The classification of algebraically closed alternative division rings of finite central dimension

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Abstract

A classical result of Noncommutative Algebra due to I. Niven, N. Jacobson and R. Baer asserts that an associative noncommutative division ring D has finite dimension over its center R and is algebraically closed (that is, every nonconstant polynomial in one indeterminate with left, or right, coefficients in D has a root in D) if and only if R is a real closed field and D is isomorphic to the ring of quaternions over R. In this paper, we extend this classification result to the nonassociative alternative case: the preceding assertion remains valid by replacing the quaternions with the octonions. As a consequence, we infer that a field k of characteristic $\neq 2$ is real closed if and only if the ring of octonions over k is an algebraically closed division ring.

1 Introduction and main theorems

The algebraically closed associative noncommutative division rings of finite vector dimension over their centers were classified in 1941 by I. Niven, N. Jacobson and R. Baer (see the Introduction of [7]): up to isomorphism, they are the rings of quaternions over real closed fields. In this paper, we extend this result to the nonassociative alternative case. We prove that, up to isomorphism, the algebraically closed nonassociative alternative division rings of finite vector dimension over their centers are the rings of octonions over real closed fields. It is somewhat mysterious why, until now, such a natural extension was never treated in the literature. In fact, the tools needed to prove the mentioned extension can be considered classic. We use Baer's argument contained in [7, p. 660] (see [6, p. 270] also), together with the strong version of the Zorn classification theorem due to R. H. Bruck and E. Kleinfeld (see [3]), to show that, if a nonassociative alternative division ring D has finite dimension over its center R and is algebraically closed, then R is a real closed field and D is isomorphic to the ring of octonions over R. In order to prove the converse, we adapt, to the case of octonions over a real closed field R, a division technique concerning the minimal polynomial of an octonion over R (see [1, Section II], [9, Section 3] and [7, p. 655]). The proof we obtain by such a division technique is simple, direct and works in the associative noncommutative case as well.

By the term "ring", we mean a nonempty set D equipped with two binary operations, called addition $D \times D \ni (a,b) \longmapsto a+b \in D$ and multiplication $D \times D \ni$

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$(a, b) \longmapsto ab \in D$, given in such a way that D is an abelian group with respect to addition and, for each $a, b, c \in D$, $(a+b)c = ac+bc$ and $a(b+c) = ab+ac$. We assume that D has the multiplicative identity 1 and 1 is different from the null element 0. Recall that such a ring is said to be commutative if $ab = ba$ for each $a, b \in D$ and associative if $(ab)c = a(bc)$ for each $a, b, c \in D$. We say that D is *noncommutative* if it is not commutative. Likewise, we say that D is *nonassociative* if it is not associative. The ring D is called *alternative* if, for each $a, b \in D$, $(aa)b = a(ab)$ and $b(aa) = (ba)a$. A subring of D is a subset of D, which turns out to be a ring with the operations induced by D. A theorem of E. Artin ([8, p. 29]) asserts that the subring generated by any pair of elements of an alternative ring is associative. The ring D is called *division ring* if, for each $a, b \in D$ with $a \neq 0$, there exist, and are unique, $c, d \in D$ (possibly equal) such that $ac = b$ and $da = b$. The *center* R of the ring D is defined as the set of all elements c of D such that $(ab)c = a(bc) = (ca)b$ (and hence $ac = ca$) for each $a, b \in D$. Observe that R is an associative commutative subring of D. If D is a division ring, then its center R is a field. In this case, we define the *central dimension of* D as the dimension of D, viewed as a vector space over R. We say that the ring D is *algebraically closed* if, for each positive integer n and for each $(n+1)$–uple $(a_0, a_1, \ldots, a_n)$ of elements of D with $a_n \neq 0$, both polynomials $\sum_{j=0}^n a_j X^j$ and $\sum_{j=0}^n X^j a_j$ have a root in D; that is, there exist $\alpha, \beta \in D$ (possibly equal) such that:

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0 = 0 = \beta^n a_n + \beta^{n-1}a_{n-1} + \ldots + \beta a_1 + a_0.$$

Likewise, we say that A is *centrally algebraically closed* if the same is true when all the coefficients a_i are choosen in the center of D.

Given a field k of characteristic $\neq 2$, we denote by $\mathbb{H}_k$ the ring of quaternions over k and by $\mathbb{O}_k$ the ring of octonions over k. We refer the reader to [8, 4] and [9, Section 2] for the construction (via the Cayley–Dickson process) and the main properties of these rings. It is worth recalling that $\mathbb{H}_k$ is an associative noncommutative ring with center k and central dimension 4, and $\mathbb{O}_k$ is a nonassociative alternative ring with center k and central dimension 8. The field k is said to be *real* if it admits a total ordering compatible with its ring operations. This is equivalent to say that, if a finite sum $\sum_{i=1}^n x_i^2$ of squares of k is null, then each x_i is null. The latter property implies that, if k is real, then it has characteristic 0 and both $\mathbb{H}_k$ and $\mathbb{O}_k$ are division rings. Finally, we recall that a field is said to be *real closed* if it is real and it has no proper real algebraic extensions. We refer the reader to Chapter 1 of [2] for the basic properties of these fields.

Our main result is as follows (see also Remark 2.2 at the end of the paper).

Theorem 1.1 *Let D be a nonassociative alternative division ring. The following assertions are equivalent:*

(1) *D has finite central dimension and is algebraically closed.*

(2) *D has finite central dimension and is centrally algebraically closed.*

(3) *The center R of D is a real closed field and D is isomorphic to $\mathbb{O}_R$.*

In particular, we infer:

Theorem 1.2 *Up to isomorphism, the algebraically closed alternative division rings of finite central dimension are either the algebraically closed fields or the rings of quaternions over real closed fields or the rings of octonions over real closed fields.*

Another immediate consequence of Theorem 1.1 is the following characterization of real closed fields: the new equivalence is (1) $\Longleftrightarrow$ (4).

Corollary 1.3 *Let k be a field of characteristic $\neq 2$. The following assertions are equivalent:*

(1) *k is real closed.*

(2) *$k[\sqrt{-1}\,] = k[X]/(X^2+1)$ is algebraically closed.*

(3) *$\mathbb{H}_k$ is an algebraically closed division ring.*

(4) *$\mathbb{O}_k$ is an algebraically closed division ring.*

The reader observes that Corollary 1.3 remains valid by replacing "algebraically closed" with "centrally algebraically closed" in points (3) and (4).

2 Proofs

Let R be a real closed field. Indicate by $\mathbb{O}_R[X]$ the ring of all polynomials with left coefficients belonging to $\mathbb{O}_R$ and by $R[X]$ the subring of $\mathbb{O}_R[X]$ consisting of all polynomials with (left) coefficients in R. Recall that the addition in $\mathbb{O}_R[X]$ is the usual one. On the contrary, the multiplication "$*$" is defined as follows. Let $P(X) = \sum_{j=0}^{n} a_j X^j$ and $Q(X) = \sum_{k=0}^{m} b_k X^k$ be polynomials in $\mathbb{O}_R[X]$, and, for each $\ell \in \{0,1,\ldots,n+m\}$, let $A_{n,m}(\ell)$ be the set of pairs $(j,k) \in \{0,1,\ldots,n\} \times \{0,1,\ldots,m\}$ such that $j+k=\ell$. We have:

$$(P*Q)(X) := \sum_{\ell=0}^{n+m} \big(\sum_{(j,k)\in A_{n,m}(\ell)} a_j b_k\big) X^\ell.$$

If $Q \in R[X]$, then we denote $P*Q$ also by PQ. In fact, in this case, $(P*Q)(\beta) = P(\beta)Q(\beta)$ for each $\beta \in \mathbb{O}_R$. If either $P \in R[X]$ or $Q \in R[X]$, then $P*Q$ is equal to $Q*P$. Let $e_0 := 1$, $e_1 := i$, $e_2 := j$, $e_3 := e_1e_2$, $e_4 := k$, $e_5 := e_1e_4$, $e_6 := e_2e_4$ and $e_7 := e_3e_4$ be the elements of the usual basis of $\mathbb{O}_R$, viewed as a vector space over R. Let $\alpha = \sum_{j=0}^{7} \alpha_j e_j \in \mathbb{O}_R$ with $\alpha_0, \alpha_1, \ldots, \alpha_7 \in R$. Define: $\bar{\alpha} := \alpha_0 e_0 - \sum_{j=1}^{7} \alpha_j e_j$, $\bar{P} \in \mathbb{O}_R[X]$ by $\bar{P}(X) := \sum_{j=0}^{n} \bar{a}_j X^j$, $N(P) \in R[X]$ by $N(P) := P * \bar{P} = \bar{P} * P$, $t_\alpha := \alpha + \bar{\alpha} = 2\alpha_0 \in R$, $n_\alpha := \alpha\bar{\alpha} = \bar{\alpha}\alpha = \sum_{j=0}^{7} \alpha_j^2 \in R$ and $\Delta_\alpha(X) := N(X-\alpha) = X^2 - t_\alpha X + n_\alpha \in R[X]$. It is well–known that the conjugacy class $\mathbb{S}_\alpha$ of α is ugual to the set $\{\beta \in \mathbb{O}_R \,|\, t_\beta = t_\alpha,\ n_\beta = n_\alpha\}$, which coincides with the zero set of Δ_α (see [9]).

We need the following technical lemma.

Lemma 2.1 *Let $P(X) = \sum_{j=0}^{n} a_j X^j$ be a polynomial in $\mathbb{O}_R[X]$ and let $\alpha \in \mathbb{O}_R$. The following statements hold.*

(1) *If $\alpha \in R$, then $N(P)(\alpha) = n_{P(\alpha)}$.*

(2) *Suppose $\alpha \notin R$. Then there exist $a, b \in \mathbb{O}_R$ such that*

$$P(\beta) = a\beta + b \quad \textit{and} \quad N(P)(\beta) = (\bar{a}b + \bar{b}a + n_a t_\alpha)\beta + n_b - n_a n_\alpha$$

for each $\beta \in \mathbb{S}_\alpha$.

Proof. (1) Let $\alpha \in R$. Since α belongs to the center R of $\mathbb{O}_R$, we infer that $\bar{\alpha} = \alpha$ and

$$\begin{aligned} n_{P(\alpha)} &= P(\alpha)\overline{P(\alpha)} = \big(\textstyle\sum_{j=0}^{n} a_j \alpha^j\big)\big(\sum_{k=0}^{n} \bar{a}_k \alpha^k\big) = \\ &= \textstyle\sum_{\ell=0}^{2n} \big(\sum_{(j,k)\in A_{n,n}(\ell)} a_j \bar{a}_k\big)\alpha^\ell = N(P)(\alpha). \end{aligned}$$

(2) Let us prove by induction on the degree n of P (the degree of the null polynomial is considered to be equal to 0) that there exist $a, b \in \mathbb{O}_R$ and $H \in \mathbb{O}_R[X]$ such that $P(X) = H(X)\Delta_\alpha(X) + aX + b$. If $n \in \{0, 1\}$, then the assertion is evident. Suppose $n \geq 2$. Define the polynomial $Z \in \mathbb{O}_R[X]$ by $Z(X) := (a_n t_\alpha)X^{n-1} - (a_n n_\alpha)X^{n-2} + \sum_{j=0}^{n-1} a_j X^j$. It holds: $P(X) = (a_n X^{n-2})(\Delta_\alpha(X) + t_\alpha X - n_\alpha) + \sum_{j=0}^{n-1} a_j X^j = (a_n X^{n-2})\Delta_\alpha(X) + Z(X)$. Since the degree of Z is $\leq n-1$, by induction, there exist $a, b \in \mathbb{O}_R$ and $K \in \mathbb{O}_R[X]$ such that $Z(X) = K(X)\Delta_\alpha(X) + aX + b$ and hence $P(X) = (a_n X^{n-2} + K(X))\Delta_\alpha(X) + aX + b$ as desired. In this way, we can write: $P(X) = H(X)\Delta_\alpha(X) + aX + b$ for some $a, b \in \mathbb{O}_R$ and $H \in \mathbb{O}_R[X]$. It follows immediately that $P(\beta) = a\beta + b$ for each $\beta \in \mathbb{S}_\alpha$. Moreover, for a suitable polynomial Y in $\mathbb{O}_R[X]$, we have that

$$\begin{aligned} N(P)(X) &= \big(\bar{H}(X)\Delta_\alpha(X) + \bar{a}X + \bar{b}\big) * \big(H(X)\Delta_\alpha(X) + aX + b\big) = \\ &= Y(X)\Delta_\alpha(X) + (\bar{a}X + \bar{b}) * (aX + b) = \\ &= Y(X)\Delta_\alpha(X) + n_a X^2 + (\bar{a}b + \bar{b}a)X + n_b = \\ &= Y(X)\Delta_\alpha(X) + n_a(\Delta_\alpha(X) + t_\alpha X - n_\alpha) + (\bar{a}b + \bar{b}a)X + n_b = \\ &= \big(Y(X) + n_a\big)\Delta_\alpha(X) + \big(\bar{a}b + \bar{b}a + n_a t_\alpha\big)X + n_b - n_a n_\alpha \end{aligned}$$

and hence $N(P)(\beta) = \big(\bar{a}b + \bar{b}a + n_a t_\alpha\big)\beta + n_b - n_a n_\alpha$ for each $\beta \in \mathbb{S}_\alpha$. $\square$

Proof of Theorem 1.1. (1) $\Longrightarrow$ (2) This implication is evident.

(2) $\Longrightarrow$ (3) The original argument of R. Baer applies in this context (see [6, p. 270] and [7, pp. 660–661]). It gives that R is a real closed field. In particular, the characteristic of R (and hence of D) is zero. By Theorem A of [3], D is isomorphic to $\mathbb{O}_R$.

(3) $\Longrightarrow$ (1) Let P be a nonconstant polynomial in $\mathbb{O}_R[X]$. Consider the simple algebraic extension $C := R(e_1)$ of R, viewed as a subset of $\mathbb{O}_R$. It is well–known that C is the algebraic closure of R (see Theorem 1.2.2 of [2]). In this way, there exists $\alpha \in C$ such that $N(P)(\alpha) = 0$. If $\alpha \in R$, then Lemma 2.1(1) implies that $n_{P(\alpha)} = 0$. Since R is a real field, it follows that $P(\alpha) = 0$. Suppose $\alpha \notin R$. By Lemma 2.1(2), there exist $a, b \in \mathbb{O}_R$ with $P(\beta) = a\beta + b$ for each $\beta \in \mathbb{S}_\alpha$ and $(\bar{a}b + \bar{b}a + n_a t_\alpha)\alpha + n_b - n_a n_\alpha = N(P)(\alpha) = 0$. On the other hand, $\bar{a}b + \bar{b}a + n_a t_\alpha$ and $n_b - n_a n_\alpha$ belong to R so we infer that

$$\bar{a}b + \bar{b}a + n_a t_\alpha = 0 = n_b - n_a n_\alpha. \tag{1}$$

If $a = 0$, then $n_b = 0$. It follows that $b = 0$ and $P(\beta) = 0$ for each $\beta \in \mathbb{S}_\alpha$. Suppose $a \neq 0$. Define $\beta := -a^{-1}b$. Observe that $n_a n_\beta = n_a((n_a)^{-1}n_b) = n_b$ and $-n_a t_\beta = n_a(a^{-1}b + \bar{b}\bar{a}^{-1}) = (n_a a^{-1})b + \bar{b}(\bar{a}^{-1}n_a) = \bar{a}b + \bar{b}a$. Equations (1) imply that $n_\beta = n_\alpha$ and $t_\beta = t_\alpha$; that is, $\beta \in \mathbb{S}_\alpha$. We infer that $P(\beta) = a\beta + b = 0$. We have just proved that any nonconstant polynomial in one indeterminate with left coefficients in $\mathbb{O}_R$ has a root in $\mathbb{O}_R$. It remains to show that the same is true replacing "left" with "right". This is very simple. Let $\mathcal{Q}(X) := \sum_{j=0}^{d} X^j c_j$ be a nonconstant polynomial with right coefficients in $\mathbb{O}_R$ and let $\alpha \in \mathbb{O}_R$ be a root of the polynomial $Q(X) := \sum_{j=0}^{d} \bar{c}_j X^j$. Then $\bar{\alpha}$ is a root of $\mathcal{Q}$: $\mathcal{Q}(\bar{\alpha}) = \overline{Q(\alpha)} = 0$. $\square$

Remark 2.2 Replacing "nonassociative alternative" with "associative noncommutative" and "$\mathbb{O}_R$" with "$\mathbb{H}_R$" in the statement of Theorem 1.1, we obtain exactly the statement of the main result of [7]. A simplification of the original proof of that main result was given in [6]. More precisely, Theorem 16.15 of [6] contains Baer's argument we used above to show implication (2) $\Longrightarrow$ (3). In Theorem 16.14 of the same book, it is proved that, given a real closed field R, the ring $\mathbb{H}_R$ is algebraically closed. This proof fails in the nonassociative case. In fact, it uses the following result (see [6, Proposition 16.3]): "*Let* $g, h \in \mathbb{H}_R[X]$ *and let* $x \in \mathbb{H}_R$ *such that* $a := h(x) \neq 0$. *Then* $(g * h)(x) = g(axa^{-1}) \cdot a$". However, if g is the constant polynomial e_1 in $\mathbb{O}_R[X]$, $h \in \mathbb{O}_R[X]$ is defined by $h(X) := e_2X - e_4$ and $x := e_6$, then $a := h(x) = -2e_4$ and $(g*h)(x) = 0 \neq -2e_5 = g(axa^{-1}) \cdot a$. For further results regarding this "nonassociative phenomenon", we refer the reader to Section 3 of [5].

References

[1] B. Beck, Sur les équations polynomiales dans les quaternions, Enseign. Math. (2) 25 (1979), no. 3–4, 193–201.

[2] J. Bochnak, M. Coste, M.–F. Roy, Real algebraic geometry, Springer–Verlag, Berlin, 1998.

[3] R. H. Bruck, E. Kleinfeld, The structure of alternative division rings, Proc. Amer. Math. Soc. 2 (1951), no. 6, 878–890.

[4] H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel, R. Remmert, Numbers, Springer–Verlag, New York, 1991.

[5] R. Ghiloni, A. Perotti, Zeros of regular functions of quaternionic and octonionic variable: a division lemma and the camshaft effect, preprint arXiv:0904.2667.

[6] T.Y. Lam, A first course in noncommutative rings, Springer–Verlag, New York, 1991.

[7] I. Niven, Equations in quaternions. Amer. Math. Monthly 48, (1941), 654–661.

[8] R. D. Schafer, An introduction to nonassociative algebras, Academic Press, New York–London, 1966.

[9] R. Serôdio, On octonionic polynomials, Adv. Appl. Clifford Algebr. 17 (2007), no. 2, 245–258.