On the weak completeness of a fragment of linear temporal logic

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Abstract

The main contribution of this work is an expanded and detailed version of a rather sketchy proof, which first appeared in Finger and Gabbay (1992, *J. Logic Lang. Inf.*, **1**, 203–233), of a weak completeness theorem for the until-free fragment LTL_ of linear temporal logic. More precisely we show that LTL_ is determined by the linearly ordered frame of the natural numbers. As a minor contribution, we also show that, under an ad hoc semantics and with a significant restriction on the admissible valuation maps, LTL_ can be regarded as a logic for the discrete first quadrant.

Keywords: Linear temporal logic, weak completeness.

1 Introduction

We deal with the until-free fragment of linear temporal logic (with future operators only). We denote such fragment by LTL_ and the entire logic by LTL. The latter has been successfully used in many applications, including software and hardware verification, protocol design and autonomous systems. In this note we are not concerned with the applications, but rather with the weak completeness of the fragment LTL_. We point out that the proof that we present below extends to LTL.

We assume that the reader is familiar with the syntax and the semantics of LTL. Nevertheless, for sake of a self-contained presentation and for setting up terminology and notation, in Section 2 we recall some basic facts about LTL. We refer the reader, for instance, to [3] for a full account on modal and temporal logics.

Sections 3 and 4 contain the main contribution of this work, namely, a self-contained and fully detailed proof of the weak completeness of LTL_. By the latter, we mean the equivalence of provability and logical consequence when a finite set of assumptions is involved. More precisely, we show that LTL_ is determined by the frame $(\mathbb{N}, <)$, where $<$ is the standard ordering on the set $\mathbb{N}$ of natural numbers. We closely follow the content of [1, section 3], where a very streamlined proof sketch of weak completeness can be found. Indeed, in the latter reference, even the proofs of the major results are just stated, with basically no details.

Detailed proofs of the weak completeness of LTL already appear in the literature (see, for instance, [3, chapter 9]). Nevertheless, we were not able to find any expanded version of [1, section 3]. This is rather surprising for what is the first proof of a seminal and highly cited result.

In Section 5 we show that, under suitable assumptions, the LTL_ axioms can be regarded as a logic for the frame formed by the first discrete quadrant $\mathbb{N} \times \mathbb{N}$, equipped with a very natural accessibility relation, here denoted by $\lhd$. For such system we still use the acronym LTL_, as there will be no ambiguity. Since $(\mathbb{N} \times \mathbb{N}, \lhd)$ is not linearly ordered, we need to suitably define the semantics of the 'next time' operator and to impose a significant constraint on the family of the admissible valuation maps. We show that the novel LTL_ is determined by (namely, it is weakly complete with respect

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to) the frame $(\mathbb{N} \times \mathbb{N}, \lhd)$. The ad hoc definition of the semantics and the constraint on the valuation maps allow us to use the previously established weak completeness result for the original LTL_. While this result does not have an obvious practical application, it is related to an apparently less difficult problem: to find whether there is some purely modal logic that is determined by the frame $(\mathbb{N} \times \mathbb{N}, \lhd)$. To the best of our knowledge, the latter is still an open problem.

2 Preliminaries

As already anticipated, we assume familiarity with the syntax and the semantics of LTL and, more in general, with the basic notions and techniques from modal logic. Some of them are recalled in the following. We also assume familiarity with [1, section 3].

For the reader's convenience, we anticipate that, in Section 3, we detail the sketch of the counter-model construction that appears in the left column of [1, p. 172]. In Section 4, we detail the proofs of the unproven statements that are stated in the right column of [1, p. 172].

As a general rule, the proofs that are standard or routine are omitted.

In this section we set up our notation and we point out some notational differences with [1].

We briefly recall that LTL_ is a propositional normal modal logic that is equipped with an additional operator $\bigcirc$ (next-time). As for the modal component of LTL_ , we denote the 'universal future' (or 'globally future') operator by $\Box$. The operator $\Diamond$ (existential future) abbreviates $\neg\Box\neg$.

We use greek letters to denote the LTL_-formulas. They are obtained by closing a countably infinite set V of propositional variables (the atomic formulas) under application of the connectives of classical propositional logic and under the following formation rules: whenever α is a formula so are $\Box\alpha$ and $\bigcirc\alpha$.

Regarding the LTL_-semantics, in the following text we fix the frame $(\mathbb{N}, <)$, where $<$ is the standard ordering on the set of natural numbers. For the reader's convenience, we next specialize the notions of truth and validity to that frame. In order to assign a truth value to all formulas at each point of the frame $(\mathbb{N}, <)$, we fix a map $v : \mathbb{N} \to P(V)$. We call such v a valuation. We write $v \models_n \alpha$ when a formula α is true at n under v. We define recursively $v \models_n \alpha$ as follows:

- if p is a propositional variable then $v \models_n p \ \Leftrightarrow \ p \in v(n)$;
- if α is of the form $\beta \star \gamma$, where $\star$ is a binary propositional connective, or is of the form $\neg\beta$ then $v \models_n \alpha$ is defined according to the semantics of the corresponding propositional connective;
- if α is of the form $\Box\beta$ then $v \models_n \alpha \Leftrightarrow$ for all $n < k$, $v \models_k \beta$;
- if α is of the form $\bigcirc\beta$ then $v \models_n \alpha \Leftrightarrow v \models_{n+1} \beta$.

We extend the relation $\models_n$ to a set Γ of formulas as usual: $v \models_n \Gamma$ if and only if $v \models_n \gamma$, for all $\gamma \in \Gamma$. We write $v \models \Gamma$ if, for all $n \in \mathbb{N}$, $v \models_n \Gamma$.

The notion of logical consequence is defined as usual. Namely, for each set $\Gamma \cup \{\varphi\}$ of formulas we say that φ is a logical consequence of Γ (notation: $\Gamma \models \varphi$) if

$$\text{for all } v \ (v \models \Gamma \ \Rightarrow \ v \models \varphi).$$

Two formulas α and β are logically equivalent (notation: $\alpha \approx \beta$) if $\alpha \models \beta$ and $\beta \models \alpha$.

We adopt the axiomatization of LTL_ in [1], but we replace the rule R1 in [1] with the axiom schema A0 below. In detail, the LTL_ axioms are as follows:

A0 the closure under substitution of each propositional tautology
A1 $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
A2 $\bigcirc(\neg\varphi) \leftrightarrow \neg \bigcirc \varphi$

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A3 $\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$
A4 $\Box\varphi \to (\bigcirc\varphi \wedge \bigcirc\Box\varphi)$
A5 $\Box(\varphi \to \bigcirc\varphi) \to (\bigcirc\varphi \to \Box\varphi)$

The LTL_ rules are as follows:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \ \text{(MP)} \qquad \frac{\varphi}{\Box\varphi} \ \text{(N)}$$

The applications of the N-rule must obey the constraint that the proof of φ does not depend on any extralogical assumption. The notion of derivation, in Hilbert style, and the provability relation $\vdash$ are standardly defined. More in detail, a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas is a derivation of formula φ_n from a set Γ of formulas if, for each $1 \leq i \leq n$, one of the following holds:

1. φ_i is an axiom;
2. $\varphi_i \in \Gamma$;
3. there exist $j, k < i$ such that φ_j is $\varphi_k \to \varphi_i$;
4. φ_i is of the form $\Box\varphi_j$ for some $j < i$ and, for all $k \leq j$, if φ_k belongs to Γ then it is some logical axiom.

A set Γ of formulas is consistent if there is some formula α such that $\Gamma \nvdash \alpha$.

Under the definition of provability above, it is straightforward to establish the following:

PROPOSITION 2.1 (Deduction Theorem).
For each set $\Gamma \cup \{\varphi, \psi\}$ of formulas we have that

$$\Gamma, \varphi \vdash \psi \ \Leftrightarrow \ \Gamma \vdash \varphi \to \psi.$$

PROOF (Sketch). The left-to-right implication above is proved by induction on derivations. As for the case relative to the necessitation rule, we assume that ψ is $\Box\eta$ and $\Gamma, \varphi \vdash \eta$. By arguing on the subderivation proving η from Γ, φ and by recalling the constraint in 4 above, we get $\vdash \eta$. Hence $\vdash \psi$. We conclude by using the axiom $\psi \to (\varphi \to \psi)$ and by applying MP. □

Proposition 2.1 above is useful in presence of extralogical assumptions.

With a routine proof by induction on the length of a derivation, and by recalling the constraint on the application of the rule (N), we easily get the following.

PROPOSITION 2.2 (Soundness Theorem).
For every set $\Gamma \cup \{\varphi\}$ of formulas,

$$\Gamma \vdash \varphi \ \Rightarrow \ \Gamma \models \varphi.$$

It is well known that the converse implication of the Soundness Theorem (Strong Completeness) does not hold in general for LTL_. Here is a counterexample: let p be a propositional variable and let $\Gamma = \{\bigcirc p, \bigcirc\bigcirc p, \ldots, \bigcirc^n p, \ldots\}$. Then $\Gamma \models \Box p$ (recall that the logical consequence relation is restricted to the frame of natural numbers). On the other hand, there is no $0 < n \in \mathbb{N}$ such that $\{\bigcirc p, \bigcirc\bigcirc p, \ldots, \bigcirc^n p\} \vdash \Box p$. Hence, by the finitary nature of the provability relation, $\Gamma \nvdash \Box p$.

The converse implication of the Soundness Theorem holds when $\Gamma = \emptyset$ (equivalently: when Γ is finite, by Proposition 2.1). As we stated above, in the first part of this note we provide the details of the original proof in [1] of the latter statement (we will refer to it as the Weak Completeness theorem). More precisely, we prove that LTL_ is weakly complete with respect to the family of models that are based on the frame $(\mathbb{N}, <)$.

First of all we need to recall some LTL_ theorems that are relevant for the rest of this work.

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REMARK 1.
The following are LTL_ (meta)theorems that we will repeatedly use. For sake of completeness, we provide commented sketches of their proofs. The missing or uncommented proof steps are routine.

1a. $\vdash \triangle(\varphi \wedge \psi) \leftrightarrow \triangle\varphi \wedge \triangle\psi$, with $\triangle \in \{\bigcirc, \Box\}$.
The left-to-right implication is straightforward. The proof of the converse implication uses the instance $\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$ of A0 and A1 or A3.

1b. $\vdash \varphi \rightarrow \bigcirc\varphi \;\Rightarrow\; \bigcirc\varphi \rightarrow \Box\varphi$.

$\vdash \varphi \rightarrow \bigcirc\varphi$	
$\vdash \Box(\varphi \rightarrow \bigcirc\varphi)$	(N)
$\vdash \Box(\varphi \rightarrow \bigcirc\varphi) \rightarrow (\bigcirc\varphi \rightarrow \Box\varphi)$	(A5)
$\vdash \bigcirc\varphi \rightarrow \Box\varphi$	(MP)

1c. $\vdash \Box\varphi \rightarrow \Box\Box\varphi$.

$\vdash \Box\varphi \rightarrow \bigcirc\Box\varphi$	(A4)
$\vdash \Box(\Box\varphi \rightarrow \bigcirc\Box\varphi)$	(N)
$\vdash \Box(\Box\varphi \rightarrow \bigcirc\Box\varphi) \rightarrow (\bigcirc\Box\varphi \rightarrow \Box\Box\varphi)$	(A5)
$\vdash \bigcirc\Box\varphi \rightarrow \Box\Box\varphi$	(MP)
$\vdash (\Box\varphi \rightarrow \bigcirc\Box\varphi) \rightarrow (\bigcirc\Box\varphi \rightarrow \Box\Box\varphi) \rightarrow (\Box\varphi \rightarrow \Box\Box\varphi)$	(A0)
$\vdash \Box\varphi \rightarrow \Box\Box\varphi$	(MP, twice)

Hence every frame in which the LTL_ axioms are valid must be equipped with a transitive accessibility relation.

1d. $\vdash (\bigcirc\varphi \wedge \bigcirc\Box\varphi) \rightarrow \Box\varphi$.

$\vdash \bigcirc\varphi \wedge \bigcirc\Box\varphi \rightarrow \bigcirc(\bigcirc\varphi \wedge \bigcirc\Box\varphi)$	(A4 and A3)
$\vdash \bigcirc(\bigcirc\varphi \wedge \bigcirc\Box\varphi) \rightarrow \Box(\bigcirc\varphi \wedge \bigcirc\Box\varphi)$	(1b)
$\vdash \bigcirc\varphi \wedge \bigcirc\Box\varphi \rightarrow \Box(\bigcirc\varphi \wedge \bigcirc\Box\varphi)$	
$\vdash \bigcirc\varphi \wedge \bigcirc\Box\varphi \rightarrow \Box \bigcirc \varphi$	(1a)
$\vdash \Box \bigcirc \varphi \rightarrow \Box(\varphi \rightarrow \bigcirc\varphi)$	(A0 and N)
$\vdash \bigcirc\varphi \wedge \bigcirc\Box\varphi \rightarrow \Box(\varphi \rightarrow \bigcirc\varphi)$	
$\vdash \Box(\varphi \rightarrow \bigcirc\varphi) \wedge \bigcirc\varphi \rightarrow \Box\varphi$	(A5)
$\vdash \bigcirc\varphi \wedge \bigcirc\Box\varphi \rightarrow \Box\varphi$	

We will refer to the proof in Remark 1c, by saying 'by transitivity', as the schema in 1c captures the class of transitive frames.

3 Towards the proof of weak completeness

In this section we begin the proof of weak completeness of LTL_ . As already anticipated, we will follow the structure of the proof sketched in [1, section3]. The various proof steps and the motivations leading to the construction that we detail in this section (with special reference to the definitions of the set $\bar{S}$ and of the relations ρ_+ and $\rho_<$ on $\bar{S}$ below) can be found in the left column of [1, p. 172]. Here we just point out that the authors of [1] needed to modify the construction of the so-called canonical model of a modal logic, in order to take into account the presence of the operator $\bigcirc$ and, at the same time, to construct a model that is eventually periodic. In the weak completeness proof, two properties are crucial: $\rho_<$ is the transitive closure of ρ_+ and the model is eventually periodic. The

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former property is proved in Proposition 4.1, with the help of remarks 3 and 4. The latter property is exploited in the $\Box$-case of the proof of Theorem 4.2.

We also point out the following:

Fact 1. For each formula β there is a finite (up to logical equivalence) set Γ_β of formulas containing β and satisfying the following:

c1 if $\Box\alpha \in \Gamma_\beta$ then $\bigcirc\alpha \in \Gamma_\beta$ and $\bigcirc\Box\alpha \in \Gamma_\beta$;
c2 if $\bigcirc\alpha \in \Gamma_\beta$ then $\bigcirc\neg\alpha \in \Gamma_\beta$;
c3 Γ_β is closed under subformulas and under application of the propositional connectives.

PROOF. By induction on β, recalling that a Boolean algebra with n generators has cardinality $\leq 2^{2^n}$. The Boolean algebra involved here is a subalgebra of the Lindenbaum algebra associated with the logical equivalence relation $\approx$ on the set of formulas. More formally, we deal with $\approx$-equivalence classes of formulas, rather than with formulas, and we pick a representative in each equivalence class. Keeping this in mind, the proof sketch below should be sufficiently clear.

The case of an atomic β is is trivial. If β is of the form $\Box\eta$, by inductive assumption there is some finite set Γ_η satisfying c1–c3 above. Any set Γ_β of representatives, one for each element of the Boolean algebra generated by the $\approx$-equivalence classes of the formulas in the finite set $\{\beta, \bigcirc\eta, \bigcirc\beta\} \cup \Gamma_\eta$, satisfies the required properties.

The other cases are similar. $\Box$

Notice also that, in [1], the property corresponding to c1 above reads

if $\Diamond\alpha \in \Gamma$ then $\bigcirc\alpha \in \Gamma$ and $\bigcirc\Diamond\alpha \in \Gamma$.

It is easy to verify that the latter is equivalent to c1, under the assumptions that c2 and c3 hold for Γ.

Convention. In the following text we will just say that Γ_β as in Fact 1 is finite.

In order to motivate the construction Γ_β, we anticipate that its finiteness is a necessary requirement in the proof of Theorem 4.2 (from which the weak completeness of LTL_ follows easily). Indeed, recall that right after Proposition 2.2 we showed that LTL_ is not strongly complete. On the other hand, if Theorem 4.2 were true even when we replace a single formula with an infinite consistent set of formulas, we would get the strong completeness of LTL_. In addition to the finiteness requirement, the closure properties c1–c3 above will make possible to proceed by induction on formulas in the proof of Theorem 4.2.

Let S be the family of maximal consistent sets of formulas. We define the relation $<$ on S as follows:

$$\Delta < \Theta \;\Leftrightarrow\; \text{for all } \alpha\ (\Box\alpha \in \Delta \Rightarrow \alpha \in \Theta).$$

By transitivity (Remark 1.c), the relation $<$ is transitive on S.

If $\Delta \in S$, we let $\Delta^+ = \{\alpha : \bigcirc\alpha \in \Delta\}$. It is easy to see that $\Delta^+ \in S$. Maximality of Δ^+ is a consequence of the axiom schema A2. Furthermore, we have $\Delta < \Delta^+$, by the axiom schema A4.

REMARK 2.
Let $\Delta \in S$ be such that $\Box\beta \notin \Delta$, for some formula β. Then there exists some $\Delta < \Theta$ such that $\neg\beta \in \Theta$. It suffices to prove that the set $\Theta_0 = \{\alpha : \Box\alpha \in \Delta\} \cup \{\neg\beta\}$ is consistent. We argue by contraposition. If Θ_0 is inconsistent, there are $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n$ in Θ_0 such that $\alpha_1, \ldots, \alpha_n \vdash \beta$.

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Hence

$$\vdash \alpha_1 \to (\alpha_2 \to (\cdots \to (\alpha_n \to \beta)\cdots)).$$

By repeated applications of the N-rule and of axiom A1 we get

$$\vdash \Box\alpha_1 \to (\Box\alpha_2 \to (\cdots \to (\Box\alpha_n \to \Box\beta)\cdots)).$$

As all the $\Box\alpha_i$'s are in Δ, by the closure properties of Δ we get $\Box\beta \in \Delta$, contradicting to the initial assumption on $\Box\beta$. Hence Θ_0 is consistent. Any maximal consistent extension Θ of Θ_0 satisfies the required properties.

From now on, β is a fixed consistent formula. By a routine application of Zorn's Lemma there exists some $\Delta_0 \in S$ such that $\beta \in \Delta_0$. We fix a set Γ_β as in Fact 1 and, for notational simplicity, we omit the subscript β. The set $\bar{S} = \{\Delta \cap \Gamma : \Delta \in S\}$ is finite. The elements of $\bar{S}$ will be denoted by $s, t, u \ldots$, possibly indexed.

On $\bar{S}$ we define two relations as follows:

$$s\,\rho_+\,t \Leftrightarrow \text{ there is some } \Delta \in S \text{ such that } s = \Delta \cap \Gamma \text{ and } t = \Delta^+ \cap \Gamma;$$

$$s\,\rho_<\,t \Leftrightarrow \text{ there are some } \Delta < \Theta \text{ in } S \text{ such that } s = \Delta \cap \Gamma \text{ and } t = \Theta \cap \Gamma.$$

The transitivity of $<$ implies that of $\rho_<$. Moreover, from $\Delta < \Delta^+$ it follows that $s\,\rho_+\,t$ implies $s\,\rho_<\,t$, for all $s, t \in \bar{S}$. Hence the transitive closure ρ_+^* of ρ_+ is contained in $\rho_<$.

We consider the pair $(\bar{S}, \rho_+)$. We say that $t \in \bar{S}$ is an immediate successor of $s \in \bar{S}$ if $s\,\rho_+\,t$. Notice that some $s \in \bar{S}$ may have more that one immediate successor, as there might be Θ and Σ in S such that $\Theta \cap \Gamma = s = \Sigma \cap \Gamma$ and $\Theta^+ \cap \Gamma \neq \Sigma^+ \cap \Gamma$.

We recall that Δ_0 is an element of S with the property that $\beta \in \Delta_0$. We let $s_0 = \Delta_0 \cap \Gamma$. Hence $\beta \in s_0$.

We define a sequence $\sigma = (s_n)_{n\in\mathbb{N}}$ of elements of $\bar{S}$, with s_0 as above, such that $s_n\,\rho_+\,s_{n+1}$, for all $n \in \mathbb{N}$. Furthermore we require that, for each $n \in \mathbb{N}$, if s_n occurs infinitely many times in σ, then the same property holds for all the t's in $\bar{S}$ such that $s_n\,\rho_+\,t$.

The latter property can be fulfilled as follows. For each $s \in \bar{S}$, we cyclically order the (finite and nonempty) set $\rho_+[s] = \{t \in \bar{S} : s\,\rho_+\,t\}$. For each $\rho_+[s]$ we set a pointer to some element of $\rho_+[s]$. We use the pointers in the following way: if at stage n of the construction of σ we have $s_n = s$, we let s_{n+1} be the element of $\rho_+[s]$ that is marked by the pointer and we update the pointer so that it points at the next element in the cyclic order of $\rho_+[s]$.

By finiteness of $\bar{S}$ and by construction, there is exactly one strongly connected component C of $(\bar{S}, \rho_+)$ such that the sequence σ is eventually inside C. In particular, each element of C occurs infinitely many times in σ and σ is ultimately periodic.

4 Weak completeness

The assumptions and notation of Section 3 are in force.

A crucial step in proving weak completeness is to show that the relation $\rho_<$ is the transitive closure ρ_+^* of ρ_+. This is the content of the unproven [1, lemma 2], whose proof we provide below, after some preliminary observations.

REMARK 3.
Let $\Delta, \Sigma \in S$ be such that $\Delta \cap \Gamma \subseteq \Sigma \cap \Gamma$. Then $\Delta \cap \Gamma = \Sigma \cap \Gamma$. For, suppose not and let φ be witnessing proper inclusion. Then $\neg\varphi \in \Sigma \cap \Gamma$, by maximal consistency of Σ and by the closure

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properties of Γ (see Fact 1), contradicting the consistency of Σ. It follows that, if $s \in \bar{S}$, the formula $\varphi_s \equiv \bigwedge s$ has the property that

$$\text{for all } \Delta \in S \ (\Delta \cap \Gamma = s \Leftrightarrow \varphi_s \in \Delta).$$

Furthermore, for each $\emptyset \neq T \subseteq \bar{S}$, the formula $\varphi_T \equiv \bigvee\{\varphi_s : s \in T\}$ has the property that, for all $\Theta \in S$, $\Theta \cap \Gamma \in T$ if and only if $\varphi_T \in \Theta$.

REMARK 4.
Let $\Delta \in S$ and let $T_\Delta = \{t \in \bar{S} : \Delta \cap \Gamma \ \rho_+^* \ t\}$. Let φ_Δ be the formula relative to T_Δ obtained as in Remark 3. Then the formula

$$\psi_\Delta \equiv \Box(\varphi_\Delta \rightarrow \bigcirc\varphi_\Delta)$$

belongs to Δ. In order to prove the latter statement, we assume for sake of contradiction that $\psi_\Delta \notin \Delta$. By Remark 2, there is some $\Delta < \Sigma \in S$ such that $\neg(\varphi_\Delta \rightarrow \bigcirc\varphi_\Delta) \in \Sigma$. Hence $\varphi_\Delta \wedge \neg\bigcirc\varphi_\Delta$ belongs to Σ. So, by the property of φ_Δ, on one hand we get $(\Delta \cap \Gamma)\ \rho_+^*\ (\Sigma \cap \Gamma)$, hence $(\Delta \cap \Gamma)\ \rho_+^*\ (\Sigma^+ \cap \Gamma)$, as $\Sigma \cap \Gamma\ \rho_+ \Sigma^+ \cap \Gamma$. On the other hand, from $\neg \bigcirc \varphi_\Delta \in \Sigma$ we get $\varphi_\Delta \notin \Sigma^+$. Therefore it is not the case that $(\Delta \cap \Gamma)\ \rho_+^*\ (\Sigma^+ \cap \Gamma)$. We have reached a contradiction. Therefore $\psi_\Delta \in \Delta$.

PROPOSITION 4.1.
The relation $\rho_<$ is the transitive closure of ρ_+.

PROOF. We have already pointed out in Section 3 that $\rho_+^* \subseteq \rho_<$. Let $\Delta < \Theta$ in S be such that $(\Delta \cap \Gamma)\ \rho_<(\Theta \cap \Gamma)$. By Remark 4, the formula ψ_Δ belongs to Δ. By transitivity, $\psi_\Delta \in \Theta$. Hence $(\Delta \cap \Gamma)\ \rho_+^*(\Theta \cap \Gamma)$, by the property of ψ_Δ. □

Recall that β is assumed to be a consistent formula.

THEOREM 4.2.
There is some valuation $v : \mathbb{N} \rightarrow \mathcal{P}(V)$ such that $v \models_0 \beta$.

PROOF. We fix a a sequence $\sigma = (s_n)_{n\in\mathbb{N}}$ of elements of $\bar{S}$ constructed as in Section 3 relative to β. We define v as follows:

$$\text{for all } k \in \mathbb{N} \text{ and all } p \in V, \quad p \in v(k) \ \Leftrightarrow \ p \in s_k.$$

As v is uniquely determined by σ, in the following we write $s_k \models \alpha$ for $v \models_k \alpha$ and we prove by induction on $\alpha \in \Gamma$ that

$$(!) \qquad \text{for all } k \in \mathbb{N}, \ (\alpha \in s_k \ \Leftrightarrow \ s_k \models \alpha).$$

If $\alpha \in \Gamma$ is atomic then (!) holds by definition of v. The cases relative to the propositional connectives are straightforward. Indeed, in each such case, property c3 of Γ (see Fact 1) allows us to apply the inductive assumption to the subformulas of the formula under consideration.

If $\alpha \in \Gamma$ is of the form $\bigcirc\gamma$, again by c3 we have that $\gamma \in \Gamma$. Let $k \in \mathbb{N}$. By definition of $(s_n)_{n\in\mathbb{N}}$ and by (!) applied to γ we get

$$\bigcirc\gamma \in s_k \ \Leftrightarrow \ \gamma \in s_{k+1} \ \Leftrightarrow \ s_{k+1} \models \gamma \ \Leftrightarrow \ s_k \models \bigcirc\gamma.$$

Eventually, let $\alpha \in \Gamma$ be of the form $\Box\gamma$. Notice that γ, $\neg\gamma$ and $\neg\Box\gamma$ are in Γ, by c3.

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Let $k \in \mathbb{N}$ be such that $\Box\gamma \in s_k$. By transitivity, $\Box\gamma \in s_{k+l}$ for all $l \in \mathbb{N}$. Hence, by axiom A4, $\gamma \in s_{k+l+1}$, for all $l \in \mathbb{N}$. By inductive assumption applied to γ we get $s_{k+l+1} \models \gamma$, for all $l \in \mathbb{N}$. Hence $s_k \models \Box\gamma$.

Therefore, for all $k \in \mathbb{N}$, if $\Box\gamma \in s_k$ then $s_k \models \Box\gamma$. It remains to prove the converse implication. We fix k and we assume that $s_k \models \Box\gamma$. We let m be the least natural number such that s_m occurs infinitely many times in σ (such m does exist by finiteness of $\bar{S}$). We claim that $\Box\gamma \in s_m$. We argue by contraposition, by proving that $\Box\gamma \notin s_m$ implies $s_k \not\models \Box\gamma$. So we assume $\Box\gamma \notin s_m$. Then $\neg\Box\gamma \in s_m$. Let $s_m = \Delta \cap \Gamma$, for some $\Delta \in S$. By Remark 2, there is $\Theta \in S$ such that $\Delta < \Theta$ and $\neg\gamma \in \Theta$. Hence $s_m \rho_< t$, where $t = \Theta \cap \Gamma$. Since $\rho_<$ is the transitive closure of ρ_+ there are $0 < l \in \mathbb{N}$ and $t_1, \ldots, t_l$ in $\bar{S}$ such that $s_m \rho_+ t_1 \rho_+ \ldots \rho_+ t_l = t$. By construction of σ, each t_i occurs infinitely many times in σ. In particular so does t. Let $h > \max(m, k)$ be such that $t = s_h$. From $\neg\gamma \in s_h$, by inductive hypothesis applied to γ we get that $s_h \models \neg\gamma$, hence $s_k \not\models \Box\gamma$. Therefore, assuming $s_k \models \Box\gamma$, by transitivity we get

$$(!!) \quad \Box\gamma \in s_n, \text{ for all } n \geq m.$$

If $k \geq m$ we are done. So we are left with the case when $k < m$. Let $s_k = \Sigma \cap \Gamma$, for some $\Sigma \in S$. We claim that $\Box\gamma \in s_k$. For, if not, by Remark 1d, axiom A2 and by the maximal consistency of Σ we get that $\bigcirc\neg\gamma \in s_k$ or $\bigcirc\neg\Box\gamma \in s_k$. The former alternative does not hold because, otherwise, $\neg\gamma \in s_{k+1}$ and so, by inductive hypothesis, $s_{k+1} \models \neg\gamma$, contradicting to $s_k \models \Box\gamma$. Hence $\neg\Box\gamma \in s_{k+1}$. Moreover, we have $s_h \models \Box\gamma$, for all $h \geq k$. Hence we can repeat the previous argument up to m and we get $\neg\Box\gamma \in s_m$, contradicting to $(!!)$. Hence $\Box\gamma \in s_k$, as required.

Finally, since $\beta \in s_0$, we get from $(!)$ that $s_0 \models \beta$. □

COROLLARY 4.3 (Weak completeness).
LTL_ is weakly complete with respect to the frame $(\mathbb{N}, <)$. Hence, for all formulas α,

$$(*) \qquad \models \alpha \Rightarrow \vdash \alpha.$$

PROOF. First of all, by a purely propositional reasoning we get that if $\neg\alpha$ is inconsistent then $\vdash \alpha$. We prove $(*)$ by contraposition. We assume $\nvdash \alpha$. Thus $\neg\alpha$ is consistent. By Theorem 4.2 there exists some valuation v such that $v \models_0 \neg\alpha$. Therefore $\not\models \alpha$. □

5 LTL_ as a logic for the first quadrant

Admittedly, at first reading the title of this section is rather puzzling. On one hand, in Section 4 we proved that LTL_ is weakly complete with respect to the linearly ordered frame $(\mathbb{N}, <)$, where $<$ denotes the standard ordering on the set of natural numbers (here it is worth pointing out that the latter result extends to LTL). In this regard, notice that axiom A2 formalizes the linearity (totality) property of $<$ on $\mathbb{N}$. On the other hand, the discrete first quadrant $\mathbb{N} \times \mathbb{N}$, equipped with the partial ordering $\lhd$ defined as follows:

$$(m, n) \lhd (h, k) \Leftrightarrow m \leq h, n \leq k \text{ and } (m, n) \neq (h, k), \tag{1}$$

is not linearly ordered.

In this section we show that, with a suitable semantics for the 'next time' operator and under a significant restriction on the set of valuations, LTL_ is the logic of a suitable class of models based on the frame $(\mathbb{N} \times \mathbb{N}, \lhd)$.

We stress that we do not want to devise a two-dimensional temporal logic for the frame $(\mathbb{N} \times \mathbb{N}, \lhd)$ by combining LTL_ with itself, possibly according to one of the constructions introduced, for

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instance, in [4, 5, 6]. We rather want to regard the one-dimensional temporal logic LTL_ as a logic for a class of models based on a two-dimensional frame.

We also point out a related result concerning the purely modal framework: in [2] the author proves that the modal logic S4.2 is the logic of the frame $(\mathbb{R} \times \mathbb{R}, \lhd)$, where $\lhd$ is defined in the same way as in (1) above. We believe that finding the modal logic of the frame $(\mathbb{N} \times \mathbb{N}, \lhd)$ is still an open problem.

We adopt the axiomatization of LTL_ and the notion of provability introduced in Section 2.

In (1) we already introduced the accessibility relation on $\mathbb{N}^2$. Notice that $\lhd$ is antireflexive and transitive and that $\mathbf{N}^2 = (\mathbb{N}^2, \lhd)$ is a non-linearly ordered frame.

As usual, a valuation map is a function $v : \mathbb{N}^2 \to P(V)$, where V is the countable set of propositional variables. We say that the triple $(\mathbb{N}^2, \lhd, v)$ is a structure.

With the exception of $\bigcirc$, the semantics of the connectives and the temporal operators is the standard one of LTL_. As each structure is uniquely determined by its own valuation, we write $v \models_{m,n} \varphi$ when the formula φ is true at $(m, n) \in \mathbb{N} \times \mathbb{N}$ under the valuation v. In particular we have that

$$v \models_{m,n} \Box\psi \quad \Leftrightarrow \quad \text{for all } (m,n) \lhd (h,k) \text{ we have that } v \models_{h,k} \psi.$$

We already mentioned in the Introduction that the semantics of connective $\bigcirc$ must be defined in a suitable way. We prescribe that the $\lhd$ -immediate successors of $(m, n) \in \mathbb{N} \times \mathbb{N}$ are $(m + 1, n)$ and $(m, n + 1)$ and we define the semantics of $\bigcirc$ as follows:

$$v \models_{m,n} \bigcirc\psi \;\Leftrightarrow\; v \models_{m+1,n} \psi \;\text{ and }\; v \models_{m,n+1} \psi. \tag{2}$$

From now on we restrict the family of valuations to the set $\mathcal{W}$ defined as follows:

$$v \in \mathcal{W} \;\Leftrightarrow\; \text{for all } (m,n) \in \mathbb{N}^2 \quad v(m+1,n) = v(m,n+1) \tag{3}$$

and we relativize the notions of validity and of logical consequence to $\mathcal{W}$ (notation: $\models_{\mathcal{W}}$).

REMARK 5.
Let $v \in \mathcal{W}$ and let φ be a formula. Then

$$\text{for all } (m,n) \in \mathbb{N}^2, \quad v \models_{m+1,n} \varphi \;\Leftrightarrow\; v \models_{m,n+1} \varphi \tag{4}$$

The statement above can be proved by induction on φ. The only non-trivial case is that when φ is of the form $\Box\psi$, under the inductive assumption that (4) holds for ψ.

We prove the left-to-right implication in (4), as the proof of the converse implication is similar. We assume $v \models_{m+1,n} \Box\psi$. We must prove that, for all $(m, n + 1) \lhd (h, k)$, we have $v \models_{h,k} \psi$.

If $m < h$ and $n + 1 \leq k$, then $m + 1 \leq h$ and $n < k$. Therefore $v \models_{h,k} \psi$ holds by the assumption $v \models_{m+1,n} \Box\psi$.

If $m = h$ and $n + 1 < k$, then $k = l + 1$, for some $l \geq n + 1$. By inductive assumption applied to ψ we have $v \models_{m,l+1} \psi \Leftrightarrow v \models_{m+1,l} \psi$ and the right-hand side of the equivalence is true by the assumption $v \models_{m+1,n} \Box\psi$. Therefore $v \models_{m,k} \psi$.

Hence, for all $(m, n + 1) \lhd (h, k)$, we have $v \models_{h,k} \psi$. Namely, $v \models_{m,n+1} \Box\psi$.

We say that a set Γ of formulas is $\mathcal{W}$-valid if for all $v \in \mathcal{W}$ we have that $v \models_{\mathcal{W}} \Gamma$.

PROPOSITION 5.1 ($\mathcal{W}$-soundness).
For each set $\Gamma \cup \{\varphi\}$ of formulas we have that

$$\Gamma \vdash \varphi \;\Rightarrow\; \Gamma \models_{\mathcal{W}} \varphi \tag{5}$$

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PROOF. By induction on the length of a derivation $\mathcal{D}$ we prove that if $\mathcal{D}$ is a derivation of φ from Γ then $\Gamma \models_{\mathcal{W}} \varphi$. We note that each LTL_-axiom is $\mathcal{W}$-valid. In particular, axiom A2 is $\mathcal{W}$-valid by Remark 5.

Regarding the $\mathcal{W}$-validity of the induction schema A5, let $v \in \mathcal{W}$ and let us assume that

$$v \models_{m,n} \Box(\varphi \to \bigcirc\varphi) \wedge \bigcirc\varphi \tag{6}$$

We prove the following by induction on $m + n < l$:

$$\text{for all } (h,k) \in \mathbb{N}^2, \text{ if } (m,n) \lhd (h,k) \text{ and } h + k = l, \text{ then } v \models_{h,k} \varphi \tag{7}$$

We assume that (7) holds for all $(m,n) \lhd (h',k')$, with $h'+k' < l$. Let $(m,n) \lhd (h,k)$, with $h+k = l$. We examine the case when $h = h'+1$. If $(h',k) = (m,n)$, then $v \models_{h,k} \varphi$ follows from the assumption (6). If $(m,n) \lhd (h',k)$, then by (6) and by the inductive assumption we get that $v \models_{h',k} \varphi \to \bigcirc\varphi$ and $v \models_{h',k} \varphi$, respectively. Therefore $v \models_{h',k} \bigcirc\varphi$ and $v \models_{h,k} \varphi$ follows. The case when $k = k' + 1$ is similar.

The cases relative to the application of a deduction rule are straightforward. □

We are now ready to prove the following:

THEOREM 5.2 ($\mathcal{W}$-weak completeness).
For each formula α we have that

$$\models_{\mathcal{W}} \alpha \;\Rightarrow \vdash \alpha.$$

PROOF. By contraposition. We assume $\not\vdash \alpha$. Let β be the consistent formula $\neg\alpha$. By proceeding as in Sections 3 and 4, we get a sequence $s = (s_n)_{n\in\mathbb{N}}$ of consistent sets and a map $v : \mathbb{N} \to P(V)$ satisfying the following.

1. Formula β belongs to s_0.
2. Each s_n is obtained by intersecting some maximal consistent set of formulas with a finite set Γ, the latter satisfying suitable closure properties, up to logical equivalence.
3. The valuation $v : \mathbb{N} \to P(V)$ defined by $v(n) = \{p \in V : p \in s_n\}$ has the property that

$$\text{for all } n \in \mathbb{N} \text{ and all } \psi \in \Gamma \; (v \models_n \psi \Leftrightarrow \psi \in s_n). \tag{8}$$

We define $(t_{m,n})_{m,n\in\mathbb{N}}$ and $w : \mathbb{N}^2 \to P(V)$ as follows: for all $(m,n) \in \mathbb{N}^2$,

$$\begin{aligned} t_{m,n} &= s_{m+n}; \\ w(m,n) &= \{p \in V : p \in t_{m,n}\}. \end{aligned}$$

Notice that $w \in \mathcal{W}$. We claim that, for all $\psi \in \Gamma$, we have

$$\text{for all } (m,n) \in \mathbb{N}^2 \; (w \models_{m,n} \psi \;\Leftrightarrow\; v \models_{m+n} \psi). \tag{9}$$

We examine in detail the cases of the inductive proof relative to the temporal operators $\Box$ and $\bigcirc$ only, the other cases being straightforward.

1. $\psi \in \Gamma$ is of the form $\bigcirc\eta$. Formula η belongs to Γ because the latter is closed under subformulas. Hence the inductive hypothesis (9) applies to η. For all $(m,n) \in \mathbb{N}^2$ we get

$$\begin{aligned} w \models_{m,n} \psi \quad &\Leftrightarrow \quad w \models_{m+1,n} \eta \text{ and } w \models_{m,n+1} \eta \\ &\Leftrightarrow \quad \text{(by inductive hypothesis)} \quad v \models_{m+n+1} \eta \\ &\Leftrightarrow \quad v \models_{m+n} \psi. \end{aligned}$$

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2. $\psi \in \Gamma$ is of the form $\Box\eta$. Same as in the previous case, $\eta \in \Gamma$. The inductive hypothesis (9) applied to η yields that, for all $(m,n) \in \mathbb{N}^2$,

$$\begin{array}{rcl} w \models_{m,n} \psi & \Leftrightarrow & \text{for all } (m,n) \lhd (h,k),\ w \models_{h,k} \eta \\ & \Leftrightarrow & \text{for all } m+n < l \in \mathbb{N},\ v \models_l \eta \\ & \Leftrightarrow & v \models_{m+n} \psi. \end{array}$$

From (8) and (9) we get that, for all $\psi \in \Gamma$ and all $(m,n) \in \mathbb{N}^2$,

$$w_{m,n} \models \psi \ \Leftrightarrow\ \psi \in s_{m+n}.$$

Since $\beta \in s_0$ we conclude that $w \not\models_{0,0} \alpha$. Hence $\not\models_{\mathcal{W}} \alpha$. □

6 Conclusions

The aim of the first part of this note was to provide evidence that the authors of [1] were right in their intuitions.

Regarding the second part, admittedly Theorem 5.2 above depends upon ad hoc assumptions. We refer in particular to the prescription that $(m+1,n)$ and $(m,n+1)$ are the only immediate successors of $(m,n) \in \mathbb{N}^2$. By retaining the accessibility relation on $\mathbb{N}^2$ defined in (1), it would be natural to include $(m+1,n+1)$ among the immediate successors of (m,n) and to modify the semantics of operator $\bigcirc$ introduced in (2) accordingly.

Moreover, the definition of set $\mathcal{W}$ of valuations given in (3) is based on a technical requirement that allows us to build on the results in [1, section 3]. It would worth investigating the case when $\mathcal{W}$ is the entire set of valuations, with the aim of finding which the logic of the frame $(\mathbb{N}^2, \lhd)$ is, if any.

Clearly, the right-to-left implication in axiom A2 is not valid in the frame $(\mathbb{N}^2, \lhd)$ with respect to all valuations from $\mathbb{N}^2$ with values in $P(V)$, under the natural definition of the set of immediate successors of a pair of natural numbers that we mentioned above. In such case, a temporal logic of the frame $(\mathbb{N}^2, \lhd)$ should include the axioms A1, A3, A4, A5 of LTL_ and the axiom $\bigcirc(\neg\varphi) \rightarrow \neg\bigcirc\varphi$, together with the deduction rules of Necessitation and Modus Ponens.

To the best of our knowledge, the problem of finding the temporal logic of the frame $(\mathbb{N}^2, \lhd)$ has not been addressed yet. Actually, as we pointed out in the introduction, the apparently simpler problem of determining the modal logic of such frame is still open.

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