JCSS Workshop Tongji University

Tail-sensitive design point and reliability index: A refined approach to structural reliability design and assessment

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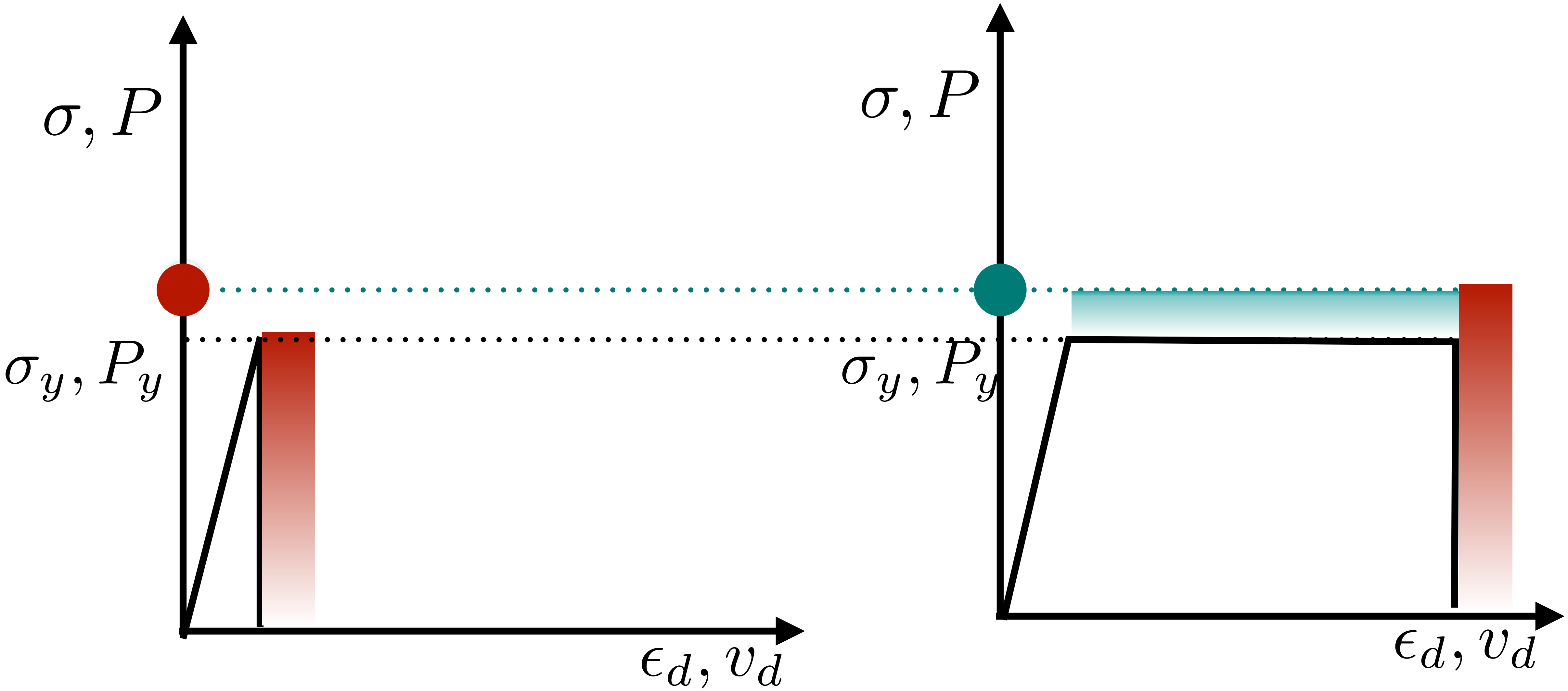
outline

- Motivation-History
- Review classical structural reliability theory
- Beyond Design point, Tail-Dependent Design (/Failure) Point
- Beyond Reliability index, Tail-Dependent Reliability Index
- Conclusion

Motivation

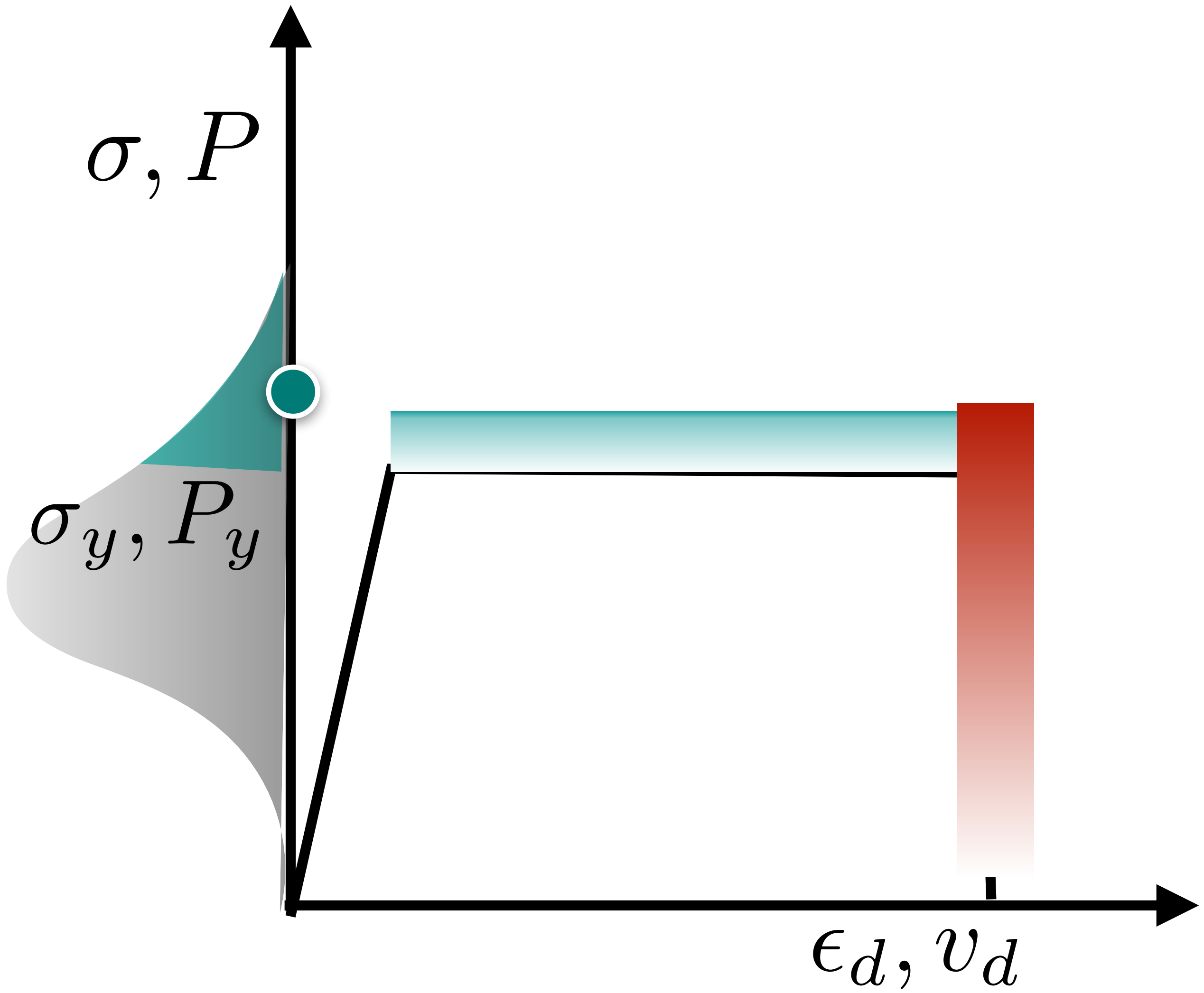
Design to failure to not fail to design

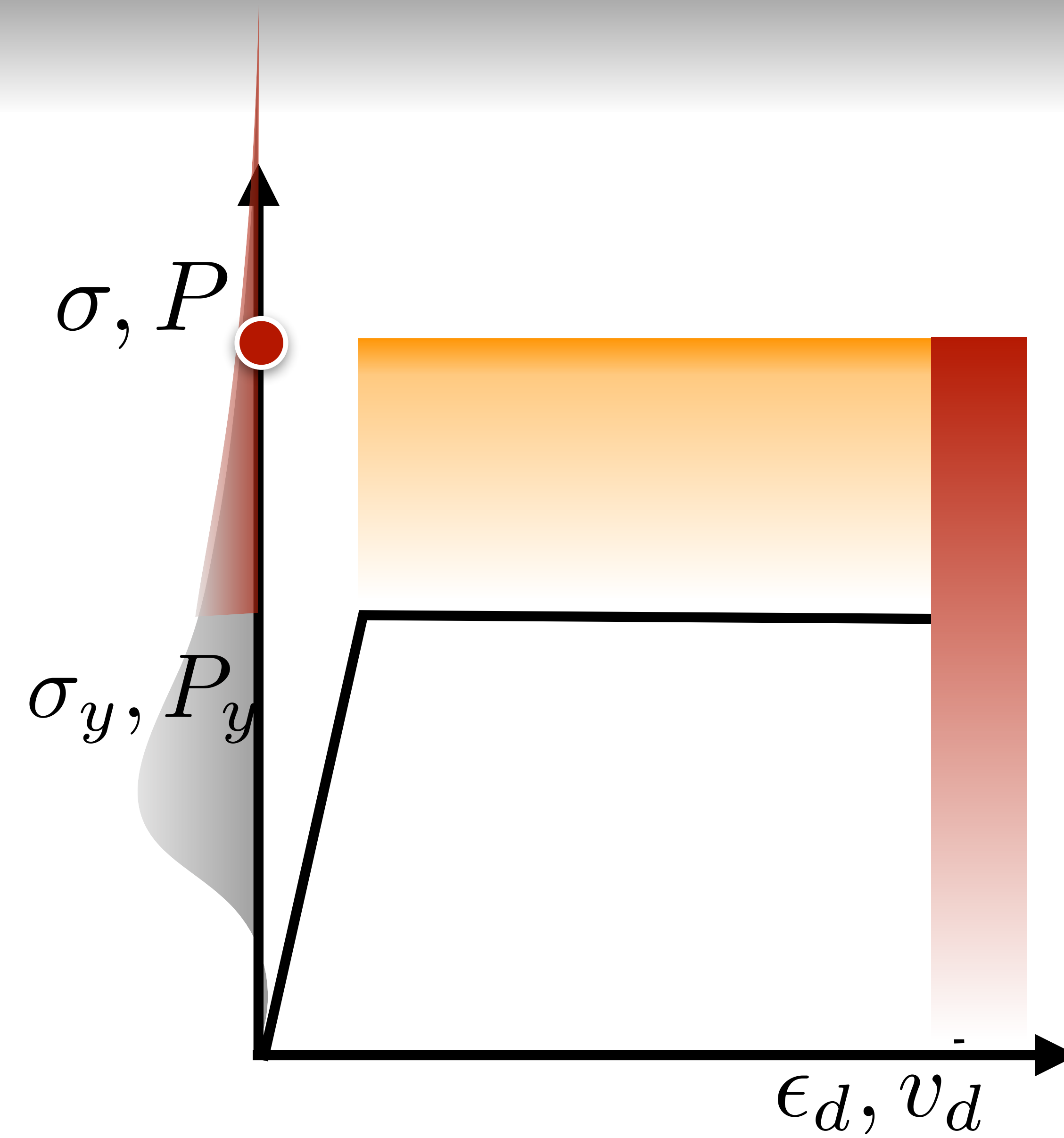


$$\ddot{v}_b = \frac{P_u}{m}$$

$$\ddot{v}_d = \frac{P_u - P_y}{m}$$

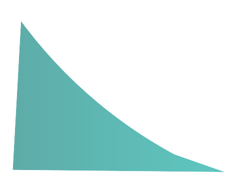
$$\frac{\ddot{v}_d}{\ddot{v}_b} = 1 - \frac{P_y}{P_u} \rightarrow 0$$





$$\ddot{v}_d = \frac{P_u - P_y}{m}$$

$$\ddot{v}_? = \frac{P_u - P_y}{m}$$

 $=$ 

$$P_f = P_f$$

Are the two design acceptable?

Do they guarantee the same safety level?

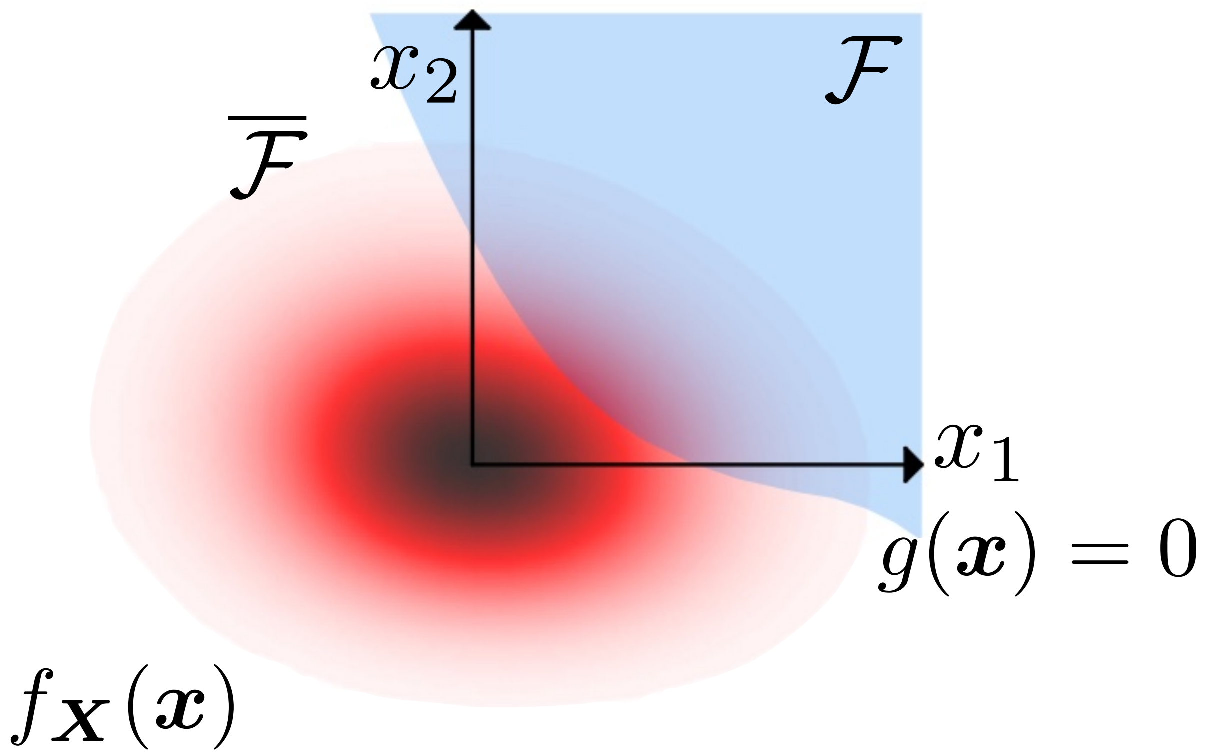
Review Classical Structural Reliability

Classical concepts

$$Y = g(\boldsymbol{X})$$

$$\boldsymbol{x} \in \mathbb{R}^{n_x}, y \in \mathbb{R}$$

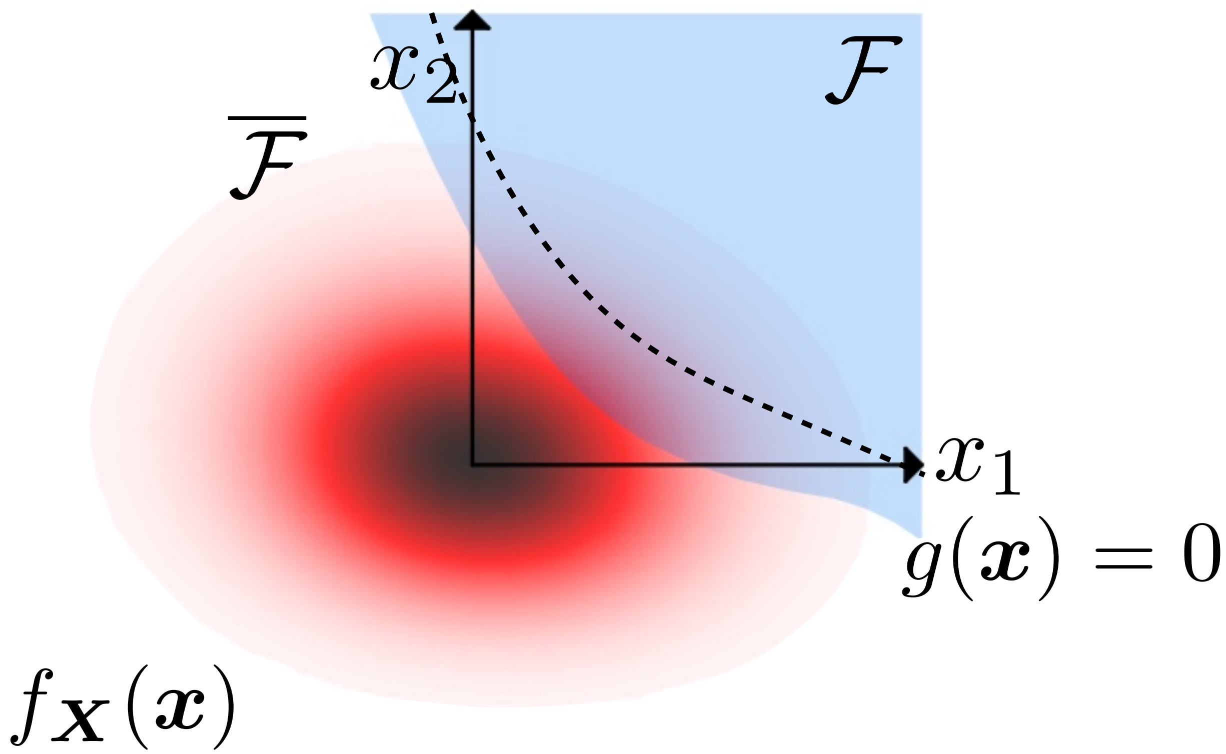
$$\mathcal{F} := \{g(\boldsymbol{x}) \leq 0\}$$

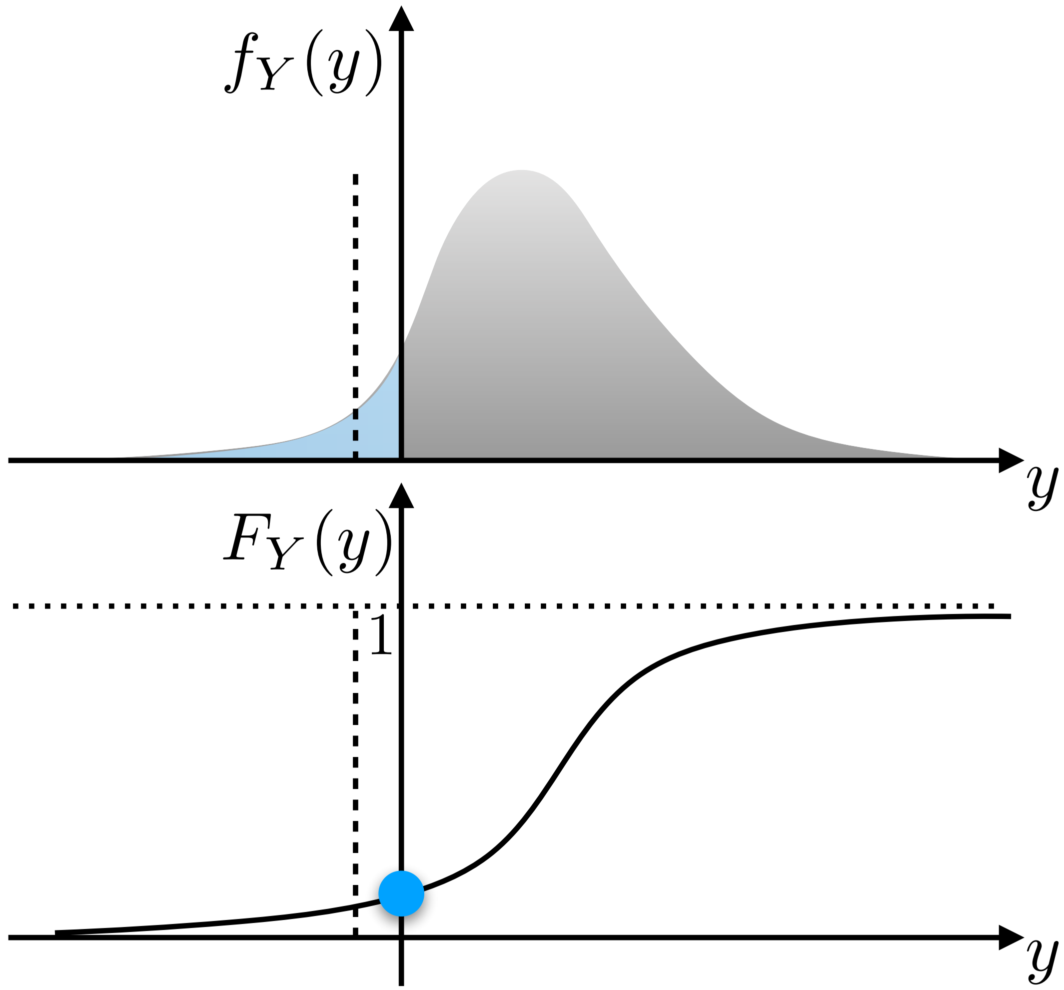


Classical concepts

$$P_{\mathcal{F}} = \int_{\mathbb{R}^n} \mathbb{I}(g(\boldsymbol{x})) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}$$

$$F_Y(y) = \int_{\mathbb{R}^n} \mathbb{I}(g(\boldsymbol{x}) - y) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}$$





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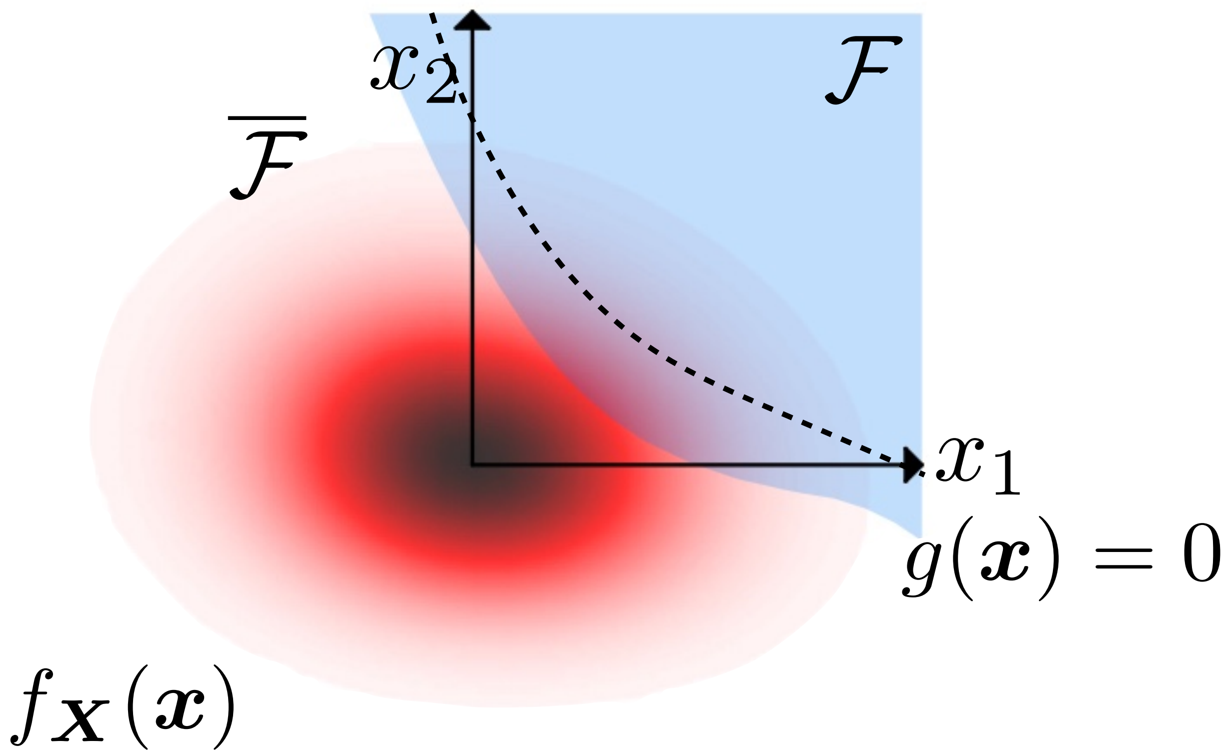

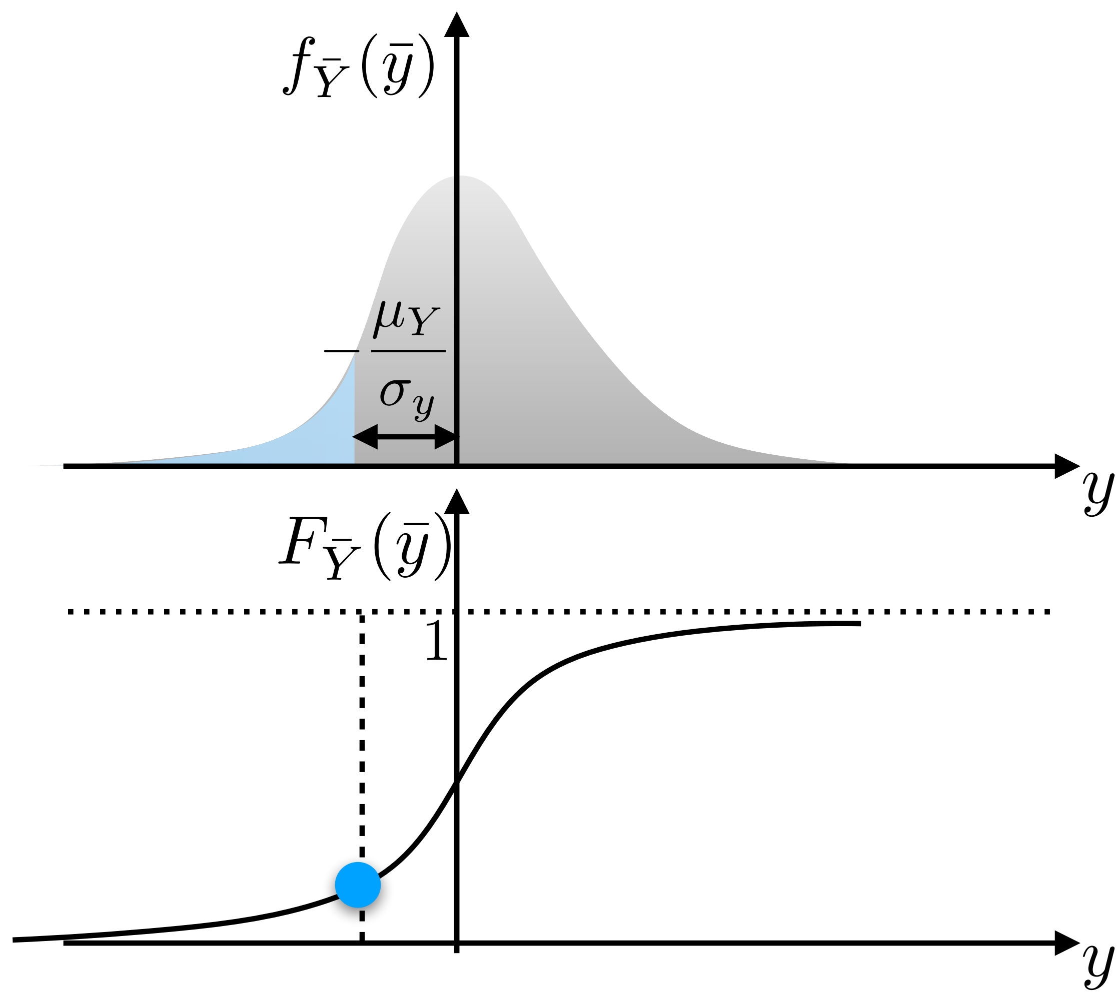

Classical concepts

$$P_{\mathcal{F}} = \int_{\mathbb{R}^n} \mathbb{I}(G(\boldsymbol{z}))\varphi(\boldsymbol{z})d\boldsymbol{z}$$

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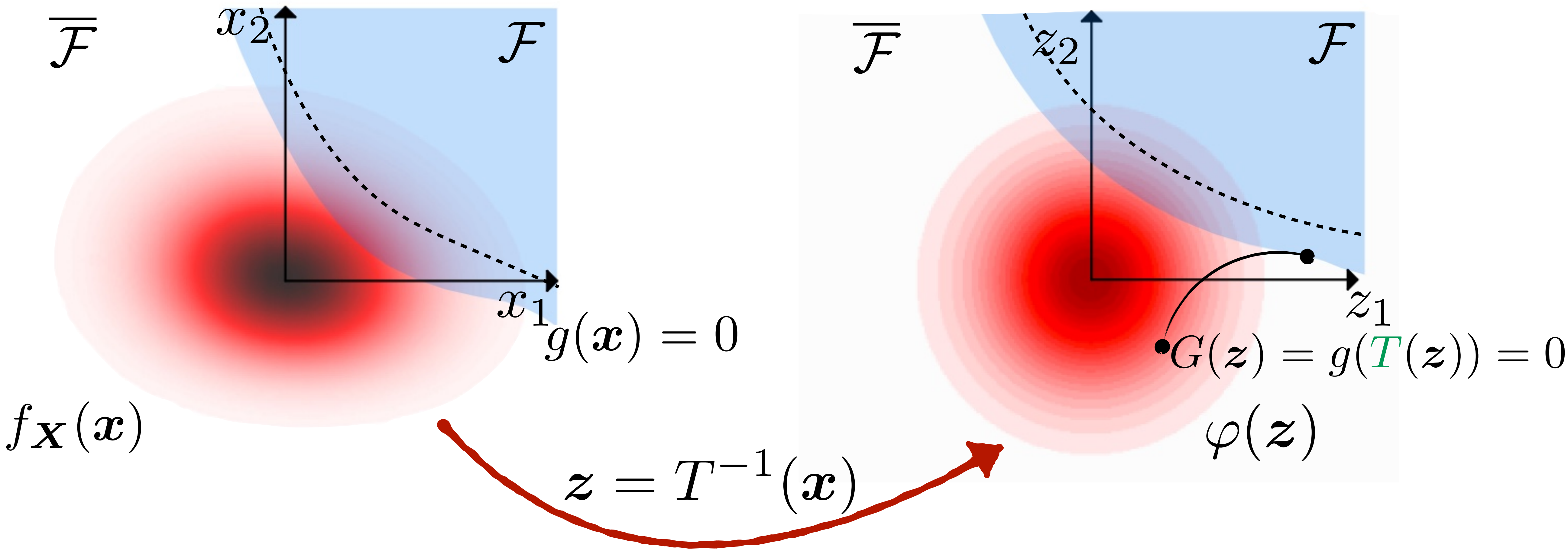

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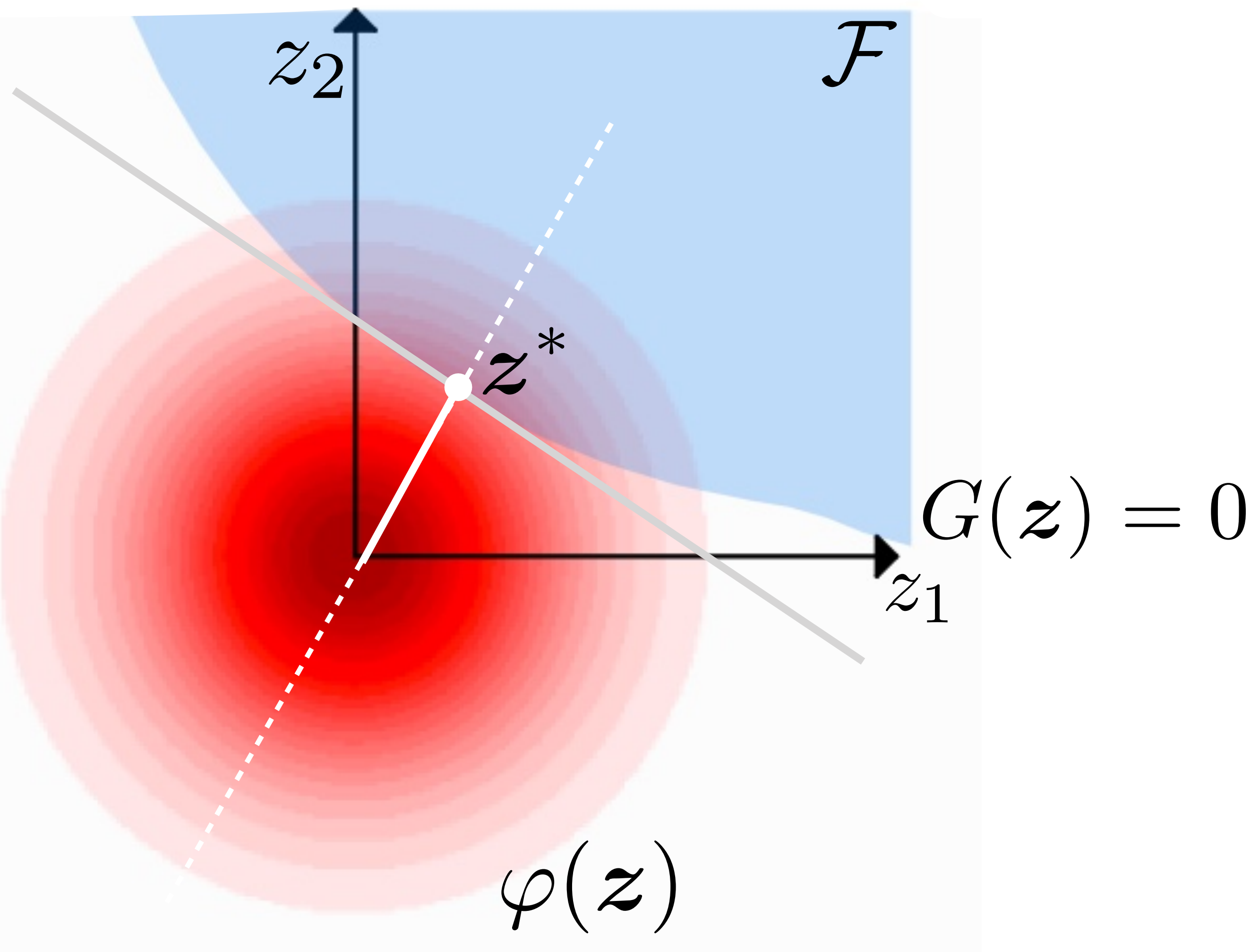
$$\boldsymbol{z}^* = \arg\min_{\boldsymbol{z}} \left[\|\boldsymbol{z}\| \middle| G(\boldsymbol{z}) = 0 \right]$$

$$\beta_{HL} = \|\boldsymbol{z}^*\|$$

$$\beta = -\Phi^{-1}(P_{\mathcal{F}})$$



Beyond Design point, Tail-Dependent Design (/Failure) Point

Tail-Dependent Design (/Failure) Point

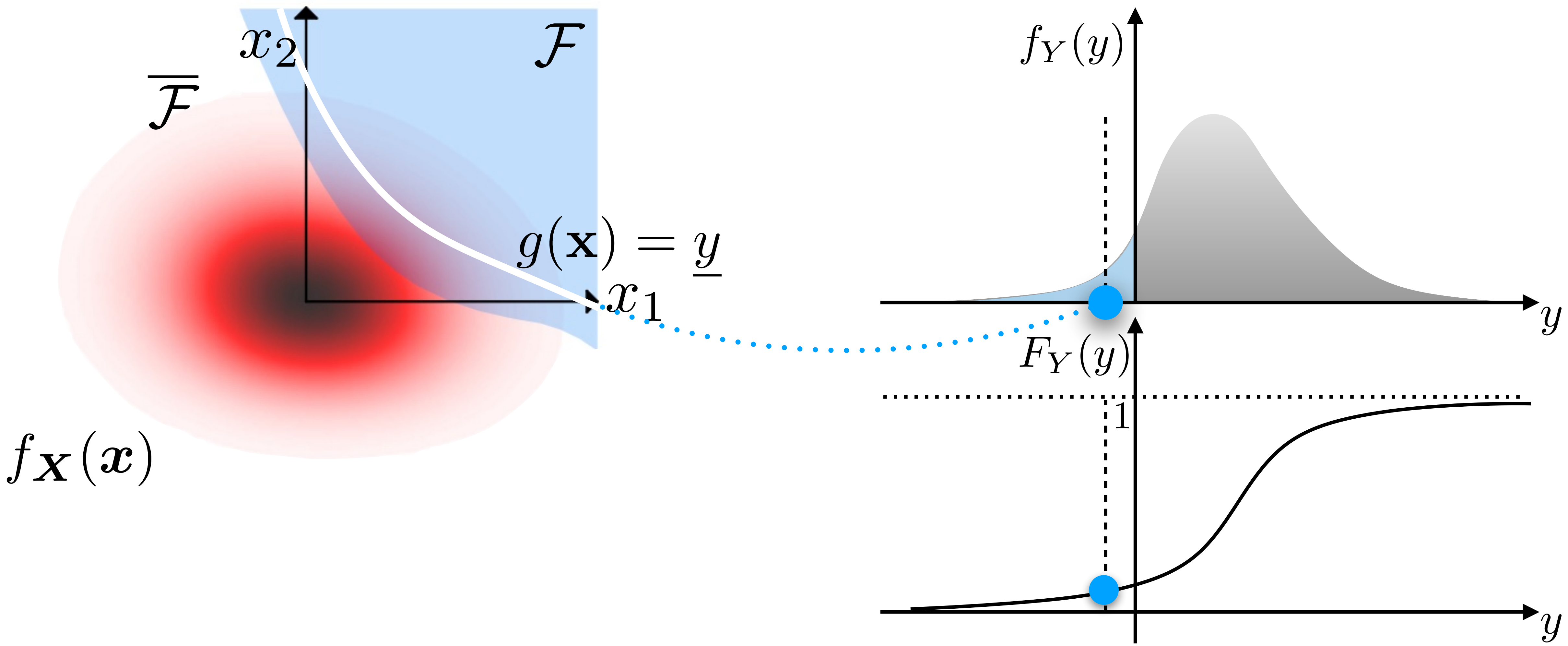
- Separate the design point from the probability of failure

- Which point in the failure domain shall we choose?

  - Think beyond the limit state and look into the failure domain
  - The failure point (among all the failure points) that minimize mean square error

Tail-Dependent Design (/Failure) Point

$$\min_y \mathbb{E}\left[(Y-y)^2|Y\leq 0\right] = \mathbb{E}[Y|Y\leq 0] = \underline{y} \leq 0$$



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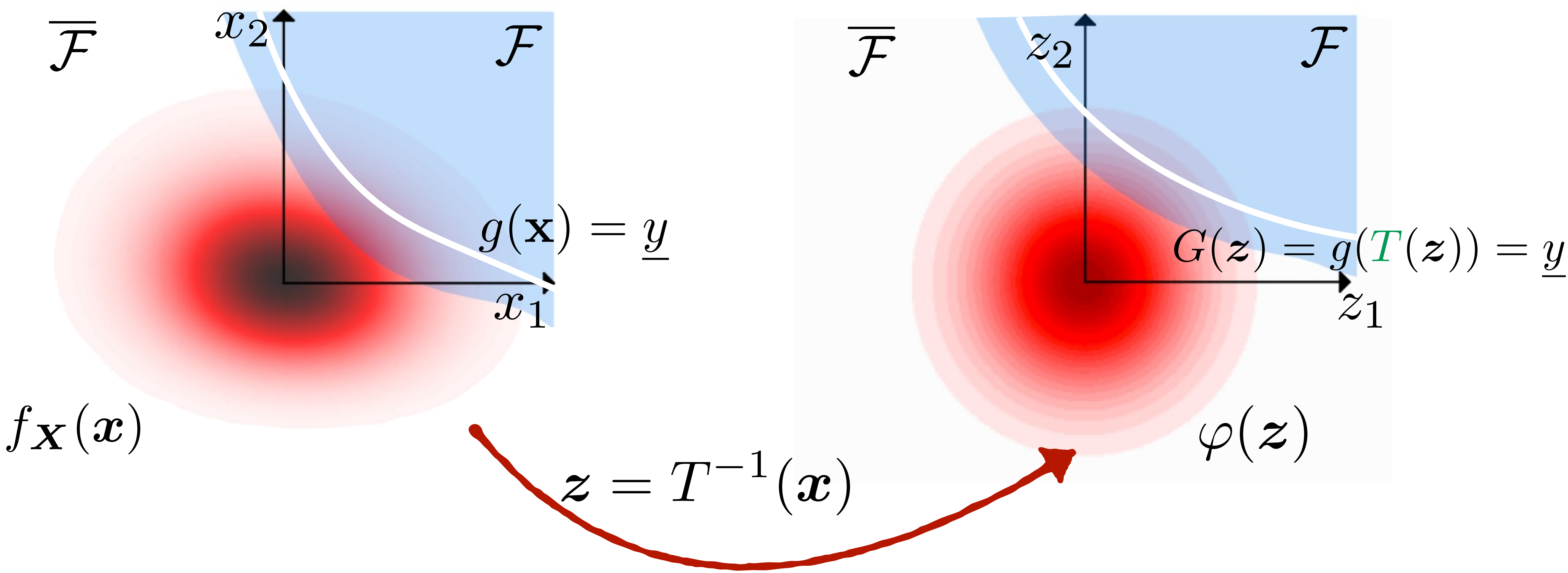
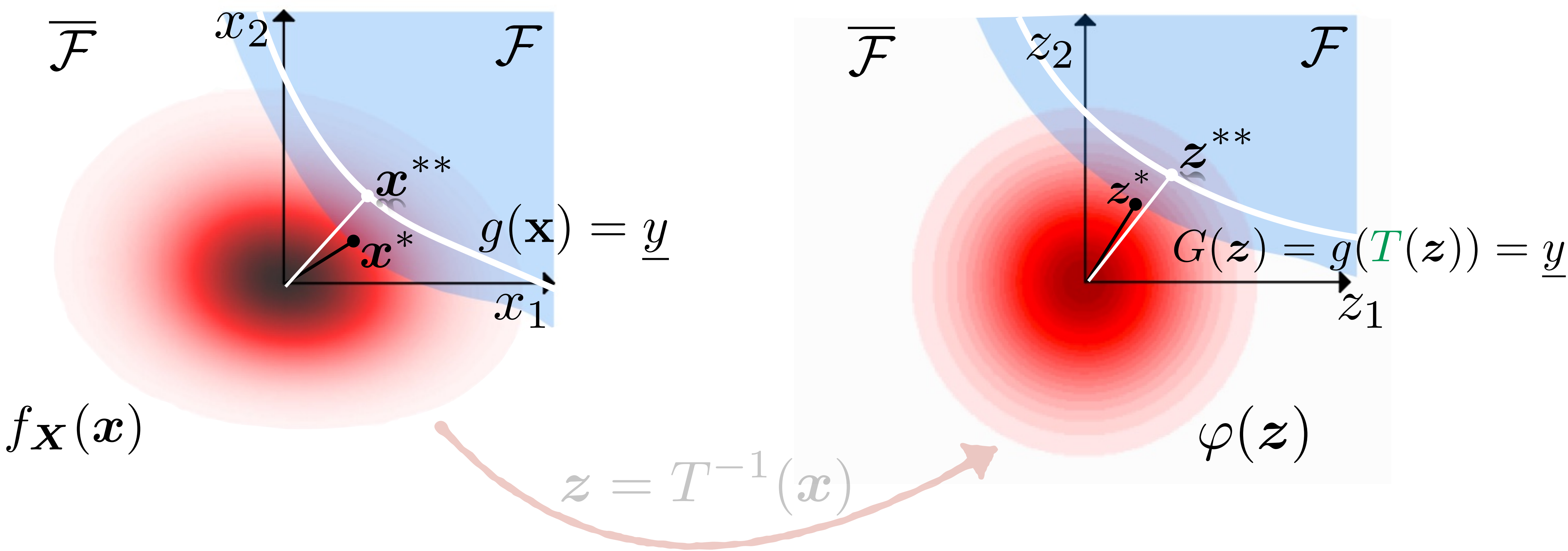


Tail-Dependent Design (/Failure) Point

$$\boldsymbol{x}^{**} = \mathbb{E}\left[\boldsymbol{X} \middle| g(\boldsymbol{x}) = \underline{y}\right] \qquad \boldsymbol{z}^{**} = \mathbb{E}\left[\boldsymbol{Z} \middle| G(\boldsymbol{z}) = \underline{y}\right]$$



Tail-Dependent Design (/Failure) Point

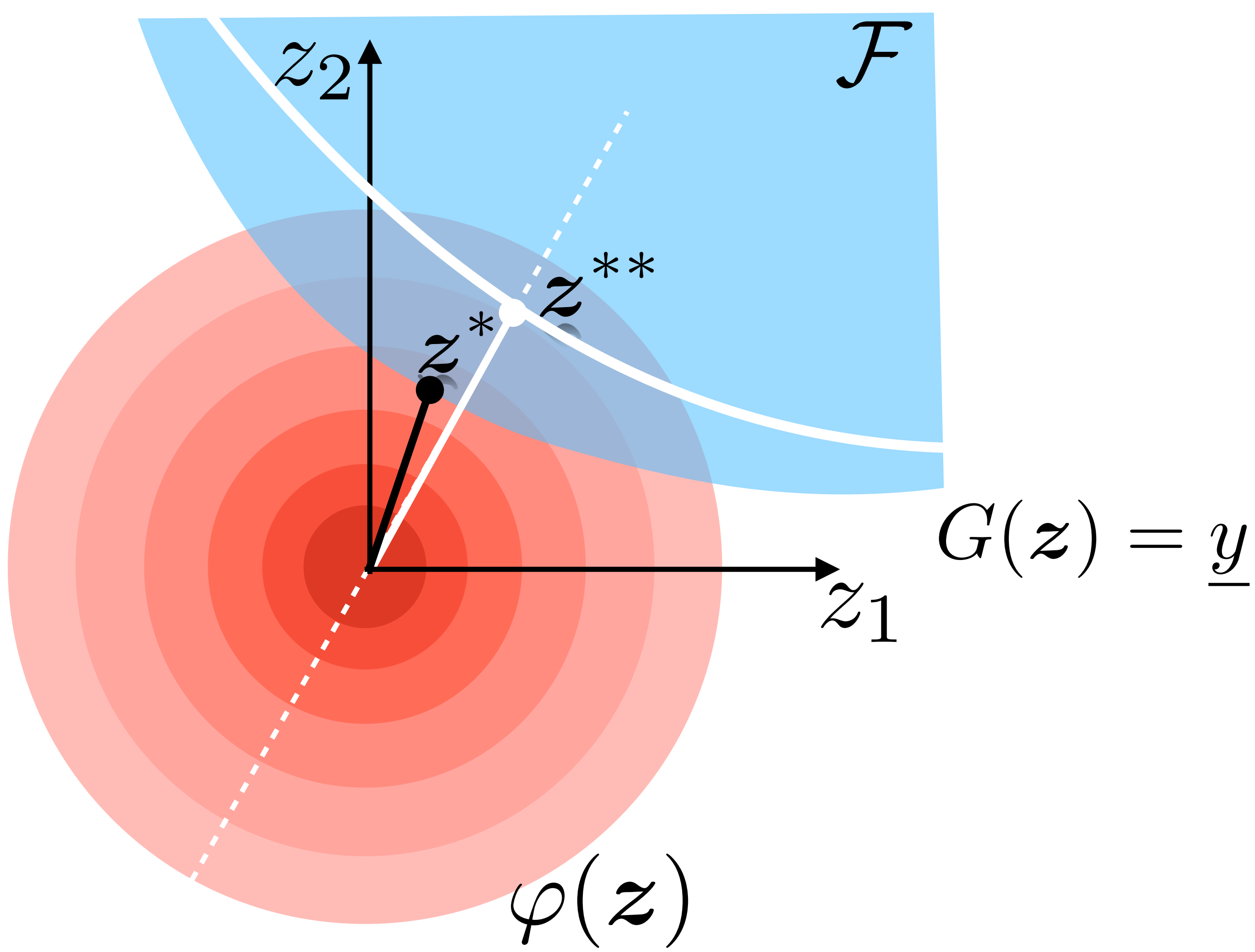
$$\boldsymbol{z}^{**} = \mathbb{E}[\boldsymbol{Z} \mid G(\boldsymbol{z}) = \underline{y}] = \frac{\int_{\mathbb{R}^n} \boldsymbol{z}\, \delta(G(\boldsymbol{z}) - \underline{y})\, \varphi(\boldsymbol{z})\, d\boldsymbol{z}}{\int_{\mathbb{R}^n} \delta(G(\boldsymbol{z}) - \underline{y})\, \varphi(\boldsymbol{z})\, d\boldsymbol{z}}$$

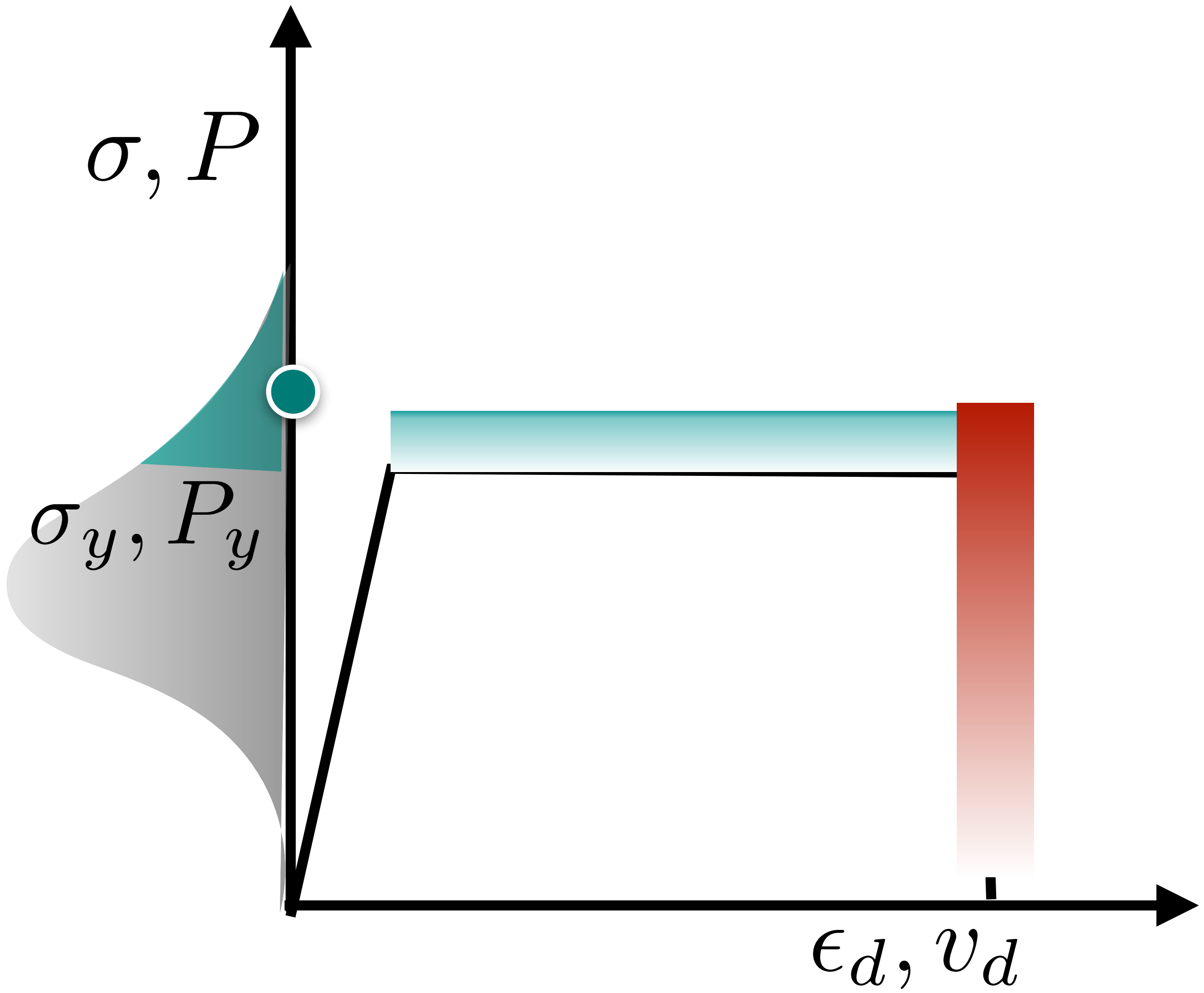
$$\boldsymbol{z}^{**} = \mathbb{E}[\boldsymbol{Z} \mid G(\boldsymbol{z}) = \underline{y}] = \frac{\int_{\Sigma_{\underline{y}}} \boldsymbol{z} \, \frac{\varphi(\boldsymbol{z})}{\|\nabla G(\boldsymbol{z})\|}\, d\Sigma_{\underline{y}}(\boldsymbol{z})}{\int_{\Sigma_{\underline{y}}} \frac{\varphi(\boldsymbol{z})}{\|\nabla G(\boldsymbol{z})\|}\, d\Sigma_{\underline{y}}(\boldsymbol{z})}$$

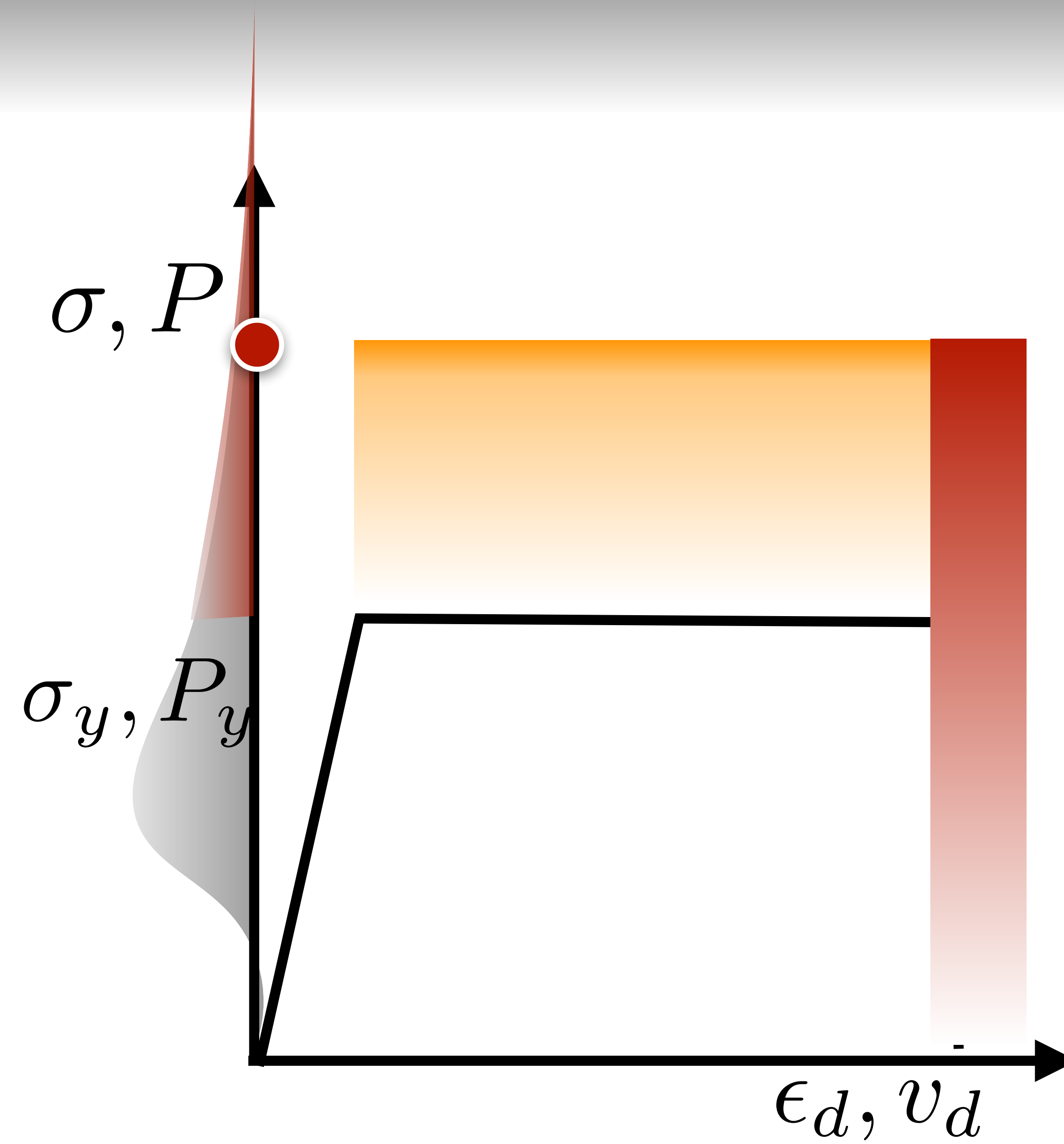
Tail-Dependent Design (/Failure) Point

$$\boldsymbol{z}^{**} = \mathbb{E}[\boldsymbol{Z} \mid G(\boldsymbol{z}) = \underline{y}] \approx \arg\min_{\boldsymbol{z}} \left[\|\boldsymbol{z}\| \middle| G(\boldsymbol{z}) = \underline{y} \right]$$

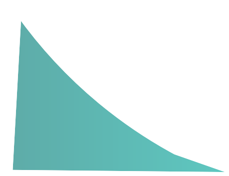
$$\boldsymbol{x}^{**} = T(\boldsymbol{z}^{**})$$



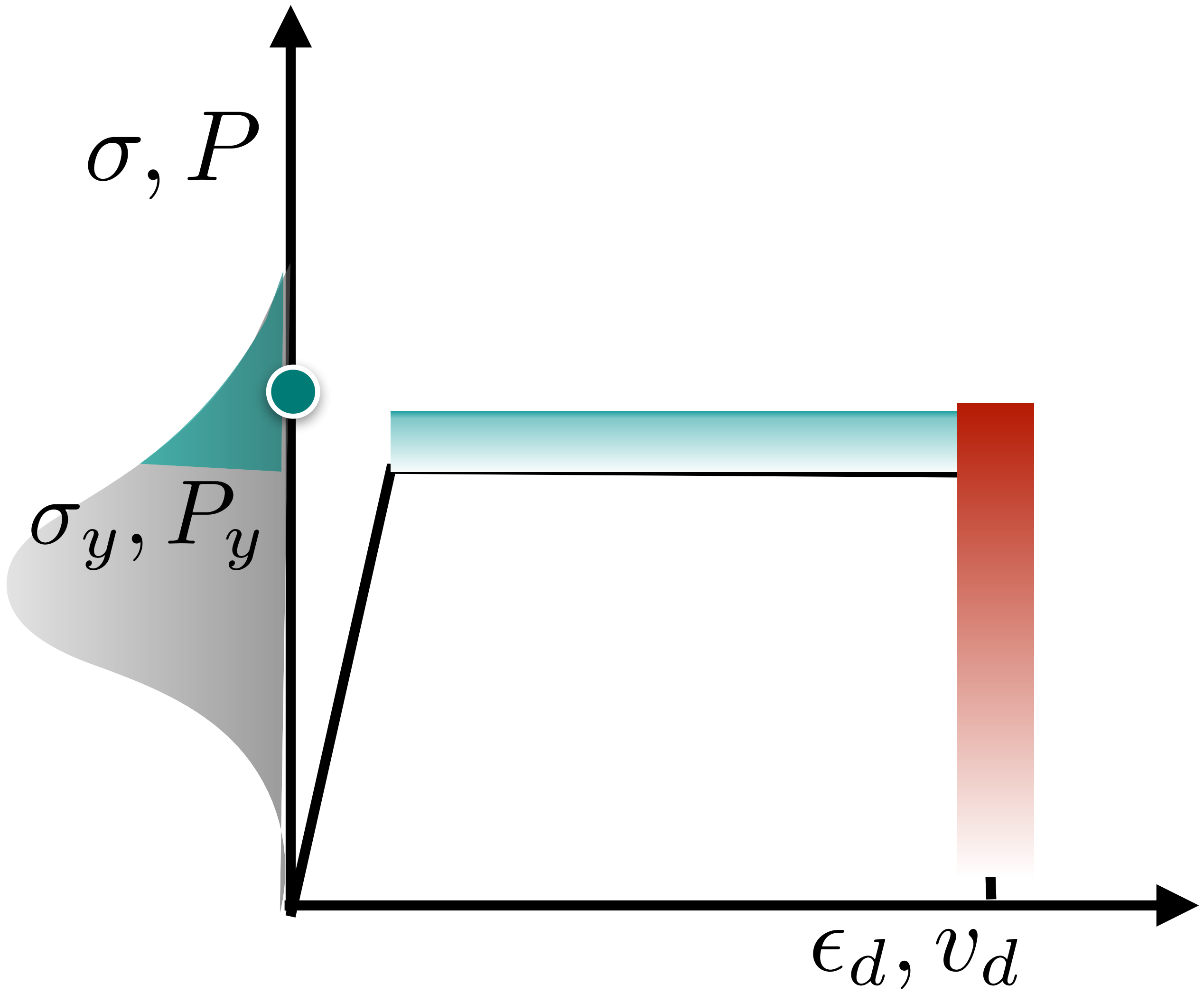


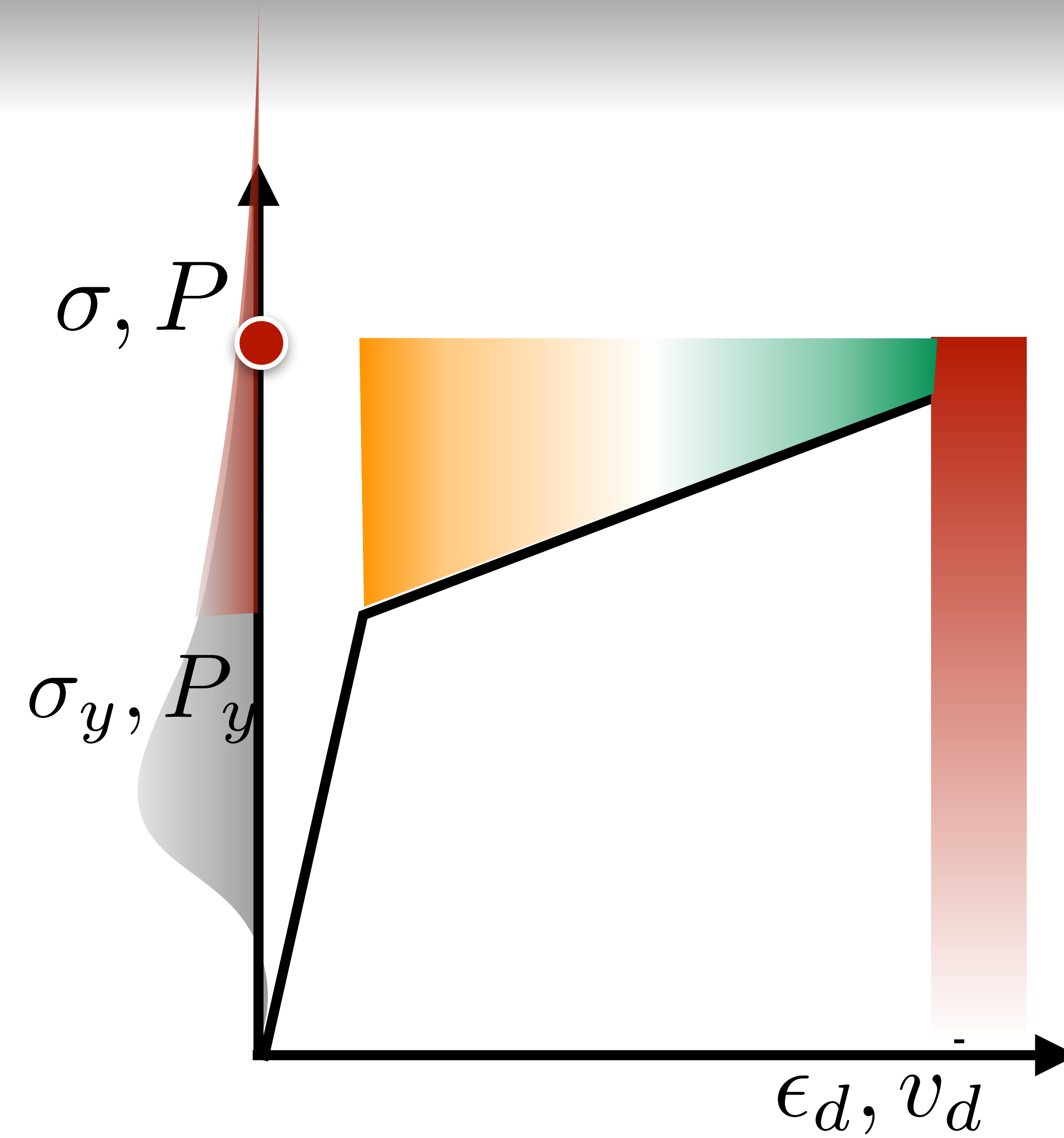


$$\ddot{v}_d = \frac{P_u - P_y}{m} \qquad \ddot{v}_? = \frac{P_u - P_y}{m}$$

 = 

$P_f = P_f$





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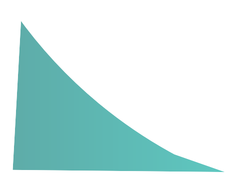

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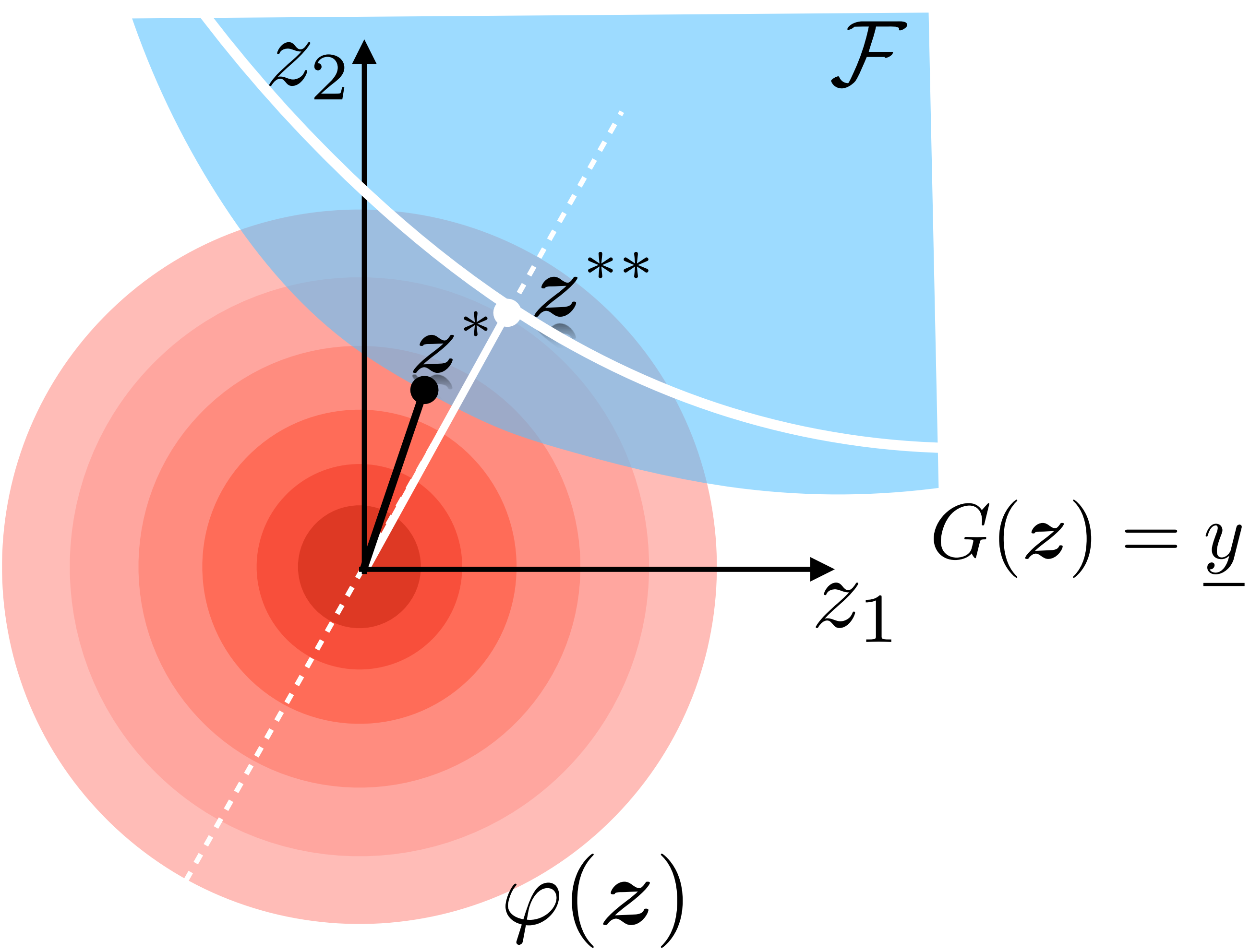
$$P_f = P_f$$

Tail-Dependent Design (/Failure) Point

$$\boldsymbol{z}^{**} = \mathbb{E}[\boldsymbol{Z} \mid G(\boldsymbol{z}) = \underline{y}] \approx \arg\min_{\boldsymbol{z}} \left[\|\boldsymbol{z}\| \middle| G(\boldsymbol{z}) = \underline{y} \right]$$

$$\boldsymbol{x}^{**} = T(\boldsymbol{z}^{**})$$



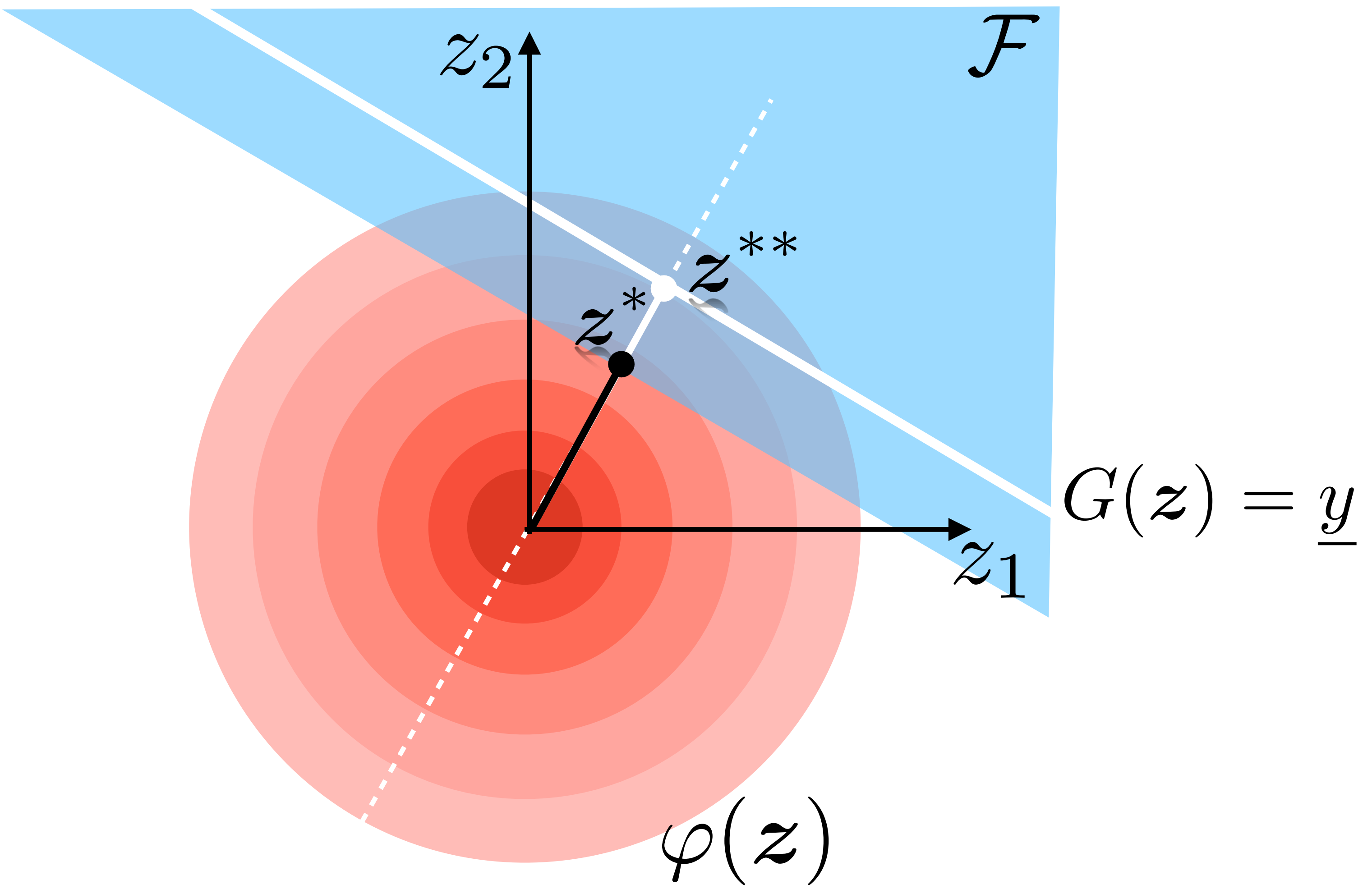
Linear Gaussian case

$$\boldsymbol{z}^{**} = \mathbb{E}[\boldsymbol{Z} \mid G(\boldsymbol{z}) = \underline{y}] = \arg\min_{\boldsymbol{z}} \left[\|\boldsymbol{z}\| \middle| G(\boldsymbol{z}) = \underline{y} \right]$$

$$G(\boldsymbol{Z}) = b_0 + \boldsymbol{b}^T \boldsymbol{Z}$$

$$\|\boldsymbol{z}^*\| = \frac{b_0}{\|\boldsymbol{b}\|} = \beta$$

$$\boldsymbol{z}^* = \frac{b_0}{\|\boldsymbol{b}\|} \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|} = \beta \boldsymbol{\alpha}$$



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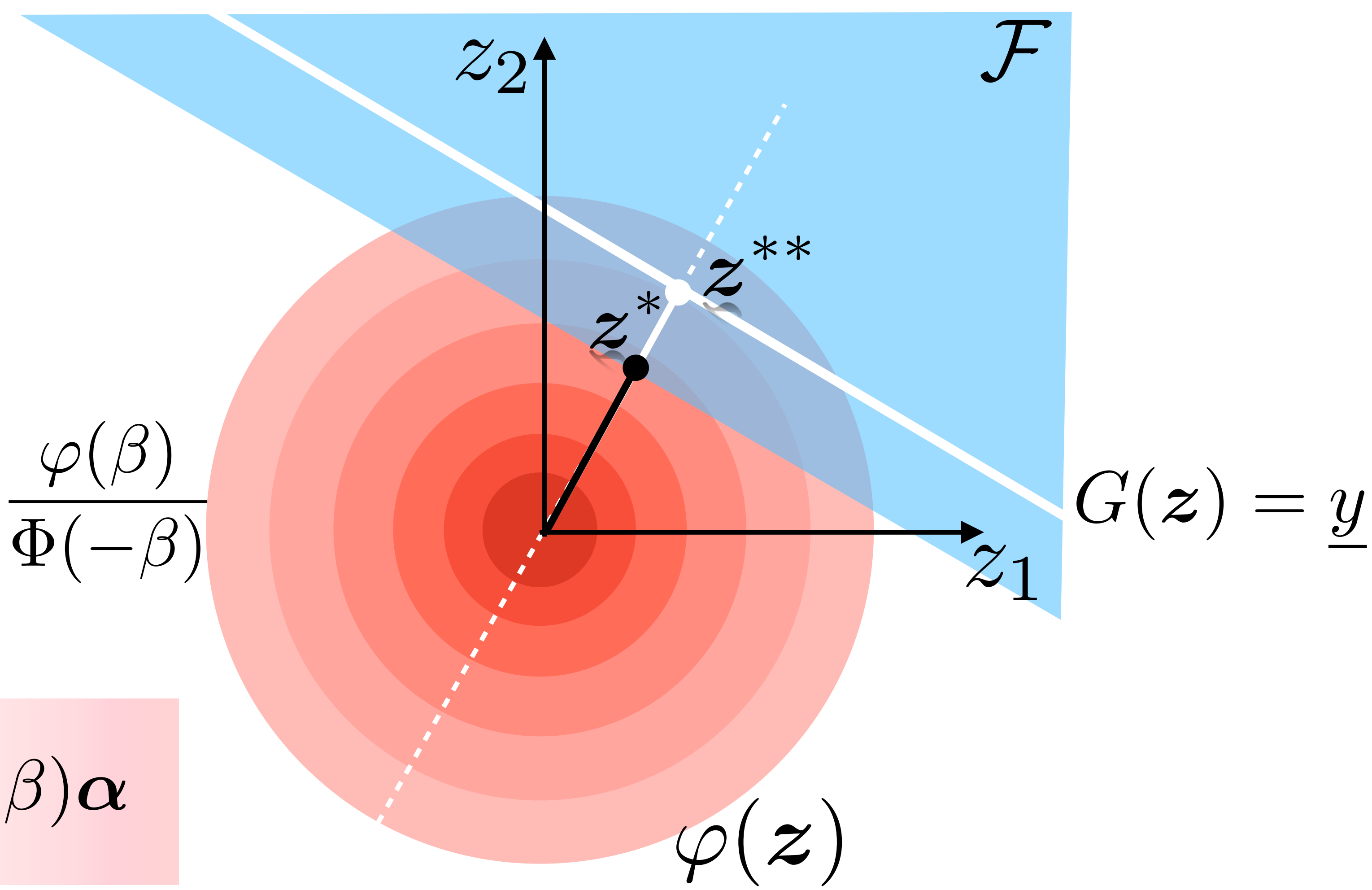
$$G(\boldsymbol{Z}) = b_0 + \boldsymbol{b}^T \boldsymbol{Z}$$

$$\|\boldsymbol{z}^{**}\| = \mathbb{E}[Z \mid Z > \beta] = \frac{\varphi(\beta)}{\Phi(-\beta)}$$

$$\boldsymbol{z}^{**} = \frac{\varphi(\beta)}{\Phi(-\beta)} \boldsymbol{\alpha} = \lambda(\beta) \boldsymbol{\alpha}$$



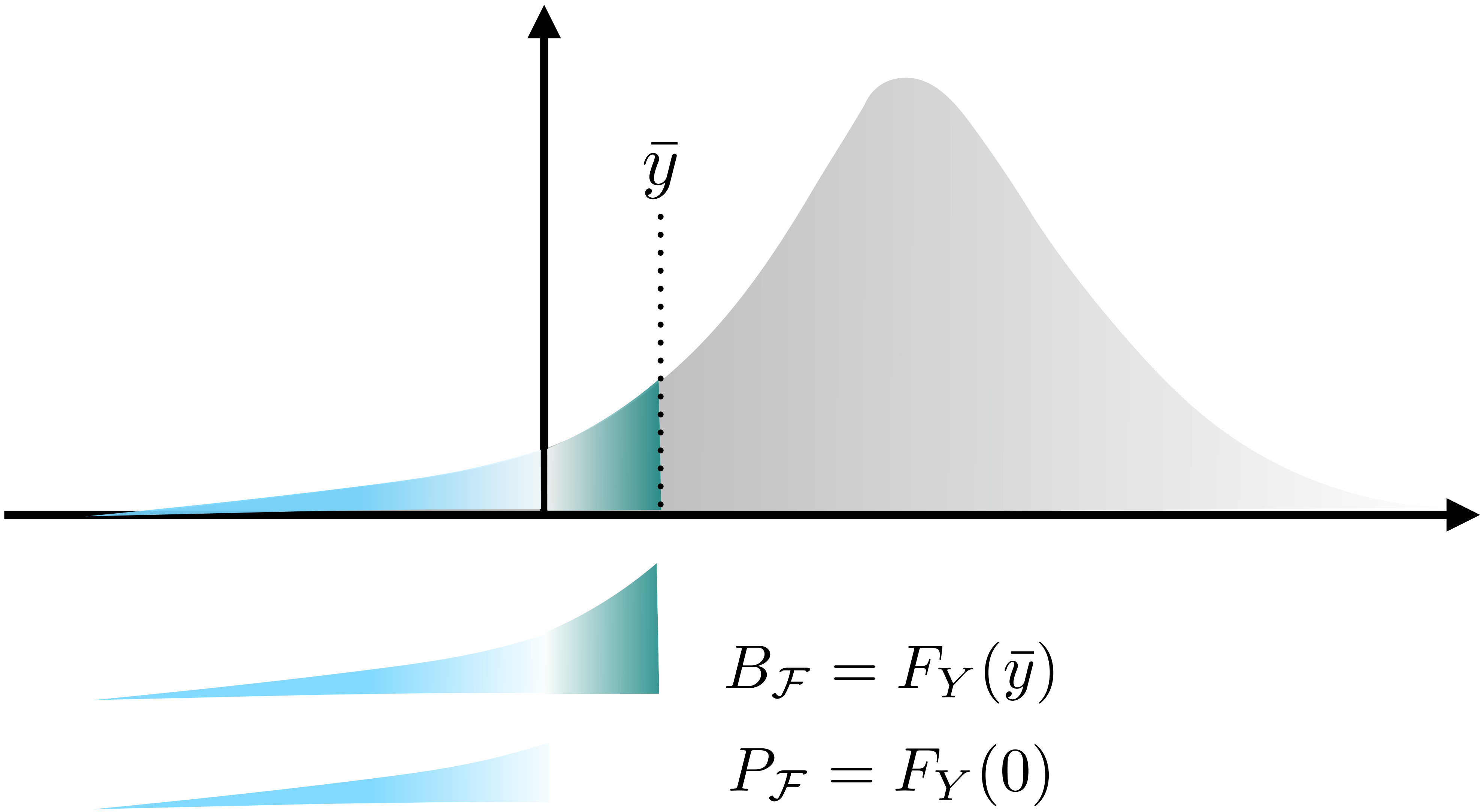
Beyond Reliability index, Tail-Dependent Reliability Index

Tail-dependent Reliability Assessment

- Separate the tail-dependent reliability index from prob. of failure
- Structures with "more dangerous" failure point m.h. lower rel.
- Use the same principle of conditional expectation
  - consider the conditional expectation as the failure event
  - compute the threshold associated with this cond. expectation
  - define a tail dependent reliability index

Tail-dependent Reliability Assessment

$$\mathbb{E}[Y|Y \leq \bar{y}] = 0 \Leftrightarrow \min_y \mathbb{E}\left[(Y-y)^2|Y \leq \bar{y}\right] = 0$$



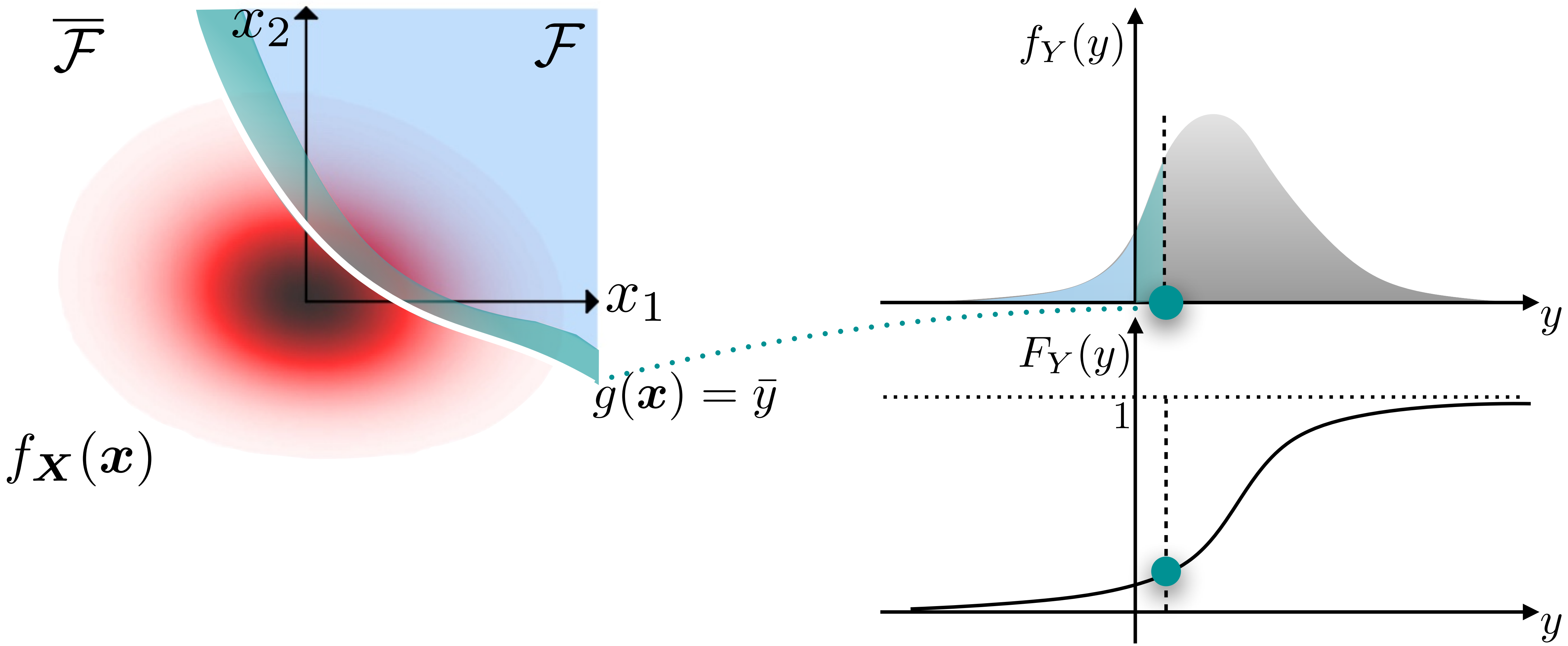
$$B_{\mathcal{F}} = F_Y(\bar{y})$$

$$P_{\mathcal{F}} = F_Y(0)$$

Rockafellar & Royset, On buffered failure probability in design and optimization of structures, RESS, 2010

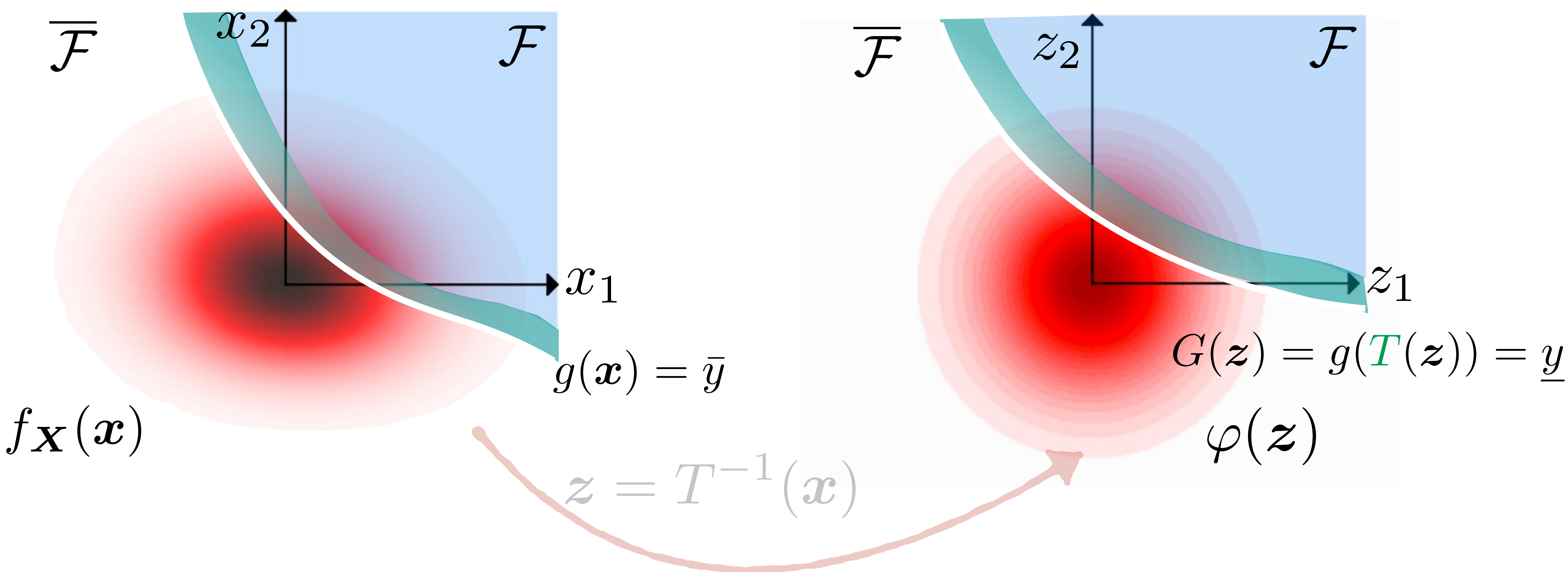
Tail-dependent Reliability Assessment

$$\mathbb{E}[Y|Y \leq \bar{y}] = 0, \bar{y} \geq 0$$



Tail-dependent Reliability Assessment

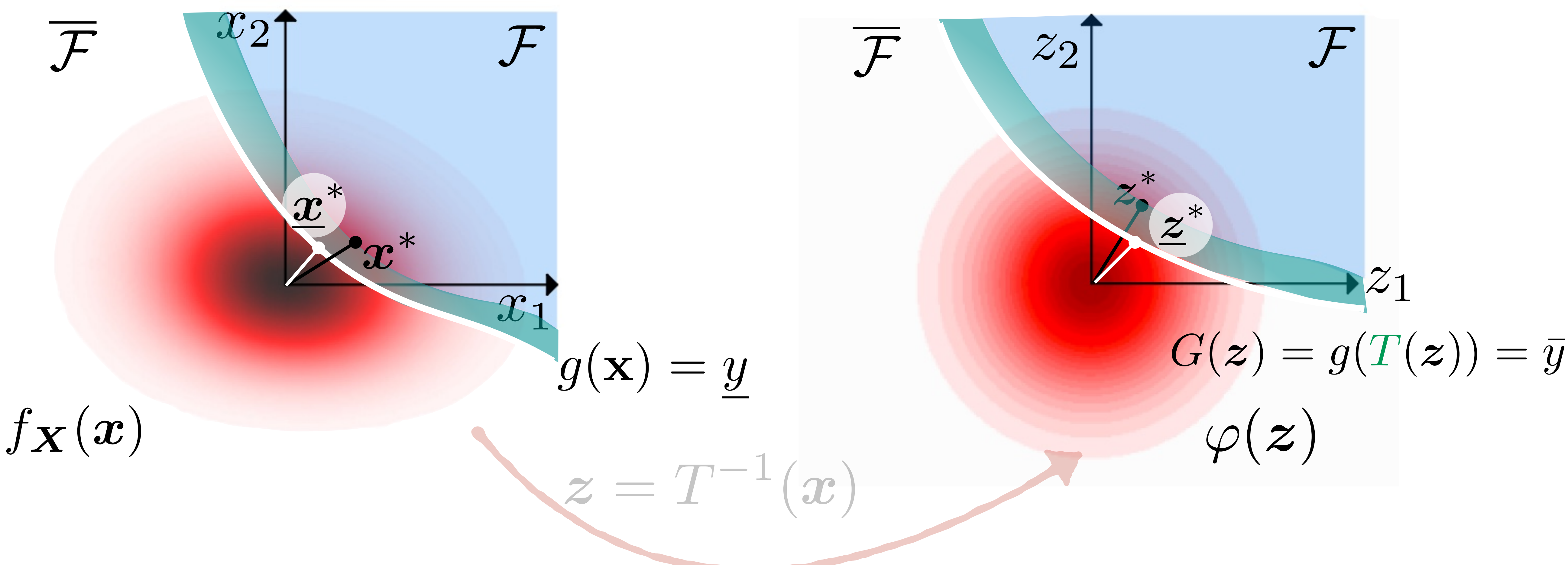
$$\mathbb{E}[Y|Y \leq \bar{y}] = 0, \bar{y} \geq 0$$



Tail-dependent Reliability Assessment

$$\underline{\boldsymbol{x}}^* = T(\underline{\boldsymbol{z}}^*)$$

$$\underline{\boldsymbol{z}}^* = \arg\min_{\boldsymbol{z}} \left[\|\boldsymbol{z}\| \middle| G(\boldsymbol{z}) = \bar{y} \right]$$



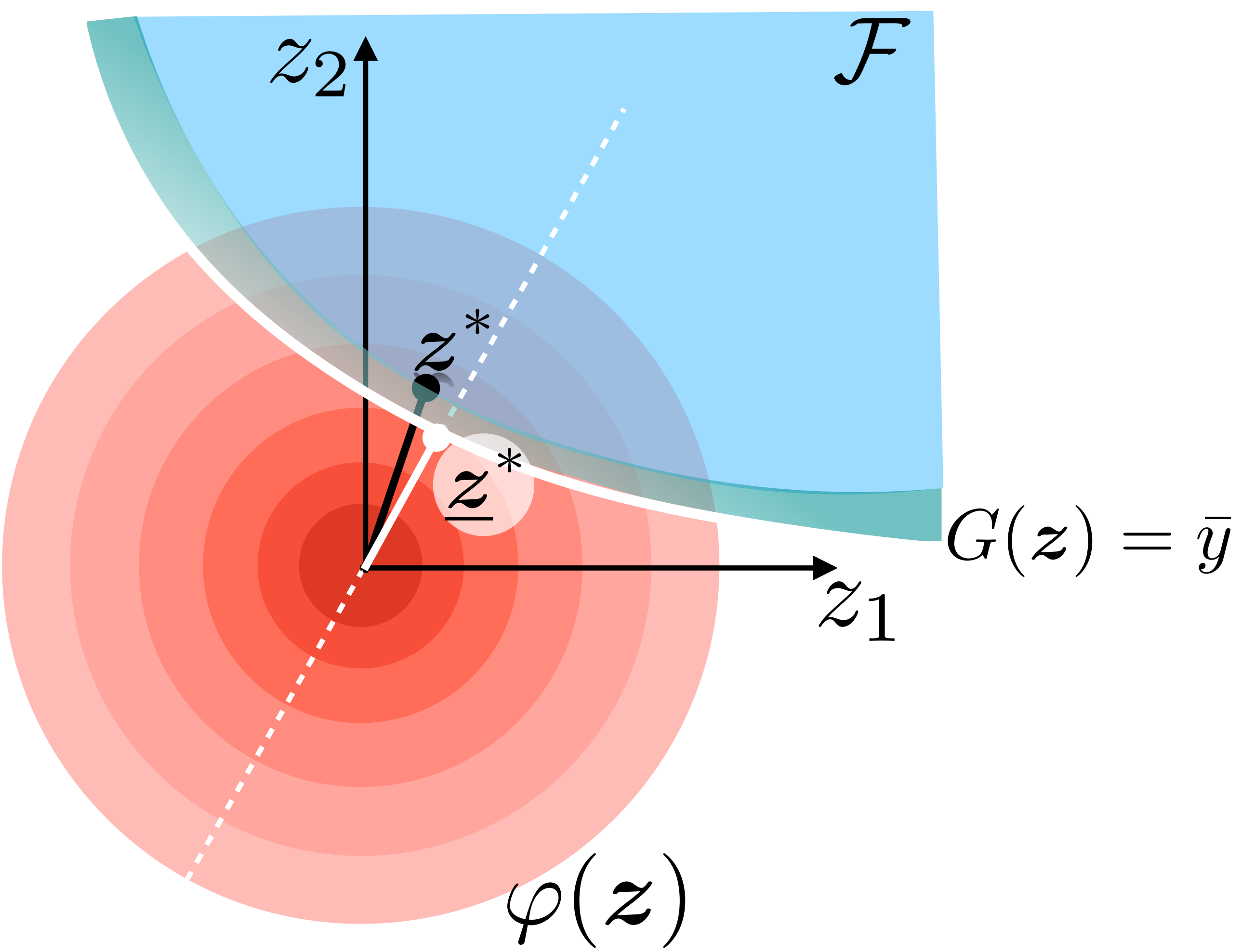
Tail-dependent reliability index

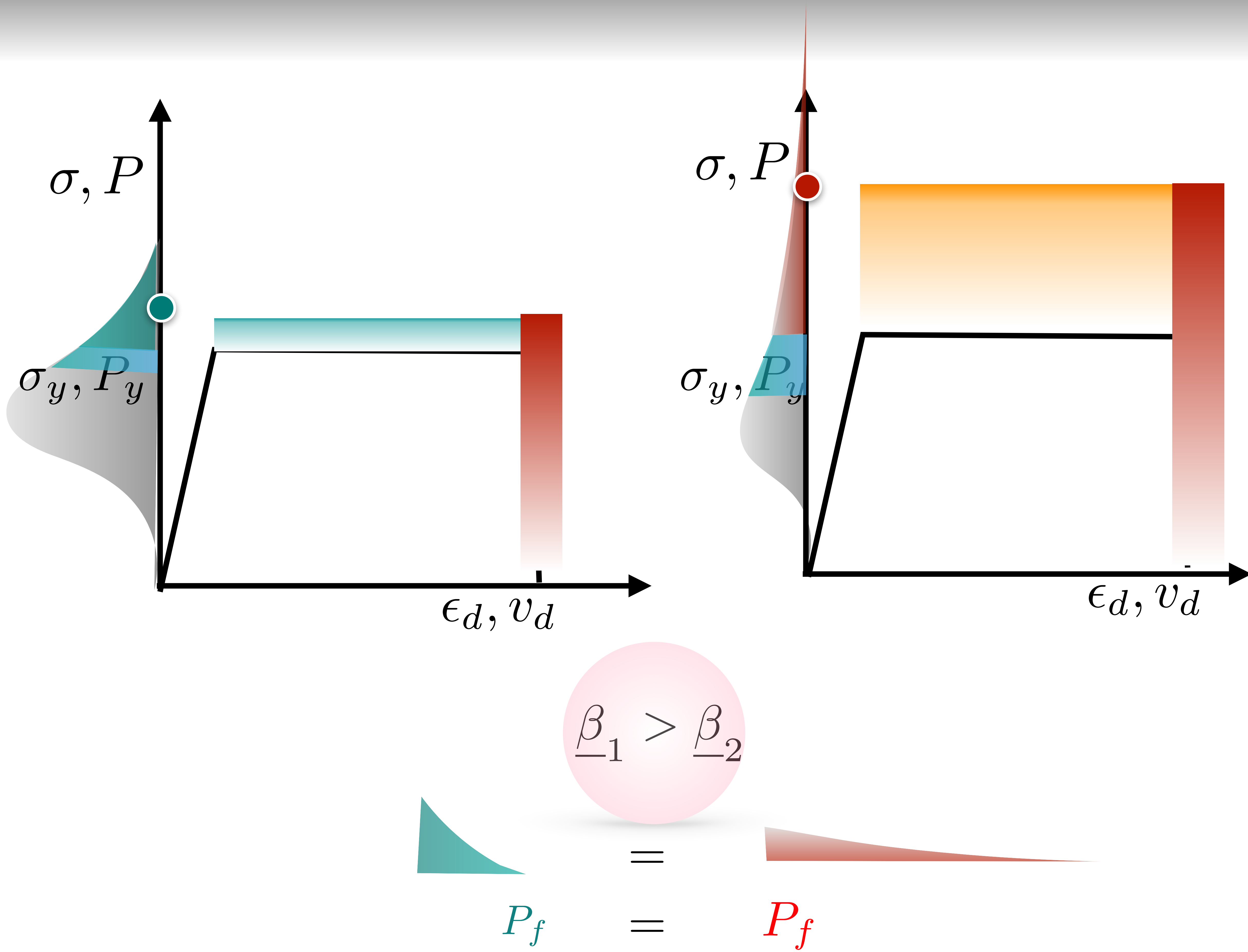
$$B_F = \int_{\mathbb{R}^n} \mathbb{I}(G(\boldsymbol{z}) - \bar{y})\varphi(\boldsymbol{z})d\boldsymbol{z}$$

$$\underline{\beta} = -\Phi^{-1}(B_F) \leq \beta$$

$$\underline{\beta}_{HL} = \|\underline{\boldsymbol{z}}^*\| \leq \underline{\beta}_{HL}$$

$$\underline{\beta} \approx \underline{\beta}_{HL}$$




σ, P
σy, Py
ϵd, vd
σ, P
σy, Py
ϵd, vd
β1 > β2
=
Pf = Pf

Tail-dependent reliability index

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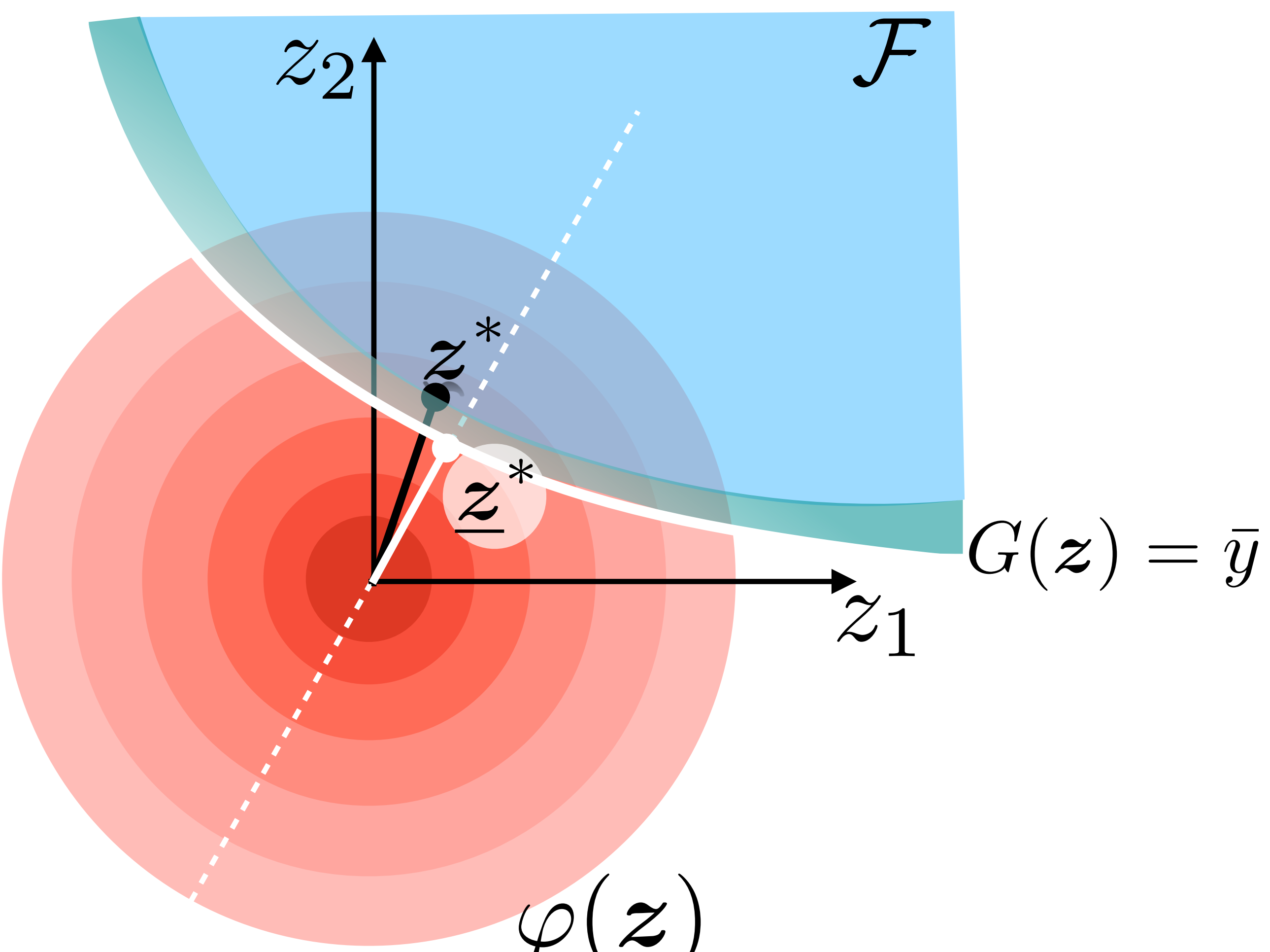


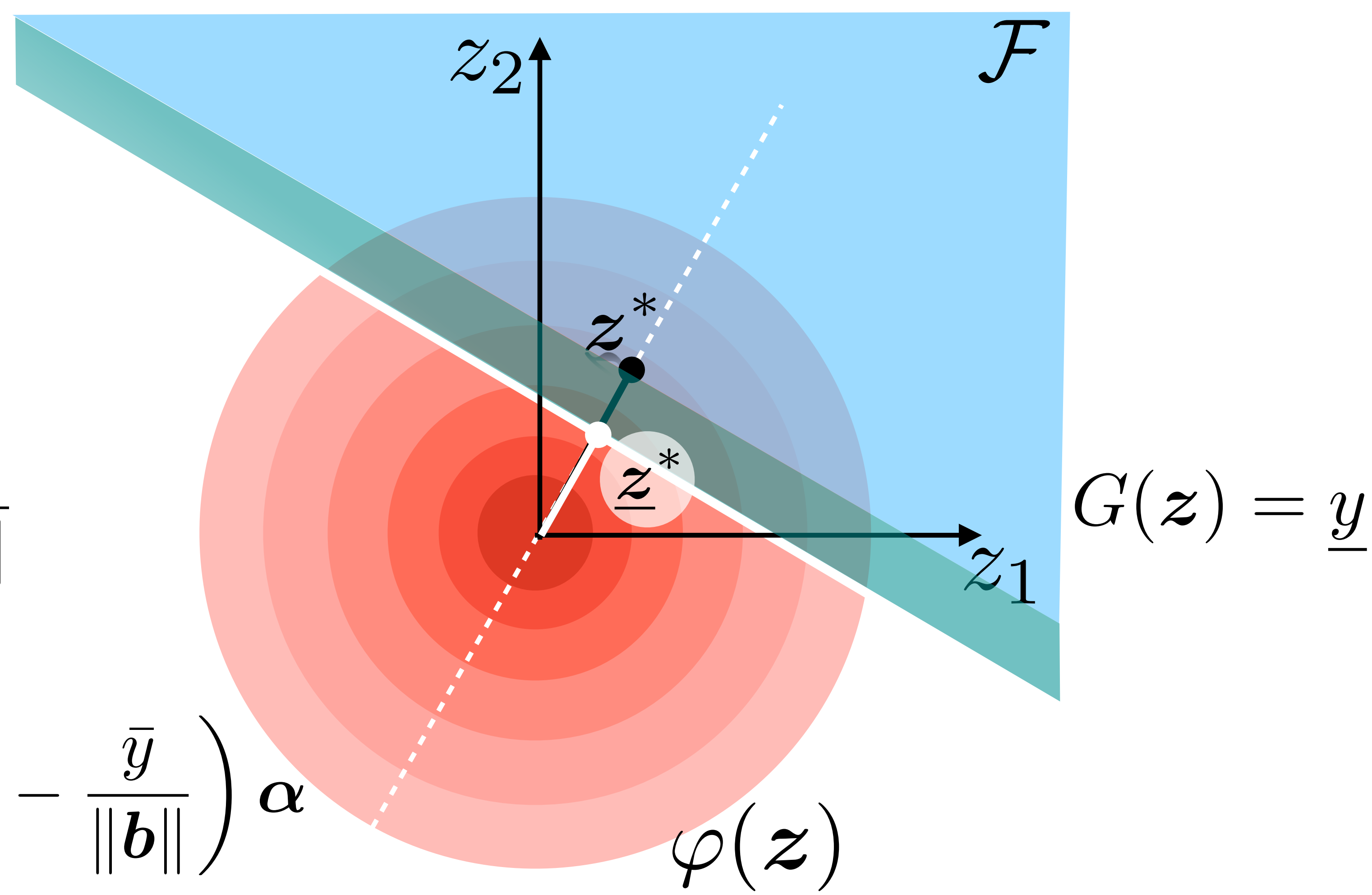
Linear Gaussian case

$$B_F = \int_{\mathbb{R}^n} \mathbb{I}(G(\boldsymbol{z}) - \bar{y})\varphi(\boldsymbol{z})d\boldsymbol{z} = \Phi(-\underline{\beta}) = \Phi(-\underline{\beta}_{HL})$$

$$G(\boldsymbol{Z}) = b_0 + \boldsymbol{b}^T \boldsymbol{Z}$$

$$||\underline{\boldsymbol{z}}^*|| = \frac{b_0 - \bar{y}}{||\boldsymbol{b}||} = \beta - \frac{\bar{y}}{||\boldsymbol{b}||}$$

$$\underline{\boldsymbol{z}}^* = \frac{b_0 - \bar{y}}{||\boldsymbol{b}||}\frac{\boldsymbol{b}}{||\boldsymbol{b}||} = \left(\beta - \frac{\bar{y}}{||\boldsymbol{b}||}\right)\boldsymbol{\alpha}$$



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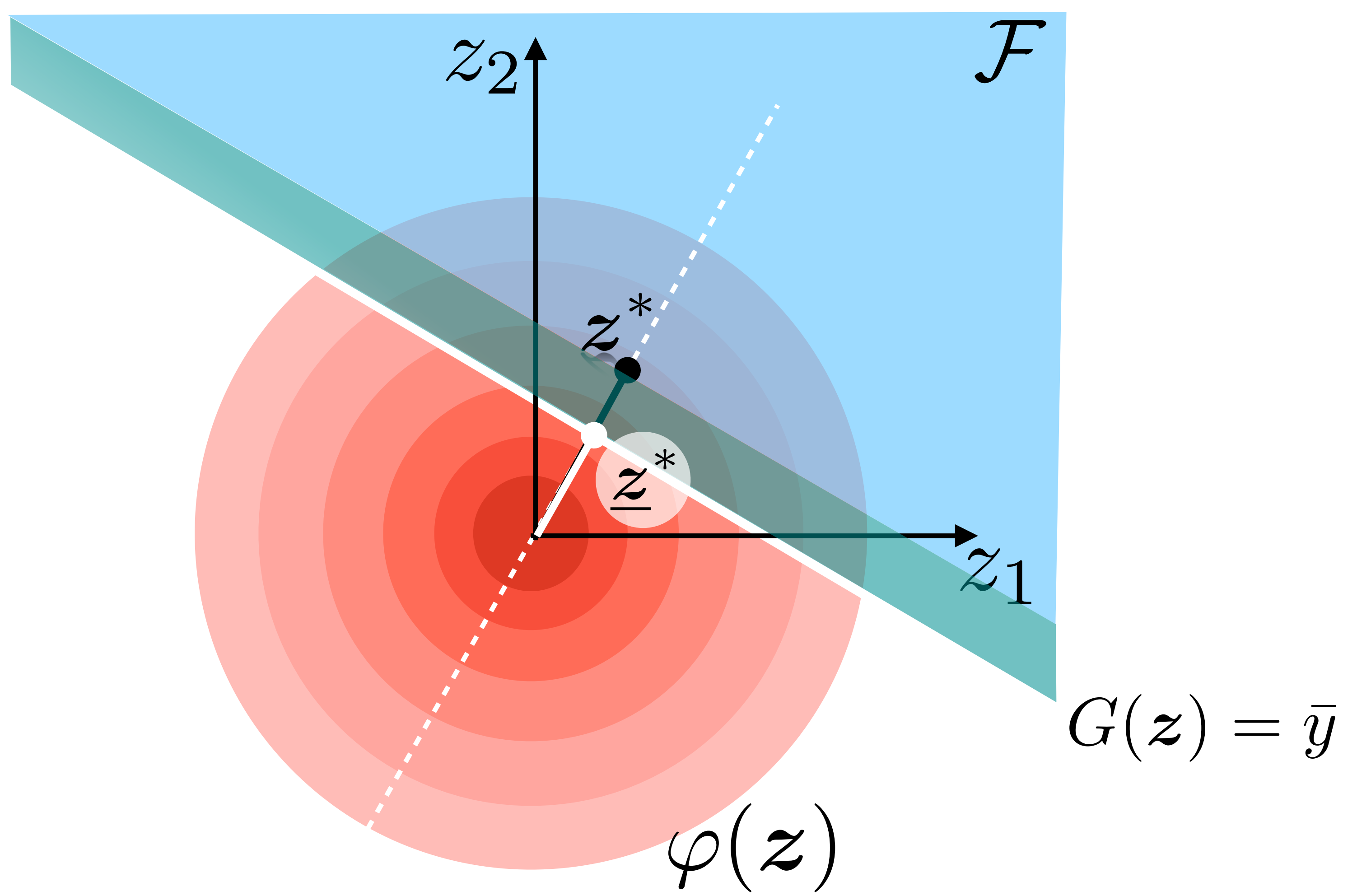
$$G(\boldsymbol{Z}) = b_0 + \boldsymbol{b}^T \boldsymbol{Z}$$

$$\mathbb{E}[Z \mid Z > \underline{\beta}] = \beta$$

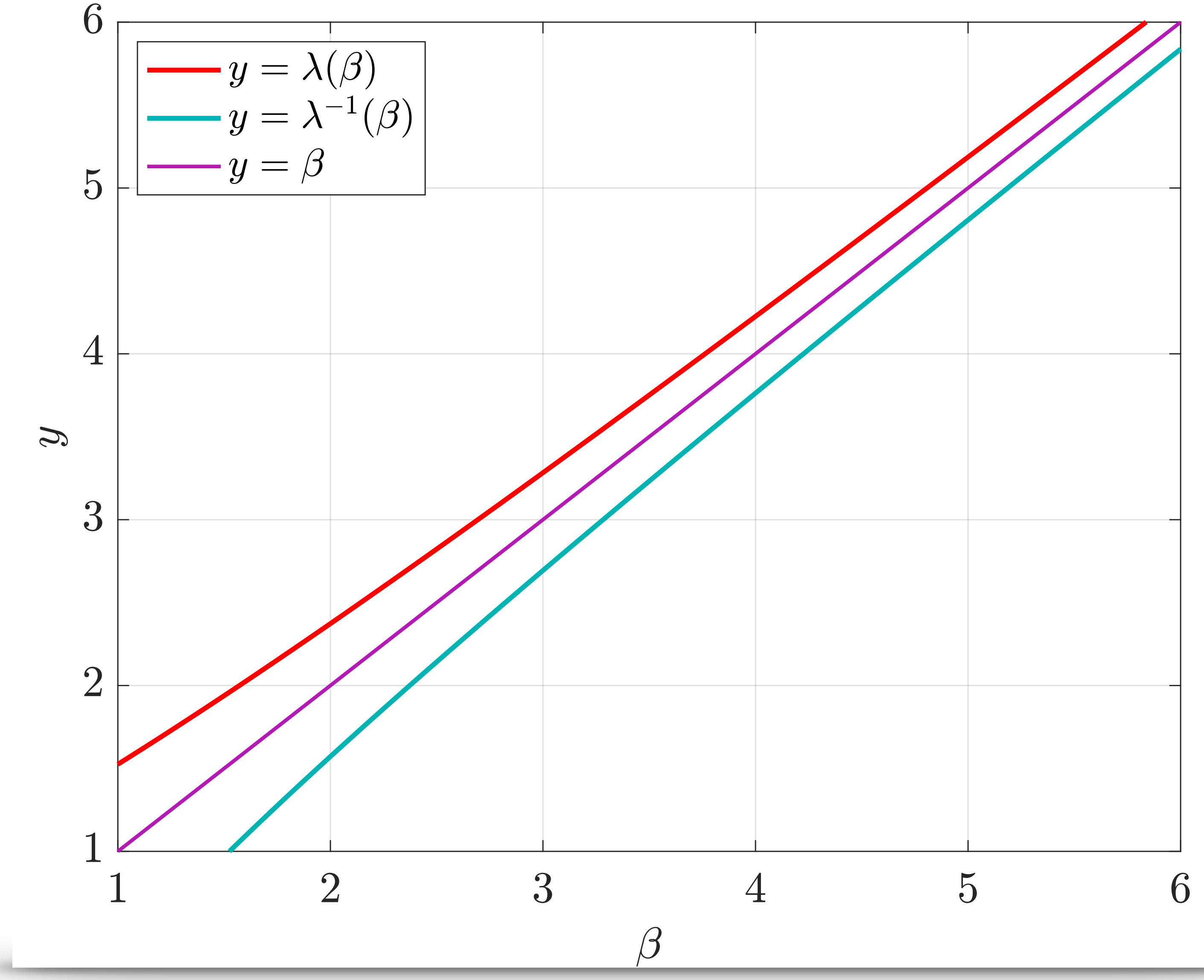
$$\lambda(\underline{\beta}) = \frac{\varphi(\underline{\beta})}{\Phi(-\underline{\beta})} = \beta$$

$$\underline{\beta} = \lambda^{-1}(\beta)$$



Linear case, Gaussian case

$$\underline{\beta} = \lambda^{-1}(\beta) < \beta < \lambda(\beta)$$



Structural Reliability problem $(P_f, \boldsymbol{z}^{**}, \underline{\beta})$

$$Y = g(\boldsymbol{X}) \quad \xrightarrow{\boldsymbol{Z} = T(\boldsymbol{X})} \quad Y = G(\boldsymbol{Z})$$

Probability of failure or classical reliability index

$$P_f \quad \beta = -\Phi^{-1}(P_f) \quad (\approx \beta_{HL})$$

- measure "how likely" a failure occurs

Tail-dependent design (/failure) point

$$\boldsymbol{x}^{**} = T(\boldsymbol{z}^{**})$$

- understand how failure occurs (minimize the MSE among all failure events)

Tail-dependent reliability index

$$\underline{\beta} \approx \underline{\beta}_{HL}$$

- measure reliability taking account on how failure occurs

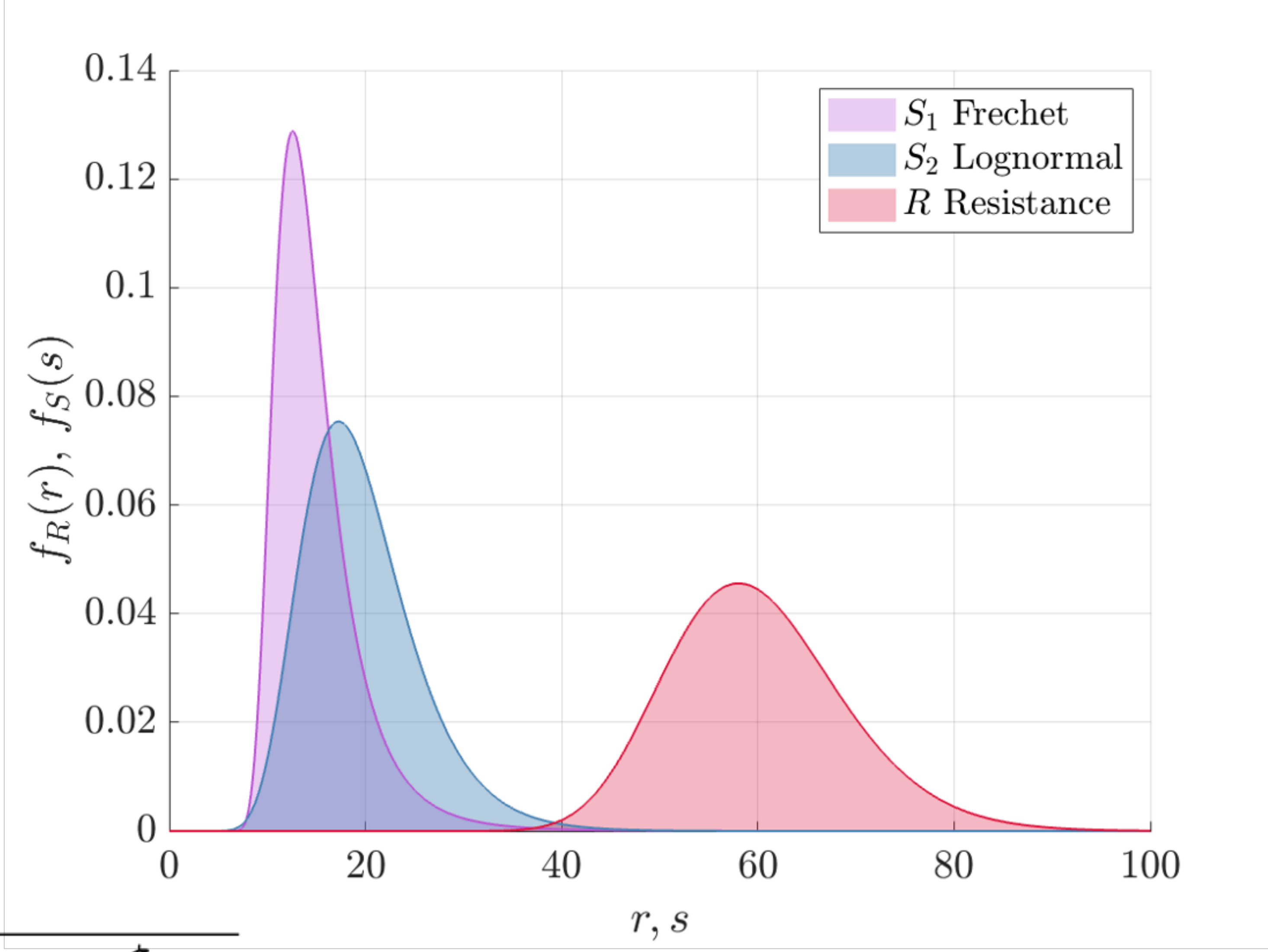
Example 1

$$g(r, s_i) = r - s_i$$

$$S_1 \sim \text{GEV}(\mu, \sigma, \xi)$$

$$S_2 \sim \ln\mathcal{N}(\mu, \sigma)$$

$$R \sim \ln\mathcal{N}(\mu, \sigma)$$



	mean	c.o.v	μ	σ	ξ
R	60.000	0.15	4.083	0.149	–
S_1	15.000	0.30	12.932	2.878	0.125
S_2	19.656	0.30	2.935	0.293	–

Example 1

$$g(r, s_i) = r - s_i$$

$$S_1 \sim \mathrm{GEV}(\mu, \sigma, \xi)$$

$$S_2 \sim \ln\mathcal{N}(\mu, \sigma)$$

$$R \sim \ln\mathcal{N}(\mu, \sigma)$$

$$\mathbb{P}(g(r, s_1) \leq 0) = 2.4501e - 04$$

$$\mathbb{P}(g(r, s_2) \leq 0) = 2.4501e - 04$$

$$\beta = 3.4862$$

	mean	c.o.v	μ	σ	ξ
R	60.000	0.15	4.083	0.149	–
S_1	15.000	0.30	12.932	2.878	0.125
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Example 1

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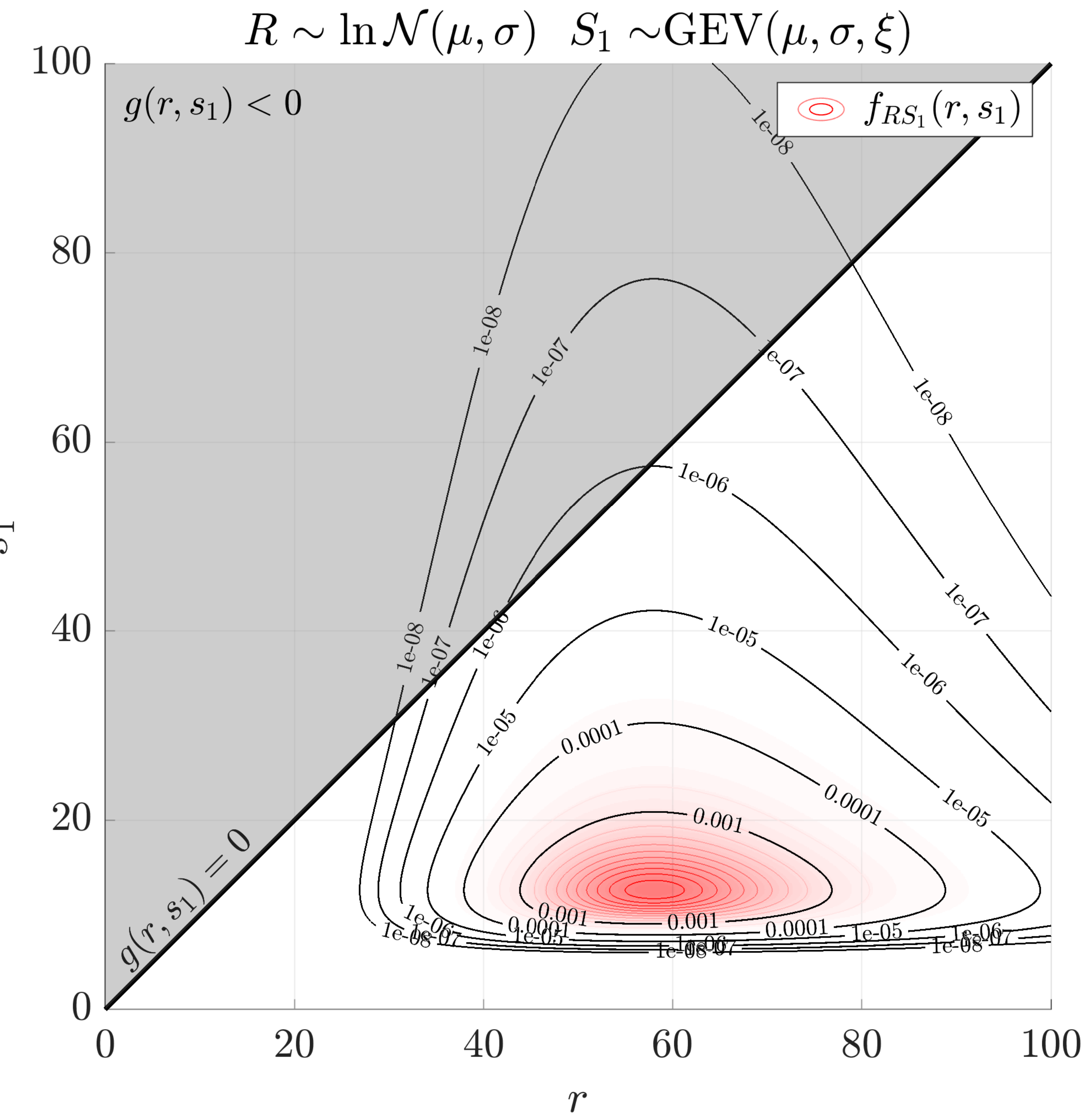
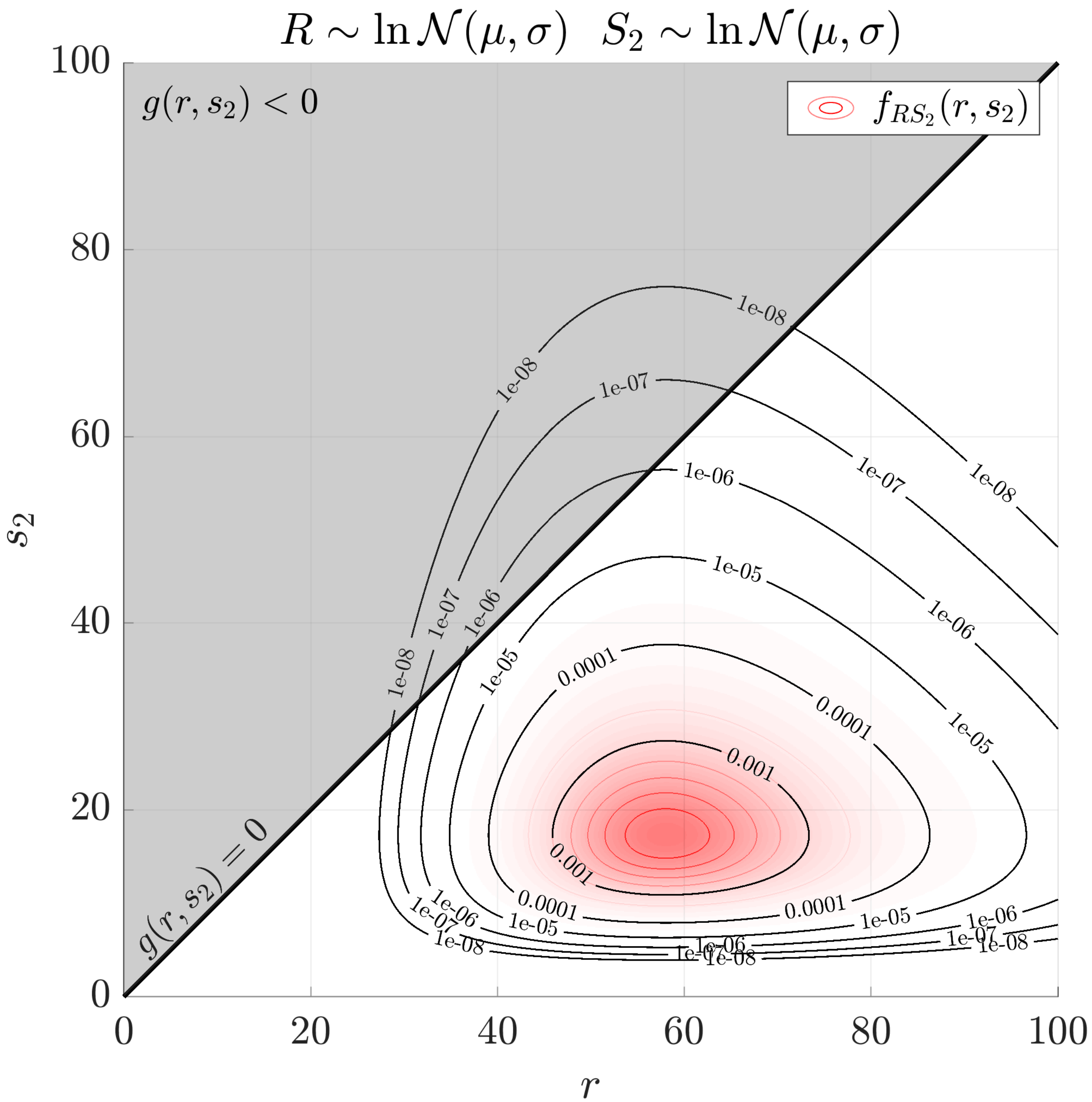
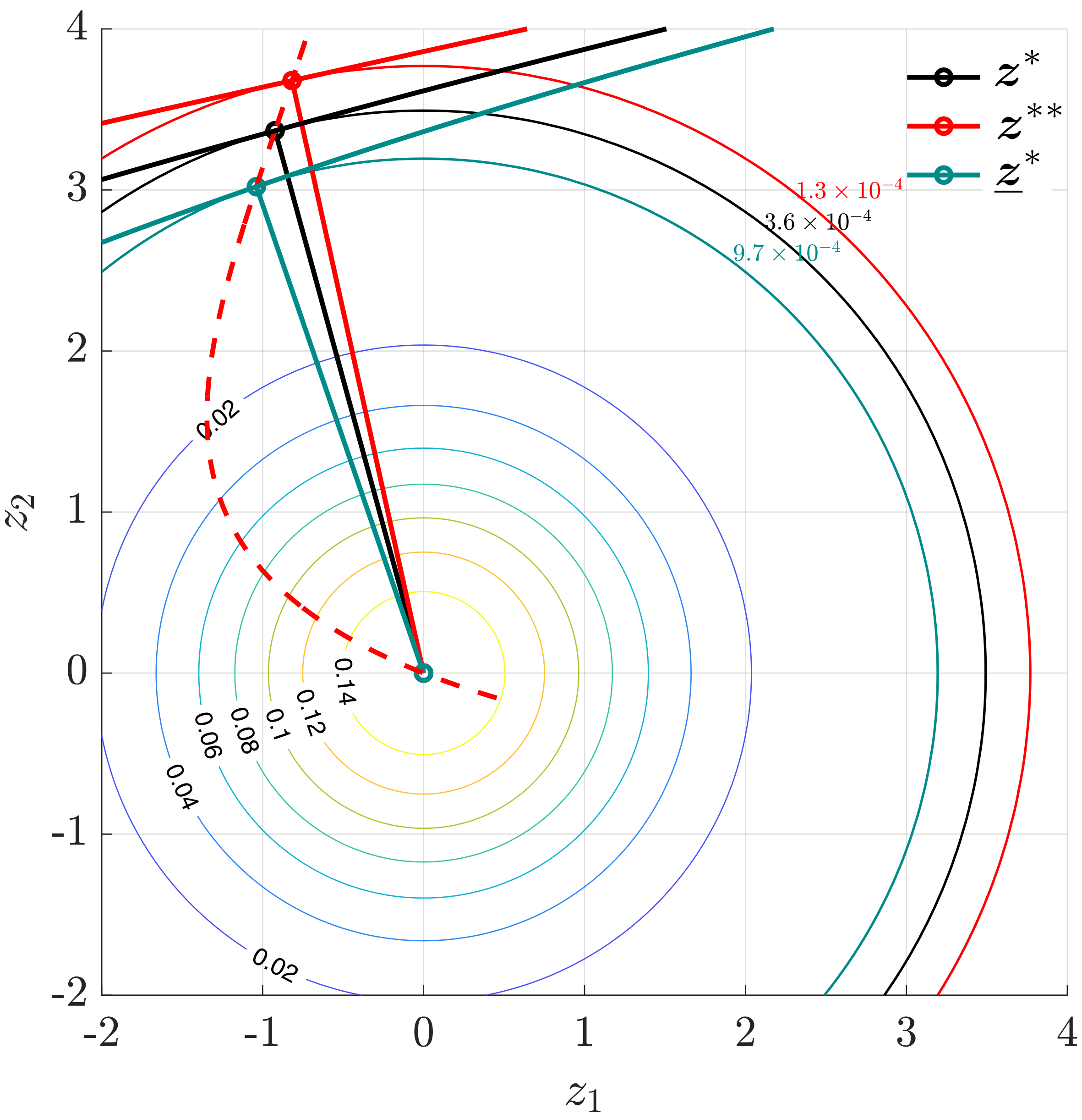
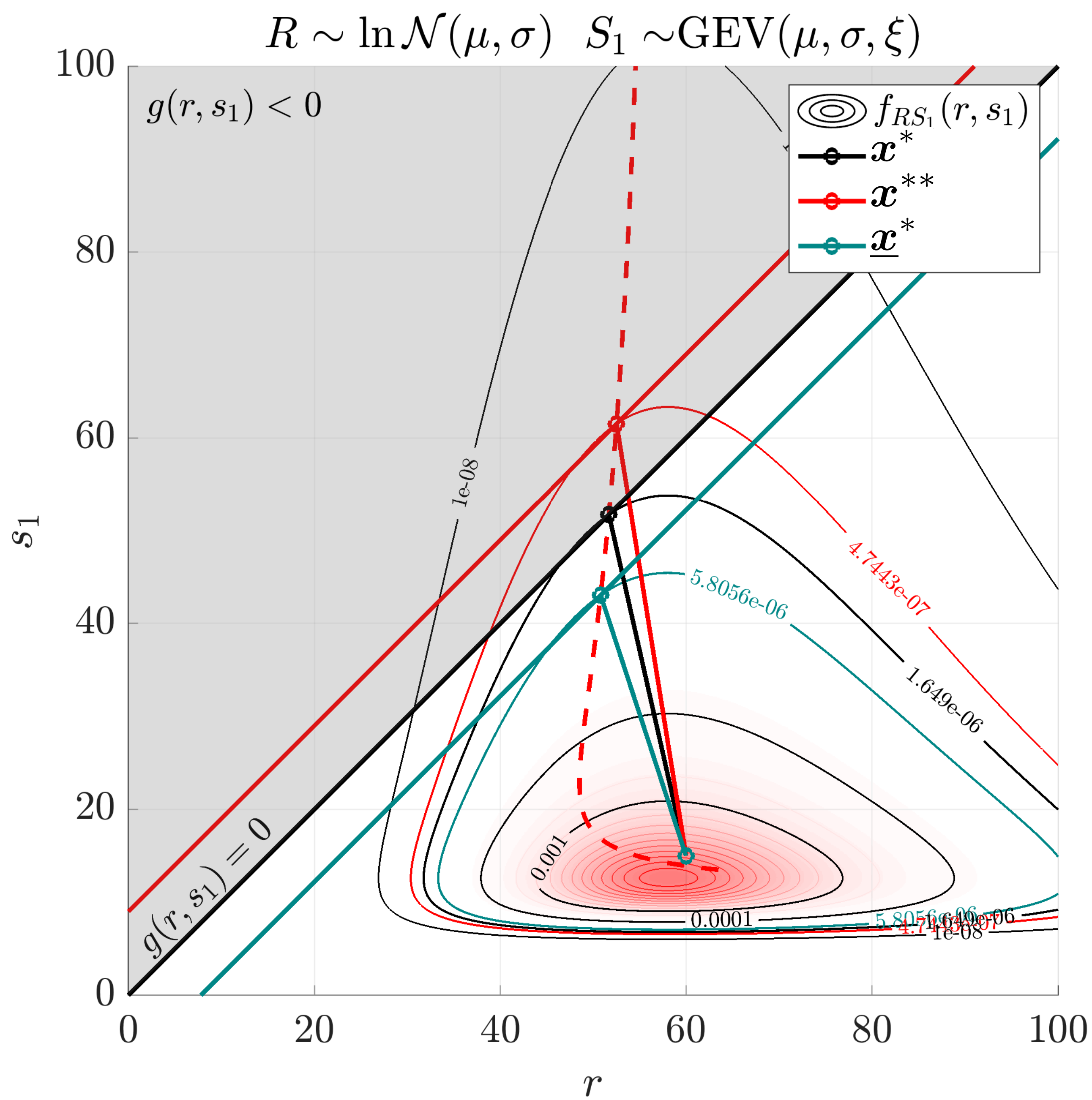




Example 1

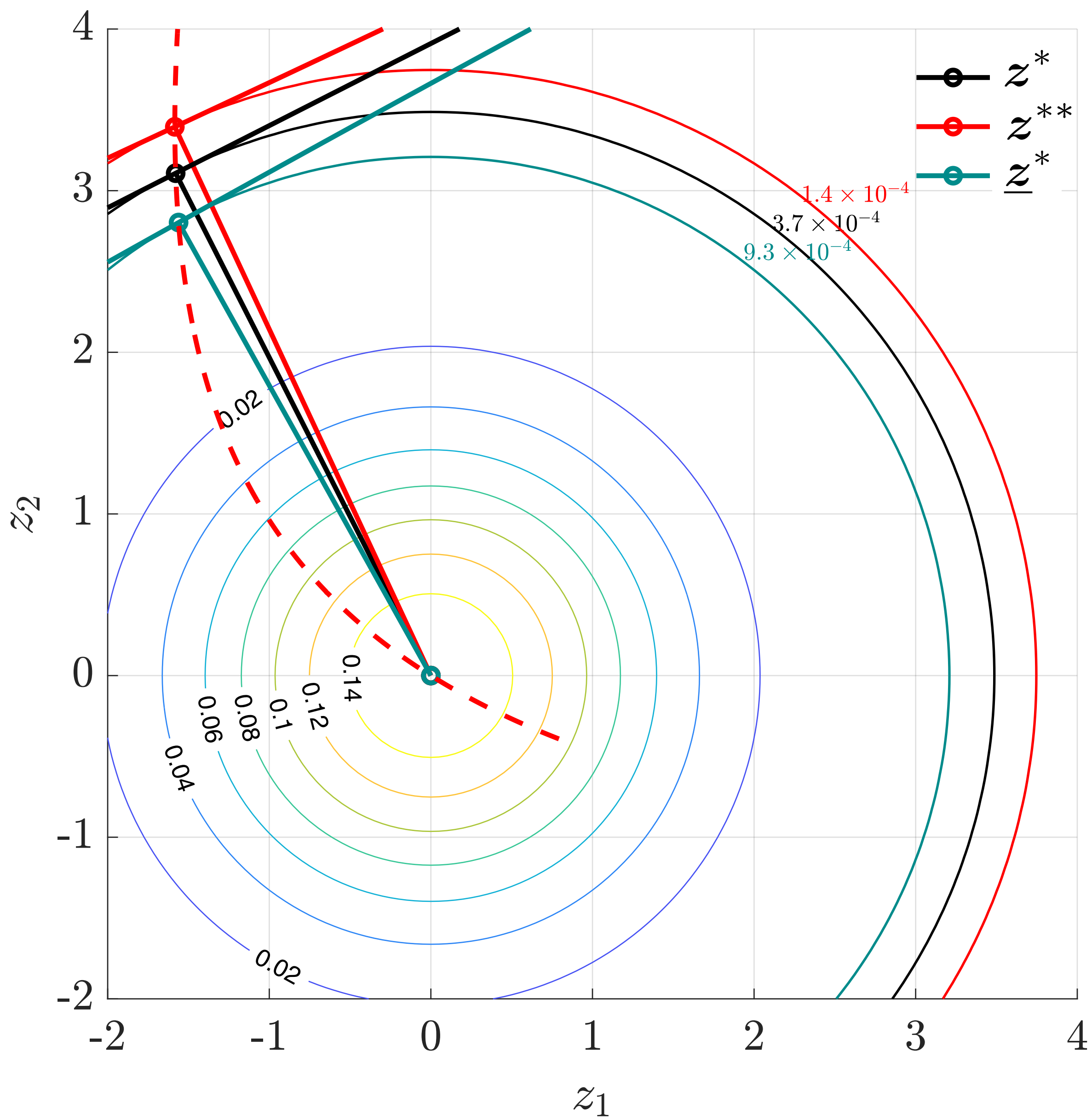
$S_1 \sim \mathrm{GEV}(\mu, \sigma, \xi)$

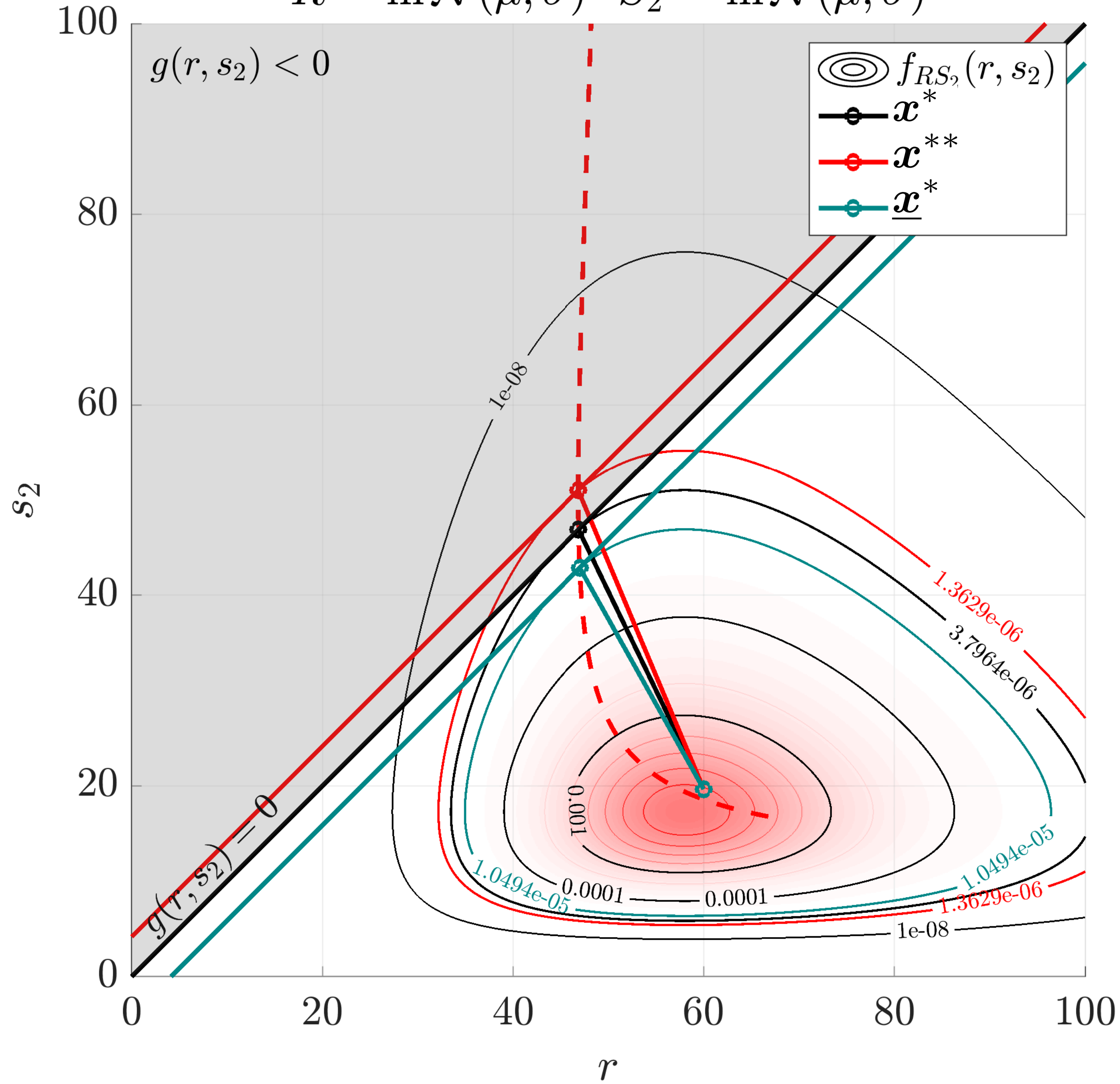




Example 1

$S_2 \sim \ln\mathcal{N}(\mu,\sigma)$





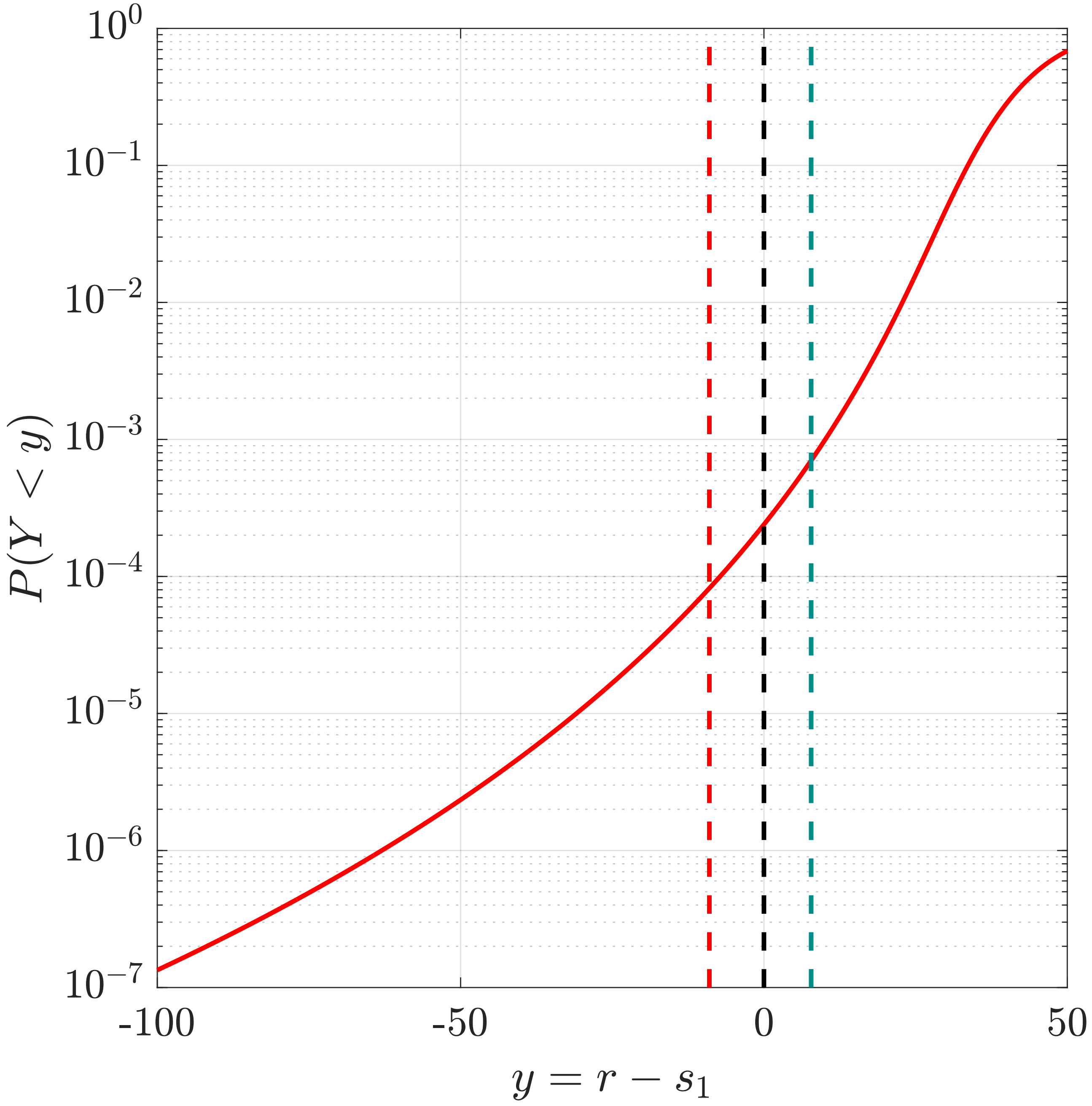
Example 1

$S_1 \sim \text{GEV}(\mu, \sigma, \xi)$

$\underline{\beta} = ||\underline{\boldsymbol{z}}^*|| = 3.1940$

$\beta = 3.4862$

$||\boldsymbol{z}^{**}|| = 3.7687$



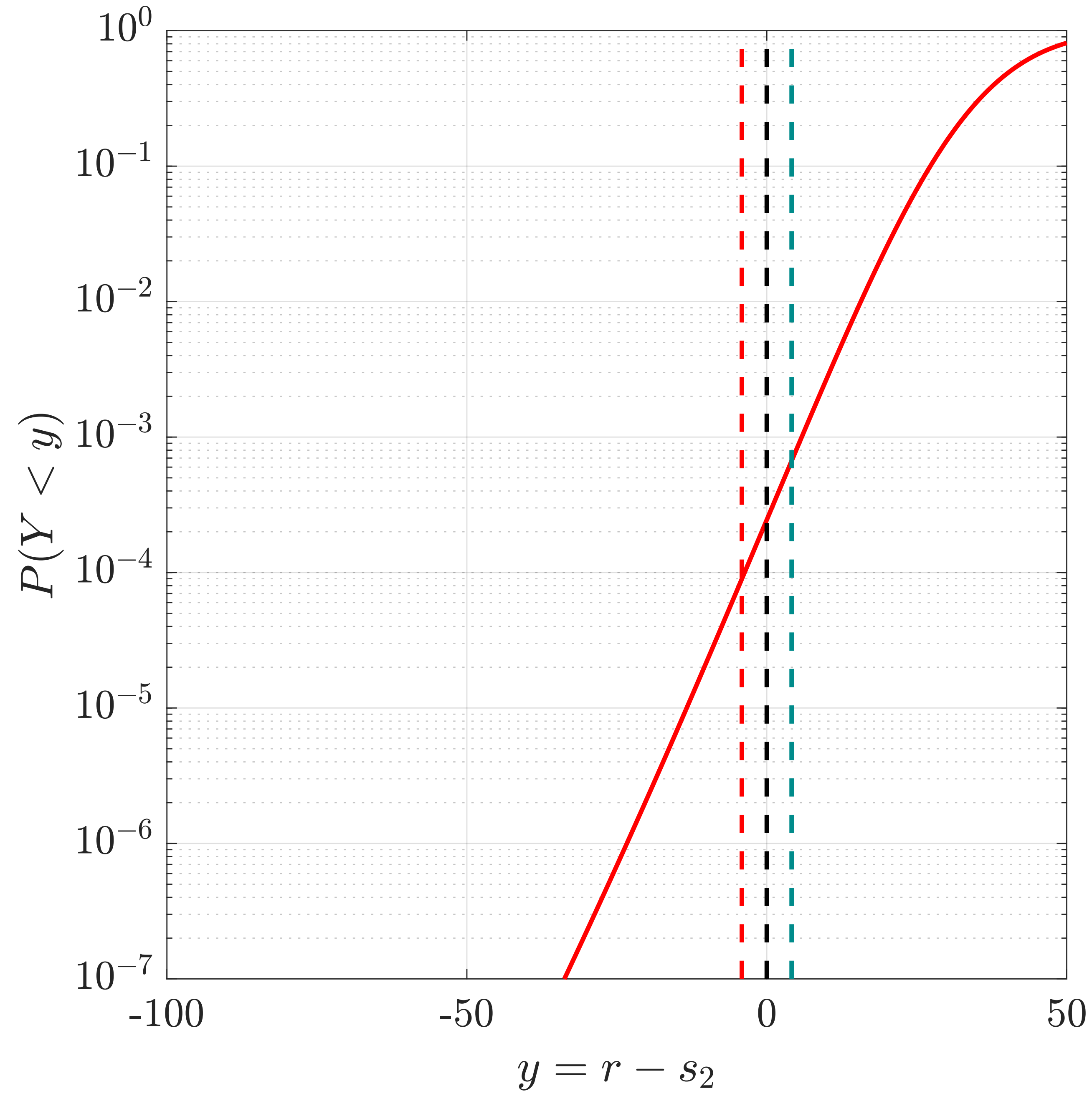
Example 1

$S_2 \sim \ln \mathcal{N}(\mu, \sigma)$

$$\underline{\beta} = ||\underline{\boldsymbol{z}}^*|| = 3.2083$$

$$\beta = 3.4862$$

$$||\boldsymbol{z}^{**}|| = 3.7463$$



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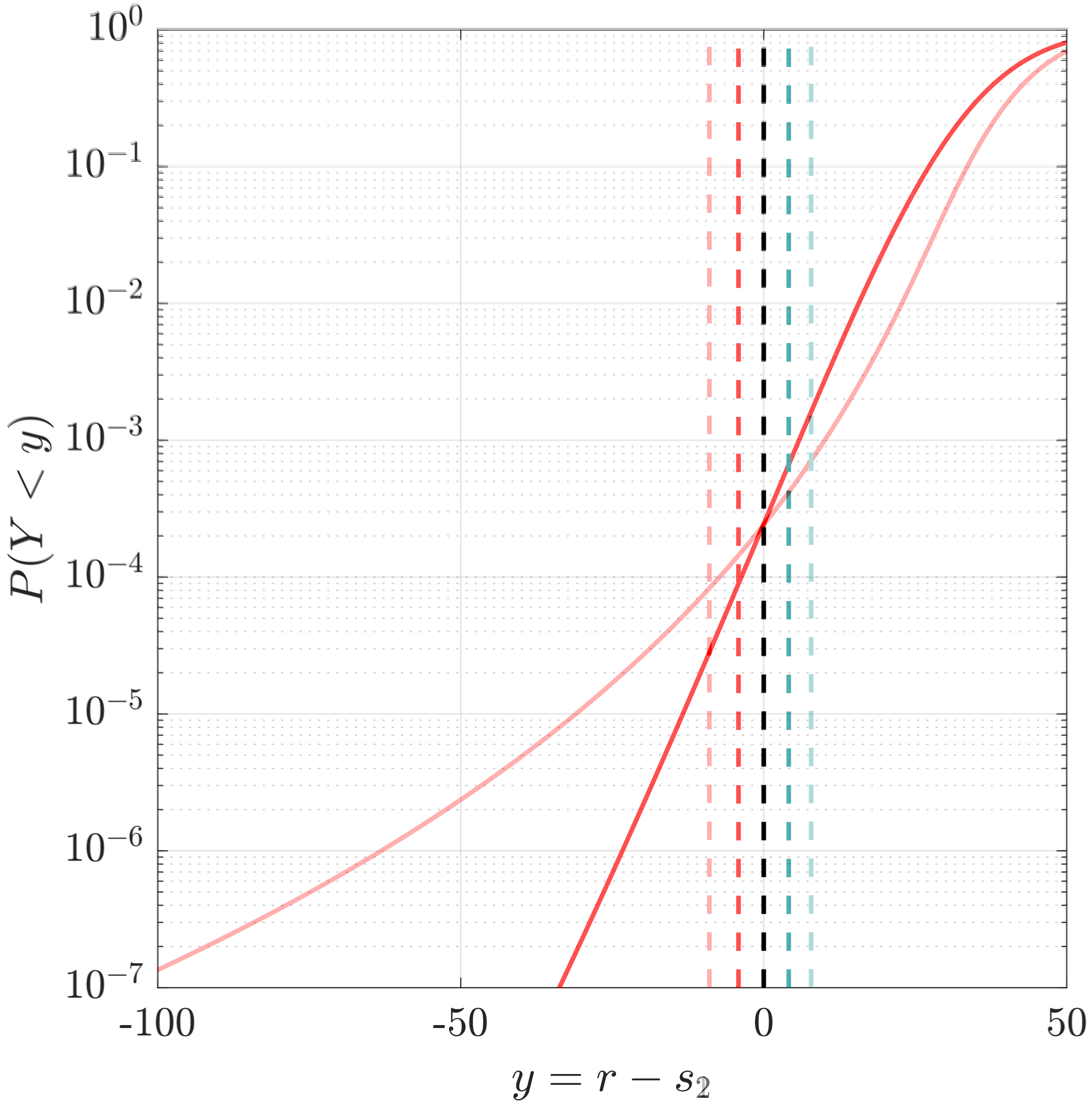


Example 2, nonlinear function with Nataf distr.

$$g(\boldsymbol{x}) = \sqrt{x_1} - x_2 - 6$$

$$X_1 \sim \ln\mathcal{N}(\mu_{\log}, \sigma_{\log}) \qquad X_2 \sim \text{Gamma}(k, \theta)$$

$$f_{X_1,X_2}(x_1, x_2) = \phi_{\mathbf{R}_0}(Z_1, Z_2) \cdot \frac{f_{X_1}(x_1)}{\phi(Z_1)} \cdot \frac{f_{X_2}(x_2)}{\phi(Z_2)}$$

$$Z_1 = \Phi^{-1}(F_{X_1}(x_1)) \qquad Z_2 = \Phi^{-1}(F_{X_2}(x_2))$$

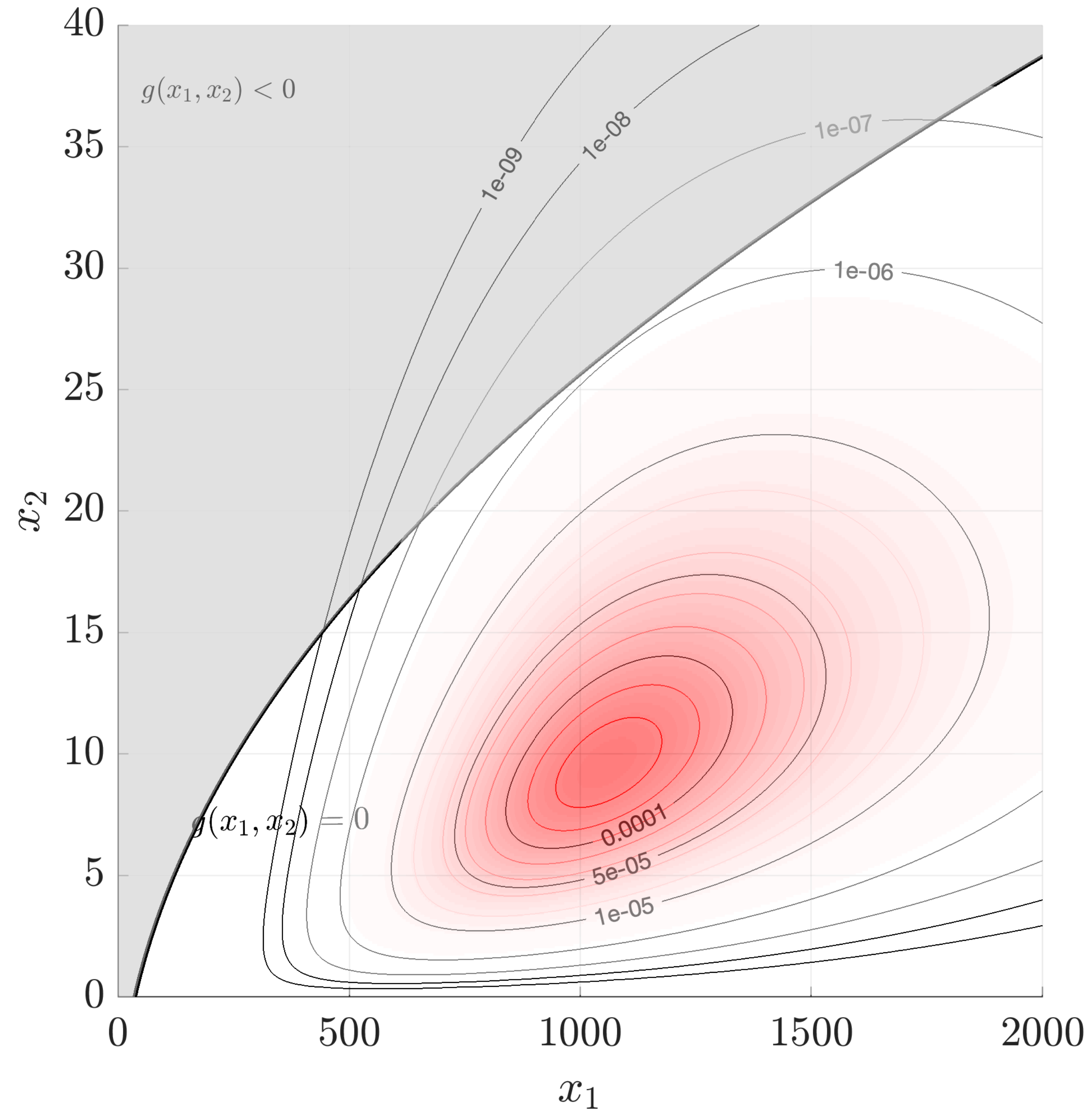
Example 2, nonlinear function with Nataf distr.

$$g(\boldsymbol{x}) = \sqrt{x_1} - x_2 - 6$$

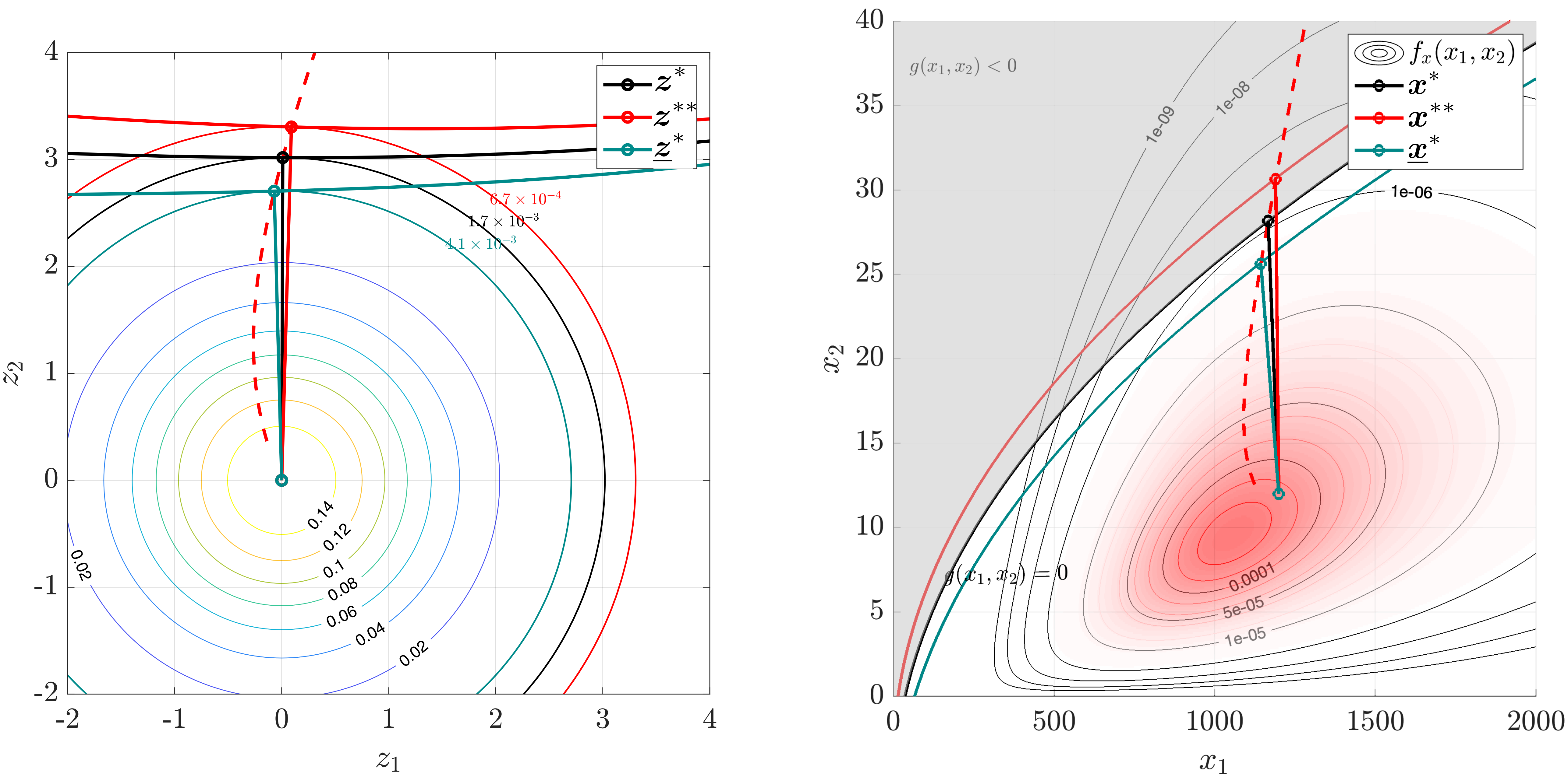
$$\boldsymbol{\mu} = \begin{bmatrix} 1200 \\ 12 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 300 & 0 \\ 0 & 4.8 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$



Example 2, nonlinear function with Nataf distr.

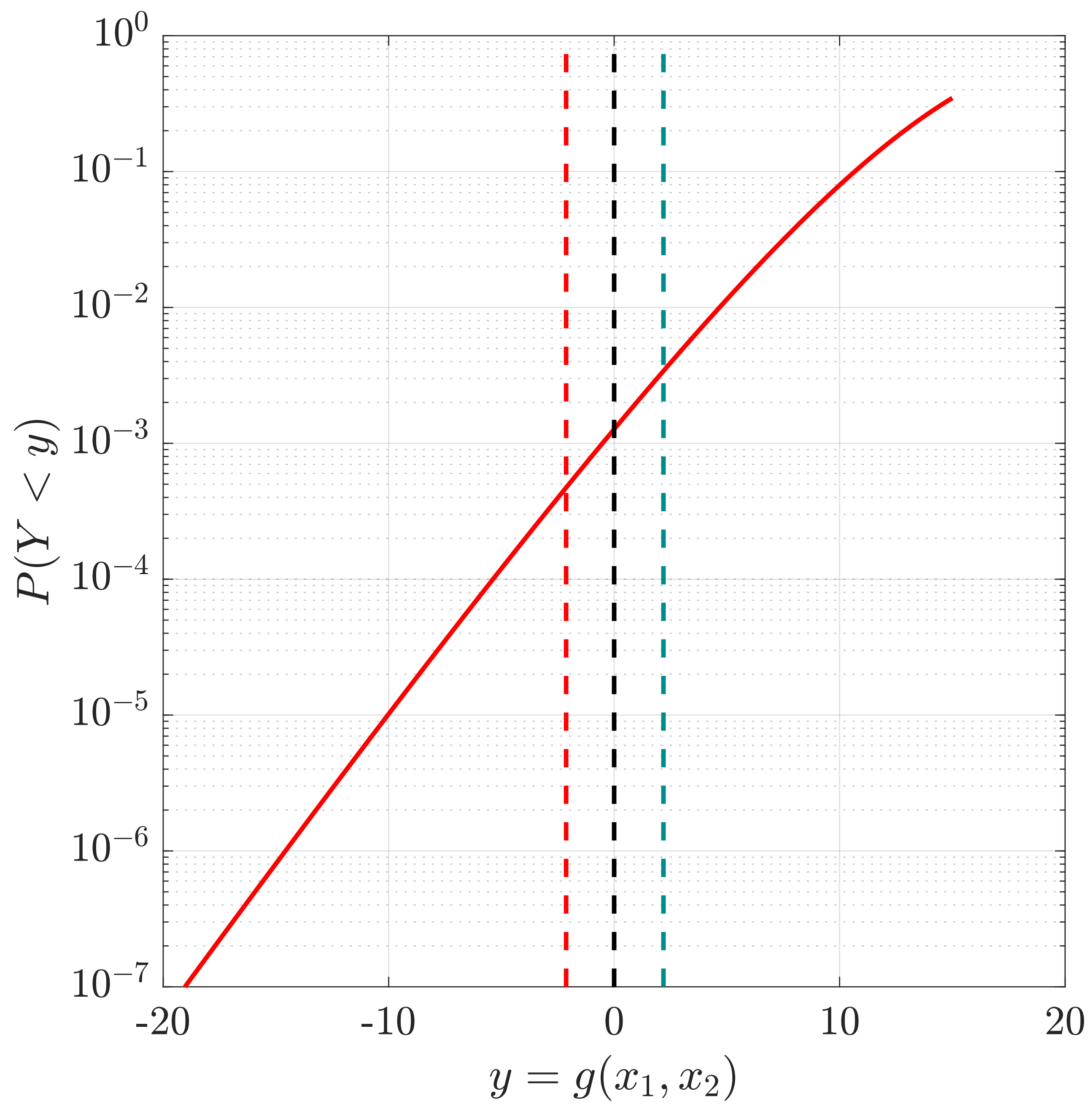


Example 2, nonlinear function with Nataf distr.

$$\underline{\beta} = ||\underline{\boldsymbol{z}}^*|| = 2.70561$$

$$\beta = 3.0185$$

$$||\boldsymbol{z}^{**}|| = 3.3067$$



Conclusion

- $(P_f, \boldsymbol{z}^{**}, \underline{\beta})$
- Basel III, connection with Expected Shortfall, Super-quantiles
- Computational developments (to be studied)
- Partial safety factors (to be studied)
- Sensitivity analysis (to be studied)
- Much more….

- end

Buffered failure probability

$$\mathbb{E}[Y|Y > y] = \int_y^{\infty} x f_Y(x)dx = [F_Y(x)x]_y^{\infty} - \int_y^{\infty} F_Y(x)dx$$

$$\mathbb{E}[Y|Y > y] = \int_y^{\infty} x f_Y(x)dx = -[G_Y(x)x]_y^{\infty} + \int_y^{\infty} G_Y(x)dx$$

$$\mathbb{E}[Y|Y > y] = \int_y^{\infty} x f_Y(x)dx = G_Y(y)y + \int_y^{\infty} G_Y(x)dx$$

$$\mathbb{E}[Y|Y < 0] = \int_{-\infty}^{0} x f_Y(x)dx = [F_Y(x)x]_{\infty}^{0} - \int_{-\infty}^{0} F_Y(x)dx$$

Buffered failure probability

$$\mathbb{E}[Y|Y<0] = \frac{1}{F_Y(0)}\int_{-\infty}^{0} x f_Y(x)dx = \frac{1}{F_Y(0)}[F_Y(x)x]_{-\infty}^{0} - \frac{1}{F_Y(0)}\int_{-\infty}^{0} F_Y(x)dx$$

$$\mathbb{E}[Y|Y<y] = \frac{1}{F_Y(y)}\int_{-\infty}^{y} x f_Y(x)dx = \frac{1}{F_Y(y)}[F_Y(x)x]_{-\infty}^{y} - \frac{1}{F_Y(y)}\int_{-\infty}^{0} F_Y(x)dx$$

$$\mathbb{E}[Y|Y<y] = \frac{1}{F_Y(y)}\int_{-\infty}^{y} x f_Y(x)dx = y - \frac{1}{F_Y(y)}\int_{-\infty}^{y} F_Y(x)dx$$

$$\mathbb{E}[Y|Y\leq y] = y - \frac{\mathbb{E}[\mathrm{ReL}(y-Y)]}{F(y)} = \frac{\mathbb{E}[\mathbb{I}(y-Y)Y]}{\mathbb{E}[\mathbb{I}(y-Y)]}$$

$$\mathbb{E}[Y|Y\leq 0] = -\frac{\mathbb{E}[\mathrm{ReL}(-Y)]}{F(0)} = \frac{\mathbb{E}[\mathbb{I}(-Y)Y]}{\mathbb{E}[\mathbb{I}(-Y)]}$$

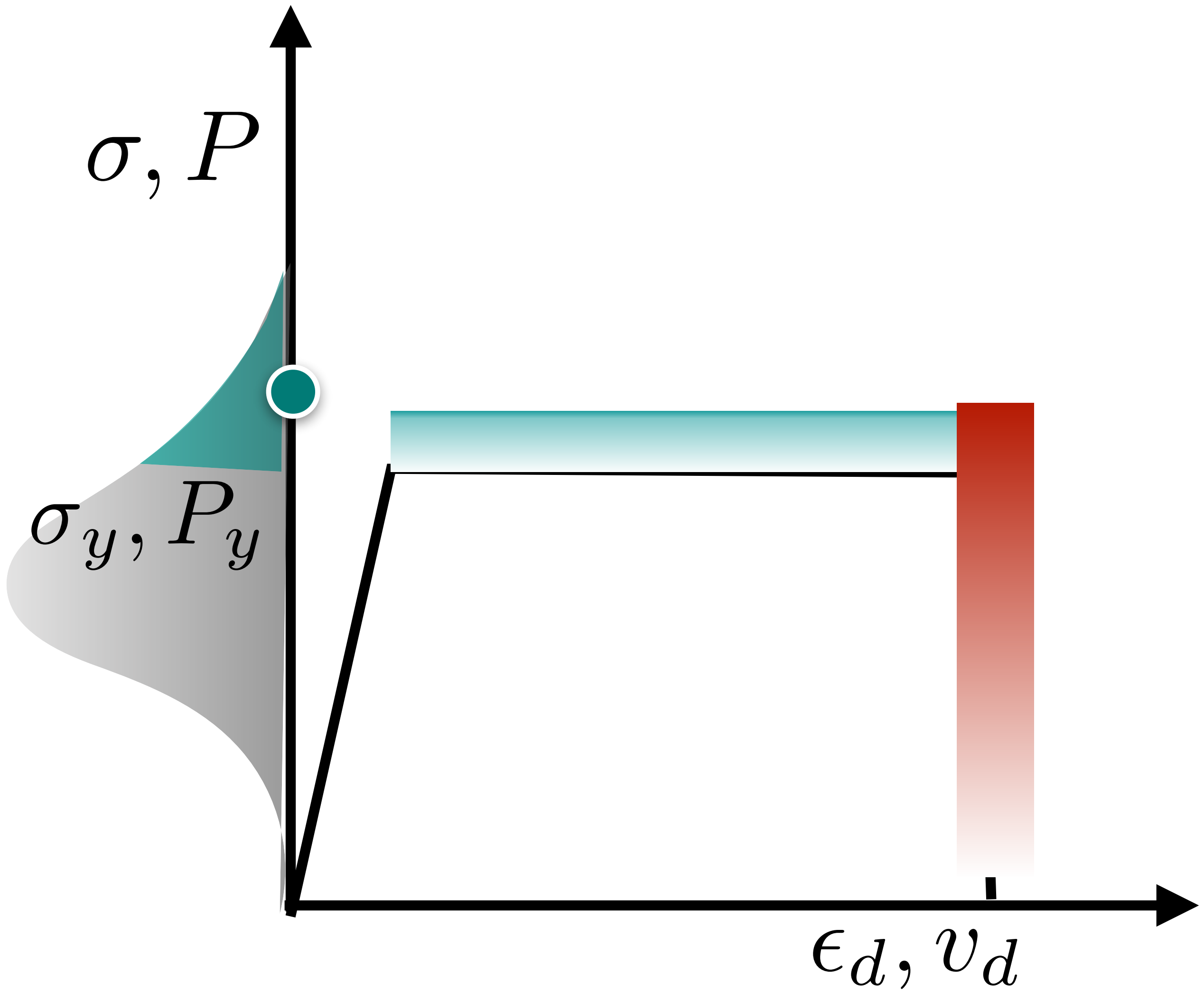
Buffered failure probability

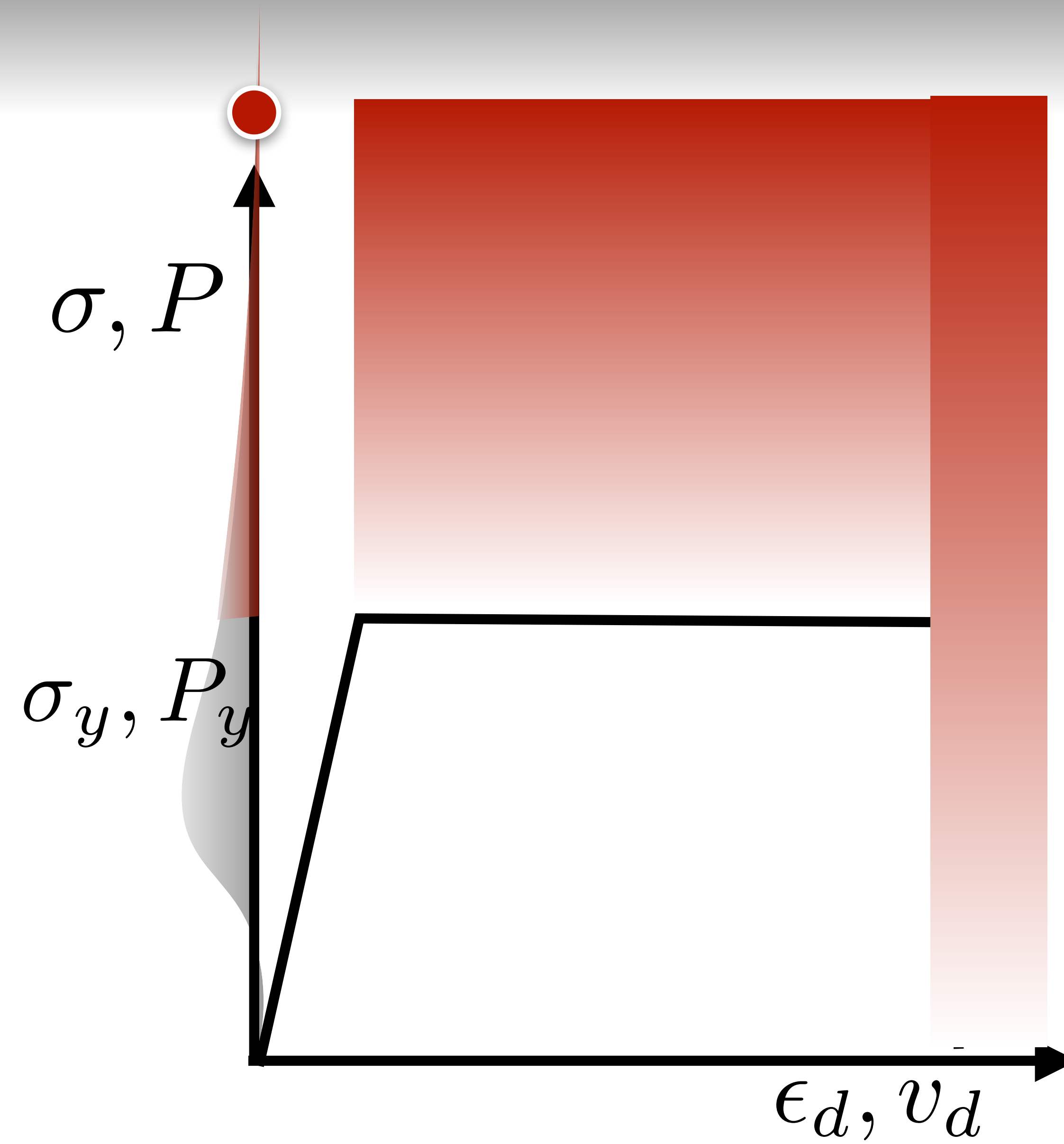
$$\mathbb{E}[Y|Y > y] = \int_y^{\infty} x f_Y(x)dx = G_Y(y)y + \int_y^{\infty} G_Y(x)dx$$

$$\mathbb{E}[Y|Y > y] = y + \frac{\mathbb{E}[\mathrm{ReL}(Y - y)]}{G(y)} = \frac{\mathbb{E}[\mathbb{I}(Y - y)Y]}{\mathbb{E}[\mathbb{I}(Y - y)]}$$

$$\mathbb{E}[Y|Y \leq y] = y - \frac{\mathbb{E}[\mathrm{ReL}(y - Y)]}{F(y)} = \frac{\mathbb{E}[\mathbb{I}(y - Y)Y]}{\mathbb{E}[\mathbb{I}(y - Y)]}$$

$$\mathbb{E}[Y|Y < 0] = \int_{-\infty}^{0} x f_Y(x)dx = [F_Y(x)x]_{\infty}^{0} - \int_{-\infty}^{0} F_Y(x)dx$$

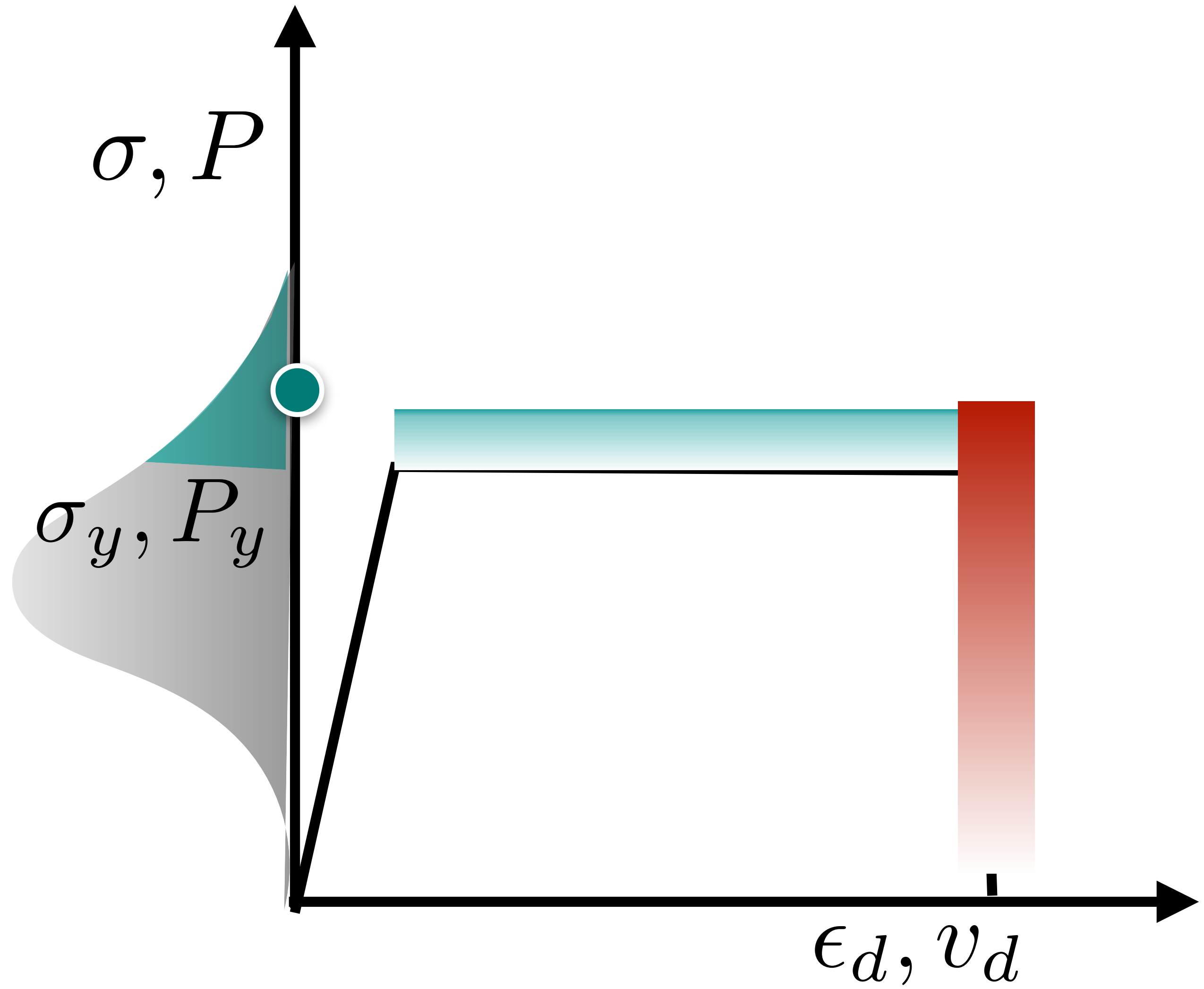


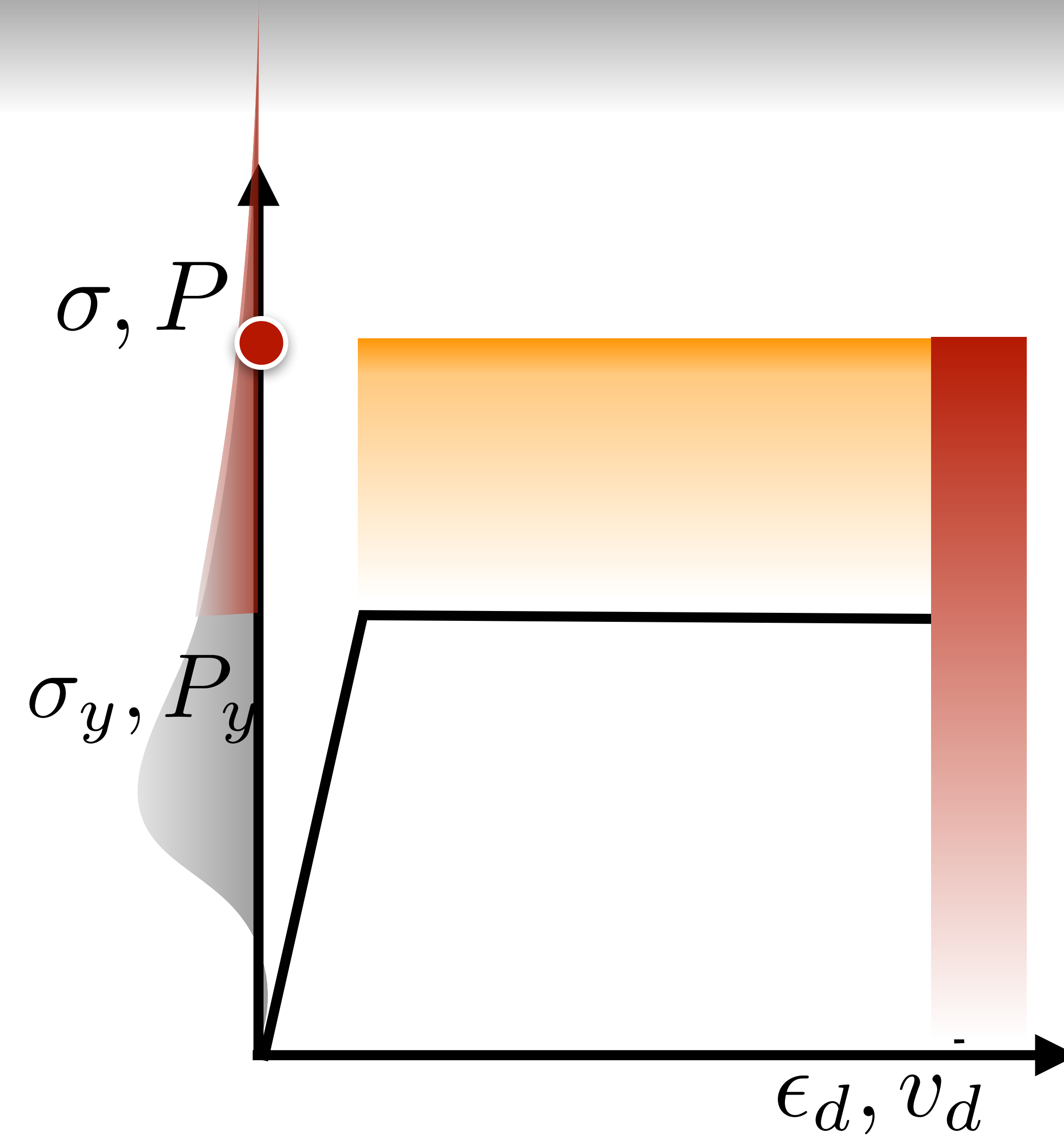


$$\ddot{v}_d = \frac{P_u - P_y}{m}$$

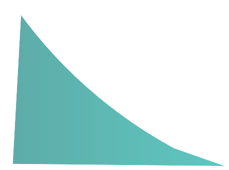
$$\ddot{v}_? = \frac{P_u - P_y}{m}$$

$$\frac{\ddot{v}_d}{\ddot{v}_b} = 1 - \frac{P_y}{P_u} \rightarrow 1$$





$$\ddot{v}_d = \frac{P_u - P_y}{m} \qquad \ddot{v}_? = \frac{P_u - P_y}{m}$$

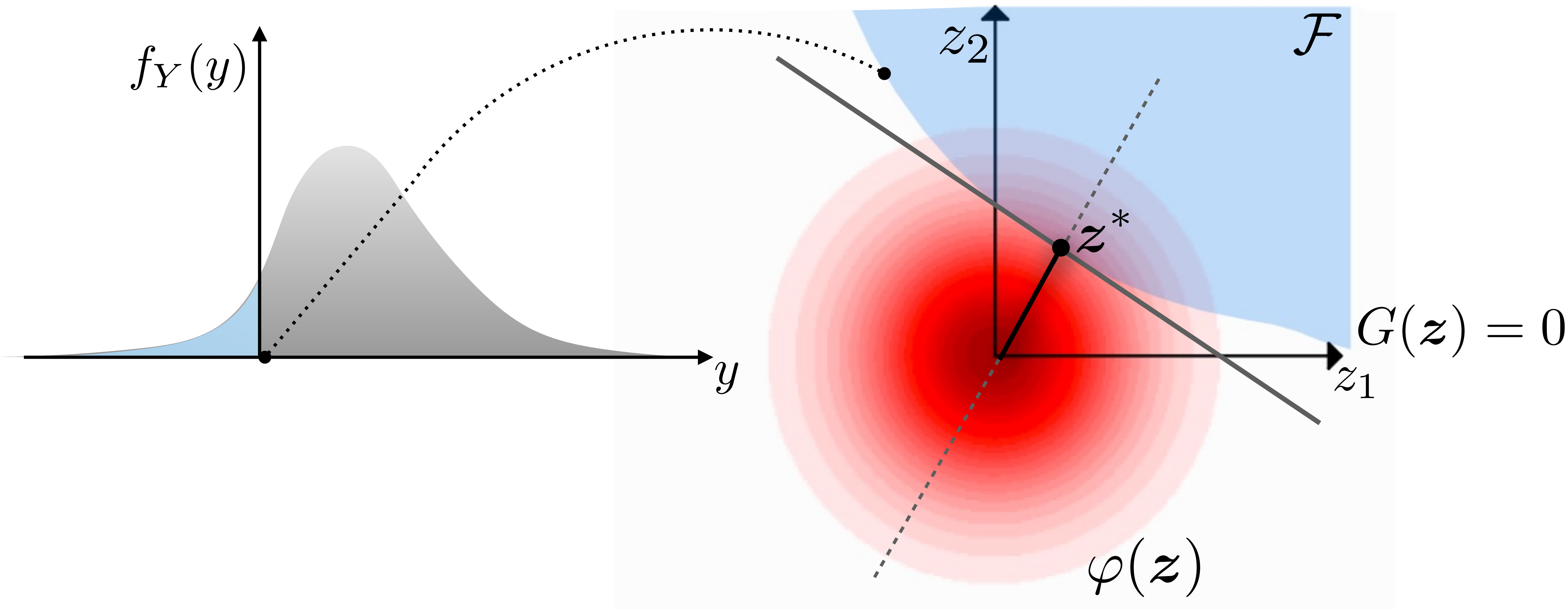
 = 

$$P_f = P_f$$

Classical concepts

$$P_{\mathcal{F}} = \int_{\mathbb{R}^n} \mathbb{I}(G(\boldsymbol{z}))\varphi(\boldsymbol{z})d\boldsymbol{z}$$

$$F_Y(y) = \int_{\mathbb{R}^n} \mathbb{I}(G(\boldsymbol{z}) - y)\varphi(\boldsymbol{z})d\boldsymbol{z}$$



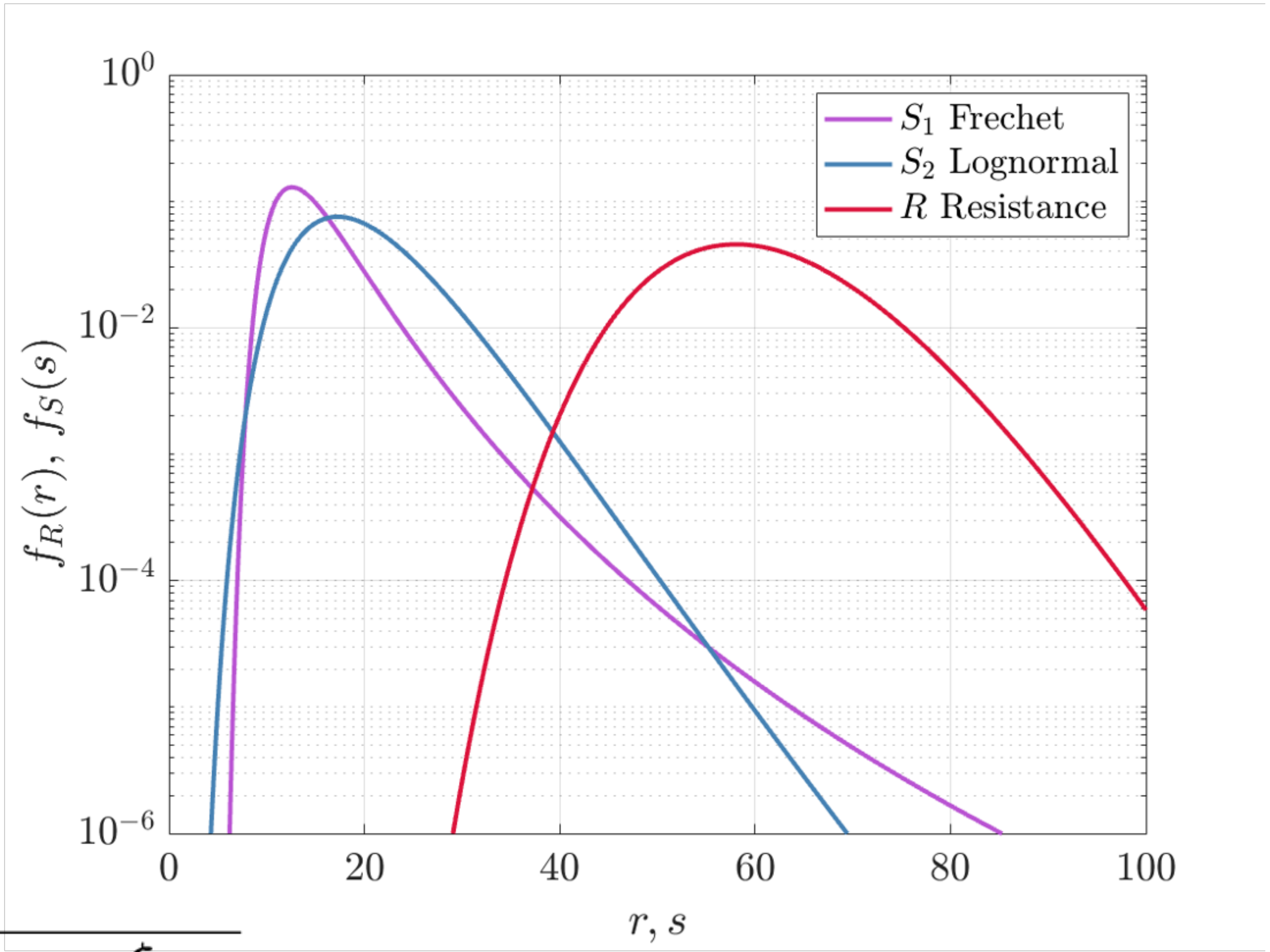
Example 1

$$g(r, s_i) = r - s_i$$

$$S_1 \sim \mathrm{GEV}(\mu, \sigma, \xi)$$

$$S_2 \sim \ln\mathcal{N}(\mu, \sigma)$$

$$R \sim \ln\mathcal{N}(\mu, \sigma)$$



	mean	c.o.v	μ	σ	ξ
R	60.000	0.15	4.083	0.149	–
S_1	15.000	0.30	12.932	2.878	0.125
S_2	19.656	0.30	2.935	0.293	–