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GEOMETRIC SIMPLE CONNECTIVITY AND LOW-DIMENSIONAL TOPOLOGY

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to the memory of L.V. Keldysh

1. Introduction

I feel very honoured to have been invited to contribute with a paper to this volume dedicated to the memory of Ljudmila Vsevolodovna Keldysh who was an important mathematician and also a very remarkable person indeed. I got to know her in 1961 when, as a young mathematician from Rumania, I have spent some time in Moscow. I gave some lectures in the topology seminar of Ljudmila Keldysh, I have been several times been a guest in her home and, in that same period, I became a friend of her son Serghei Novikov. It was quite an exciting episode of my life.

The main part of this largely informal paper is to explain the present status of my program for the Poincaré Conjecture. Of course, I am fully aware of the announcement made by Grisha Perelman of a proof of the complete Thurston geometrization Conjecture (I have even lectured to my Italian colleagues about it, or rather about the very little I understand of the work of Perelman and Hamilton.) But the point is that my own proof of the Poincaré Conjecture is, modulo very serious verification (in particular of the nevralgic [PoV-B], see below) finished too. Moreover the argument have hardly anything to do with dimension three, they are all 4-dimensional and so, even if they may only allow some partial steps in the direction of the full Thurston Conjecture, they do have potential applications indeed, in the realm of 4-dimensional topology where, to the best of my knowledge the Ricci flow does not say anything very strong (except, possibly, in the Kähler context.)

So, you may also take this paper, if you wish to do so, as kind of a very informal announcement. Bibliography has been kept to a minimum; only references to papers containing proofs for (some of) the statement I make have been given. But there one can also find the other references which one might wish to have,too.

Thanks are due to David Gabai, Frank Quinn and Barry Mazur for their helps and encouragements. David actually did much more than that, since some of the results stated below are result of joint work with him.

A very condensed version of this paper has been presented as a 45 minutes lecture at the April 2003 meeting of the AMS in New York.

2. A Program for the 3-dimensional Poincaré Conjecture

My program is an almost entirely 4-dimensional construction and I will start by explaining why this is not such a completely unreasonable idea, to begin with.

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Let Δ^3 be a homotopy 3-ball and, starting with it, consider the following compact bounded 4-manifold

$$X_0^4 = (\Delta^3 \times I)\#p\#(S^2 \times D^2); \tag{2.2.1}$$

in all this paper "#" will mean connected sum along the boundary. Assume now that we would know (which we actually don't, hence the question mark added to the formula below) that

$$X_0^4 \overset{?}{=} B^4\#p\#(S^2 \times D^2). \tag{2.2.?}$$

This would imply the Poincaré Conjecture. One can see this as follows.

By taking the boundaries of the two terms in formula (2.2.?) we get

$$2(\Delta^2\#p\#(S^2 \times D^2)) = 2(B^3\#p\#(S^2 \times D^2)),$$

and hence there is an embedding of Δ^3 into the universal covering

$$2(B^3\#p\#(S^2 \times D^2))^{\sim},$$

a standard object which embeds into R^3. But any smooth embedding $S^2 \subset R^3$ (and in all this discussion we have tacitly assumed that we are in the DIFF category) bounds a 3-ball, according to an old theorem of Alexander. For the little argument above, it is essential that we stick to dimension four, and do not go to dimensions > 4; in dimension five or more, for instance, the analogue of (2.2.?) is known be true, but this does not help at all with the Poincaré Conjecture itself.

Before going on, I will explain what the "geometric simple connectivity" from the title of this paper means. A smooth manifold M^n (which may be with $\partial M \neq \emptyset$ and not necessarily compact) is said to be ***geometrically simply connected*** if it admits a smooth handlebody decomposition without handles of index $\lambda = 1$. Alternatively there exists a smooth proper function $M^n \xrightarrow{f} R_+$ all the singularities of which are in intM^n and of Morse type, such that

i) There are no singularities of index $\lambda = 1$.
ii) The restriction $f|\partial M^n$ is also Morse and all of its "non fake " singularities are also with $\lambda \neq 1$. Here "non fake" means that they command actual changes in topology for the set $f^{-1}(-\infty, c]$.

Clearly if M^n is g.s.c. (by which we will mean geometrically simply-connected) then $\pi_1 M^n = 0$ and it is instructive to see what is the answer for the following natural question

$$\pi_1 M^n = 0 \longrightarrow M^n \ \text{ g.s.c. ?} \tag{Q}$$

If M^n is compact, the answer to (Q) is YES if $n \geq 5$ (Smale), NO if $n = 4$ (there is a subtle obstruction discovered by Casson, when M^4 is contractile, connected with the representations of $\pi_1 \partial M^4$) while if $n = 3$ the arrow in (Q) is equivalent to the Poincaré Conjecture. Notice that Casson's result quoted above implies that the non trivial factors of the 5-cube, discovered more than forty years ago by Barry Mazur and by the present author (Po-Mazur manifolds) are not g.s.c. If M^n is open and if $\pi_1^\infty M^n = 0$ then the answer to (Q) is YES again, provided that $n \geq 5$ (see [PoTa1]). If M^n is both non-compact and with $\partial M^n \neq \emptyset$ then there is no reasonable theorem, to the best of my knowledge.

With this my program for the Poincaré Conjecture (a 1994 overview of which can be found in [Ga]) consists of three steps

STEP I Go from Δ^3 to the following smooth open 4-manifold

$$X^4 \overset{\text{def}}{=} \text{int}[(\Delta^3 \times I)\#\infty\#(S^2 \times D^2)]. \tag{2.3}$$

For this we have now the following theorem

Theorem 1. *X^4 is geometrically simply-connected.*

The proof of this theorem about which more things will be said later in this paper, is contained in [PoI], [PoII], [PoIII], [PoIV-A and B], and [PoV-A]. And so we move to the next Step II.

STEP II This step is written down, at least in a preliminary version, but it still require a lot of checking as well as a finalized version, reason for which I state it here with a question mark. We consider any homotopy 3-ball Δ^3 for which we construct the X^4 from (2.3). The next step takes the form of an implication which is totally independent of theorem 1.

Theorem 2 (?). *If the open X^4 is geometrically simply-connected, then so is the compact $\Delta^3 \times I$.*

We will refer to this second step, which still remains to be completely firmly established , as "PoV-B".

We move now to the third and last step.

STEP III Just like step II this third step also takes the form of an implication, which is totally independent of Theorem 2 (?) above.

Theorem 3. *Let Δ^3 be a homotopy 3-ball which is such that $\Delta^3 \times I$ is geometrically simply-connected. Then $\Delta^3 = B^3$.*

It should be obvious that, together, the three theorems above imply the Poincaré Conjecture. Before saying more things concerning the three statements above, I will make a general comment. Everything said above is supposed to hold in the DIFF category and this is not just a piece of pedantry. Of course for 3-manifolds, the TOP, PL and DIFF categories are all equivalent. But we are in dimension four in this approach and at several crucial points some form or other of Hauptvermutung has to be involved. And as we know from gauge theory, Hauptvermutung, as such, is violently false in dimension four. For instance, R^4 admits uncountably many distinct PL structures. But according to an old theorem of J.H.C. Whitehead there is uniqueness for the PL structures compatible with a given DIFF structure, irrespective of the dimensions involved. So, we can use this result, provided we never leave the DIFF category.

We will discuss now Step I above. I will start by introducing the "***sort of links***". By definition these are non compact manifolds W^4 with non-empty boundary, such that for some $1 \leq \alpha \leq \infty$

$$\text{int}W^4 = R^4_{\text{standard}}, \quad \partial W^4 = \sum_1^\infty S^1_i \times \text{int}D^2_i (\text{null-framing is assumed here}). \tag{2.4}$$

Here is the most obvious example of a sort of link (and, at the same time, the justification for the name). Consider, to begin with, a smooth pair, and this is how I like to think about links or knots

$$\left(B^4, \sum_1^{N<\infty} S^1_i \times D^2_i \subset \partial B^4\right) \tag{2.5}$$

(with B^4, from now on, the standard smooth 4-ball). With this, the manifold

$$W^4 = B^4 - \Big(\partial B^4 - \sum_1^N S_i^1 \times \mathrm{int} D_i^2\Big) \tag{2.6}$$

is, indeed, a sort of link in the sense of the definition which we just gave. Moreover, a sort of link of this particular type will be said to be *smoothly tame*.

Here is an example of a sort of link which is, I believe, *smoothly wild*. Start with the Whitehead pair of solid tori $T_0 \subset T_1$ and then iterate indefinitely the same embedding

$$T_0 \subset T_1 \subset T_2 \subset T_3 \subset \cdots$$

This leads to the classical Whitehead manifold

$$Wh^3 = \bigcup_0^\infty T_n.$$

It is a standard fact that for this open contractible manifold (which is not R^3, since $\pi_1^\infty Wh^3 \neq 0$) we have the diffeomorphism

$$Wh^3 \times (0,1) = \bigcup_0^\infty T_n \times (0,1) = R^4_{\text{standard}}.$$

With a slight modification of this formula, we can get a sort of link, namely

$$W^4 \overset{\text{def}}{=} \{(\mathrm{int} T_0) \times (0,1]\} \cup \bigcup_1^\infty T_n \times (0,1),$$

with $\partial W^4 = (\mathrm{int} T_0) \times 1$. It turns out that this "sort of knot" is a Casson handle which, I believe, is known to be smoothly wild. Of course all the Casson handles are sort of links (with $\alpha = 1$) and it is known from Freedman's work that all Casson handles are topologically standard, and hence topologically tame. Via Donaldson's and Gompf's work, Casson handles which are not smoothly standard (and hence not smoothly tame either) have to exist.

The sort of links (2.4) with $\alpha < \infty$ are, in certain sense, natural extensions of the classical links and knots. But the distinction smoothly tame versus smoothly wild is absent in the usual word of knot theory. One can speculate whether smooth wildness is detectable by appropriate topological quantum field theory (TQFT), but I will not go into that here.

The first result on the road to Theorem 1 is the following result which, at face value, is a very weak variant of (2.2.?).

Theorem 4. (The smooth tameness theorem). *For every homotopy 3-ball one can find a sort of link V^4 with infinitely many boundary components, i.e. such that*

$$\partial V^4 = \sum_1^\infty S_i^1 \times \mathit{int} D_i^2,$$

and with the following two properties

(A) *We have a diffeomorphism of open 4-manifolds*

$$\mathit{int}((\Delta^3 \times I)\#\infty\#(S^2 \times D^2)) = V^4 + \{\textit{the infinitely many 2-handles } \sum_1^\infty D_i^2 \times \mathit{int} D_i^2 \textit{ corresponding to } \partial V^4\}. \tag{2.7}$$

The 2-handles concerned here are without lateral boundary.

(B) *For any finite $N \in Z_+$, the truncation*

$$V^4|N \stackrel{\text{def}}{=} V^4 - \sum_{N+1}^{\infty} S_i^1 \times intD_i^2 \tag{2.8}$$

can be smoothly compactified into a copy of B^4_{standard}, i.e. it is smoothly tame.

The proof of theorem 4, is contained in [PoIV-A and B] (a long paper which, in turn, relies [PoI] to [PoIII]). This proof which we will not even sketch here, makes heavy use of an *infinite process*, which is quite different from the one used by M. Freedman for the topological 4-dimensional Poincaré Conjecture. For instance, since we want to stay inside the smooth category, no Bing type shrinking can ever be used in our context.

The infinite process via which Theorem 4 can be proved, constructs V^4 by putting together infinitely many compact non simply-connected pieces, a bit like for Wh^3. Also, for each individual finite truncation $V^4|N$, the same infinite process adds infinitely pieces at infinity, until the boundary extends to a copy of R^3. It turns out that this is enough for compactifying $V^4|N$ into a copy of B^4. The reason is the following easy *fact*: In the smooth category, for $n \geq 4$ (in particular for $n = 4$ which is the case of interest for us), there is a unique way to glue R^{n-1} to the infinity of R^n, i.e.

$$R^n \cup R^{n-1} = R^n_+$$

(all the euclidean spaces considered here are standard).

Notice, on the other hand, that for $n = 3$ the analogous fact is false, due to the existence of the Artin-Fox arcs. This will turn out to be a major difficulty, to be overcome, for the proof of the strange compactification below. It is not being claimed, in the context of theorem 4 that the compactifications of the various $V^4|N$ are compatible with each other.

I will give now the main applications of the smooth tameness theorem. Some definitions will be necessary. Let Y^n be a smooth n-manifold (which is not closed). We will say that Y^n is geometrically simply-connected *at long distance*, if for every compact subset $K \subset \text{int}Y^n$ we can find a compact geometrically simply-connected submanifold $M^n \subset Y^n$, such that $K \subset M^n \subset Y^n$. In the special case when Y^n is itself compact bounded, there is an equivalent form of this definition, where only compact objects occur. Let $\partial Y^n \times [0,1] \subset Y^n$ be a collar of the boundary $\partial Y^n = \partial Y^n \times 1$; define

$$Y^n_{\text{small}} \subset M^n - \partial Y^n \times (0,1],$$

i.e. Y^n_{small} is another version of Y^n, canonically embedded inside $\text{int}Y^n$. Then Y^n is geometrically simply-connected at long distance iff there exists a compact geometrically simply-connected submanifold $M^n \subset Y^n$, such that

$$Y^n_{\text{small}} \subset M^n \subset Y^n.$$

COROLLARY 5.A *The open manifold (see (2.3))*

$$X^4 = int((\Delta^3 \times I)\#\infty\#(S^2 \times D^2)),$$

is geometrically simply-connected at long distance.

Proof. Consider a compact subset $K \subset X^4$. We can always find a $N < \infty$, such that

$$K \subset (V^4|N) + \left\{\text{the 2-handles } \sum_1^N D_i^2 \times \text{int} D_i^2\right\} \subset X^4. \tag{2.9}$$

Now, since the sort of link $V^4|N$ is smoothly tame, the

$$Z^4 \overset{\text{def}}{=} (V^4|N) + \sum_1^N D_i^2 \times \text{int} D_i^2$$

which appears in (2.9), is the interior of a *compact* bounded geometrically simply-connected manifold M^4. This M^4 can be slightly pulled inside its interior, so as to get the desired engulfing of K by M^4

$$K \subset M^4 \subset Z^4 \subset Y^4.$$

Corollary 5-B. *For any homotopy 3-ball Δ^3, the 4-manifold $\Delta^3 \times I$ is geometrically simply-connected at long distance.*

Proof. We define $A_n = (\Delta^3 \times I)\# n\#(S^2 \times D^2)$, and consider the standard inclusions $A_n \subset \text{int} A_{n+1}$. With this, our X^4 has the compact exhaustion

$$X^4 = \bigcup_0^\infty A_n.$$

We consider the compact subset $A_0 = \Delta^3 \times I \subset Y^4$, to which we apply corollary 5-A; this gives us a compact geometrically simply-connected $M^4 \subset Y^4$, such that

$$A_0 = \Delta^3 \times I \subset M^4 \subset Y^4.$$

By compactness, there is a finite n such that we also have

$$A_0 \subset M^4 \subset A_n.$$

Since the inclusion $A_0 \subset A_n$ is standard, if we kill the $\# n\#(S^2 \times D^2)$ of A_n with 3-handles, so as to get $\Delta^3 \times I$, the resulting inclusion $A_0 \subset \Delta^3 \times I$ is just our $(\Delta^3 \times I)_{\text{small}} \subset \Delta^3 \times I$, and hence the last formula also yields

$$(\Delta^3 \times I)_{\text{small}} \subset M^4 \subset \Delta^3 \times I.$$

This ends the proof of Corollary 5-B of the smooth theorem. We still need another big step before we can get to theorem 1. This is the following result, the proof of which is to be found in [PoV-A].

Theorem 6. *Let Δ^3 be a smooth compact 4-manifold which is such that $\partial\Delta^4$ is a homology sphere and which, moreover is itself geometrically simply-connected* ***at long distance****. Then the smooth open 4-manifold*

$$\mathit{int}(\Delta^4 \# \infty \# (S^2 \times D^2))$$

is geometrically simply-connected.

Clearly (Corollary 5-B)+ (Theorem 6) $\Longrightarrow$ Theorem 1.

This is about as much as we will say concerning Step I, here, we will come back to the more problematic Step II in the next section, and so we move now to a short discussion concerning the Step III. By assumption our Δ^3 is now such that $\Delta^3 \times I$ is geometrically simply connected.

Let us consider any smooth manifold pair like (2.5), call it now

$$\text{LK} = \big(B^4, \sum_1^{k<\infty} S_i^1 \times D_i^2 \subset \partial B^4\big) \tag{2.10}$$

with null-framing. (This is really what one would like to be a LINK, as opposed to a pair $\sum_1^k S_i^1 \subset M^3$ which one would call a "classical link".)

Starting with (2.10) we perform the following 2-stage construction:

(2.11.a) We fist perform an infinite connected sum along the boundary, far from $\sum_1^k S_i^1 \times D_i^2$, call this process

$$B^4 \implies B^4 \# \infty \# (S^2 \times D^2).$$

(2.11.b) Then we erase any piece of boundary, for $B^4\#\infty\#(S^2 \times D^2)$, except for $\sum_1^k S_i^1 \times \text{int} D_i^2$ itself, which produces the following non-compact 4-manifold (with non-empty boundary)

$$V^4(\text{LK}) \overset{\text{def}}{=} \text{int}(B^4\#\infty\#(S^2 \times D^2)) \cup \sum_1^k S_i^1 \times \text{int} D_i^2.$$

Notice that while

$$B^4\#\infty\#(S^2 \times D^2) \tag{2.12}$$

is not a uniquely defined object, as long as the end-point structure is not specified, the $V^3(\text{LK})$, which is called a *stable sort of link*, is unique, once LK (2.10) is given. We state now the main result of the present section

Theorem 7. (THE STRANGE COMPACTIFICATION THEOREM). *Let Δ^3 be a homotopy 3-ball which is such that $\Delta^3 \times I$ is geometrically simply-connected.*

Then there exists a link LK (2.10) with the following two properties.

A) *If we use LK in order to add k handles of index two to B^4, then we get a diffeomorphism*

$$(\Delta^3 \times I)\#(k\#(S^2 \times D^2)) = \tag{2.13}$$

$$= B^4 + \big\{\textit{the } k \textit{ handles of index two } \sum_1^k D_j^2 \times D_j^2 \textit{ defined by LK}\big\},$$

an equality which re-expresses the geometric simple connectivity of $\Delta^3 \times I$.

B) *For the stable sort of link V^4(LK) which is attached to our LK we can find a noncompact smooth 4-dimensional W^4 with a connected and simply-connected boundary $W^4 = \partial W^4$ and also a smooth embedding*

$$(V^4(\text{LK}), \partial V^4(\text{LK})) \xrightarrow{\xi} (W^4, W^3) \tag{2.14}$$

such that:

B-1) *The restriction $\xi | intV^4(\text{LK})$ is a diffeomorphism $intV^4(\text{LK}) \approx intW^4$.*

B.2) *If we consider the core curves*

$$\Gamma \overset{\text{def}}{=} \sum_1^k S_i^1 \subset \sum_1^k S_i^1 \times D_i^2 = \partial V^4(\text{LK}), \tag{2.15}$$

then the classical link $(W^3, \xi(\Gamma))$ is trivial.

Before discussing the statement, I will give right away its main application.

COROLLARY 8. *Let Δ^3 be a homotopy 3-ball which is such that $\Delta^3 \times I$ is geometrically simply-connected. Then $\Delta^3 = B^3$.*

Sketch of Proof. Since $\Delta^3 \times I$ is geometrically simply-connected, we have, according to Theorem 7, an LK (2.10) with the properties A), B). We also have the classical link $\Gamma \subset S^3 = \partial B^4$ which comes along with (2.10); we denote

$$\pi = \pi_1(S^3 - \Gamma).$$

We consider now the fundamental group at infinity for the non-compact space $V^4(\text{LK})$, namely

$$\pi_1^\infty V^4(\text{LK}) = \varprojlim_C \pi_1(V^4(\text{LK}) - C),$$

where the inverse limit runs over the compact sets $C \subset V^4(\text{LK})$; it turns out that, with some care, we can handle the base-point problems, and this huge topological *pro-group* is well-defined.

The non-compact space $V^4(\text{LK})$ has two, completely distinct, compactifications, namely

(2.16.a) The *standard compactification* which is, by definition the one-point (Alexandrov) compactification of (2.12).

(2.16.b) The one-point compactification $W^4 \cup \{\infty\}$, which we call *strange*, provided by theorem 7.

Each of then two compactifications gives us a way for computing $\pi_1^\infty V^4(\text{LK})$. If we use the standard compactification, we get

$$\pi_1^\infty V^4(\text{LK}) = \varprojlim (\pi \star F_n) \tag{2.17}$$

when "$\star$" means free product and F_n means the free group with n generators. If we use the strange compactification, then we get, on the other hand

$$\pi_1^\infty V^4(\text{LK}) = \varprojlim F_n. \tag{2.18}$$

This *pro-free* group is a very complicated topological group, which is actually *not* free; but it can shown (via Grushko) that every finitely generated subgroup is free. Since (2.17) combined with (2.18) gives us an injection $\pi \subset \varprojlim F_n$, it follows that π is free. So (according to a well-known theorem of Papakyriakopoulos the classical link (S^3, Γ) is trivial. It follows then from (2.13) that

$$(\Delta^3 \times I)\#k\#(S^2 \times D^2) = B^4\#k\#(S^2 \times D^2). \tag{2.19}$$

But this last formula is the same as (2.2) (without question mark now !) and so proceeding exactly as we have done in the beginning of this section we get $\Delta^3 = B^3$.

I will go back now to theorem 7. Extending the boundary $\sum S_i^1 \times \text{int} D_i^2$ of $V^4(\text{LK})$) to the connected $W^3 = \partial W^4$ involves again an infinite process not unlike, the one via which one proves theorem 4. I will not discuss here, at all, the very delicate process of gluing W^3 to the infinity of $V^4(\text{LK})$, but rather talk about the part of the story concerning the pair (W^3, Γ) itself.

The infinite process turns out to provide two "canonical structures" for (W^3, Γ).

I) *The first structure.* The inclusion $\Gamma \subset W^3$ extends to a **PROPER** embedding

$$\sum_1^k (D_i^2 - C_i) \hookrightarrow W^3 \tag{2.20}$$

where each $C_i \subset \text{int} D_i^2$ is a Cantor set and where

$$\sum_1^k \partial(D_i^2 - C_i) = \Gamma.$$

We will say that Γ *can be pulled to the infinity of* W^3. Here is an easy fact concerning this notion.

Lemma 9. *Let X^3 be an open simply-connected 3-manifold which is simply-connected at infinity (for which we will use the notation $\pi_1^\infty X^3 = 0$).*

If $\Gamma \subset X^3$ is a classical link which can be pulled at infinity, then (X^3, Γ) is trivial.

Proof. Inside each $D_i^2 - C_i$ we choose a finite system of simple closed curves $\{\gamma_i\}$ which split off from $D_i - C_i$ a disk with finitely many holes d_i^2, such that $\partial d_i^2 = \Gamma \cap (D_i^2 - C_i) + \{\gamma_i\}$. Since $\pi_1^\infty X^3 = 0$, if we choose $\{\gamma_i\}$ sufficiently close to the infinity of X^3, then we can find an extension g of $\Gamma \subset X^3$ appearing in the following diagram (the triangle of which is not commutative)

$$\begin{array}{ccc} \Gamma \subset \sum_1^k d_i^2 & \longrightarrow & X^3 \\ \downarrow & \swarrow_{g} & \\ \sum_1^k D_i^2 & & \end{array}$$

where $\Gamma = \sum_1^k \partial D_i^2$ and where the singular map g has the Dehn property $\Gamma \cap M_2(g) = \emptyset$, where we have denote by $M_2(g) \subset \sum_1^k D_i^2$ the set of double points of g, i.e. the set of those $x \in \sum_1^k D_i^2$ such that card $g^{-1}g(x) > 1$. Using Dehn's lemma we can produce another extension of $\Gamma \subset X^3$ to an embedded system $\sum_1^k D_i^2 \subset X^3$.

Without the π_1^∞ hypothesis, lemma 9 no longer holds; one can use the Whitehead Wh^3 in order to see this.

We go back now to (2.20); our W^3, which is build by an infinite process, is a *wild* manifold just like Wh^3, in the sense that

$$\pi_1^\infty Wh^3 \neq 0.$$

So, (2.20) above cannot guarantee us that (W^3, Γ) is trivial. [Incidentally, also, $\pi_2 W^2$ is very large, but we will ignore this fact in the present paper].

Fortunately for us, the infinite process produces also a second "canonical structure" for (W^3, Γ).

II) *The second structure.* This is a *lamination* $\mathcal{L}$ of W^3, by planes.

I will start by briefly recalling some basic facts concerning laminations, a notion which is due to W. Thurston. So let X^3 be a 3-manifold which is with $\partial X^3 = \emptyset$. A lamination $\mathcal{L}$ of X^3 is defined by a closed subset

$$K = K(\mathcal{L}) \subset X^3$$

such that X^3 admits a smooth atlas $X^3 = \cup_\alpha U_\alpha$ with coordinate charts

$$(U_\alpha, U_\alpha \cap K) = (R^2 \times R, R^2 \times \{\text{Cantor set}\}).$$

This is analogous to a foliation $\mathcal{F}$ of X^3, except that now the transverse structure is a Cantor set, instead of being the real line. Like for foliations we can talk about plaquettes $R^2 \times \{x\}$ with $x \in$ Cantor, out of which one builds up connected (2-dimensional) leaves L^2 for $\mathcal{L}$. Again like for foliation, the leaves L^2 have extrinsic topologies induced from $L^2 \subset X^3$ but also *intrinsic* 2-manifold topologies. The connected components of the open subset $X^3 - K(\mathcal{L})$ will be called 3-dimensional leaves of $\mathcal{L}$. Such a 3-dimensional leaf admits a natural completion $\overline{L}^3$, with $L^3 = \text{int}\overline{L}^3$ and $\partial\overline{L}^3 = \{$the union of L^2's adjacent to $L^2\}$. For $\overline{L}^3$ one has again to distinguish between extrinsic topology (the only one to be considered here.)

What I have just defined is a lamination *without singulation* Our lamination $\mathcal{L}$ of W^3 will have singularities which are somehow similar to the 1-prong singularities for measured foliation of surfaces with positive Euler characteristic ([FLP]); they turn out to be relatively benign and we will ignore them here.

It would be wrong to think of laminations as being just another variation on the theme of foliations. Here is an instance of a specific phenomenon for which there is no foliation counterpart. It is possible that all L^2's be contractible without the inclusion map $H_*(L^3) \longrightarrow H_*(X^3)$ being injective. This turns out to be one of the many obstacles which one has to overcome, in the proof of the strange compactification theorem.

So, for our W^3, the infinite process creates a lamination $\mathcal{L}$ (with mild local singularities). Here is a list of **Properties of the Laminations** $\mathcal{L}$ of W^3.

(2.21.1) Each (non-singular) leaf L^2 of $\mathcal{L}$ is a plane.

(2.21.2) Each 3-dimensional leaf L^3 of $\mathcal{L}$ has

$$\pi_1 L^3 = \pi_1^\infty L^3 = 0.$$

(2.21.3) For the link $\Gamma \subset W^3$ we have

$$\Gamma \cap K(\mathcal{L}) = \emptyset.$$

All the three properties above are very good, but the next one is not.

(2.21.4) The completed 3-dimensional leaves $\overline{L}^3$ of $\mathcal{L}$ can be *wild*, in a sense which I will explain now.

Notice, first, that each connected component $L^2 \subset \partial\overline{L}^3$ has its location at the infinity of L^3 completely determined by a **PROPER** embedded arc

$$[0,1) \overset{\alpha}{\longrightarrow} L^3,$$

which I call the *wick* of L^2 (inside L^3), and which is defined as follows. Consider a tubular neighbourhood $L^2 \times [0,1] \subset \overline{L}^3$, with $L^2 \times 1 = L^2 \subset \partial\overline{L}^3$ and a point $x \in L^2 = R^2$. Then $\alpha[0,1) = x \times [0,1)$.

The point is that inside the very nice L^3 the wicks can be Artin-Fox type arcs and this is what makes $\overline{L}^3$ wild (and it also accounts for $\pi_1^\infty W^3 \neq 0$.)

Fortunately, the good properties (2.21.1) to (2.21.3) are enough to save the day, since we have the following

Lemma 10. *1) As a consequence of (2.21.1) to (2.21.3) all the links* $(L^3, (L^3 \cap \Gamma)$ *are trivial.*

2) As an immediate consequence of 1), (W^3, Γ) *is also trivial.*

Proof. Because of (2.21.3) each component S_i^1 of Γ (2.15) falls completely inside a 3-dimensional leaf L^3. We will assume $L_i^3 \neq L_j^3$ now, but the general case is not much more difficult. Inside L_i^3 the S_i^1 can again be pulled to infinity and, since $\pi_1^\infty L_i^3 = 0$ (2.21.2), by lemma 9, the classical link (L_i^3, S_i^1) is trivial.

All this should give vague idea about how one achieves B-2) in Theorem 7.

The proof of theorem7 is contained in the long series of papers [PoVI] and the construction of the good LK and of $W^4 \supset V^4(\text{LK})$ which these paper give, is too long and intricate to be given here. By contrast the *description* of $(W^3, \mathcal{L})$ alone is quite short. Here is how it goes.

The infinite process produces (among many other things), the following data.

(2.22.1) A non-compact 3-manifold T^3, which is an infinite connected sum (along the boundary) of elementary pieces, each of which is an infinite connected sum (along the boundary) of elementary pieces, each of which is either $S^1 \times D^2$ or $S^2 \times I$. There are infinitely many pieces of both kinds and also the end-structure is complicated, (the space of ends is a Cantor set.)

(2.22.2) Inside ∂T^3 we have an injection

$$\sum_1^\infty \gamma_i \subset \partial T^3,$$

where each γ_i is a simple closed loop, and the set of γ_i's is conjugated to a free basis of $\pi_1 T^3$.

(2.22.3) There is also a second injection

$$\sum_i^\infty (\Delta_j^2, \partial\Delta_j^2) \subset (T^3, \partial T^3),$$

where the $\Delta_j^{"}$ is an embedded disk, with $\text{int}\Delta_j^2 \subset \text{int}T^3$ and with

$$\partial\Delta_j^2 \cap \gamma_k = \begin{cases} \text{a unique transversal intersection point}, & \text{if } j = k, \\ \emptyset, & \text{if } j \neq k. \end{cases}$$

With this data, we have

$$W^3 = \text{int}\{T^3 + [\text{the infinitely many handles of index } \lambda = 2, \text{ defined by } \sum_1^\infty \gamma_i \subset \partial T^3]\}, \tag{2.23}$$

and this W^3 comes also equipped with a natural embedding

$$\Gamma \subset \text{int}T^3 \subset W^3,$$

which is the one from the theorem, and which can be pulled to infinity (2.20).

Notice, before anything else, that *if* the data above would not concern an infinite situation, but a finite one, then the analogue of (2.23) would be just $R^3 - \{\text{a finite set}\}$. But the data (2.22.1) to (2.22.3) as produced by the infinite process, are not only infinite, but also *wild*, in the sense that neither $\sum_1^\infty \gamma_j \subset \partial T^3$ nor $\sum_1^\infty \Delta_j^2 \subset T^3$ are *closed* subsets. [*Exercise*: give a description like (2.23) for the Whitehead manifold Wh^3.]

We would have reached a dead end, if it were not for the following basic fact:

$$\text{Only } \textit{finitely} \text{ many } \Delta_j^2\text{'s touch } \Gamma. \tag{2.24}$$

Achieving (2.24) puts actually enormous constraints on the infinite process which creates all these things.

The lamination $\mathcal{L}$ is defined by taking the limit points of (2.22.3). More specifically

$$K(\mathcal{L}) = \{\text{the set of points } x_\infty \in \text{int}T^3 \text{ for which} \tag{2.25}$$

$$\text{there exists an infinite sequence } x_i \in \text{int}\Delta_{n_i}, \\ \text{with } n_i \to \infty \text{ and } \lim x_i = x_\infty\} \subset W^3.$$

With this, the crucial property (2.21.3) is an immediate consequence of (2.24) above.

Remark. It is essential that the $\sum_1^\infty \text{int}\Delta_i \subset \text{int}T^3$ accumulate nicely, making (2.25) a lamination. By contrast, the accumulation pattern of $\sum_1^\infty \Delta_i^2 \subset T^3$ is very wild on ∂T^3. It is actually very hard for a 3-dimensional lamination to exhibit good boundary behavior.

3. On Step II of the Poincaré Program, and other loose ends

As we have already said Step II of the program consists of the following implication, a first version of the proof of which is already written, but which still needs careful checking, as well as a more complete finalized writing up

$$\text{(3.1)} \qquad \textit{IF } X^4 = \text{int}[(\Delta^3 \times I)\#\infty\#(S^2 \times D^2)] \text{ is g.s.c.} \quad \Longrightarrow \ \Delta^3 \times I \text{ is g.s.c.}.$$

This [PoV-B], i.e. the (3.1) above (but do not look for the corresponding reference, although it appears between brackets, it is only a manuscript not ready for circulation, yet) has been the object of a long collaboration with David Gabai and so has been another project, which as an outgrowth of it, and about which I will say a few words now.

Let us consider a smooth embedding of S^{n-1} into S^n and remember that according to the classical work of Barry Mazur (supplemented by contribution from Smale, Kervaire and Milnor) this is standard in the DIFF category if $n \neq 4$. We talk here of the celebrated Schoenflies problem, of course. Moreover Barry has also shown that, even if $n = 4$, this is certainly so in the TOP category and even almost so in the DIFF context. More precisely, let $\Delta^4_{\text{Schoenflies}}$, also called a smooth 4-dimensional "Schoenflies ball", be the closure of one of the two regions into which the smoothly embedded S^3 splits S^4. The Barry has shown, long ago, that

$$\text{(3.2)} \qquad \Delta^4_{\text{Schoenflies}} - \{\text{a boundary point}\} \overset{\text{DIFF}}{=\!=} B^4 - \{\text{a boundary point }\},$$

which, of course, immediately implies that

$$\text{(3.3)} \qquad \text{int}\Delta^4_{\text{Schoenflies}} \overset{\text{DIFF}}{=\!=} R^4_{\text{standard}}.$$

There is now an idea (which was triggered by a suggestion which Michael Freedman made some years ago to David Gabai and myself) that the techniques of [PoV-B], if they really work, could be very useful with the 4-dimensional smooth Schoenflies problem too. Here is a first item in the direction and I only give it here with a big question mark.

Proposition 11 (??). *Let $\Delta^4_{\textit{Schoenflies}}$ be a smooth 4-dimensional Schoenflies ball which is geometrically simply-connected. Then this $\Delta^4_{\textit{Schoenflies}}$ is standard, i.e. diffeomorphic to the standard 4-ball.*

The alleged proof is a mixture of 4-dimensional and 2-dimensional topology; Barry's old result above play a big role too. Now much more recently (but this really is by now "work in progress") I have realized that it might be possible to twist around the arguments of [PoV-B] and use the obvious consequence of (3.3) that $\text{int}\Delta^4_{\text{Schoenflies}}$ is geometrically simply connected, so as to, hopefully, show that any smooth 4-dimensional Schoenflies ball is g.s.c.. In the [PoV-B] i.e. (3.1) a great use is made of the collar "with defects" $X^4 - \text{int}(\Delta^3 \times I)$. For $\Delta^4 = \Delta^4_{\text{Schoenflies}}$ one is supposed to use now, in a similar fashion, the bona fide collar $\text{int}\Delta^4 - \text{int}\Delta^4_{\text{small}}$. For many years, the fact that the product structure $\Delta^3 \times I$ (very much used in [PoV-B]) is absent for $\Delta^4_{\text{Schoenflies}}$ was a stumbling block but there seems now to be a way which might allow us to go around this. Together with Proposition 11 (??), this would completely settle the smooth 4-dimensional Schoenflies problem indeed.

There are other outgrowths of the (techniques of the) Poincaré program and I will end this section with some hints concerning my program for "$\pi_1^\infty \widetilde{M}^3 = 0$". The issue now is to show that for any closed 3-manifold M^3, the universal covering space $\widetilde{M}^3$ is simply-connected at infinity. Some time ago I had put out a preprint which was supposed to contain a complete proof of $\pi_1^\infty \widetilde{M}^3 = 0$, but then a gap was detected in one part of that paper, last year. By now I hope that I have managed to fill in completely the gap in question; but I have not written up, yet, a new paper. The point is that, ***if*** the full Thurston geometrization conjecture is proved, then this would vastly supersedes the $\pi_1^\infty \widetilde{M}^3 = 0$. [Putting together the Poincaré Conjecture and $\pi_1^\infty \widetilde{M}^3$ would only show that for any closed M^3,

$$\widetilde{M}^3 = S^3 - \{\text{ends of } \pi_1 M^3\},$$

where in the Cantor case the embedding of the end space into S^3 is tame. I call this very weakened form of Thurston Conjecture, the "topological uniformization".] So I am not in a great hurry to write these things up, right now. On the other hand, my approach to $\pi_1^\infty \widetilde{M}^3 = 0$ is mostly high-dimensional (not just $n = 4$ but with an n al least 7 or 8) and so I am trying to extract, from my proof, final results. going beyond the dimensional $n = 3$.

By now I should explain a bit the whole issue, from scratch. Many years ago, Serghei Novikov asked a number of questions concerning closed n-manifold M^n which are $K(\pi, 1)$. It might have him who first raised the issue of the simple connectivity of their $\widetilde{M}^n$'s. But then, about 1982 M. Davis proved that for all $n \geq 4$ there are closed $M^n = K(\pi, 1)$ with $\pi_1^\infty \widetilde{M}^n \neq 0$. For $n = 3$ the issue is equivalent to our "$\pi_1^\infty \widetilde{M}^3 = 0$", for all closed M^3's. About 1990 myself, A. Casson and others have shown that if $\pi_1 M^3$ satisfies various nice geometric conditions (like almost-convexity, which implies Gromov hyperbolicity, combability, which implies "automatic", a.s.o.), then indeed $\pi_1^\infty \widetilde{M}^3 = 0$. About the same time, I started connecting the issue of π_1^∞ to geometric simple connectivity, and it is to this that I will come now, as it befits the title of this paper.

The diagram below gives various notions extending geometric simple-connectivity, the arrows meaning here "implies"

$$\begin{array}{c} \text{g.s.c. (for manifolds)} \\ \Downarrow \\ \text{g.s.c. at long distance (for manifolds)} \\ \Downarrow \\ \text{weakly geometrically simply-connected} \\ \Downarrow \\ \text{Dehn exhaustibility.} \end{array} \tag{3.4}$$

The notion of g.s.c. at long distance has already been used in the corollary 5-A (and 5-B) above. The last two notions make sense for any locally finite simplicial complex (and even for general locally compact spaces); it is even a meaningful question what they mean for a finitely presented group G. We will say that X is weakly g.s.c. if it has an exhaustion by finite simply-connected subcomplexes. We will say that X is Dehn-exhaustible if for any compact $k \subset X$ there is a commutative diagram

$$\begin{array}{ccc} k & \overset{i}{\hookrightarrow} & K \\ & \searrow \quad \swarrow_{g} & \\ & X & \end{array} \tag{3.5}$$

where

a) K is compact, simply connected.
b) The map g is an immersion, for which we denote by $M_2(g) \subset K$ the set of points $x \in K$ such that card $g^{-1}(g(x)) > 1$.
c) The following "Dehn condition " is satisfied

$$(ik) \cap M_2(g) = \emptyset.$$

The connection with π_1^∞ (and, incidentally, with dimension $n = 3$ too) is the following implication

Proposition 12. *If V^3 is an open simply-connected 3-manifold which is Dehn-exhaustible, then $\pi_1^\infty V^3 = 0$.*

This is the classical Dehn lemma, in disguise. The connections involved in (3.4) are even more interesting. Under same condition which I will not specify more here, the following things happen. Consider a locally finite complex X and some high-dimensional regular neighbourhood $N^n(X)$. Assume that in between X and $N^n(X)$ we can sandwich an n-manifold W^n

$$X \hookrightarrow W^n \underset{\mathcal{J}}{\hookrightarrow} N^n(X) \tag{3.6}$$

such that W^n is weakly g.s.c. and the inclusion $\mathcal{J}$ is PROPER ($\mathcal{J}^{-1}$(compact) = compact). Then X is Dehn-exhaustible. It should be stressed that here the singularities (=non-manifold points) of X are not allowed to be too nasty. Notice also that this allows us to read the third arrow in (3.4) not only as "implies" by also as "descends to".

With all this, the very general idea of the π_1^∞ program is to construct a (singular) 3-dimensional object which is such that, on the one hand it enters in a diagram like (3.6) above and, on the other hand, Dehn exhaustibility can descend from X to

$$\widetilde{M}^3 - \{\text{a closed, totally disconnected, tame subset.}\}$$

The most interesting issue, concerning this circle of ideas is where the symmetry of $\widetilde{M}^3$ comes in, i.e. why should $\pi_1^\infty \widetilde{M}^3$ be zero when $\pi_1^\infty Wh^3 \neq 0$. I cannot go into this here and now and only refer to the items [Po7] (which contains, in a first approximation, a vague description of the π_1^∞ program) and to the more technical [PoTa2].

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