The data that supports the findings of this study are available within the article.

The non-holonomic Herglotz variational problem

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The geometric approach to the study of the Herglotz problem developed in [1] is extended to the case in which the evolution of the system is subject to a set of non-holonomic constraints. The original setup is suitably adapted to the case in study. Various aspects of the problem are considered: the direct derivation of the evolution equations; the super-lagrangian approach; the resulting super-Hamiltonian and its relation with Pontryagin's maximum principle; the abnormality index of the extremals; the invariance properties of the theory and the consequent existence of Herglotz Lagrangians gauge equivalent to ordinary ones.

Keywords: Variational principles in Physics, Herglotz variational problem, Gauge structure of Lagrangian Mechanics, Non-holonomic constraints.

INTRODUCTION

In 1930 Herglotz proposed a variational principle in which the Lagrangian $L(t, q^i, u, \dot{q}^i)$ involved in the definition of the action functional can also depend on the instantaneous value $u(t) = \int_{t_0}^{t} L dt$ of the action itself[2,3].

In recent years, Herglotz's variational principle and its application have attracted significant interest[4–13]. A revisitation of this principle has recently been proposed[1], based on the introduction of the principal fibre bundle $P \to \mathcal{V}_{n+1}$ formed by the totality of *affine scalars* over the configuration manifold.

In addition to the velocity space $j_1(\mathcal{V}_{n+1})$ and the phase space $\Pi(\mathcal{V}_{n+1})$, the geometric setup involves the first jet spaces $j_1(P, \mathbb{R})$, $j_1(P, \mathcal{V}_{n+1})$ associated with the fibrations $P \to \mathbb{R}$ and $P \to \mathcal{V}_{n+1}$, as well as four quotient spaces $\mathfrak{L}(\mathcal{V}_{n+1})$, $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$, $\mathcal{H}(\mathcal{V}_{n+1})$ and $\mathcal{H}^{(c)}(\mathcal{V}_{n+1})$, respectively called the *lagrangian*, *co-lagrangian*, *hamiltonian* and *co-hamiltonian* bundles[14,15].

In the resulting context, the Herglotz Lagrangian $L(t, q^i, u, \dot{q}^i)$ is interpreted as the representation of a section of the co-lagrangian bundle $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$ into the first jet space $j_1(P, \mathbb{R})$.

On this basis, various approaches to the study of the extremals are developed. Among these we recall the direct derivation of the Herglotz equations both in lagrangian and in hamiltonian form, the super-lagrangian and super-hamiltonian formulation of the equations and their comparison with the ordinary prescriptions of constrained variational calculus (Lagrange multipliers, Pontryagin's maximum principle), the evaluation of the abnormality index of the extremals, the study of the invariance properties of the Herglotz functional.

In this paper we extend the analysis to the case in which the presence of kinetic constraints restricts the space of admissible velocities to a submanifold $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$.

Similar restrictions affect the first jet space $j_1(P, \mathbb{R})$ and the associated lagrangian and co-lagrangian bundles, reducing them to sub-bundles $j_1^{(\mathcal{A})}(P, \mathbb{R}) \to j_1(P, \mathbb{R})$, $\mathfrak{L}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \to \mathfrak{L}(\mathcal{V}_{n+1})$ and $\mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \to \mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$.

The geometrical setup, as well as a formulation of the resulting *non-holonomic Herglotz problem*, are described in Sec. I.

A direct characterization of the extremals is presented in Sec. II. The behaviour of the extremality conditions under arbitrary changes of fibred coordinates is discussed.

The identification of $j_1(P, \mathbb{R})$ with an affine subbundle of the tangent bundle $T(P)$ opens the way to a representation of the Herglotz functional in terms of a corresponding *super-Lagrangian*. Through the latter, the original problem is converted into a standard non-holonomic variational problem, solvable by means of the Pontryagin algorithm[16–19].

This aspect is discussed in Sec. III: denoted by $\tilde{H}$ the super-Hamiltonian, the extremals of the Herglotz functional are characterized as the solutions of the Hamilton equations determined by $\tilde{H}$, obeying the first integral $\tilde{H} = 0$.

A subsequent comparison with Maupertuis' Least Action Principle, performed in Sec. IV, completes the analysis. In the same Section it is observed that, as it happens in the holonomic case, the Herglotz extremals are always *abnormal*, with abnormality index ≥ 1.

Finally, in Sec. V, the invariance properties of the Herglotz functional established in [1] are revisited and extended to the case in study, showing the existence of a group of gauge transformations isomorphic to the group of diffeomorphisms $\kappa : P \to P$ fibred over the identity map of $\mathcal{V}_{n+1}$. As in the holonomic case, the result is applied to the characterization of the class of Herglotz Lagrangians gauge-equivalent to ordinary ones, i.e. of Lagrangians giving rise to evolution equations in $\mathcal{V}_{n+1}$ not involving the variable u.

I. PRELIMINARIES

The analysis of the Herglotz problem developed in [1] is naturally extended to systems whose behaviour is subject to restrictions of non-holonomic nature.

The argument will be analysed in variational terms, with the action integral regarded as a functional acting on the totality of curves satisfying the constraints. Technically, this means working with an intrinsic Lagrangian $L(t, q^i, u, z^A)$ depending on t, q^i, u and on a set of control variables z^A, $A = 1, \ldots, r$, and adopting the standard algorithm of control theory (variational equation, Pontryagin maximum principle) to characterize the extremals.

An alternative approach, specifically related to possible applications of the Herglotz principle in the framework of non-holonomic mechanics, will be presented in a forthcoming contribution. For a thorough discussion on the distinction between the variational and the mechanical approach to the study of non-holonomic systems see e.g. [20].

(i) The presence of kinetic constraints does not modify the configuration manifold $\mathcal{V}_{n+1}$ or the bundle of affine scalars $P \to \mathcal{V}_{n+1}$, but restricts the space of admissible kinetic states to a sub-bundle $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$. Similar restrictions affect the overlying bundles $j_1(P,\mathbb{R})$, $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$, $\mathfrak{L}(\mathcal{V}_{n+1})$, reducing them to sub-bundles $j_1^{(\mathcal{A})}(P,\mathbb{R}) \to j_1(P,\mathbb{R})$, $\mathfrak{L}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \to \mathfrak{L}(\mathcal{V}_{n+1})$, $\mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \to \mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$, all sharing a coordinate representation of the form

$$\dot{q}^i = \psi^i(t, q^i, z^A), \qquad i = 1,\ldots,n,\ A = 1,\ldots,r. \tag{1}$$

In all cases the restriction process commutes withe the group actions induced by the vector fields $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial \dot{u}}$, giving rise to the commutative diagram

$$\begin{array}{ccc} j_1^{(\mathcal{A})}(P,\mathbb{R}) & \longrightarrow & \mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \\ \downarrow & & \downarrow \\ \mathfrak{L}_{(\mathcal{A})}(\mathcal{V}_{n+1}) & \longrightarrow & \mathcal{A} \end{array} \tag{2}$$

formally analogous to one valid in the holonomic case.[1]

No changes occur in the hamiltonian setup: the jet-bundle $j_1(P,\mathcal{V}_{n+1})$, as well as the diagram

$$\begin{array}{ccc} j_1(P,\mathcal{V}_{n+1}) & \longrightarrow & \mathcal{H}^{(c)}(\mathcal{V}_{n+1}) \\ \downarrow & & \downarrow \\ \mathcal{H}(\mathcal{V}_{n+1}) & \longrightarrow & \Pi(\mathcal{V}_{n+1}) \end{array} \tag{3}$$

summarizing the definition of the hamiltonian bundle $\mathcal{H}(\mathcal{V}_{n+1})$, of the co-hamiltonian bundle $\mathcal{H}^{(c)}(\mathcal{V}_{n+1})$ and of the phase space $\Pi(\mathcal{V}_{n+1})$, preserve their original meaning.[1]

One point to be noticed for future reference is the fact that the manifold $j_1(P,\mathcal{V}_{n+1})$, viewed as the affine bundle formed by the totality of 1-forms $\omega = du + p_0\,dt + p_i\,dq^i$ over P, coincides with the space of linear connections over the principal fibre bundle $P \to \mathcal{V}_{n+1}$.

(ii) In the lagrangian environment (2), let $\ell : \mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \to j_1^{(\mathcal{A})}(P,\mathbb{R})$ denote a section, described in coordinates as $\dot{u} = L(t, q^i, u, z^A)$. Also, let $\sigma : \mathcal{V}_{n+1} \to P$ denote a section of the bundle $P \to \mathcal{V}_{n+1}$, described as $u = s(t, q^i)$. The (gauge-dependent) function $L = \ell^*(\dot{u})$ is called the non-holonomic Herglotz Lagrangian.

By means of ℓ and σ, every curve $\gamma : q^i = q^i(t)$, $t_0 \le t \le t_1$ in $\mathcal{V}_{n+1}$ can be raised to a curve $\hat{\gamma} : q^i = q^i(t), u = u(t)$ in P, with the function $u(t)$ satisfying the differential equation

$$\frac{du}{dt} = L\left(t, q^i(t), u, \frac{dq^i}{dt}\right) \tag{4}$$

with initial condition $u(t_0) = s(t_0, q^i(t_0))$.

We focus on the functional $I[\gamma] = u(t_1) - u(t_0) = \int_{t_0}^{t_1} L dt$ and look for its extremals. The issue is clearly equivalent to studying the extremals of the functional

$$\hat{I}\,[\hat{\gamma}] = \int_{t_0}^{t_1} L\left(t, q^i, u, z^A\right) dt$$

within the family of curves $\hat{\gamma} : [t_0, t_1] \to P$ which join the initial point $\hat{\gamma}(t_0) = \sigma(\gamma(t_0))$ with a final point $\hat{\gamma}(t_1)$ varying along the fiber $\pi^{-1}(\gamma(t_1))$ and which satisfy, in addition to the kinetic constraints, the further requirement $\frac{du}{dt} = L$.

Strictly speaking, the latter is not a variational problems with fixed endpoints. However, since its aim is the determination of curves along which the first variation of the difference $\hat{I}\,[\hat{\gamma}] = u(t_1) - u(t_0)$ vanishes, it is not surprising that its solution relies on a set of equations identical to those attainable with the standard tools of constrained variational calculus.

We will return to this point in Subsection III. As in the holonomic case, the analysis will show that the fact that the vanishing of $\delta u(t_1)$ is not due to the assignment of the value $u(t_1)$ but to the requirement of stationarity of $\hat{I}\,[\hat{\gamma}]$ does not affect the characterization of the extremals, but their *normality*, assigning them an abnormality index ≥ 1.

II. THE LAGRANGIAN SETUP

(i) Given a curve $\gamma : q^i = q^i(t)$, every admissible deformation with fixed endpoints of γ, lifted to a deformation $q^i = q^i(t,\xi), z^A = z^A(t,\xi)$ of $j_1(\gamma)$, determines a deformation $u = u(t,\xi)$ of the function $u(t)$, fulfilling the integral equation

$$u(t,\xi) = \int_{t_0}^{t} L\big(t, q^i(t,\xi), u(t,\xi), z^A(t,\xi)\big) dt.$$

Setting by $X^i(t) := \left.\frac{\partial q^i}{\partial \xi}\right|_{\xi=0}$, $Z^A(t) := \left.\frac{\partial z^A}{\partial \xi}\right|_{\xi=0}$, $U(t) := \left.\frac{\partial u}{\partial \xi}\right|_{\xi=0}$, we have the variational equations

$$\frac{dX^i}{dt} = \frac{\partial \psi^i}{\partial q^k} X^k + \frac{\partial \psi^i}{\partial z^A} Z^A, \tag{5a}$$

$$\frac{dU}{dt} = \frac{\partial L}{\partial q^k} X^k + \frac{\partial L}{\partial z^A} Z^A + \frac{\partial L}{\partial u} U. \tag{5b}$$

with initial data $X^i(t_0) = U(t_0) = 0$, and with the functions $Z^A(t)$ bound by the requirement that the solutions $X^i(t)$ of eq. (5a) vanish at $t = t_1$.

Eq. (5b) entails the identification

$$U(t_1) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^k} X^k + \frac{\partial L}{\partial z^A} Z^A + \frac{\partial L}{\partial u} U \right) dt = \left. \frac{dI[\gamma_\xi]}{d\xi} \right|_{\xi=0},$$

consistent with the intuitive perception of the extremals of the Herglotz functional as curves $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ along which the solution of eq. (5b) with initial value $U(t_0) = 0$ satisfies $U(t_1) = 0$ for all infinitesimal deformations $X = X^i(t) \left(\frac{\partial}{\partial q^i} \right)_\gamma$ vanishing at the endpoints.

As in [1], along each curve γ we introduce the auxiliary function $g(t) = e^{-\int_{t_0}^{t} \frac{\partial L}{\partial u} dt}$. We have then the identities

$$\frac{dg}{dt} = -g \frac{\partial L}{\partial u}, \tag{6}$$

$$\frac{d}{dt}(gU) = -g \frac{\partial L}{\partial u} U + g \frac{dU}{dt} = g \left(\frac{\partial L}{\partial q^k} X^k + \frac{\partial L}{\partial z^A} Z^A \right), \tag{7}$$

whence also

$$g(t_1)\, U(t_1) = \int_{t_0}^{t_1} g \left(\frac{\partial L}{\partial q^k} X^k + \frac{\partial L}{\partial z^A} Z^A \right) dt.$$

The extremality condition $U(t_1) = 0$ is therefore equivalent to the requirement

$$\int_{t_0}^{t_1} g \left(\frac{\partial L}{\partial q^k} X^k + \frac{\partial L}{\partial z^A} Z^A \right) dt = 0 \tag{8}$$

for all infinitesimal deformations of γ satisfying $X^k(t_0) = X^k(t_1) = 0$.

To analyse the implications of eq. (8), we prolong γ to a curve $\tilde{\gamma} : q^i = q^i(t)$, $u = u(t)$, $p_i = p_i(t)$ in $\mathcal{H}^{(c)}(\mathcal{V}_{n+1})$, with the functions $p_i(t)$ satisfying the evolution law [21]

$$\frac{d}{dt}(g p_i) + g p_k \frac{\partial \psi^k}{\partial q^i} - g \frac{\partial L}{\partial q^i} = 0. \tag{9}$$

Taking eqs. (5a), (9) into account, eq. (8) may be rewritten as

$$0 = \int_{t_0}^{t_1} \left[\left(\frac{d}{dt}(g p_i) + g p_k \frac{\partial \psi^k}{\partial q^i} \right) X^i + g \frac{\partial L}{\partial z^A} Z^A \right] dt =$$

$$= \int_{t_0}^{t_1} g \left(-p_i \frac{dX^i}{dt} + p_k \frac{\partial \psi^k}{\partial q^i} X^i + \frac{\partial L}{\partial z^A} Z^A \right) dt = \int_{t_0}^{t_1} g \left(-p_i \frac{\partial \psi^i}{\partial z^A} + \frac{\partial L}{\partial z^A} \right) Z^A dt. \tag{10}$$

Eq. (10) is required to hold for all choices of the functions $Z^A(t)$ for which the solutions $X^i(t)$ of eq. (5a) vanishing at $t = t_0$ vanish also at $t = t_1$. To formalize this aspect we observe that the solutions of eq. (5a) with initial data $X^i(t_0) = 0$ can be expressed as

$$X^i(t) = (M^{-1})^i{}_k(t) \int_{t_0}^{t} M^k{}_r \frac{\partial \psi^r}{\partial z^A} Z^A dt,$$

${M^i}_k(t)$ being a non-singular matrix fulfilling the evolution equation

$$\frac{d{M^k}_r}{dt} + {M^k}_p \frac{\partial \psi^p}{\partial q^r} = 0 \qquad \left(\Longleftrightarrow \ \frac{d}{dt}{(M^{-1})^i}_k = \frac{\partial \psi^i}{\partial q^s}\,{(M^{-1})^s}_k \right). \tag{11}$$

The request $X^i(t_1) = 0$ is therefore equivalent to the condition

$$\int_{t_0}^{t_1} {M^k}_r \frac{\partial \psi^r}{\partial z^A}\, Z^A\, dt = 0. \tag{12}$$

Denoting by $\mathfrak{M}$ the infinite dimensional vector space formed by r-tuples of functions $\underset{\sim}{Z} = \big(Z^1(t), \dots, Z^r(t)\big)$, we may regard the left-hand side of eq. (12) as the representation of a linear map $\Upsilon : \mathfrak{M} \to \mathbb{R}^n$, and the right-hand side of eq. (10) as the representation of a linear functional $d\mathcal{I} : \mathfrak{M} \to \mathbb{R}$.

The extremality condition (10) requires the vanishing of $d\mathcal{I}$ on the totality of vectors $\underset{\sim}{Z} \in \mathfrak{M}$ satisfying eq. (12), i.e. the validity of the inclusion $\ker \Upsilon \subset (d\mathcal{I})^0$.

As a consequence of the latter, the functional $d\mathcal{I}$ induces a linear functional $\hat{\eta}$ on the image space $\Upsilon(\mathfrak{M}) \subset \mathbb{R}^n$, based on the prescription $\langle \hat{\eta}, X \rangle := \big\langle d\mathcal{I}, \underset{\sim}{Z} \big\rangle \ \forall\, \underset{\sim}{Z} \in \Upsilon^{-1}(X)$.

Extending $\hat{\eta}$ to a linear functional $\eta : \mathbb{R}^n \to \mathbb{R}$ yields the relation $\big\langle d\mathcal{I}, \underset{\sim}{Z} \big\rangle = \big\langle \eta, \Upsilon(\underset{\sim}{Z}) \big\rangle$ $\forall\, \underset{\sim}{Z} \in \mathfrak{M}$, mathematically equivalent to the factorization $d\mathcal{I} = \eta \circ \Upsilon$.

Expressing everything in coordinates we have thus proved that, if γ is an extremal of the Herglotz functional, for any solution $gp_i(t)$ of eq. (9) there exists at least one co-vector $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^{n*}$ satisfying the equality [22]

$$\int_{t_0}^{t_1} g\left(-p_i \frac{\partial \psi^i}{\partial z^A} + \frac{\partial L}{\partial z^A} \right) Z^A dt \ = \ \eta_k \int_{t_0}^{t_1} {M^k}_i \frac{\partial \psi^i}{\partial z^A}\, Z^A\, dt,$$

written equivalently as

$$\int_{t_0}^{t_1} \left[-\big(gp_i + \eta_k {M^k}_i\big) \frac{\partial \psi^i}{\partial z^A} + g\, \frac{\partial L}{\partial z^A} \right] Z^A dt = 0 \qquad \forall\, \underset{\sim}{Z} \in \mathfrak{M}.$$

On the other hand, according to eq. (11), if $gp_i(t)$ is a solution of eq. (9), $gp_i + \eta_k {M^k}_i$ is also a solution. There exists therefore a (possibly non-unique) choice of the functions $p_i(t)$ for which the extremality condition reads

$$\int_{t_0}^{t_1} g\left(-p_i \frac{\partial \psi^i}{\partial z^A} + \frac{\partial L}{\partial z^A} \right) Z^A = 0 \quad \forall\, \underset{\sim}{Z} \in \mathfrak{M} \quad \Longrightarrow \quad p_i \frac{\partial \psi^i}{\partial z^A} = \frac{\partial L}{\partial z^A}\,.$$

Summing up we conclude that the characterization of the extremals of the non-holonomic

Herglotz functional relies on the system of $2n+r+1$ equations

$$\frac{dq^i}{dt} = \psi^i(t, q^i, z^A) \tag{13a}$$

$$\frac{du}{dt} = L(t, q^i, u, z^A) \tag{13b}$$

$$\frac{dp_i}{dt} + p_k \frac{\partial \psi^k}{\partial q^i} - p_i \frac{\partial L}{\partial u} = \frac{\partial L}{\partial q^i} \tag{13c}$$

$$p_i \frac{\partial \psi^i}{\partial z^A} = \frac{\partial L}{\partial z^A} \tag{13d}$$

for the unknowns t, q^i, u, z^A, p_i.

Under the regularity condition $\det\Big[p_i \frac{\partial^2 \psi^i}{\partial z^A \partial z^B} - \frac{\partial^2 L}{\partial z^A \partial z^B}\Big] \neq 0$, eq. (13d) can be solved for the z^A's as functions $z^A = z^A(t, q^i, u, p_i)$. In this way, introducing the expression

$$H(t, q^i, u, p_i) := p_i\, \psi^i\big(t, q^i, z^A(t, q^i, u, p_i)\big) - L\big(t, q^i, u, z^A(t, q^i, u, p_i)\big), \tag{14}$$

one can easily verify the relations

$$\frac{\partial H}{\partial t} = p_k \frac{\partial \psi^k}{\partial t} - \frac{\partial L}{\partial t}\,, \qquad \frac{\partial H}{\partial q^i} = p_k \frac{\partial \psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i}\,, \qquad \frac{\partial H}{\partial u} = -\frac{\partial L}{\partial u}\,, \qquad \frac{\partial H}{\partial p_i} = \psi^i.$$

In view of these and of eq. (14), eqs. (13a,b,c) take the form

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} - p_i \frac{\partial H}{\partial u}\,, \qquad \frac{du}{dt} = -H + p_i \psi^i = -H + p_i \frac{\partial H}{\partial p_i}\,, \tag{15}$$

formally identical to the one valid in the absence of kinetic constraints[1].

For later use we point out the relationship

$$L = p_i \psi^i - H = p_i \frac{\partial H}{\partial p_i} - H, \tag{16}$$

already employed in eq. (15), and the evolution law

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial u}\frac{du}{dt} = \frac{\partial H}{\partial t} - H\frac{\partial H}{\partial u} = -\frac{\partial L}{\partial t} + p_k \frac{\partial \psi^k}{\partial t} + H\frac{\partial L}{\partial u}\,. \tag{17}$$

The gauge dependent expression (14) will be called the non-holonomic Herglotz Hamiltonian. Under an arbitrary change of trivialization $\bar{u} = u + f(t, q^i)$, it undergoes the transformation law

$$\bar{H} = \bar{p}_i \psi^i - \bar{L} = \left(p_i + \frac{\partial f}{\partial q^i}\right)\psi^i - L - \frac{df}{dt} = H - \frac{\partial f}{\partial t}\,.$$

From this, recalling the transformation law $\bar{p}_0 = p_0 + \frac{\partial f}{\partial t}$ for the coordinate function p_0 in $j_1(P, \mathcal{V}_{n+1})$, we conclude that the sum $\varphi = p_0 + H$ is independent of the choice of trivialization of P. As such, φ is a scalar function over $j_1(P,\mathbb{R})$, satisfying the condition $\frac{\partial \varphi}{\partial p_0}=1$, i.e. a trivialization of the bundle $j_1(P,\mathcal{V}_{n+1}) \to \mathcal{H}^{(c)}(\mathcal{V}_{n+1})$. The map $h:\mathcal{H}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P,\mathcal{V}_{n+1})$ described in coordinates as $p_0=-H(t,q^i,u,p_i)$ is therefore a section of this bundle.

(ii) An alternative representation of the Herglotz equations relies on the use of the operation of time derivative of 1-forms along admissible curves [19,23,24].

Referring to [19] for details we recall that, given a curve $\hat{\gamma}: q^i=q^i(t),\ u=u(t)$ in P satisfying the constraints $\frac{dq^i}{dt}=\psi^i$, $\frac{du}{dt}=L$ and a 1-form $\omega=\omega_0\,dt+\omega_i\,dq^i+\omega_u\,du$ along $\hat{\gamma}$, the time derivative of ω is the 1-form

$$\frac{d\omega}{dt}=\frac{d\omega_0}{dt}\,dt+\frac{d\omega_i}{dt}\,dq^i+\omega_i\,d\psi^i+\frac{d\omega_u}{dt}\,du+\omega_u\,dL$$

along the curve $q^i=q^i(t)$, $u=u(t)$, $z^A=z^A(t)$, lift of $\hat{\gamma}$ to $\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$.[25]

Recalling that $j_1(P,\mathcal{V}_{n+1})$ may be identified with the bundle of linear connections over the principal bundle $P \to \mathcal{V}_{n+1}$, we can then state

Theorem 1. *Let γ be an admissible curve in $\mathcal{V}_{n+1}$, prolonged to a curve $\hat{\gamma}:[t_0,t_1] \to P$ through the algorithm (4). Let $\dot{\hat{\gamma}}$ denote the tangent vector to $\hat{\gamma}$. Then γ is an extremal of the Herglotz functional if and only if there exists a connection 1-form $\omega=du-p_0(t)dt-p_i(t)dq^i$ along $\hat{\gamma}$ satisfying the properties*

$$\langle \omega, \dot{\hat{\gamma}} \rangle = L-p_0-p_i\,\psi^i=0 \qquad \Big[\Longleftrightarrow \hat{\gamma} \text{ is the horizontal lift of } \gamma\Big], \tag{18a}$$

$$\frac{d}{dt}\big[g(t)\,\omega\big]=-\frac{\partial L}{\partial u}\,g\,\omega+g\,\frac{d\omega}{dt}=g\left(-\frac{\partial L}{\partial u}\,\omega+dL-\frac{dp_0}{dt}\,dt-\frac{dp_i}{dt}\,dq^i-p_i\,d\psi^i\right)=0. \tag{18b}$$

Proof. Assigning a connection 1-form $du-p_0dt-p_idq^i$ along $\hat{\gamma}$ means prolonging $\hat{\gamma}$ to a curve $\tilde{\gamma}: q^i=q^i(t), u=u(t), p_0=p_0(t), p_i=p_i(t)$ in $j_1(P,\mathcal{V}_{n+1})$. On account of eq. (14), eq. (18a) is then equivalent to the relation

$$p_0+H_{|\tilde{\gamma}}=0.$$

In view of the latter, eq. (18b) takes the form

$$\left(p_0\,\frac{\partial L}{\partial u}+\frac{\partial L}{\partial t}-\frac{dp_0}{dt}-p_i\,\frac{\partial \psi^i}{\partial t}\right)dt+\left(p_i\,\frac{\partial L}{\partial u}+\frac{\partial L}{\partial q^i}-\frac{dp_i}{dt}-p_k\,\frac{\partial \psi^k}{\partial q^i}\right)dq^i+\left(\frac{\partial L}{\partial z^A}-p_i\,\frac{\partial \psi^i}{\partial z^A}\right)dz^A=0,$$

reproducing the content of eqs. (13c,d), (17). □

Notice that, although suggestive, the characterization (18) of the extremals of the Herglotz functional is partially redundant: as already pointed out, the evolution law (17) for the Hamiltonian H is in fact a consequence of eqs. (13c,d).

(ii) An important application of Theorem 1 concerns the analysis of the behaviour of the Herglotz equations under arbitrary change of the coordinate along the fibres of P.

So far, the function u has been identified with a trivialization of P, completing t, q^i to a coordinate system "adapted" to the principal bundle structure of $P \to \mathcal{V}_{n+1}$.

Starting with any such coordinate system, let's now perform a transformation $\bar{u} = G(t, q^i, u)$, with $\frac{\partial G}{\partial u} \neq 0$, and denote by $u = N(t, q^i, \bar{u})$ the corresponding inverse.

We have then the obvious relations $u = G\big(t, q^i, N(t, q^i, u)\big) = N\big(t, q^i, G(t, q^i, u)\big)$, and the consequent equalities

$$1 = \frac{\partial G}{\partial u}\,\frac{\partial N}{\partial \bar{u}}\,, \qquad 0 = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial u}\,\frac{\partial N}{\partial t}\,, \qquad 0 = \frac{\partial G}{\partial q^i} + \frac{\partial G}{\partial u}\,\frac{\partial N}{\partial q^i}\,. \tag{19}$$

In the new coordinates, the description of the section $\ell : \mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \to j_1^{(\mathcal{A})}(P, \mathbb{R})$ takes the form $\dot{\bar{u}} = \bar{L}$, with

$$\bar{L}(t, q^i, \bar{u}, z^A) = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\psi^k + \frac{\partial G}{\partial u}\,L = \frac{1}{\frac{\partial N}{\partial \bar{u}}}\left[L(t, q^i, N(t, q^i, \bar{u}), z^A) - \frac{\partial N}{\partial t} - \frac{\partial N}{\partial q^k}\,\psi^k\right]. \tag{20}$$

Setting $\bar{g} = g\,\frac{\partial N}{\partial \bar{u}}\,\left(\Longleftrightarrow\ g = \bar{g}\,\frac{\partial G}{\partial u}\right)$, eqs. (6), (20) yield the identity

$$0 = \frac{d\bar{g}}{dt}\,\frac{\partial G}{\partial u} + \bar{g}\,\frac{d}{dt}\,\frac{\partial G}{\partial u} + \bar{g}\,\frac{\partial G}{\partial u}\,\frac{\partial L}{\partial u} = \frac{d\bar{g}}{dt}\,\frac{\partial G}{\partial u} + \bar{g}\,\frac{\partial \bar{L}}{\partial u} = \frac{\partial G}{\partial u}\left(\frac{d\bar{g}}{dt} + \bar{g}\,\frac{\partial \bar{L}}{\partial \bar{u}}\right),$$

mathematically equivalent to

$$\frac{d\bar{g}}{dt} + \bar{g}\,\frac{\partial \bar{L}}{\partial \bar{u}} = 0.$$

The function $\bar{g}$ is therefore related to $\frac{\partial \bar{L}}{\partial \bar{u}}$ in the same way as $g(t)$ is related to $\frac{\partial L}{\partial u}$.

Going back to the characterization of the extremals given in Theorem 1 we observe that, as a consequence of the coordinate transformation in P, the fibre coordinates in $\mathcal{H}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1})$ undergo the transformation law

$$\bar{p}_0 = \frac{\partial G}{\partial u}\,p_0 + \frac{\partial G}{\partial t}\,, \qquad \bar{p}_i = \frac{\partial G}{\partial u}\,p_i + \frac{\partial G}{\partial q^i}\,.$$

Restoring the notation $\omega = du - p_0\,dt - p_i\,dq^i$, this entails the equality

$$d\bar{u} - \bar{p}_0\,dt - \bar{p}_i\,dq^i = \frac{\partial G}{\partial u}\,du + \left(\frac{\partial G}{\partial t} - \bar{p}_0\right)dt + \left(\frac{\partial G}{\partial q^i} - \bar{p}_i\right)dq^i = \frac{\partial G}{\partial u}\,\omega,$$

whence also

$$\bar{g}\left(d\bar{u} - \bar{p}_0\, dt - \bar{p}_i\, dq^i\right) = \frac{\partial G}{\partial u}\,\bar{g}\,\omega = g\,\omega.$$

Comparing with Theorem 1, we conclude that the determination of the extremals in the new coordinates rests on equations that are formally identical to (13c,d), (17), the only differences being the replacement of u with $\bar{u}$ and the identification $\bar{L} = \ell^*(\dot{\bar{u}})$.

As in the holonomic case, the previous argument highlights the fact that in the formulation of the Herglotz problem the function u is not subject to any restriction, apart from the condition $\dot{u} = L$.

The reason why, except for special needs, the use of coordinates adapted to the principal bundle structure of P is advisable is that, with this choice, the difference $\varphi = \dot{u} - L$ is a trivialization of the bundle $j_1^{(\mathcal{A})}(P,\mathbb{R}) \to \mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1})$, thus allowing to identify $L = \ell^*(\dot{u})$ with the Lagrangian of the system. In the general case, one would have only the proportionality relation $\dot{\bar{u}} - \bar{L} = \frac{\partial \bar{u}}{\partial u}(\dot{u} - L) = \frac{\partial \bar{u}}{\partial u}\,\varphi$.

III. THE SUPER-LAGRANGIAN

Another way of looking at the Herglotz problem comes from the identification of the first jet bundle $j_1(P,\mathbb{R})$ with the affine subbundle of the tangent space $T(P)$ formed by the totality of vectors X satisfying $\langle X, dt\rangle = 1$.[1]

Referring $T(P)$ to natural coordinates t, q^i, u, t', q'^i, u', this gives rise to a projection $\nu : T_+(P) \to j_1(P,\mathbb{R})$ of the submanifold $T_+(P) = \{X \,|\, X \in T(P), t'(X) > 0\}$ onto $j_1(P,\mathbb{R})$, described in coordinates as $\dot{q}^i = \frac{q'^i}{t'}$, $\dot{u} = \frac{u'}{t'}$.

The inverse image $\nu^{-1}\big(j_1^{(\mathcal{A})}(P,\mathbb{R})\big)$ is then a submanifold of $T_+^{(\mathcal{A})}(P)$, referred to coordinates t, q^i, u, t', z^A, u' and represented by the equations

$$q'^i = t'\psi^i(t, q^i, z^A), \qquad i = 1,\ldots,n,\ A = 1,\ldots,r. \tag{21}$$

Similar considerations apply to the first jet bundle $j_1(P,\mathcal{V}_{n+1})$, viewed as the affine subbundle of the cotangent space $T^*(P)$ formed by the totality of 1-forms η satisfying $y_u(\eta) = \langle \eta, \frac{\partial}{\partial u}\rangle = 1$. Once again, this determines a projection $\nu : T^*_-(P) \to j_1(P,\mathcal{V}_{n+1})$ of the submanifold $T^*_-(P) = \{\omega \in T^*(P), y_u(\omega) < 0\}$ onto $j_1(P,\mathcal{V}_{n+1})$.

Referring $T^*(P)$ — and $T^*_-(P)$ as well — to coordinates t, q^i, u, y_0, y_i, y_u, the map ν is described by the equations

$$p_i = -\frac{y_i}{y_u}\,, \quad p_0 = -\frac{y_0}{y_u}\,, \tag{22}$$

the minus sign reflecting the fact that the coordinates p_0, p_i in $j_1(P, \mathcal{V}_{n+1})$ are defined according to the identification $\omega = du - p_0(\omega)dt - p_i(\omega)dq^i$.

After these premises, let us return to the study of the Herglotz problem. Given a section $\ell : \mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \to j_1^{(\mathcal{A})}(P, \mathbb{R})$ and denoted by $\varphi = \dot{u} - L(t, q^i, u, z^A) \in \mathscr{F}(j_1^{(\mathcal{A})}(P, \mathbb{R}))$ the corresponding trivialization, we lift the opposite $-\varphi$ to a function

$$\tilde{L}(t, q^i, u, t', z^A, u') := -\nu^*(\varphi) = L(t, q^i, u, z^A) - \frac{u'}{t'} \in \mathscr{F}(T^{\mathcal{A}}_{+}(P)).$$

We call $\tilde{L}$ the *super-Lagrangian*, and shift our attention to the variational problem based on the action integral $\int \tilde{L} d\tau$ defined on the totality of curves $t = t(\tau), q^i = q^i(\tau)$, $u = u(\tau)$ satisfying the constraints (21).

This is an ordinary non-holonomic variational problem, whose solutions can be found making use of the Pontryagin algorithm, namely looking for the extremals of the functional

$$\int_{\tau_0}^{\tau_1} \left[\tilde{L} + y_0\left(\frac{dt}{d\tau} - t'\right) + y_i\left(\frac{dq^i}{d\tau} - t'\psi^i\right) + y_u\left(\frac{du}{d\tau} - u'\right)\right] d\tau$$

in the independent variables $t, q^i, u, t', z^A, u', y_0, y_i, y_u$, thought of as coordinates in the fibred product $T^{\mathcal{A}}_{+}(P) \times_P T^*(P)$, regarded as a bundle over the cotangent space $T^*(P)$.

The procedure is well known: setting

$$\tilde{\mathfrak{H}} := -\tilde{L} + y_0 t' + y_i t' \psi^i + y_u u' = -L + y_0 t' + y_i t' \psi^i + \left(y_u + \frac{1}{t'}\right) u' \tag{23}$$

and observing the identities

$$\frac{\partial \tilde{\mathfrak{H}}}{\partial u'} = y_u + \frac{1}{t'} = 0\,, \quad \frac{\partial \tilde{\mathfrak{H}}}{\partial t'} = y_0 + y_i \psi^i - \frac{u'}{t'^2} = 0\,, \quad \frac{\partial \tilde{\mathfrak{H}}}{\partial z^A} = -\frac{\partial L}{\partial z^A} + y_i t' \frac{\partial \psi^i}{\partial z^A} = 0, \tag{24}$$

the characterization of the extremals relies on the equations

$$\frac{du}{d\tau} = \frac{\partial \tilde{\mathfrak{H}}}{\partial y_u} = u', \quad \frac{dq^i}{d\tau} = \frac{\partial \tilde{\mathfrak{H}}}{\partial y_i} = t'\psi^i, \quad \frac{dt}{d\tau} = \frac{\partial \tilde{\mathfrak{H}}}{\partial y_0} = t', \quad \frac{dy_u}{d\tau} = -\frac{\partial \tilde{\mathfrak{H}}}{\partial u} = \frac{\partial L}{\partial u}\,, \tag{25a}$$

$$\frac{dy_i}{d\tau} = -\frac{\partial \tilde{\mathfrak{H}}}{\partial q^i} = \frac{\partial L}{\partial q^i} - y_k\, t' \frac{\partial \psi^k}{\partial q^i}\,, \quad \frac{dy_0}{d\tau} = -\frac{\partial \tilde{\mathfrak{H}}}{\partial t} = \frac{\partial L}{\partial t} - y_k\, t' \frac{\partial \psi^k}{\partial t}\,. \tag{25b}$$

Eqs. (24) entail the relations

$$t' = -\frac{1}{y_u}\,, \qquad u' = \frac{1}{y_u^2}\big(y_0 + y_i\psi^i\big), \qquad \frac{\partial L}{\partial z^A} = -\frac{y_i}{y_u}\frac{\partial \psi^i}{\partial z^A}\,. \tag{26}$$

Under the regularity assumption $\det\left(\frac{\partial^2 L}{\partial z^A\,\partial z^B} + \frac{y_i}{y_u}\,\frac{\partial^2\psi^i}{\partial z^A\,\partial z^B}\right) \neq 0$, these allow to express the variables t', z^A, u' in terms of t, q^i, u, y_0, y_i, y_u, thus defining — at least locally — a section $\psi : T^*(P) \to T^A_+(P) \times_P T^*(P)$, described in coordinates as

$$z^A = z^A\left(t, q^i, u, -\frac{y_i}{y_u}\right), \qquad t' = -\frac{1}{y_u}, \qquad u' = \frac{1}{y_u}\left[\frac{y_0}{y_u} + \frac{y_i}{y_u}\,\psi^i\left(t, q^i, z^A\left(t, q^i, u, -\frac{y_i}{y_u}\right)\right)\right]. \tag{27}$$

Substituting in eq. (23), we get the super-Hamiltonian

$$\tilde H := \psi^*(\mathfrak{H}) = -L - \frac{y_0}{y_u} - \frac{y_i}{y_u}\,\psi^i = -\tilde L, \tag{28}$$

meant as a function over $T^*(P)$. Eqs. (26) - (28) imply the relations

$$\frac{\partial\tilde H}{\partial y_0} = -\frac{1}{y_u}\,, \qquad \frac{\partial\tilde H}{\partial y_i} = -\frac{\psi^i}{y_u}\,, \qquad \frac{\partial\tilde H}{\partial y_u} = \frac{1}{y_u^2}\left(y_0 + y_i\,\psi^i\right),$$
$$\frac{\partial\tilde H}{\partial t} = -\frac{\partial L}{\partial t} - \frac{y_i}{y_u}\,\frac{\partial\psi^i}{\partial t}\,, \qquad \frac{\partial\tilde H}{\partial u} = -\frac{\partial L}{\partial u}\,, \qquad \frac{\partial\tilde H}{\partial q^i} = -\frac{\partial L}{\partial q^i} - \frac{y_k}{y_u}\,\frac{\partial\psi^k}{\partial q^i}\,.$$

In view of these, eqs. (25) take the form

$$\frac{dt}{d\tau} = \frac{\partial\tilde H}{\partial y_0}\,, \quad \frac{dq^i}{d\tau} = t'\psi^i = -\frac{\psi^i}{y_u} = \frac{\partial\tilde H}{\partial y_i}\,, \quad \frac{du}{d\tau} = u' = \frac{1}{y_u^2}\left(y_0 + y_i\,\psi^i\right) = \frac{\partial\tilde H}{\partial y_u}\,, \tag{29a}$$

$$\frac{dy_0}{d\tau} = \frac{\partial L}{\partial t} + \frac{y_i}{y_u}\,\frac{\partial\psi^i}{\partial t} = -\frac{\partial\tilde H}{\partial t}, \quad \frac{dy_i}{d\tau} = \frac{\partial L}{\partial q^i} + \frac{y_k}{y_u}\,\frac{\partial\psi^k}{\partial q^i} = -\frac{\partial\tilde H}{\partial q^i}, \quad \frac{dy_u}{d\tau} = \frac{\partial\tilde L}{\partial u} = -\frac{\partial\tilde H}{\partial u}. \tag{29b}$$

The vanishing of $\frac{\partial\tilde H}{\partial\tau} = 0$ implies the Jacobi equation $\frac{d\tilde H}{d\tau} = 0$, ensuring that the function $\tilde H = -\tilde L = -L + \frac{u'}{t'}$ is conserved along the extremals.

Exactly as it happens in the absence of kinetic constraints, the request $L = \frac{u'}{t'} = \dot u$ is thus converted into a condition on the initial data.[1]

To verify that the previous algorithm restores the content of eqs. (13a,b,c) we observe that eqs. (14), (22), (28) yield the identification

$$\tilde H = -L + p_0 + p_i\,\psi^i = p_0 + H(t, q^i, u, p_i) = -\frac{y_0}{y_u} + H\left(t, q^i, u, -\frac{y_i}{y_u}\right). \tag{30}$$

In view of the latter and of eq. (16) it is easily verified that eqs. (29), restricted to the class of solution satisfying the first integral $\tilde H = 0$, can be rewritten as

$$\frac{dq^i}{d\tau} = \frac{\partial\tilde H}{\partial y_i} = -\frac{1}{y_u}\,\frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial p_i}\,\frac{dt}{d\tau} \qquad\Longrightarrow\qquad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\,,$$
$$\frac{dp_i}{d\tau} = \frac{1}{y_u}\,\frac{\partial\tilde H}{\partial q^i} + \frac{p_i}{y_u}\,\frac{\partial\tilde H}{\partial u} = -\frac{dt}{d\tau}\left(\frac{\partial H}{\partial q^i} + p_i\,\frac{\partial H}{\partial u}\right) \quad\Longrightarrow\quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} - p_i\,\frac{\partial H}{\partial u}\,,$$
$$\frac{du}{d\tau} = \frac{\partial\tilde H}{\partial y_u} = -\frac{\partial H}{\partial p_i}\,\frac{p_i}{y_u} - \frac{p_0}{y_u} = \frac{dt}{d\tau}\left(p_i\,\frac{\partial H}{\partial p_i} - H\right) \quad\Longrightarrow\quad \frac{du}{dt} = p_i\,\frac{\partial H}{\partial p_i} - H = L\,.$$

IV. RELATION TO PONTRYAGIN'S MAXIMUM PRINCIPLE

Given the super-Hamiltonian $\tilde{H}$, Maupertuis' principle of least action allows to regard the solutions of the Hamilton equations fulfilling the first integral $\tilde{H} = 0$ as extremals of the functional $\int y_0\, dt + y_i\, dq^i + y_u\, du$ restricted to the submanifold $\tilde{H} = 0$ of $T^*(P)$.

To profit from this fact, we denote by $V^*(P)$ the bundle of virtual 1-forms over P, identified with the quotient of the cotangent space $T^*(P)$ with respect to the equivalence relation $\omega \sim \omega' \Longleftrightarrow \omega - \omega' \propto dt$, and referred to fibred coordinates t, q^i, u, y_i, y_u.

The quotient map $\omega \to [\omega]$ makes $T^*(P)$ into a principal bundle over $V^*(P)$, and thus also $T^*_-(P)$ into a principal bundle over the submanifold $V^*_-(P) := \big\{[\omega] \in V^*(P),\, y_u([\omega]) < 0\big\}$.

We now observe that, in view of the identification of $j_1(P, \mathcal{V}_{n+1})$ with the submanifold $y_u = -1$ in $T^*_-(P)$, the section $h : \mathcal{H}^{(c)}(\mathcal{V}_{n+1}) \to j_1(P, \mathcal{V}_{n+1})$ described by the Hamiltonian (14) — and, more generally, any section of the bundle $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}^{(c)}(\mathcal{V}_{n+1})$ — determines a $(2n+2)$-dimensional hypersurface $\Sigma : y_u = -1,\, y_0 = -H(t, q^i, u, y_i)$ in $T^*_-(P)$.

Dragging Σ by means of the 1-parameter group of diffeomorphisms φ_ξ generated by the Liouville field $Y = y_0 \frac{\partial}{\partial y_0} + y_i \frac{\partial}{\partial y_i} + y_u \frac{\partial}{\partial y_u}$ in $T^*_-(P)$ produces a $(2n+3)$-dimensional submanifold $S \subset T^*_-(P)$, formed by the totality of points $\big\{z = \varphi_\xi(x),\, x \in \Sigma, \xi \in \mathbb{R}\big\}$.

In coordinates we have the equations

$$y_u(z) = -e^\xi, \qquad y_i(z) = e^\xi y_i(x),$$
$$y_0(z) = e^\xi y_0(x) = -e^\xi H\big(t(x), q^i(x), u(x), y_i(x)\big) = y_u(z) H\left(t(z), q^i(z), u(z), -\frac{y_i(z)}{y_u(z)}\right),$$

assigning to S the cartesian representation

$$y_0 = y_u\, H\left(t, q^i, u, -\frac{y_i}{y_u}\right) := -H_P(t, q^i, u, y_i, y_u). \tag{31}$$

According to the latter, the submanifold S is the image of a section, henceforth denoted by $h_P : V^*_-(P) \to T^*_-(P)$, entirely determined by the original section h. In terms of trivializations, the relation between h and h_P is expressed by the equality

$$y_0 + H_P(t, q^i, u, y_i, y_u) = -y_u\, \nu^*\big(p_0 + H(t, q^i, u, p_i)\big),$$

$\nu : T^*_-(P) \to j_1(P, \mathcal{V}_{n+1})$ denoting the projection (22). For reasons that will be clear soon, the function H_P defined by eq. (31) is called the *Pontryagin Hamiltonian*.

The previous arguments have a direct relevance in the analysis of Maupertuis' principle.

The restriction of Maupertuis' functional to the submanifold $h_P(V_-^*(P))$ reads in fact $\int -H_P\,dt + y_i dq^i + y_u du$, i.e. it is identical to the functional giving rise to the Hamilton equations determined by H_P on the manifold $V_-^*(P)$.

The resulting extremality conditions are therefore

$$\frac{du}{dt} = \frac{\partial H_P}{\partial y_u} = -H - y_u\,\frac{\partial H}{\partial p_i}\,\frac{y_i}{y_u^2} = -H + \frac{\partial H}{\partial p_i}\,p_i = L \tag{32a}$$

$$\frac{dq^i}{dt} = \frac{\partial H_P}{\partial y_i} = \frac{\partial H}{\partial p_i} \tag{32b}$$

$$\frac{dy_u}{dt} = -\frac{\partial H_P}{\partial u} = y_u\,\frac{\partial H}{\partial u} \tag{32c}$$

$$\frac{dy_i}{dt} = -\frac{\partial H_P}{\partial q^i} = y_u\,\frac{\partial H}{\partial q^i} \tag{32d}$$

Eqs. (32c,d) may be replaced by

$$\frac{dp_i}{dt} = -\frac{dy_i}{dt}\,\frac{1}{y_u} + \frac{y_i}{y_u^2}\,\frac{dy_u}{dt} = -\frac{\partial H}{\partial q^i} + \frac{y_i}{y_u}\,\frac{\partial H}{\partial u} = -\frac{\partial H}{\partial q^i} - p_i\,\frac{\partial H}{\partial u}$$

and by the Jacobi equation

$$\frac{dH_P}{dt} = \frac{\partial H_P}{\partial t} \quad\Longrightarrow\quad -\frac{dy_u}{dt}\,H - y_u\,\frac{dH}{dt} = -y_u\,\frac{\partial H}{\partial t} \quad\Longrightarrow\quad \frac{dH}{dt} = \frac{\partial H}{\partial t} - H\,\frac{\partial H}{\partial u}\,.$$

In order to compare the previous algorithm with Pontryagin's maximum principle, we modify the original variational problem, including the requirement $\dot u = L$ among the kinetic constraints and focussing on the action functional

$$\int_{t_0}^{t_1}\left[L + y_i\left(\frac{dq^i}{dt} - \psi^i\right) + \big(y_u + 1\big)\left(\frac{du}{dt} - L\right)\right]dt\,. \tag{33}$$

with the functions $y_i, y_u + 1$ playing the role of Lagrange multipliers.

Subtracting a total time derivative $\frac{du}{dt}$ and setting

$$\mathfrak{H} := y_i\,\psi^i + y_u\,L\,, \tag{34}$$

the functional (33) takes the form

$$\int_{t_0}^{t_1}\left[-\mathfrak{H} + y_i\,\frac{dq^i}{dt} + y_u\,\frac{du}{dt}\right]dt\,. \tag{35}$$

The corresponding extremals are determined by the equations

$$\frac{\partial\mathfrak{H}}{\partial z^A} = y_i\,\frac{\partial\psi^i}{\partial z^A} + y_u\,\frac{\partial L}{\partial z^A} = 0 \qquad [\textit{maximum principle}], \tag{36}$$

$$\frac{dq^i}{dt} = \frac{\partial\mathfrak{H}}{\partial y_i}\,, \qquad \frac{du}{dt} = \frac{\partial\mathfrak{H}}{\partial y_u}\,, \qquad \frac{dy_i}{dt} = -\frac{\partial\mathfrak{H}}{\partial q^i}\,, \qquad \frac{dy_u}{dt} = -\frac{\partial\mathfrak{H}}{\partial u}\,. \tag{37}$$

Under the regularity condition $\det\left[\frac{y_i}{y_u}\frac{\partial^2\psi^i}{\partial z^A\partial z^B}+\frac{\partial^2 L}{\partial z^A\partial z^B}\right]\neq 0$, eq. (36), identical to the last equation (26), determines the variables z^A as functions $z^A(t,q^i,u,p_i)$, with $p_i=-\frac{y_i}{y_u}$. Substituting into eq. (34) yields the expression

$$\mathfrak{H}(t,q^i,u,z^A(t,q^i,u,p_i),y_u,y_i)=-y_u\,(p_i\,\psi^i-L)=-y_u H=H_P\,,$$

identical to the representation (31) of the Pontryagin Hamiltonian. The rest of the comparison follows easily from this identification.

As it happens in the holonomic case[1], also in the presence of kinetic constraints the extremals of the Herglotz functional have an *abnormal* character. To enlighten this aspect we consider the homogenous system

$$\rho_i\,\frac{\partial\psi^i}{\partial z^A}+\rho_u\,\frac{\partial L}{\partial z^A}=0 \tag{38a}$$

$$\frac{d\rho_i}{dt}+\rho_u\,\frac{\partial\psi^k}{\partial q^i}+\rho_u\,\frac{\partial L}{\partial q^i}=0 \tag{38b}$$

$$\frac{d\rho_u}{dt}+\rho_u\,\frac{\partial L}{\partial u}=0 \tag{38c}$$

in the unknowns $\rho_i(t),\rho_u(t)$ along a generic extremal γ, and discuss its solvability.

Eq. (38c) determines ρ_u up to a multiplicative factor, in the form $\rho_u=A\,e^{-\int\frac{\partial L}{\partial u}\,dt}$. Furthermore, on account of eq. (38c), eqs. (38a,b) give rise to the system

$$\frac{\rho_i}{\rho_u}\,\frac{\partial\psi^i}{\partial z^A}+\frac{\partial L}{\partial z^A}=0\,,\qquad \frac{d}{dt}\,\frac{\rho_i}{\rho_u}-\frac{\rho_i}{\rho_u}\,\frac{\partial L}{\partial u}+\frac{\rho_k}{\rho_u}\,\frac{\partial\psi^k}{\partial q^i}+\frac{\partial L}{\partial q^i}=0\,. \tag{39}$$

in the unknowns $\frac{\rho_i}{\rho_u}$.

Comparing with eqs. (13c,d) we see that, along any extremal, eqs. (39) admit the solution $\frac{\rho_i}{\rho_u}=-p_i(t)$. The uniqueness or non-uniqueness of this solution depends on the nature of the kinetic constraints (1) imposed on the system. In any case, the system (38) admits at least ∞^1 solutions $\rho_u=Ae^{-\int\frac{\partial L}{\partial u}\,dt}$, $\rho_i=-A\,p_i\,e^{-\int\frac{\partial L}{\partial u}\,dt}$, $A\in\mathbb{R}$.

The abnormality index of the extremals is therefore ≥ 1.

V. GAUGE STRUCTURE OF THE HERGLOTZ EQUATIONS

Let's finally discuss the possibility that different sections $\ell:\mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1})\to j_1^{(\mathcal{A})}(P,\mathbb{R})$ determine the same extremals in $\mathcal{V}_{n+1}$. The analysis is an adaptation to the case in study of the procedure outlined in [1]. The relevant points are summarized below: let

$$\begin{array}{ccc} P & \xrightarrow{\kappa} & P \\ \big\downarrow & & \big\downarrow \\ \mathcal{V}_{n+1} & = & \mathcal{V}_{n+1} \end{array} \tag{40}$$

be a diffeomorphism of the manifold P, fibred over the identity map of $\mathcal{V}_{n+1}$ and described in coordinates as $\kappa^*(u) = G(t, q^i, u)$, with $\frac{\partial G}{\partial u} > 0$. [26]

The inverse diffeomorphism κ^{-1} is described by $(\kappa^{-1})^*(u) = N(t, q^i, u)$, with the functions G, N related by the identity

$$u = G\big(t, q^i, N(t, q^i, u)\big) = N\big(t, q^i, G(t, q^i, u)\big). \tag{41}$$

The map κ (as well as κ^{-1}) can be raised to a bundle diffeomorphism

$$\begin{array}{ccc} \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) & \xrightarrow{\kappa} & \mathfrak{L}^{(c)}(\mathcal{V}_{n+1}) \\ \big\downarrow & & \big\downarrow \\ j_1(\mathcal{V}_{n+1}) & = & j_1(\mathcal{V}_{n+1}) \end{array}$$

and to a diffeomorphism $\delta\kappa : j_1(P, \mathbb{R}) \to j_1(P, \mathbb{R})$, fibred over the previous one and described in coordinates as

$$(\delta\kappa)^*(u) = G(t, q^i, u)\,, \qquad (\delta\kappa)^*(\dot{u}) = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\dot{q}^k + \frac{\partial G}{\partial u}\,\dot{u}\,. \tag{42}$$

In the presence of kinetic constraints everything holds unchanged, with the diffeomorphisms κ and $\delta\kappa$ restricted to the submanifolds $\mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1})$ and $j_1^{(\mathcal{A})}(P, \mathbb{R})$. The resulting situation is summarized in the fibred diagram

$$\begin{array}{ccc} j_1^{(\mathcal{A})}(P, \mathbb{R}) & \xrightarrow{\delta\kappa} & j_1^{(\mathcal{A})}(P, \mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ \mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) & \xrightarrow{\kappa} & \mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{A} & = & \mathcal{A} \end{array} \tag{43}$$

with

$$(\delta\kappa)^*(u) = G(t, q^i, u)\,, \qquad (\delta\kappa)^*(\dot{u}) = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\psi^k + \frac{\partial G}{\partial u}\,\dot{u}\,.$$

In this setup, let $\ell : \mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \to j_1^{(\mathcal{A})}(P,\mathbb{R})$ be a section, represented by the Lagrangian $L(t,q^i,u,z^A) := \ell^*(\dot{u})$. Then, the composite map $\ell' = \delta\kappa \cdot \ell \cdot \kappa^{-1} : \mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \to j_1^{(\mathcal{A})}(P,\mathbb{R})$ is itself a section, represented by the Lagrangian $L'(t,q^i,u,z^A) = \ell'^*(\dot{u})$, with

$$L' = (\kappa^{-1})^* \cdot \ell^* \cdot (\delta\kappa)^*(\dot{u}) = (\kappa^{-1})^*\left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial q^k}\,\psi^k + \frac{\partial G}{\partial u}\,L\right) = \frac{\partial G}{\partial t}\big(t,q^i,N(t,q^i,u)\big) + \\ + \frac{\partial G}{\partial q^k}\big(t,q^i,N(t,q^i,u)\big)\,\psi^k + \frac{\partial G}{\partial u}\big(t,q^i,N(t,q^i,u)\big)\,L\big(t,q^i,N(t,q^i,u),z^A\big). \quad (44)$$

Theorem 2. *The diffeomorphism $\kappa : P \to P$ transforms the extremals $\hat{\gamma} : [t_0,t_1] \to P$ of the Herglotz functional determined by the section ℓ into extremals $\hat{\gamma}' = \kappa \cdot \hat{\gamma}$ of the functional determined by the section $\ell' = \delta\kappa \cdot \ell \cdot \kappa^{-1}$.*

Proof. Consider the coordinate transformation $\bar{q}^i = q^i$, $\bar{u} = G(t,q^i,u)$ with inverse $q^i = \bar{q}^i$, $u = N(t,\bar{q}^i,u)$, G being the function involved in the representation of the diffeomorphism κ.

Comparing eq. (20) with eq. (44) we see that the representation of the section ℓ in the new coordinates involves a function $\bar{L} = \ell^*(\dot{\bar{u}})$ whose dependence on the variables $t, \bar{q}^i, \bar{u}, \bar{z}^A$ is identical to the dependence of the Lagrangian L' on the variables t, q^i, u, z^A.

In view of the result established in Sec. II on the invariance in form of the Herglotz equations under arbitrary transformations of the coordinate u in P this means that, whenever $\hat{\gamma}' : q^i = f^i(t), u = h(t)$ is a solution of the Herglotz problem determined by the Lagrangian L', the curve $\hat{\gamma} : \bar{q}^i = f^i(t), \bar{u} = h(t)$ is an extremal of the functional $\int_{t_0}^{t_1} \bar{L}dt$, i.e. a solution of the Herglotz problem determined by the section $\ell : \mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}) \to j_1^{(\mathcal{A})}(P,\mathbb{R})$, expressed in the coordinate system $t, q^i, \bar{u}$.

Returning to the coordinates t, q^i, u through the inverse transformation, the same evolution $\hat{\gamma}$ is described by the equations $q^i = f^i(t), u = N(t, f^i(t), h(t))$, identical to the equations describing of the curve $\kappa^{-1} \cdot \hat{\gamma}'$. This proves the equality $\hat{\gamma} = \kappa^{-1} \cdot \hat{\gamma}'$, whence the thesis. □

Theorem 2 extends the notion of gauge invariance to the Herglotz problems of non-holonomic type, pointing out the dynamical equivalence between sections ℓ, ℓ' related by (lifts of) fibred diffeomorphisms $\kappa : P \to P$. As in the holonomic case, this allows to convert ordinary Lagrangians into dynamically equivalent Herglotz Lagrangians and, conversely, to establish when a Herglotz Lagrangian is dynamically equivalent to an ordinary one.[1]

In this connection, we have the following

Corollary 1. *The most general Lagrangian $L(t,q^i,u,z^A)$ gauge-equivalent to an ordinary Lagrangian $L_C(t,q^i,z^A)$ is necessarily of the form*

$$L=\frac{1}{\frac{\partial G}{\partial u}}\left[L_C-\frac{\partial G}{\partial t}-\frac{\partial G}{\partial q^k}\,\psi^k\right] \tag{45}$$

with $G(t,q^i,u)\in\mathscr{F}(P)$ satisfying the condition $\frac{\partial G}{\partial u}\neq 0$. A necessary and sufficient condition for L to admit the representation (45) *is the existence of a function $G(t,q^i,u)\in\mathscr{F}(P)$ satisfying the equations*

$$\frac{\partial G}{\partial u}\neq 0\,,\qquad \frac{\partial}{\partial u}\left(\frac{\partial G}{\partial t}+\frac{\partial G}{\partial q^k}\,\psi^k+\frac{\partial G}{\partial u}\,L\right)=0\,. \tag{46}$$

Proof. Both assertions are straightforward consequences of eq. (44) and of the identity $\kappa^*(f)=f$, valid for any function $f\in\mathscr{F}(\mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1}))$ satisfying $\frac{\partial f}{\partial u}=0$. More specifically: if L is gauge-equivalent to an ordinary Lagrangian L_C, eq. (44) entails the relation

$$\frac{\partial G}{\partial t}+\frac{\partial G}{\partial q^k}\,\psi^k+\frac{\partial G}{\partial u}\,L=\kappa^*(L_C)=L_C$$

mathematically equivalent to eq. (45); conversely, if there exists a function G fulfilling eqs. (46), L is gauge equivalent to the Lagrangian

$$L_C=(\kappa^{-1})^*\left(\frac{\partial G}{\partial t}+\frac{\partial G}{\partial q^k}\,\psi^k+\frac{\partial G}{\partial u}\,L\right)=\frac{\partial G}{\partial t}+\frac{\partial G}{\partial q^k}\,\psi^k+\frac{\partial G}{\partial u}\,L \tag{47}$$

which does not depend on the variable u. □

Remark 1. *In view of the identities*

$$1=\frac{\partial G}{\partial u}\,\frac{\partial N}{\partial u}\,,\qquad 0=\frac{\partial G}{\partial t}+\frac{\partial G}{\partial u}\frac{\partial N}{\partial t}\,,\qquad 0=\frac{\partial G}{\partial q^i}+\frac{\partial G}{\partial u}\,\frac{\partial N}{\partial q^i}\,.$$

resulting from eq. (41), eq. (45) may be written in the equivalent form

$$L=\frac{\partial N}{\partial t}+\frac{\partial N}{\partial q^k}\,\psi^k+\frac{\partial N}{\partial u}\,L_C$$

formally identical to eq. (47), with κ^{-1} in place of κ.

Remark 2. *Corollary 1 includes the gauge-equivalence between ordinary Lagrangians. To this end it suffices to restrict the choice of the diffeomorphism $\kappa:P\to P$ to the class of principal bundle isomorphisms, by imposing the condition $G=u-f(t,q^i)$. Eq.* (45) *takes then the familiar form*

$$L=L_C+\frac{\partial f}{\partial t}+\frac{\partial f}{\partial q^k}\,\psi^k:=L_C+\frac{df}{dt}\,.$$

Another instance of the classical Lagrangian gauge is present in eq. (47): if $G(t,q^i,u)$ is a solution of the differential equation

$$\frac{\partial}{\partial u}\left(\frac{\partial G}{\partial t}+\frac{\partial G}{\partial q^k}\,\dot q^k+\frac{\partial G}{\partial u}\,L\right)=0\,,$$

any other function $G'=G+f(t,q^i)$ is also a solution. The ordinary Lagrangian L_C is therefore defined up to a transformation $L_C\to L_C+\frac{df}{dt}$.

VI. AN EXAMPLE

Most of the topics covered in the text are illustrated in the following example.

In a time interval $[t_0,t_1]$ a vehicle P, starting from an initial position $\mathbf{x}_0$ and moving with constant speed v, must reach a final position $\mathbf{x}_1$. The fuel consumption rate is proportional to the mass of P and increases linearly with the coordinate $x^3(P)$. Problem: determine the evolution $\mathbf{x}=\mathbf{x}(t)$ that extremizes the fuel consumption.

Denoting by $u(t)$ the fuel burned in the interval $[t_0,t]$ and by $m(t)=m_0-u(t)$ the resulting mass of the vehicle, we have the relation

$$\frac{du}{dt}=a+bx^3+\alpha(m_0-u)$$

Leaving aside the unessential constants and introducing the Herglotz Lagrangian

$$L=bx^3-\alpha u \tag{48}$$

what we are looking for are the extremals of the functional $\int_{t_0}^{t_1}L\,dt$ subject to the conditions $\frac{du}{dt}=L$ and to the non-holonomic constraint $\sqrt{\delta_{ij}\,\dot x^i\dot x^j}=v=$const.

Keeping the notation $\mathcal{V}_{n+1}$ (here equal to $E_3\times\mathbb{R}$) for the configuration manifold, the embedding $\mathfrak{L}^{(c)}_{(\mathcal{A})}(\mathcal{V}_{n+1})\to\mathfrak{L}^{(c)}(\mathcal{V}_{n+1})$ is represented in terms of the two generalized velocities ϑ,φ by the equations $\dot x^i=\psi^i$, with

$$\psi^1=v\sin\vartheta\cos\varphi,\qquad \psi^2=v\sin\vartheta\sin\varphi,\qquad \psi^3=v\cos\vartheta.$$

To formulate the Herglotz equations determined by L, we express the variables ϑ,φ in terms of the momenta p_i through the equations

$$p_i\,\frac{\partial\psi^i}{\partial\vartheta}=\frac{\partial L}{\partial\vartheta}=0\quad\Longrightarrow\quad p_1\cos\vartheta\cos\varphi+p_2\cos\vartheta\sin\varphi-p_3\sin\vartheta=0,$$

$$p_i\,\frac{\partial\psi^i}{\partial\varphi}=\frac{\partial L}{\partial\varphi}=0\quad\Longrightarrow\quad -p_1\sin\vartheta\sin\varphi+p_2\sin\vartheta\sin\varphi=0.$$

A straightforward calculation yields the expressions for the control variables

$$\tan\varphi = \frac{p_2}{p_1}\,, \qquad \tan\vartheta = \frac{\sqrt{{p_1}^2 + {p_2}^2}}{p_3}$$

and for the Hamiltonian

$$H = p_i \psi^i - L = v\sqrt{{p_1}^2 + {p_2}^2 + {p_3}^2} - bx^3 + \alpha u. \tag{49}$$

The Herglotz equations (15) read

$$\frac{dx^i}{dt} = \frac{v p_i}{\sqrt{{p_i}^2 + {p_2}^2 + {p_3}^2}}\,, \qquad \frac{dp_i}{dt} = -\alpha p_i + b\delta_i^3, \qquad \frac{du}{dt} = bx^3 - \alpha u,$$

whence, in particular,

$$p_i = C_i\, e^{-\alpha t} + \frac{b}{\alpha}\,\delta_i^3\,. \tag{50}$$

From eq. (49) we derive the Pontryagin Hamiltonian

$$H_P = -y_u H\left(t, x^i, u, -\frac{y_i}{y_u}\right) = -v\sqrt{{y_1}^2 + {y_2}^2 + {y_3}^2} + \big(bx^3 - \alpha u\big)\, y_u$$

with $p_i = -\frac{y_i}{y_u}$. The resulting Hamilton equations read

$$\frac{dx^i}{dt} = \frac{\partial H_P}{\partial y_i} = \frac{-v y_i}{\sqrt{{y_1}^2 + {y_2}^2 + {y_3}^2}}\,, \qquad \frac{du}{dt} = \frac{\partial H_P}{\partial y_u} = bx^3 - \alpha u, \tag{51a}$$

$$\frac{dy_i}{dt} = -\frac{\partial H_P}{\partial x^i} = -b y_u\, \delta_i^3, \qquad \frac{dy_u}{dt} = -\frac{\partial H_P}{\partial u} = \alpha y_u. \tag{51b}$$

These imply the relations

$$y_u = \beta_u e^{\alpha t}, \qquad y_i = \beta_i - \frac{b}{\alpha}\, \beta_u e^{\alpha t} \delta_i^3,$$

consistent with eq. (50), with $C_i = -\frac{\beta_i}{\beta_u}$.

The super-Lagrangian associated with L reads

$$\tilde{L} = bx^3 - \alpha u - \frac{u'}{t'}$$

while the corresponding super-Hamiltonian is given by $\tilde{H} = -\tilde{L}$, with the variables $t', u', \vartheta, \varphi$ expressed in terms of t, x^i, u, y_0, y_i, y_u through eqs. (26), namely[27]

$$\tilde{H} = -bx^3 + \alpha u - \frac{1}{y_u}\big(y_0 - v\sqrt{{y_1}^2 + {y_2}^2 + {y_3}^2}\big).$$

The resulting Hamilton equations are

$$\frac{dt}{d\tau} = -\frac{1}{y_u}, \qquad \frac{du}{d\tau} = \frac{1}{y_u^2}\Big(y_0 - v\sqrt{{y_1}^2 + {y_2}^2 + {y_3}^2}\Big), \qquad \frac{dx^i}{d\tau} = \frac{v}{y_u}\,\frac{y_i}{\sqrt{{y_1}^2 + {y_2}^2 + {y_3}^2}}, \tag{52a}$$

$$\frac{dy_0}{d\tau} = 0, \qquad \frac{dy_u}{d\tau} = -\alpha, \qquad \frac{dy_i}{d\tau} = b\delta_i^3. \tag{52b}$$

On account of the identity $\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = -y_u \frac{d}{d\tau}$ it is easily seen that eqs. (52), formulated in terms of the independent variable t and restricted to the class of solutions satisfying $\tilde{H} = 0$, have the same content as eqs. (51).

As a final remark we observe that the function $G(t, x^i, u) = ue^{\alpha t}$ satisfies the relation

$$\frac{\partial}{\partial u}\left[\frac{\partial G}{\partial t} + \frac{\partial G}{\partial x^i}\,\dot{x}^i + \frac{\partial G}{\partial u}\,L\right] = \frac{\partial}{\partial u}\left(e^{\alpha t}\,bx^3\right) = 0\,,$$

indicating that the Lagrangian (48) is gauge-equivalent to the ordinary Lagrangian

$$L_c = bx^3 e^{\alpha t}.$$

The Pontryagin Hamiltonian H_c associated with L_c is expressed in terms of t, x^i and of three momenta π_i (not to be confused with p_i or y_i) by the equation

$$H_c = v\sqrt{{\pi_1}^2 + {\pi_2}^2 + {\pi_3}^2} - bx^3 e^{\alpha t}.$$

The corresponding Hamilton equations read

$$\frac{dx^i}{dt} = \frac{\partial H_C}{\partial \pi_i} = \frac{v\pi_i}{\sqrt{{\pi_1}^2 + {\pi_2}^2 + {\pi_3}^2}}\,, \qquad \frac{d\pi_i}{dt} = -\frac{\partial H_C}{\partial x^i} = be^{\alpha t}\,\delta_i^3 \tag{53}$$

whence, in particular,

$$\pi_i = C_i + \frac{b}{\alpha}\,e^{\alpha t}\,\delta_i^3 = e^{\alpha t}\left(C_i e^{-\alpha t} + \frac{b}{\alpha}\,\delta_i^3\right).$$

Consistently with the notion of gauge equivalence, eqs. (53) and (51) determine the same curves $x^i = x^i(t)$ in $\mathcal{V}_{n+1}$. On the contrary, the momenta π_i and p_i are different, and are related to each other by the dilation $\pi_i = e^{\alpha t} p_i$.

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[21]For completeness, let us examine the behaviour of eq. (9) under an arbitrary change $\bar{u} = u + f(t, q^i)$ of the trivialization of P. The latter implies the transformation laws $\bar{p}_i = p_i + \frac{\partial f}{\partial q^i}$, $\bar{L}(t, q^i, \bar{u}, z^A) = L(t, q^i, \bar{u} - f, z^A) + \frac{df}{dt}$, whence also

$$\frac{\partial \bar{L}}{\partial \bar{u}} = \frac{\partial L}{\partial u}, \qquad \frac{\partial \bar{L}}{\partial q^i} = \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial u}\frac{\partial f}{\partial q^i} + \frac{\partial}{\partial q^i}\frac{df}{dt} = \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial u}\frac{\partial f}{\partial q^i} + \frac{d}{dt}\frac{\partial f}{\partial q^i} + \frac{\partial f}{\partial q^k}\frac{\partial \psi^k}{\partial q^i}.$$

From these, it is a straightforward matter to verify the equalities

$$\bar{g}(t) = g(t), \qquad \frac{d}{dt}\left(\bar{g}\bar{p}_i\right) + \bar{g}\bar{p}_k \frac{\partial \psi^k}{\partial q^i} - \bar{g}\frac{\partial \bar{L}}{\partial q^i} = \frac{d}{dt}\left(g p_i\right) + g p_k \frac{\partial \psi^k}{\partial q^i} - g\frac{\partial L}{\partial q^i}.$$

[22]More specifically, as shown by the previous discussion, the co-vector η is unique if and only if the map $\Upsilon : \mathfrak{M} \to \mathbb{R}^n$ is surjective.

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[25]Actually, $\frac{d}{dt}$ is the restriction to $\hat{\gamma}$ of the operation of *symbolic time derivative* of differential forms, making sense in any bundle $M \xrightarrow{t} \mathbb{R}$, and identical to Tulczyjew's operator d_T.

[26]As everywhere in the paper, the coordinate function u is assumed to be a trivialization of the principal bundle $P \to \mathcal{V}_{n+1}$. The diffeomorphism (40) is a principal bundle isomorphism if and only if the function G satisfies $\frac{\partial G}{\partial u} = 1$.

[27]The minus signs in front of the square root comes from the relation $y_i \psi^i = -y_u p_i \psi^i = = -y_u v \sqrt{{p_i}^2 + {p_2}^2 + {p_3}^2} = -v\sqrt{{y_i}^2 + {y_2}^2 + {y_3}^2}$.