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COMPLEXITY OF CONTEXTUAL REASONING

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Complexity of Contextual Reasoning

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Abstract

The aim of this paper is to delineate the computational complexity of propositional multi-context systems. We establish NP-completeness by translating multi-context systems into bounded modal K_n, and obtain more refined complexity results by achieving the so-called bounded model property: the number of local models needed to satisfy a set of formulas in a multi-context system is bounded by the number of formulas in that set plus the number of bridge rules of the system.

Exploiting this property of multi-context systems, we are able to encode contextual satisfiability problems into propositional ones, providing for the implementation of contextual reasoners based on specialized SAT solvers.

We apply our results to improve on complexity bounds for McCarthy's propositional logic of context – we show that satisfiability in this framework can be settled in non-deterministic polynomial time $O(|\varphi|^2)$.

Content areas: Contextual reasoning, computational complexity.

1 Introduction

The establishment of a solid paradigm for contextual knowledge representation and contextual reasoning is of paramount importance for the development of sophisticated theory and applications in AI.

McCarthy [?] pleaded for a formalization of context as a possible solution to the problem of generality; Giunchiglia [?] emphasized that reasoning based on large (common sense) knowledge bases could only be effectively pursued if confined to a manageable subset (context) of that knowledge base.

Contextual knowledge representation has been formalized in several ways. Most notable are the propositional logic of context (PLC) developed by McCarthy, Buvač and Mason [?, ?], and the multi-context systems (MCS) introduced by Giunchiglia and Serafini [?], which became associated with the local model semantics (LMS) devised by Giunchiglia and Ghidini [?]. Recently, MCS/LMS has been proven strictly more general than PLC [?].

Contexts were first implemented as microtheories into the notorious common sense knowledge base CYC [?, ?]. However, while in CYC the notion of local microtheories was a choice, in contemporary settings like that of the semantic web the notion of local, distributed knowledge is a must. Modern architectures impose highly scattered, heterogeneous knowledge fragments, which a central reasoner is not able to deal with. This engenders a high demand for distributed, contextual reasoning procedures.

More recently, the emerging idea of *grid computing* [?] fostered the development of distributed reasoning systems [?, ?]. These approaches show, from the practical point of view, that implementing a logical reasoner as a cooperative system of autonomous local reasoners, can improve performance.

The *complexity* of contextual reasoning, however, has so far received little attention. Massacci [?] accomplished a tableaux-based decision procedure for PLC, which establishes NP-completeness but leaves open a substantial number of efficiency issues. The same goes for a SAT-based procedure for MCS/LMS, to be provided by the authors of the current paper.

The goal of this paper is exactly this: to characterize the computational complexity of contextual reasoning. The lion's share of our analysis regards reasoning based on MCS/LMS. Towards the end of the paper, however, our results are shown to be applicable to PLC as well.

We proceed as follows. After defining MCS/LMS and explicating the contextual satisfiability problem we establish an equivalence result with bounded modal K_n, which directly entails NP-completeness. In pursuit of more specific upper bounds, we subsequently embark upon a more direct analysis of contextual satisfiability. Next, we encode the contextual satisfiability problem into a purely propositional one. This encoding paves the way for the implementation of contextual reasoning systems based on already existing SAT solvers. At last, we show how our results can be applied to obtain improved complexity results for PLC. We conclude with a concise recapitulation of our achievements, and some pointers to future research avenues.

2 Multi-Context Systems

A simple illustration of the intuitions underlying MCS/LMS is provided by the so-called "magic box" example [?], depicted below.

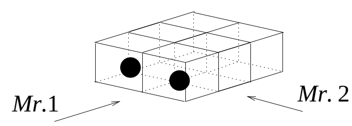

Figure 1: The magic box

Example 1 *Mr.1 and Mr.2 look at a box, which is called "magic" because neither of the observers can make out its depth. Both Mr.1 and Mr.2 maintain a local representation of what they see. These representations must be coherent – if Mr.1's sees a ball, for instance, then Mr.2's must see some ball too.*

We will now demonstrate how such interrelated local representations can be captured formally. Our point of departure is a set of indices I. Each index $i \in I$ denotes a *context*, which is described by a corresponding formal (in this case standard propositional) language L_i. To state that a propositional formula φ in the language L_i holds in context i we utilize so-called *labeled formulas* of the form $i : \varphi$ (when no ambiguity arises we will simply refer to *labeled formulas* as *formulas*). Formulas that apply to different contexts may be related by so-called *bridge rules*. These are expressions of the form:

$$i_1 : \phi_1, \ldots, i_n : \phi_n \rightarrow i : \varphi \tag{1}$$

where $i_1, \ldots, i_n, i \in I$ and $\phi_1, \ldots, \phi_n, \varphi$ are formulas. Note that "$\rightarrow$" does not denote implication (we'll use "$\supset$" for this purpose). Also note that our language does not include expressions like $\neg(i : \varphi)$ and $(i : \varphi \wedge j : \psi)$. $i : \varphi$ is called the *consequence* and $i_1 : \phi_1, i_n : \phi_n$ are called *premises* of bridge rule (**??**). We write *cons*(br) and *prem*(br) for the consequence and the set of all premises of a bridge rule br, respectively.

Definition 1 (Propositional Multi-Context System) *A propositional multi context system $\langle \{L_i\}_{i\in I}, \mathbb{BR}\rangle$ over set of indices I consists of a set of propositional languages $\{L_i\}_{i\in I}$ and a set of bridge rules $\mathbb{BR}$.*

In this paper, we assume I to be (at most) countable and $\mathbb{BR}$ to be finite. Note that the latter assumption does not apply to MCSs with *schematic* bridge rules, such as provability - and multi-agent belief systems [**?**]. The question whether our results may be generalized to capture these cases as well is subject to further investigation.

Example 2 *The MCS that formalizes the situation in example* **??** *consists of two contexts* 1 *and* 2*, described by $L_1 = L(\{l, r\})$ and $L_2 = L(\{l, c, r\})$, respectively. The constraint that Mr.2 must see a ball if Mr.1 sees one, is formalized by the following bridge rule:*

$$1 : l \vee r \quad \rightarrow \quad 2 : l \vee c \vee r$$

Let M_i be the class of classical interpretations of L_i. An interpretation $m \in M_i$ is called a *local model* of L_i. Interpretations of entire MCSs are called *chains*. They are constructed from sets of local models.

Definition 2 (Chain) *A chain c over a set of indices I is a sequence $\{c_i\}_{i\in I}$, where each $c_i \subseteq M_i$ is a set of local models of L_i. c is i-consistent if c_i is nonempty. It is point-wise if $|c_i| \leq 1$ for all $i \in I$; set-wise otherwise.*

A chain can be thought of as a set of "epistemic states", each corresponding to a certain context (or agent). The fact that c_i contains more than one local model amounts to L_i being interpretable in more than one unique way. So, set-wise chains correspond to partial knowledge; point-wise chains to complete knowledge.

Example 3 *Consider the situation depicted in Figure* **??**. *Both agents have complete knowledge, corresponding to a point-wise chain $\{\{\{l, r\}\}, \{\{l, \neg c, \neg r\}\}\}$.*

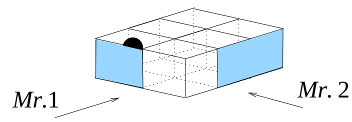

Figure 2: The partially hidden magic box

We can imagine a scenario however, in which Mr.1 and Mr.2's views are restricted to the right half and the left-most section of the box, as depicted in Figure **??**.
Now, both Mr.1 and Mr.2 have only partial knowledge; their observations may be interpreted in different ways. This is reflected by the set-wise chain:

$$\left\{ \begin{array}{c} \{\{l, \neg r\}, \{\neg l, \neg r\}\}, \\ \{\{l, \neg c, \neg r\}, \{l, \neg c, r\}, \{l, c, \neg r\}, \{l, c, r\}\} \end{array} \right\}$$

The epistemic states that a chain consists of concern *one and the same* situation. Therefore, arbitrary sets of local models may not always constitute a "sensible" chain. The somewhat vague conception of "sensibility" is captured by the more formal notion of "bridge rule compliance" specified below.

Definition 3 (Compliance and Satisfiability) *Let c be a chain, φ a formula over L_i, and br an element of the set of bridge rules $\mathbb{BR}$ of a multi-context system MS.*

1. *$c \models i : \varphi$ if $m \models \varphi$ in a classical sense for all local models $m \in c_i$. We say that c satisfies $i : \varphi$.*
2. *c complies with br if either $c \models cons(br)$ or $c \not\models i : \xi$ for some $i : \xi \in prem(br)$. c complies with $\mathbb{BR}$ if it complies with every $br \in \mathbb{BR}$.*
3. *c satisfies $i : \varphi$ in compliance with $\mathbb{BR}$ if c satisfies $i : \varphi$ and complies with $\mathbb{BR}$.*
4. *If there is an i-consistent chain c that satisfies $i : \varphi$ in compliance with $\mathbb{BR}$, we say that $i : \varphi$ is satisfiable in MS.*

The contextual satisfiability problem, then, is to settle whether or not a set of labeled formulas Φ is satisfiable in a multi-context system MS.

Example 4 *Consider an MCS with contexts 1 and 2, described by $L(\{p\})$ and $L(\{q\})$, respectively, and subject to the following bridge rules:*

$$\begin{array}{rcl} 1 : p & \rightarrow & 2 : q \\ 1 : \neg p & \rightarrow & 2 : q \end{array}$$

The formula $2 : \neg q$ is satisfied in this system by the following chain:

$$\left\{ \begin{array}{c} \{\{p\}, \{\neg p\}\}, \\ \{\{\neg q\}\} \end{array} \right\}$$

This example reflects that multi-context systems can not be encoded into propositional logic, simply by indexing propositions – such an encoding of the above system would be inconsistent.

In the following we will refer to the set of bridge rules of MS as $\mathbb{BR}$, and to the set of contexts involved by formulas in Φ as J.

3 Encoding Into Modal K_n

A first insight regarding the complexity of contextual SAT may be obtained by investigating its encoding into modal K_n satisfiability. In this section we show that any contextual satisfiability problem may be reduced to that of satisfying some formula in K_n, whose modal depth is at most equal to one. This problem is well-known to be NP-complete [**?**, **?**].

We define a translation $(.)^*$ of labeled formulas into modal formulas:

$$(i : \phi)^* = \Box_i \phi$$

For bridge rules we have:

$$\begin{array}{c} (i_1 : \phi_1, \ldots, i_n : \phi_n \rightarrow i : \phi)^* = \\ (i_1 : \phi_1)^* \wedge \ldots \wedge (i_n : \phi_n)^* \supset (i : \phi)^* \end{array}$$

A j-consistency constraint is captured by:

$$(j\text{-cons})^* = \neg \Box_j \bot$$

Theorem 1 *There is a kripke model $K = \langle W, \pi, \mathbf{R} \rangle$ such that $K, w_0 \models \psi$ for some $w_0 \in W$ and:*

$$\psi = \bigwedge_{i:\phi \in \Phi} (i : \phi)^* \wedge \bigwedge_{j \in J} (j\text{-}cons)^* \wedge \bigwedge_{br \in \mathbb{BR}} (br)^*$$

if and only if there is a J-consistent chain c^K that satisfies Φ in compliance with $\mathbb{BR}$.

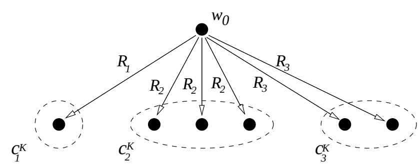

Figure 3: A schematic kripke model for ψ.

Proof. ($\Rightarrow$) We show how to construct c^K from K. Let m_w be the interpretation of $\bigcup_{i\in I} L_i$ associated to a world $w \in W$; for any $i \in I$, let $m_w|_i$ be the restriction of m_w to L_i and let $c_i^K = \{m_w|_i \; | w_0 R_i w\}$.

As $K, w_0 \models \Box_i \phi$, we have that $w \models \phi$ for any w with $w_0 R w$. As $\phi \in L_i$, we have that $m_w|_i \models \phi$. This implies that $c \models i : \phi$. Bridge rule compliance and J-consistency are established likewise.

($\Leftarrow$) From c^K we obtain a suitable kripke model K. Let W consist of a world w_0 plus one world w_{m_i} for each local model m_i of every component c_i^K of c^K. Let every $w_{m_i} \in W/\{w_0\}$ evaluate L_i according to m_i, and the rest of $\bigcup_{i\in I} L_i$ to $True$. Let w_0 evaluate every atomic proposition to $True$. For all $i \in I$, let:

$$R_i = \{\langle w_0, w_{m_i}\rangle \mid w_{m_i} \text{ corresponds to } m_i \in c_i^K\}$$

The resulting model is schematically depicted above. One can easily verify that $K, w_0 \models \psi$. $\Box$

Contextual satisfiability clearly subsumes classical SAT, and is therefore NP-hard [**?**]. From the above result, and the fact that satisfiability for bounded modal K_n is NP-complete [**?**], it follows that contextual satisfiability is NP-complete.

Moreover, the syntax of the formula that results from our translation is highly constrained: we obtain a conjunction of disjunctions of (negated) boxed formulas. Each disjunction comprises at most one boxed formula that is not negated, and furthermore, each boxed formula is purely propositional. This form strongly alludes to the existence of relatively efficient ways to solve the contextual satisfiability problem. Therefore, to obtain a more nuanced understanding of its complexity, we proceed with a more direct analysis.

4 Firsthand Analysis

Let us first introduce some notation and terminology. We denote the size of a labeled formula $i : \varphi$ by $|i : \varphi|$. Let $P(i : \varphi)$ and $P(\Phi)$ be the set of propositional atoms appearing in a formula $i : \varphi$ or a set of formulas Φ. Let G_i be the number of local models contained by the i^{th} component of a chain c, and let G be the total number of local models contained by c. Let $\Xi(br)$ and $\Xi(\mathbb{BR})$ consist of the premises and the consequence(s) of a bridge rule br or a set of bridge rules $\mathbb{BR}$. Finally, let N be the total size of the formulas in Φ and $\Xi(\mathbb{BR})$:

$$N = \sum_{i:\varphi\in\Phi} |i : \varphi| + \sum_{i:\xi\in\Xi(\mathbb{BR})} |i : \xi|$$

We first consider the *model checking problem*, that is, the problem of determining whether a given chain c satisfies a set of formulas Φ in a multi-context system MS. This task can be split into three sub-tasks:

1. Check whether c satisfies Φ;
2. Check whether c complies with $\mathbb{BR}$;
3. Check whether c is J-consistent.

Theorem 2 *Model checking can be performed deterministically in time:*

$$O(\sum_{i:\varphi\in\Phi\cup\Xi(\mathbb{BR})} G_i \times |\varphi|)$$

Proof. First consider sub-task 1. Checking whether a particular formula $i : \varphi \in \Phi$ is satisfied by c can be done as follows. Let $\varphi_1, \ldots, \varphi_k$ be an ordering of the subformulas of φ, such that $\varphi_k = \varphi$ and if φ_i is a subformula of φ_j, then $i < j$. Since φ has at most $|\varphi|$ subformulas, we have $k < |\varphi|$. By induction on k' we can label each local model m in c_i with either φ_j or $\neg\varphi_j$, for $j = 1, \ldots, k'$, depending on whether or not $m \models \varphi_j$, in time $O(G_i \times k')$. As a result, checking whether c satisfies Φ can be carried out in time $O(\sum_{i:\varphi\in\Phi} G_i \times |\varphi|)$.

Sub-task 2 takes time $O(\sum_{i:\xi\in\Xi(\mathbb{BR})} G_i \times |\xi|)$, as in the worst case it involves checking whether all the consequences and premises of every bridge rule in $\mathbb{BR}$ are satisfied or not. Sub-task 3 merely consists in checking whether c_j is nonempty, for $j \in J$. This can be done in $O(|J|)$ timesteps. The result follows directly. □

Next, we consider satisfiability. We first show that MCSs enjoy the so-called *bounded model property*. More specifically, we establish that if a chain satisfies Φ in MS, then it can be reduced to a chain that contains at most $|\Phi| + |\mathbb{BR}|$ local models and still satisfies Φ. Using this result, we reprove contextual satisfiability to be NP-complete, and establish an upper bound for the amount of time it requires.

Theorem 3 (Bounded Model Property) *A set of formulas Φ is satisfiable in a multi-context system MS iff there exists a J-consistent chain that contains at most $|\Phi| + |\mathbb{BR}|$ local models and satisfies Φ in compliance with $\mathbb{BR}$.*

Proof. Take any chain c that satisfies Φ in compliance with $\mathbb{BR}$. Let $\mathbb{BR}^* \subseteq \mathbb{BR}$ be the set of bridge rules whose consequences are not satisfied by c. Every $br \in \mathbb{BR}^*$ must have a premise which is not satisfied in some local model $m(br)$ contained by c. On the other hand, every formula $i : \varphi \in \Phi$ must be satisfied in at least one local model $m(i : \varphi)$ in c_i. The chain c^* obtained from c by eliminating all local models except:

$$\bigcup_{br \in \mathbb{BR}^*} m(br) \cup \bigcup_{i:\varphi \in \Phi} m(i : \varphi)$$

satisfies Φ in compliance with $\mathbb{BR}$ and contains at most $|\Phi| + |\mathbb{BR}^*| \leq |\Phi| + |\mathbb{BR}|$ local models. □

Theorem 4 *Contextual satisfiability is NP-complete. It requires non-deterministic time:*

$$O((|\Phi| + |\mathbb{BR}|) \times N)$$

Proof. We already observed that contextual satisfiability is NP-hard. Now, to determine satisfiability we may proceed as follows. First, we non-deterministically appoint a set $Cons$ of bridge rule consequences, and a set $Prem$ of bridge rule premises, such that for every $br \in \mathbb{BR}$, either br's consequence is in $Cons$, or one of br's premises is in $Prem$. Let I_Φ, I_{Cons}, and I_{Prem} be the set of contexts involved by Φ, $Cons$, and $Prem$ respectively. Furthermore, let Φ_i, $Cons_i$, and $Prem_i$ be the set of i-formulas contained by Φ, $Cons$, and $Prem$ respectively. Without loss of generality, we assume that $|\Phi_i| = 1$ for all $i \in I$. We construct a chain c, such that:

- For all $i \in I_{Prem}$, c_i contains exactly $|Prem_i|$ local models;
- For all $i \in I_\Phi / I_{Prem}$, c_i contains exactly one local model;
- For all $i \notin I_\Phi \cup I_{Prem}$, c_i is empty;
- For all $i \in I$, each local model in c_i evaluates the atomic propositions *not* appearing in $\Phi_i \cup Cons_i \cup Prem_i$ to $True$.

The only "guessing" involved in constructing c is the choice of $Cons$ and $Prem$, and the truth values to which each local model in c_i evaluates the atomic propositions in $P(\Phi_i \cup Cons_i \cup Prem_i)$. Notice that c contains at most $|\Phi| + |Prem| \leq |\Phi| + |\mathbb{BR}|$ local models. These local models are distributed over those components c_i of c with $i \in I_\Phi \cup I_{Prem}$. All the other components of c are empty. Consider a local model m contained in c_i for some $i \in I_\Phi \cup I_{Prem}$. The number of atomic propositions $|P(\Phi_i \cup Cons_i \cup Prem_i)|$ that m should "explicitly" evaluate is in any case smaller than $|P(\Phi_i)| + |P(Cons_i \cup Prem_i)|$, which, in turn, is bounded by

$$\max_{i:\varphi \in \Phi} |P(i : \varphi)| + \sum_{br \in \mathbb{BR}} \max_{i:\xi \in \Xi(br)} |P(i : \xi)| \leq N$$

We need to choose at most $|\Phi| + |\mathbb{BR}|$ such "explicit" valuations (one for each model in c), so c can be constructed in non-deterministic time $O((|\Phi| + |\mathbb{BR}|) \times N)$.

It remains to check whether c is J-consistent, satisfies Φ, and complies with $\mathbb{BR}$. By theorem **??** this can be done in deterministic time $O((|\Phi| + |\mathbb{BR}|) \times N)$.

Theorem **??** assures that, if Φ is satisfiable in MS, then guessing a chain as described above is bound to result in a suitable one. Thus, satisfiability of Φ in MS can be determined in non-deterministic polynomial time $O((|\Phi| + |\mathbb{BR}|) \times N)$. □

5 Encoding Into Propositional Sat

As contextual satisfiability is NP-complete, it must be tractably reducible to purely propositional SAT. In this section we provide such a reduction. In doing so we may loose the particular structure of our problem, but do lay the groundwork for an implementation of a purely SAT-based contextual reasoner, which could benefit from already existing well-advanced techniques.

To obtain a purely propositional representation of multi-contextual satisfiability problems, we exploit the understanding we obtained while establishing the bounded model property in the previous section. The key insight there was that a set of formulas Φ is satisfied by a chain c iff it is satisfied by chain c^b such that:

- For every formula $i : \varphi \in \Phi$, c_i^b contains at least one local model m that satisfies φ.
- For every bridge rule $br \in \mathbb{BR}$ whose consequence is not satisfied by c, there is at least one premise $j : \xi$ of br, such that c_j^b contains at least one local model m that satisfies $\neg\xi$.

Notice that to meet these requirements, the number of local models in each component of c^b can be kept down to $|\mathbb{BR}|$ (we assume that $|\mathbb{BR}| \geq 1$). Also, if a non-empty component of c^b contains less than $|\mathbb{BR}|$ local models it can be extended to comprise exactly $|\mathbb{BR}|$ models, simply by adding duplicates of already existing models. So we may say that Φ is satisfiable in MS iff it is satisfied by a chain c^* all of whose components are either empty or contain exactly $|\mathbb{BR}|$ local models.

Now, we construct a propositional formula ψ, which is satisfiable iff such a chain c^* exists. We express this formula in a language which contains an atomic proposition p_i^k for each $p \in L_i$, and each $k = 1, \ldots, |\mathbb{BR}|$. Intuitively, the truth value assigned to p_i^k by a propositional model of ψ corresponds to the truth value assigned to p by the k^{th} local model in c_i^*. The language also contains an atomic proposition e_i for each index $i \in I$. Intuitively, e_i being assigned $True$ corresponds to c_i^* being empty.

Let us write $K = \{1, \ldots, |\mathbb{BR}|\}$. For any formula φ, $i \in I$ and $k \in K$ let φ_i^k denote the formula that results from substituting every atomic proposition p in φ with p_i^k. Furthermore, let us write $\varphi_i^K = \bigwedge_{k \in K} \varphi_i^k$. Now, the translation of a labeled formula reads:

$$(i : \varphi)^* = e_i \vee \varphi_i^K$$

For bridge rules we have:

$$(i_1 : \varphi_1, \ldots, i_n : \varphi_n \rightarrow i : \phi)^* =$$
$$(i_1 : \phi_1)^* \wedge \ldots \wedge (i_n : \phi_n)^* \supset (i : \phi)^*$$

A j-consistency constraint is captured by:

$$(j\text{-cons})^* = \neg e_i$$

Theorem 5 *There is an assignment V to the propositions $\{p_i^k | i \in I$ and $k = 1, \ldots, |\mathbb{BR}|\} \cup \{e_i | i \in I\}$ that satisfies:*

$$\psi = \bigwedge_{i:\phi \in \Phi} (i : \phi)^* \wedge \bigwedge_{j \in J} (j\text{-}cons)^* \wedge \bigwedge_{br \in \mathbb{BR}} (br)^*$$

if and only if there is a J-consistent chain c^V that satisfies Φ in compliance with $\mathbb{BR}$.

Proof ($\Rightarrow$) From V construct a chain c^V, such that each component c_i^V is empty if $V(e_i) = True$ and contains exactly $|\mathbb{BR}|$ local models otherwise. In the latter case, let the k^{th} local model of c_i^V evaluate each atomic proposition $p \in L_i$ to $True$ iff $V(p_i^k) = True$. It is easy to see that c^V is J-consistent and satisfies Φ in compliance with $\mathbb{BR}$.

($\Leftarrow$) If there is a J-consistent chain c that satisfies Φ in compliance with $\mathbb{BR}$, there must also be a J-consistent chain c^* each of whose components is either empty or contains exactly $|\mathbb{BR}|$ local models, and which still satisfies Φ in compliance with $\mathbb{BR}$.

From c^* we obtain an assignment V as follows. To an atomic proposition e_i, V assigns $True$ iff c_i^* is empty. To an atomic proposition p_i^k, V assigns $True$ iff the k^{th} local model of c_i^* satisfies p, and any truth value iff c_i^* is empty. It is easy to see that V satisfies ψ. □

6 Application to PLC

In this section, we apply the results presented so far to improve current complexity bounds for McCarthy's propositional logic of context. The best result so far is due to Massacci [**?**]. He described a tableaux-based decision procedure, which determines satisfiability of a PLC formula φ in non-deterministic time $O(|\varphi|^4)$.

We translate a PLC formula φ into a labeled formula $\epsilon : \varphi$ and a multi-context system MCS(φ), so that φ is satisfiable in PLC iff $\epsilon : \varphi$ is satisfiable in MCS(φ). Furthermore, we show that determining whether or not $\epsilon : \varphi$

is satisfiable in MCS(φ) can be done in non-deterministic polynomial time $O(|\varphi|^2)$.

We proceed as follows. For each nesting pattern $ist(k_1, \dots ist(k_n, \psi) \dots)$ in φ, let MCS(φ) contain a context labeled with the sequence $k_1 \dots k_n$. Let the language of context $k_1 \dots k_n$ consist of all the atomic propositions in ψ, plus a new atomic proposition for each formula of the form $ist(k, \chi)$ occurring in ψ. At last, equip MCS(φ) with the following bridge rules[1]:

$$\begin{array}{l} \bar{k}k : \psi \rightarrow \bar{k} : ist(k, \psi) \\ \bar{k} : ist(k, \psi) \rightarrow \bar{k}k : \psi \\ \bar{k} : \neg ist(k, ist(h, \psi)) \rightarrow \bar{k}k : \neg ist(h, \psi) \\ \bar{k} : \neg ist(k, \neg ist(h, \psi)) \rightarrow \bar{k}k : ist(h, \psi) \end{array}$$

where $\bar{k} = k_1 \dots k_n$ refers to any context of MCS(φ), whose language contains $ist(k, \psi)$ or $ist(k, ist(h, \chi))$.

Example 5 *Consider $\varphi = p \vee ist(k, q \supset (ist(h, r \wedge s) \supset ist(j, q)))$. MCS($\varphi$) consists of four contexts labeled ϵ (the empty sequence), k, kh, and kj. The language of ϵ, L_ϵ, contains two propositions, p and $ist(k, q \supset (ist(h, r \vee s) \supset ist(j, q)))$; L_k contains two proposition, q and $ist(h, r \wedge s)$; $L_{kh} = L\{r, s\}$ and $L_{kj} = L\{q\}$. The bridge rules of MCS(φ) are as stated above.*

Theorem 6 ([?], 2003) *φ is satisfiable in PLC if and only if $\epsilon : \varphi$ is satisfiable in MCS(φ).*

Theorem 7 *Satisfiability of φ in PLC can be computed in non-deterministic polynomial time $O(|\varphi|^2)$.*

Proof. By theorem **??** any satisfiability problem in PLC can be transformed into a satisfiability problem in MCS. This transformation can be established in linear time.

Every bridge rule MCS(φ) involves at least one proposition of the form $ist(k, \psi)$. Every such proposition is involved in at most four bridge rules. Every subformula of φ of the form $ist(k, \psi)$ (and nothing else) results in a proposition of the form $ist(k, \psi)$ in the language of exactly one context in MCS(φ). The number of subformulas of φ of the form $ist(k, \psi)$ is bounded

[1] The first two bridge rules correspond to the notions of *entering* and *exiting* contexts [?], while the last two bridge rules correspond to the Δ axiom introduced in [?].

by $|\varphi|$. From these observations, we may conclude that the number of bridge rules $|\mathbb{BR}|$ in MCS(φ) is bounded by $4 \times |\varphi|$. Furthermore, by construction, the sum of the lengths of the formulas involved in any bridge rule of MCS(φ) is at most four.

By theorem **??**, satisfiability of $\epsilon : \varphi$ in MCS(φ) can be determined in time:

$$O((|\Phi| + |\mathbb{BR}|) \times (\sum_{i:\varphi \in \Phi} |i : \varphi| + \sum_{br \in \mathbb{BR}} \sum_{i:\xi \in \Xi(br)} |i : \xi|))$$

In the light of the above observations, and keeping in mind that Φ merely consists of $\epsilon : \varphi$, we may rewrite this in terms of φ as:

$$O(|\varphi|^2)$$

□

7 Conclusion

We have characterized the complexity of contextual reasoning based on propositional multi-context systems with finite sets of bridge rules.

A first insight was obtained by establishing an encoding of contextual satisfiability into satisfiability in multi-modal K_n, which is known to be NP-complete.

Next, we accomplished a more fine-grained upper bound for the complexity of contextual satisfiability by a direct investigation of its semantical properties. Herein we observed that multi-context systems enjoy the bounded model property.

Also, we provided a tractable encoding of contextual satisfiability problems into purely propositional ones. In doing so, we laid the groundwork for a SAT-based implementation of contextual reasoning systems.

Finally, we obtained improved complexity results for the satisfiability problem in McCarthy's propositional logic of context, by translating it into the satisfiability problem that we have considered in this paper.

Future work will encompass experimenting with both native and SAT-based contextual reasoning systems. Also, we are interested to what extent our results may be generalized so as to apply to multi-context systems with schematic bridge rules as well.