On the period function of planar systems with unknown normalizers

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Abstract

A sufficient condition for the period function's monotonicity on a period annulus is given. The approach is based on normalizers' theory, but is applicable without actually knowing a normalizer. Some applications to polynomial and hamiltonian systems are presented. [0]

1 Introduction

In this paper we are concerned with plane differential systems,

$$z' = V(z), \qquad z \in \Omega \subset \mathbb{R}^2, \tag{1}$$

with Ω open connected, $V(z) = (V_1(z), V_2(z)) \in C^2(\Omega, \mathbb{R}^2)$, $z = (x, y) \in \Omega$. We denote by $\phi_V(t, z)$ the local flow defined by (1). A connected subset A of Ω is said to be a *period annulus* of (1) if every orbit of V contained in A is a non-trivial cycle of (1). In some cases the inner boundary of A is a single point O, called *center*, and the largest connected punctured neighbourhood N_O of O covered with non-trivial cycles is called *central region*. If A is a period annulus, we can define on A the *period function* T by assigning to each point $z \in A$ the minimal period $T(z)$ of the cycle γ_z passing at z. We say that the period function T is *increasing* if larger cycles have larger periods. When T is constant, we say that A is *isochronous*.

The monotonicity of the period function is important in approaching several problems related to (1). It is related to existence and uniqueness of solutions of some boundary value, bifurcation or perturbation problems (see [1], [2], [7], [9], [11]). Moreover, isochronicity has a strong relationship to stability, since a periodic solution contained in A is Liapunov stable if and only if it has an isochronous neighbourhood [3].

A recent result presenting a new approach to the monotonicity of T is based on the properties of a suitable class of auxiliary systems, called *normalizers* [4].

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Let us denote by $[V, U] = \partial_V U - \partial_U V$ the Lie brackets of V and U. A vector field U, transversal to V, is said to be a *non-trivial normalizer* of V on a set $B \subset \Omega$ if $[V, U] \wedge V = 0$ on B. If U is a normalizer of V, there exists a C^1 function μ, defined on A, such that $[V, U] = \mu V$. Let $\phi_U(s, z)$ be the local flow defined by the solutions of

$$z' = U(z). \tag{2}$$

In [4], theorem 1, it was proved that

$$\partial_U T(z^*) = \left[\frac{d}{ds} T(\phi_U(s, z^*)) \right]_{s=0} = \int_0^T \mu(\phi_V(t, z^*))dt. \tag{3}$$

In the same paper a non-trivial normalizer was found for hamiltonian systems with separable variables

$$x' = -F'(y), \qquad y' = G'(x).$$

The monotonicity of the period function was studied in detail for centers of such systems. Such an approach proves to be very useful and effective when a normalizer is known. In general, it is not known how to find a normalizer of a given system.

In this paper we present an extension of theorem 1 in [4], that allows to replace a normalizer with a transversal vector field W. This overcomes the difficulty of finding a normalizer, but usually leads to more complex computations. In spite of this, we show that some previous results can be re-proved by the technique presented here, and we give some new results. For instance, we prove that if the function

$$\Lambda_{PQ} = -(P_y + Q_x)P^2 + 2(P_x - Q_y)PQ + (P_y + Q_x)Q^2$$

has positive sign on an orbit γ, then T is increasing at γ. This extends the classical result giving isochronicity of systems satisfying $P_x = Q_y$, $P_y = -Q_x$. Moreover, we prove that if Λ_{PQ} vanishes on a single orbit γ, then γ is a critical orbit.

2 Results

Let us set $V \wedge U := V_1 U_2 - V_2 U_1$. If U is a non-trivial normalizer of V on the period annulus A, then $V \wedge U$ vanishes nowhere on A. Moreover, the proportionality function μ satisfies

$$\mu = \frac{[V, U] \cdot V}{|V|^2},$$

so that it has the same regularity as V and U. U is a normalizer of V if and only if the local flow $\phi_U(s, z)$ locally takes arcs of orbits of (1) into arcs of orbits of (1) [4]. If V and U are normalizers of each other, they are said to be *commutators*.

A period annulus A of (1) is *isochronous*, that is all of its cycles have the same period, if and only if V has a commutator on A [8].

If U is a normalizer of V on A, then the derivative of T at γ_{z^*} along the solution $\phi_U(s,z^*)$ is given by the integral in (3). The value of such an integral depends on the normalizer U, but the sign of $\partial_U T$ is the same for all normalizers crossing γ_{z^*} in the same direction, for instance outwards, as is usually done.

We do not use the words *increasing* and *decreasing* in a strict sense. When dealing with strict mononicity properties, this will be explicitly stated.

In next a lemma we show that on every period annulus A, V has a non-trivial normalizer.

Lemma 1 *Let A be a period annulus of (1). Then V has a non-trivial normalizer $U \in C^2(A, \mathbb{R}^2)$.*

Proof. For $z \in A$, let us consider the new vector field $V(z)T(z)$. If $\gamma_z(t)$ is a T(z)-periodic solution to (1), then $\gamma_z(tT(z))$ is a 1-periodic solution to

$$z' = V(z)T(z).$$

hence the annulus A is an isochronous annulus for such a system, and the construction of lemma 1 in [8] generates a vector field U commuting with V. Since every commutator is a normalizer, the statement is proved. ♣

In next theorem we consider an arbitrary vector field W transversal to V defined on a period annulus A. By the transversality assumption, there exist functions η, ν, such that $[V,W] = \eta V + \nu W$. If $W \in C^2(A, \mathbb{R}^2)$, then $\eta, \nu \in C^2(A, \mathbb{R}^2)$, since:

$$\eta = \frac{[V,W] \wedge W}{V \wedge W}, \qquad \nu = \frac{[V,W] \wedge V}{W \wedge V}. \tag{4}$$

We denote by $\phi_W(r,z)$ the local flow defined by the differential system:

$$z' = W(z).$$

We denote by $\partial_W T(z^*)$ the derivative of the period function T with respect to the solution $\phi_W(r,z^*)$ at the point z^*.

Theorem 1 *Let A be a period annulus of (1) and $z^* \in A$. Let $W \in C^2(A, \mathbb{R}^2)$ be a vector field transversal to V on A, with $[V,W] = \eta V + \nu W$. Then, for every T-periodic V-cycle $\gamma(t) = \phi_V(t,z^*)$ contained in A, one has:*

$$\int_0^T \nu(\gamma(\tau)) \, d\tau = 0. \tag{5}$$

Moreover, setting

$$\beta(\gamma(\tau)) = \beta(\gamma(0)) \exp\left(-\int_0^t \nu(\gamma(\tau)) \, d\tau\right), \tag{6}$$

one has:

$$\partial_W T(z^*) = \frac{1}{\beta(z^*)} \int_0^T \eta\big(\gamma(\tau)\big)\beta\big(\gamma(\tau)\big)\ d\tau. \tag{7}$$

Proof. Let U be the normalizer of V existing by lemma 1. There exist $\sigma, \beta \in C^2(A, \mathbb{R})$ such that $U = \sigma V + \beta W$, $\beta \neq 0$ by the transversality of U and V. As observed above, the regularity of σ and β comes from the equalities

$$\sigma = \frac{U \wedge W}{V \wedge W}, \qquad \beta = \frac{U \wedge V}{W \wedge V}.$$

One has:

$$\mu V = [V, U] = [V, \sigma V + \beta W] = \partial_V(\sigma V) - \sigma \partial_V V + \partial_V(\beta W) - \beta \partial_W V =$$
$$(\partial_V \sigma)V + (\partial_V \beta)W + \beta(\partial_V W) - \beta(\partial_W V) = (\partial_V \sigma)V + (\partial_V \beta)W + \beta[V, W] =$$
$$(\partial_V \sigma)V + (\partial_V \beta)W + \beta(\eta V + \nu W) = (\partial_V \sigma + \eta\beta)V + (\partial_V \beta + \beta\nu)W.$$

By the transversality of V and W, this implies

$$\partial_V \beta = -\beta\nu, \qquad \mu = \partial_V \sigma + \eta\beta. \tag{8}$$

From the first equation, for every orbit γ of (1) contained in A, one has:

$$\beta(\gamma(t)) = \beta(\gamma(0)) exp\left(-\int_0^t \nu(\gamma(\tau))\ d\tau\right), \qquad \beta(\gamma(0)) \neq 0.$$

The condition $\beta(\gamma(0)) \neq 0$ comes from the transversality of V and U. Since $\gamma(T) = \gamma(0)$, one has:

$$0 = \beta(\gamma(T)) - \beta(\gamma(0)) = \beta(\gamma(0)) \left(exp\left(-\int_0^T \nu(\gamma(\tau))\ d\tau\right) - 1\right),$$

hence $\int_0^T \nu(\gamma(\tau))\ d\tau = 0$, proving the first statement.

The field βW is a normalizer of V. In fact, one has

$$[V, \beta W] = \partial_V \beta W - \partial_{\beta W} V = (\partial_V \beta + \beta\nu)W + \beta\eta V = \beta\eta V.$$

By theorem 1 in [4], the derivative of T at z^* along a solution of $z' = \beta(z)W(z)$ is

$$\partial_{\beta W} T(z^*) = \int_0^T \eta(\gamma(\tau))\beta(\gamma(\tau))\ d\tau,$$

where $\gamma(0) = z^*$. The statement comes observing that $\partial_{\beta W} T(z^*) = \beta(z^*)\partial_W T(z^*)$. ♣

The above proof also shows that for every vector field W transversal to V, there exist a function $\beta \neq 0$, such that βW is a normalizer of V.

In general, computing the integral in (7) is not easy, even when W is itself a normalizer of a hamiltonian system with separable variables [4]. Anyway, β does not change sign on a single orbit, so that if also η has constant sign, then its sign is that of $\partial_W T$.

Corollary 1 *Let γ be a cycle of (1), contained in a period annulus A. Let there exist a vector field $W \in C^2(A, \mathbb{R}^2)$, such that $V \wedge W \neq 0$ on γ.*

(i) If $[V,W] \wedge W \geq 0$ on γ, then T is increasing at γ; if there $\exists \bar{z} \in \gamma$ such that $([V,W] \wedge W)(\bar{z}) > 0$, then $\partial_W T > 0$ on γ.

(ii) If $[V,W] \wedge W = 0$ on γ, then γ is a critical orbit.

Proof.

(i) From the first equality in (4), if (i) holds, then $\eta \geq 0$ on γ. The presence of a point $\bar{z} \in \gamma$ such that $\eta(\bar{z}) > 0$ implies that the integral in (7) is positive.

(ii) As in (i), because in this case $\eta(\bar{z}) \equiv 0$ on γ. ♣

If case (ii) occurs on all of a period annulus, then every orbit is critical, so that A is an isochronous annulus. This gives a different proof of the main result in [5], [10].

If the orbits of A are star-shaped with respect to the origin, we can choose $W(x,y) = (x,y)$ as a transversal vector field. Then for a vector field $V(x,y) = (P(x,y), Q(x,y))$, we have

$$\begin{array}{l} \eta(V \wedge W) = [V,W] \wedge W = \\ (P - xP_x - yP_y, Q - xQ_x - yQ_y) \wedge (x,y) = \\ yP - xQ - y^2 P_y + xy(Q_y - P_x) + x^2 Q_x. \end{array} \tag{9}$$

If we take $P(x,y) = y$, $Q(x,y) = x + \kappa x^3$, $\kappa \neq 0$, then $[V,W] \wedge W = -\kappa x^4$, that implies the strict monotonicity of T. A similar formula can be written for ν:

$$\nu(W \wedge V) = [V,W] \wedge V = P(xQ_x + yQ_y) - Q(xP_x + yP_y).$$

If P and Q are homogeneous of the same degree, then $[V,W] \wedge V = 0$, so that W is a normalizer of V.

We can apply the formula (9) to Liénard systems,

$$x' = y, \qquad y' = -g(x) - yf(x),$$

Every Liénard system can be written in the following form for a suitable choice of $B(x)$ and $C(x)$ [9],

$$x' = y - xB(x), \qquad y' = -C(x) - yB(x).$$

Choosing $W(x,y) = (x,y)$ one has

$$[V,W] \wedge W = xC(x) - xC'(x).$$

Applying corollary 1, one can re-prove the results of [9].

If (1) is a hamiltonian system,

$$x' = H_y, \qquad y' = -H_x,$$

choosing $W(x,y) = (x,y)$ gives:

$$[V,W] \wedge W = yH_y + xH_x - y^2 H_{yy} - 2xyH_{xy} - x^2 H_{xx}.$$

If H is analytic, we can write it as follows,

$$H(z) = \sum_0^\infty H_n(z),$$

where $H_n(z)$ is an n-degree homogeneous polynomial. Then we have, from the properties of homogeneous functions:

$$\begin{array}{l} [V,W] \wedge W = \\ \sum_0^\infty \left(yH_{ny} + xH_{nx} - y^2 H_{nyy} - 2xyH_{nxy} - x^2 H_{nxx}\right) = \\ \sum_0^\infty (2n - n^2) H_n. \end{array}$$

If H has an extremum at the origin O, such a formula shows that H_2 has no influence on the monotonicity of T, and that the lowest degree term $H_{\bar n}$, $\bar n > 2$, determines the monotonicity of T at O, if semi-definite in sign. In particular, if $H_{\bar n} \geq 0$, then T is decreasing in a neigbourhood of O. Moreover, a necessary condition for isochronicity is that $\min\{n > 2 : H_n \not\equiv 0\}$ be odd.

If we take $W(z) = z$ transversality only holds for orbits bounding a star-shaped region. The simplest choice of a globally transversal field consists in taking the orthogonal one, as in next corollary. Let us define a function Λ_{PQ} as follows,

$$\Lambda_{PQ} = -(P_y + Q_x)P^2 + 2(P_x - Q_y)PQ + (P_y + Q_x)Q^2$$

Corollary 2 *Let γ be a cycle of (1), contained in a period annulus A.*
(i) If $\Lambda_{PQ} \geq 0$ on γ, then T is increasing at γ; if there $\exists \bar z \in \gamma$ such that $\Lambda_{PQ}(\bar z) > 0$ at $\bar z$, then $\partial_W T > 0$ at γ.
(ii) If $\Lambda_{PQ} = 0$ on γ, then γ is a critical orbit.

Proof. Let us choose $W(z) = (-Q(z), P(z))$. Then

$$[V,W] \wedge W = -(P_y + Q_x)P^2 + 2(P_x - Q_y)PQ + (P_y + Q_x)Q^2 = \Lambda_{PQ}$$

and the statement is an immediate consequence of corollary 1. ♣

If P and Q are conjugate harmonic functions, then $\eta = 0$, so that the period annulus is isochronous. Point (i) gives another proof of a celebrated theorem (see **??**, §6, or [3], §6). Every center of such systems is isochronous, that is equivalent to say that every orbit is critical.

Point (ii) extends such a result to single orbits.

Choosing W as the orthogonal system gives a simple form also for ν's numerator,

$$[V,W] \wedge V = (Q_y - P_x)P^2 - 2(P_y + Q_x)PQ - (Q_y - P_x)Q^2.$$

If one considers Λ_{PQ} as a quadratic form in (P, Q), then Λ_{PQ} cannot be definite, because the coefficients of P^2 and Q^2 have opposite sign. On the other hand, there exist systems whose Λ_{PQ} has constant sign on some period annuli, or vanish on single orbits. An example can be constructed by considering systems of the type

$$x' = yD(x, y), \qquad y' = -xD(x, y).$$

Applying corollary 2, one has

$$\Lambda_{PQ} = -(y^2 + x^2)D(x, y)^2 \big(x\partial_x D(x, y) + y\partial_x D(x, y)\big).$$

Let us choose $D(x, y) = (x^2 + y^2)^2 - (x^2 + y^2) + 1$. Then $x\partial_x D(x, y) + y\partial_x D(x, y) = 2(x^2 + y^2)\big((x^2 + y^2) - 1\big)$ vanishes only at $(x^2 + y^2) - 1$, which is a critical orbit of the system. The period function is increasing for $(x^2 + y^2) < 1$, decreasing for $(x^2 + y^2) > 1$.

If (1) is a hamiltonian system, choosing W as the orthogonal system gives

$$\Lambda_H = [V, W] \wedge W = (H_{xx} - H_{yy})H_y^2 - 4H_{xy}H_xH_y + (H_{yy} - H_{xx})H_x^2.$$

One can see such a function as the sum of two quadratic forms,

$$\Lambda_H = \left(H_{xx}H_y^2 - 2H_{xy}H_xH_y + H_{yy}H_x^2\right) - \left(H_{yy}H_y^2 + 2H_{xy}H_xH_y + H_{xx}H_x^2\right).$$

In the special case of hamiltonian system with separable variables, $H(x, y) = G(x) + F(y)$,

$$x' = F'(y), \qquad y' = -G'(x). \tag{10}$$

We assume $G(x)$ and $F(y)$ to have isolated minima at 0.

If there exists a function $\psi \in C^1(\mathbb{R}, \mathbb{R})$ such that ψ' does not change sign, and

$$F'' = \psi(F'^2), \qquad G'' = \psi(G'^2), \tag{11}$$

then T is monotone. In fact, one has

$$-(P_y + Q_x)P^2 + 2(P_x - Q_y)PQ + (P_y + Q_x)Q^2 = (F'' - G'')(G'^2 - F^2) =$$
$$\big(\psi(F'^2) - \psi(G'^2)\big)(G'^2 - F'^2) = -\psi'\big(\xi(F'^2, G'^2)\big)(G'^2 - F'^2)^2,$$

where $\xi(F'^2, G'^2)$ is a suitable point in the interval (F'^2, G'^2) or in (G'^2, F'^2).

A simple example is given by $\psi(r) = \kappa \tan(r)$, $\kappa \in \mathbb{R}$, $\kappa \neq 0$, which gives the system

$$x' = \kappa \tan(y), \qquad y' = -\kappa \tan(x),$$

which has strictly decreasing period function. Similarly for $\psi(r) = \kappa \tanh(r)$, which gives a strictly increasing period function. In general, if $R(x)$ is an odd polynomial, then $R(\tan(x))$ and $R(\tanh(x))$ satisfy (11).

Systems of the type (10) were considered in [4], were a normalizer was given,

$$x' = \frac{G(x)}{G'(x)}, \qquad y' = \frac{F(y)}{F'(y)} \tag{12}$$

Such a normalizer is obviously defined where $G'(x)F'(y)$ does not vanish. In particular, if $G(x) = x^h + o(x^h)$, $F(y) = x^k + o(x^k)$, h, k even positive integers, then (12) is defined also at the origin. On the other hand, studying the period function on period annuli surrounding several critical points, it may happen that one of the above fractions diverges. This is the case of $H(x,y) = (x^2-1)^2 + y^2$, whose hamiltonian system

$$x' = 2y, \qquad y' = -4x(x^2-1),$$

has two centers at $(-1,0)$ and $(1,0)$, a saddle point at the origin, and a period annulus A surrounding the two central regions. The curve $(x^2-1)^2 + y^2 = 1$, consisting of a critical point and two homoclinics, is the inner boundary of A. The cycles contained in A are star-shaped with respect to the origin, since $(x,y) \wedge (2y, -4x(x^2-1)) = -4x^2(x^2-1) - y^2 < 0$ out of $(x^2-1)^2 + y^2 = 1$. One cannot apply theorem 1 in [4], since the system

$$x' = \frac{G(x)}{G'(x)} = \frac{x^2-1}{4x}, \qquad y' = \frac{F(y)}{F'(y)} = \frac{y}{2},$$

is not defined on the line $x = 0$. In this case one can apply theorem 1, choosing $W(x,y) = (x,y)$. Then one has

$$[V,W] \wedge W = -8x^4, \tag{13}$$

that gives the strict monotonicity of T.

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