,a,h)$ and $u_2(\lambda,a,h)$, in particular, from (3.3.50), we have

$$\begin{aligned} u_1(\lambda,a,h) &= H_{10}(\lambda,a,h) - \frac{1}{K_1}\frac{h}{6}q(\lambda,a) + \int_0^a V_1(\lambda,a-\tau)q(\lambda,\tau)d\tau+ \\ &+\int_0^{a_\dagger}F_1(\lambda,a,\tau)q(\lambda,\tau)d\tau, \end{aligned} \tag{3.3.51}$$

where

$$V_1(\lambda,\alpha) = \frac{2}{h}e^{-\lambda\alpha}\sum_{n=1}^{\infty}e^{-\lambda_nK_1\alpha},$$

and

$$F_1(\lambda,a,\tau) = \frac{2}{h}e^{-\lambda a}\sum_{n=1}^{\infty}e^{-\lambda_nK_1a}\gamma_1^{(n)}(\lambda,\tau).$$

On the other hand,

$$\begin{aligned} u_2(\lambda,a,h) &= H_{20}(\lambda,a,h) + \frac{1}{K_2}\frac{h}{6}q(\lambda,a) - \int_0^a V_2(\lambda,a-\tau)q(\lambda,\tau)d\tau+ \\ &-\int_0^{a_\dagger}F_2(\lambda,a,\tau)q(\lambda,\tau)d\tau, \end{aligned} \tag{3.3.52}$$

where

$$V_2(\lambda,\alpha) = \frac{2}{h}e^{-\lambda\alpha}\sum_{n=1}^{\infty}e^{-\lambda_nK_2\alpha},$$

and

$$F_2(\lambda,a,\tau) = \frac{2}{h}e^{-\lambda a}\sum_{n=1}^{\infty}e^{-\lambda_nK_2a}\gamma_2^{(n)}(\lambda,\tau).$$

Now, imposing condition (3.3.24), we end up with the following integral equation for $q(\lambda, a)$

$$C_0 q(\lambda, a) = H(\lambda, a) + \int_0^a V(\lambda, a-\tau) q(\lambda, \tau) d\tau + \int_0^{a_\dagger} F(\lambda, a, \tau) q(\lambda, \tau) d\tau, \tag{3.3.53}$$

where

$$C_0 = \frac{h}{6}\left(\frac{1}{K_1} + \frac{1}{K_2}\right), \quad H(\lambda, a) = H_{10}(\lambda, a, h) - H_{20}(\lambda, a, h), \tag{3.3.54}$$

$$V(\lambda, \alpha) = V_1(\lambda, \alpha) + V_2(\lambda, \alpha), \quad F(\lambda, a, \tau) = F_1(\lambda, a, \tau) + F_2(\lambda, a, \tau). \tag{3.3.55}$$

Summing up, we have reduced the problem (3.3.1)-(3.3.6) to the integral equation (3.3.53) for the new variable $q(\lambda, a)$; this equation contains both a Volterra type and a Fredholm type term. In the next section we shall prove an existence result for equation (3.3.53) by writing it as a fixed point problem. Moreover, the results of the following section will give a meaning to the formal calculations we developed up to now.

### 3.3.5 Existence result

As specified at the end of the previous section, we now concentrate on the equation

$$C_0 u(a) = H(\lambda, a) + \int_0^a V(\lambda, a-\tau) u(\tau) d\tau + \int_0^{a_\dagger} F(\lambda, a, \tau) u(\tau) d\tau, \tag{3.3.56}$$

aiming to prove that it has a unique solution $u$ for every fixed $\lambda$ large enough. In order to prove this existence result we write the integral equation (3.3.56) as a fixed point problem. Namely, for every fixed $\lambda$, we consider the following operator

$$(\mathcal{T}(\lambda)u)(a) = \frac{1}{C_0}\left[H(\lambda, a) + \int_0^a V(\lambda, a-\tau) u(\tau) d\tau + \right. \\ \left. + \int_0^{a_\dagger} F(\lambda, a, \tau) u(\tau) d\tau\right], \tag{3.3.57}$$

where $C_0$, $H$, $V$ and $F$ are defined in (3.3.54) and (3.3.55). In particular we shall prove that $\mathcal{T}(\lambda) : L^2(0, a_\dagger) \to L^2(0, a_\dagger)$ for every fixed $\lambda$ and that $\mathcal{T}(\lambda)$ is a contraction for $\lambda$ large enough. This will guarantee the fact that $\mathcal{T}(\lambda)$ has a fixed point, i.e. equation (3.3.56) has a unique solution, at least for $\lambda$ large.
First of all we prove the following

**Lemma 3.3.1** *Assume that $f_1(\cdot, x) \in L^2(0, a_\dagger)$ and $f_2(\cdot, x) \in L^2(0, a_\dagger)$, then $\mathcal{T}(\lambda) : L^2(0, a_\dagger) \to L^2(0, a_\dagger)$.*

**Proof.**
For a fixed $u \in L^2(0, a_\dagger)$ we have to prove that every term of the right hand side of (3.3.57) is in $L^2(0, a_\dagger)$.

In order to prove that the convolution term $\int_0^a V(\lambda, a-\tau)u(\tau)d\tau$ is in $L^2(0,a_\dagger)$, we show that the kernel $V(\lambda,\cdot) \in L^1(0,a_\dagger)$. Indeed

$$\begin{aligned}
\int_0^{a_\dagger} |V(\lambda,\alpha)|d\alpha &\leq \frac{2}{h}\int_0^{a_\dagger}\sum_{n=1}^{\infty}\left(e^{-\lambda_n K_1 \alpha} + e^{-\lambda_n K_2 \alpha}\right) d\alpha = \\
&= \frac{2}{h}\sum_{n=1}^{\infty}\left(\frac{1-e^{-\lambda_n K_1 a_\dagger}}{\lambda_n K_1} + \frac{1-e^{-\lambda_n K_2 a_\dagger}}{\lambda_n K_2}\right) \leq \\
&\leq \frac{2h}{\pi^2}\left(\frac{1}{K_1}+\frac{1}{K_2}\right)\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{h}{3}\left(\frac{1}{K_1}+\frac{1}{K_2}\right),
\end{aligned}$$

proving that $V$ is an $L^1$ function with respect to the variable $a$, as claimed.
Even if the argument that allows us to conclude that, if $V(\lambda,\cdot) \in L^1(0,a_\dagger)$ and $u \in L^2(0,a_\dagger)$, then the convolution $\int_0^a V(\lambda, a-\tau)u(\tau)d\tau$ is in $L^2(0,a_\dagger)$ is standard, we describe it in details. We first observe that

$$\begin{aligned}
\left|\int_0^a V(\lambda,a-\tau)u(\tau)d\tau\right| &\leq \int_0^a |V(\lambda,a-\tau)||u(\tau)|d\tau = \\
&= \int_0^a |V(\lambda,a-\tau)|^{\frac{1}{2}}|V(\lambda,a-\tau)|^{\frac{1}{2}}|u(\tau)|d\tau \leq \\
&\leq \|V(\lambda,\cdot)\|_{L^1(0,a_\dagger)}^{\frac{1}{2}}\left(\int_0^a |V(\lambda,a-\tau)||u(\tau)|^2 d\tau\right)^{\frac{1}{2}},
\end{aligned}$$

and then

$$\begin{aligned}
\int_0^{a_\dagger}\left|\int_0^a V(\lambda,a-\tau)u(\tau)d\tau\right|^2 da &\leq \|V(\lambda,\cdot)\|_{L^1(0,a_\dagger)}\int_0^{a_\dagger}\int_0^a |V(\lambda,a-\tau)||u(\tau)|^2 d\tau da = \\
&= \|V(\lambda,\cdot)\|_{L^1(0,a_\dagger)}\int_0^{a_\dagger}|u(\tau)|^2\int_\tau^{a_\dagger}|V(\lambda,a-\tau)|da d\tau \leq \\
&\leq \|V(\lambda,\cdot)\|_{L^1(0,a_\dagger)}^2\|u\|_{L^2(0,a_\dagger)}^2 < \infty.
\end{aligned} \tag{3.3.58}$$

Concerning the Fredholm term $\int_0^{a_\dagger} F(\lambda,a,\tau)u(\tau)d\tau$, we first prove that

$$\int_0^{a_\dagger}|F(\lambda,a,\tau)|^2 da \leq \frac{C_F}{\lambda}, \tag{3.3.59}$$

where $C_F = C_F(h, K_1, K_2, \beta_+, a_\dagger)$ is a constant. In order to prove this estimate, we notice that, from the definition of $\gamma_j^{(n)}(\lambda, \tau)$ (see (3.3.41)), we have, for $j = 1, 2$, $n \geq 0$,

$$\begin{aligned} \gamma_j^{(n)}(\lambda, \tau) &= \frac{\widehat{\eta}(\lambda + \lambda_n K_j, \tau)}{1 - \widehat{m}(\lambda + \lambda_n K_j)} \leq \\ &\leq \frac{\widehat{\eta}(\lambda + \lambda_n K_j, \tau)}{1 - \beta_+ a_\dagger} = \\ &= \frac{1}{1 - \beta_+ a_\dagger} \int_0^{a_\dagger} e^{-(\lambda + \lambda_n K_j)\sigma} m(\sigma + \tau) d\sigma \leq \\ &\leq \frac{\beta_+}{1 - \beta_+ a_\dagger} \frac{1 - e^{-(\lambda + \lambda_n K_j) a_\dagger}}{\lambda + \lambda_n K_j} \leq \frac{c}{\lambda + \lambda_n K_j}, \end{aligned} \tag{3.3.60}$$

where $c = \dfrac{\beta_+}{1 - \beta_+ a_\dagger}$. Here we have used the fact that, being a probability, $\Pi(a) \leq 1$. Thus we have

$$\begin{aligned} \int_0^{a_\dagger} |F(\lambda, a, \tau)|^2 da = \frac{4}{h^2} \int_0^{a_\dagger} e^{-2\lambda a} \left| \sum_{n=1}^{\infty} \left( e^{-\lambda_n K_1 a} \gamma_1^{(n)}(\lambda, \tau) + e^{-\lambda_n K_2 a} \gamma^n(\lambda, \tau) \right) \right|^2 da \leq \\ \leq \frac{4c^2}{h^2} \left[ \sum_{n=1}^{\infty} \left( \frac{1}{\lambda + \lambda_n K_1} + \frac{1}{\lambda + \lambda_n K_2} \right) \right]^2 \int_0^{a_\dagger} e^{-2\lambda a} da = \\ = \frac{4c^2 h^2}{\pi^4} \left( \frac{1}{K_1} + \frac{1}{K_2} \right)^2 \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^2 \frac{1 - e^{-2\lambda a_\dagger}}{2\lambda} \leq \\ \leq \frac{c^2 h^2}{18} \left( \frac{1}{K_1} + \frac{1}{K_2} \right)^2 \frac{1}{\lambda}, \end{aligned}$$

i.e. we have obtained (3.3.59) with $C_F = \dfrac{c^2 h^2}{18} \left( \dfrac{1}{K_1} + \dfrac{1}{K_2} \right)^2$. Now we conclude with the following estimate

$$\begin{aligned} \int_0^{a_\dagger} \left| \int_0^{a_\dagger} F(\lambda, a, \tau) u(\tau) d\tau \right|^2 da &\leq a_\dagger \int_0^{a_\dagger} \int_0^{a_\dagger} |F(\lambda, a, \tau)|^2 |u(\tau)|^2 d\tau da = \\ &= a_\dagger \int_0^{a_\dagger} |u(\tau)|^2 \int_0^{a_\dagger} |F(\lambda, a, \tau)|^2 da d\tau \leq \\ &\leq \frac{a_\dagger C_F}{\lambda} \|u\|_{L^2(0, a_\dagger)} < \infty. \end{aligned} \tag{3.3.61}$$

Finally we have to prove that $H(\lambda,\cdot) \in L^2(0,a_\dagger)$. From (3.3.54), (3.3.46) and (3.3.47) we have

$$\begin{aligned} H(\lambda,a) &= \int_0^a \sum_{n=0}^{\infty} \Big( e^{-(\lambda+\lambda_n K_1)(a-\tau)} X_1^{(n)}(h) f_1^{(n)}(\tau) + \\ &\quad - e^{-(\lambda+\lambda_n K_2)(a-\tau)} X_2^{(n)}(h) f_2^{(n)}(\tau) \Big)\, d\tau + \\ &\quad + \int_0^{a_\dagger} \sum_{n=0}^{\infty} \Big( e^{-(\lambda+\lambda_n K_1)a} \gamma_1^{(n)}(\lambda,\tau) X_1^{(n)}(h) f_1^{(n)}(\tau) + \\ &\quad - e^{-(\lambda+\lambda_n K_2)} \gamma_2^{(n)}(\lambda,\tau) X_2^{(n)}(h) f_2^{(n)}(\tau) \Big)\, d\tau. \end{aligned}$$

Thus it is enough to observe that, for $j = 1,2$,

$$\begin{aligned} &\left\| \int_0^{\cdot} \left( \sum_{n=0}^{\infty} e^{-(\lambda+\lambda_n K_j)(\cdot-\tau)} X_j^{(n)}(h) f_j^{(n)}(\tau) \right) d\tau \right\|^2_{L^2(0,a_\dagger)} \leq \\ &\leq \int_0^{a_\dagger} \left| \int_0^a \left( \sum_{n=0}^{\infty} X_j^{(n)}(h) f_j^{(n)}(\tau) \right) d\tau \right|^2 da \\ &= \int_0^{a_\dagger} \left| \int_0^a f_j(\tau,h) d\tau \right|^2 da \leq {a_\dagger}^2 \|f_j(\cdot,h)\|^2_{L^2(0,a_\dagger)}, \end{aligned}$$

and

$$\begin{aligned} &\left\| \int_0^{a_\dagger} \left( \sum_{n=0}^{\infty} e^{-(\lambda+\lambda_n K_j)\cdot} \gamma_j^{(n)}(\lambda,\tau) X_j^{(n)}(h) f_j^{(n)}(\tau) \right) \right\|^2_{L^2(0,a_\dagger)} \leq \\ &\leq c^2 \int_0^{a_\dagger} \left| \int_0^{a_\dagger} \sum_{n=0}^{\infty} \frac{e^{-(\lambda+\lambda_n K_j)a}}{\lambda+\lambda_n K_j} X_j^{(n)}(h) f_j^{(n)}(\tau) d\tau \right|^2 da \leq \\ &\leq c^2 \int_0^{a_\dagger} \left| \int_0^{a_\dagger} \left( \sum_{n=0}^{\infty} X_j^{(n)}(h) f_j^{(n)}(\tau) \right) d\tau \right|^2 da \leq c^2 {a_\dagger}^2 \|f_j(\cdot,h)\|^2_{L^2(0,a_\dagger)}, \end{aligned}$$

we observe that in the last estimate we have used (3.3.60). Thus, since $f_1(\cdot,x)$, $f_2(\cdot,x) \in L^2(0,a_\dagger)$, we have that $H(\lambda,\cdot) \in L^2(0,a_\dagger)$, completing the proof.

□

As specified before, the next step consists in showing that the operator $\mathcal{T}(\lambda)$ is a contraction on $L^2(0,a_\dagger)$, at least for $\lambda$ large. This result is proved in the following

**Theorem 3.3.1** *The operator $\mathcal{T}(\lambda)$ defined in* (3.3.57) *is a contraction on $L^2(0,a_\dagger)$ for $\lambda$ large.*

**Proof.** Let $u, \widetilde{u} \in L^2(0, a_\dagger)$, we have to prove that

$$\|\mathcal{T}(\lambda)u - \mathcal{T}(\lambda)\widetilde{u}\|_{L^2(0,a_\dagger)} \leq C_\lambda \|u - \widetilde{u}\|_{L^2(0,a_\dagger)},$$

where $C_\lambda$ is a constant such that $C_\lambda < 1$ for $\lambda$ large.
For every fixed $\lambda$, we have

$$\begin{aligned}\|\mathcal{T}(\lambda)u - \mathcal{T}(\lambda)\widetilde{u}\|_{L^2(0,a_\dagger)} \leq \; & \frac{1}{C_0}\left\|\int_0^{\cdot} V(\lambda, \cdot - \tau)(u(\tau) - \widetilde{u}(\tau))d\tau\right\|_{L^2(0,a_\dagger)} + \\ & + \frac{1}{C_0}\left\|\int_0^{a_\dagger} F(\lambda, \cdot, \tau)(u(\tau) - \widetilde{u}(\tau))d\tau\right\|_{L^2(0,a_\dagger)}.\end{aligned} \tag{3.3.62}$$

From (3.3.58), we have that

$$\left\|\int_0^{\cdot} V(\lambda, \cdot - \tau)(u(\tau) - \widetilde{u}(\tau))d\tau\right\|_{L^2(0,a_\dagger)} \leq \|V(\lambda, \cdot)\|_{L^1(0,a_\dagger)}\|u - \widetilde{u}\|_{L^2(0,a_\dagger)} \leq$$

$$\leq \frac{2}{h}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda + \lambda_n K_1} + \frac{1}{\lambda + \lambda_n K_2}\right)\|u - \widetilde{u}\|_{L^2(0,a_\dagger)},$$

while, from (3.3.61),

$$\left\|\int_0^{a_\dagger} F(\lambda, \cdot, \tau)(u(\tau) - \widetilde{u}(\tau))d\tau\right\|_{L^2(0,a_\dagger)} \leq \sqrt{\frac{a_\dagger C_F}{\lambda}}\|u - \widetilde{u}\|_{L^2(0,a_\dagger)}.$$

Thus, using these two estimates in (3.3.62), we end up with

$$\|\mathcal{T}(\lambda)u - \mathcal{T}(\lambda)\widetilde{u}\|_{L^2(0,a_\dagger)} \leq$$

$$\leq \frac{1}{C_0}\left[\frac{2}{h}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda + \lambda_n K_1} + \frac{1}{\lambda + \lambda_n K_2}\right) + \sqrt{\frac{a_\dagger C_F}{\lambda}}\right]\|u - \widetilde{u}\|_{L^2(0,a_\dagger)},$$

and then we conclude that $\mathcal{T}(\lambda)$ is a contraction if $\lambda$ is large enough, namely if $\lambda$ is such that

$$\frac{2}{h}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda + \lambda_n K_1} + \frac{1}{\lambda + \lambda_n K_2}\right) + \sqrt{\frac{a_\dagger C_F}{\lambda}} < 1.$$

□

Thus the following existence result holds

**Theorem 3.3.2** *Assume that $f_1(\lambda, \cdot),\ f_2(\lambda, \cdot) \in L^2(0, a_\dagger)$ for every $\lambda$, then the integral equation*

$$C_0 u(\lambda, a) = H(\lambda, a) + \int_0^a V(\lambda, a - \tau)u(\tau)d\tau + \int_0^{a_\dagger} F(\lambda, a, \tau)u(\tau)d\tau, \tag{3.3.63}$$

*has an unique solution $u \in L^2(0, a_\dagger)$ for $\lambda$ large enough.*

The proof of Theorem 3.3.2 immediately follows from Lemma 3.3.1 and Theorem 3.3.1, in fact from these results we have that the operator $\mathcal{T}(\lambda) : L^2(0, a_\dagger) \to L^2(0, a_\dagger)$ is a contraction for $\lambda$ large, and then it has a unique fixed point which is actually the solution of equation (3.3.63).

Theorem 3.3.2 guarantees the existence and the uniqueness of the solution $q(\lambda, \cdot)$ to equation (3.3.53), now, by substituting $q$ in (3.3.51) and (3.3.52), we obtain an expression for $u_1$ amd $u_2$ solutions to problem (3.3.13)-(3.3.17).

# Chapter 4

# Pattern formation and population dynamics

## 4.1 The effect of diffusion

In this section we analyze the effect of spatial diffusion on the spatially homogeneous steady states of nonlinear reaction-diffusion systems. In particular, in Section 4.1.1 we consider a model presented by Fisher describing the spatial spreading of an advantageous gene in a population and show how the presence of diffusion leads to the stabilization of an equilirium that is unstable in the absence of diffusion. On the other hand, in Section 4.1.2, we introduce the so-called Turing mechanism for a system of two chemical species which are diffusing and reacting. According to this mechanism the stability of a spatially homogenous steady state of the system changes when spatial diffusion occurs, in particular an equilibrium that is stable in the absence of diffusion shows the so called diffusion driven instability. Following Turing ideas, we see how this mechanism leads to the formation of a stable spatial pattern.

### 4.1.1 Fisher equation

In [19] Fisher considered a one-dimensional deterministic version of a stochastic model for the spatial diffusion of an advantageous gene in a population. The equation of Fisher model reads

$$\frac{\partial p}{\partial t} = kp(1-p) + D\frac{\partial^2 p}{\partial x^2}, \tag{4.1.1}$$

where $k$ and $D$ are positive parameters. As we have already seen in Chapter 1 (equation (4.1.1) is actually equation (1.1.15) when the population density is expressed in units of carrying capacity, i.e. $K = 1$), (4.1.1) is a nonlinear reaction-diffusion equation representing the natural extension of the classical logistic equation in the case of a population spreading in a linear way.

For simplicity we rescale (4.1.1) by writing

$$t^* = kt, \quad x^* = x\left(\frac{k}{D}\right)^{1/2}, \tag{4.1.2}$$

thus, omitting the asterics for notational simplicity, (4.1.1) becomes

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + p(1-p). \tag{4.1.3}$$

We are interested in understanding the effect that diffusion produces on the spatially homogeneous steady states of this equation.

Let us suppose that the population lives in an interval $[0, L]$ and, for mathematical completeness, add the following initial condition

$$p(x,0) = p_0(x), \qquad \text{for } x \in [0, L], \tag{4.1.4}$$

and the Dirichlet boundary conditions

$$p(0,t) = p(t,L) = 0, \qquad \text{for } t \geq 0. \tag{4.1.5}$$

It is easy to see that $p \equiv 0$ is a spatially homogeneous steady state of the equation (4.1.3), i.e. it is an equilibrium of equation (4.1.3) when spatial diffusion is absent. If we define $F(p) = p(1-p)$, then the stability of the homogeneous steady state $p \equiv 0$ depends on the sign of $F'(0)$. Being $F'(0) = 1$, we conclude that $p \equiv 0$ is an unstable steady state.

Let us now come back to the problem with diffusion and linearize equation (4.1.3) in $p \equiv 0$, we obtain

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + F'(0)p. \tag{4.1.6}$$

Studying equation (4.1.6) with conditions (4.1.4) and (4.1.5) using separation of variables, one can see that the solution is the sum of infinite modes, namely

$$p(x,t) = \sum_{n=1}^{\infty} c_n \exp\left[\left(-\frac{n^2\pi^2}{L^2} + 1\right)t\right] \sin\left(\frac{n\pi}{L}x\right), \tag{4.1.7}$$

where the coefficients $c_n$ are the coefficients of the Fourier expansion of the initial datum $p_0$. For every $n$ such that $n^2 > \frac{L^2}{\pi^2}$ the $n$-th mode is stable, being the stability of the $n$-th mode depending on the sign of $-\frac{n^2\pi^2}{L^2} + 1$. Thus we conclude that, in this case, diffusion leads to a stabilization of these modes, while a finite number of modes remain unstable. Moreover we observe that this stabilization strictly depends on the length $L$ of the spatial domain (namely, if $L$ is large, then there are less modes that become stable).

### 4.1.2 Turing mechanism

As we have seen in the previous section, usually spatial diffusion leads to stabilization, actually this has been the common idea up to 1952 when, in [49], Turing described a phenomenon that he called *diffusion driven instability* and that now is known as *Turing mechanism*.

Turing's idea of diffusion driven instability came out within the theory of morphogenesis, the part of biology concerning the formation of patterns and forms during the growth of an embryo (the most common examples are the development of patterns on animal coat or on butterfly wings). Turing's work consisted in understanding how, starting from a certain number of completely equal cells, these cells differentiate according to where they are at a given time. Turing suggested that the presence of chemical substances (that he called *morphogens*, without giving any very exact meaning to this word, but using it only "to convey the idea of a form producer"), which are reacting and diffusing, leads to the formation of a pattern within the concentrations of the morphogens themself and then, as a consequence, the cells differentiate depending on the concentration of the morphogens they "feel".

Here we analyze Turing mechanism in depth, following the description made by Murray in [43], Chapter 2, Section 2.2. Let us consider the following reaction-diffusion system (see Section 1.1.5)

$$\frac{\partial \mathbf{c}}{\partial t} = \mathbf{f}(\mathbf{c}) + D\Delta\mathbf{c}, \tag{4.1.8}$$

where $\mathbf{c}$ is the vector of the concentrations of the morphogens (namely an inibitor and a reactant), $f$ represents the reaction and, in all concrete problems, is a nonlinear function and $D$ is the diagonal matrix of positive constant diffusion coefficients. In order to simplify the discussion, let us focus on models for two chemical species $u(t,x)$ and $v(t,x)$ diffusing and reacting in a one-dimensional environment (i.e. $x \in [0,L]$). In particular problem (4.1.8) now reads

$$u_t = f(u,v) + d_u u_{xx}, \tag{4.1.9}$$

$$v_t = g(u,v) + d_v v_{xx}, \tag{4.1.10}$$

here we have used the subscripts to indicate the partial derivatives. Following Turing's idea, we now want to show that if, in the absence of diffusion (i.e. $d_u = d_v = 0$), $u$ and $v$ tend to a linearly stable uniform steady state then, under certain conditions, which we shall derive, spatially inhomogeneous patterns can evolve by diffusion driven instability if $d_u \neq d_v$.

First of all we rescale equations (4.1.9) and (4.1.10), namely we define

$$t^* = \frac{d_u}{L^2}t, \quad x^* = \frac{x}{L}, \tag{4.1.11}$$

and we obtain the following equations

$$u_t = \gamma f(u,v) + u_{xx}, \tag{4.1.12}$$

$$v_t = \gamma g(u,v) + d v_{xx}, \tag{4.1.13}$$

where $\gamma = \dfrac{L^2}{d_u}$ indicates the reaction intensity and $d = \dfrac{d_v}{d_u}$ and where, for the writing simplicity, we omit the symbol $*$. Notice that now the spatial variable is $x \in [0,1]$.
Concerning initial conditions, we impose

$$u(0,x) = u_0(x), \tag{4.1.14}$$

$$v(0,x) = v_0(x), \tag{4.1.15}$$

where $u_0$ and $v_0$ are supposed to be given.
Finally the choice of the boundary conditions depends on the fact that we are interested in self-organization of patterns and then we impose zero-flux boundary conditions, which guarantee that there are no external inputs, namely we take

$$u_x(t,0) = u_x(t,1) = 0, \tag{4.1.16}$$

$$v_x(t,0) = v_x(t,1) = 0. \tag{4.1.17}$$

We aim to show that problem (4.1.12)-(4.1.17) has a spatially homogeneous steady state which is linearly stable in the absence of diffusion and whose asymptotic behaviour changes when the two reactants diffuse. In particular, we shall give conditions on $f$, $g$ and $d$ that imply spatial pattern formation.

An homogeneous steady state of problem (4.1.12)-(4.1.17) is a couple $(u^*, v^*)$ solving the system

$$f(u^*, v^*) = 0,$$

$$g(u^*, v^*) = 0.$$

We suppose that there exists at least one equilibrium $(u^*, v^*)$ and linearize the homogeneous problem about $(u^*, v^*)$, namely we take

$$\mathbf{w} = \begin{pmatrix} u - u^* \\ v - v^* \end{pmatrix},$$

then the linerized problem reads

$$\mathbf{w}_t = \gamma \mathcal{J}(f,g)\mathbf{w},$$

where $\mathcal{J}(f,g)$ is the Jacobian matrix of $(f,g)$ evaluated in the steady state $(u^*, v^*)$, i.e.

$$\mathcal{J}(f,g) = \begin{pmatrix} f_u(u^*, v^*) & f_v(u^*, v^*) \\ \\ g_u(u^*, v^*) & g_v(u^*, v^*) \end{pmatrix}.$$

It is well known that linear stability of $(u^*, v^*)$ is guaranteed if

$$\mathrm{tr}[\gamma \mathcal{J}(f,g)] < 0, \qquad \text{i.e. } f_u + g_v < 0, \tag{4.1.18}$$

$$\det[\gamma \mathcal{J}(f,g)] > 0, \qquad \text{i.e. } f_u g_v - f_v g_u > 0. \tag{4.1.19}$$

Now we suppose that these inequalities are satisfied by an equilibirum $(u^*, v^*)$ and concentrate on the non-homogeneous problem (4.1.12)-(4.1.17). By linearizing it about the steady state $(u^*, v^*)$ we obtain

$$\mathbf{w}_t = \gamma \mathcal{J}(f,g)\mathbf{w} + D\mathbf{w}_{xx}, \tag{4.1.20}$$

where $D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$.
We solve system (4.1.20) by separating the variables, following the procedure used in the previous

section for the Fisher equation. Since we are now dealing with Neumann boundary conditions, the solution $\mathbf{w} = (u, v)$ to system (4.1.20) can be written as the sum of infinite modes, namely

$$\mathbf{w}(x,t) = \begin{pmatrix} u(x,t) \\ \\ v(x,t) \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{n=0}^{\infty} c_n T_n(t) \cos(n\pi x) \\ \\ \displaystyle\sum_{n=0}^{\infty} d_n T_n(t) \cos(n\pi x) \end{pmatrix},$$

where the constants $c_n$ and $d_n$ are respectively the Fourier coefficients of the initial conditions $u_0(x)$ and $v_0(x)$. For every $n \geq 0$, $T_n$ satisfies

$$\frac{T_n'(t)}{T_n(t)} = \lambda_n + \gamma f_u + \gamma f_v \frac{d_n}{c_n} = \alpha_n, \tag{4.1.21}$$

$$\frac{T_n'(t)}{T_n(t)} = d\lambda_n + \gamma g_u \frac{c_n}{d_n} + \gamma g_v = \alpha_n, \tag{4.1.22}$$

where $\lambda_n = -n^2\pi^2$ and the derivatives $f_u$, $f_v$, $g_u$ and $g_v$ are evaluated at the equilibrium $(u^*, v^*)$. From (4.1.21) and (4.1.22) we obtain that $T_n(t) = e^{\alpha_n t}$ and that $\alpha_n$ solves

$$c_n \alpha_n = c_n(\lambda_n + \gamma f_u) + d_n \gamma f_v,$$

$$d_n \alpha_n = c_n \gamma g_u + d_n(d\lambda_n + \gamma g_v),$$

in other words, $\alpha_n$ is an eigenvalue of

$$M_n = \lambda_n D + \gamma \mathcal{J} = \begin{pmatrix} \lambda_n + \gamma f_u & \gamma f_v \\ \\ \gamma g_u & d\lambda_n + \gamma g_v \end{pmatrix}.$$

To have diffusion driven instability we expect at least one of the modes to be unstable, obviously this happens if, for a finite number of $n \geq 0$, $\alpha_n$ has positive real part, and this happens if either $\mathrm{tr} M_n > 0$ and $\det M_n > 0$ or $\det M_n < 0$. We observe that, since $f_u + g_v < 0$ from condition (4.1.18), we have

$$\mathrm{tr} M_n = \lambda_n(1+d) + \gamma(f_u + g_v) < 0,$$

thus the only way $\mathcal{R}e(\alpha_n)$ can be positive is if $\det M_n < 0$, i.e. if

$$d\lambda_n^2 + \lambda_n \gamma(g_v + d f_u) + \gamma^2 \det \mathcal{J}(f,g) < 0. \tag{4.1.23}$$

Since $\lambda_n < 0$, $\gamma > 0$ and $\det \mathcal{J}(f,g) > 0$, we conclude that a necessary condition in order $\det M_n$ to be negative is that

$$d f_u + g_v > 0. \tag{4.1.24}$$

We notice that condition (4.1.24) can not be satisfied if $d = 1$, in fact in that case we would have both $f_u + g_v < 0$, from condition (4.1.18), and $f_u + g_v > 0$, from condition (4.1.24).

Moreover, by comparing (4.1.18) and (4.1.24), we conclude that $f_u$ and $g_v$ must have opposite signs, and, as a consequence, we also must have $f_v g_u < 0$ (otherwise (4.1.19) could not be satisfied).

We stress the fact that condition (4.1.24) is necessary but not sufficient for $\mathcal{R}e\alpha_n > 0$ and we now look for a stronger hypothesis, to come to this end we shall analyze equation (4.1.23) in terms of $n^2$. If we define

$$\begin{aligned} h(n^2) &= d\lambda_n^2 + \lambda_n \gamma (df_u + g_v) + \gamma^2 \det \mathcal{J}(f,g) = \\ &= d\pi^4 n^4 - \pi^2 \gamma (df_u + g_v) n^2 + \gamma^2 \det \mathcal{J}(f,g), \end{aligned}$$

we have that condition (4.1.23) is satisfied for each $n$ such that $h(n^2) < 0$, in particular the minimum $h_{\min}$ of $h$ must be negative. From the definition of $h(n^2)$, elementary differentiation with respect to $n^2$ shows that the minimum $h_{\min}$ is achieved in

$$n^2 = \frac{\gamma}{2d}\frac{1}{\pi^2}(df_u + g_v),$$

and then

$$h_{\min} = \gamma^2 \left[ \det \mathcal{J}(f,g) - \frac{1}{4d}(df_u + g_v)^2 \right],$$

thus $h_{\min} < 0$ if and only if

$$\frac{(df_u + g_v)^2}{4d} > \det \mathcal{J}(f,g), \qquad \text{i.e. } (df_u + g_v)^2 - 4d(f_u g_v - f_v g_u) > 0. \tag{4.1.25}$$

Thus we conclude that, if condition (4.1.25) is satisfied, then there exist $n_1$, $n_2 \in \mathbb{N}$, solutions of $h(n^2) = 0$, such that, for every $n$ with $n_1^2 \le n^2 \le n_2^2$, we have $\mathcal{R}e\alpha_n > 0$. In other words, if

$$\begin{aligned} n_1^2 = \gamma \frac{(df_u + g_v) - \left\{(df_u + g_v)^2 - 4d \det \mathcal{J}(f,g)\right\}^{1/2}}{2d\pi^2} \le \\ \le n^2 \le \gamma \frac{(df_u + g_v) + \left\{(df_u + g_v)^2 - 4d \det \mathcal{J}(f,g)\right\}^{1/2}}{2d\pi^2} = n_2^2, \end{aligned}$$

then the $n$-th mode is unstable.

Observe that condition (4.1.25) is strictly dependent on the ratio $d = d_v/d_u$ of the diffusion coefficients, in particular there exists a critical ratio $d_c$ defined by

$$(d_c f_u + g_v)^2 = 4d_c \det \mathcal{J}(f,g),$$

such that for $d > d_c$ there exists at least one $n$ such that the corresponding $\alpha_n$ has positive real part and then the $n$-th mode is unstable.

We conclude that, for $t$ large, the solution to the problem (4.1.12)-(4.1.17) is given by

$$u(t,x) \sim \sum_{n=n_1}^{n_2} c_n e^{\alpha_n t} \cos(n\pi x), \tag{4.1.26}$$

$$v(t,x) \sim \sum_{n=n_1}^{n_2} d_n e^{\alpha_n t} \cos(n\pi x). \tag{4.1.27}$$

It is important to observe that the linearly unstable eigenfunctions in (4.1.26) and in (4.1.27), which are growing exponentially with time, will eventually be bounded by the nonlinear terms in the reaction diffusion system and the ultimate spatially inhomogeneous steady state will emerge.

We can summarize the conclusions obtained above in the following

**Theorem 4.1.1** *Let us consider a two-species reaction diffusion mechanism of the form* (4.1.12)-(4.1.17). *Let* $(u^*, v^*)$ *be a spatially homogeneous steady state of the system, i.e. a solution of* $f(u^*, v^*) = g(u^*, v^*) = 0$. *Assume that the functions* $f$ *and* $g$ *and the constant* $d$ *satisfy the following conditions:*

$$f_u + g_v < 0, \quad f_u g_v - f_v g_u > 0, \tag{4.1.28}$$

$$d f_u + g_v > 0, \quad (d f_u + g_v)^2 - 4d(f_u g_v - f_v g_u) > 0, \tag{4.1.29}$$

*where all the derivatives of* $f$ *and* $g$ *are evaluated at* $(u^*, v^*)$. *Then* $(u^*, v^*)$ *is stable to small perturbations in the absence of diffusion, but unstable to small spatial perturbations when diffusion is present. Due to this diffusion driven instability, the homogeneous steady states evolves to a stable spatial pattern.*

**Remark 4.1.1** *Some calculations on the second condition in* (4.1.29) *imply that it is equivalent to the following one:*

$$d > \frac{\left(\sqrt{f_u g_v - f_v g_u} + \sqrt{-f_v g_u}\right)^2}{f_u^2}. \tag{4.1.30}$$

*In particular in order to satisfy* (4.1.29) *it is enough to choose* $d$ *such that*

$$d > \max\left\{ -\frac{g_v}{f_u}, \frac{\left(\sqrt{f_u g_v - f_v g_u} + \sqrt{-f_v g_u}\right)^2}{f_u^2} \right\}. \tag{4.1.31}$$

### 4.1.3 Examples from population dynamics

In this section we briefly discuss some examples of pattern formation in population dynamics, following the discussion given by Okubo and Levin in [46].

The first problem we consider is a model presented by Levin and Segel in [36] describing the interaction between phytoplankton and herbivore. Namely, if we denote by $P$ and $H$ the densities of phytoplankton and herbivore respectively, we consider the following equations

$$P_t = aP + bP^2 - c_1 PH + d_P P_{xx},$$

$$H_t = -eH^2 + c_2 PH + d_H H_{xx},$$

where $a$, $b$, $c_1$, $c_2$ and $e$ are positive constants describing the dynamics of the population and $d_P$ and $d_H$ are the diffusion coefficients of phytoplankton and herbivore respectively. The unique non trivial spatially homogeneous equilibrium of this system is given by

$$P^* = \frac{ae}{c_1 c_2 - be},$$

$$H^* = \frac{ac_2}{c_1 c_2 - be},$$

obviously, since $P^*$ and $H^*$ must be positive, we assume $c_1 c_2 - be > 0$.
Let us denote by

$$f(P, H) = aP + bP^2 - c_1 PH,$$
$$g(P, H) = -eH^2 + c_2 PH,$$

it is known from the previous section that the system exibits diffusion driven instability if $f$ and $g$ satisfy (4.1.28) and (4.1.29). In our case, simple calculations lead to the following conditions on $a$, $b$, $c_1$, $c_2$, $e$, and $d = d_H/d_P$

$$c_1 c_2 - be > 0, \quad c_2 > b, \quad d > \frac{c_2}{b^2}\left(\sqrt{c_1 c_2 - be} + \sqrt{c_1 c_2}\right).$$

Another interaction model that lead to pattern formation was described in [35], by Levin, who took into account the following prey-predator system with functional response

$$H_t = aH - \varphi(H)P + d_H H_{xx},$$
$$P_t = -\psi(P)P + b\varphi(H)P,$$

where now $H$ and $P$ denote the densities of prey and predator populations respectively, $a$ and $b$ are positive constants, $\varphi(H)$ is the number of preys eaten by a single predator in the unit time (the so-called functional response) and $\psi(P)$ is the density-dependent death rate of the predator. Levin observed that the functional response $\varphi$ often possesses the property that $\dfrac{\varphi(H^*)}{H^*} > \varphi'(H^*)$, where $H^*$ is the density of preys at spatially homogeneous equilibrium. If this is the case, diffusive instability require self-damping in the predator, $\psi'(P) > 0$, and a sufficiently high rate of dispersal of the predator.

Another model concerning phytoplankton was investigated by Okubo in [44] and [45]. He cosidered the following problem for the interaction between phytoplancton and nutrient

$$N_t = Q - \alpha(N)P + d_N N_{xx},$$
$$P_t = -Z + \beta(N)P + d_P P_{xx},$$

where $N$ is the nutrient concentration, $P$ is the phytoplankton density, $Q$ and $Z$ are, respectively, constant rates of nutrient supply and grazing, and $\alpha$ and $\beta$ are arbitrary functions of $N$ only. These equations provide a model of a well-mixed upper layer of the ocean, where nutrient is supplied at a constant rate from the lower layer. Grazing is in general depending on both phytoplankton density and herbivore density; for simplicity here we consider the grazing term a constant sink. Actually, a more realistic model would include the herbivores in the system, leading to a three-species system.
By analyzing the assumptions that lead to pattern formation, Okubo concluded that diffusion driven instability is possible if the following conditions hold:

(i) the ratio of nutrient supply per unit concentration of nutrient and grazing loss per unit density of phytoplankton must be greater than the ratio of the mean slope of the uptake rate $\alpha$ to the gradient of the uptake coefficient at equilibrium;

(ii) the latter ratio must also be greater than the ratio of the mean slope of the growth rate $\beta$ to the gradient of the growth coefficient at equilibrium;

(iii) the ratio of diffusivities of nutrient and phytoplankton must be greater than a certain critical value.

Our idea is now to consider population dynamics models describing the interacion within two classes of the same population, namely young and adult individuals, this kind of models are analyzed in the next section.

## 4.2 Turing mechanism in juveniles-adults interaction

In this section we look for conditions for pattern formation according to Turing mechanism in a system describing the dynamics of a population divided into two separate classes. In particular, in view of possible future applications to age-structured population models, we shall consider a population in which the separation into classes is due to age, namely we shall divide the population into a juvenile class and an adult class. Under the hypoteses that juveniles and adults diffuse with different velocities, we look for conditions on the functions describing the dynamics of the population and on the diffusion coefficients that lead to diffusion driven instability and than to pattern formation.
In particular, in Section 4.2.1 and Section 4.2.2 we consider a general model for this population and we obtain conditions for pattern formation according to Turing mechanism and to the conditions obtained in Section 4.1.2. However, these conditions are general and quite complicated, thus, in Section 4.2.3, we consider a particular case and we analyze the diffusion driven instability in depth.

### 4.2.1 Description of the model

As explained at the end of the previous section, we now consider a population composed by two separate classes of individuals (namely juvenile and adults), diffusing in a one-dimensional region of space (the interval $[0, L]$). We denote by $J(t, x)$ the density of young individuals and by $A(t, x)$ the density of adults standing in the position $x$ at time $t$. The dynamics of such a population is then described by the following general equations

$$J_t = \beta(J, A)A + \sigma(J, A)J - \mu(J, A)J - \alpha J + D_J J_{xx}, \tag{4.2.1}$$

$$A_t = \alpha J - \eta(J, A)A + D_A A_{xx}, \tag{4.2.2}$$

together with the following initial conditions

$$J(0, x) = J_0(x), \tag{4.2.3}$$

$$A(0, x) = A_0(x), \tag{4.2.4}$$

and the homogeneous Neumann boundary conditions

$$J_x(t, 0) = J_x(t, L) = 0,$$

$$A_x(t,0) = A_x(t,L) = 0.$$

In particular we assume that juveniles and adults reproduce with different reproduction rates, $\sigma(J,A)$ and $\beta(J,A)$ respectively, and die with different mortality rates, $\mu(J,A)$ and $\eta(J,A)$ respectively. Moreover in (4.2.1) and (4.2.2) $\alpha > 0$ is the number of juveniles that pass to the adult stage per unit time and $D_J$ and $D_A$ are the diffusion coefficients of juveniles and adults respectively. The initial distributions $J_0(x)$ and $A_0(x)$ are supposed to be known.
Concerning the functions describing the dynamics of the population we assume that $\beta$, $\sigma$, $\mu$ and $\eta$ are positive functions of $J$ and $A$, moreover we assume that

$$\beta_J(J,A) > 0, \quad \beta_A(J,A) \le 0, \quad \sigma_J(J,A) \le 0, \quad \sigma_A(J,A) \le 0, \tag{4.2.5}$$

$$\mu_J(J,A) > 0, \quad \mu_A(J,A) \ge 0, \quad \eta_J(J,A) \ge 0, \quad \eta_A(J,A) \ge 0. \tag{4.2.6}$$

We observe that, according to these hypotheses, the presence of a large number of juveniles implies the increasing of the fertility of adults, while this fertility decreases with the number of adults. In this model also young individuals reproduce with a fertility $\sigma(J,A)$ which is decreasing with both the number of juveniles and adults. The mortalities $\mu(J,A)$ and $\eta(J,A)$ amount to describe a competition between juveniles and adults, which can be view as a sort of cannibalism, the individuals belonging to the same species.

Before starting the analysis of this model we rescale equations (4.2.1) and (4.2.2) by performing the following change of variables (see (4.1.11))

$$t^* = \frac{D_J}{L^2}t, \quad x^* = \frac{x}{L}, \tag{4.2.7}$$

ending up with

$$J_t = \gamma[\beta(J,A)A + \sigma(J,A)J - \mu(J,A)J - \alpha J] + J_{xx}, \tag{4.2.8}$$

$$A_t = \gamma[\alpha J - \eta(J,A)A] + dA_{xx}, \tag{4.2.9}$$

where $\gamma = \dfrac{L^2}{D_J}$ and $d = \dfrac{D_J}{D_A}$, and we have omitted the symbol $*$. The initial conditions read exactly like (4.2.3)-(4.2.4), while the boundary conditions are now replaced by

$$J_x(t,0) = J_x(t,1) = 0, \tag{4.2.10}$$

$$A_x(t,0) = A_x(t,1) = 0. \tag{4.2.11}$$

Our goal is to find conditions on $\beta$, $\sigma$, $\mu$, $\eta$, $\alpha$ and $d$ that imply the formation of spatial patterns, according to Turing mechanism. These conditions are obtained in the next section.

### 4.2.2 Conditions for pattern formation

We define the following functions

$$f(J,A) = \beta(J,A)A + \sigma(J,A)J - \mu(J,A)J - \alpha J,$$

$$g(J,A) = \alpha J - \eta(J,A)A,$$

then the system (4.2.8)-(4.2.9) can be rewritten as

$$J_t = \gamma f(J, A) + J_{xx}, \tag{4.2.12}$$

$$A_t = \gamma g(J, A) + dA_{xx}, \tag{4.2.13}$$

whose uniform steady states $(J^*, A^*)$ are determined by $f(J^*, A^*) = g(J^*, A^*) = 0$, and then by

$$A^* = \frac{\alpha J^*}{\eta(J^*, A^*)}, \tag{4.2.14}$$

$$\alpha[\beta(J^*, A^*) - \eta(J^*, A^*)] = \eta(J^*, A^*)[\mu(J^*, A^*) - \sigma(J^*, A^*)]. \tag{4.2.15}$$

Obviously $J^* \equiv 0$, $A^* \equiv 0$ satisfy $f(0,0) = g(0,0) = 0$ and then $(0,0)$ is a spatially homogeneous steady state of (4.2.12)-(4.2.13), anyway we shall not take into account this equilibrium because it can not evolve to a linearly stable spatial pattern. In fact, as we have observed in Section 4.1.2, a necessary (but not sufficient) condition for pattern formation is $f_A(J^*, A^*) \cdot g_J(J^*, A^*) < 0$, while for the trivial case $(J^*, A^*) = (0,0)$ we have $f_A(0,0) \cdot g_J(0,0) = \alpha \cdot \beta(0,0) > 0$.

Let us suppose that there exist at least a solution $(J^*, A^*)$ of (4.2.14)-(4.2.15) with $J^* > 0$ and $A^* > 0$, we first look for conditions on $\beta$, $\sigma$, $\mu$, $\eta$ and $\alpha$ such that the functions $f$ and $g$ satisfy (4.1.28). Let us evaluate the derivatives $f_J$, $f_A$, $g_J$ and $g_A$ at the steady state $(J^*, A^*)$. We have

$$\begin{aligned} f_J(J^*, A^*) &= \beta_J(J^*, A^*)A^* + [\sigma_J(J^*, A^*) - \mu_J(J^*, A^*)]J^* + \sigma(J^*, A^*) - \mu(J^*, A^*) - \alpha = \\ &= \frac{\alpha}{\eta(J^*, A^*)}[J^*\beta_J(J^*, A^*) - \beta(J^*, A^*)] + [\sigma_J(J^*, A^*) - \mu_J(J^*, A^*)]J^*, \end{aligned} \tag{4.2.16}$$

$$f_A(J^*, A^*) = \beta_A(J^*, A^*)A^* + \beta(J^*, A^*) + [\sigma_A(J^*, A^*) - \mu_A(J^*, A^*)]J^*, \tag{4.2.17}$$

$$g_J(J^*, A^*) = \alpha - \eta_J(J^*, A^*)A^* = \frac{\alpha}{\eta(J^*, A^*)}[\eta(J^*, A^*) - J^*\eta_J(J^*, A^*)], \tag{4.2.18}$$

$$g_A(J^*, A^*) = -\eta_A(J^*, A^*)A^* - \eta(J^*, A^*), \tag{4.2.19}$$

notice that, in the above calculations, we have used (4.2.14) and (4.2.15).

We recall that, in order to have diffusion driven instability (i.e. in order to satisfy both (4.1.28) and (4.1.29)), it must be

$$f_J(J^*, A^*) \cdot g_A(J^*, A^*) < 0.$$

We observe that, under our hypotheses (namely (4.2.6)), we have that $g_A(J, A) < 0$ for each $(J, A)$ with $J, A > 0$, thus $f_J(J^*, A^*)$ must be positive, i.e. we require

$$\alpha\big[J^*\beta_J(J^*, A^*) - \beta(J^*, A^*)\big] + J^*\eta(J^*, A^*)\big[\sigma_J(J^*, A^*) - \mu_J(J^*, A^*)\big] > 0, \tag{4.2.20}$$

in particular we notice that, being $\sigma_J(J^*,A^*) - \mu_J(J^*,A^*) < 0$ (see (4.2.5)), it must be $J^*\beta_J(J^*,A^*) - \beta(J^*,A^*) > 0$, and then $\beta$ can not be a linear function of $J$.
Moreover, in order to satisfy the first inequality in (4.1.28) (i.e. $\mathrm{tr}\mathcal{J}(f,g) < 0$) we have to impose

$$\begin{aligned}\alpha\big[J^*\beta_J(J^*,A^*) - \beta(J^*,A^*)\big] + J^*\eta(J^*,A^*)\big[\sigma_J(J^*,A^*) - \mu_J(J^*,A^*)\big]+ \\ -\eta(J^*,A^*)\big[\eta_A(J^*,A^*)A^* + \eta(J^*,A^*)\big] < 0,\end{aligned} \tag{4.2.21}$$

thus, combining (4.2.20) and (4.2.21), we obtain

$$\begin{aligned}0 < \alpha[J^*\beta_J(J^*,A^*) - \beta(J^*,A^*)] + J^*\eta(J^*,A^*)\big[\sigma_J(J^*,A^*) - \mu_J(J^*,A^*)\big] < \\ < \eta(J^*,A^*)\big[\eta_A(J^*,A^*)A^* + \eta(J^*,A^*)\big].\end{aligned}$$

It remains to find conditions such that $\det\mathcal{J}(f,g) > 0$, obviously, as already observed in Section 4.1.2, this inequality can be satisfied only if $f_A(J^*,A^*)\cdot g_J(J^*,A^*) < 0$. By using (4.2.16)-(4.2.19) and the fact that $(J^*,A^*)$ is a steady state satisfying (4.2.14)-(4.2.15), the condition $\det\mathcal{J}(f,g) > 0$ can be equivalently written as

$$\begin{aligned}\eta(J^*,A^*)\det\mathcal{J}(f,g) = \alpha^2J^*\big[\beta(J^*,A^*)\eta_A(J^*,A^*) - \eta(J^*,A^*)\beta_A(J^*,A^*)\big]+ \\ +J^*\eta(J^*,A^*)\big[A^*\eta_A(J^*,A^*) + \eta(J^*,A^*)\big]\big[\mu_J(J^*,A^*) - \sigma_J(J^*,A^*)\big]+ \\ -\alpha J^*\beta_J(J^*,A^*)\big[\eta(J^*,A^*) + A^*\eta_A(J^*,A^*)\big]+ \\ +\alpha J^*\eta_J(J^*,A^*)\big[\beta(J^*,A^*) + A^*\beta_A(J^*,A^*)\big]+ \\ +\alpha J^*\big[\eta(J^*,A^*) - J^*\eta_J(J^*,A^*)\big][\mu_A(J^*,A^*) - \sigma_A(J^*,A^*)] > 0.\end{aligned} \tag{4.2.22}$$

We observe that, since we are assuming that $\beta_A(J,A) \le 0$, $\sigma_J(J,A) \le 0$ and $\eta(J,A) \ge 0$ for every $J, A > 0$, the first and the second term in (4.2.22) are positive, while the third one is negative since $\beta_J(J,A) > 0$ for every $J, A > 0$. However we can not say anything about the sign of the fourth and the fifth term without adding assumptions.

Finally we have to impose also the condition $f_A(J^*,A^*)\cdot g_J(J^*,A^*) < 0$, which is equivalent to

$$\big[\eta(J^*,A^*) - J^*\eta_J(J^*,A^*)\big]\big[\beta(J^*,A^*) + A^*\beta_A(J^*,A^*) - J^*[\mu_A(J^*,A^*) - \sigma_A(J^*,A^*)]\big] < 0.$$

Summing up, we conclude with the following result

**Theorem 4.2.1** *Assume that the positive functions $\beta$, $\sigma$, $\mu$ and $\eta$ satisfy* (4.2.5)-(4.2.6)*, let $(J^*,A^*)$ be a positive stationary equilibrium of the system* (4.2.12)-(4.2.13) *(endowed with inital conditions given by* (4.2.3)-(4.2.4) *and boundary conditions given by* (4.2.10)-(4.2.11)*). If the following conditions are satisfied*

$$\begin{aligned}0 < \alpha[J^*\beta_J(J^*,A^*) - \beta(J^*,A^*)] + J^*\eta(J^*,A^*)\big[\sigma_J(J^*,A^*) - \mu_J(J^*,A^*)\big] < \\ < \eta(J^*,A^*)\big[\eta_A(J^*,A^*)A^* + \eta(J^*,A^*)\big],\end{aligned} \tag{4.2.23}$$

$$\begin{aligned}
&\alpha^2 J^*\left[\beta(J^*,A^*)\eta_A(J^*,A^*)-\eta(J^*,A^*)\beta_A(J^*,A^*)\right]+\\
&+J^*\eta(J^*,A^*)\left[A^*\eta_A(J^*,A^*)+\eta(J^*,A^*)\right]\left[\mu_J(J^*,A^*)-\sigma_J(J^*,A^*)\right]+\\
&-\alpha J^*\beta_J(J^*,A^*)\left[\eta(J^*,A^*)+A^*\eta_A(J^*,A^*)\right]+\\
&+\alpha J^*\eta_J(J^*,A^*)\left[\beta(J^*,A^*)+A^*\beta_A(J^*,A^*)\right]+\\
&+\alpha J^*\left[\eta(J^*,A^*)-J^*\eta_J(J^*,A^*)\right]\left[\mu_A(J^*,A^*)-\sigma_A(J^*,A^*)\right]>0,
\end{aligned} \tag{4.2.24}$$

$$\left[\eta(J^*,A^*)-J^*\eta_J(J^*,A^*)\right]\left[\beta(J^*,A^*)+A^*\beta_A(J^*,A^*)-J^*[\mu_A(J^*,A^*)-\sigma_A(J^*,A^*)]\right]<0, \tag{4.2.25}$$

*then $(J^*,A^*)$ is linearly stable in the absence of diffusion, while it is unstable to small spatial perturbations when juveniles $J$ and adults $A$ diffuse with different coefficients $D_J$ and $D_A$ such that $d=\dfrac{D_A}{D_J}>1$ (i.e. adults diffuse faster than juveniles).*

### 4.2.3 First specific model

In this section we analyze a particular case of the model (4.2.8)-(4.2.9), namely we consider the following equation describing the dynamics of the population

$$J_t=[\beta_1 J^2+\beta_3]A+[\sigma-\mu J]J-\alpha J+J_{xx}, \tag{4.2.26}$$

$$A_t=\alpha J-\varepsilon[\beta_1 J^2+\beta_3]A+dA_{xx}, \tag{4.2.27}$$

where $\beta_1$, $\beta_3$, $\sigma$, $\mu$, $\alpha$ and $\varepsilon$ are positive constants. Notice that adults' fertility is increasing with the number of young individuals, according to hypothesis (4.2.5), and in particular it is super-linear, as requested in the previous section. The term $[\sigma-\mu J]J$ describes a logistic behaviour of juveniles, while, as before, $\alpha$ represents the transition of youngs to the adult stage. In particular, using the notation introduced in the previous section, we have

$$\beta(J)=\beta_1 J^2+\beta_3,\quad \sigma(J)\equiv\sigma,\quad \mu(J)=\mu J,\quad \eta(J)=\varepsilon\beta(J). \tag{4.2.28}$$

In other words, we now deal with the system

$$J_t=f(J,A)+J_{xx}, \tag{4.2.29}$$

$$A_t=g(J,A)+dA_{xx}, \tag{4.2.30}$$

where the reaction terms are

$$f(J,A)=[\beta_1 J^2+\beta_3]A+[\sigma-\mu J]J-\alpha J,$$

and

$$g(J,A)=\alpha J-\varepsilon[\beta_1 J^2+\beta_3]A,$$

with the following initial conditions

$$J(0,x) = J_0(x), \quad A(0,x) = A_0(x),$$

and homogeneous Neumann boundary conditions

$$J_x(t,0) = J_x(t,1) = A_x(t,0) = A_x(t,1) = 0,$$

(recall that, having performed the change of variables (4.2.7), the space domain is now the interval $[0,1]$).

It is not difficult to show that there exists a unique spatially homogeneous steady state of the system (4.2.29)-(4.2.30), given by

$$J^* = \frac{\alpha + \varepsilon(\sigma - \alpha)}{\varepsilon\mu}, \quad A^* = \frac{\mu\alpha(\alpha + \varepsilon(\sigma - \alpha))}{\beta_1(\alpha + \varepsilon(\sigma - \alpha))^2 + \beta_3\varepsilon^2\mu^2},$$

since we are interested in positive equilibria, we observe that, if $\sigma - \alpha > 0$, $J^* > 0$ and $A^* > 0$, while if $\sigma - \alpha < 0$ we have to require $\varepsilon < \dfrac{\alpha}{\alpha - \sigma}$. The derivatives of the functions $f$ and $g$ evaluated in $(J^*, A^*)$ are

$$f_J = 2\beta_1 J^* A^* - 2\mu J^* + \sigma - \alpha, \tag{4.2.31}$$

$$f_A = \beta_1 {J^*}^2 + \beta_3, \tag{4.2.32}$$

$$g_J = \alpha - \varepsilon 2\beta_1 J^* A^*, \tag{4.2.33}$$

$$g_A = -\varepsilon[\beta_1 {J^*}^2 + \beta_3]. \tag{4.2.34}$$

Now, it is possible to prove that, under suitable assumptions on the coefficients, $(J^*, A^*)$ is a linaearly stable equilibrium in the absence of spatial diffusion, namely we have

**Lemma 4.2.1** *Assume $\alpha + \varepsilon(\sigma - \alpha) > 0$, then the system (4.2.29)-(4.2.30) has a unique spatially homogeneous steady state $(J^*, A^*)$ with $J^* > 0$, $A^* > 0$. Moreover if we denote by $K = \beta_1(\alpha + \sigma\varepsilon - \alpha\varepsilon)^2 + \beta_3\varepsilon^2\mu^2$ and we assume that the constants $\beta_1$, $\beta_3$, $\sigma$, $\mu$, $\alpha$ and $\varepsilon$ satisfy the following inequality*

$$K^2 + \mu^2(\alpha + \varepsilon(\sigma - \alpha))K - \alpha\mu^2\left[\beta_1(\alpha + \varepsilon(\sigma - \alpha))^2 - \beta_3\varepsilon^2\mu^2\right] > 0, \tag{4.2.35}$$

*then $(J^*, A^*)$ is linearly stable when spatial diffusion is absent.*

**Proof.** We have already observed that, if $\alpha + \sigma\varepsilon - \alpha\varepsilon > 0$, the system (4.2.26)-(4.2.29) has a unique space independent equilibrium $(J^*, A^*)$ with $J^* > 0$, $A^* > 0$. It remains to show that, if $J$ and $A$ do not diffuse, this equilibrium is linearly stable, in other words we have to prove that

$$\mathrm{tr}\mathcal{J}(f,g) < 0, \quad \det \mathcal{J}(f,g) > 0,$$

where $\mathcal{J}(f,g)$ is the Jacobian matrix evaluated at the equilibrium $(J^*, A^*)$. It is not difficult to see that

$$\mathrm{tr}\mathcal{J}(f,g) = \frac{\alpha\mu^2\left[\beta_1(\alpha + \varepsilon(\sigma - \alpha))^2 - \beta_3\varepsilon^2\mu^2\right] - \mu^2(\alpha + \varepsilon(\sigma - \alpha))K - K^2}{\varepsilon\mu^2 K},$$

and then $\mathrm{tr}\mathcal{J}(f,g) < 0$ if and only if condition (4.2.35) holds.
On the other hand, using (4.2.31)-(4.2.34), we have that

$$\det \mathcal{J}(f,g) = \left[\beta_1 J^{*2} + \beta_3\right] \cdot [2\mu\varepsilon J^* - \alpha - \varepsilon(\sigma-\alpha)] =$$

$$= \left[\beta_1 J^{*2} + \beta_3\right] \cdot [\alpha + \varepsilon(\sigma-\alpha)] = \frac{K(\alpha+\varepsilon(\sigma-\alpha))}{\varepsilon^2\mu^2},$$

and then $\det \mathcal{J}(f,g) > 0$, since, by assumption, $\alpha + \sigma\varepsilon - \alpha\varepsilon > 0$. □

We now look for conditions on $\beta_1$, $\beta_3$, $\sigma$, $\mu$, $\alpha$, $\varepsilon$ and $d$ that imply diffusion driven instability. We first have to impose $f_J \cdot g_A < 0$ and $f_A \cdot g_J < 0$. In particular we have that $g_A < 0$ and then $f_J$ must be positive, from (4.2.31) simple calculations lead to

$$f_J = \frac{-K(\sigma-\alpha) - 2\beta_3\alpha\varepsilon\mu^2}{K}, \tag{4.2.36}$$

and then we require

$$K(\sigma-\alpha) + 2\beta_3\alpha\varepsilon\mu^2 < 0, \tag{4.2.37}$$

in particular we observe that it must be $\sigma < \alpha$. On the other hand $f_A > 0$ and then $g_J$ must be negative, indeed, since we are imposing $f_J > 0$, i.e. $2\beta_1 J^* A^* > 2\mu J^* - \sigma + \alpha$, we have

$$g_J = \alpha - 2\varepsilon\beta_1 J^* A^* < \alpha - \varepsilon(2\mu J^* - \sigma + \alpha) = -(\alpha + \varepsilon(\sigma-\alpha)), \tag{4.2.38}$$

which is negative under our assumptions.

The next step consist in finding conditions on the parameters in order to satisfy (see (4.1.29))

$$df_J + g_A > 0, \quad (df_J + g_A)^2 - 4d(f_J g_A - f_A g_J) > 0, \tag{4.2.39}$$

actually, once that the constants $\beta_1$, $\beta_3$, $\sigma$, $\mu$, $\alpha$ and $\varepsilon$ satisfy $\alpha + \sigma\varepsilon - \alpha\varepsilon > 0$, the inequality (4.2.35) and the condition (4.2.38), as already observed in Remark 4.1.1, it is enough to chose $d > 1$ large enough to satisfy (4.1.31).
We summarize the necessary conditions for pattern formation in system (4.2.26)-(4.2.29) in the following

**Theorem 4.2.2** *Assume $\alpha+\sigma\varepsilon-\alpha\varepsilon > 0$, denote by $K = \beta_1(\alpha+\sigma\varepsilon-\alpha\varepsilon)^2 + \beta_3\varepsilon^2\mu^2$ and suppose that the following conditions hold*

$$0 < \mu^2 \left[\alpha\left(\beta_1(\alpha+\varepsilon(\sigma-\alpha))^2 - \beta_3\varepsilon^2\mu^2\right) - K(\alpha+\varepsilon(\sigma-\alpha))\right] < K^2, \tag{4.2.40}$$

*and*

$$d > d_{max} = \max\left\{\frac{K^2}{\mu^2\left[K(\sigma-\alpha)+2\beta_3\alpha\varepsilon\mu^2\right]}, \right.$$
$$\left. \frac{K^2}{\varepsilon^2\mu^2}\frac{\left[\sqrt{K(\alpha+\varepsilon(\sigma-\alpha))} + \sqrt{-\alpha(K-2\beta_1(\alpha+\varepsilon(\sigma-\alpha))^2)}\right]^2}{\left[K(\sigma-\alpha)+2\beta_3\alpha\varepsilon\mu^2\right]^2}\right\}, \tag{4.2.41}$$

*then the unique spatially homogeneous steady state $(J^*, A^*)$ of the system (4.2.26)-(4.2.29) is linearly stable in the absence of diffusion, but it evolves to a spatially inhomogeneous steady state when spatial diffusion is present.*

**Proof.** The hypoteses of Lemma 4.2.1 are satisfied (in particular the second inequality in (4.2.40) is actually hypothesis (4.2.35)), thus we have that $(J^*, A^*)$ is a linearly stable equilibrium when diffusion does not occur.
Moreover it is not difficul to see that the first inequality in condition (4.2.40) is actually (4.2.37), and then $f_J > 0$. Finally by choosing $d$ satisfying (4.2.41) we have that $df_J + g_A > 0$ and $(df_J + g_A)^2 - 4d(f_J g_A - f_A g_J) > 0$, i.e. we guarantee that the conditions for diffusion driven instability (4.1.29) are satified, and then a stable spatially inhomogeneos pattern will evolve from the steady state $(J^*, A^*)$. □

In Figure 4.1 and Figure 4.2 we have, respectively, the juveniles' and the adults' profile and the evolution of the stable spatial pattern in the case in which $\beta_1 = 0.3$, $\beta_3 = 0.8$, $\sigma = 0.3$, $\mu = 0.1$, $\alpha = 2$, $\varepsilon = 1$, $D_J = 0.001$ and $D_A = 0.1$. The choice $\varepsilon = 1$ implies that the fertility and the mortality of adults are the same). In this case the homogenous spatially steady state is given by $J^* = 3$, $A^* = 1.7143$ and in Figure 4.1 we see both the random initial data $J_0(x)$ and $A_0(x)$ (obtained in this simulation by a perturbation of $J^*$ and $A^*$) and the stable spatial pattern that evolve from it.

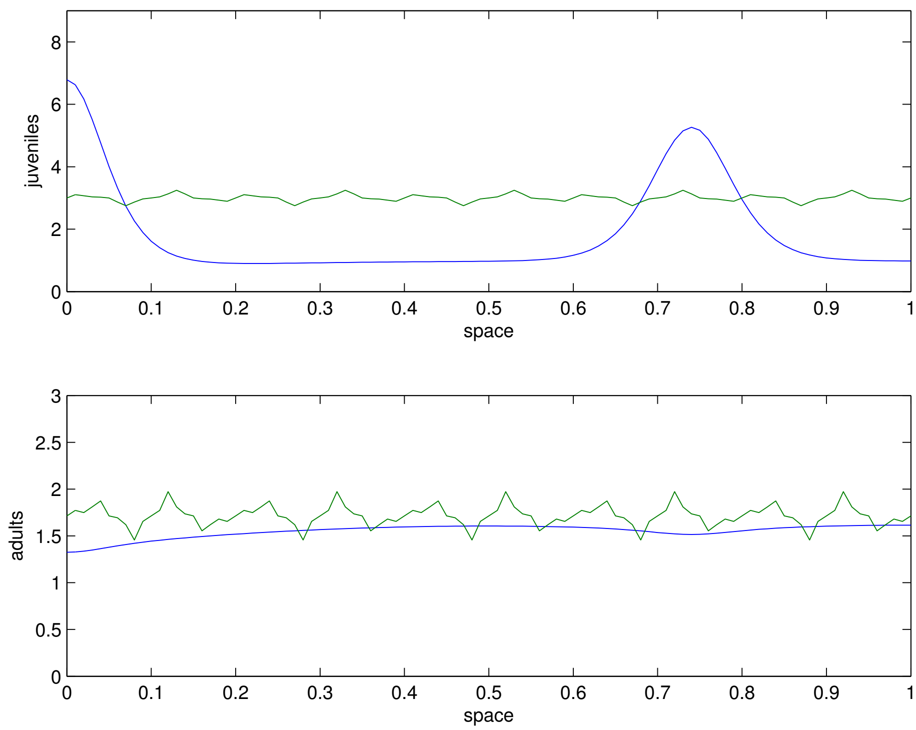


Figure 4.1: Juveniles' and adults' profiles, $\varepsilon = 1$

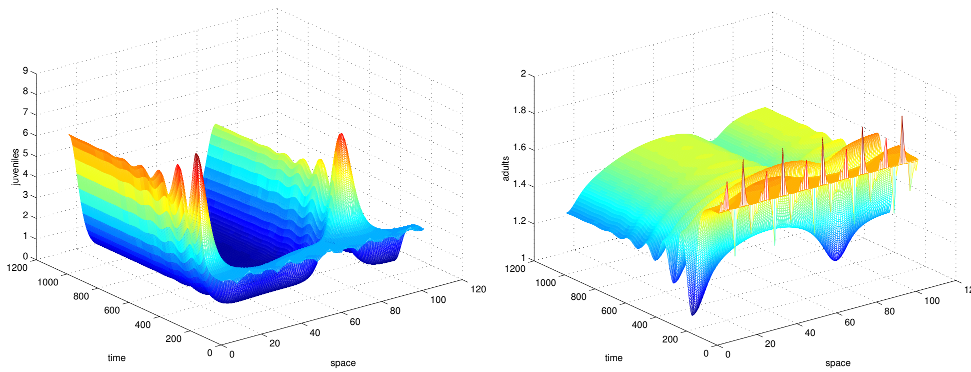


Figure 4.2: Evolution of the pattern, $\varepsilon = 1$

Now we consider the case in which the fertility and the mortality of adults are different but proportional, this means that we take $\varepsilon \neq 1$, in particular we take $\varepsilon = 0.8$ (in Figure 4.3 and in Figure 4.4) and $\varepsilon = 0.9$ (in Figure 4.5 and in Figure 4.6). Now $J^* = 8$ and $A^* = 1$ when $\varepsilon = 0.8$ while $J^* = 5.2222$ and $A^* = 1.2921$ when $\varepsilon = 0.9$.

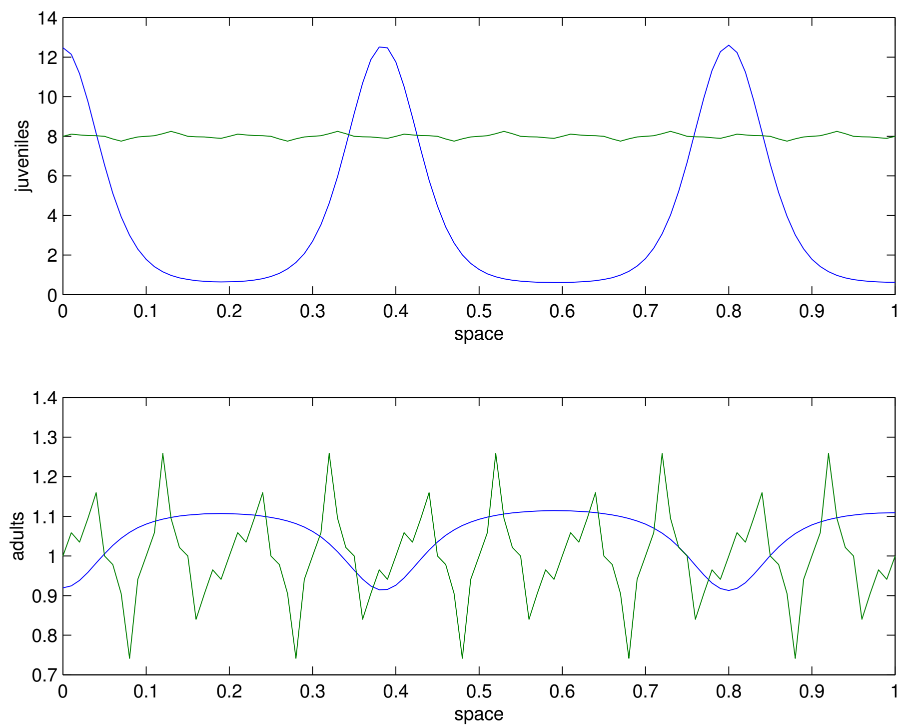


Figure 4.3: Juveniles' and adults' profiles, $\varepsilon = 0.8$

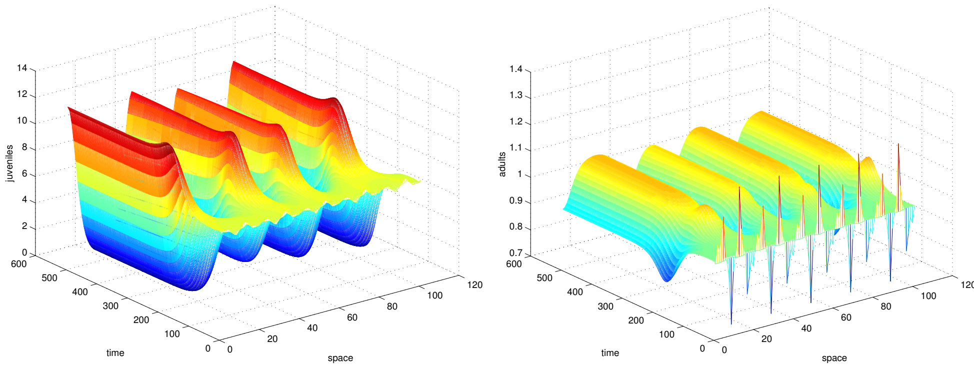


Figure 4.4: Evolution of the pattern, $\varepsilon = 1$

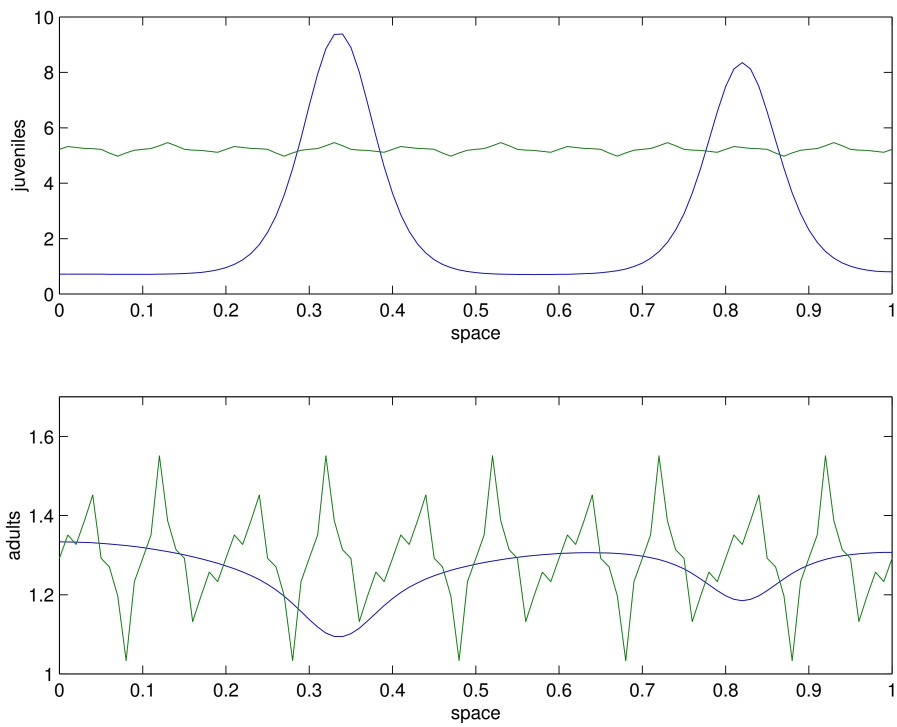


Figure 4.5: Juveniles' and adults' profiles, $\varepsilon = 0.9$

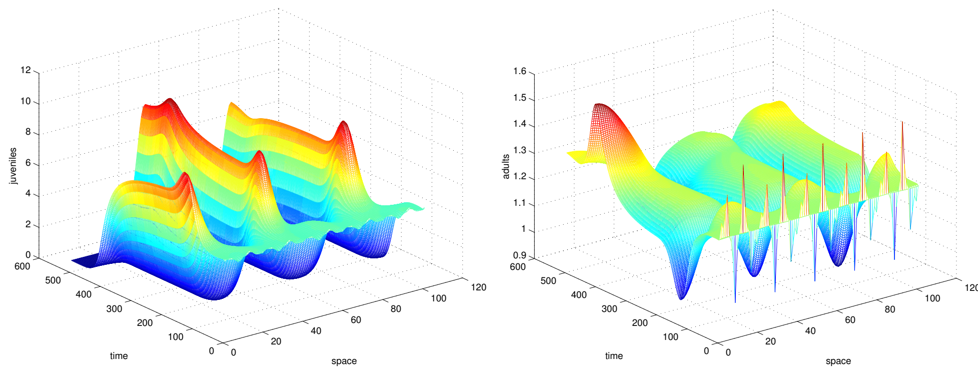


Figure 4.6: Evolution of the pattern, $\varepsilon = 1$

### 4.2.4 Second specific model

Now we consider a slight modification of the model consider in the previous section, in particular we assume that there is a competion between juveniles and adults that lead to an increase of juveniles' mortality due to the presence of adults and a decrease of adults' mortality as a consequence. We deal with the following problem

$$J_t = [\beta_1 J^2 - \beta_2 J + \beta_3]A + [\sigma - \mu J]J - \alpha J + J_{xx}$$

$$A_t = \alpha J - \varepsilon[\beta_1 J^2 - \beta_2 J + \beta_3]A + dA_{xx},$$

where $\beta_1$, $\beta_2$, $\beta_3$, $\sigma$, $\mu$, $\alpha$ and $\varepsilon$ are positive constants. Moreover we assume that $4\beta_1\beta_3 > \beta_2^2$, in fact, if this condition is not satisfied, the mortality of adults could become negative for some values of $J > 0$ and this is nonsense. In particular, using the notation introduced in Section 4.2.2, we now have

$$\beta(J) = \beta_1 J^2 - \beta_2 J + \beta_3, \quad \sigma(J) \equiv \sigma, \quad \mu(J) = \mu J, \quad \eta(J) = \varepsilon\beta(J).$$

In other words, we now deal with the system

$$J_t = f(J, A) + J_{xx}, \tag{4.2.42}$$

$$A_t = g(J, A) + dA_{xx}, \tag{4.2.43}$$

where the reaction terms are

$$f(J, A) = [\beta_1 J^2 - \beta_2 J + \beta_3]A + [\sigma - \mu J]J - \alpha J,$$

and

$$g(J, A) = \alpha J - \varepsilon[\beta_1 J^2 - \beta_2 J + \beta_3]A,$$

with the following initial conditions

$$J(0, x) = J_0(x), \quad A(0, x) = A_0(x),$$

and homogeneous Neumann boundary conditions

$$J_x(t,0) = J_x(t,1) = A_x(t,0) = A_x(t,1) = 0,$$

(recall that, having performed the change of variables (4.2.7), the space domain is now the interval $[0,1]$).

The unique spatially homogeneous steady state of system (4.2.42)-(4.2.43) is given by

$$J^* = \frac{H}{\varepsilon\mu}, \quad A^* = \frac{\alpha\mu H}{K},$$

where we have used the following definitions in order to simplify the notation:

$$H = \alpha + \varepsilon(\sigma - \alpha), \quad K = \beta_1 H^2 - \beta_2 \varepsilon\mu H + \beta_3 \varepsilon^2 \mu^2.$$

As in the previous case we have to assume that $\alpha + \varepsilon(\sigma - \alpha) > 0$, in order $J^*$ and $A^*$ to be positive (notice that $K > 0$ because we are assuming $4\beta_1\beta_3 > \beta_2^2$).

The derivatives of the functions $f$ and $g$ evaluated at the steady state $(J^*, A^*)$ are

$$f_J = (2\beta_1 J^* - \beta_2)A^* + \sigma - 2\mu J^* - \alpha, \tag{4.2.44}$$

$$f_A = (\beta_1 J^{*2} - \beta_2 J^* + \beta_3), \tag{4.2.45}$$

$$g_J = \alpha - \varepsilon(2\beta_1 J^* - \beta_2)A^*, \tag{4.2.46}$$

$$g_A = -\varepsilon(\beta_1 J^{*2} - \beta_2 J^* + \beta_3). \tag{4.2.47}$$

Now, it is possible to prove that, under suitable assumptions on the coefficients, the equilibrium $(J^*, A^*)$ is linearly stable in the absence of diffusion, namely we have

**Lemma 4.2.2** *Assume $4\beta_1\beta_3 > \beta_2^2$ and $\alpha + \varepsilon(\sigma - \alpha) > 0$, then the system (4.2.42)-(4.2.43) has a unique spatially homogeneous steady state $(J^*, A^*)$ with $J^* > 0$, $A^* > 0$. Moreover, if we denote by $H = \alpha + \varepsilon(\sigma - \alpha)$ and $K = \beta_1 H^2 - \beta_2 \varepsilon\mu H + \beta_3\varepsilon^2\mu^2$, and assume that the constants $\beta_1$, $\beta_2$, $\beta_3$, $\sigma$, $\mu$, $\alpha$ and $\varepsilon$ satisfy the following inequality*

$$K^2 + \mu^2 HK - \alpha\mu^2[\beta_1 H^2 - \beta_3\varepsilon^2\mu^2] > 0, \tag{4.2.48}$$

*then $(J^*, A^*)$ is linearly stable when spatial diffusion is absent.*

**Proof.** We have already observed that, if $4\beta_1\beta_3 > \beta_2^2$ and $H = \alpha + \varepsilon(\sigma - \alpha) > 0$, the system (4.2.42)-(4.2.43) has a unique space independent equilibrium $(J^*, A^*)$ with $J^* > 0$, $A^* > 0$. It remains to show that, if $J$ and $A$ do not diffuse, this equilibrium is linarly stable, in other words we have to prove that

$$\mathrm{tr}\mathcal{J}(f,g) < 0, \quad \det \mathcal{J}(f,g) > 0,$$

where $\mathcal{J}(f,g)$ is the Jacobian matrix evaluated at the equilibrium $(J^*, A^*)$. It is not difficult to see that

$$\mathrm{tr}\mathcal{J}(f,g) = \frac{\alpha\mu^2[\beta_1 H^2 - \beta_3\varepsilon^2\mu^2] - \mu^2 HK - K^2}{\varepsilon\mu^2 K},$$

and then it is negative if condition (4.2.48) is satisified.
On the other hand, using (4.2.44)-(4.2.47), we have that

$$\det \mathcal{J}(f,g) = \left[\beta_1 J^{*2} - \beta_2 J^* + \beta_3\right] H = \frac{KH}{\varepsilon^2 \mu^2},$$

and then it is positive since, by assumption $H > 0$. □

We now look for conditions on the parameters $\beta_1$, $\beta_2$, $\beta_3$, $\sigma$, $\mu$, $\alpha$, $\varepsilon$ and $d$ that imply diffusion driven instability. We first impose $f_J \cdot g_A < 0$ and $f_A \cdot g_J < 0$. In particular, under the assumption that $4\beta_1\beta_3 > \beta_2^2$, we have that $g_A < 0$ and then $f_J$ must be positive, from (4.2.44) simple calculations imply that we must require

$$\alpha\left[\beta_1 H^2 - \beta_3 \varepsilon^2 \mu^2\right] - HK > 0. \tag{4.2.49}$$

On the other hand, $f_A > 0$ and then $g_J$ must be negative, indeed, since we are imposing $f_J > 0$, i.e. $(2\beta_1 J^* - \beta_2)A^* > 2\mu J^* - \sigma + \alpha$, we have

$$g_J = \alpha - \varepsilon(2\beta_1 J^* - \beta_2) < \alpha - \varepsilon(2\mu J^* - \sigma + \alpha) = -H,$$

which is negative under our assumptions.

We are now ready to state the following

**Theorem 4.2.3** *Assume $4\beta_1\beta_3 > \beta_2^2$ and $\alpha + \varepsilon(\sigma - \alpha) > 0$, denote by $H = \alpha + \varepsilon(\sigma - \alpha)$ and by $K = \beta_1 H^2 - \beta_2 \varepsilon \mu H + \beta_3 \varepsilon^2 \mu^2$ and suppose that the following conditions hold*

$$0 < \mu^2\left[\alpha(\beta_1 H^2 - \beta_3 \varepsilon^2 \mu^2) - HK\right] < K^2, \tag{4.2.50}$$

*and*

$$d > d_{max} = \max\left\{\frac{K^2}{\mu^2\left[\alpha(\beta_1 H^2 - \beta_3\varepsilon^2\mu^2) - HK\right]},\right.$$
$$\left.\frac{K^2}{\varepsilon^2\mu^2}\frac{\left[\sqrt{KH} + \sqrt{-\alpha(K - 2\beta_1 H^2 + \beta_2 \varepsilon \mu H)}\right]^2}{\left[K(\sigma - \alpha) + 2\beta_3 \alpha \varepsilon \mu^2\right]^2}\right\}, \tag{4.2.51}$$

*then the unique spatially homogeneous steady state $(J^*, A^*)$ of the system (4.2.26)-(4.2.29) is linearly stable in the absence of diffusion, but it evolves to a spatially inhomogeneous steady state when spatial diffusion is present.*

In Figure 4.7 and in Figure 4.8 we have, respectively, the juveniles' and the adults' profile and the evolution of the stable spatial pattern in the case in which $\beta_1 = 0.3$, $\beta_2 = 0.2$, $\beta_3 = 0.8$, $\sigma = 0.3$, $\mu = 0.1$, $\alpha = 2$, $\varepsilon = 1$, $D_J = 0.001$ and $D_A = 0.1$. In this case the homogenous spatially steady state is given by $J^* = 3$, $A^* = 2.0690$.

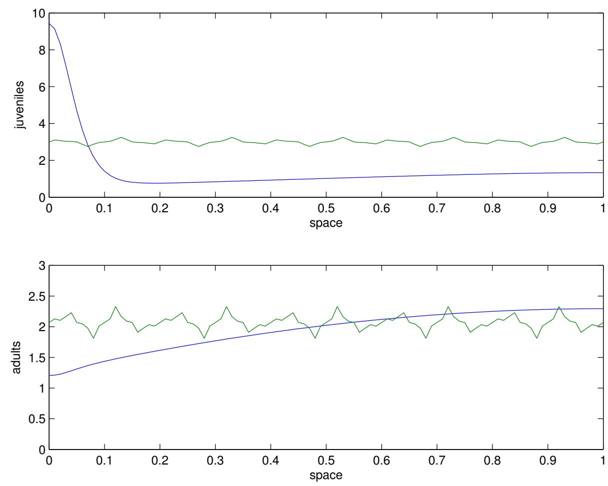


Figure 4.7: Juveniles' and adults' profiles, $\varepsilon = 1$

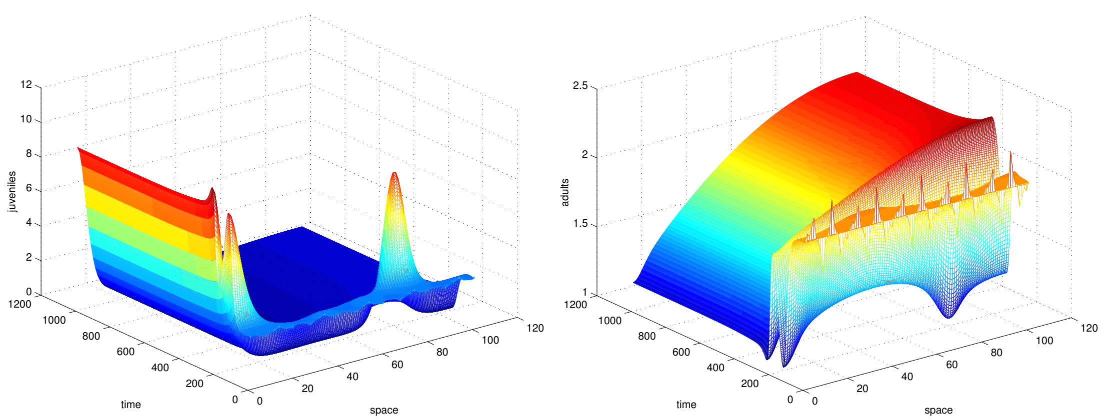


Figure 4.8: Evolution of the pattern, $\varepsilon = 1$

Now we consider the case in which $\varepsilon \neq 1$, as in the previous case who chose $\varepsilon = 0.8$ (Figure 4.9 and Figure 4.10) and $\varepsilon = 0.9$ (Figure 4.11 and Figure 4.11). Now $J^* = 8$ and $A^* = 1$ when $\varepsilon = 0.8$ while $J^* = 5.2222$ and $A^* = 1.2921$ when $\varepsilon = 0.9$.

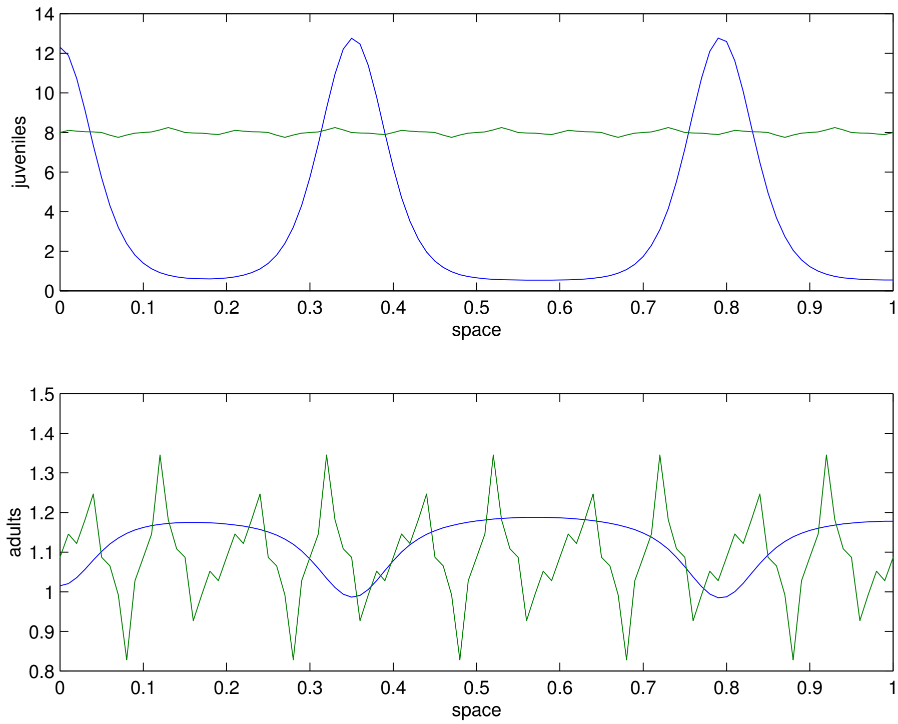


Figure 4.9: Juveniles' and adults' profiles, $\varepsilon = 0.8$

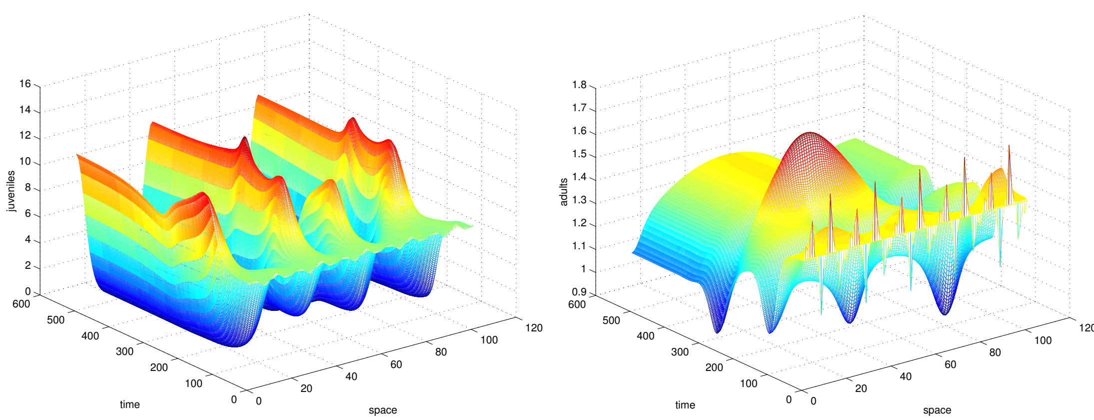


Figure 4.10: Evolution of the pattern, $\varepsilon = 0.8$

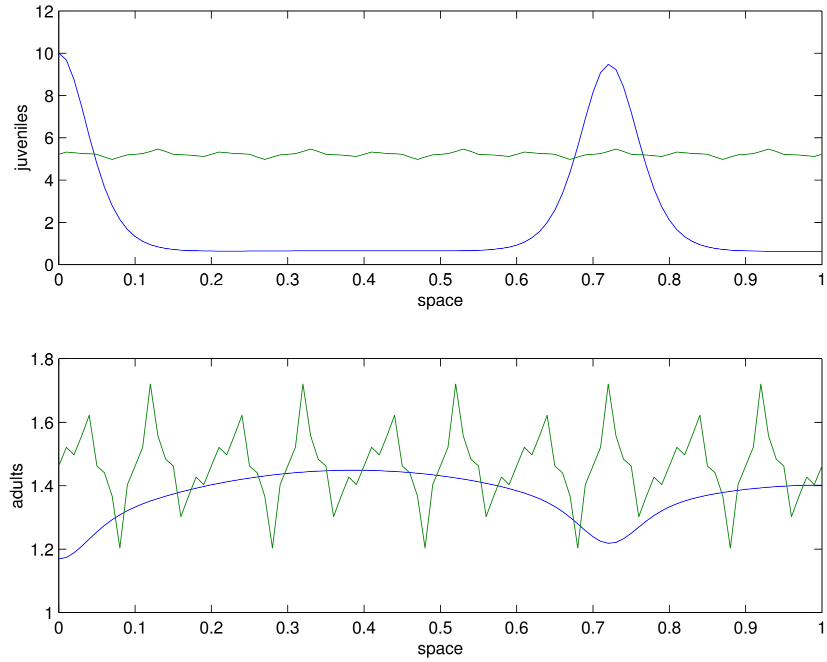


Figure 4.11: Juveniles' and adults' profiles, $\varepsilon = 0.9$

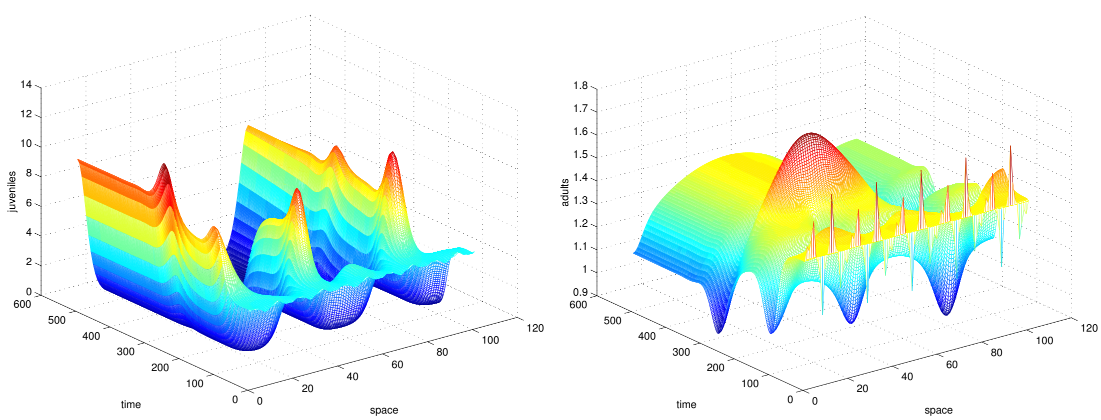


Figure 4.12: Evolution of the pattern, $\varepsilon = 0.9$

### 4.2.5 Age-structured models and pattern formation: future perspectives

The models described in the previous sections have been analyzed in view of future studies concerning the formation of pattern due to the age-structure of a population. In this Section we construct two specific models, based on the problems considered in Section 4.2.3 and in Section 4.2.4, for the diffusion of an age-structured population in a bounded region $\Omega \subset \mathbb{R}^n$ which is supposed to be divided into two separate classes depending on the age of individuals, namely:

juveniles and adults.

For semplicity we focus on the one-dimensional case, by taking $\Omega = [0,1]$. Using the notation introduced in Chapter 2, we denote by $p(t,a,x)$ the density, per unit volume and age, at time t of the population, where $t \geq 0$, $a \in [0, a_\dagger]$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega$. Moreover we denote by $a^*$ the maturation age, i.e. the age at which an individual becomes adult. Thus the integrals

$$J(t,x) = \int_0^{a^*} p(t,a,x)da$$

and

$$A(t,x) = \int_{a^*}^{a_\dagger} p(t,a,x)da$$

give, respectively, the number of juveniles and the number of adults standing in the position $x$ at time $t$.

We consider the following problem

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu(a, J(t,x), A(t,x))p = D(a)\frac{\partial^2 p}{\partial x^2}, \qquad \text{for } t \in (0, +\infty),\ a \in (0, a_\dagger),\ x \in (0,1),$$

$$p(0,a,x) = p_0(a,x), \qquad \text{for } a \in (0, a_\dagger),\ x \in (0,1),$$

$$p(t,0,x) = \int_0^{a_\dagger} \beta(a, J(t,x), A(t,x))p(t,a,x)da, \qquad \text{for } t \in (0, +\infty),\ x \in (0,1),$$

$$\frac{\partial p}{\partial x}(t,a,0) = \frac{\partial p}{\partial x}(t,a,1) = 0, \qquad \text{for } t \in (0, +\infty),\ a \in (0, a_\dagger).$$

Here $\mu(a, J(t,x), A(t,x))$ is the age-specific mortality which depends on age but also on $J(t,x)$ and $A(t,x)$, i.e. on the presence of juveniles and adults. In the same way, also the age-specific fertility $\beta(a, J(t,x), A(t,x))$. The function $D(a)$ represents the diffusivity and it is supposed to depend on $a$ only. The initial distribution $p_0(a,x)$ is supposed to be given, while the homogeneous boundary condition amounts to assume that the individuals do not cross the boundary.
On the basis of the models discussed in the previous section, we can take

$$\mu(a, J(t,x)) = \mu_0(a) + m_J J(t,x)\gamma_1(a) + m_A(J(t,x))\gamma_2(a),$$

and

$$\beta(a, J(t,x)) = b_J \gamma_1(a) + b_A(J(t,x))\gamma_2(a),$$

where $\mu_0$ represents an intrinsic mortality, which does not depend on the interaction between individuals, but on their age only, moreover $\gamma_1 = \chi_{[0,a^*]}$ and $\gamma_2 = \chi_{[a^*, a_\dagger]}$ denote the carachteristic functions of $[0, a^*]$ and $[a^*, a_\dagger]$ respectively.
With this choice for $\mu$ and $\beta$, we have that juveniles' mortality depends on their number in a linear way, while, on the basis of the observation made in the previous sections, we expect $m_A(J(t,x))$ and $b_A(J(t,x))$ to be super-linear functions of $J$.

The goal of our future work is to analyze in depth this model, aiming to find conditions on the functions $b_A(J)$ and $m_A(J)$ that guarantee the formation of stable spatial pattern in the juveniles' and in the adults' profile, under the hypothesis that individuals diffuse in a different way according to their age. To come to this end, we shall first need to look for the existence of at least one spatially homogeneous steady state $p^*(a)$ of the problem and to analyze its stability, both in the absence and in the presence of spatial diffusion, by finding the roots of the corresponding characteristic equation. In order to study this characteristic equation some computational methods are needed, in this direction some results have already been obtained in collaboration with Breda, Iannelli, Maset and Vermiglio in [6] and [5].

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