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On a Consensus Measure in a Group Multi-Criteria Decision Making Problem

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October 1990*

Abstract

A method for consensus measuring in a group decision problem is presented for the multiple criteria case.
The decision process is supposed to be carried out according to Saaty's Analytic Hierarchy Process, and hence using pairwise comparison among the alternatives.
Using a suitable distance between the experts' judgements, a scale transformation is proposed which allows a fuzzy interpretation of the problem and the definition of a consensus measure by means of fuzzy tools as linguistic quantifiers.
Sufficient conditions on the expert's judgements are finally presented, which guarantee any a priori fixed consensus level to be reached.

Keywords: group decision making, multiple criteria, degree of consensus, fuzzy preferences.

JEL Classification Codes: D70, D81, C63.

MSC Classification Codes: 90B50, 03B52.

1 Introduction

An interesting issue within the group decision theory is that of measuring the consensus inside the group.

*This is a slightly revised version, with some minor text corrections, of the paper "On a Consensus Measure in a Group MCDM Problem" published in Kacprzyk and Fedrizzi [1990].
This paper can be found in our institutional repository http://repec.cs.unitn.it/People/fedrizzi_michele.html or else use http://eprints.biblio.unitn.it/perl/advsearch?authors="Fedrizzi, Michele" .

We consider the decision problem in which m experts have to express their judgements on n different alternatives on the basis of p criteria; from these judgements, priorities are then derived to be assigned to the alternatives.

In section 2 the structure of the problem and the resolution method are briefly described, both referring essentially to the Analytic Hierarchy Process (AHP) introduced by T.L. Saaty in the 70's [Saaty, 1977, 1980].

The aim of this paper is to analyze the consensus problem and to supply a suitable consensus measure in this particular framework.

The choice of studying the consensus measuring problem in this context is due to the wide interest the AHP has produced both in scientific literature and in its practical applications (several US - government agencies, consulting firms and corporations are currently using the AHP).

In section 3 a dissimilarity measure between the opinions of the experts is introduced, coherent with Saaty's 1 to 9 ratio scale on which the opinions are expressed. This dissimilarity measure induces, in a natural way, a scale transformation, thus allowing a fuzzy formulation of the problem; a soft measure of the consensus is then defined for each criterion, according to the approach of Fedrizzi, Kacprzyk, and Zadrożny [1988]. More precisely, a fuzzy-logic-based calculus of linguistic quantified propositions is used to derive a measure of consensus that expresses the degree to which, for example, "almost all experts agree with the group's opinions concerning the most important alternatives".

To synthesize the degrees of consensus, which refer to the different criteria, a parametrized operator due to Zimmermann and Zysno [1980, 1983] is used, which allows the choice of different values of compensation among the criteria.

In section 4 some conditions on the experts' judgements are supplied, which guarantee an a-priori fixed level of consensus.

2 The Decision Problem and the Resolution Method

This section briefly describes the decision problem and the method used to derive the priorities to be assigned to the alternatives.

The main features of the Analytic Hierarchy Process are supposed to be known , and only brief references will therefore be given.

Let $S = \{s_1, \ldots, s_n\}$ be a set of alternatives and $C = \{c_1, \ldots, c_p\}$ a set of criteria; each expert out of a group of m formulates his judgements on the alternatives by pairwise comparisons on the basis of the given criteria.

It is required to calculate, by means of these judgements, the priority vector $\mathbf{w} = (w_1, \ldots, w_n)$, where w_i indicates the weight, or priority, the group assigns to s_i.

In the pairwise comparison matrices

$$\mathbf{A}^{kh} = [a_{ij}^{kh}] \quad k = 1, \ldots, m\,;\; h = 1, \ldots, p$$

the element a_{ij}^{kh} represents the ratio between the priority of s_i and that of s_j, as subjectively judged by expert k, according to criterion c_h.

Saaty suggests measuring this intensity of preference a_{ij}^{kh} using a ratio scale, and precisely the 1 to 9 scale: $a_{ij}^{kh} = 1$ indicates indifference between s_i and s_j, $a_{ij}^{kh} = 9$ indicates that s_i is absolutely preferred to s_j, and $a_{ij}^{kh} \in \{2, 3, \dots, 8\}$ indicates intermediate evaluations.

$\mathbf{A}^{kh}$ is completed by putting, for the remaining elements,

$$a_{ij}^{kh} = \frac{1}{a_{ji}^{kh}} \,.$$

Matrices $\mathbf{A}^{kh}$ are therefore called positive reciprocal matrices.

If the following equality holds

$$a_{ij}^{kh}\, a_{jl}^{kh} = a_{il}^{kh} \qquad \forall i, j, l = 1, \dots, n \tag{1}$$

that is, if expert k is perfectly coherent in his judgements, matrix $\mathbf{A}^{kh}$ is said to be consistent.

Being (1) unrealistic for subjective evaluations, consistency is not required for the matrices $\mathbf{A}^{kh}$ (the measure of inconsistency of the matrices is nevertheless an important aspect of the theory).

The experts will also perform similar pairwise comparison in set $\mathbf{C}$ of criteria, thus providing m additional matrices $\mathbf{B}^k = [b_{ij}^k], k = 1, \dots, m.$ The quantities b_{ij}^k, will therefore indicate the ratio between the relevance of criterion c_i and that of criterion c_j according to the opinion of expert k.

Matrices $\mathbf{A}^{kh}$ and $\mathbf{B}^k$ (for $k = 1, \dots, m$) are aggregated by means of the geometric mean in order to derive the following matrices, which express the opinions of the group:

$$A^h = [a_{ij}^h],\ h = 1, \dots, p;\ B = [b_{ij}] \,,$$

where

$$a_{ij}^h = \left(\prod_{k=1}^{m} a_{ij}^{kh} \right)^{1/m} \qquad \text{and} \qquad b_{ij} = \left(\prod_{k=1}^{m} b_{ij}^{k} \right)^{1/m} . \tag{2}$$

For what concerns the functional properties of the geometric mean and the suitability of this kind of aggregation operator for the problem under examination, see the interesting paper by Aczél and Saaty [1983]. As an example, it is easy to verify that the geometric mean preserves the reciprocity:

$$a_{ji}^h = \frac{1}{a_{ij}^h},\ b_{ji} = \frac{1}{b_{ij}},\ h = 1, \dots, p;\ i, j = 1, \dots, n$$

By means of group matrices $\mathbf{A}^h$ and $\mathbf{B}$, the problem can be hierarchically structured, and the AHP method can be applied to calculate priority vector $\mathbf{w}$.

More precisely, let us consider the 3-levels group hierarchy where the third level contains the alternatives and the second one the criteria, the first being, as usual, simply the vertex of the hierarchy.

Following the AHP, the normalized eigenvector (say $\mathbf{w}^h$) corresponding to the maximum eigenvalue of $\mathbf{A}^h$ is calculated for $h = 1, \dots, p$, thus obtaining the so-called local priority vectors.

Analogously, the normalized eigenvector $\mathbf{b} = (b_1, ..., b_p)$ is calculated, which corresponds to the maximum eigenvalue of $\mathbf{B}$.

The global priority vector $\mathbf{w}$ is calculated according to the principle of the hierarchical composition:

$$\mathbf{w} = \sum_{h=1}^{p} b_h \mathbf{w}^h \,.$$

3 Evaluation of the Experts' Agreement

In comparing alternatives it is a crucial point to determine not simply whether different opinions agree or not, but, also, how close the judgements are.

A suitable definition of the dissimilarity among experts' opinions is therefore a prerequisite to derive a consistent measure of consensus in the group.

3.1 A dissimilarity measure among experts' opinions

Taking into account the meaning of the subjective estimates $a_{ij}^{kh} \in [1/9, 9]$, it is evident that the euclidean distance $|a_{ij}^{k_1 h} - a_{ij}^{k_2 h}|$, for instance, is not a suitable dissimilarity measure between the judgement of the two experts k_1 and k_2: it is obvious, for example, that the two estimates $a_{ij}^{k_1 h} = 1$ and $a_{ij}^{k_2 h} = 2$ are much more dissimilar than the estimates $a_{ij}^{k_1 h} = 8$ and $a_{ij}^{k_2 h} = 9$.

Having indicated by $d(a_{ij}^{k_1 h}, a_{ij}^{k_2 h})$ the dissimilarity measure we are looking for, let us consider the following set of conditions that function d must satisfy. Let us, for simplicity, indicate by x and y the arguments of d, assuming, in the following, that x and y are positive real numbers.

(i) $d(x, y)$ is continuous for $x, y > 0$ [1]

(ii) d is a distance:

$$\left.\begin{array}{l} d(x,y) = d(y,x) \\ d(x,y) = 0 \iff x = y \\ d(x,z) \leq d(x,y) + d(y,z) \end{array}\right\} \forall x, y, z > 0$$

(iii) $d(x, y) = d(kx, ky) \quad \forall k > 0,\ \forall x, y > 0$

(iv) $d(1/x, y) = d(1/x, 1) + d(1, y) \quad \forall x, y \geq 1$

(v) $d(1/9, 9) = 1$

A short explanation is needed for conditions (iii) $\rightarrow$ (v). Condition (iii) states that equal distance is assigned to pairs of judgements with equal ratio; (iv) states that if two judgements are disagreeing, in the sense that the first (say $1/x$) prefers s_j to s_i and the second one (say y) prefers s_i to s_j, then the distance between them is the sum of the distance between the first and the indifference (that is 1) and the distance between the indifference and the second one. Finally (v) is a normalization condition.

By the following theorem a dissimilarity measure is supplied, which will be widely used in the rest of the paper.

Theorem 1.
The only function d satisfying conditions (i) $\rightarrow$ (v) is the following:

$$d(x, y) = \tfrac{1}{2}\, |\, \log_9 x - \log_9 y \,|\,, \tag{3}$$

[1] The condition (i) is implied by (ii), however it has been specified in order to emphasize this assumption that will be used in the proof of theorem 1.

Proof
From (iii), taking $k = 1/y$, it is $d(x,y) = d(x/y,1)$. This means that, in order to calculate the distance between two points, it is sufficient to have the expression of the distance of an arbitrary point from point 1. Let us therefore indicate by $f(z)$ the distance of z from 1:

$$f(z) = d(z,1) .$$

From (iii), (ii) and (iv) it is then, for $x, y \geq 1$

$$d(xy,1) = d(x,1) + d(y,1) ,$$

or

$$f(xy) = f(x) + f(y) . \tag{4}$$

Taking into account (i), the general solution of functional equation (4) in $[1 + \infty[$ is [see Aczél, 1966]:

$$f(z) = k_1 \ln z, \quad k_1 \in \mathbb{R} ,$$

where $\ln z$ is the natural logarithm of z.
Analogously, for $x, y \geq 1$, we obtain

$$f(\frac{1}{x}\frac{1}{y}) = f(\frac{1}{x}) + f(\frac{1}{y}) ,$$

or

$$f(xy) = f(x) + f(y) \qquad \text{for} \qquad 0 < x, y \leq 1 .$$

Solving the previous equation in $]0,1]$ we obtain again

$$f(z) = k_2 \ln z, \quad k_2 \in \mathbb{R} .$$

For $z \geq 1$, condition $f(z) \geq 0$ implies $k_1 > 0$, because $k_1 = 0$ violates (ii). For the same reason from $0 < z < 1$ it follows that $k_2 < 0$.
It is easy to verify that $k_2 = -k_1$: in fact, from (iii) it is $d(1/z,1) = d(1,z)$, and hence $f(1/z) = f(z)$.
For example, by choosing $z \geq 1$, it is therefore

$$k_2 \ln(1/z) = k_1 \ln z ,$$

and then

$$-k_2 = k_1 .$$

It is now possible to express f more synthetically (k stays for k_1):

$$f(z) = k|\ln z|, \quad k > 0,\ z > 0 .$$

Function d therefore takes the following expression:

$$d(x,y) = d(\frac{x}{y},1) = f(\frac{x}{y}) = k\,|\ln(\frac{x}{y})| ,$$

where constant k is determined by (v):

$$1 = d(\frac{1}{9}, 9) = d(\frac{1}{9}, 1) + d(1, 9) = 2d(1, 9) = 2f(9) = 2k|\ln 9| \,,$$

and then

$$k = \frac{1}{2\ln 9} \,.$$

Expression (3) is finally obtained:

$$d(x, y) = \frac{1}{2}\left|\frac{\ln \frac{x}{y}}{\ln 9}\right| = \frac{|\log_9 x - \log_9 y|}{2} \,.$$

It is immediate to verify that conditions (ii) are all satisfied:

$$d(x, z) = \frac{|\log_9 x - \log_9 z|}{2} = \frac{|\log_9 x - \log_9 y + \log_9 y - \log_9 z|}{2} \leq d(x, y) + d(y, z).$$

□

3.2 Obtaining fuzzy preference relations

Expression (3) suggests to interpret the distance d between two points x and y of the interval $[1/9, 9]$ as the usual euclidean distance between $g(x)$ and $g(y)$ where g represents a transformation of logarithmic type. More precisely, it is possible to rewrite (3) as

$$d(x, y) = |g(x) - g(y)| \,,$$

where

$$g(x) = \tfrac{1}{2}(1 + \log_9 x) \,.$$

Function g maps $[1/9, 9]$ into $[0, 1]$ with some interesting properties. Putting

$$r_{ij}^{kh} = \mu_{\mathbf{R}^{kh}}(s_i, s_j) = g(a_{ij}^{kh}) \,,$$

a fuzzy preference relation $\mathbf{R}^{kh} = [r_{ij}^{kh}]$ is defined for each expert k and each criterion c_h.

The following properties point out how matrices $\mathbf{A}^{kh}$ and $\mathbf{R}^{kh}$ relates:

$$\begin{array}{rcll} a_{ij}^{kh} = \frac{1}{9} & \iff & r_{ij}^{kh} = 0 & \text{stating that } s_j \text{ is absolutely preferred to } s_i \,. \\ a_{ij}^{kh} = 9 & \iff & r_{ij}^{kh} = 1 & \text{stating that } s_i \text{ is absolutely preferred to } s_j \,. \\ a_{ij}^{kh} = 1 & \iff & r_{ij}^{kh} = 0.5 & \text{stating indifference between } s_i \text{ and } s_j \,. \\ a_{ij}^{kh}\, a_{ji}^{kh} = 1 & \iff & r_{ij}^{kh} + r_{ji}^{kh} = 1 & . \end{array} \tag{5}$$

The last property is rather interesting, as it shows that g transforms the "multiplicative" reciprocity of Saaty's matrices in the "additive" reciprocity, usually requested for the fuzzy preference relations [see Tanino, 1988].

Another property of function g is pointed out by applying it also to the elements a_{ij}^h of the group matrices $\mathbf{A}^h$, thus defining the group fuzzy preference relations $\mathbf{R}^h = [r_{ij}^h]$:

$$r_{ij}^h = \mu_{\mathbf{R}^h}(s_i, s_j) = g(a_{ij}^h) = \tfrac{1}{2}(1 + \log_9 a_{ij}^h) \ .$$

It is now easy to verify that r_{ij}^h is just the arithmetic mean of r_{ij}^{kh} for $k = 1, \ldots, m$:

$$r_{ij}^h = \frac{\sum_{k=1}^m r_{ij}^{kh}}{m}$$

(remember that a_{ij}^h was obtained as the geometric mean of a_{ij}^{kh} for $k = 1, \ldots, m$).

The following diagram synthesizes the previous results

$$\begin{array}{rcccl} & a_{ij}^{kh} & \xrightarrow{\;g\;} & r_{ij}^{kh} & \\ \text{geometric mean} & \downarrow & & \downarrow & \text{arithmetic mean} \\ & a_{ij}^{h} & \xrightarrow[\;g\;]{} & r_{ij}^{h} & \end{array}$$

To conclude, it can be said that by means of function g it is possible to transform, in a certain sense, a "multiplicative" formulation of the problem into an "additive" one.

3.3 A consensus measure

In this subsection a fuzzy-logic-based calculus of linguistically quantified propositions is used, for which we refer to Fedrizzi, Kacprzyk, and Zadrożny [1988].

Being $d(\cdot,\cdot) \in [0,1]$ a dissimilarity measure (as defined in subsection 3.1), expression

$$v_{ij}^h(k) = 1 - d(a_{ij}^{kh}, a_{ij}^h), \qquad h = 1, \ldots, p;\ k = 1, \ldots, m;\ i, j = 1, \ldots, n$$

will therefore express the degree of agreement between expert k and the group as to their preferences between alternatives s_i and s_j on the basis of criterion c_h.

In order to measure the consensus on vector $\mathbf{w}^h$, let us define the relevance of the pair of alternatives (s_i, s_j) as

$$\beta_{ij}^h = \frac{w_i^h + w_j^h}{2}\ , \qquad h = 1, \ldots, p;\ i, j = 1, \ldots, n\ .$$

The degree of agreement between expert k and the group as to their preferences between all the relevant pairs of alternatives can then be expressed by

$$V^h(k) = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} v_{ij}^h(k) * \beta_{ij}^h}{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{ij}^h}\ , \qquad h = 1, \ldots, p \tag{6}$$

where $*$ is a t-norm. Note that denominator of (6) adds to $(n-1)/2$. Let Q_1 be a linguistic quantifier, that is a fuzzy set in $[0,1]$, and let us consider, for instance, $Q_1 =$ "most".

It is therefore possible to define the degree of agreement between expert k and the group as to their preferences between Q_1 relevant pairs of alternatives:

$$V_{Q_1}^h(k) = \mu_{Q_1}(V^h(k)) \,, \qquad h = 1, \ldots, p; \; k = 1, \ldots, m \,.$$

The degree of agreement of all experts with the group as to their preferences between Q_1 relevant pairs of alternatives is

$$V_{Q_1}^h = \frac{\sum_{k=1}^m V_{Q_1}^h(k)}{m} \,, \qquad h = 1, \ldots, p$$

Let Q_2 be another linguistic quantifier similar to Q_1, (for example Q_2 = "almost all"); the degree of agreement of Q_2 experts with the group as to their preferences between Q_1 relevant pairs of alternatives is

$$E_h = \mu_{Q_2}(V_{Q_1}^h) \,, \qquad h = 1, \ldots, p \tag{7}$$

In order to obtain a global measure (say F) of the consensus which synthesizes the degrees of agreement (7), let us aggregate them by means of a parametrized operator due to Zimmermann and Zysno [1983]:

$$F = \left(\prod_{h=1}^{p} E_h^{b_h} \right)^{1-\gamma} \left[1 - \prod_{h=1}^{p} (1 - E_h)^{b_h} \right]^{\gamma} . \tag{8}$$

As the compensation parameter γ varies from 0 to 1, the operator describes the whole class of operators between "AND" and "OR". It is therefore possible to choose the desired compensation among the degrees of agreement corresponding to the different criteria.

Note that (8) takes into account the relevance b_h of the criteria as determined by the AHP. Since the Zimmermann-Zysno operator requires that $b_1 + \ldots + b_p = p$, every b_h must therefore be multiplied by p. For simplicity the same notation b_h is maintained in (8).

Finally F can be interpreted as the degree of agreement of almost all (Q_2) experts, on the basis of the p criteria, as to their preferences between most (Q_1) pairs of alternatives which have turned out to be relevant through the AHP (remember that the degrees of agreement were updated by the weights w_i^h).

It can then be said that F is a consensus measure on the priority vector $\mathbf{w}$.

4 Sufficient Conditions for the Consensus

Some conditions on the matrices $\mathbf{A}^{kh}$ are presented, which guarantee an a-priori fixed level of consensus.

4.1 The one-criterion case

Let us consider, for a fixed criterion c_h, the following condition on the elements of matrices $\mathbf{A}^{kh}$:

$$\exists\, \delta > 0 : \frac{a_{ij}^{k_1 h}}{a_{ij}^{k_2 h}} \leq \delta, \qquad \forall k_1, k_2 \in \{1, \dots, m\};\ \forall i, j \in \{1, \dots, n\}. \tag{9}$$

Condition (9) states that, for each pair of alternatives, no estimate can be so different from any other, to be more than δ times greater.

The next theorem shows that for each desired consensus level $z^* \in [0, 1])$ it is possible to find a suitable value of δ which guarantees, under assumption (9), the consensus E_h, to be not less than z^*.

Before formulating the theorem, let us assume that membership functions μ_{Q_1} and μ_{Q_2} have the following expressions:

$$\mu_{Q_1}(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x \leq \alpha \\ \dfrac{x - \alpha}{\beta - \alpha} & \text{for} \quad \alpha < x < \beta \\ 1 & \text{for} \quad \beta \leq x \leq 1 \end{cases}, \qquad \mu_{Q_2}(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x \leq \alpha' \\ \dfrac{x - \alpha'}{\beta' - \alpha'} & \text{for} \quad \alpha' < x < \beta' \\ 1 & \text{for} \quad \beta' \leq x \leq 1 \end{cases}. \tag{10}$$

Theorem 2.
Under the following assumptions

(i) *functions μ_{Q_1} and μ_{Q_2} are defined by (10),*

(ii) *the product is chosen as t-norm in (6),*

(iii) *for any $z* \in [0, 1]$, (9) is satisfied by taking*

$$\delta = 9^{\sigma(z^*)}, \tag{11}$$

where

$$\sigma(z^*) = 2\{1 - \alpha - (\beta - \alpha)[\alpha' + z^*(\beta' - \alpha')]\}\,,$$

the following inequality holds

$$E_h \geq z^*\,.$$

Proof
From (9) it follows

$$\frac{1}{\delta} \leq \frac{a_{ij}^{k_1 h}}{a_{ij}^{k_2 h}} \leq \delta \qquad \forall k_1, k_2 \in \{1, \dots, m\};\ \forall i, j \in \{1, \dots, n\}$$

and thus, being a_{ij}^h a mean,

$$\frac{1}{\delta} \leq \frac{a_{ij}^{kh}}{a_{ij}^{h}} \leq \delta \qquad \forall k \in \{1, \dots, m\};\ \forall i, j \in \{1, \dots, n\}\,.$$

Taking the logarithms we have

$$-\log_9 \delta \;\leq\; \log_9 a_{ij}^{kh} - \log_9 a_{ij}^{h} \;\leq\; \log_9 \delta \;,$$

and then

$$d(a_{ij}^{kh}, a_{ij}^{h}) = \tfrac{1}{2} \left|\log_9 a_{ij}^{kh} - \log_9 a_{ij}^{h}\right| \leq \tfrac{1}{2} \log_9 \delta \;.$$

We therefore obtain

$$v_{ij}^{h} \geq 1 - \tfrac{1}{2} \log_9 \delta \;,$$

and after some calculations, from (11) we have

$$V_{Q_1}^{h}(k) \geq \mu_{Q_1}\left(1 - \tfrac{1}{2}\log_9 \delta\right) = \mu_{Q_1}\left(\alpha + (\beta - \alpha)[\alpha' + z^*(\beta' - \alpha')]\right) \;.$$

Taking into account (10) it is therefore

$$V_{Q_1}^{h} \geq \alpha' + z^*(\beta' - \alpha');$$

from (7) and (10) we finally obtain

$$E_h = \mu_{Q_2}\left(V_{Q_1}^{h}\right) \geq \mu_{Q_2}\left(\alpha' + z^*(\beta' - \alpha')\right) \geq z^* \;.$$

□

4.2 Global consensus

Let us now take into account all the criteria $c_1, \ldots, c_p$; the following theorem, which is analogous to theorem 2, supplies conditions which guarantee a fixed level t^* for the global consensus F given by (8).

Theorem 3.
Under assumptions (i) and (ii) of theorem 2, for any $t^* \in [0,1]$, *if (9) is satisfied by taking*

$$\delta = 9^{\phi(t^*)}, \tag{12}$$

where

$$\phi(t^*) = 2\{1 - \alpha - (\beta - \alpha)[\alpha' + (t^*)^{1/p}(\beta' - \alpha')]\} \;,$$

then

$$F \geq t^* \;.$$

Proof
Being

$$\phi(t^*) = \sigma\left((t^*)^{1/p}\right) \;,$$

it follows from theorem 2,

$$E_h \geq (t^*)^{1/p} \qquad h = 1, \ldots, p \;.$$

From (8) it is then (remember that $b_1 + \ldots + b_p = p$)

$$F \geq (t^*)^{1-\gamma} \left\{1 - [1 - (t^*)^{1/p}]^p\right\}^\gamma \, .$$

It is now sufficient to prove that for any $t \in [0,1]$, $\gamma \in [0,1]$, and $p \in \mathbb{N}$ it is

$$(t^*)^{1-\gamma} \left\{1 - [1 - (t^*)^{1/p}]^p\right\}^\gamma \geq t^*. \tag{13}$$

Taking the logarithms in (13) we have, after some calculations,

$$t^* + \left[1 - (t^*)^{1/p}\right]^p \leq 1$$

which can be written as

$$\left[(t^*)^{1/p}\right]^p + \left[1 - (t^*)^{1/p}\right]^p \leq 1 \, .$$

The last inequality is easily verified to be true:

$$\left[(t^*)^{1/p}\right]^p + \left[1 - (t^*)^{1/p}\right]^p \leq (t^*)^{1/p} + 1 - (t^*)^{1/p} = 1 \, .$$

Inequality (13) and consequently theorem 3 are therefore proved.

□

It can be noted that inversion of (12) allows an alternative formulation of theorem 3. Instead of fixing a level t^* of the consensus, and consequently derive sufficient conditions which guarantee the level t^* to be reached, it is possible to determine, by direct examination of matrices $\mathbf{A}^{kh}$, the minimum value of δ, say δ^*, which satisfies (12). A lower bound for the global consensus F is then obtained:

$$F \geq \left[\frac{(1 - \alpha - \frac{1}{2}\log_9 \delta^*)/(\beta - \alpha) - \alpha'}{\beta' - \alpha'}\right]^p \, .$$

5 Concluding Remarks

(a) Assumption (ii) of theorems 2 and 3 can be modified by choosing "min" as t-norm; more generally, any t-norm $*$ such that $x * y \geq xy \ \forall \, x, y \in [0,1]$ can be chosen. Statements of the theorems still hold, since the degree of agreement $V^h(k)$ given by (6) does not decrease by this substitution.

(b) If number 9, which Saaty proposes as the maximum value of the ratio scale in the AHP, is substituted with any other value $n \in \{2, 3, \ldots\}$, all the results of the previous paragraphs still hold, with obvious changes in the formulas.

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