University of Trento

Department of Mathematics

PhD Thesis in Mathematics

Harmonicity in Slice Analysis: Almansi decomposition and Fueter theorem for several hypercomplex variables

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Cycle:XXXVI

Yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth[...], the fundamental formula with the symbols, i, j, k; namely,

$$i^2 = j^2 = k^2 = ijk = -1$$

which contains the Solution of the Problem, but of course, as an inscription, has long since mouldered away.

W.R. Hamilton

Greetings

Doverosi ringraziamenti vanno ai miei genitori Loris ed Alessandra e a mio fratello Marco, che da sempre mi hanno dato fiducia, dimostrando di credere in me.

Ringrazio Amanda, la mia ragazza. Questo traguardo, come ogni altro, è in parte anche tuo.

Ringrazio il mio supervisore, Alessandro, il quale mi ha aiutato ad entrare nel mondo della ricerca dell'analisi ipercomplessa, dandomi spunti da approfondire e la libertà di concentrarmi su ciò che più mi interessava.

Ringrazio i referee, per la disponibilità, per i gentili commenti e i preziosi suggerimenti riguardo alla tesi.

Ringrazio anche le università di Trento e di Verona, per l'ospitalità che mi hanno riservato in questi anni.

Ringrazio i miei colleghi dottorandi e ricercatori, per il tempo passato assieme e le discussioni su come affrontare il mondo reale con un approccio scientifico.

Ringrazio i miei compagni ed amici, coi quali una risata non manca mai.

Ed infine ringrazio te, Matematica, che non smetterai mai di meravigliare.

Contents

1 Introduction

Hypercomplex analysis is a branch of mathematics that studies functions with domain in algebras that extend complex numbers and aims to expand the richness of complex analysis to more general settings. The first example of these extensions was discovered in 1843 by the Irish mathematician W. R. Hamilton after long and fruitless endeavors to extend the theory of complex numbers to a three dimensional algebra [62]. Hamilton realized that no multiplication without zero divisors could be defined in $\mathbb{R}^3$, forcing him to add another dimension and renouncing to the commutative character of multiplication. In this way he created the real associative, non commutative, division algebra of quaternions, denoted with $\mathbb{H}$ in his honour. After that, new number systems were rapidly discovered, through an iterative process: each step of the process consisted in doubling the dimension of the algebra, by introducing an independent imaginary unit. This necessarily entailed the loss of certain properties. In this way, quaternions could be obtained by complex numbers, introducing the imaginary unit j; the following step would produce Octonions $\mathbb{O}$, that formed a real non associative and non commutative division algebra of dimension 8 and so on. The number systems obtained through this construction were called Cayley-Dickson algebras [13, 29]. A complete different direction was taken by W. K. Clifford. In his work [15] he produced the so called geometric algebras, by defining an associative, anticommutative product between an arbitrary number of imaginary units. These geometric algebras were named Clifford algebras and its elements Clifford numbers. They were combinations of scalars, vectors and k-vectors, generalizing exterior algebras previously introduced by Grassmann [58]. Beside all this apparent flourish of hypercomplex algebras, mathematicians began to ask how much freedom do generalizations of complex numbers have and which system of hypercomplex numbers exist that preserved the basic properties of complex numbers. The answer was given by Frobenius [38] and Zorn [83], setting important limits to these algebraic constractions. Indeed, apart from $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, no other real quadratic division algebras could exist.

After the discovery of all these hypercomplex algebras, there was the need to invent a proper calculus involving these numbers, since their inventors didn't delimit any special class of regular functions. Indeed, that issue was not so easy to achieve, because the two fundamental definitions of holomorphic functions, the existence of a complex derivative and the series expansion, could not define a class of regular functions in the hypercomplex algebras that lead to satisfactory theories. For example, in $\mathbb{H}$, the requirement of the existence of a quaternionic (left) derivative is too strong, since quaternionic affine functions of the form $f(q) = qa + b$, with $a, b \in \mathbb{H}$, are the unique functions to possess that property. On the other hand, any quaternionic function can be writeen as quaternionic series and so the second condition does not pose restrictions at all, leading to a theory equivalent to differentiable functions in $\mathbb{R}^4$ [82].

The first of these new theories that successfully reproduced the richness of complex analysis was conceived in 1935 by R. Fueter [39], by extending the complex Wirtinger operator $\partial/\partial\overline{z} = \frac{1}{2}(\partial_\alpha + i\partial_\beta)$ to the three imaginary units of quaternions, by $\partial/\partial\overline{q} = \frac{1}{2}(\partial_\alpha + i\partial_\beta + j\partial_\gamma + k\partial_\delta)$, known as Cauchy-Riemann-Fueter operator, where we represent $q = \alpha + i\beta + j\gamma + k\delta$. As holomorphic functions belong to the kernel of $\partial/\partial\overline{z}$, he proposed as regular those quaternionic valued functions that belong to the kernel of $\partial/\partial\overline{q}$, now known as Fueter regular functions. This class of functions mimic many properties of holomorphic functions, such as Cauchy's theorem, Cauchy's integral formula, Liouville's theorem and Laurent series expansion [61, 26, 76]. The idea of Fueter was then carried forward to Clifford algebras, by considering regular (now called monogenic) those functions that belong to the kernel of the Dirac (sometimes called Weyl) operator $\overline{\partial} = \partial_{x_0} + \sum_{i=1}^m e_i\partial_{x_i}$. The study of monogenic functions is known as Clifford analysis, which is a very well developed theory (see for example [12, 28, 27, 60, 59]) and can be considered a refinement of harmonic analysis, as the Dirac operator factorizes the Laplacian operator Δ_{m+1} of

$\mathbb{R}^{m+1}$ by $\Delta_{m+1} = \partial\overline{\partial}$, where $\partial = \partial_{x_0} - \sum_{i=1}^m e_i\partial_{x_i}$. While these theories are extremely successful in replicating many important properties of holomorphic functions, the major disappointment is that even the identity, and therefore polynomials and series of the form $\sum_n x^n a_n$, fails to be regular in the sense described above. Furthermore, unlike the complex case, the class of Fueter regular or monogenic functions does not form an algebra, as they are not closed with respect to the pointwise product. Hence, even producing examples of these classes of functions became an issue. In order to solve the latter problem, Fueter devised a two steps method to generate Fueter regular functions from holomorphic functions defined on open sets of the upper complex half plane. Given an holomorphic function $F : D \subset \mathbb{C}^+ \to \mathbb{C}$, $F(\alpha + i\beta) = F_0(\alpha, \beta) + iF_1(\alpha, \beta)$, the first step of Fueter's machinery produces the quaternionic valued function $f = \mathcal{I}(F)$, defined by

$$f(q) = f(q_0 + \mathrm{Im}(q)) = F_0(q_0, |\,\mathrm{Im}(q)|) + \frac{\mathrm{Im}(q)}{|\,\mathrm{Im}(q)|} F_1(q_0, |\,\mathrm{Im}(q)|), \tag{1}$$

for any $q \in \mathbb{H}$ such that $q_0 + i|\,\mathrm{Im}(q)| \in D$. The second step of Fueter's costruction consists in applying the four dimensional Laplacian to the induced quaternionic functions: Fueter's theorem states that $\Delta_4 f$ is a Fueter regular function. This construction was then extended by Sce [79] to Clifford algebras $\mathbb{R}_m$, with an even number of imaginary units, replacing Δ_4 with $\Delta_{m+1}^{\frac{m-1}{2}}$ (the constant $\frac{m-1}{2}$ is called Sce exponent of the algebra $\mathbb{R}_m$, [21]) and finally completed by Qian [77] in the odd case, where he used techniques of Fourier analysis dealing with fractional powers of the Laplacian operator. The relevance of this construction is also attested by the amount of studies related to the Fueter mapping, see e.g. [30, 80, 17, 18, 4]

However, the algebraic poorness of the above-cited functions theories urged mathematicians to propose a notion of regular hypercomplex function that contained polynomials, preserving also the algebraic richness of complex analysis. For example, Leutwiler, Eriksson and their collaborators developed a modification of classical Clifford analysis that incorporates the powers of the variable into the kernel of the modified Cauchy-Riemann-Fueter and Dirac operator, respectively in the quaternionic and Clifford setting [35, 67]. Another functions theory that overcomes the algebraic challenge encountered by monogenic functions is the theory of holomorphic Cliffordian functions, founded by Laville and Ramadanoff [66, 65], which are null solutions of a higher order differential operator. More precisely, for any odd number m, holomorphic Cliffordian functions are sufficiently differentiable function $f : \mathbb{R}^{m+1} \to \mathbb{R}_m$ in the kernel of $\overline{\partial}\Delta_{m+1}^{\frac{m-1}{2}}$. The class of holomorphic Cliffordian functions contains the monogenic functions (by Fueter-Sce theorem), together with polynomials of the form $\sum_n x^n a_n$ of any degree. Recently, a broader class of functions has been considered: holomorphic Cliffordian functions of order k [24], i.e. functions in the kernel of $\overline{\partial}\Delta_{m+1}^k$, for $0 \le k \le \frac{m-1}{2}$.

In 2006-2007, a new hypercomplex function theory was introduced by Gentili and Struppa [41, 42], exploiting the complex-slice structure of $\mathbb{H}$. Among quaternion-valued functions, they selected the class of slice regular functions: real differentiable functions, which are slice by slice holomorphic. They initially refered to this class as C-regular functions, in honor of Cullen, who previously conceived the idea [25] and the class of slice regular functions was restricted to functions whose domains are Euclidean balls with real centre. The main novelty of the theory was that convergent power series, in particular polynomials, of the form $\sum_{n=0}^{\infty} x^n a_n$ are (left)-slice regular on such open balls. This new subject has attracted considerable interest and experienced a rapid growth over the last years. In fact, the theory was soon generalized to more general domains of definition, the so called slice domains [16] and extended to octonions [44] and Clifford algebras [23]. A new viewpoint took place after the work of Ghiloni and Perotti [52] with the introduction of the notion of stem functions, which are complex instrinsic functions, closely related to the holomorphic functions considered by Fueter in the first step of his construction.

Indeed, the quaternionic functions defined in (1) are examples of slice-preserving slice regular functions. This approach gave slice analysis a crucial development, extending the theory to any real alternative *-algebra with unity, embodying the aforementioned generalizations. Moreover, from this formulation, it was possible to give a definition of slice function, without requiring any regularity assumption. The theory of slice regular functions of one variable is nowadays well-established and has found significant applications in the study of quaternionic quantum mechanics and spectral theory of several operators, see e.g.[64, 22]. Without any claim of completeness, we refer the reader to the monograph [40] or to [46, 47, 54, 31, 69, 49, 43, 56, 33, 20, 2] and the references therein, for a comprehensive treatment of the theory of slice regular functions. The stem functions' approach paved the way to achieve an analogous theory in several variables in the foundational paper [50], which is now of great interest (see e.g. [55, 51, 9, 19, 57, 32, 14]).

Despite the theory of monogenic and slice regular functions are very skew, namely only locally constant functions join both theories (Corollary 8.2), they present some connections [73, 45, 81], as Fueter construction in the first instance suggests. Indeed, beside including slice regular functions in the class of holomorphic Cliffordian functions, we can infer in modern terminology, that Fueter-Sce theorem is a bridge between the slice and the monogenic worlds, which is constitued by the Fueter-Sce mapping $\Delta_{m+1}^{\frac{m-1}{2}}$. Another link is provided by the relation between the spherical derivative and the Dirac operator applied to slice regular functions: they coincide up to a multiplicative constant which depends uniquely on the dimension of the algebra. Thus, another formulation of Fueter-Sce theorem is that the spherical derivative of an $\mathbb{R}_m$-valued slice regular function is polyharmonic of degree exactly the Sce exponent. But polyharmonicity is not limited to spherical derivatives, but applies to slice regular functions, too. Indeed, since by Fueter-Sce theorem, slice regular functions belongs to the kernel of $\overline{\partial}\Delta_{m+1}^{\frac{m-1}{2}}$, they are in particular $\frac{m+1}{2}$-polyharmonic.

It is clear that polyharmonicity plays an importan role in slice analysis and harmonic properties of slice regular functions has been intensively studied [74, 11, 3]. A peculiar property of polyharmonic functions was proven in 1899, when E. Almansi [1] proved that any polyharmonic function f of degree m, defined on a star-like domain centered at the origin of some $\mathbb{R}^n$ could be written as a combination of m harmonic functions $\{\mathcal{S}_i(f)\}_{i=0}^{m-1}$ as

$$f(x) = \mathcal{S}_0(f)(x) + |x|^2\mathcal{S}_1(f)(x) + \cdots + |x|^{2(m-1)}\mathcal{S}_{m-1}(f)(x).$$

This is a very important theorem that establishes a bridge between harmonic and polyharmonic functions [5]. Generalizations of Almansi decomposition have been studied both concerning other type of iterated differential operators (e.g. in [78] for the Dunkl Laplacian and in [36] for a discrete version in umbral calculus), both for more general classes of functions, especially in the hypercomplex settings of slice regular [70, 71] and monogenic functions [68].

The aim of this work is to study in more depth the harmonic properties that concern the theory of slice regular functions, both in one and several variables. As regards the theory in one variable, we have found several formulas that express the iterative application of the Laplacian, evaluated both on the spherical derivative and on the slice regular function itself. This makes their harmonic properties evident, at least in the quaternionic case and in Clifford algebras generated by an odd number of imaginary units. Furthermore, we investigate which functions can be both slice regular and Cliffordian holomorphic of a degree lower than the critical index determined by Sce exponent. At this problem we find a rather limited answer, namely that only slice regular polynomials that are annihilated by the degree of the differential operator can belong to both classes. However, the major contribution concerns the theory in several variables. First, we study the properties of partial sliceness and extend the concepts of spherical value and derivative. Then, we provide Almansi-type decompositions for slice regular functions of several quaternionic and Clifford variables. Finally we extend Fueter and Fueter-Sce theorems in the context of several variables.

We describe the structure of the work. Beside this introduction, in the following section we give the algebraic preliminaries we will use throughout the notes. We recall the definitions of the quaternionic and Clifford algebras and more generally of a real associative $*$-algebra. We stress the definition of quadratic cone, which is the general setting in which slice analysis can be studied.

The third section is dedicated to the definition of slice regular functions, in the context of a general real associative $*$-algebra, exploiting the universality of the stem functions language. We first recall the one variable theory, with the main definitions and a few results. Then we introduce the several variables theory, that presents a heavier notation, but analogue ideas of the one variable counterpart. Again, the definition of slice function goes back to stem functions, where the even-odd properties of each component are preserved with respect to every variable.

In the fourth section we study partial sliceness of slice functions in several variables, namely sliceness properties with respect to a specific variable or sets of variables. In particular, we study conditions the stem functions must satisfy for their induced slice functions to be slice, slice regular or circular with respect to some variables. For sliceness and circularity, these conditions are given by the annihilation of some components of the inducing stem functions (Propositions 4.1, 4.4), while for slice regularity, holomorphicity, together with sliceness in such variables is required (Proposition 4.2).

The study of these partial slice properties is useful to understand the generalization of spherical value and derivative for several variables functions. Indeed, we introduce partial spherical values and derivatives for slice functions in several variables by defining the inducing stem functions. Thanks to the previous characterizations, it is immediate to see that partial spherical values and derivatives of slice functions have circular and partial slice properties. Moreover, we extend some features of their one variable analogues. The section ends recalling an important characterization of slice regularity in several variables, that exploits the notion of partial slice regularity. Indeed, it is possible to interpret the slice regularity of an n-variables slice function in terms of the one-variable slice regularity of $2^n - 1$ slice functions [50, Theorem 3.23], known as truncated spherical derivatives, which are iterations of partial spherical values and derivatives. This result establishes a bridge between the one and several variables theories, which has been frequently used, for example in [75], where local slice analysis was naturally extended from one to several quaternionic variables. A similar characterization will be given in section 6, through the components of the Almansi decomposition.

Section 5 deals with harmonic properties of slice regular functions of one and several variables. First of all, we recall that the spherical derivative of a quaternionic-valued slice regular function is harmonic (Proposition 5.1). Furthermore, we compute the k-th power of the Laplacian applied to the spherical derivative of a $\mathbb{R}_m$-valued slice regular function (Proposition 5.2). As a consequence, we deduce the polyharmonicity of such spherical derivative, in Clifford algebras generated by an odd number of imaginary units and the degree of polyharmonicity is exactly the Sce exponent. Furthermore, we compute arbitrary powers of the Laplacian of a slice regular function, deducing its polyharmonic character (Theorem 5.6). Finally, we propose a method to generate polyharmonic functions from harmonic functions in the plane (Proposition 5.8). This will allow to prove the same polyharmonic properties of slice regular functions in several variables (Corollaries 5.9, 5.10).

In Section 6 we study Almansi decompositions for slice functions of one and several hypercomplex variables. First, we recall the classical Almansi decomposition for polyharmonic functions (Theorem 6.1), that can be applied to slice regular functions, thanks to the computation of the previous section. Then, we give an Almansi-type decomposition (we will call it slice-Almansi decomposition to distinguish it from the classical one) for slice functions of one quaternionic and Clifford variable (Theorems 6.3, 6.4). The components are given through spherical derivatives. Then, we present slice-Almansi decompositions for slice functions of several variables in a unified way. The extension to higher dimensions poses some new challenges, one of which is the exponential growth of all possible decompositions. Indeed, for slice functions of n variables, we obtain 2^n decompositions (Theorem 6.6), as the cardinality of all possible choices of variables between $x_1, ..., x_n$. Every component is circular with respect to the chosen variables that determine the decomposition; if, moreover, the decomposing function is slice regular, they are harmonic in the same variables, too. Every component of each decomposition is given explicitly and it is completely determined by the original function through its partial spherical derivatives, as in their one variable counterpart. We also prove the unique character of these decompositions (Proposition 6.10), namely the functions performing the decomposition are unique, if specific symmetry properties are required. Among these, we point out the class of ordered decompositions, corresponding to integers intervals of the form $\{1, 2, ..., m\}$ (Corollary 6.11). These components have already been exploited to define strongly slice regular functions of several variables, which are a generalization of slice regular functions defined on a not necessarily axially symmetric domain [75]. Such special components are sufficient for characterizing the slice regularity of the slice function they decompose (Proposition 6.12). We then show that the two characterizations are essentially equivalent (Lemma 6.13). Furthermore, by exploiting the harmonicity of the components of the slice-Almansi decompositions we give some mean value and Poisson formulas in the quaternionic case.

Section 7 gives an interpretation of our studies through Clifford analysis, by exploiting that spherical derivatives agree with the Dirac operator on slice regular functions. Thus, we can prove Fueter and Fueter-Sce theorem by using the results of Section 5. Furthermore, we extend these known results in the several variables setting splitting the cases of quaternions and Clifford algebras. In both situations, we give two proofs, one makes use of the results of Section 5, the other uses the ordered slice-Almansi decompositions, that can be written also in terms of the Dirac operator $\overline{\partial}_{x_h}$ applied iteratively, instead of the partial spherical derivatives.

In Section 8 we investigates the kernel of $\overline{\partial}\Delta_{m+1}^k$, restricted to slice regular functions. It is found (Theorem 8.1) that the Sce exponent is a critical index, indeed, for any k less than $\frac{m-1}{2}$, polynomial of degree at most $2k$ are the only slice regular holomorphic Cliffordian functions of order k. Instead, for $k \geq \frac{m-1}{2}$, every slice regular function is holomorphic Cliffordian of order k as Fueter-Sce theorem provides.

Finally, in Section 9, we outline possible directions for future research.

2 Preliminaries

We follow [34, 60] for a breaf presentation of quaternions and Clifford algebras, then we adapt the notions of [52, 47] in the associative setting.

2.1 Quaternions

Quaternions were introduced by Hamilton in 1843 by adding a multiplicative structure to $\mathbb{R}^4$. Any quaternion can be represented as $q = 1\alpha + i\beta + j\gamma + k\delta$, where $\{1, i, j, k\}$ forms a basis of $\mathbb{R}^4$ and multiplication on the basis elements is defined as follows: 1 is the unity of the algebra, while the other three elements satisfy

$$i^2 = j^2 = k^2 = ijk = -1.$$

The product is extended to the whole algebra by bylinearity. The vector space $\mathbb{R}^4$, endowed with this product is known as the algebra of quaternions, denoted with $\mathbb{H}$. Note that $\mathbb{H}$ is a non-commutative, associative algebra. We can embed $\mathbb{R} \subset \mathbb{H}$ as the subspace generated by 1. Moreover, if $\mathrm{Im}(\mathbb{H}) = \mathrm{span}(i, j, k)$, we have that

$$\mathbb{H} = \mathbb{R} \oplus \mathrm{Im}(\mathbb{H}).$$

$\mathrm{Im}(\mathbb{H})$ is also characterized as $\mathrm{Im}(\mathbb{H}) = \{q \in \mathbb{H} : q^2 \in \mathbb{R}^-\} \cup \{0\}$. Moreover, let us define $\mathbb{S}_{\mathbb{H}} = \{q \in \mathrm{Im}(\mathbb{H}) \mid q^2 = -1\}$.

We can define the conjugation of a quaternion, namely if $q = \alpha + i\beta + j\gamma + k\delta$, then $\overline{q} = \alpha - i\beta - j\gamma - k\delta$. This gives $\mathbb{H}$ the structure of a real associative $*$-algebra. Moreover, we have $\mathrm{Im}(\mathbb{H}) = \{q \in \mathbb{H} : \overline{q} = -q\}$.

2.2 Clifford algebras

[34, 60] Let $m \in \mathbb{N}$ and let $\{e_0, e_1, ..., e_m\}$ be an orthonormal basis of $\mathbb{R}^{m+1}$. Let us define the following product rule

$$\begin{aligned} &e_i \cdot e_0 = e_0 \cdot e_i = e_i, \qquad \forall i = 1, ..., m \\ &e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}, \qquad \forall i, j = 1, ..., m. \end{aligned} \tag{2}$$

The Clifford algebra $\mathbb{R}_m$ is the vector space of dimension 2^m generated by

$$\begin{aligned} &\{e_0; e_1, ..., e_m; e_1 \cdot e_2, \ldots; e_1 \cdot e_2 \cdot e_3, \ldots; \ldots; e_1 \cdot ... \cdot e_m\} \\ &= \{e_A = e_{\{i_1,...,i_k\}} : A = \{i_1, ..., i_k\} \in \mathcal{P}(\{1, ..., m\}), 1 \leq i_1 < \cdots < i_k \leq m\}, \end{aligned}$$

i.e. by all the possible ordered products of $e_0, \ldots, e_m$ and it is endowed with the product defined in (2), extended by associativity and bilinearity to the all algebra. For $m > 2$, $\mathbb{R}_m$ is an associative, non commutative algebra, which is not an integral domain. For $m = 1$, it reduces to the complex numbers, while $\mathbb{R}_2 \cong \mathbb{H}$.

Any Clifford number $x \in \mathbb{R}_m$ can be uniquely written as

$$x = \sum_{A \in \mathcal{P}(\{1,...,m\})} x_A e_A,$$

where $x_A \in \mathbb{R}$ and if $A = \{i_1, ..., i_k\}$, with $1 \leq i_1 < \cdots < i_k$, $e_A \coloneqq e_{i_1} \cdot \cdots \cdot e_{i_k}$. We write $e_0 = e_\emptyset = 1$, the unity of the algebra. We can decompose the Clifford algebra $\mathbb{R}_m$ as

$$\mathbb{R}_m = \bigoplus_{k=1}^{m} \mathbb{R}_m^k,$$

where

$$\mathbb{R}_m^k = \{x = [x]_k = x_A e_A \in \mathbb{R}_m : |A| = k\}.$$

With this decomposition, any Clifford number can be respresented as $x = [x]_0 + [x]_1 + \dots + [x]_m$, where $[x]_k$ is the projection of x in $\mathbb{R}_m^k$. Elements contained in $\mathbb{R}_m^0 = \mathrm{span}(e_0)$ or $\mathbb{R}_m^1 = \mathrm{span}(e_1, \dots, e_m)$ will respectively be called real numbers and vectors. Elements in their direct sum $\mathbb{R}_m^0 \oplus \mathbb{R}_m^1 = \mathrm{span}(e_0, e_1, \dots, e_m)$ are called paravectors and they are of the form $x = x_0 + \sum_{|A|=1} x_A e_A = x_0 + \sum_{j=1}^m x_j e_j$. The subspace of $\mathbb{R}_m$ consisting of paravectors is isomorphic to $\mathbb{R}^{m+1}$ by the isomorphism

$$\mathbb{R}^{m+1} \ni (x_0, x_1, \dots, x_m) \mapsto x_0 + \sum_{j=1}^m x_j e_j \in \mathbb{R}_m^0 \oplus \mathbb{R}_m^1$$

For this reason, we will simply denote the subspace of paravectors with $\mathbb{R}^{m+1}$.

We can define a conjugations on Clifford numbers, too. If $x = [x]_0 + [x]_1 + \dots + [x]_m$, then

$$\overline{x} = [x]_0 - [x]_1 - [x]_2 + [x]_3 + [x]_4 - \dots \tag{3}$$

This makes $\mathbb{R}_m$ a real associative $*$-algebra.

2.3 Real associative *-algebras

Let A be a finite dimensional real associative algebra,[1]. Suppose that A has a unity 1 and denote with $\mathbb{R}$ the subspace of A generated by 1, i.e. $\mathbb{R} = \mathrm{span}(1)$. Let $\mathrm{Im}(A) \coloneqq \{x \in A \setminus \mathbb{R} \mid x^2 \in \mathbb{R}^-\} \cup \{0\}$. In general, $\mathrm{Im}(A)$ is not a subspace of A, but for finite dimensional associative algebras A, $\mathrm{Im}(A)$ is a subspace of A and we can decompose A as [Frobenius Lemma, 10.8.2.1]

$$A = \mathbb{R} \oplus \mathrm{Im}(A).$$

Let us endow A with an involutory antiautomorphism (or antiinvolution), namely a linear map $A \ni x \mapsto x^c \in A$ such that

1. $(x^c)^c = x$, $\forall x \in A$;
2. $x^c = x$, $\forall x \in \mathbb{R}$;
3. $(xy)^c = y^c x^c$, $\forall x, y \in A$.

The couple $(A, ^c)$ is called *-algebra.

We can define two maps through the antiinvolution. For any $x \in A$, define its trace and its (squared) norm respectively as

$$t(x) \coloneqq x + x^c, \qquad n(x) \coloneqq x x^c.$$

Now, we can select the set of imaginary units compatible with the antiinvolution. Define

$$\mathbb{S}_{(A,^c)} \coloneqq \{J \in A : t(J) = 0, n(J) = 1\} = \{J \in A : J^c = -J, J^2 = -1\} \subset \mathrm{Im}(A).$$

Even though $\mathbb{S}_{(A,^c)}$ depends on the antiinvolution ^{c}, we will simply use the symbol $\mathbb{S}_A$. Note that for any $J \in \mathbb{S}_A$, the subspace $\mathbb{C}_J \coloneqq \mathrm{span}(1, J)$ is a *-algebra isomorphic to $\mathbb{C}$, via the *-algebra isomorphism

$$\phi_J : \mathbb{C} \to \mathbb{C}_J, \qquad \phi_J(a + ib) \coloneqq a + Jb. \tag{4}$$

[1] In this notes we will consider only the cases $A = \mathbb{H}, \mathbb{R}_m$, but everything can be set in real alternative $*$-algebras

Finally, we can define the quadratic cone of A as

$$\mathcal{Q}_A \coloneqq \bigcup_{J\in\mathbb{S}_A} \mathbb{C}_J.$$

The quadratic cone truly relfects the book structure with complex pages of the algebra. Indeed, for any $x \in \mathcal{Q}_A \setminus \mathbb{R}$, there exist unique $\alpha, \beta \in \mathbb{R}$, with $\beta > 0$ and $J \in \mathbb{S}_A$ such that $x = \alpha + J\beta$.

Example 1. Let $A = \mathbb{H}$, endowed with the quaternionic conjugation $\overline{q} = q_0 - iq_1 - jq_2 - kq_3$. Then $t(q) = 2q_0 = 0$ if and only if $q \in \mathrm{Im}(\mathbb{H})$ and $n(q) = 1$ implies that $q_1^2 + q_2^2 + q_3^2 = 1$. Hence,

$$\mathbb{S}_{\mathbb{H}} = \{iq_1 + jq_2 + kq_3 \in \mathrm{Im}(\mathbb{H}) : q_1^2 + q_2^2 + q_3^2 = 1\} \cong \mathbb{S}^2.$$

In particular, $\mathcal{Q}_{\mathbb{H}} = \mathbb{H}$.

Example 2. Assume again $A = \mathbb{H}$, but consider the antiinvolution given by the reflection on the hyperplane orthogonal to a given $\alpha \in \mathrm{Im}(\mathbb{H})$:

$$q^c \coloneqq q - 2\frac{< \alpha, q > \alpha}{|\alpha|^2} = \alpha\overline{q}\alpha^{-1}.$$

It is easy to see that $q \mapsto q^c$ is an antiinvolution and $t(q) = 0$ if and only if $q \in \mathrm{span}(\alpha)$. Thus, in this case we have

$$\mathbb{S}_{(\mathbb{H},^c)} = \pm\{\alpha/|\alpha|\}, \qquad \mathcal{Q}_{(\mathbb{H},^c)} = \mathbb{C}_\alpha.$$

As the previous examples underlines, the sphere of imaginary units and the quadratic cone of the algebra A heavily depend on the antiinvolution. Moreover, in general it holds $\mathcal{Q}_A \subsetneq A$. It can be proven that only for exceptional cases equality can hold, namely, if A is a (non necessarily associative) $*$-algebra, we have

$$\mathcal{Q}_A = A \iff (A,^c) \cong (\mathbb{C},^-), (\mathbb{H},^-), (\mathbb{O},^-),$$

where $^-$ denotes the usual conjugation of complex numbers, quaternions or octonions respectively.

Example 3. Let $A = \mathbb{R}_3$, endowed with the conjugation defined in (3). By definition,

$$\mathbb{S}_{\mathbb{R}_3} = \{x \in \mathbb{R}_3 : t(x) = 0, n(x) = 1\} = \{x \in \mathbb{R}_3 : x + \overline{x} = 0, x\overline{x} = 1\}.$$

Let us represent any $x \in \mathbb{R}_3$ as

$$x = x_0 + [x]_1 + [x]_2 + [x]_3,$$

then

$$\overline{x} = x_0 - [x]_1 - [x]_2 + [x]_3$$

and so

$$t(x) = 0 \iff x_0 = [x]_3 = 0.$$

Thus, if $x \in \mathbb{S}_{\mathbb{R}_3}$, $x = [x]_1 + [x]_2 = \sum_{i=1}^3 x_1 e_1 + \sum_{i<j=1}^3 x_{ij} e_{ij}$. Moreover,

$$\begin{aligned} x\overline{x} = -x^2 = &- \left(\sum_{i=1}^3 x_1 e_1 + \sum_{i<j=1}^3 x_{ij} e_{ij} \right) \\ &= x_1^2 + x_2^2 + x_3^2 + x_{12}^2 + x_{13}^2 + x_{23}^2 - 2(x_1 x_{23} - x_2 x_{13} + x_3 x_{12}) e_{123}, \end{aligned}$$

thus

$$x\overline{x}=1 \iff \begin{cases} x_1x_{23}-x_2x_{13}+x_3x_{12}=0 \\ x_1^2+x_2^2+x_3^2+x_{12}^2+x_{13}^2+x_{23}^2=1. \end{cases}$$

We have then

$$\mathbb{S}_{\mathbb{R}_3}=\left\{\sum_{i=1}^{3}x_1e_1+\sum_{i<j=1}^{3}x_{ij}e_{ij}:x_1x_{23}-x_2x_{13}+x_3x_{12}=0,x_1^2+x_2^2+x_3^2+x_{12}^2+x_{13}^2+x_{23}^2=1\right\}.$$

Moreover,

$$\mathcal{Q}_{\mathbb{R}_3}=\bigcup_{J\in\mathbb{S}_{\mathbb{R}_3}}\mathbb{C}_J=\left\{\sum_{i=1}^{3}x_1e_1+\sum_{i<j=1}^{3}x_{ij}e_{ij}:x_1x_{23}-x_2x_{13}+x_3x_{12}=0\right\}.$$

Note that, the paravector space $\mathbb{R}^4=\left\{x=x_0+\sum_{i=1}^{3}x_ie_1\right\}$ is contained in $\mathbb{Q}_{\mathbb{R}_3}$.

Remark 1. We can prove that the paravector space $\mathbb{R}^{m+1}$ is always contained in the quadratic cone of $\mathbb{R}_m$. Indeed, the sphere

$$\mathbb{S}^m=\left\{\sum_{i=1}^{m}x_ie_i:\sum_{i=1}^{m}x_1^2=1\right\}\subset\mathbb{S}_{\mathbb{R}_m},$$

since, if $x\in\mathbb{S}^m$, $x+\overline{x}=0$ and $x\overline{x}=1$. Moreover,

$$\mathbb{R}^{m+1}=\bigcup_{J\in\mathbb{S}^m}\mathbb{C}_J\subset\bigcup_{J\in\mathbb{S}_{\mathbb{R}_m}}\mathbb{C}_J=\mathcal{Q}_{\mathbb{R}_m}.$$

Important remark. For the rest of the paper, $(A,^c)$ will stand for $(\mathbb{H},^-)$ or $(\mathbb{R}_m,^-)$.

3 Slice regular functions on real associative *-algebras

3.1 One variable theory

We present the theory of slice regular functions of one variable as in [52], reducing it to an associative setting.

Let $(A,^c)$ be a $*$-algebra with unity and let $\{1, e_1\}$ denote a basis of $\mathbb{R}^2$. Consider the algebra $A \otimes \mathbb{R}^2 = \{a + e_1 b : a, b \in A\}$, where 1 is the unity of $A \otimes \mathbb{R}^2$ and $e_1^2 = -1$, thus the product of any elements of $A \otimes \mathbb{R}^2$ is defined by bilinearity as

$$(a + e_1 b)(\alpha + e_1 \beta) = a\alpha - b\beta + e_1(a\beta + b\alpha),$$

where $a\alpha$ is just the product of A, whenever $a, \alpha \in A$. Equip $A \otimes \mathbb{R}^2$ with the conjugation

$$\overline{a + e_1 b} = a - e_1 b.$$

This makes $(A \otimes \mathbb{R}^2, ^-)$ a $*$-algebra.

Definition 3.1. A set $D \subset \mathbb{C}$ is called symmetric if it is invariant with respect to conjugation, i.e.

$$z \in D \iff \overline{z} \in D.$$

A function $F : D \subset \mathbb{C} \to A \otimes \mathbb{R}^2$, where $D \subset \mathbb{C}$ is an open symmetric set, is called a stem function if it is complex instrinsic, i.e. it satisfies

$$F(\overline{z}) = \overline{F(z)}, \qquad \forall z \in D. \tag{5}$$

Note that if $F = F_0 + e_1 F_1$, with $F_0, F_1 : D \to A$, it is equivalent to require the components of F satisfy the following even-odd conditions with respect to the imaginary part of z:

$$F_0(\overline{z}) = F_0(z), \quad F_1(\overline{z}) = -F_1(z), \qquad \forall z \in D.$$

The set of stem functions over D is denoted by $Stem(D)$.

Multiplication by e_1 defines a complex structure on $A \otimes \mathbb{R}^2$. Given a stem function $F \in \mathcal{C}^1(D)$, consider the following Wirtinger operators

$$\frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial F}{\partial \alpha} - e_1 \frac{\partial F}{\partial \beta}\right), \qquad \frac{\partial F}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial \alpha} + e_1 \frac{\partial F}{\partial \beta}\right).$$

A stem function is said to be holomorphic if $F \in \ker(\partial/\partial\overline{z})$. This is equivalent to require its components F_0, F_1 satisfy the following Cauchy-Riemann equations:

$$\frac{\partial F_0}{\partial \alpha} = \frac{\partial F_1}{\partial \beta}, \qquad \frac{\partial F_0}{\partial \beta} = -\frac{\partial F_1}{\partial \alpha}.$$

Note that $\partial F/\partial z$ and $\partial F/\partial \overline{z}$ are stem functions, too.

Definition 3.2. Given a symmetric set $D \subset A$, we define its circularization Ω_D as

$$\Omega_D \coloneqq \bigcup_{J \in \mathbb{S}_A} \phi_J(D) = \{\alpha + J\beta : \alpha + i\beta \in D, J \in \mathbb{S}_A\} \subset \mathcal{Q}_A.$$

A set Ω is called circular, or axially symmetric, if $\Omega = \Omega_D$ for some symmetric set $D \subset \mathbb{C}$. An axially symmetric set Ω_D is called slice domain if $D \cap \mathbb{R} \neq \emptyset$, while product domain if $D \cap \mathbb{R} = \emptyset$.

Every stem function $F : D \to A \otimes \mathbb{R}^2$ induces (uniquely) a function $f : \Omega_D \to A$ as follows:

Definition 3.3. Let $F = F_0 + e_1 F_1 : D \to A \otimes \mathbb{R}^2$ be a stem function. Given any $x = \alpha + J\beta = \phi_J(z) \in \Omega_D$, define

$$f(x) = F_0(z) + JF_1(z). \tag{6}$$

We will say that f is induced by F ($f = \mathcal{I}(F)$) and such induced functions are called (left) slice functions. We denote by $\mathcal{S}(\Omega_D)$ the set of slice functions over Ω_D and

$$\mathcal{I} : \mathit{Stem}(D) \to \mathcal{S}(\Omega_D)$$

the map sending a stem function to its induced slice function.

Remark 2. We can also define right slice functions as functions satisfying $f = F_0 + F_1 J$, instead of (6), for some stem function $F_0 + e_1 F_1$. Obviously, the theory of left and right slice functions are equivalent. In this work we will only consider left slice functions, that will be simply called slice functions.

Definition 3.4. Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ be a slice function. We say that f is slice regular if F is holomorphic. $\mathcal{SR}(\Omega_D)$ will denote the set of slice regular functions over Ω_D.

Remark 3. Note that the even-odd properties of the components of stem functions make slice functions well defined. Indeed, let $f = \mathcal{I}(F)$, with $F = F_0 + e_1 F_1$, then we can represent any $x = \alpha + J\beta = \alpha + (-J)(-\beta)$, then

$$\begin{aligned} f(x) = f(\alpha + (-J)(-\beta)) &= F_0(\alpha - i\beta) + (-J)F_1(\alpha - i\beta) = F_0(\alpha + i\beta) + (-J)(-F_1(\alpha + i\beta)) \\ &= F_0(\alpha + i\beta) + JF_1(\alpha + i\beta) = f(\alpha + J\beta) = f(x). \end{aligned}$$

We can also define slice derivatives of a slice function $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D) \cap \mathcal{C}^1(\Omega_D)$ as

$$\frac{\partial f}{\partial x} = \mathcal{I}\left(\frac{\partial F}{\partial z}\right), \qquad \frac{\partial f}{\partial x^c} = \mathcal{I}\left(\frac{\partial F}{\partial \overline{z}}\right).$$

Note that they are well defined, since $\partial F/\partial z$ and $\partial F/\partial \overline{z}$ are stem functions. Forthermore, a slice function f is slice regular if and only if $\partial f/\partial x^c = 0$.

Proposition 3.1 ([52], Proposition 8)**.** *Let $f \in \mathcal{C}^1(\Omega_D)$ be a slice function, then f is slice regular if and only if it is slice by slice holomorphic, namely its restriction*

$$f_J : \mathbb{C}_J \cap \Omega_D \ni (\alpha + J\beta) \mapsto f(\alpha + J\beta) \in A$$

is holomorphic for every $J \in \mathbb{S}_A$ with respect to the complex structure defined by left multiplication by J, i.e. it satisfies

$$\left(\frac{\partial}{\partial \alpha} + J\frac{\partial}{\partial \beta}\right) f_J = 0.$$

This is equivalent to the original definition of slice regular functions defined over slice domains of the quaternionic algebra [42].

Alternatively, we can define slice functions through a commutative diagram too. For any $J \in \mathbb{S}_A$ define

$$\Phi_J : A \otimes \mathbb{R}^2 \ni a + e_1 b \mapsto a + Jb \in A.$$

Given $F \in \mathit{Stem}(D)$, its induced slice function $f = \mathcal{I}(F)$ is defined as the unique slice function that makes the following diagram commutative for any $J \in \mathbb{S}_A$:

$$\begin{tikzcd} D \arrow[r, "F"] \arrow[d, "\phi_J"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes \mathbb{R}^2 \arrow[d, "\Phi_J"] \\ \mathcal{Q}_A \supset \Omega_D \arrow[r, "f"'] & A. \end{tikzcd}$$

Indeed, for any $z \in D$ and any $J \in \mathbb{S}_A$, (6) means exactly $f(\phi_J(z)) = \Phi_J(F(z))$.

Every slice function is uniquely determined by its value on two distinct half planes $\mathbb{C}_J^+$ and $\mathbb{C}_K^+$, if $J - K$ is invertible.

Proposition 3.2 ([52], Proposition 6)**.** *Let $f \in \mathcal{S}(\Omega_D)$, define its restriction on the complex half plane $f_J^+ := f|_{\mathbb{C}_J^+ \cap \Omega_D}$, then, for any $x = \alpha + I\beta \in \Omega_D$ we have*

$$f(\alpha + I\beta) = (I - K)(J - K)^{-1} f_J^+(\alpha + J\beta) - (I - J)(J - K)^{-1} f_K^+(\alpha + K\beta).$$

In particular, taking $K = -J$, we recover any slice function by its value on a complex plane $\mathbb{C}_J$:

$$f(\alpha + I\beta) = \frac{1}{2}\left(f_J^+(\alpha + J\beta) + f_J^+(\alpha - J\beta)\right) - \frac{1}{2} IJ \left(f_J^+(\alpha + J\beta) - f_J^+(\alpha - J\beta)\right). \tag{7}$$

Important remark. When $A = \mathbb{R}_m$, we know (Remark 1) that $\mathcal{Q}_A \supset \mathbb{R}^{m+1}$. In this case, we can consider the restriction of any slice functions on domains $\Omega = \Omega_D \cap \mathbb{R}^{m+1}$, for any symmetric domain $\Omega_D \subset \mathcal{Q}_{\mathbb{R}_m}$. Thanks to Proposition 3.2, this restriction uniquely determines the slice function. We will use the same symbol to denote the slice function and its restriction, since no confusion arises. Similarly, we will denote $\Omega_D = \bigcup_{J \in \mathbb{S}^m} \phi_J(D) \subset \mathbb{R}^{m+1}$ for the restriction to the paravector space of the domain of a slice function.

The previous formulas are known as representation formulas. On the slice $\mathbb{C}_J$, the previous formula reduces to

$$f(x) = \frac{1}{2}\left(f(x) + f(x^c)\right) + \frac{1}{2}\left(f(x) - f(x^c)\right),$$

with $x = \alpha + J\beta$ and $x^c = \alpha - J\beta$. Note that, if $x = \phi_J(z)$, it holds

$$\frac{1}{2}\left(f(x) + f(x^c)\right) = F_0(z)$$

and if $\operatorname{Im}(x) \neq 0$,

$$[2\operatorname{Im}(x)]^{-1}\left(f(x) - f(x^c)\right) = F_1(z)/\beta$$

with $F_0(z)$ and $F_1(z)/\beta$ A-valued stem functions. This leads to the following

Definition 3.5. Given $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$, with $F = F_0 + e_1 F_1$, define the spherical value and the spherical derivative of f respectively as

$$\begin{aligned} f_s^\circ(x) &:= \mathcal{I}(F_0)(x) = \frac{1}{2}\left(f(x) + f(x^c)\right), \qquad \forall x \in \Omega_D \\ f_s'(x) &:= \mathcal{I}(F_1/\operatorname{Im}(z))(x) = [2\operatorname{Im}(x)]^{-1}\left(f(x) - f(x^c)\right), \qquad \forall x \in \Omega_D \setminus \mathbb{R}. \end{aligned}$$

Since $F_0(z)$ and $F_1(z)/\operatorname{Im}(z)$ are A-valued, $f_s^\circ(x)$ and $f_s'(x)$ depends only on $\operatorname{Re}(x)$ and $|\operatorname{Im}(x)|$. This means that f_s° and f_s' are constant on every sphere $\mathbb{S}_{\alpha,\beta} = \{\alpha + I\beta : I \in \mathbb{S}_A\}$. Moreover, they decompose f through

$$f(x) = f^\circ(x) + \operatorname{Im}(x) f_s'(x).$$

Definition 3.6. Let $f = \mathcal{I}(F)$ be a slice function. Suppose that $F = F_0$, i.e. F is an A-valued stem function. Then we call f a circular slice function. Namely, circular slice functions are slice functions which are constant over "spheres" $\mathbb{S}_{\alpha,\beta} = \{\alpha + I\beta : I \in \mathbb{S}_A\}$.

Remark 4. Note that if $A = \mathbb{H}$, then $\mathbb{S}_{\alpha,\beta}$ are actually spheres. When $A = \mathbb{R}_m$, they are not in general. However, if we consider imaginary units in the paravector space, $\mathbb{S}_{\alpha,\beta} = \{\alpha + I\beta : I \in \mathbb{S}^m\}$ are spheres, too.

By definition, the spherical value and the spherical derivative of slice function are circular slice functions.

Proposition 3.3. *Let f be a circular slice function. Then $f_s^\circ = f$ and $f_s' = 0$. In particular, for any slice function f it holds $(f_s')_s' = (f_s^\circ)_s' = 0$, $(f_s^\circ)_s^\circ = f_s^\circ$ and $(f_s')_s^\circ = f_s'$.*

We can give stem functions and slice functions the structure of algebras, by defining a product of stem functions, that will induce one on slice functions.

Definition 3.7. Let $F, G \in Stem(D)$, with $F = F_0 + e_1F_1$ and $G = G_\emptyset + e_1G_1$. Define

$$F \otimes G := F_0G_\emptyset - F_1G_1 + e_1(F_0G_1 + F_1G_0).$$

It is easy to prove that $F \otimes G$ is a stem function, hence $(Stem(D), \otimes)$ is an algebra. Now, if $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$, define

$$f \odot g = \mathcal{I}(F \otimes G).$$

Thus, $(\mathcal{S}(\Omega_D), \odot)$ is an algebra, too and $\mathcal{I} : (Stem(D), \otimes) \to (\mathcal{S}(\Omega_D), \odot)$ is an algebra isomorphism.

With respect to this product, the spherical derivative satisfies a Lebniz rule, in which evaluation is replaced by spherical value:

$$(f \odot g)_s' = f_s' \odot g_s^\circ + f_s^\circ \odot g_s'.$$

3.2 Several variables theory

We follow [50] for the several variables version of the theory of slice regular functions. Note that we restrict our attention to associative algebras.

Let $n \in \mathbb{N}^*$ be a positive integer and let $\mathcal{P}(n) := \mathcal{P}(\{1, ..., n\})$ denote all possible subsets of $\{1, ..., n\}$. Given a sequence $x = (x_1, \dots x_n) \in A^n$, define its ordered product as $[x] = x_1$ if $n = 1$ and for $n \geq 2$,

$$[x] = [x_1, \dots x_n] = x_1 \cdot x_2 \cdots x_{n-1} \cdot x_n.$$

Moreover, given $y \in A$, we denote

$$[x, y] = [x_1, \dots, x_n, y] = x_1 \cdot x_2 \cdots x_{n-1} \cdot x_n \cdot y.$$

Let $K = \{k_1, ..., k_p\} \in \mathcal{P}(n)$ be an ordered set of indexes, with $k_1 < \cdots < k_p$. If $K = \emptyset$, then set $x_K = \emptyset$ and $[x_K] = 1$; if $K \neq 0$, define $x_K = (x_{k_1}, \dots, x_{k_p}) \in A^p$, so by the definition above

$$[x_K] = [x_{k_1}, \dots x_{k_p}] = x_{k_1} \cdot x_{k_2} \cdots x_{k_{p-1}} \cdot x_{k_p}$$

and

$$[x_K, y] = [x_{k_1}, \dots x_{k_p}, y] = x_{k_1} \cdot x_{k_2} \cdots x_{k_{p-1}} \cdot x_{k_p} \cdot y.$$

Definition 3.8. Given $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $h \in \{1, ..., n\}$, define

$$\overline{z}^h := (z_1, \ldots, z_{h-1}, \overline{z}_h, z_{h+1}, \ldots, z_n).$$

A set $D \subset \mathbb{C}^n$ is called symmetric if it is invariant with respect to any complex conjugation, i.e. if

$$z \in D \iff \overline{z}^h \in D, \qquad \forall h = 1, ..., n.$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ and denote with $\{e_K\}_{K \in \mathcal{P}(n)}$ a basis of $\mathbb{R}^{2^n}$.

Definition 3.9. Let $D \subset \mathbb{C}^n$ be an open symmetric set and consider a function $F : D \subset \mathbb{C}^n \to A \otimes \mathbb{R}^{2^n}$, then there exist unique A-valued functions $F_K : D \to A$, such that $F(z) = \sum_{K \in \mathcal{P}(n)} e_K F_K(z)$. We call F a stem function if $F_K(\overline{z}^h) = (-1)^{|K \cap \{h\}|} F_K(z)$ or equivalently

$$F_K(\overline{z}^h) = \begin{cases} F_K(z) & \text{if } h \notin K \\ -F_K(z) & \text{if } h \in K, \end{cases} \tag{8}$$

for every $z \in D$, every $K \in \mathcal{P}(n)$ and any $h \in \{1, \ldots, n\}$. Again, we use the symbol $Stem(D)$ to denote the set of stem functions $F : D \to A \otimes \mathbb{R}^{2^n}$.

Equip $\mathbb{R}^{2^n}$ with the family of commutative complex structures $\mathcal{J} = \left\{\mathcal{J}_h : \mathbb{R}^{2^n} \to \mathbb{R}^{2^n}\right\}_{h=1}^n$, where each $\mathcal{J}_h$ is defined over any basis element e_K of $\mathbb{R}^{2^n}$ as

$$\mathcal{J}_h(e_K) := (-1)^{|K \cap \{h\}|} e_{K \Delta \{h\}} = \begin{cases} e_{K \cup \{h\}} & \text{if } h \notin K \\ -e_{K \setminus \{h\}} & \text{if } h \in K, \end{cases}$$

where $K \Delta H = (K \cup H) \setminus (K \cap H)$ and extend it by linearity to all $\mathbb{R}^{2^n}$. $\mathcal{J}$ induces a family of commutative complex structure on $A \otimes \mathbb{R}^{2^n}$ (by abuse of notation, we use the same symbol) $\mathcal{J} = \left\{\mathcal{J}_h : A \otimes \mathbb{R}^{2^n} \to A \otimes \mathbb{R}^{2^n}\right\}_{h=1}^n$ according to the formula

$$\mathcal{J}_h(x \otimes a) := x \otimes \mathcal{J}_h(a) \qquad \forall x \in A, \quad \forall a \in \mathbb{R}^{2^n}.$$

We can associate two Cauchy-Riemann operators to each complex structure $\mathcal{J}_h$.

Definition 3.10. Given a stem function $F \in Stem(D) \cap \mathcal{C}^1(D)$, we define

$$\partial_h F := \frac{1}{2}\left(\frac{\partial F}{\partial \alpha_h} - \mathcal{J}_h\left(\frac{\partial F}{\partial \beta_h}\right)\right), \qquad \overline{\partial}_h F := \frac{1}{2}\left(\frac{\partial F}{\partial \alpha_h} + \mathcal{J}_h\left(\frac{\partial F}{\partial \beta_h}\right)\right).$$

$F = \sum_{K \in \mathcal{P}(n)} e_K F_K$ is called h-holomorphic with respect to $\mathcal{J}$ if $F \in \ker \overline{\partial}_h$ and it is called holomorphic if it is h-holomorphic for every $h = 1, ..., n$.

We can give the definition of holomorphic stem function through a system of Cauchy-Riemann equations.

Proposition 3.4 ([50],Lemma 3.12)**.** *Let F be a stem function. Then F is h-holomorphic if and only if*

$$\frac{\partial F_K}{\partial \alpha_h} = \frac{\partial F_{K \cup \{h\}}}{\partial \beta_h}, \qquad \frac{\partial F_K}{\partial \beta_h} = -\frac{\partial F_{K \cup \{h\}}}{\partial \alpha_h}, \qquad \forall K \in \mathcal{P}(n), h \notin K. \tag{9}$$

For any $J_1, \ldots J_n \in \mathbb{S}_A$, define

$$\phi_{J_1} \times ... \times \phi_{J_n} : \mathbb{C}^n \ni (z_1, ..., z_n) \mapsto (\phi_{J_1}(z_1), ..., \phi_{J_n}(z_n)) \in \mathbb{H}^n,$$

where ϕ_J is defined in (4).

Definition 3.11. Given a symmetric set $D \subset \mathbb{C}^n$, we define its circularization $\Omega_D \subset (\mathcal{Q}_A)^n$ as $\Omega_D \coloneqq \bigcup_{(J_1,\dots,J_n)\in(\mathbb{S}_A)^n}(\phi_{J_1}\times\cdots\times\phi_{J_n})(D)$, namely

$$\Omega_D = \{(\alpha_1 + J_1\beta_1, \dots, \alpha_n + J_n\beta_n) : (\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n) \in D, J_1, \dots, J_n \in \mathbb{S}_A\}.$$

A subset $\Omega \subset \mathcal{Q}_A$ is called circular if it is the circularization of a symmetric set D, i.e. $\Omega = \Omega_D$ for some $D \subset \mathbb{C}^n$.

Definition 3.12. A map $f : \Omega_D \subset (\mathcal{Q}_A)^n \to A$ is called a slice function if there exist a stem function $F = \sum_{K\in\mathcal{P}(n)} e_K F_K : D \to A \otimes \mathbb{R}^{2^n}$, such that, for every $x \in \Omega_D$

$$f(x) = \sum_{K\in\mathcal{P}(n)} [J_K, F_K(z)],$$

where $x = (\phi_{J_1}\times\cdots\times\phi_{J_n})(z)$ and $J = (J_1, \dots, J_n)$. A slice regular function is a slice function induced by a holomorphic stem function. A slice function is called slice preserving whenever the components of its inducing stem function are real valued. We denote with $\mathcal{S}(\Omega_D)$, $\mathcal{SR}(\Omega_D)$ and $\mathcal{S}_{\mathbb{R}}(\Omega_D)$ respectively the set of slice, slice regular and slice preserving functions on $\Omega_D \subset (\mathcal{Q}_A)^n$ and $\mathcal{I} : Stem(D) \to \mathcal{S}(\Omega_D)$ will be the map sending a stem function to its induced slice function.

Remark 5 ([50], §2.1)**.** Note that (8) is necessary to make slice functions well defined, indeed any $x = (\alpha_1 + J_1\beta_1, \dots, \alpha_n + J_n\beta_n)$ can be represented, for any $H \in \mathcal{P}(n)$ as

$$x_H = (\alpha_1 + (\epsilon_1 J_1)(\epsilon_1\beta_1), \dots, \alpha_n + (\epsilon_n J_n)(\epsilon_n\beta_n)),$$

where $\epsilon_h = -1$, if $h \in H$, otherwise $\epsilon_h = 1$. Then, in order for f to be well defined we must have $f(x) = f(x_H)$, for any $H \in \mathcal{P}(n)$. In fact, let $z = (\alpha_1 + i\beta_1, \dots, \alpha_n + J_n\beta_n)$, then it holds

$$\begin{aligned} f(x_H) &= \sum_{K\in\mathcal{P}(n)} [(-1)^{|K\cap H|} J_K, F_K(\overline{z}^H)] = \sum_{K\in\mathcal{P}(n)} [(-1)^{|K\cap H|} J_K, (-1)^{|K\cap H|} F_K(z)] \\ &= \sum_{K\in\mathcal{P}(n)} [J_K, F_K(z)] = f(x). \end{aligned}$$

If F is a stem function, so are $\partial_h F$ and $\overline{\partial}_h F$ [50, Lemma 3.9], thus, if $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D) := \mathcal{I}(Stem(D) \cap \mathcal{C}^1(D))$, we can define the partial derivatives for every $h = 1, ..., n$

$$\frac{\partial f}{\partial x_h} := \mathcal{I}\left(\partial_h F\right), \qquad \frac{\partial f}{\partial x_h^c} := \mathcal{I}\left(\overline{\partial}_h F\right).$$

and $f \in \mathcal{SR}(\Omega_D)$ if and only if $\frac{\partial f}{\partial x_h^c} = 0$ for every $h = 1, ..., n$.

Equivalently, we can define f as the unique slice function that makes the following diagram commutative for any $J_1, ..., J_n \in \mathbb{S}_{\mathbb{H}}$:

$$\begin{tikzcd} D \arrow[r, "F"] \arrow[d, "\phi_{J_1}\times\ldots\times\phi_{J_n}"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes \mathbb{R}^{2^n} \arrow[d, "\Phi_{J_1,\ldots,J_n}"] \\ \Omega_D \arrow[r, "f"'] & A, \end{tikzcd}$$

where
$$\Phi_{J_1,\dots,J_n}: A\otimes\mathbb{R}^{2^n}\ni\sum_{K\in\mathcal{P}(n)}e_K a_K\mapsto\sum_{K\in\mathcal{P}(n)}[J_K,a_K]\in A.$$

Every slice function can be fully recovered by its values on n-slices.

Proposition 3.5 ([50], Proposition 2.12)**.** *Let $f\in\mathcal{S}(\Omega_D)$ and fix $I_1,\dots,I_n\in\mathbb{S}_A$. Then, for any $x=(\alpha_1+J_1\beta_1,\dots,\alpha_n+J_n\beta_n)\in\Omega_D$ it holds*
$$f(x)=2^{-n}\sum_{H,K\in\mathcal{P}(n)}(-1)^{|H\cap K|}\left[J_K,\left[I_K^{-1}f\left(\overline{y}^H\right)\right]\right],\tag{10}$$
with $y=(\alpha_1+I_1\beta_1,\dots,\alpha_n+I_n\beta_n)$. Furthermore, if $f=\mathcal{I}(F)$, with $F=\sum_{K\in\mathcal{P}(n)}e_KF_K$, it holds
$$F_K(z)=2^{-n}\sum_{H\in\mathcal{P}(n)}(-1)^{|K\cap H|}\left[I_K^{-1},f\left(\overline{y}^H\right)\right].$$
In particular, any slice function is induced by a unique stem function and so $\mathcal{I}:Stem(D)\to\mathcal{S}(\Omega_D)$ is injective.

The following results correspond to [50, Corollaries 2.13, 2.16]

Corollary 3.6 (Identity principle)**.** *Let $f,g\in\mathcal{S}(\Omega_D)$ and suppose there exist $I_1,\dots I_n\in\mathbb{S}_A$ such that $f=g$ on $\Omega_D\cap(\mathbb{C}_{I_1}\times\cdots\times\mathbb{C}_{I_n})$. Then $f\equiv g$.*

Corollary 3.7 (Sliceness criterion)**.** *Let $f:\Omega_D\to A$ be a function. Then f is slice if and only if there exist $I_1,\dots,I_n\in\mathbb{S}_A$ such that f satisfies* (10) *for any $x=(\alpha_1+J_1\beta_1,\dots,\alpha_n+J_n\beta_n)$, with $y=(\alpha_1+I_1\beta_1,\dots,\alpha_n+I_n\beta_n)$.*

Equip $\mathbb{R}^{2^n}$ with the product $\otimes:\mathbb{R}^{2^n}\times\mathbb{R}^{2^n}\to\mathbb{R}^{2^n}$, defined on each basis element as
$$e_H\otimes e_K:=(-1)^{|H\cap K|}e_{H\Delta K},$$
and extended by linearity to all $\mathbb{R}^{2^n}$. This product induces a product on $A\otimes\mathbb{R}^{2^n}$: given $a,b\in A\otimes\mathbb{R}^{2^n}$, $a=\sum_{H\in\mathcal{P}(n)}e_Ha_H$ and $b=\sum_{K\in\mathcal{P}(n)}e_Kb_K$, with $a_H,b_K\in A$, define
$$a\otimes b:=\sum_{H,K\in\mathcal{P}(n)}(e_H\otimes e_K)(a_Hb_K)=\sum_{H,K\in\mathcal{P}(n)}(-1)^{|H\cap K|}e_{H\Delta K}a_Hb_K,$$
where a_Hb_K is just the usual product of A. Furthermore, we can define a product between stem functions as the pointwise product induced by $\otimes$.

Definition 3.13. Let $F,G\in Stem(D)$, define $(F\otimes G)(z):=F(z)\otimes G(z)$. More precisely, if $F=\sum_{H\in\mathcal{P}(n)}e_HF_H$ and $G=\sum_{K\in\mathcal{P}(n)}e_KG_K$,
$$(F\otimes G)(z):=\sum_{H,K\in\mathcal{P}(n)}(-1)^{|H\cap K|}e_{H\Delta K}F_H(z)G_K(z).$$
The advantage of this definition is that the product of two stem functions is again a stem function [50, Lemma 2.34] and this allows to define a product on slice functions, too. Let $f,g\in\mathcal{S}(\Omega_D)$, with $f=\mathcal{I}(F)$ and $g=\mathcal{I}(G)$, then define the slice tensor product $f\odot g$ between f and g as
$$f\odot g:=\mathcal{I}(F\otimes G).$$
In particular, $\mathcal{I}:(Stem(D),\otimes)\to(\mathcal{S}(\Omega_D),\odot)$ is an algebra isomorphism.

4 Partial slice regularity

The notion of partial sliceness was already given in [50]. The results of this section are taken from [9].

Let $f : \Omega_D \subset (\mathcal{Q}_A)^n \to A$ and $h = 1, ..., n$. For any $y = (y_1, ..., y_n) \in \Omega_D$, let

$$\Omega_{D,h}(y) := \{x \in A \mid (y_1, ..., y_{h-1}, x, y_{h+1}, ..., y_n) \in \Omega_D\} \subset \mathcal{Q}_A.$$

It is easy to see ([50, §2]) that $\Omega_{D,h}(y)$ is a circular set of $\mathcal{Q}_A$, more precisely $\Omega_{D,h}(y) = \Omega_{D_h(z)}$, where

$$D_h(z) := \{w \in \mathbb{C} \mid (z_1, ..., z_{h-1}, w, z_{h+1}, ..., z_n) \in D\},$$

and $z = (z_1, ..., z_n)$ is such that $y \in \Omega_{\{z\}}$.

Definition 4.1. We say that a slice function $f \in \mathcal{S}(\Omega_D)$ is *slice* (resp. *slice regular*) *with respect to* x_h if, $\forall y \in \Omega_D$, its restriction

$$f_h^y : \Omega_{D,h}(y) \to A, \; f_h^y(x) := f(y_1, ..., y_{h-1}, x, y_{h+1}, ..., y_n)$$

is a one variable slice (resp. slice regular) function, as defined in §3.1. We denote by $\mathcal{S}_h(\Omega_D)$ (resp. $\mathcal{SR}_h(\Omega_D)$) the set of slice functions from Ω_D to A that are slice (resp. slice regular) with respect to x_h. For $H \in \mathcal{P}(n)$, define also

$$\mathcal{S}_H(\Omega_D) := \bigcap_{h\in H} \mathcal{S}_h(\Omega_D), \qquad \mathcal{SR}_H(\Omega_D) := \bigcap_{h\in H} \mathcal{SR}_h(\Omega_D)$$

Note that, by definition, $\mathcal{SR}_H(\Omega_D) \subset \mathcal{S}_H(\Omega_D) \subset \mathcal{S}(\Omega_D)$.

We say that f is *circular with respect to* x_h if $\forall y = (y_1, ..., y_n) \in \Omega_D$, f_h^y is constant on $\mathbb{S}_{y_h} \subset \mathcal{Q}_A$. The set of slice functions which are circular with respect to x_h will be denoted by $\mathcal{S}_{c,h}(\Omega_D) \subset \mathcal{S}(\Omega_D)$. Moreover, if $H \in \mathcal{P}(n)$, set $\mathcal{S}_{c,H}(\Omega_D) := \bigcap_{h\in H} \mathcal{S}_{c,h}(\Omega_D)$. Note that in euclidean spaces, circularity can be characterized through invariance with respect to orthogonal transformations. Indeed, if $\Omega_D \subset \mathbb{H}^n$, or $\Omega_D \subset (\mathbb{R}^{m+1})^n$, f is circular with respect to x_h if and only if for every orthogonal transformation $T : \mathbb{H} \to \mathbb{H}$ or $T : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ that fixes 1, it holds

$$f(x_1, ..., x_{h-1}, T(x_h), x_{h+1}, ..., x_n) = f(x_1, ..., x_n),$$

for any $(x_1, ..., x_n) \in \Omega_D$. In this case, if $x_h = \alpha_h + J_h\beta_h$, f does not depend on J_h.

Every slice function is, in particular, slice with respect to the first variable [50, Proposition 2.23], i.e. $\mathcal{S}_1(\Omega_D) = \mathcal{S}(\Omega_D)$, but in general $\mathcal{S}_h(\Omega_D) \subsetneq \mathcal{S}(\Omega_D)$. The next proposition characterizes the set $\mathcal{S}_H(\Omega_D)$ for any $H \in \mathcal{P}(n)$ in terms of stem functions.

Proposition 4.1. *For every* $H \in \mathcal{P}(n)$ *it holds*

$$\mathcal{S}_H(\Omega_D) = \left\{ \mathcal{I}(F) : F \in Stem(D), \; F = \sum_{K\in H^c} e_H F_K + \sum_{h\in H} e_{\{h\}} \sum_{Q\subset\{h+1,...,n\}\setminus H} e_Q F_{\{h\}\cup Q} \right\}. \tag{11}$$

In particular, for any $h \in \{1, ..., n\}$,

$$\mathcal{S}_h(\Omega_D) = \left\{ \mathcal{I}(F) : F \in Stem(D), \; F = \sum_{K\in\mathcal{P}(n), h\notin K} e_H F_K + e_{\{h\}} \sum_{Q\subset\{h+1,...,n\}} e_Q F_{\{h\}\cup Q} \right\}. \tag{12}$$

Equivalently, $f = \mathcal{I}(F) \in \mathcal{S}_H(\Omega_D)$ *if and only if* $F_{P\cup\{h\}\cup Q} = 0$, $\forall h \in H$, $\forall Q \subset \{h+1, ..., n\}$, $\forall P \in \mathcal{P}(h-1)$ *with* $P \neq \emptyset$.

Proof. Since $\mathcal{S}_H(\Omega_D) := \bigcap_{h\in H} \mathcal{S}_h(\Omega_D)$, it is sufficient to assume $H = \{h\}$ for some $h = 1, ..., n$.

$\Rightarrow$) $f \in \mathcal{S}_h(\Omega_D)$ means that $\forall y \in \Omega_D$, the one-variable function f_h^y is slice, thus, it must satisfies representation formula (7): namely, if $x = a + Ib \in \Omega_{D,h}(y)$ and $J \in \mathbb{S}_A$, it holds

$$f_h^y(x) = \frac{1}{2}\left(f_h^y(a+Jb) + f_h^y(a-Jb)\right) - \frac{IJ}{2}\left(f_h^y(a+Jb) - f_h^y(a-Jb)\right). \tag{13}$$

Set $z = (z_1, ..., z_n)$, $z' = (z_1, ..., z_{h-1})$, $z" = (z_{h+1}, ..., z_n)$, $y = (\phi_{J_1} \times ... \times \phi_{J_n})(z)$, for some $J_1, ..., J_n \in \mathbb{S}_A$, $w = a + ib$, $x = \phi_I(w)$, $L_s = M_s = J_s$ for $s \neq h$, $L_h := I$ and $M_h := J$. Then we have

$$f_h^y(x) = \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', w, z'')] + \sum_{K\in\mathcal{P}(n),h\notin K} [L_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')], \tag{14}$$

$$f_h^y(a+Jb) = \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', w, z'')] + \sum_{K\in\mathcal{P}(n),h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')]$$

and

$$\begin{aligned} f_h^y(a-Jb) &= \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', \overline{w}, z'')] + \sum_{K\in\mathcal{P}(n),h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', \overline{w}, z'')] \\ &= \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', w, z'')] - \sum_{K\in\mathcal{P}(n),h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')], \end{aligned}$$

where we have used (8). Thus, the right hand side of (13) becomes

$$\begin{aligned} &\frac{1}{2}\left(f_h^y(a+Jb) + f_h^y(a-Jb)\right) - \frac{I}{2}\left[J\left(f_h^y(a+Jb) - f_h^y(a-Jb)\right)\right] = \\ &= \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', w, z'')] - IJ \sum_{K\in\mathcal{P}(n),h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')]. \end{aligned} \tag{15}$$

Comparing (14) and (15), (13) is satisfied if and only if

$$\sum_{K\in\mathcal{P}(n),h\notin K} [L_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')] = -IJ \sum_{K\in\mathcal{P}(n),h\notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')]. \tag{16}$$

Since (13) is assumed to be true for every $I, J, J_1, ..., J_n \in \mathbb{S}_A$ and every z', w, z'', (16) holds if and only if $\forall K \subset \{1, ..., n\} \setminus \{h\}$

$$[L_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')] = -IJ[M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')]. \tag{17}$$

Indeed, if (17) were not true, there would be a $K \subset \mathcal{P}(\{1, ..., n\} \setminus \{h\})$ such that

$$[L_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')] \neq -IJ[M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z'')],$$

but for $J_1 = ... = J_n = J = I$ we would have

$$(-1)^{|K\cup\{h\}|} F_{K\cup\{h\}}(z', w, z'') \neq (-1)^{|K\cup\{h\}|} F_{K\cup\{h\}}(z', w, z''),$$

which is false. Let us represent $\{K \in \mathcal{P}(n) \mid h \notin K\} = \{P \sqcup Q \mid P \in \mathcal{P}(h-1),\ Q \subset \{h+1, ..., n\}\}$. Suppose $P \neq \emptyset$, then $\forall Q \subset \{h+1, ..., n\}$), (17) becomes

$$[L_{(P\cup\{h\}\cup Q)}, F_{P\cup\{h\}\cup Q}(z', w, z'')] = -IJ[M_{(P\cup\{h\}\cup Q)}, F_{P\cup\{h\}\cup Q}(z', w, z'')]$$

and this implies that $F_{P\cup\{h\}\cup Q} \equiv 0$. Indeed, if $F_{P\cup\{h\}\cup Q} \neq 0$, the previous equation would reduce to $J_P I = -IJJ_PJ$ which does not hold for every choice of I, J, J_P.

$\Leftarrow$) Vice versa, suppose F takes the form

$$F = \sum_{K\in\mathcal{P}(n), h\notin K} e_K F_K + e_h \sum_{Q\subset\{h+1,\dots,n\}} e_Q F_{\{h\}\cup Q}.$$

Following the notation above, it holds

$$f_h^y(x) = \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', w, z'')] + I \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')].$$

Thus, consider the function $G_h^y = G_{1,h}^y + e_1 G_{2,h}^y$, with

$$G_{1,h}^y(w) := \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', w, z'')], \quad G_{2,h}^y(w) := \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')].$$

G_h^y is a one-variable stem function, indeed,

$$\begin{aligned} G_h^y(\overline{w}) &= \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', \overline{w}, z'')] + e_1 \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', \overline{w}, z'')] \\ &= \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', w, z'')] - e_1 \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')] = \overline{G_h^y(w)} \end{aligned}$$

and $f_h^y = \mathcal{I}(G_h^y)$, by construction, so $f \in \mathcal{S}_h(\Omega_D)$.

□

Remark 6. By the previous proof, we can better understand the set $\mathcal{S}_H(\Omega_D)$: let $f = \mathcal{I}(F) \in \mathcal{S}_H(\Omega_D)$, then for any $x \in \Omega_D$ with $x = (\phi_{J_1} \times \dots \times \phi_{J_n})(z)$, $f(x)$ takes the form

$$f(x) = \sum_{K\in H^c} [J_K, F_K(z)] + \sum_{h\in H} J_h \sum_{Q\subset\{h+1,\dots,n\}\setminus H} \left[J_Q, F_{\{h\}\cup Q}(z)\right].$$

Moreover, for any $h \in H$ and any $y = (y_1, \dots, y_n)$, f_h^y is a one-variable slice function, induced by the stem function G_h^y, with components

$$G_{1,h}^y(w) := \sum_{K\in\mathcal{P}(n), h\notin K} [J_K, F_K(z', w, z'')], \qquad G_{2,h}^y(w) := \sum_{Q\subset\{h+1,\dots,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')], \tag{18}$$

where $z = (z', z_h, z")$ and $y = (\phi_{J_1} \times \dots \times \phi_{J_n})(z)$.

Now, we deal with partial slice regularity.

Proposition 4.2. *For every $H \in \mathcal{P}(n)$ it holds*

$$\mathcal{SR}_H(\Omega_D) = \mathcal{S}_H(\Omega_D) \cap \bigcap_{h\in H} \ker(\partial/\partial x_h^c).$$

Proof. Since $\mathcal{SR}_H(\Omega_D) := \bigcap_{h\in H} \mathcal{SR}_h(\Omega_D)$, it is sufficient to assume $H = \{h\}$ for some $h = 1, \dots, n$.

$\subset$) By definition, $\mathcal{SR}_h(\Omega_D) \subset \mathcal{S}_h(\Omega_D)$, so let $f = \mathcal{I}(F)$, with

$$F = \sum_{K\in\mathcal{P}(n),h\notin K} e_K F_K + e_h \sum_{Q\subset\{h+1,...,n\}} e_Q F_{\{h\}\cup Q}, \tag{19}$$

thanks to (12). For any $y \in \Omega_D$, f_h^y is induced by the stem function $G_h^y = G_{1,h}^y + e_1 G_{2,h}^y$, with

$$G_{1,h}^y(w) := \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z', w, z'')], \quad G_{2,h}^y(w) := \sum_{Q\subset\{h+1,...,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')].$$

By definition, $f \in \mathcal{SR}_h(\Omega_D)$ means that $\forall y \in \Omega_D$, the stem function G_h^y is holomorphic, i.e. recalling (18) it must hold that for every $z = (z', z_h, z") \in D$, $w \in D_h(z)$ and $\forall J_j \in \mathbb{S}_A$ that

$$\begin{cases} \sum_{P,Q}[J_{P\cup Q}, \partial_{\alpha_h} F_{P\cup Q}(z', w, z'')] = \sum_Q [J_Q, \partial_{\beta_h} F_{\{h\}\cup Q}(z', w, z'')] \\ \sum_{P,Q}[J_{P\cup Q}, \partial_{\beta_h} F_{P\cup Q}(z', w, z'')] = -\sum_Q [J_Q, \partial_{\alpha_h} F_{\{h\}\cup Q}(z', w, z'')], \end{cases}$$

where in the above sums $P \in \mathcal{P}(h-1)$ and $Q \subset \{h+1, ..., n\}$. Now, since that system is true for every choice of imaginary unit J_j, proceeding as in the proof of Proposition 4.1 we can deduce that an equivalence between each term of the sum holds. Let any $Q \subset \{h+1, ..., n\}$: if $P \neq \emptyset$, equality can sussist only if $\partial_{\alpha_h} F_{P\cup Q} = \partial_{\beta_h} F_{P\cup Q} = 0$ and this trivially proves that the components $F_{P\cup Q}$ satisfies (9), since $F_{P\cup\{h\}\cup Q} = 0$, by (12). Otherwise, let $P = \emptyset$, then the previous system becomes

$$\begin{cases} \partial_{\alpha_h} F_Q = \partial_{\beta_h} F_{\{h\}\cup Q} \\ \partial_{\beta_h} F_Q = -\partial_{\alpha_h} F_{\{h\}\cup Q} \end{cases}$$

and (9) are satisfied too. This proves that F is h-holomorphic, which means that $f \in \ker(\partial/\partial x_h^c)$.

$\supset$) Suppose $f \in \mathcal{S}_h(\Omega_D) \cap \ker(\partial/\partial x_h^c)$, then F satisfies (19) and (9). As in the proof of Proposition 4.1, represent $K = P \sqcup Q$, with $P \in \mathcal{P}(h-1)$ and $Q \subset \{h+1, ..., n\}$. Since, by (19), $F_{P\cup\{h\}\cup Q} \equiv 0$, $\forall P \in \mathcal{P}(h-1) \setminus \{\emptyset\}$, $\forall Q \subset \{h+1, ..., n\}$ the h-holomorphicity of F reduces to the following conditions:

$$\begin{cases} \partial_{\alpha_h} F_{P\cup Q} = \partial_{\beta_h} F_{P\cup Q} = 0 \\ \partial_{\alpha_h} F_Q = \partial_{\beta_h} F_{\{h\}\cup Q} \\ \partial_{\beta_h} F_Q = \partial_{\alpha_h} F_{\{h\}\cup Q}. \end{cases} \tag{20}$$

On the other hand, $f \in \mathcal{SR}_h(\Omega_D)$ if and only if G_h^y is a slice regular function $\forall y \in \Omega_D$, which means that $\partial_\alpha G_{1,h}^y = \partial_\beta G_{2,h}^y$ and $\partial_\beta G_{1,h}^y = -\partial_\alpha G_{2,h}^y$, which, by definition of G_h^y is equivalent to

$$\begin{cases} \partial_{\alpha_h} \sum_{K\in\mathcal{P}(n),h\notin K}[J_K, F_K(z)] = \partial_{\beta_h} \sum_{Q\subset\{h+1,...,n\}}[J_Q, F_{\{h\}\cup Q}(z)] \\ \partial_{\beta_h} \sum_{K\in\mathcal{P}(n),h\notin K}[J_K, F_K(z)] = -\partial_{\alpha_h} \sum_{Q\subset\{h+1,...,n\}}[J_Q, F_{\{h\}\cup Q}(z)], \end{cases}$$

where $y = (\phi_{J_1} \times ... \times \phi_{J_n})(z)$, $z = (z_1, ..., z_n)$, $z_j = \alpha_j + i\beta_j$. Let us prove the first row of the system. Using the first two equation of (20) and splitting $K = P \sqcup Q$, we can write the left side as

$$\begin{aligned} &\partial_{\alpha_h} \sum_{P\in\mathcal{P}(h-1),Q\subset\{h+1,...,n\}} [J_{P\cup Q}, F_{P\cup Q}(z', w, z'')] \\ &= \sum_{P\in\mathcal{P}(h-1),Q\subset\{h+1,...,n\}} [J_{P\cup Q}, \partial_{\alpha_h} F_{P\cup Q}(z', w, z'')] = \sum_{Q\subset\{h+1,...,n\}} [J_Q, \partial_{\alpha_h} F_Q(z', w, z'')] \\ &= \sum_{Q\subset\{h+1,...,n\}} [J_Q, \partial_{\beta_h} F_{\{h\}\cup Q}(z', w, z'')] = \partial_{\beta_h} \sum_{Q\subset\{h+1,...,n\}} [J_Q, F_{\{h\}\cup Q}(z', w, z'')]. \end{aligned}$$

The second equation is proved in the same way.

$\square$

Corollary 4.3. *Let $f \in \mathcal{SR}(\Omega_D)$ and $H \in \mathcal{P}(n)$. Then $f \in \mathcal{S}_H(\Omega_D)$ if and only if $f \in \mathcal{SR}_H(\Omega_D)$.*

Proof. The "if" part is trivial. Viceversa, note that by [50, Proposition 3.13], $f \in \mathcal{SR}(\Omega_D)$, implies $\partial f/\partial x_h^c = 0$, $\forall h = 1, ..., n$, hence $\mathcal{S}_H(\Omega_D) \cap \mathcal{SR}(\Omega_D) \subset \mathcal{S}_H(\Omega_D) \cap \bigcap_{h\in H} \ker(\partial/\partial x_h^c) = \mathcal{SR}_H(\Omega_D)$, by Proposition 4.2. $\square$

Finally, we characterize circularity.

Proposition 4.4. *For every $H \in \mathcal{P}(n)$ it holds*

$$\mathcal{S}_{c,H}(\Omega_D) = \left\{ \mathcal{I}(F) : F \in Stem(D), F = \sum_{K \subset H^c} e_K F_K \right\}. \tag{21}$$

In particular, $\mathcal{S}_{c,H}(\Omega_D) \subset \mathcal{S}_H(\Omega_D)$.

Proof. Since $\mathcal{S}_{c,H}(\Omega_D) = \bigcap_{h\in H} \mathcal{S}_{c,h}(\Omega_D)$, it is sufficient to assume $H = \{h\}$ for some $h = 1, ..., n$. Let any $y = (y_1, ..., y_n) \in \Omega_D$, with $y_j := \alpha_j + J_j\beta_j$, $z_j := \alpha_j + i\beta_j$, set $z' = (z_1, ..., z_{h-1})$ and $z" = (z_{h+1}, ..., z_n)$. $f \in \mathcal{S}_{c,h}(\Omega_D)$ if for every $x = a + Ib$, $f_h^y(x)$ does not depend on I. Let $w := a + ib$, $M_p := J_p$ if $p \neq h$ and $M_h = I$, then

$$f_h^y(x) = \sum_{K \in \mathcal{P}(n), h \notin K} [J_K, F_K(z', w, z")] + \sum_{K \in \mathcal{P}(n), h \notin K} [M_{K\cup\{h\}}, F_{K\cup\{h\}}(z', w, z")].$$

It is clear that $f_h^y(a + Ib)$ does not depend on I if and only if $F_{K\cup\{h\}} = 0$ for every $K \in \mathcal{P}(n)$. Finally, comparing (11) and (21) we see that $\mathcal{S}_{c,H}(\Omega_D) \subset \mathcal{S}_H(\Omega_D)$. $\square$

Note that functions of the form (21) were introduced in [50] as H^c-reduced slice functions, hence we can say that $f \in \mathcal{S}_{c,H}(\Omega_D)$ if and only if it is H^c-reduced. It is easy now to prove the following property.

Corollary 4.5. *For every $H \in \mathcal{P}(n)$, the set $\mathcal{S}_{c,H}(\Omega_D)$ is a real subalgebra of $(\mathcal{S}(\Omega_D), \odot)$.*

Proof. We need to show that if $f, g \in \mathcal{S}_{c,H}(\Omega_D)$, then $f \odot g \in \mathcal{S}_{c,H}(\Omega_D)$. Let $f = \mathcal{I}(F)$ and $g \in \mathcal{I}(G)$, with $F = \sum_{K\subset H^c} e_K F_K$ and $G = \sum_{T\subset H^c} e_T G_T$, by (21). Then

$$F \otimes G = \sum_{K,T \subset H^c} (-1)^{|K\cap T|} e_{K\Delta T} F_K G_T,$$

with $K\Delta T = (K \cup T) \setminus (K \cap T) \subset K \cup T \subset H^c$. Then, again (21) implies $f \odot g \in \mathcal{S}_{c,H}(\Omega_D)$. $\square$

Note that the previous result does not apply to $\mathcal{S}_H(\Omega_D)$, nor $\mathcal{SR}_H(\Omega_D)$, unless for $\mathcal{S}_1(\Omega_D) = \mathcal{S}(\Omega_D)$ and $\mathcal{SR}_1(\Omega_D)$. Indeed, for example, $x_1, x_2 \in \mathcal{SR}_2(\Omega_D)$, while $x_1 \odot x_2 \notin \mathcal{S}_2(\Omega_D)$.

Slice regularity and circularity are hardly compatible.

Proposition 4.6. *Let $f \in \mathcal{S}_{c,h}(\Omega_D) \cap \mathcal{SR}_h(\Omega_D)$. Then f is locally constant with respect to x_h.*

Proof. Let $x_h = a_h + J_h b_h$ and $f = \mathcal{I}(F)$. Since $f \in \mathcal{S}_{c,h}(\Omega_D)$, f does not depend on J_h and $F_{K\cup\{h\}} = 0$ for any $K \in \mathcal{P}(n)$. Moreover, $f \in \mathcal{SR}_h(\Omega_D) \subset \ker(\partial/\partial x_h^c)$, by Proposition 4.2, so by (9)

$$\frac{\partial F_K}{\partial \alpha_h} = \frac{\partial F_{K\cup\{h\}}}{\partial \beta_h} = 0 = \frac{\partial F_{K\cup\{h\}}}{\partial \alpha_h} = -\frac{\partial F_K}{\partial \beta_h}.$$

Thus, f does not depend neither on α_h and β_h and so it is locally constant with respect to x_h. □

Example 4. Consider the following polynomial function $f : \mathbb{H}^3 \to \mathbb{H}$, $f(x_1, x_2, x_3) := x_1x_3 + x_2x_3^2k$, which happens to be a slice regular function, [50, Proposition 3.14]. We claim that $f \in \mathcal{SR}_2(\Omega_D)$. Let us explicit the components of the stem function inducing f: let $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, with $z_j := \alpha_j + i\beta_j$, then $f = \mathcal{I}(F)$, with $F = \sum_{K\in\mathcal{P}(3)} e_K F_K$, where

$$\begin{aligned}
&F_\emptyset(z) = \alpha_1\alpha_3 + \alpha_2(\alpha_3^2 - \beta_3^2)k, \qquad F_{\{1\}}(z) = \beta_1\alpha_3, \qquad F_{\{2\}}(z) = \beta_2(\alpha_3^2 - \beta_3^2)k,\\
&F_{\{3\}}(z) = \alpha_1\beta_3 + 2\alpha_2\alpha_3\beta_3k, \quad F_{\{1,2\}}(z) = 0, \qquad F_{\{1,3\}}(z) = \beta_1\beta_3, \qquad F_{\{2,3\}}(z) = 2\beta_2\alpha_3\beta_3k,\\
&F_{\{1,2,3\}}(z) = 0.
\end{aligned}$$

Thus, F has the structure required by (12) for $h = 2$, so $f \in \mathcal{S}_2(\Omega_D)$. Moreover, for $K = \emptyset, \{1\}, \{3\}, \{1,3\}$ it holds

$$\frac{\partial F_K}{\partial \alpha_2} = \frac{\partial F_{K\cup\{2\}}}{\partial \beta_2}, \qquad \frac{\partial F_K}{\partial \beta_2} = -\frac{\partial F_{K\cup\{2\}}}{\partial \alpha_2},$$

so $f \in \ker(\partial/\partial x_2^c)$ and so $f \in \mathcal{SR}_2(\Omega_D) = \mathcal{S}_2(\Omega_D) \cap \ker(\partial/\partial x_2^c)$.

We could have proven the claim by definition, through Remark 6, which explicitly gives us the stem function that induces the corresponding one variable slice function, for every choice of y. Fix any $y = (y_1, y_2, y_3) \in \mathbb{H}^3$, then f_2^y is a slice regular function, induced by the holomorphic stem function $G_2^y = G_{1,2}^y + e_1G_{2,2}^y$, with

$$G_{1,2}^y(\alpha + i\beta) = y_1y_3 + \alpha y_3^2k, \qquad G_{2,2}^y(\alpha + i\beta) = \beta y_3^2k.$$

4.1 Partial spherical values and derivatives

For $h \in \{1, ..., n\}$, define $\mathbb{R}_h := \{(x_1, ..., x_n) \mid x_h \in \mathbb{R}\}$ and for $H \in \mathcal{P}(n)$, $\mathbb{R}_H := \bigcup_{h\in H} \mathbb{R}_h$.

Definition 4.2. Let $F : D \subset \mathbb{C}^n \to A \otimes \mathbb{R}^{2^n}$ be a stem function. Define for $h = 1, ..., n$ and for $H = \{h_1, ..., h_p\} \in \mathcal{P}(n)$

$$\begin{aligned}
F_h^\circ(z) &:= \sum_{K\in\mathcal{P}(n), h\notin K} e_K F_K(z),\\
F_H^\circ(z) &:= \sum_{K\subset H^c} e_K F_K(z) = \left(\dots(F_{h_1}^\circ)_{h_2}^\circ \dots\right)_{h_p}^\circ(z)
\end{aligned}$$

and

$$F_h'(z) := \beta_h^{-1} \sum_{K\in\mathcal{P}(n), h\notin K} e_K F_{K\cup\{h\}}(z), \qquad \text{if } z \in D \setminus \mathbb{R}_h \tag{22}$$

$$F_H'(z) := \beta_H^{-1} \sum_{K\subset H^c} e_K F_{K\cup H}(z) = \left(\dots(F_{h_1}')_{h_2}' \dots\right)_{h_p}'(z), \qquad \text{if } z \in D \setminus \mathbb{R}_H, \tag{23}$$

where $z = (z_1, ..., z_n)$ with $z_j = \alpha_j + i\beta_j$ and $\beta_H = \prod_{h\in H} \beta_h$.

Lemma 4.7. *For every $H \in \mathcal{P}(n)$, F_H° and F_H' are well defined stem functions on D and $D \backslash \mathbb{R}_H$, respectively.*

Proof. Firstly, let us prove that F_H° and F_H' are well defined, i.e. their definition does not depend on the order of the elements of H. Indeed, for any $i, j = 1, ..., n$ it holds

$$(F_i')_j'(z) = \sum_{K \in \mathcal{P}(n), i,j \notin K} e_K \beta_j^{-1} \beta_i^{-1} F_{K \cup \{i,j\}}(z) = (F_j')_i'(z)$$

and analogously for $(F_i^\circ)_j^\circ$. Without loss of generality, assume $H = \{h\}$, for some $h = 1, ..., n$. F_h° is trivially a stem function because its non zero components are the same of F. Let us explicit $F_h' = \sum_{K \in \mathcal{P}(n)} e_K G_K$, with

$$G_K(z) = \begin{cases} \beta_h^{-1} F_{K \cup \{h\}} & \text{if } h \notin K \\ 0 & \text{if } h \in K, \end{cases}$$

we will show that every component of F_h' satisfies (8). Let us consider only the components G_K, with $h \notin K$, otherwise (8) is trivial. For any $m \neq h$ we have

$$G_K(\overline{z}^m) = \beta_h^{-1} F_{K \cup \{h\}}(\overline{z}^m) = \beta_h^{-1} (-1)^{|K \cap \{m\}|} F_{K \cup \{h\}}(z) = (-1)^{|K \cap \{m\}|} G_K(z),$$

while, for $m = h$

$$G_K(\overline{z}^h) = (-\beta_h^{-1}) F_{K \cup \{h\}}(\overline{z}^h) = (-\beta_h^{-1})(-F_{K \cup \{h\}}(z)) = \beta_h^{-1} F_{K \cup \{h\}}(z) = G_K(z).$$

□

The previous Lemma allows to make the following

Definition 4.3. Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$. For $h \in \{1, ..., n\}$, we define its *spherical x_h-value and x_h-derivative* rispectively as

$$f_{s,h}^\circ := \mathcal{I}(F_h^\circ), \qquad f_{s,h}' := \mathcal{I}(F_h').$$

Analogously, for $H \in \mathcal{P}(n)$, define

$$f_{s,H}^\circ := \mathcal{I}(F_H^\circ), \qquad f_{s,H}' := \mathcal{I}(F_H').$$

Note that $f_{s,H}^\circ \in \mathcal{S}(\Omega_D)$, while $f_{s,H}' \in \mathcal{S}(\Omega_{D_H})$, where $\Omega_{D_H} := \Omega_D \setminus \mathbb{R}_H$.

The following proposition justifies the names given to $f_{s,h}^\circ$ and $f_{s,h}'$, comparing them to their one-variable analogues.

Proposition 4.8. *Let $f \in \mathcal{S}_h(\Omega_D)$ and $h = 1, ..., n$. Then it holds*

1. $\forall x = (x_1, ..., x_n) \in \Omega_D$
$$f_{s,h}^\circ(x) = \frac{1}{2}\left(f(x) + f\left(\overline{x}^h\right)\right) = (f_h^x)_s^\circ(x_h);$$

2. $\forall x \in \Omega_D \setminus \mathbb{R}_h$
$$f_{s,h}'(x) = [2\operatorname{Im}(x_h)]^{-1}\left(f(x) - f(\overline{x}^h)\right) = (f_h^x)_s'(x_h). \tag{24}$$
In particular, if we assume $f \in \mathcal{S}^1(\Omega_D)$, then we can extend the definition of $f_{s,h}'$ to all Ω_D, thanks to [52, Proposition 7, (2)].

Proof. Let $f = \mathcal{I}(F)$, with $F = \sum_{K\in\mathcal{P}(n)} e_K F_K$. Then for any $z \in D$ and $x = (\phi_{J_1} \times ... \times \phi_{J_n})(z)$ we get

$$f(x) + f(\overline{x}^h) = \sum_{K\in\mathcal{P}(n)} \left([J_K, F_K(z)] + [J_K, F_K(\overline{z}^h)]\right)$$
$$= \sum_{K\in\mathcal{P}(n)} \Big([J_K, F_K(z)] + (-1)^{|K\cap\{h\}|}[J_K, F_K(z)]\Big) = \sum_{K\in\mathcal{P}(n), h\notin K} (2[J_K, F_K(z)]) = 2f^\circ_{s,h}(x).$$

Now, since $f \in \mathcal{S}_h(\Omega_D)$, then by (12)

$$f(x) = \sum_{h\notin K} [J_K, F_K(z)] + J_h \sum_{Q\subset\{h+1,...,n\}} \left[J_Q, F_{\{h\}\cup Q}(z)\right]$$

and so

$$f'_{s,h}(x) = \sum_{Q\subset\{h+1,...,n\}} [J_Q, \beta_h^{-1} F_{\{h\}\cup Q}(z)].$$

On the other hand, let $x = (\phi_{J_1} \times ... \times \phi_{J_n})(z)$, then by (8) we have

$$\begin{aligned} f(x) - f\left(\overline{x}^h\right) = &\sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(z)] + J_h \sum_{Q\subset\{h+1,...,n\}} [J_Q, F_{\{h\}\cup Q}(z)] + \\ &- \sum_{K\in\mathcal{P}(n),h\notin K} [J_K, F_K(\overline{z}^h)] - J_h \sum_{Q\subset\{h+1,...,n\}} [J_Q, F_{\{h\}\cup Q}(\overline{z}^h)] \\ = &\, 2J_h \sum_{Q\subset\{h+1,...,n\}} [J_Q, F_{\{h\}\cup Q}(z)], \end{aligned}$$

from which

$$[2\operatorname{Im}(x_h)]^{-1}\left(f(x) - f\left(\overline{x}^h\right)\right) = [2J_h\beta_h]^{-1}\left(2J_h \sum_{Q\subset\{h+1,...,n\}} [J_Q, F_{\{h\}\cup Q}(z)]\right)$$
$$= \sum_{Q\subset\{h+1,...,n\}} [J_Q, \beta_h^{-1} F_{\{h\}\cup Q}(z)] = f'_{s,h}(x).$$

□

Remark 7. Point 1 of the previous Proposition holds for any $f \in \mathcal{S}(\Omega_D)$. Indeed, in the proof we didn't used the hypothesis $f \in \mathcal{S}_h(\Omega_D)$.

The next proposition presents some properties of partial spherical values and derivatives peculiar of the several variables setting.

Proposition 4.9. *Let $f \in \mathcal{S}(\Omega_D)$, $h \in \{1, ..., n\}$ and $H \in \mathcal{P}(n)$, with $p = \min H^c$ if $H \neq \{1, ..., n\}$. Then*

1. $f^\circ_{s,H} \in \mathcal{S}_{c,H}(\Omega_D) \cap \mathcal{S}_p(\Omega_D)$ *and* $f'_{s,H} \in \mathcal{S}_{c,H}(\Omega_{D_H}) \cap \mathcal{S}_p(\Omega_{D_H})$;
2. *if* $f \in \mathcal{S}_h(\Omega_D)$, $f'_{s,h} \in \mathcal{S}_{h+1}(\Omega_{D_H}) \cap \mathcal{S}_{c,\{1,...,h\}}(\Omega_{D_H})$;
3. *if* $f \in \mathcal{S}_{c,h}(\Omega_D)$, $f^\circ_{s,h} = f$ *and* $f'_{s,h} = 0$;
4. *if* $h \in H$, $H \cap \{1, ..., h-1\} \neq \emptyset$ *and* $f \in \mathcal{S}_h(\Omega_D)$, *then* $f'_{s,H} = 0$;

5. $(f^\circ_{s,h})^\circ_{s,h} = f^\circ_{s,h}$ *and* $(f'_{s,h})'_{s,h} = 0$.

Proof. 1. If $f = \mathcal{I}(F)$, by definition $f^\circ_{s,h} = \sum_{K\subset H^c}[J_K, F_K]$, hence by Proposition 4.4, $f^\circ_{s,H} \in \mathcal{S}_{c,H}(\Omega_D)$. Moreover, we can write it as

$$f^\circ_{s,h} = \sum_{K\subset (H\cup p)^c} [J_K, F_K] + J_p \sum_{K\subset (H\cup p)^c} [J_K, F_{K\cup p}],$$

so $f^\circ_{s,h} \in \mathcal{S}_p(\Omega_D)$. In the same way one can prove that $f'_{s,H} \in \mathcal{S}_{c,H}(\Omega_{D_H}) \cap \mathcal{S}_p(\Omega_{D_H})$.

2. By Proposition 4.1, F takes the form

$$F = \sum_{K\in\mathcal{P}(n), h\notin K} e_K F_K + e_{\{h\}} \sum_{Q\subset\{h+1,\dots,n\}} e_Q F_{\{h\}\cup Q},$$

hence,

$$F'_h = \beta_h^{-1} \sum_{Q\subset\{h+1,\dots,n\}} e_Q F_{\{h\}\cup Q}.$$

This shows that $f'_{s,h} \in \mathcal{S}_{c,\{1,\dots,h\}}(\Omega_{D_h})$, by Proposition 4.4. Finally, by Proposition 4.1, $f'_{s,h} \in \mathcal{S}_{h+1}(\Omega_{D_h})$.

3. By Proposition 4.4, $F = \sum_{K\in\mathcal{P}(n),h\notin K} e_K F_K$, so $F'_h = 0$ and $F^\circ_h = F$.

4. Let $i \in H \cap \{1,\dots,h-1\} \neq \emptyset$, since $f \in \mathcal{S}_h(\Omega_D)$, by (2) $f'_{s,h} \in \mathcal{S}_{c,i}(\Omega_{D_i})$ and by (3) $(f'_{s,h})'_{s,i} = 0$. In particular, $f'_{s,H} = 0$.

5. It follows from (1) and (3).

□

Partial values or spherical derivatives do not affect regularity in other variables.

Proposition 4.10. *Let $f \in \mathcal{S}^1(\Omega_D)$. Suppose that $f \in \ker(\partial/\partial x^c_t)$ for some $t = 1,\dots,n$, then $f^\circ_{s,h}, f'_{s,h} \in \ker(\partial/\partial x^c_t)$, $\forall h \neq t$.*

Proof. Let $f = \mathcal{I}(F)$, with $F = \sum_{K\in\mathcal{P}(n)} e_K F_K$, so $f'_{s,h} = \mathcal{I}(F'_h)$, with $F'_h = \sum_{K\in\mathcal{P}(n)} e_K G_K$, $G_K = 0$, if $h \in K$ and $G_K = \beta_h^{-1} F_{K\cup\{h\}}$, if $h \notin K$. Let $K \in \mathcal{P}(n)$, with $h, t \notin K$, then by the regularity of F it holds

$$\begin{cases} \dfrac{\partial G_K}{\partial \alpha_t} = \dfrac{\partial \beta_h^{-1} F_{K\cup\{h\}}}{\partial \alpha_t} = \beta_h^{-1}\dfrac{\partial F_{K\cup\{h\}}}{\partial \alpha_t} = \beta_h^{-1}\dfrac{\partial F_{K\cup\{h\}\cup\{t\}}}{\partial \beta_t} = \dfrac{\partial G_{K\cup\{t\}}}{\partial \beta_t} \\ \dfrac{\partial G_K}{\partial \beta_t} = \dfrac{\partial \beta_h^{-1} F_{K\cup\{h\}}}{\partial \beta_t} = \beta_h^{-1}\dfrac{\partial F_{K\cup\{h\}}}{\partial \beta_t} = -\beta_h^{-1}\dfrac{\partial F_{K\cup\{h\}\cup\{t\}}}{\partial \alpha_t} = -\dfrac{\partial G_{K\cup\{t\}}}{\partial \alpha_t}. \end{cases}$$

This proves that F'_h is t-holomorphic, hence $f'_{s,h} \in \ker(\partial/\partial x^c_t)$. The spherical value case is analogue. □

As recalled in §3.1, every one variable slice function f can be decomposed as

$$f(x) = f^\circ_s(x) + \mathrm{Im}(x) f'_s(x).$$

We now give a similar decomposition for every variable, through the slice product.

Proposition 4.11. *Let $f \in \mathcal{S}(\Omega_D)$, then for any $h = 1, ..., n$ we can decompose*

$$f = f^\circ_{s,h} + \operatorname{Im}(x_h) \odot f'_{s,h}. \tag{25}$$

Equivalently, if $f = \mathcal{I}(F)$, it holds

$$F = F^\circ_h + \operatorname{Im}(Z_h) \otimes F'_h, \tag{26}$$

where $\operatorname{Im}(Z_h)(\alpha_1 + i\beta_1, ..., \alpha_n + i\beta_n) := e_h\beta_h$ is the stem function inducing $\operatorname{Im}(x_h)$.

Proof. Let $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$. Suppose first $x \in \mathbb{R}_h$, i.e. $\operatorname{Im}(x_h)(x) = 0$, then by (8), with the usual notation, we have

$$f(x) = \sum_{K \in \mathcal{P}(n)} [J_K, F_K(z)] = \sum_{K \in \mathcal{P}(n), h \notin K} [J_K, F_K(z)] = f^\circ_{s,h}(x).$$

Now, suppose $x \in \Omega_D \setminus \mathbb{R}_h$ and define . Then

$$\begin{aligned} F^\circ_h + \operatorname{Im}(Z_h) \otimes F'_h &= \sum_{K \in \mathcal{P}(n), h \notin K} e_K F_K + (e_h\beta_h) \otimes \left(\sum_{K \in \mathcal{P}(n), h \notin K} e_K \beta_h^{-1} F_{K \cup \{h\}} \right) \\ &= \sum_{K \in \mathcal{P}(n), h \notin K} e_K F_K + \sum_{K \in \mathcal{P}(n), h \notin K} e_{K \cup \{h\}} F_{K \cup \{h\}} = F. \end{aligned}$$

Finally, $f = \mathcal{I}(F) = \mathcal{I}(F^\circ_h + \operatorname{Im}(Z_h) \otimes F'_h) = f^\circ_{s,h} + \operatorname{Im}(x_h) \odot f'_{s,h}$. □

Next proposition shows that the partial spherical derivatives satisfies a Leibniz-type formula, analogue to the one-dimensional case.

Proposition 4.12 (Leibniz rule)**.** *Let $f, g \in \mathcal{S}(\Omega_D)$. For any $h \in \{1, \dots, n\}$, it holds*

$$(f \odot g)'_{s,h} = f'_{s,h} \odot g^\circ_{s,h} + f^\circ_{s,h} \odot g'_{s,h}. \tag{27}$$

Equivalently, if $F, G \in Stem(D)$, with $f = \mathcal{I}(F), g = \mathcal{I}(G)$ it holds

$$(F \otimes G)'_h = F'_h \otimes G^\circ_h + F^\circ_h \otimes G'_h. \tag{28}$$

Proof. Let $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$ and $G = \sum_{K \in \mathcal{P}(n)} e_K G_K$. We have to show that $(F \otimes G)'_h = F'_h \otimes G^\circ_h + F^\circ_h \otimes G'_h$. By [50, Lemma 2.34] we have $F'_h \otimes G^\circ_h = \sum_{K \in \mathcal{P}(n), h \notin K} e_K (F'_h \otimes G^\circ_h)_K$, where

$$(F'_h \otimes G^\circ_h)_K = \sum_{K_1, K_2, K_3 \in \mathcal{D}(K)} (-1)^{|K_3|} (F'_h)_{K_1 \cup K_3} (G^\circ_h)_{K_2 \cup K_3},$$

and $\mathcal{D}(K) := \{(K_1, K_2, K_3) \in \mathcal{P}(n)^3 \mid K = K_1 \sqcup K_2, K_3 \cap K = \emptyset\}$. By definition of F'_h and G°_h, the previous equation reduces to

$$(F'_h \otimes G^\circ_h)_K = \sum_{K_1, K_2, K_3 \in \mathcal{D}'_h(K)} (-1)^{|K_3|} F_{K_1 \cup K_3 \cup \{h\}} G_{K_2 \cup K_3},$$

with $\mathcal{D}'_h(K) := \{(K_1, K_2, K_3) \in \mathcal{P}(n)^3 \mid K = K_1 \sqcup K_2, K_3 \cap (K \cup \{h\}) = \emptyset\}$. In the very same way, we get

$$(F^\circ_h \otimes G'_h)_K = \sum_{K_1, K_2, K_3 \in \mathcal{D}'_h(K)} (-1)^{|K_3|} F_{K_1 \cup K_3} G_{K_2 \cup K_3 \cup \{h\}},$$

hence

$$F'_h \otimes G^\circ_h + F^\circ_h \otimes G'_h = \sum_{K\in\mathcal{P}(n), h\notin K} e_K \sum_{K_1,K_2,K_3\in\mathcal{D}'_h(K)} (-1)^{|K_3|} \big(F_{K_1\cup K_3\cup\{h\}} G_{K_2\cup K_3} + \\ + F_{K_1\cup K_3} G_{K_2\cup K_3\cup\{h\}}\big).$$

On the other hand, $F\otimes G = \sum_{K\in\mathcal{P}(n)} e_K (F\otimes G)_K$, where

$$(F\otimes G)_K = \sum_{K_1,K_2,K_3\in\mathcal{D}(K)} (-1)^{|K_3|} F_{K_1\cup K_3} G_{K_2\cup K_3}.$$

Thus

$$\begin{aligned}(F\otimes G)'_h &= \sum_{K\in\mathcal{P}(n), h\notin K} e_K \beta_h^{-1} (F\otimes G)_{K\cup\{h\}} \\ &= \sum_{K\in\mathcal{P}(n), h\notin K} e_K \sum_{K_1,K_2,K_3\in\mathcal{D}(K\cup\{h\})} (-1)^{|K_3|} F_{K_1\cup K_3} G_{K_2\cup K_3}.\end{aligned}$$

Note that

$$\begin{aligned}\mathcal{D}(K\cup\{h\}) &= \{(K_1,K_2,K_3)\in\mathcal{P}(n)^3 \mid K\cup\{h\} = K_1\sqcup K_2, K_3\cap(K\cup\{h\})=\emptyset\} \\ &= \{(K_1\cup\{h\},K_2,K_3),(K_1,K_2\cup\{h\},K_3) \mid (K_1,K_2,K_3)\in\mathcal{D}'_h(K)\}\end{aligned}$$

so

$$\begin{aligned}(F\otimes G)'_h &= \sum_{K\in\mathcal{P}(n), h\notin K} e_K \sum_{K_1,K_2,K_3\in\mathcal{D}(K\cup\{h\})} (-1)^{|K_3|} F_{K_1\cup K_3} G_{K_2\cup K_3} \\ &= \sum_{K\in\mathcal{P}(n), h\notin K} e_K \sum_{K_1,K_2,K_3\in\mathcal{D}'_h(K)} (-1)^{|K_3|} \big(F_{K_1\cup\{h\}\cup K_3} G_{K_2\cup K_3} + F_{K_1\cup K_3} G_{K_2\cup\{h\}\cup K_3}\big) \\ &= F'_h\otimes G^\circ_h + F^\circ_h \otimes G'_h.\end{aligned}$$

□

Corollary 4.13. *Let $f\in\mathcal{S}(\Omega_D)$ and $g\in\mathcal{S}_{c,H}(\Omega_D)$ for some $H\in\mathcal{P}(n)$, then*

$$(f\odot g)'_{s,H} = f'_{s,H}\odot g.$$

Proof. We proceed by induction over $|H|$. Suppose first $|H|=1$, then it follows from Proposition 4.12 and Proposition 4.9 (3). Now, suppose by induction that $(f\odot g)'_{s,H} = f'_{s,H}\odot g$ and let $h\notin H$, then in the same way we have

$$(f\odot g)'_{s,H\cup\{h\}} = (f'_{s,h}\odot g^\circ_{s,h} + f^\circ_{s,h}\odot g'_{s,h})'_{s,H} = (f'_{s,h}\odot g)'_{s,H} = f'_{s,H\cup\{h\}}\odot g.$$

□

4.2 One variable interpretation of slice regularity

The one variable interpretation of slice regularity is given in [50, §2.3 and 3.4]

We stress that the terms spherical value and spherical derivatives have been already used in [50, §2.3] in the context of slice functions of several quaternionic variables, but they refer to different objects. With respect to our definition, spherical values and derivatives are more related to the truncated spherical derivatives.

Definition 4.4 (Definition 2.24,[50])**.** Let $\Omega_D \subset (\mathcal{Q}_A)^n$ and let $f \in \mathcal{S}(\Omega_D)$. For any $h = 1, \dots, n$ and $\epsilon : \{1, \dots, h\} \to \{0, 1\}$, define the truncated spherical ϵ-derivative of f of order h, $\mathcal{D}^h_\epsilon(f) : \Omega_D \setminus \mathbb{R}_{\epsilon^{-1}(1)} \to A$ as

$$\mathcal{D}^h_\epsilon(f) \coloneqq \mathcal{D}^{\epsilon(h)}_{x_h} \cdots \mathcal{D}^{\epsilon(1)}_{x_1}(f),$$

with

$$\mathcal{D}^0_{x_l}(f) = f^\circ_{s,l}, \qquad \mathcal{D}^1_{x_l}(f) = f'_{s,l}.$$

Alternatively, for given $H \in \mathcal{P}(h)$, we call the truncated spherical H-derivative of f the truncated spherical χ_H-derivative of f, where χ_H is the characteristic function of H, i.e. $\chi_H(j) = 0$ if $j \notin H$ and $\chi_H(j) = 1$ if $j \in H$. Namely, we set

$$\mathcal{D}^h_H(f) = \mathcal{D}^h_{\chi_H}(f).$$

Remark 8. For any given $h = 1, \dots, n$ and any $\epsilon : \{1, \dots, h\} \to \{0, 1\}$, denote with $H = \epsilon^{-1}(1)$ and $K = \epsilon^{-1}(0) = \{1, \dots, h\} \setminus H$. Then it holds

$$\mathcal{D}^h_\epsilon(f) = \mathcal{D}^h_H(f) = (f'_{s,H})^\circ_{s,K}.$$

Explicitely, if $f = \mathcal{I}(F)$, with $F = \sum_{K \in \mathcal{P}(n)} e_K F_K$, for every $h = 1, \dots, n$ and any $H \in \mathcal{P}(H)$, it holds $\mathcal{D}^h_H = \mathcal{I}(D^h_H)$, with

$$D^h_H = \beta_H^{-1} \sum_{K \subset \{h+1, \dots, n\}} e_K F_{K \cup H}.$$

From Proposition 4.2 and Proposition 4.10 it is easy to see that if $f \in \mathcal{SR}(\Omega_D)$, then $\mathcal{D}^h_K(f) \in \mathcal{SR}_{h+1}(\Omega_D)$, for any $h = 1, \dots, n$ and $H \in \mathcal{P}(h)$. Next Theorem tells that also the converse holds true.

Theorem 4.14 (One variable characterization of slice regularity, Theorem 3.23 [50])**.** *Let $\Omega_D \subset (\mathbb{Q}_A)^n$ and let $f \in \mathcal{S}(\Omega_D)$. Then, for any $h = 1, \dots, n-1$ and any $K \in \mathcal{P}(h)$, the truncated spherical K-derivative of order h, $\mathcal{D}^h_K(f)$ is a slice functions with respect to x_{h+1}. Moreover, $f \in \mathcal{SR}(\Omega_D)$ if and only if $f \in \mathcal{SR}_1(\Omega_D)$ and $\mathcal{D}^h_{\chi_K}(f) \in \mathcal{SR}_{h+1}(\Omega_D)$, for any $h = 1, \dots, n-1$ and any $K \in \mathcal{P}(h)$.*

5 Polyharmonicity in slice analysis

Most of the harmonic properties presented in this section are known, see [74], however we provide some new formulas that appeared in [10].

5.1 Polyharmonicity of spherical derivatives

The spherical derivative of a quaternion valued slice regular function is harmonic.

Proposition 5.1. *Let $f : \Omega_D \subset \mathbb{H} \to \mathbb{H}$ be a slice regular function. Then, f'_s is harmonic, i.e.*

$$\Delta_4 f'_s = (\partial^2_{x_0} + \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3}) f'_s = 0.$$

Even if the result is known, we explicitly compute the Laplacian of the spherical derivative of a quaternionic slice regular function, similar to the proof of [9, Proposition 4.9]. This will help for future computations.

Proof. Suppose that f is induced by the holomorphic stem function $F = F_0 + e_1 F_1$. In particular, F_1 is harmonic, i.e. $\Delta_2 F_1 = (\partial^2_\alpha + \partial^2_\beta) F_1 = 0$. Let $\beta = \sqrt{x_1^2 + x_2^2 + x_3^2}$, then

$$f'_s(x_0 + ix_1 + jx_2 + kx_3) = \beta^{-1}(x_1, x_2, x_3) F_1(x_0, \beta(x_1, x_2, x_3)).$$

Let us compute $\Delta_4 f'_s$. Immediately we get $\partial^2_{x_0} f'_s = \beta^{-1} \partial^2_{x_0} F_1$. Moreover, by

$$\partial_{x_i} \beta = x_i \beta^{-1}, \qquad \partial_{x_i} F_1 = x_i \beta^{-1} \partial_\beta F_1,$$

we find for any $i = 1, 2, 3$

$$\partial_{x_i}(\beta^{-1} F_1) = -x_i \beta^{-3} F_1 + x_i \beta^{-2} \partial_\beta F_1$$

and

$$\begin{aligned}
\partial^2_{x_i}(\beta^{-1} F_1) &= \partial_{x_i}\left(-x_i \beta^{-3} F_1 + x_i \beta^{-2} \partial_\beta F_1\right) \\
&= (3x_i^2 - \beta^2)\beta^{-5} F_1 - x_i^2 \beta^{-4} \partial_\beta F_1 + (\beta^2 - 2x_i^2)\beta^{-4} \partial_\beta F_1 + x_i^2 \beta^{-3} \partial^2_\beta F_1 \\
&= (3x_i^2 - \beta^2)\beta^{-5} F_1 + (\beta^2 - 3x_i^2)\beta^{-4} \partial_\beta F_1 + x_i^2 \beta^{-3} \partial^2_\beta F_1.
\end{aligned}$$

So

$$\left(\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3}\right)(\beta^{-1} F_1) = \beta^{-1} \partial^2_\beta F_1$$

and finally

$$\Delta_4 f'_s = \Delta_4(\beta^{-1} F_1) = \beta^{-1}\left(\partial^2_{x_0} + \partial^2_\beta\right) F_1 = \beta^{-1} \Delta_2 F_1 = 0.$$

□

The previous Proposition is a special case of a more general result. The next formulas are a slight variation of [74, Theorem 4.1].

Proposition 5.2. *Let m be odd and let $f = \mathcal{I}(F) : \Omega_D \subset \mathbb{R}^{m+1} \to \mathbb{R}_m$ be a slice regular function, with $F = F_0 + e_1 F_1$. Then, for any $k = 1, 2, \dots,$ it holds*

$$\Delta^k_{m+1} f'_s(x) = (m-3) \cdot \ldots \cdot (m-2k-1) \sum_{j=1}^{k} a_j^{(k)} \beta^{j-2k} \partial^{(j)}_\beta f'_s(x), \tag{29}$$

or equivalently

$$\Delta_{m+1}^{k} f_s'(x) = (m-3)\cdot \ldots \cdot (m-2k-1) \sum_{j=1}^{k+1} a_j^{(k+1)} \beta^{j-2k-2} \partial_\beta^{(j-1)} F_1(\mathrm{Re}(x), |\,\mathrm{Im}(x)|), \tag{30}$$

where $\Delta_{m+1} = \partial_a^2 + \sum_{j=1}^{m} \partial_{x_j}^2$ *is the Laplacian of* $\mathbb{R}^{m+1}$ *and*

$$a_j^{(k)} := \frac{(2k-j-1)!}{(j-1)!\,(k-j)!\,(-2)^{k-j}}. \tag{31}$$

In particular, f_s' *is polyharmonic of degree* $\frac{m-1}{2}$*, i.e.*

$$\Delta_{m+1}^{\frac{m-1}{2}} f_s' = 0.$$

Before giving the proof of Proposition 5.2, we need some preliminary results.

Lemma 5.3. *The coefficients* $a_j^{(k)}$ *of the previous Proposition satisfy the following relations:*

1. *we can define iteratively* $a_j^{(k)}$ *as*

$$\begin{cases} a_j^{(k+1)} = a_{j-1}^{(k)} - (2k-j) a_j^{(k)}, \qquad \forall j = 1, \ldots, k \\ \\ a_k^{(k)} = 1, \ a_0^{(k)} = 0. \end{cases} \tag{32}$$

2. *for any* $j = 0, \ldots, k$ *it holds*

$$a_{j+1}^{(k+1)} = \sum_{l=j}^{k} (-1)^{l-j} \frac{l!}{j!} a_l^{(k)}. \tag{33}$$

Proof. 1. To prove (32), we just compute

$$\begin{aligned} a_{j-1}^{(k)} + (j-2k) a_k^{(k)} &= \frac{(-2)^{j-k-1}(2k-j)!}{(j-2)!(k-j+1)!} - \frac{(-2)^{j-k}(2k-j)!}{(j-1)!(k-j)!} \\ &= \frac{(-2)^{j-k-1}(2k-j)!}{(j-1)!(k-j+1)!} \left[(j-1) - (-2)(k-j+1)\right] \\ &= \frac{(-2)^{j-k-1}(2k-j+1)!}{(j-1)!(k-j+1)!} = a_j^{(k+1)}. \end{aligned}$$

2. Note that

$$a_{j+1}^{(k+1)} = \frac{(2k-j)!}{j!(k-j)!(-2)^{k-j}} = \frac{2^{j-k}(-1)^{j-k}}{j!} \frac{(2k-j)!}{(k-j)!},$$

while, on the other hand

$$\sum_{l=j}^{k} (-1)^{l-j} \frac{l!}{j!} a_l^{(k)} = \sum_{l=j}^{k} (-1)^{l-j} \frac{l(2k-l-1)!}{j!(k-l)!(-2)^{k-l}} = \frac{2^{-k}(-1)^{j-k}}{j!} \sum_{l=j}^{k} \frac{2^l l(2k-l-1)!}{(k-l)!}.$$

Thus, (33) holds if and only if

$$\frac{2^j(2k-j)!}{(k-j)!} = \sum_{l=j}^{k} \frac{2^l l(2k-l-1)!}{(k-l)!}.$$

Consider the right hand side of the previous equation:

$$\sum_{l=j}^{k} \frac{2^l l(2k-l-1)!}{(k-l)!} = \sum_{l=j}^{k-1} \frac{2^l l(2k-l-1)!}{(k-l)!} + 2^k k!,$$

note that

$$\frac{2^l l(2k-l-1)!}{(k-l)!} = \frac{2^l(2k-l)!}{(k-l)!} - \frac{2^{l+1}(2k-(l+1))!}{(k-(l+1)!},$$

thus the sum $\sum_{l=j}^{k} \frac{2^l l(2k-l-1)!}{(k-l)!}$ is telescopic and gives

$$\sum_{l=j}^{k} \frac{2^l l(2k-l-1)!}{(k-l)!} = \frac{2^j(2k-j)!}{(k-j)!} - 2^k k!$$

and finally

$$\sum_{l=j}^{k} \frac{2^l l(2k-l-1)!}{(k-l)!} = \frac{2^j(2k-j)!}{(k-j)!} - 2^k k! + 2^k k! = \frac{2^j(2k-j)!}{(k-j)!}.$$

□

Proposition 5.4 (Theorem 4.1, [74])**.** *Let $f\colon \Omega_D \to \mathbb{R}_m$ be a slice regular function and let Δ_{m+1} the Laplacian of $\mathbb{R}^{m+1}$, then for every $k = 1, ..., [\frac{m-1}{2}]$ and any $x \in \Omega_D$, it holds*

$$\Delta_{m+1}^k f_s'(x) = 2^k(m-3)\cdot \dots \cdot (m-2k-1)\partial_2^k G(\mathrm{Re}(x), |\operatorname{Im}(x)|^2), \tag{34}$$

where $G\colon D \subset \mathbb{C} \to \mathbb{R}_m$ is defined by $G(\mathrm{Re}(x), |\operatorname{Im}(x)|^2) = f_s'(x)$

The goal of the following Proposition is to express (34) in term of derivatives of f_s'.

Proposition 5.5. *Let $f\colon \Omega_D \to \mathbb{R}_m$ be a slice function and let G as before. Then, for any $k = 1, 2, \dots$ and any $x = \alpha + J\beta \in \Omega_D$*

$$\partial_2^k G(\alpha, \beta^2) = 2^{-k} \sum_{j=1}^{k} a_j^{(k)} \beta^{j-2k} \partial_2^j f_s'(x), \tag{35}$$

Proof. Let us write $G(\alpha, \beta) = f_s'\left(\alpha, \sqrt{\beta}\right)$, since f_s' only depends on $\mathrm{Re}(x)$ and $|\operatorname{Im}(x)|$. Let us prove the result by induction over k. First, note that

$$\partial_2 G(\alpha, \beta) = \partial_\beta f_s'(\alpha, \sqrt{\beta}) = \frac{1}{2\sqrt{\beta}} \partial_\beta (f_s')(\alpha, \sqrt{\beta}),$$

and so

$$\partial_2 G(\alpha, \beta^2) = \frac{1}{2\beta} \partial_\beta (f_s')(\alpha, \beta),$$

which is (35) for $k=1$. Now, assume by induction that, for some $k \in \mathbb{N}$, it holds

$$\partial_2^k G(\alpha,\beta) = 2^{-k} \sum_{j=1}^{k} a_j^{(k)} \beta^{\frac{j-2k}{2}} \partial_2^j f_s' \left(\alpha, \sqrt{\beta}\right),$$

so

$$\begin{aligned}
\partial_2^{k+1} G(\alpha,\beta) &= \frac{\partial}{\partial\beta}\left[2^{-k} \sum_{j=1}^{k} a_j^{(k)} \beta^{\frac{j-2k}{2}} \partial_2^j f_s' \left(\alpha, \sqrt{\beta}\right)\right] \\
&= 2^{-k} \sum_{j=1}^{k} a_j^{(k)} \left[\frac{j-2k}{2} \beta^{\frac{j-2k-2}{2}} \frac{\partial^j}{\partial\beta^j} f_s' \left(\alpha, \sqrt{\beta}\right) + \frac{1}{2\sqrt{\beta}} \beta^{\frac{j-2k}{2}} \frac{\partial}{\partial\beta}^{j+1} f_s' \left(\alpha, \sqrt{\beta}\right)\right] \\
&= 2^{-k-1} \sum_{j=1}^{k} a_j^{(k)} \left[(j-2k) \beta^{\frac{j-2k-2}{2}} \frac{\partial^j}{\partial\beta^j} f_s' \left(\alpha, \sqrt{\beta}\right) + \beta^{\frac{j-2k-1}{2}} \frac{\partial^{j+1}}{\partial\beta^{j+1}} f_s' \left(\alpha, \sqrt{\beta}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\partial_2^{k+1} G(\alpha,\beta^2) &= 2^{-k-1} \sum_{j=1}^{k} a_j^{(k)} \left[(j-2k)\beta^{j-2k-2} \frac{\partial^j}{\partial\beta^j} f_s'(\alpha,\beta) + \beta^{j-2k-1} \frac{\partial^{j+1}}{\partial\beta^{j+1}} f_s'(\alpha,\beta)\right] \\
&= 2^{-k-1} \sum_{j=1}^{k} a_j^{(k)} \left[(j-2k)\beta^{j-2k-2} \frac{\partial^j}{\partial\beta^j} f_s'(\alpha,\beta)\right] + \\
&\quad + 2^{-k-1} \sum_{j=2}^{k+1} a_{j-1}^{(k)} \left[\beta^{j-2k-2} \frac{\partial^j}{\partial\beta^j} f_s'(\alpha,\beta)\right] \\
&= 2^{-k-1} \sum_{j=1}^{k+1} \left[a_j^{(k)}(j-2k) + a_{j-1}^{(k)}\right] \beta^{j-2k-2} \frac{\partial^j}{\partial\beta^j} f_s'(\alpha,\beta) \\
&= 2^{-(k+1)} \sum_{j=1}^{k+1} a_j^{(k+1)} \beta^{j-2(k+1)} \partial_\beta^j f_s'(\alpha,\beta),
\end{aligned}$$

where we have used (32). □

Proof of Proposition 5.2. (29) immediately follows from (34) and (35). Let us prove (30) from (29). Recall that $f_s' = \beta^{-1} F_1$ and that for any $j = 1, 2, \ldots$ it holds

$$\partial_\beta^{(j)} (\beta^{-1} F_1) = \sum_{l=0}^{j} \frac{j!}{l!} (-1)^{j-l} \beta^{l-j-1} \partial_\beta^{(l)} F_1.$$

Then by (29) we have

$$
\begin{aligned}
\Delta_{m+1}^k f_s' &= (m-3)\cdot \ldots \cdot (m-2k-1) \sum_{j=1}^{k} a_j^{(k)} \beta^{j-2k} \partial_\beta^{(j)} (\beta^{-1} F_1) = \\
&= (m-3)\cdot \ldots \cdot (m-2k-1) \sum_{j=1}^{k} \sum_{l=0}^{j} a_j^{(k)} \beta^{j-2k} \frac{j!}{l!} (-1)^{j-l} \beta^{l-j-1} \partial_\beta^{(l)} F_1 \\
&= (m-3)\cdot \ldots \cdot (m-2k-1) \sum_{l=0}^{k} \sum_{j=l}^{k} a_j^{(k)} \beta^{l-2k-1} \frac{j!}{l!} (-1)^{j-l} \partial_\beta^{(l)} F_1 \\
&= (m-3)\cdot \ldots \cdot (m-2k-1) \sum_{j=0}^{k} \beta^{j-2k-1} \sum_{l=j}^{k} a_l^{(k)} \frac{l!}{j!} (-1)^{l-j} \partial_\beta^{(j)} F_1 \\
&= (m-3)\cdot \ldots \cdot (m-2k-1) \sum_{j=0}^{k} a_{j+1}^{(k+1)} \beta^{j-2k-1} \partial_\beta^{(j)} F_1 \\
&= (m-3)\cdot \ldots \cdot (m-2k-1) \sum_{j=1}^{k+1} a_j^{(k+1)} \beta^{j-2k-2} \partial_\beta^{(j-1)} F_1 .
\end{aligned}
$$

□

5.2 Polyharmonicity of slice regular functions

Theorem 5.6. *Let $\Omega_D \subset \mathbb{R}^{m+1}$ and let $f \in \mathcal{SR}(\Omega_D)$, then for any $k \in \mathbb{N}$ it holds*

$$
\Delta_{m+1}^{k+1} f = -2(m-1)\cdot \cdots \cdot (m-2k-1) \sum_{j=1}^{k} a_j^{(k)} \frac{\partial}{\partial x} \left(\beta^{j-2k} \partial_\beta^{(j)} f_s' \right), \tag{36}
$$

or equivalently, if $f = \mathcal{I}(F_0 + e_1 F_1)$,

$$
\Delta_{m+1}^{k+1} f = -2(m-1)\cdot \cdots \cdot (m-2k-1) \sum_{j=1}^{k+1} a_j^{(k+1)} \frac{\partial}{\partial x} \left(\beta^{j-2k-2} \partial_\beta^{(j)} F_1 \right), \tag{37}
$$

where for $k=0$ we mean

$$
\Delta_{m+1} f = -2(m-1) \frac{\partial}{\partial x} \left(f_s' \right). \tag{38}
$$

In particular, any slice regular function $f : \Omega_D \subset \mathbb{R}^{m+1} \to \mathbb{R}_m$ is polyharmonic of degree $\frac{m+1}{2}$, i.e.

$$
\Delta_{m+1}^{\frac{m+1}{2}} f = 0.
$$

Next Lemma shows that for circular functions the Laplacian and the slice derivative commute.

Lemma 5.7. *Let $f \in \mathcal{S}_c^3(\Omega_D)$, then it holds*

$$
\Delta_{m+1} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (\Delta_{m+1} f). \tag{39}
$$

Proof. Since $f \in \mathcal{S}^3_c(\Omega_D)$, $f(x) = F(x_0, \beta(x_1, \dots, x_m))$, where $\beta(x_1, \dots, x_m) = \sqrt{x_1^2 + \dots x_m^2}$. By definition, if $x = \alpha + J\beta \in \mathbb{R}^{m+1}$,

$$\frac{\partial f}{\partial x}(x) = \frac{1}{2}\left(\partial_\alpha - J\partial_\beta\right) F(x_0,\beta) = \frac{1}{2}\left(\partial_\alpha F(x_0,\beta) - J\partial_\beta F(x_0,\beta)\right).$$

Note that for any $i = 1, \dots m$, $\partial_{x_i} J = e_i \beta^{-1} - x_i J \beta^{-2}$, so

$$\partial_{x_i}\left(\partial_\alpha F - J\partial_\beta F\right) = x_i\beta^{-1}\partial_\beta\partial_\alpha F - e_i\beta^{-1}\partial_\beta F + x_i J\beta^{-2}\partial_\beta F - x_i J\beta^{-1}\partial^2_\beta F$$

and

$$\begin{aligned}\partial^2_{x_i}\left(\partial_\alpha F - J\partial_\beta F\right) = {} & \beta^{-1}\partial_\beta\partial_\alpha F - x_i^2\beta^{-3}\partial_\beta\partial_\alpha F + x_i^2\beta^{-2}\partial^2_\beta\partial_\alpha F + x_i e_i\beta^{-3}\partial_\beta F - x_i e_i \beta^{-2}\partial^2_\beta F + \\ & + J\beta^{-2}\partial_\beta F + x_i e_i\beta^{-3}\partial_\beta F - x_i^2 J\beta^{-4}\partial_\beta F - 2x_i^2 J\beta^{-4}\partial_\beta F + x_i^2 J\beta^{-3}\partial^2_\beta F + \\ & - x_i e_i\beta^{-2}\partial^2_\beta F + x_i^2 J\beta^{-3}\partial^2_\beta F - J\beta^{-1}\partial^2_\beta F + x_i^2 J\beta^{-3}\partial^2_\beta F - x_i^2 J\beta^{-2}\partial^3_\beta F.\end{aligned}$$

Thus, if $\Delta_m = \sum_{i=1}^m \partial^2_{x_i}$, we have

$$\begin{aligned}\Delta_m(\partial_\alpha F - J\partial_\beta F) & = m\beta^{-1}\partial_\beta\partial_\alpha F - \beta^{-1}\partial_\beta\partial_\alpha F + \partial^2_\beta\partial_\alpha F + J\beta^{-2}\partial_\beta F - J\beta^{-1}\partial^2_\beta F + \\ & \quad + mJ\beta^{-2}\partial_\beta F + J\beta^{-2}\partial_\beta F - J\beta^{-2}\partial_\beta F - 2J\beta^{-2}\partial_\beta F + J\beta^{-1}\partial^2_\beta F + \\ & \quad - mJ\beta^{-1}\partial^2_\beta F + J\beta^{-1}\partial^2_\beta F - J\partial^3_\beta F \\ & = (m-1)\left[\beta^{-1}\partial_\beta\partial_\alpha F + J\beta^{-2}\partial_\beta F - J\beta^{-1}\partial^2_\beta F\right] + \partial^2_\beta\partial_\alpha F - J\partial^3_\beta F.\end{aligned}$$

So, the left hand side of (39) becomes

$$\begin{aligned}\Delta_{m+1}\left(\frac{\partial f}{\partial x}\right) = & \frac{1}{2}\left[\partial^3_\alpha F + \partial^2_\beta\partial_\alpha F - J\partial_\beta\partial^2_\alpha F - J\partial^3_\beta F + \right. \\ & \left. + (m-1)(\beta^{-1}\partial_\beta\partial_\alpha F + J\beta^{-2}\partial_\beta F - J\beta^{-1}\partial^2_\beta F)\right].\end{aligned}$$

On the other hand, $\partial_{x_i} f = x_i\beta^{-1}\partial_\beta F$ and

$$\partial^2_{x_i} f = \beta^{-1}\partial_\beta F - x_i^2\beta^{-3}\partial_\beta F + x_i^2\beta^{-2}\partial^2_\beta F,$$

so

$$\Delta_{m+1} f = \partial^2_\alpha F + (m-1)\beta^{-1}\partial_\beta F + \partial^2_\beta F$$

and finally the right hand side of (39) becomes

$$\begin{aligned}\frac{\partial}{\partial x}(\Delta_{m+1} f) & = \frac{1}{2}(\partial_\alpha - J\partial_\beta)(\partial^2_\alpha F + (m-1)\beta^{-1}\partial_\beta F + \partial^2_\beta F) \\ & = \frac{1}{2}\left[\partial^3_\alpha F + \partial^2_\beta\partial_\alpha F - J\partial_\beta\partial^2_\alpha F - J\partial^3_\beta F + \right. \\ & \quad \left. + (m-1)(\beta^{-1}\partial_\beta\partial_\alpha F + J\beta^{-2}\partial_\beta F - J\beta^{-1}\partial^2_\beta F)\right].\end{aligned}$$

This proves (39). □

Proof of Theorem 5.6. The proof of (38) can be find in [72, Proposition 9, (f)]. Let us prove (36). By (29), if $f \in \mathcal{SR}(\Omega_D)$ we have

$$\Delta^k_{m+1} f'_s = (m-3)\cdot\dots\cdot(m-2k-1)\sum_{j=1}^k a_j^{(k)}\beta^{j-2k}\partial^{(j)}_\beta f'_s.$$

Now, using (38) and Lemma 5.7, which applies, since f'_s is circular, we have

$$\Delta_{m+1}^{k+1} f = \Delta_{m+1}^{k}(\Delta_{m+1} f) = -2(m-1)\Delta_{m+1}^{k}\left(\frac{\partial f'_s}{\partial x}\right) = -2(m-1)\frac{\partial}{\partial x}\left(\Delta_{m+1}^{k} f'_s\right)$$
$$= -2(m-1)(m-3)\cdot\dots\cdot(m-2k-1)\sum_{j=1}^{k} a_j^{(k)}\frac{\partial}{\partial x}\left(\beta^{j-2k}\partial_\beta^{(j)} f'_s\right).$$

Finally, (37) follows analogously by using (30), instead of (29).

□

5.3 Polyharmonicity in several variables

We can actually prove something more, namely a method to construct polyharmonic functions, starting by harmonic functions in the plane.

Proposition 5.8. *Let m be odd and let $F : D \subset \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}_m$ be harmonic function, i.e. $\Delta_2 F(a,b) = (\partial_a^2 + \partial_b^2)F(a,b) = 0$ and let $f : \Omega_D \subset \mathbb{R}^{m+1} \to \mathbb{R}_m$,*

$$f(a, x_1, \dots, x_m) := \frac{1}{\sqrt{x_1^2 + \dots x_m^2}} F\left(a, \sqrt{x_1^2 + \dots x_m^2}\right).$$

Then, for any $k \in \mathbb{N}$ it holds

$$\Delta_{m+1}^{k} f(a, x_1, \dots, x_m) = (m-3)\cdot\dots\cdot(m-2k-1)\sum_{j=1}^{k+1} a_j^{(k+1)} b^{j-2k-2}\partial_b^{(j-1)} F(a, b(x_1, \dots, x_m)),$$

where $\Delta_{m+1} = \partial_a^2 + \sum_{j=1}^{m}\partial_{x_j}^2$ is the Laplacian of $\mathbb{R}^{m+1}$. In particular, f is polyharmonic of degree $\frac{m-1}{2}$, i.e.

$$\Delta_{m+1}^{\frac{m-1}{2}} f = 0.$$

Proof. Note that for $k = 0$ we get $f = \frac{1}{\sqrt{x_1^2+\dots x_m^2}}F$. Now, suppose by induction that for some k it holds

$$\Delta_{m+1}^{k-1} f = (m-3)\cdot\dots\cdot(m-2k+1)\sum_{j=1}^{k} a_j^{(k)} b^{j-2k}\partial_b^{(j-1)} F$$

then let us compute $\partial_{x_i}^2(b^{j-2k}\partial_b^{(j-1)}F)$ for $i = 1, \dots, n$. Recalling that

$$\partial_{x_i}(b^n) = n x_i b^{n-2}, \qquad \partial_{x_i} F = x_i b^{-1}\partial_b F,$$

we have

$$\partial_{x_i}(b^{j-2k}\partial_b^{(j-1)}F) = (j-2k)x_i b^{j-2k-2}\partial_b^{(j-1)}F + x_i b^{j-2k-1}\partial_b^{(j)}F,$$

and

$$\partial_{x_i}^2(b^{j-2k}\partial_b^{(j-1)}F) = (j-2k)b^{j-2k-2}\partial_b^{(j-1)}F + (j-2k)(j-2k-2)x_i^2 b^{j-2k-4}\partial_b^{(j-1)}F +$$
$$+ (j-2k)x_i^2 b^{j-2k-3}\partial_b^{(j)}F + b^{j-2k-1}\partial_b^{(j)}F +$$
$$+ (j-2k-1)x_i^2 b^{j-2k-3}\partial_b^{(j)}F + x_i^2 b^{j-2k-2}\partial_b^{(j+1)}F.$$

Now, since $b^2 = \sum_{i=1}^m x_i^2$

$$\sum_{i=1}^{m} \partial_{x_i}^2 (b^{j-2k} \partial_b^{(j-1)} F) = m(j-2k) b^{j-2k-2} \partial_b^{(j-1)} F + (j-2k)(j-2k-2) b^{j-2k-2} \partial_b^{(j-1)} F +$$
$$+ (j-2k) b^{j-2k-1} \partial_b^{(j)} F + m b^{j-2k-1} \partial_b^{(j)} F +$$
$$+ (j-2k-1) b^{j-2k-1} \partial_b^{(j)} F + b^{j-2k} \partial_b^{(j+1)} F$$
$$= (m+j-2k-2)(j-2k) b^{j-2k-2} \partial_b^{(j-1)} F + (m+2j-4k-1) b^{j-2k-1} \partial_b^{(j)} F +$$
$$+ b^{j-2k} \partial_b^{(j+1)} F$$

and by the harmonicity of F,

$$\Delta_{m+1}(b^{j-2k} \partial_b^{(j-1)} F) = \left(\partial_a^2 + \sum_{i=1}^{m} \partial_{x_i}^2 \right) (b^{j-2k} \partial_b^{(j-1)} F)$$
$$= (m+j-2k-2)(j-2k) b^{j-2k-2} \partial_b^{(j-1)} F + (m+2j-4k-1) b^{j-2k-1} \partial_b^{(j)} F +$$
$$+ b^{j-2k} \partial_b^{(j-1)} (\partial_b^2 F + \partial_a^2 F)$$
$$= (m+j-2k-2)(j-2k) b^{j-2k-2} \partial_b^{(j-1)} F + (m+2j-4k-1) b^{j-2k-1} \partial_b^{(j)} F.$$

Let us split $m+j-2k-2 = m-2k-1+j-1$ and $m+2j-4k-1 = m-2k-1+2j-2k$, so we have

$$\Delta_{m+1}(b^{j-2k} \partial_b^{(j-1)} F) = (m-2k-1)[(j-2k) b^{j-2k-2} \partial_b^{(j-1)} F + b^{j-2k-1} \partial_b^{(j)} F] +$$
$$+ (j-1)(j-2k) b^{j-2k-2} \partial_b^{(j-1)} F + 2(j-k) b^{j-2k-1} \partial_b^{(j)} F$$

and considering the whole function f we have

$$\Delta_{m+1}^k f = \Delta_{m+1}(\Delta_{m+1}^{k-1} f) = (m-3) \cdot ... \cdot (m-2k+1) \sum_{j=1}^{k} a_j^{(k)} \Delta_{m+1}(b^{j-2k} \partial_b^{(j-1)} F)$$
$$= (m-3) \cdot ... \cdot (m-2k+1)(m-2k-1) \sum_{j=1}^{k} a_j^{(k)} [(j-2k) b^{j-2k-2} \partial_b^{(j-1)} F + b^{j-2k-1} \partial_b^{(j)} F] +$$
$$+ (m-3) \cdot ... \cdot (m-2k+1) \sum_{j=1}^{k} a_j^{(k)} [(j-1)(j-2k) b^{j-2k-2} \partial_b^{(j-1)} F + 2(j-k) b^{j-2k-1} \partial_b^{(j)} F].$$

Let us focus on the second sum and let us prove that it is actually zero. Indeed, we have

$$\sum_{j=1}^{k} a_j^{(k)} (j-1)(j-2k) b^{j-2k-2} \partial_b^{(j-1)} F + \sum_{j=1}^{k} a_j^{(k)} 2(j-k) b^{j-2k-1} \partial_b^{(j)} F$$
$$= \sum_{j=2}^{k} a_j^{(k)} (j-1)(j-2k) b^{j-2k-2} \partial_b^{(j-1)} F + \sum_{j=2}^{k} a_{j-1}^{(k)} 2(j-k-1) b^{j-2k-2} \partial_b^{(j-1)} F$$
$$= \sum_{j=2}^{k} [a_{j-1}^{(k)} 2(j-k-1) + a_{j-1}^{(k)} 2(j-k-1)] b^{j-2k-2} \partial_b^{(j-1)} F,$$

but, by definition of $a_j^{(k)}$

$$\begin{aligned}
&a_{j-1}^{(k)}2(j-k-1)+a_{j-1}^{(k)}2(j-k-1)\\
&=\frac{(2k-j-1)!}{(j-1)!(k-j)!(-2)^{k-j}}(j-1)(j-2k)+\frac{(2k-j)!}{(j-2)!(k-j+1)!(-2)^{k-j+1}}2(j-k-1)\\
&=\frac{-(2k-j)!}{(j-2)!(k-j)!(-2)^{k-j}}+\frac{(2k-j)!}{(j-2)!(k-j)!(-2)^{k-j}}=0.
\end{aligned}$$

So, finally

$$\begin{aligned}
\Delta_{m+1}^k f &= (m-3)\cdot\ldots\cdot(m-2k+1)(m-2k-1)\sum_{j=1}^{k}a_j^{(k)}[(j-2k)b^{j-2k-2}\partial_b^{(j-1)}F+b^{j-2k-1}\partial_b^{(j)}F]\\
&=(m-3)\cdot\ldots\cdot(m-2k-1)\left[\sum_{j=1}^{k+1}a_j^{(k)}(j-2k)b^{j-2k-2}\partial_b^{(j-1)}F+\sum_{j=1}^{k+1}a_{j-1}^{(k)}b^{j-2k-2}\partial_b^{(j-1)}F\right]\\
&=(m-3)\cdot\ldots\cdot(m-2k-1)\sum_{j=1}^{k}a_j^{(k+1)}b^{j-2k-2}\partial_b^{(j-1)}F,
\end{aligned}$$

where we have used the property $a_j^{(k+1)}=a_{j-1}^{(k)}+(j-2k)a_j^{(k)}$ and that $a_j^{(k)}=0$ if $j\notin\{1,\ldots,k\}$. □

Remark 9. Note that, f may not be a slice function. Indeed, consider $F(a,b)=a^4-6a^2b^2+b^4$, then

$$f(x_0,x_1,x_2,x_3)=\frac{x_0^4}{\sqrt{x_1^2+x_2^2+x_3^2}}-6x_0^2\sqrt{x_1^2+x_2^2+x_3^2}+(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}=\frac{(x^4)_s^\circ}{\sqrt{x_1^2+x_2^2+x_3^2}}$$

is an harmonic function, which is not slice. This follows from the unicity of the stem function and that f would be induced by F, which is not a Stem function, since it does not satisfy (5).

Corollary 5.9. *Let $\Omega_D\subset\mathbb{H}^n$ and let $f\in\mathcal{S}^1(\Omega_D)$. Suppose that $\frac{\partial f}{\partial x_h^c}=0$, for some $h=1,\ldots,n$, then the partial spherical x_h-derivative of f is harmonic with respect to x_h, namely*

$$\Delta_h f'_{s,h}=0.$$

Proof. Let $F=\sum_{K\in\mathcal{P}(n)}e_KF_K$ such that $f=\mathcal{I}(F)$. Since $f\in\ker(\partial/\partial x_h^c)$, $\Delta_{4,h}F_K=0$, for any $K\in\mathcal{P}(n)$. For any $y=(y_1,\ldots,y_n)$, it holds $\Delta_{4,h}f'_{s,h}(y)=\Delta_4(f_h^y)'_s(y_h)$ and for any fixes $y=\phi_{J_1,\ldots,J_n}(z',w,z")$, it holds

$$(f_h^y)'_s(a,x_1,x_2,x_3)=\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}G_h^y\left(a,\sqrt{x_1^2+x_2^2+x_3^2}\right),$$

with

$$G_h^y\left(a,\sqrt{x_1^2+x_2^2+x_3^2}\right)=\sum_{K\in\mathcal{P}(n),h\notin K}[J_K,F_{K\cup\{h\}}(z',w,z")].$$

In particular, $\Delta_2G_h^y=0$, and so, by Proposition 5.8, applied with $m=3$, we get that $\Delta_hf'_{s,h}=0$. □

Corollary 5.10. *Let $\Omega_D \subset (\mathbb{R}^{m+1})^n$ and let $f \in \mathcal{S}^1(\Omega_D) \cap \ker(\partial/\partial x_h^c)$, for some $h = 1, \ldots, n$. Then it holds*

$$\Delta_{m+1,h}^{\frac{m-1}{2}} f'_{s,h} = 0.$$

Proof. The proof is analogue to the proof of the previous Corollary, where we apply Proposition 5.8 for any m. $\square$

6 Almansi decomposition

6.1 Classical Almansi decomposition

We present the Classical Almansi Theorem [1]. We give the proof, which is taken from [5, Proposition 1.3], to understand the properties of the components in the classical decomposition.

Theorem 6.1. *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a polyharmonic function, i.e. $\Delta_n^p f = 0$ for some p, in a star-like domain D with centre 0. Then, there exist unique harmonic functions $h_0, \dots h_{p-1}$ in D such that*

$$f(x) = h_0(x) + |x|^2 h_1(x) + \dots |x|^{2p-2} h_{p-1}(x) = \sum_{j=0}^{p-1} |x|^{2j} h_j(x). \tag{40}$$

Proof. Let us prove the theorem by induction over p. For $p = 1$, it means that f is harmonic, so we can take $h_0 = f$. Now, suppose that $\Delta_n^p f = 0$ and that the theorem holds for $p - 1$. In particular, since $\Delta f \in \ker \Delta_n^{p-1}$, there exist unique harmonic functions $g_0, \dots g_{p-2}$ such that

$$\Delta_n f = \sum_{j=0}^{p-2} r^{2j} g_j(x) = \sum_{j=1}^{p-1} r^{2j-2} g_{j-1}(x), \qquad \forall x \in D,$$

where $r = r(x) = \sqrt{\sum_{i=1}^n x_i^2}$. Define for $j = 1, \dots, p-1$

$$h_j(x) = \frac{1}{4j} \int_0^1 \xi^{j-2+\frac{n}{2}} g_{j-1}(\xi x) d\xi \tag{41}$$

and

$$h_0(x) = f(x) - \sum_{j=1}^{p-1} r^{2j} h_j(x). \tag{42}$$

Then, by (42) we have $f(x) = \sum_{j=0}^{p-1} r^{2j} h_j(x)$ and, by (41), $\Delta_n h_j = 0$ for any $j = 1, \dots, p-1$. We only need to prove that h_0 is harmonic as well, or equivalently that

$$\Delta_n \left(r^{2j} h_j(x) \right) = r^{2j-2} g_{j-1}(x). \tag{43}$$

Note that

$$\begin{aligned}
\Delta_n \left(r^{2j} h_j(x) \right) &= \Delta_n(r^{2j}) h_j(x) + r^{2j} \Delta_n h_j(x) + 2 \sum_{i=1}^n \frac{\partial r^{2j}}{\partial x_i} \frac{\partial h_j}{\partial x_i} \\
&= \Delta_n(r^{2j}) h_j(x) + 2 \sum_{i=1}^n \frac{\partial r^{2j}}{\partial x_i} \frac{\partial h_j}{\partial x_i},
\end{aligned}$$

since h_j is harmonic. Let us compute the other terms:

$$\frac{\partial r^{2j}}{\partial x_i} = 2j x_j r^{2j-2}, \quad \frac{\partial^2 r^{2j}}{\partial x_i^2} = 2j r^{2j-2} + 2j(2j-2) x_j^2 r^{2j-4}$$

and so $\Delta_n r^{2j} = \sum_{j=1}^n 2j r^{2j-2} + 2j(2j-2) x_j^2 r^{2j-4} = 2j(n + 2j - 2) r^{2j-2}$. Thus, the previous equation becomes

$$\begin{aligned}
\Delta_n \left(r^{2j} h_j(x) \right) &= 2j(n + 2j - 2) r^{2j-2} h_j + 2 \sum_{i=1}^n 2j x_j r^{2j-2} \frac{\partial h_j}{\partial x_i} \\
&= r^{2j-2} \left[4j \left(j - 1 + \frac{n}{2} \right) h_j + 4jr \frac{\partial h_j}{\partial r} \right],
\end{aligned}$$

since $\frac{\partial}{\partial r} = \sum_{i=1}^{n} \frac{x_i}{r} \frac{\partial}{\partial x_i}$. In particular, (43) reduces to

$$4j\left(j-1+\frac{n}{2}\right)h_j + 4jr\frac{\partial h_j}{\partial r} = g_{j-1}.$$

Let us multiply both sides of the previous equation by $r^{j-2+\frac{n}{2}}$ and integrating with respect to r, which is allowed since D is star-like, it becomes

$$\left(j-1+\frac{n}{2}\right)\int_0^r \rho^{j-2+\frac{n}{2}} h_j(\rho\theta)d\rho + \int_0^r \rho^{j-1+\frac{n}{2}} \frac{\partial}{\partial\rho} h_j(\rho\theta)d\rho = \frac{1}{4j}\int_0^r \rho^{j-2+\frac{n}{2}} g_{j-1}(\rho\theta)d\rho.$$

We integrate by parts the second integral

$$\begin{aligned}\int_0^r \rho^{j-1+\frac{n}{2}} \frac{\partial}{\partial\rho} h_j(\rho\theta)d\rho &= \left[h_j(\rho\theta)\rho^{j-1+\frac{n}{2}}\right]_0^r - \left(j-1+\frac{n}{2}\right)\int_0^r \rho^{j-2+\frac{n}{2}} h_j(\rho\theta)d\rho \\ &= h_j(r\theta)r^{j-1+\frac{n}{2}} - \left(j-1+\frac{n}{2}\right)\int_0^r \rho^{j-2+\frac{n}{2}} h_j(\rho\theta)d\rho,\end{aligned}$$

so the previous equation reduces to

$$h_j(r\theta)r^{j-1+\frac{n}{2}} = \frac{1}{4j}\int_0^r \rho^{j-2+\frac{n}{2}} g_{j-1}(\rho\theta)d\rho,$$

now, let $\xi = \rho/r$, then (43) holds if and only if

$$h_j(r\theta)r^{j-1+\frac{n}{2}} = \frac{1}{4j}\int_0^1 (\xi r)^{j-2+\frac{n}{2}} g_{j-1}(\xi r\theta) r d\xi = \frac{1}{4j} r^{j-1+\frac{n}{2}} \int_0^1 \xi^{j-2+\frac{n}{2}} g_{j-1}(\xi r\theta)d\xi,$$

and setting $x = r\theta$, we conclude by (41). □

Corollary 6.2. *Let $\Omega_D \subset \mathbb{R}^{m+1}$ and let f be a polyharmonic slice function. Then, the components of the Almansi decomposition are slice functions, too. Analogously, if f is a circular slice function, the components of the Almansi decomposition are circular functions.*

Proof. Recall that if $f \in \mathcal{S}(\Omega_D)$, then $\Delta_{m+1} f \in \mathcal{S}(\Omega_D)$ and $\int_0^1 \xi^l f(\xi x)d\xi \in \mathcal{S}(\Omega_D)$, for any l. Then, the result follows, since the components are given by taking powers of Laplacian of f and line integrals as in (41). □

6.1.1 Examples

Example 5. Let $f : \mathbb{R}^6 \to \mathbb{R}_5$, $f(x) = x^4 = \alpha^4 - 6\alpha^2\beta^2 + \beta^4 + J\beta(4\alpha^3 - 4\alpha\beta^2)$. By Theorem 5.6, $f \in \ker \Delta_6^3$, so $\Delta_6^2 f$ is harmonic. Note that $f_s'(x) = 4\alpha^3 - 4\alpha\beta^2$ By (36), it holds

$$\begin{aligned}\Delta_6 f(x) &= -2\cdot 4\frac{\partial}{\partial x}(4\alpha^3 - 4\alpha\beta^2) = -8\mathcal{I}\left(\frac{\partial}{\partial z}(4\alpha^3 - 4\alpha\beta^2)\right) \\ &= -8\mathcal{I}\left(\frac{1}{2}\left(\frac{\partial}{\partial\alpha} - i\frac{\partial}{\partial\beta}\right)(4\alpha^3 - 4\alpha\beta^2)\right) = -8\mathcal{I}(6\alpha^2 - 2\beta^2 + 4i\alpha\beta) \\ &= -16(3\alpha^2 - \beta^2 + 2\alpha\beta J) \in \ker \Delta_6^2,\end{aligned}$$

$$\Delta_6^2 f(x) = -2\cdot 4\cdot 2\frac{\partial}{\partial x}\left(\beta^{-1}\partial_\beta(4\alpha^3 - 4\alpha\beta^2)\right) = -16\frac{\partial}{\partial x}(-8\alpha) = 64 \in \ker \Delta_6,$$

Since $\Delta_6^2 f$ is harmonic, it has triavial Almansi decomposition. Let us find the Almansi decomposition of $\Delta_6 f \in \ker \Delta_6^2$, following twice the step of the proof of Theorem 6.1. $\Delta_6 f = h_0(x) + |x|^2 h_1(x)$, with

$$h_1(x) = \frac{1}{4}\int_0^1 \xi^{1-2+3}\Delta_6^2 f(\xi x)d\xi = \frac{1}{4}\cdot 64\cdot\frac{1}{3} = \frac{16}{3}$$

and

$$h_0(x) = \Delta_6 f(x) - (\alpha^2+\beta^2)h_1(x) = -\frac{160}{3}\alpha^2 + \frac{32}{3}\beta^2 - 32\alpha\beta J.$$

Now, let us set $g_0 := h_0$ and $g_1 := h_1$ and let us find the Almansi decomposition of f:

$$f(x) = h_0(x) + |x|^2 h_1(x) + |x|^4 h_2(x),$$

with

$$h_2(x) = \frac{1}{8}\int_0^1 \xi^2 - 2 + 3g_1(\xi x)d\xi = \frac{1}{8}\cdot\frac{16}{3}\cdot\frac{1}{4} = \frac{1}{6},$$

$$h_1(x) = \frac{1}{4}\int_0^1 \xi^{1-2+3} g_0(\xi x)d\xi = \frac{1}{4}\left(-\frac{160}{3}\alpha^2 + \frac{32}{3}\beta^2 - 32\alpha\beta J\right)\frac{1}{5} = -\frac{8}{3}\alpha^2 + \frac{8}{15}\beta^2 - \frac{8}{5}\alpha\beta J$$

and finally,

$$\begin{aligned}
h_0(x) &= f(x) - |x|^2 h_1(x) - |x|^4 h_2(x)\\
&= \alpha^4 - 6\alpha^2\beta^2 + \beta^4 + J\beta(4\alpha^3 - 4\alpha\beta^2) - (\alpha^2+\beta^2)\left(-\frac{8}{3}\alpha^2 + \frac{8}{15}\beta^2 - \frac{8}{5}\alpha\beta J\right) +\\
&- (\alpha^4 + \beta^4 + 2\alpha^2\beta^2)\frac{1}{6}\\
&= \frac{7}{2}\alpha^4 - \frac{21}{5}\alpha^2\beta^2 + \frac{3}{10}\beta^4 + \beta J\left(\frac{28}{5}\alpha^3 - \frac{12}{5}\alpha\beta^2\right).
\end{aligned}$$

Explicitly, for any $\alpha + J\beta \in \mathbb{R}^{m+1}$, we have

$$f(x) = h_0(x) + |x|^2 h_1(x) + |x|^4 h_2(x),$$

with

$$\begin{cases} h_2(x) = \frac{1}{6},\\ h_1(x) = -\frac{8}{3}\alpha^2 + \frac{8}{15}\beta^2 - \frac{8}{5}\alpha\beta J,\\ h_0(x) = \frac{7}{2}\alpha^4 - \frac{21}{5}\alpha^2\beta^2 + \frac{3}{10}\beta^4 + \beta J\left(\frac{28}{5}\alpha^3 - \frac{12}{5}\alpha\beta^2\right). \end{cases} \tag{44}$$

Example 6. Consider $f : \mathbb{R}^6 \to \mathbb{R}_5$, $f(x) = x^5 = \alpha^5 - 10\alpha^3\beta^2 + 5\alpha\beta^4 + 5\alpha^4\beta J - 10\alpha^2\beta^3 J + \beta^5 J$. Then by Theorem 5.6, $f \in \ker \Delta_6^3$. Thus, Theorem 6.1 states that there exist h_0, h_1, h_2 harmonic such that

$$f(x) = h_0(x) + |x|^2 h_1(x) + |x|^2 h_2(x).$$

We can find h_0, h_1, h_2 by following the proof of Theorem 6.1. Since $f \in \ker \Delta_6^3$, $\Delta_6^2 f$ is harmonic, so its Almansi decomposition is trivial. Thus, we can find the Almansi decomposition of $\Delta_6 f = h_0 + |x|^2 h_1$, where

$$h_1(x) = \frac{1}{4}\int_0^1 \xi^{1-2+3}\Delta_6^2 f(\xi x)d\xi, \qquad h_0(x) = f(x) - |x|^2 h_1(x).$$

Let us compute $\Delta_6 f$ and $\Delta_6^2 f$. Since f is slice regular, by (38) we have

$$\Delta_6 f = 2(1-5)\frac{\partial f_s'}{\partial x} = -8\frac{\partial f_s'}{\partial x}.$$

$f_s'(x) = 5\alpha^4 - 10\alpha^2\beta^2 + \beta^4$, where $\beta = |\operatorname{Im}(x)|$. Then, for any $x = \alpha + J\beta \in \mathbb{R}^6$, it holds

$$\begin{aligned}\frac{\partial f_s'}{\partial x} &= \mathcal{I}\left(\frac{\partial}{\partial z}(5\alpha^4 - 10\alpha^2\beta^2 + \beta^4)\right) = \mathcal{I}\left(\frac{1}{2}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right)(5\alpha^4 - 10\alpha^2\beta^2 + \beta^4)\right)\\ &= \mathcal{I}\left(10\alpha^3 - 10\alpha\beta^2 - i(-10\alpha^2\beta + 2\beta^3)\right) = 10\alpha^3 - 10\alpha\beta^2 + J\beta(10\alpha^2 - 2\beta^2),\end{aligned}$$

and so

$$\Delta_6 f = -8(10\alpha^3 - 10\alpha\beta^2 + J\beta(10\alpha^2 - 2\beta^2)).$$

Now, we can compute $\Delta_6^2 f$ thanks to (36):

$$\begin{aligned}\Delta_6^2 f &= -2(5-1)(5-3)\frac{\partial}{\partial x}(\beta^{-1}\partial_\beta f_s') = -16\frac{\partial}{\partial x}(-20\alpha^2 + 4\beta^2)\\ &= -16\mathcal{I}\left(\frac{1}{2}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right)(-20\alpha^2 + 4\beta^2)\right) = -16\mathcal{I}(-20\alpha - 4i\beta) = 64(5\alpha + J\beta).\end{aligned}$$

So, by (41), with $p = 2$ and $g_0 = \Delta_6^2 f$, we have

$$h_1(x) = \frac{1}{4}\int_0^1 \xi^2 \Delta_6^2 f(\xi x)d\xi = 16(5\alpha + J\beta)\int_0^1 \xi^3 d\xi = 4(5\alpha + J\beta) \in \ker \Delta_6$$

and

$$\begin{aligned}h_0(x) &= \Delta_6 f(x) - |x|^2 h_1(x)\\ &= -8(10\alpha^3 - 10\alpha\beta^2 + J\beta(10\alpha^2 - 2\beta^2)) - (\alpha^2 + \beta^2)4(5\alpha + J\beta)\\ &= 20(-5\alpha^3 + 3\alpha\beta^2) + 12(-7\alpha^2\beta J + \beta^3 J) \in \ker \Delta_6.\end{aligned}$$

Then, h_0 and h_1 are the components of the Almansi decomposition of $\Delta_6 f$. Now, let us set $g_0 = h_0$ and $g_1 = h_1$, then let us find the components that decompose f. By the inductive step we have

$$\begin{aligned}h_j &= \frac{1}{4j}\int_0^1 \xi^{j-2+3} g_{j-1}(\xi x)d\xi, \qquad j = 1, 2.\\ h_0 &= f(x) - (\alpha^2 + \beta^2)h_1(x) - (\alpha^2 + \beta^2)^2 h_2(x).\end{aligned}$$

Thus, it holds

$$h_2(x) = \frac{1}{8}\int_0^1 \xi^{2-2+3} g_1(\xi x)d\xi = \frac{1}{2}(5\alpha + J\beta)\int_0^1 \xi^4 d\xi = \frac{1}{10}(5\alpha + J\beta),$$

$$\begin{aligned}h_1(x) &= \frac{1}{4}\int_0^1 \xi^{1-2+3} g_0(\xi x)d\xi = (-25\alpha^3 + 15\alpha\beta^2 - 21\alpha^2\beta J + 3\beta^3 J)\int_0^1 \xi^5 d\xi\\ &= \frac{1}{6}(-25\alpha^3 + 15\alpha\beta^2 - 21\alpha^2\beta J + 3\beta^3 J)\end{aligned}$$

and

$$\begin{aligned}h_0(x) &= \alpha^5 - 10\alpha^3\beta^2 + 5\alpha\beta^4 + 5\alpha^4\beta J - 10\alpha^2\beta^3 J + \beta^5 J +\\ &- (\alpha^2+\beta^2)\frac{1}{6}(-25\alpha^3 + 15\alpha\beta^2 - 21\alpha^2\beta J + 3\beta^3 J) +\\ &- (\alpha^4 + 2\alpha^2\beta^2 + \beta^4)\frac{1}{10}(5\alpha + J\beta)\\ &= \frac{2}{3}(7\alpha^5 - 14\alpha^3\beta^2 + 3\alpha\beta^4) + \frac{2}{5}J\beta(21\alpha^4 - 18\alpha^2\beta^2 + \beta^4)\end{aligned}$$

$f(x) = h_0(x) + |x|^2 h_1(x) + |x|^4 h_2(x)$, with

$$\begin{cases} h_2(x) = \frac{1}{10}(5\alpha + J\beta),\\ h_1(x) = \frac{5}{6}(-5\alpha^3 + 3\alpha\beta^2) + \frac{1}{2}J\beta(-7\alpha^2 + \beta^2),\\ h_0(x) = \frac{2}{3}(7\alpha^5 - 14\alpha^3\beta^2 + 3\alpha\beta^4) + \frac{2}{5}J\beta(21\alpha^4 - 18\alpha^2\beta^2 + \beta^4).\end{cases}$$

Note that h_0, h_1, h_2 are slice functions.

Remark 10. The harmonic components of the Almansi decomposition of a polynomial can be obtained also through the so called Gauss or canonical decomposition. Indeed, the components of a homogeneous polynomial p_n of degree n are given by [6, §2.1]

$$h_k(x) = \frac{(m+2n-4k-1)!!}{(2k)!!(m+2n-2k-1)!!}\sum_{j=0}^{\lfloor \frac{n}{2}-k \rfloor} \frac{(-1)^j(m+2n-4k-2j-3)!!}{(2j)!!(m+2n-2k-3)!!}|x|^{2j}\Delta_{m+1}^{j+k}(p_n),$$

with $k = 1, \dots \frac{m+1}{2}$.

6.2 Slice-Almansi decomposition in $\mathbb{H}$

The following Theorem is taken from [70].

Theorem 6.3. *Let $\Omega_D \subset \mathbb{H}$ be an axially symmetric set and let $f : \Omega_D \to \mathbb{H}$ be a slice function. Then there exist two unique circular slice functions $h_1, h_2 : \Omega_D \to \mathbb{H}$ such that*

$$f(x) = h_1(x) - \overline{x}h_2(x). \tag{45}$$

If f is slice regular, h_1 and h_2 are harmonic. More precisely, the unique functions performing the decomposition are $h_1 = (xf)'_s$ and $h_2 = f'_s$.

Viceversa, let $h_1, h_2 : \Omega_D \subset \mathbb{H} \to \mathbb{H}$ be slice and circular functions such that

$$f(x) = h_1(x) - \overline{x}h_2(x), \qquad \forall x \in \Omega_D$$

be slice. Then, $h_1 = (xf)'_s$ and $h_2 = f'_s$. Moreover, $f \in \mathcal{SR}(\Omega_D)$ if and only if h_1 and h_2 satisfy the following system

$$\begin{cases} \partial_\alpha h_1 - \alpha\partial_\alpha h_2 - \beta\partial_\beta h_2 = 2h_2\\ \partial_\beta h_1 - \alpha\partial_\beta h_2 + \beta\partial_\alpha h_2 = 0,\end{cases} \tag{46}$$

where as usual $\alpha = x_0$ and $\beta = |\operatorname{Im}(x)|$. In this case, h_1 and h_2 are circular harmonic functions.

Remark 11. In literature, the term zonal harmonic is referred to harmonic functions defined on a sphere $\mathbb{S}^m$, which are constant along parallels orthogonal to some point $\eta \in \mathbb{S}^m$. Thus, in this setting, circular slice functions are zonal harmonic with respect to $\eta = 1$.

We give the first part of the proof using the stem function's language. This proves also the same decomposition for Clifford algebras (6.4).

Proof. Let us prove the first part of the Theorem. Note that $f = (xf)'_s - \overline{x}f'_s$ if and only if $F = (Z \otimes F)'_s - \overline{Z} \otimes F'_s$. Since

$$Z \otimes F = (\alpha + e_1\beta) \otimes (F_0 + e_1F_1) = \alpha F_0 - \beta F_1 + e_1(\alpha F_1 + \beta F_0),$$

it holds $(Z \otimes F)'_s = \beta^{-1}(\alpha F_1 + \beta F_0) = \alpha\beta^{-1}F_1 + F_0$. Moreover,

$$\overline{Z} \otimes F'_s = (\alpha - e_1\beta) \otimes \beta^{-1}F_1 = \alpha\beta^{-1}F_1 - e_1F_1,$$

thus

$$(Z \otimes F)'_s - \overline{Z} \otimes F'_s = \alpha\beta^{-1}F_1 + F_0 - \alpha\beta^{-1}F_1 + e_1F_1 = F.$$

Moreover, by definition, $(xf)'_s$ and f'_s are circular functions and by Propostion 5.1, $(xf)'_s$ and f'_s are harmonic if f is slice regular.

Finally, suppose that h_1, h_2 are circular functions such that $f = h_1 - \overline{x}h_2$, then

$$f'_s = (h_1 - \overline{x}h_2)'_s = (h_1)'_s - (\overline{x}h_2)'_s = -(\overline{x})'_s(h_2)^\circ_s - (\overline{x})^\circ_s(h_2)'_s = h_2$$

and

$$\begin{aligned}(xf)'_s &= [x(h_1 - \overline{x}h_2)]'_s = x'_s(h_1 - \overline{x}h_2)^\circ_s + x^\circ_s(h_1 - \overline{x}h_2)'_s \\ &= h_1 - (\overline{x}h_2)^\circ_s + \alpha[-(\overline{x})'_s(h_2)^\circ_s - (\overline{x})^\circ_s(h_2)'_s] = h_1 - \alpha h_2 + \alpha h_2 = h_1.\end{aligned}$$

Let us prove the second part. If $f \in \mathcal{S}(\Omega_D)$ and $f = h_1 - \overline{x}h_2 = (xf)'_s - \overline{x}f'_s$, by the uniqueness of decomposition (45) it must be $h_1 = (xf)'_s$ and $h_2 = f'_s$. Moreover, let $F = F_0 + e_1F_1$, with

$$F_0(\alpha + i\beta) = h_1(\alpha + J\beta) - \alpha h_2(\alpha + J\beta), \qquad F_1(z) = \beta h_2(\alpha + J\beta),$$

for any $J \in \mathbb{S}_{\mathbb{H}}$. Note that, since h_1, h_2 are circular functions, they do not depend on the choice of J, so F_0 and F_1 are well defined. Morevoer, $F_0(\overline{z}) = F_1(z)$ and $F_1(\overline{z}) = -F_1(z)$, so $F \in Stem(D)$. Moreover, for any $x = \alpha + J\beta$,

$$\mathcal{I}(F)(x) = h_1(x) - \alpha h_2(x) + J\beta h_2(x) = h_1(x) - \overline{x}h_2(x) = f(x).$$

Thus, $f \in \mathcal{SR}(\Omega_D)$ if and only if F is holomorphic. Note that (46) is equivalent to the holomorphicity of F, indeed

$$\begin{aligned}\partial_\alpha F_0 = \partial_\beta F_1 &\iff \partial_\alpha h_1 - h_2 - \alpha\partial_\alpha h_2 = h_2 + \beta\partial_\beta h_2 \\ \partial_\alpha F_1 = -\partial_\beta F_0 &\iff \beta\partial_\alpha h_1 = -\partial_\beta h_2 + \alpha\partial_\beta h_2.\end{aligned}$$

Finally, since $(xf)'_s$ and f'_s are circular harmonic functions, so are h_1 and h_2. □

6.3 Slice-Almansi decomposition in $\mathbb{R}_m$

The results in this subsection are taken from [71].

Theorem 6.4. *Let m be odd, let $\Omega_D \subset \mathbb{R}^{m+1}$ be an axially symmetric set and let $f : \Omega_D \to \mathbb{R}_m$ be a slice function. Then there exist unique circular slice functions $h_1, h_2 : \Omega_D \to \mathbb{R}_m$ such that*

$$f(x) = h_1(x) - \overline{x}h_2(x). \tag{47}$$

If f is slice regular, then $h_1, h_2 \in \ker \Delta_{m+1}^{\frac{m-3}{2}}$. As before, the unique functions performing the decomposition are $h_1 = (xf)'_s$ and $h_2 = f'_s$.

Viceversa, if $h_1, h_2 : \Omega_D \subset \mathbb{R}^{m+1} \to \mathbb{R}_m$ are slice and circular functions such that

$$f(x) = h_1(x) - \overline{x} h_2(x), \qquad \forall x \in \Omega_D$$

is slice, then, $h_1 = (xf)'_s$ and $h_2 = f'_s$. Moreover, $f \in \mathcal{SR}(\Omega_D)$ if and only if h_1 and h_2 satisfy system (46) and in this case, h_1 and h_2 are circular polyharmonic functions of order $\frac{m-1}{2}$.

Proof. The decomposition is proven in the same way of Theorem 6.3. Finally, proceed as in Theorem 6.3, but apply Proposition 5.2, instead of Proposition 5.1, to get that $h_0, h_1 \in \ker \Delta_{m+1}^{\frac{m-3}{2}}$. □

Corollary 6.5. *Suppose that $\Omega_D \subset \mathbb{R}^{m+1}$ is a star-like domain, with centre 0, let $f : \Omega_D \to \mathbb{R}_m$ be a slice regular functions and let $f(x) = h_1(x) - \overline{x} h_2(x)$ be the decomposition (47). Then, we can further decompose*

$$f(x) = \sum_{j=0}^{\frac{m-3}{2}} |x|^{2j} u_j(x) - \overline{x} \sum_{j=0}^{\frac{m-3}{2}} |x|^{2j} v_j(x),$$

with u_j, v_j circular harmonic functions, for $j = 1, \dots, \frac{m-3}{2}$. Furthermore, there exist $g_0, \dots, g_{\frac{m-3}{2}} \in \ker \overline{\partial} \Delta_{m+1}$, where $\overline{\partial}$ is defined in (66), such that

$$f(x) = \sum_{j=0}^{\frac{m-3}{2}} |x|^{2j} g_j(x). \tag{48}$$

In particular, $g_0, \dots, g_{\frac{m-3}{2}}$ are biharmonic slice functions.

Proof. By Theorem 6.4, h_1 and h_2 are circular and polyharmonic functions of degree $\frac{m-3}{2}$ on the star-like domain Ω_D, hence by Theorem 6.1 there exist $u_0, \dots, u_{\frac{m-3}{2}}, v_0, \dots, v_{\frac{m-3}{2}}$ circular and harmonic functions such that

$$h_1(x) = \sum_{j=0}^{\frac{m-3}{2}} |x|^{2j} u_j(x), \qquad h_2(x) = \sum_{j=0}^{\frac{m-3}{2}} |x|^{2j} v_j(x)$$

and so

$$f(x) = h_1(x) - \overline{x} h_2(x) = \sum_{j=0}^{\frac{m-3}{2}} |x|^{2j} u_j(x) - \overline{x} \sum_{j=0}^{\frac{m-3}{2}} |x|^{2j} v_j(x)$$

Set $g_j = u_j - \overline{x} v_j$. Thus, by the harmonicity of u_j and v_j, it holds

$$\Delta_{m+1} g_j = \Delta_{m+1}(u_j - \overline{x} v_j) = \Delta_{m+1} u_j - \Delta_{m+1}(\overline{x}) v_j - \overline{x} \Delta_{m+1} v_j - 2 \nabla \overline{x} \cdot \nabla v_j = -2 \partial v_j,$$

and so

$$\overline{\partial} \Delta_{m+1} g_j = -2 \overline{\partial} \partial v_j = -\frac{1}{2} \Delta_{m+1} v_j = 0.$$

Finally, for every $j = 0, \dots, \frac{m-3}{2}$, the functions g_j are slice, since $u_j, \overline{x}, v_j$ are slice functions and biharmonic, since $\Delta_{m+1}^2 g_j = \partial \overline{\partial} \Delta_{m+1} g_j = 0$.

□

Remark 12. We can see that (48) is formally equivalent to (40), but the components in (48) are only biharmonic. By the uniqueness of (40), they cannot be harmonic.

6.3.1 Examples

Example 7. Let $f : \mathbb{R}^6 \to \mathbb{R}_5$ be the slice regular function $f(x) = x^4$ as in Example 5. Then, the slice-Almansi decomposition of f is

$$f(x) = (x^5)'_s - \overline{x}(x^4)'_s = 5\alpha^4 - 10\alpha^2\beta^2 + \beta^4 + (-\alpha + J\beta)(4\alpha^3 - 4\alpha\beta^2).$$

By Proposition 5.2, $(x^5)'_s, (x^4)'_s \in \ker \Delta_6^2$, then there exist harmonic functions u_0, u_1, v_0, v_1 such that

$$(x^5)'_s = u_0 + |x|^2 u_1, \qquad (x^4)'_s = v_0 + |x|^2 v_1.$$

Let us find them explicitly, through the Classical Almansi decomposition. Since $(x^5)'_s \in \ker \Delta_6^2$, $\Delta_6 (x^5)'_s \in \ker \Delta_6$, with

$$\Delta_6 (x^5)'_s = (5-3)\beta^{-1}\partial_\beta (x^5)'_s = 2\beta^{-1}\partial_\beta (5\alpha^4 - 10\alpha^2\beta^2 + \beta^4) = 8(\beta^2 - 5\alpha^2),$$

then

$$u_1(x) = \frac{1}{4}\int_0^1 \xi^{1-2+3}\Delta_6(x^5)'_s(\xi x)d\xi = \frac{1}{4}8(\beta^2 - 5\alpha^2)\int_0^1 \xi^2 \cdot \xi^2 d\xi = \frac{2}{5}\beta^2 - 2\alpha^2 \in \ker \Delta_6;$$

$$\begin{aligned} u_0(x) &= (x^5)'_s(x) - |x|^2 u_1(x) = 5\alpha^4 - 10\alpha^2\beta^2 + \beta^4 - (\alpha^2+\beta^2)\frac{2}{5}(\beta^2 - 5\alpha^2) = \\ &= 7\alpha^4 - \frac{42}{5}\alpha^2\beta^2 + \frac{3}{5}\beta^4 \in \ker \Delta_6. \end{aligned}$$

In the very same way, we have $\Delta_6 (x^4)'_s \in \ker \Delta_6$, with

$$\Delta_6 (x^4)'_s = (5-3)\beta^{-1}\partial_\beta (x^4)'_s = 2\beta^{-1}\partial_\beta (4\alpha^3 - 4\alpha\beta^2) = -16\alpha,$$

then

$$v_1(x) = \frac{1}{4}\int_0^1 \xi^{1-2+3}\Delta_6(x^4)'_s(\xi x)d\xi = -\frac{1}{4}16\alpha\int_0^1 \xi^2 \cdot \xi d\xi = -\alpha \in \ker \Delta_6;$$

$$v_0(x) = (x^4)'_s(x) - |x|^2 v_1(x) = 4\alpha^3 - 4\alpha\beta^2 - (\alpha^2+\beta^2)(-\alpha) = 5\alpha^3 - 3\alpha\beta^2 \in \ker \Delta_6.$$

Thus, we have

$$f(x) = u_0(x) + |x|^2 u_1(x) - \overline{x} v_0(x) - |x|^2 \overline{x} v_1(x),$$

with harmonic components

$$\begin{cases} u_0(x) = 7\alpha^4 - \frac{42}{5}\alpha^2\beta^2 + \frac{3}{5}\beta^4 \\ u_1(x) = -2\alpha^2 + \frac{2}{5}\beta^2 \\ v_0(x) = 5\alpha^3 - 3\alpha\beta^2 \\ v_1(x) = -\alpha. \end{cases}$$

Moreover, by considering

$$g_0 = u_0 - \overline{x} v_0 = 2\alpha^4 - \frac{27}{5}\alpha^2\beta^2 + \frac{3}{5}\beta^4 + J\beta(5\alpha^3 - 3\alpha\beta^2) \in \ker \Delta_6^2$$

and

$$g_1 = u_1 - \overline{x} v_1 = -\alpha^2 + \frac{2}{5}\beta^2 - \alpha\beta J \in \ker \Delta_6^2,$$

it holds

$$f(x) = g_0(x) + |x|^2 g_1(x). \tag{49}$$

Note that g_1 and g_0 are biharmonic, moreover, as stressed in Remark 12, (49) is formally equivalent to Classical Almansi decomposition (44) and by the uniqueness of Almansi decomposition (40) we can infer that g_0, g_1 are not harmonic.

Example 8. Let $f : \mathbb{R}^6 \to \mathbb{R}_5$ be the slice regular function $f(x) = x^7$. Then, the slice-Almansi decomposition of f is

$$f(x) = (x^8)'_s - \overline{x}(x^7)'_s = 8\alpha^7 - 56\alpha^5\beta^2 + 56\alpha^3\beta^4 - 8\alpha\beta^6 + (-\alpha + J\beta)(7\alpha^6 - 35\alpha^4\beta^2 + 21\alpha^2\beta^4 - \beta^6).$$

By Proposition 5.2, $(x^8)'_s, (x^7)'_s \in \ker \Delta_6^2$, then there exist harmonic functions u_0, u_1, v_0, v_1 such that

$$(x^8)'_s = u_0 + |x|^2 u_1, \qquad (x^7)'_s = v_0 + |x|^2 v_1.$$

Let us find them explicitly, through the Classical Almansi decomposition. Since $(x^8)'_s \in \ker \Delta_6^2$, $\Delta_6 (x^8)'_s \in \ker \Delta_6$, with

$$\begin{aligned}\Delta_6 (x^8)'_s &= (5-3)\beta^{-1}\partial_\beta (x^8)'_s = 2\beta^{-1}\partial_\beta(8\alpha^7 - 56\alpha^5\beta^2 + 56\alpha^3\beta^4 - 8\alpha\beta^6)\\ &= 32(-7\alpha^5 + 14\alpha^3\beta^2 - 3\alpha\beta^4),\end{aligned}$$

then

$$\begin{aligned}u_1(x) &= \frac{1}{4}\int_0^1 \xi^{1-2+3}\Delta_6(x^8)'_s(\xi x)d\xi = \frac{1}{4}32(-7\alpha^5 + 14\alpha^3\beta^2 - 3\alpha\beta^4)\int_0^1 \xi^2 \cdot \xi^5 d\xi\\ &= -7\alpha^5 + 14\alpha^3\beta^2 - 3\alpha\beta^4 \in \ker \Delta_6;\end{aligned}$$

$$\begin{aligned}u_0(x) &= (x^8)'_s(x) - |x|^2 u_1(x)\\ &= 8\alpha^7 - 56\alpha^5\beta^2 + 56\alpha^3\beta^4 - 8\alpha\beta^6 - (\alpha^2+\beta^2)(-7\alpha^5 + 14\alpha^3\beta^2 - 3\alpha\beta^4)\\ &= 15\alpha^7 - 63\alpha^5\beta^2 + 45\alpha^3\beta^4 - 5\alpha\beta^6 \in \ker \Delta_6.\end{aligned}$$

In the very same way, we have $\Delta_6 (x^7)'_s \in \ker \Delta_6$, with

$$\Delta_6(x^7)'_s = (5-3)\beta^{-1}\partial_\beta(x^7)'_s = 2\beta^{-1}\partial_\beta(7\alpha^6 - 35\alpha^4\beta^2 + 21\alpha^2\beta^4 - \beta^6) = 4(-35\alpha^4 + 42\alpha^2\beta^2 - 3\beta^4),$$

then

$$\begin{aligned}v_1(x) &= \frac{1}{4}\int_0^1 \xi^{1-2+3}\Delta_6(x^4)'_s(\xi x)d\xi = \frac{1}{4}4(-35\alpha^4 + 42\alpha^2\beta^2 - 3\beta^4)\int_0^1 \xi^2 \cdot \xi^4 d\xi\\ &= \frac{1}{7}(-35\alpha^4 + 42\alpha^2\beta^2 - 3\beta^4) \in \ker \Delta_6;\end{aligned}$$

$$\begin{aligned}v_0(x) &= (x^7)'_s(x) - |x|^2 v_1(x)\\ &= 7\alpha^6 - 35\alpha^4\beta^2 + 21\alpha^2\beta^4 - \beta^6 - (\alpha^2+\beta^2)\frac{1}{7}(-35\alpha^4 + 42\alpha^2\beta^2 - 3\beta^4)\\ &= 12\alpha^6 - 36\alpha^4\beta^2 + \frac{108}{7}\alpha^2\beta^4 - \frac{4}{7}\beta^6 \in \ker \Delta_6.\end{aligned}$$

Thus, we have

$$f(x) = u_0(x) + |x|^2 u_1(x) - \overline{x}v_0(x) - |x|^2\overline{x}v_1(x),$$

with harmonic components

$$\begin{cases} u_0(x) = 15\alpha^7 - 63\alpha^5\beta^2 + 45\alpha^3\beta^4 - 5\alpha\beta^6 \\ u_1(x) = -7\alpha^5 + 14\alpha^3\beta^2 - 3\alpha\beta^4 \\ v_0(x) = 12\alpha^6 - 36\alpha^4\beta^2 + \frac{108}{7}\alpha^2\beta^4 - \frac{4}{7}\beta^6 \\ v_1(x) = -5\alpha^4 + 6\alpha^2\beta^2 - \frac{3}{7}\beta^4. \end{cases}$$

Moreover, by considering

$$\begin{aligned} g_0 = u_0 - \overline{x}v_0 = 3\alpha^7 - 27\alpha^5\beta^2 + \frac{207}{7}\alpha^3\beta^4 - \frac{31}{7}\alpha\beta^6 + \\ + J\beta\left(12\alpha^6 - 36\alpha^4\beta^2 + \frac{108}{7}\alpha^2\beta^4 - \frac{4}{7}\beta^6\right) \in \ker\Delta_6^2 \end{aligned}$$

and

$$g_1 = u_1 - \overline{x}v_1 = -2\alpha^5 + 8\alpha^3\beta^2 - \frac{18}{7}\alpha\beta^4 J\beta\left(-5\alpha^4 + 6\alpha^2\beta^2 - \frac{3}{7}\beta^4\right) \in \ker\Delta_6^2,$$

it holds

$$f(x) = g_0(x) + |x|^2 g_1(x).$$

6.4 Slice-Almansi decomposition in several variables

We extend also to Clifford algebras results proved in [8] for several quaternionic variables.

Definition 6.1. Let $\Omega_D \subset \mathbb{H}^n$, or $\Omega_D \subset (\mathbb{R}^{m+1})^n$, with m odd, let $f \in \mathcal{S}(\Omega_D)$ and $H \in \mathcal{P}(n)$. For every $K = \{k_1, ..., k_p\} \subset \mathbb{H}$, with $k_1 < ... < k_p$, define over Ω_{D_H} the slice functions

$$\mathcal{S}_K^H(f) := (x_K \odot f)'_{s,H} = \left(\prod_{i=1}^p x_{k_i} \odot f\right)'_{s,H},$$

and set $\mathcal{S}_\emptyset^\emptyset(f) := f$. If $H = [\![m]\!] := \{1, 2, ..., m\}$ is an integers interval from 1 to some $m \in \{1, ..., n\}$, we can write $\forall K \in \mathcal{P}(m)$

$$\mathcal{S}_K^{[\![m]\!]}(f) := (x_m^{\chi_K(m)} \dots (x_1^{\chi_K(1)} f)'_{s,1} \dots)'_{s,m},$$

where χ_K is the characteristic function of the set K. Note that, in this case, we can use the ordinary pointwise product as well as the slice product [50, Proposition 2.52]. If $f = \mathcal{I}(F)$, every $\mathcal{S}_K^H(f)$ is induced by the stem function

$$G_K^H(F) := (Z_K \otimes F)'_H,$$

where $Z_j \in Stem(\mathbb{C}^n)$ is the stem function $Z_j(\alpha_1 + i\beta_1, ..., \alpha_n + i\beta_n) := \alpha_j + e_j\beta_j$, inducing the monomial x_j, for any $j = 1, ..., n$.

We can now formulate the analogue of Theorems 6.3 and 6.4 in several variables.

Theorem 6.6. *Let $\Omega_D \subset \mathbb{H}^n$, or $\Omega_D \subset (\mathbb{R}^{m+1})^n$, with m odd, be a circular set and let $f \in \mathcal{S}(\Omega_D)$ be a slice function. Fix any $H \in \mathcal{P}(n)$, then*

1. *we can decompose* f *as*

$$f(x) = \sum_{K \subset H} (-1)^{|H \setminus K|} (\overline{x})_{H \setminus K} \odot \mathcal{S}_K^H(f)(x), \tag{50}$$

where $(\overline{x})_T = \prod_{j=1}^{s} \overline{x}_{t_j}$*, if* $T = \{t_1, \ldots, t_s\}$*, with* $1 \leq t_1 < \cdots < t_s \leq n$*.*

2. *if* $H \neq [\![n]\!]$*,* $\mathcal{S}_K^H(f) \in \mathcal{S}_{c,H}(\Omega_{D_H}) \cap \mathcal{S}_p(\Omega_{D_H})$*, for every* $K \subset H$*, where* $p = \min H^c$*, while* $\mathcal{S}_K^{[\![n]\!]} \in \mathcal{S}_{c,[\![n]\!]}(\Omega_{D_{[\![n]\!]}})$*, for every* $K \in \mathcal{P}(n)$*;*

3. *suppose* $f \in \mathcal{SR}(\Omega_D)$*, then*

   (a) *if* $\Omega_D \subset \mathbb{H}^n$*,* $\Delta_{4,h}\mathcal{S}_K^H(f) = 0$*,* $\forall h \in H$*,* $\forall K \subset H$*;*

   (b) *if* $\Omega_D \subset (\mathbb{R}^{m+1})^n$*,* $\Delta_{m+1,h}^{\frac{m-1}{2}}\mathcal{S}_K^H(f) = 0$*,* $\forall h \in H$*,* $\forall K \subset H$*;*

4. $f \in \mathcal{SR}(\Omega_D)$ *if and only if* $\mathcal{S}_K^H(f) \in \mathcal{SR}_p(\Omega_{D_H})$*,* $\forall H \in \mathcal{P}(n) \setminus \{1, \ldots, n\}$*,* $K \subset H$*,* $p = \min H^c$*;*

5. $f \in \mathcal{S}_{\mathbb{R}}(\Omega_D)$*, i.e.* f *is slice preserving (see Definition 3.12), if and only if* $\mathcal{S}_K^{[\![n]\!]}(f)$ *is real valued,* $\forall K \in \mathcal{P}(n)$*.*

Remark 13. For any $H \in \mathcal{P}(n)$ we can define the linear operator

$$\mathcal{S}^H : \mathcal{S}(\Omega_D) \ni f \mapsto \left\{\mathcal{S}_K^H(f)\right\}_{K \subset H} \in \left(\mathcal{S}_{c,H}(\Omega_{D_H}) \cap \mathcal{S}_p(\Omega_{D_H})\right)^{2^{|H|}},$$

where $p := \min H^c$.

Before proving Theorem 6.6 we need some preliminary results.

Lemma 6.7. *For every* $m = 1, ..., n$*, it holds*

1. $(x_m)'_{s,h} = (Z_m)'_h = \delta_{h,m}$ *and* $(\overline{x}_m)'_{s,h} = (\overline{Z}_m)'_h = -\delta_{h,m}$*, where* $\delta_{i,j}$ *is the Kronecker symbol;*

2. $(x_m)^\circ_{s,h} = x_m$*,* $(\overline{x}_m)^\circ_{s,h} = \overline{x}_m$*,* $(Z_m)^\circ_h = Z_m$*,* $(\overline{Z}_m)^\circ_h = \overline{Z}_m$ *if* $h \neq m$ *and* $(x_m)^\circ_{s,m}(x) = (\overline{x}_m)^\circ_{s,m}(x) = \mathrm{Re}(x_m)$*,* $(Z_m)^\circ_m(z) = (\overline{Z}_m)^\circ_m(z) = \mathrm{Re}(z_m)$*.*

Proof. Since $x_m = \mathcal{I}(Z_m)$ and $\overline{x}_m = \mathcal{I}(\overline{Z}_m)$, it is enough to prove the properties for the slice funtions or for the stem functions. By Proposition 4.9 (3), since $x_m, \overline{x}_m \in \mathcal{S}_{c,h}(\Omega_D)$, $\forall h \neq m$, immediately it holds

$$(x_m)^\circ_{s,h} = x_m, \qquad (\overline{x}_m)^\circ_{s,h} = \overline{x}_m, \qquad (x_m)'_{s,h} = (\overline{x}_m)'_{s,h} = 0.$$

Finally, by direct computation,

$$(Z_m)^\circ_m(z) = (\overline{Z}_m)^\circ_m(z) = \mathrm{Re}(z_m),$$

$$(Z_m)'_m(z) = \mathrm{Im}(z_m)^{-1}\,\mathrm{Im}(z_m) = 1$$

and

$$(\overline{Z}_m)'_m(z) = \mathrm{Im}(z_m)^{-1}(-\mathrm{Im}(z_m)) = -1.$$

$\square$

Proposition 6.8. *Let $F \in Stem(D)$ and fix $H \in \mathcal{P}(n)$. For every $K \subset H$, the functions $G_K^H(F)$ satisfy the following properties:*

1. *For every $m \notin H$, it holds*
$$G_K^H(F) = G_{K\cup\{m\}}^{H\cup\{m\}}(F) - \overline{Z}_m \otimes G_K^{H\cup\{m\}}(F). \tag{51}$$

2. *For every $h \in H$ it holds*
$$G_K^H(F) = \left(Z_h^{\chi_K(h)} \otimes G_{K\setminus\{h\}}^{H\setminus\{h\}}(F)\right)'_h.$$

3. *Explicitely, for $z = (z_1, ..., z_n) \in D \setminus \mathbb{R}_H$, with $z_j = \alpha_j + i\beta_j$, we have*
$$G_K^H(F)(z) = \sum_{T\subset H^c} e_T \left(\sum_{L\subset K} \alpha_{K\setminus L}\beta_{H\setminus L}^{-1} F_{(T\cup H)\setminus L}(z)\right). \tag{52}$$

4. *If F is holomorphic, then every $G_K^H(F)$ is m-holomorphic, $\forall m \notin H$.*

Proof. 1. Apply (26), (28) and Lemma 6.7, then

$$\begin{aligned}
&G_{K\cup\{m\}}^{H\cup\{m\}}(F) - \overline{Z}_m \otimes G_K^{H\cup\{m\}}(F) = \left(Z_{K\cup\{m\}} \otimes F\right)'_{H\cup\{m\}} - \overline{Z}_m \otimes (Z_K \otimes F)'_{H\cup\{m\}} = \\
&= \left(\left(Z_{K\cup\{m\}} \otimes F\right)'_m\right)'_H - \overline{Z}_m \otimes \left((Z_K \otimes F)'_m\right)'_H \\
&= \left(\left(Z_{K\cup\{m\}}\right)'_m \otimes F_m^\circ + \left(Z_{K\cup\{m\}}\right)^\circ_m \otimes F'_m\right)'_H - \overline{Z}_m \otimes \left((Z_K)'_m \otimes F_m^\circ + (Z_K)^\circ_m \otimes F'_m\right)'_H \\
&= (Z_K \otimes F_m^\circ + \alpha_m Z_K \otimes F'_m)'_H - \overline{Z}_m \otimes (Z_K \otimes F'_m)'_H \\
&= \left(Z_K \otimes F_m^\circ + \alpha_m Z_K \otimes F'_m - \overline{Z}_m \otimes Z_K \otimes F'_m\right)'_H \\
&= (Z_K \otimes (F_m^\circ + \mathrm{Im}(Z_m) \otimes F'_m))'_H = (Z_K \otimes F)'_H = G_K^H(F).
\end{aligned}$$

2. It follows immediately by definition of $G_K^H(F)$.
3. We procede by induction over $|H|$. Suppose first $|H| = 1$, i.e. $H = \{h\}$ for some $h \in \{1, ..., n\}$, then we have two components $G_\emptyset^{\{h\}}(F)$ and $G_{\{h\}}^{\{h\}}(F)$. Let us compute them explicitely:

$$G_\emptyset^{\{h\}}(F) = F'_h := \sum_{T\in\mathcal{P}(n), h\notin T} e_T \beta_h^{-1} F_{T\cup\{h\}},$$

$$\begin{aligned}
G_{\{h\}}^{\{h\}}(F) &= (Z_h \otimes F)'_h = (Z_h)'_h \otimes F_h^\circ + (Z_h)^\circ_h \otimes F'_h \\
&= \sum_{T\in\mathcal{P}(n), h\notin T} e_T F_T + \sum_{T\in\mathcal{P}(n), h\notin T} e_T \alpha_h \beta_h^{-1} F_{T\cup\{h\}}.
\end{aligned}$$

Now, suppose that (52) holds for some $H \in \mathcal{P}(n)$ and let us prove it for $H' = H \cup \{m\}$, for any $m \notin H$. Suppose first $m \notin K$, then

$$\begin{aligned}
G_K^{H'}(F) &= \left((Z_K \otimes F)'_H\right)'_m = \left(\sum_{T\subset H^c} e_T \left(\sum_{L\subset K} \alpha_{K\setminus L}\beta_{H\setminus L}^{-1} F_{(T\cup H)\setminus L}\right)\right)'_m \\
&= \beta_m^{-1} \sum_{T\subset H^c\setminus\{m\}} e_T \left(\sum_{L\subset K} \alpha_{K\setminus L}\beta_{H\setminus L}^{-1} F_{(T\cup H\cup\{m\})\setminus L}\right) \\
&= \sum_{T\subset (H\cup\{m\})^c} e_T \left(\sum_{L\subset K} \alpha_{K\setminus L}\beta_{(H\cup\{m\})\setminus L}^{-1} F_{(T\cup H\cup\{m\})\setminus L}\right).
\end{aligned}$$

Suppose now $m \in K$, then

$$
\begin{aligned}
G_K^{H'}(F) &= \left(Z_m \otimes \left(Z_{K\setminus\{m\}} \otimes F\right)'_H\right)'_m \\
&= \left(Z_m \otimes \sum_{T\subset H^c} e_T \left(\sum_{L\subset (K\setminus\{m\})} \alpha_{(K\setminus\{m\})\setminus L}\beta^{-1}_{H\setminus L}F_{(T\cup H)\setminus L}\right)\right)'_m \\
&= \left(\sum_{T\subset H^c} e_T \left(\sum_{L\subset (K\setminus\{m\})} \alpha_{(K\setminus\{m\})\setminus L}\beta^{-1}_{H\setminus L}F_{(T\cup H)\setminus L}\right)\right)^{\circ}_m + \\
&+ \alpha_m \left(\sum_{T\subset H^c} e_T \left(\sum_{L\subset (K\setminus\{m\})} \alpha_{(K\setminus\{m\})\setminus L}\beta^{-1}_{H\setminus L}F_{(T\cup H)\setminus L}\right)\right)'_m \\
&= \sum_{T\subset (H^c\setminus\{m\})} e_T \left(\sum_{L\subset (K\setminus\{m\})} \alpha_{(K\setminus\{m\})\setminus L}\beta^{-1}_{H\setminus L}F_{(T\cup H)\setminus L}\right) + \\
&+ \beta_m^{-1}\alpha_m \sum_{T\subset (H^c)\setminus\{m\}} e_T \left(\sum_{L\subset (K\setminus\{m\})} \alpha_{(K\setminus\{m\})\setminus L}\beta^{-1}_{H\setminus L}F_{(T\cup H\cup\{m\})\setminus L}\right) \\
&= \sum_{T\subset (H\cup\{m\})^c} e_T \left(\sum_{L\subset (K\setminus\{m\})} \alpha_{K\setminus(\{m\}\cup L)}\beta^{-1}_{(H\cup\{m\}\setminus(L\cup\{m\})}F_{(T\cup H\cup\{m\})\setminus(L\cup\{m\})}\right) + \\
&+ \sum_{T\subset (H\cup\{m\})^c} e_T \left(\sum_{L\subset (K\setminus\{m\})} \alpha_{K\setminus L}\beta^{-1}_{(H\cup\{m\})\setminus L}F_{(T\cup H\cup\{m\})\setminus L}\right) \\
&= \sum_{T\subset H'} e_T \left(\sum_{L\subset K} \alpha_{K\setminus L}\beta^{-1}_{H'\setminus L}F_{(T\cup H')\setminus L}\right).
\end{aligned}
$$

4. F and Z_K are holomorphic, so is $Z_K \otimes F$. Finally, $G_K^H(F) = (Z_K \otimes F)'_H$ is holomorphic with respect to z_m for every $m \notin H$, by Proposition 4.10. $\square$

Next Lemma will be used to prove 5. of Theorem 6.6.

Lemma 6.9. *For every $H \in \mathcal{P}(n)$ and every $K \subset H$, it holds on $D \setminus \mathbb{R}_H$*

$$\beta_K^{-1} \sum_{T\subset H^c} e_T F_{K\cup T} = \sum_{T\subset H\setminus K} (-1)^{|T|}\alpha_T G^H_{H\setminus(K\cup T)}(F), \tag{53}$$

where, if $H = \{1, ..., n\}$ we mean

$$\beta_K^{-1} F_K = \sum_{T\subset K^c} (-1)^{|T|}\alpha_T G^{[\![n]\!]}_{(K\cup T)^c}(F). \tag{54}$$

Proof. Let us proceed by induction over $|H|$. First, suppose $H = \{h\}$, for any $h = 1, ..., n$, then $K = \emptyset, \{h\}$. If $K = \emptyset$, we have

$$
\begin{aligned}
&\sum_{T\subset\{h\}} (-1)^{|T|}\alpha_T G^{\{h\}}_{\{h\}\setminus T}(F) = G^{\{h\}}_{\{h\}}(F) - \alpha_h G^{\{h\}}_{\emptyset}(F) = (Z_h \otimes F)'_h - \alpha_h F'_h \\
&= F^{\circ}_h + \alpha_h F'_h - \alpha_h F'_h = \sum_{T\subset\{h\}^c} e_T F_T,
\end{aligned}
$$

where (28) and Lemma 6.7 has been used. If $K=\{h\}$, immediately we get

$$G_{\emptyset}^{\{h\}}(F)=F_h'=\beta_h^{-1}\sum_{T\subset\{h\}^c}e_TF_{\{h\}\cup T}.$$

Now, suppose that (53) holds for some $H\in\mathcal{P}(n)$ and let us prove it for $H'=H\cup\{m\}$, with any $m\notin H$ and any $K\subset H'$. Let us split $m\in K$ and $m\notin K$. If $m\in K$, let us set $K':=K\setminus\{m\}$, then we have

$$\begin{aligned}
&\sum_{T\subset H'\setminus K}(-1)^{|T|}\alpha_TG^{H'}_{H'\setminus(K\cup T)}(F)=\sum_{T\subset H\setminus K'}(-1)^{|T|}\alpha_T\left(Z_{H\setminus(K'\cup T)}\otimes F\right)'_{H\cup\{m\}}\\
&=\left(\sum_{T\subset H\setminus K'}(-1)^{|T|}\alpha_T\left(Z_{H\setminus(K'\cup T)}\otimes F\right)'_H\right)'_m=\left(\sum_{T\subset H\setminus K'}(-1)^{|T|}\alpha_TG^H_{H\setminus(K'\cup T)}(F)\right)'_m\\
&=\left(\beta_{K'}^{-1}\sum_{T\subset H^c}e_TF_{K'\cup T}\right)'_m=\beta_K^{-1}\sum_{T\subset(H')^c}e_TF_{K\cup T}.
\end{aligned}$$

Finally, if $m\notin K$

$$\begin{aligned}
&\sum_{T\subset H'\setminus K}(-1)^{|T|}\alpha_TG^{H'}_{H'\setminus(K\cup T)}(F)=\sum_{T\subset H\setminus K}(-1)^{|T|}\alpha_TG^{H'}_{H'\setminus(K\cup T)}(F)+\\
&\quad-\sum_{T\subset H\setminus K}(-1)^{|T|}\alpha_m\alpha_TG^{H'}_{H\setminus(K\cup T)}(F)\\
&=\sum_{T\subset H\setminus K}(-1)^{|T|}\alpha_T\left(\left(Z_m\otimes\left(Z_{H\setminus(K\cup T)}\otimes F\right)\right)'_m\right)'_H-\sum_{T\subset H\setminus K}(-1)^{|T|}\alpha_m\alpha_T\left(Z_{H\setminus(K\cup T)}\otimes F\right)'_{H\cup m}\\
&=\sum_{T\subset H\setminus K}(-1)^{|T|}\alpha_T\left(\left(Z_{H\setminus(K\cup T)}\otimes F\right)^\circ_m+\alpha_m\left(Z_{H\setminus(K\cup T)}\otimes F\right)'_m\right)'_H+\\
&\quad-\sum_{T\subset H\setminus K}(-1)^{|T|}\alpha_m\alpha_T\left(Z_{H\setminus(K\cup T)}\otimes F\right)'_{H\cup m}\\
&=\left(\sum_{T\subset H\setminus K}(-1)^{|T|}\alpha_TG^H_{H\setminus(K\cup T)}(F)\right)^\circ_m=\left(\beta_K^{-1}\sum_{T\subset H^c}e_TF_{K\cup T}\right)^\circ_m=\beta_K^{-1}\sum_{T\subset(H')^c}e_TF_{K\cup T}.
\end{aligned}$$

□

Proof of Theorem 6.6. 1. Let us prove that decomposition (50) holds for stem functions, too, namely that for any $H\in\mathcal{P}(n)$ we have

$$F=\sum_{K\subset H}(-1)^{|H\setminus K|}\overline{Z}_{H\setminus K}\otimes G^H_K(F).\tag{55}$$

We proceed by induction over $|H|$. Suppose first $H=\{h\}$, for some $h=1,...,n$, then we have

$$G^{\{h\}}_{\{h\}}(F)-\overline{Z}_h\otimes G^{\{h\}}_{\emptyset}(F)=(Z_h\otimes F)'_h-\overline{Z}_h\otimes F'_h=F^\circ_h+\alpha_hF'_h-\alpha_hF'_h+\mathrm{Im}(Z_h)\otimes F'_h=F,$$

by (26) and Lemma 6.7. Now, suppose (55) holds for some $H \in \mathcal{P}(n)$, let us prove it for $H' = H \cup \{m\}$, with $m \notin H$. We have

$$\begin{aligned}
&\sum_{K\subset H'} (-1)^{|H'\setminus K|}\overline{Z}_{H'\setminus K} \otimes G_K^{H'}(F) \\
&= \sum_{K\subset H} (-1)^{|H\setminus K|}\overline{Z}_{H\setminus K} \otimes G_{K\cup\{m\}}^{H\cup\{m\}}(F) - \sum_{K\subset H} (-1)^{|H\setminus K|}\overline{Z}_{H\setminus K} \otimes \overline{Z}_m \otimes G_K^{H\cup\{m\}}(F) \\
&= \sum_{K\subset H} (-1)^{|H\setminus K|}\overline{Z}_{H\setminus K} \otimes \left(G_{K\cup\{m\}}^{H\cup\{m\}}(F) - \overline{Z}_m \otimes G_K^{H\cup\{m\}}(F)\right) \\
&= \sum_{K\subset H} (-1)^{|H\setminus K|}\overline{Z}_{H\setminus K} \otimes G_K^{H}(F) = F,
\end{aligned}$$

by (51) and the inductive hypothesis. Now (50) easily follows, indeed

$$\begin{aligned}
f = \mathcal{I}(F) &= \mathcal{I}\left(\sum_{K\subset H} (-1)^{|H\setminus K|}\overline{Z}_{H\setminus K} \otimes G_K^{H}(F)\right) \\
&= \sum_{K\subset H} (-1)^{|H\setminus K|}\mathcal{I}\left(\overline{Z}_{H\setminus K}\right) \odot \mathcal{I}\left(G_K^H(F)\right) = \sum_{K\subset H} (-1)^{|H\setminus K|}\left(\overline{x}\right)_{H\setminus K} \odot \mathcal{S}_K^H(f).
\end{aligned}$$

2. For any $K \subset H$, $\mathcal{S}_K^H(f) = (x_K \odot f)'_{s,H} \in \mathcal{S}_{c,H}(\Omega_{D_H}) \cap \mathcal{S}_p(\Omega_{D_H})$, by Proposition 4.9 (1).

3. Write $\mathcal{S}_K^H(f) = \left(x_h^{\chi_K(h)} \odot \mathcal{S}_{K\setminus\{h\}}^{H\setminus\{h\}}(f)\right)'_{s,h}$. By hypothesis, $f \in \ker(\partial/\partial x_t^c)$, $\forall t = 1, ..., n$, then, by Proposition 4.10, $\mathcal{S}_{K\setminus\{h\}}^{H\setminus\{h\}}(f) \in \ker(\partial/\partial x_h^c)$ and thanks to Leibniz formula [50, Proposition 3.25], $x_h^{\chi_K(h)} \odot \mathcal{S}_{K\setminus\{h\}}^{H\setminus\{h\}}(f) \in \ker(\partial/\partial x_h^c)$. Finally, by Corollary 5.9

$$\Delta_h \mathcal{S}_K^H(f) = \Delta_h \left(x_h^{\chi_K(h)} \odot \mathcal{S}_{K\setminus\{h\}}^{H\setminus\{h\}}(f)\right)'_{s,h} = 0.$$

4. $\Rightarrow$) By hypothesis, $x_K \odot f \in \mathcal{SR}(\Omega_D)$, then $\mathcal{S}_K^H(f) = (x_K \odot f)'_{s,H} \in \ker(\partial/\partial x_t^c)$, for any $t \notin H$. In particular, $\mathcal{S}_K^H(f) = (x_K \odot f)'_{s,H} \in \ker(\partial/\partial x_p^c) \cap \mathcal{S}_p(\Omega_D) = \mathcal{SR}_p(\Omega_D)$, by Proposition 4.2.

$\Leftarrow$) It is a particular case of Proposition 6.12, we will prove later.

5. f is slice preserving if and only if F_K is real $\forall K \in \mathcal{P}(n)$, which by (54) is equivalent for $G_K^{[\![n]\!]}(F) = \mathcal{S}_K^{[\![n]\!]}(f)$ to be real valued for every $K \in \mathcal{P}(n)$.

□

We highlight the unique character of the decomposition, indeed for every choice of $H \in \mathcal{P}(n)$, the functions $\mathcal{S}_K^H(f)$ are the only H-circular functions that realize decomposition (50).

Proposition 6.10. *Let $f \in \mathcal{S}(\Omega_D)$ and fix $H \in \mathcal{P}(n)$. Suppose that there exist functions $\{h_K\}_{K\subset H}$ such that $h_K \in \mathcal{S}_{c,H}(\Omega_D)$, $\forall K \subset H$ and*

$$f(x) = \sum_{K\subset H} (-1)^{|H\setminus K|}\left(\overline{x}\right)_{H\setminus K} \odot h_K(x).$$

Then $h_K = \mathcal{S}_K^H(f)$.

Proof. Apply (25) and the hypothesis $h_T \in \mathcal{S}_{c,H}(\Omega_D)$ in the following computation

$$\begin{aligned}\mathcal{S}_K^H(f) := (x_K \odot f)'_{s,H} &= \left(x_K \odot \sum_{T\subset H} (-1)^{|H\setminus T|} (\overline{x})_{H\setminus T} \odot h_T\right)'_{s,H} \\ &= \sum_{T\subset H} (-1)^{|H\setminus T|} \left(x_K \odot (\overline{x})_{H\setminus T} \odot h_T\right)'_{s,H} \\ &= \sum_{T\subset H} (-1)^{|H\setminus T|} \left(x_K \odot (\overline{x})_{H\setminus T}\right)'_{s,H} \odot h_T.\end{aligned}$$

Now we claim that $\left(x_K \odot (\overline{x})_{H\setminus T}\right)'_{s,H} = (-1)^{|H\setminus K|}\delta_{K,T}$, which would reduce the prevoious equation to $\mathcal{S}_K^H(f) = h_K$. Suppose first that exists $h \in T\setminus K \subset H$, then $x_K \odot (\overline{x})_{H\setminus T} \in \mathcal{S}_{c,h}(\Omega_D)$, thus in particular $\left(x_K \odot (\overline{x})_{H\setminus T}\right)'_{s,H} = 0$. Viceversa, suppose $h \in K \setminus T \subset H$, but again $x_K \odot (\overline{x})_{H\setminus T} \in \mathcal{S}_{c,h}(\Omega_D)$, indeed

$$x_K \odot (\overline{x})_{H\setminus T} = x_h\overline{x}_h \odot x_{K\setminus\{h\}} \odot (\overline{x})_{H\setminus(T\cup\{h\})} = (\alpha_h^2 + \beta_h^2)x_{K\setminus\{h\}} \odot (\overline{x})_{H\setminus(T\cup\{h\})} \in \mathcal{S}_{c,h}(\Omega_D).$$

Thus, the unique non trivial element of the sum refers to $T = K$, for which we have

$$\begin{aligned}\left(x_K \odot (\overline{x})_{H\setminus K}\right)'_{s,H} &= \left(\left(x_K \odot (\overline{x})_{H\setminus K}\right)'_{s,K}\right)'_{s,H\setminus K} = \left((x_K)'_{s,K} \odot (\overline{x})_{H\setminus K}\right)'_{s,H\setminus K} \\ &= \left((\overline{x})_{H\setminus K}\right)'_{s,H\setminus K} = (-1)^{|H\setminus K|},\end{aligned}$$

where we have used (25) and Lemma 6.7 (1). □

6.4.1 New one variable interpretation of slice regularity

We now give a one variable interpretation of slice regularity in terms of partial slice regularity of the functions $\mathcal{S}_K^{[\![m]\!]}(f)$.

From Theorem 6.6, we emphasize the particular case in which $H = [\![m]\!] = \{1, 2, ..., m\}$, that leads to what we call an ordered decomposition of f.

Corollary 6.11. *Let $f \in \mathcal{S}(\Omega_D)$ and fix any $m \in \{1, ..., n\}$, then we can orderly decompose f as*

$$f(x) = \sum_{K\in\mathcal{P}(m)} (-1)^{|K^c|} (\overline{x})_{K^c} \mathcal{S}_K^{[\![m]\!]}(f)(x), \tag{56}$$

where $K^c = [\![m]\!] \setminus K$. Moreover,

1. *if $m < n$, $\mathcal{S}_K^{[\![m]\!]}(f) \in \mathcal{S}_{c,[\![m]\!]}(\Omega_{D_{[\![m]\!]}}) \cap \mathcal{S}_{m+1}(\Omega_{D_{[\![m]\!]}})$, for any $K \in \mathcal{P}(m)$;*
2. *suppose $f \in \mathcal{SR}(\Omega_D)$, then*
   (a) if $\Omega_D \subset \mathbb{H}^n$, $\Delta_h \mathcal{S}_K^{[\![m]\!]}(f) = 0$, $\forall h \le m$, $\forall K \in \mathcal{P}(m)$;
   (b) if $\Omega_D \subset (\mathbb{R}^{m+1})^n$, $\Delta_h^{\frac{m-1}{2}} \mathcal{S}_K^{[\![m]\!]}(f) = 0$, $\forall h \le m$, $\forall K \in \mathcal{P}(m)$.

.

We point out that formula (56) holds with the ordinary pointwise product [50, Proposition 2.52]. On the contrary, in (50) the slice product is necessary.

Proposition 6.12. *Let $f \in \mathcal{S}(\Omega_D)$, then $f \in \mathcal{SR}(\Omega_D)$ if and only if $\mathcal{S}_K^{[\![m]\!]}(f) \in \mathcal{SR}_{m+1}(\Omega_D)$, $\forall m = 0, ..., n-1$, $K \in \mathcal{P}(m)$.*

Proof of Proposition 6.12.

$\Longrightarrow$) We have already proved in Theorem 6.6 (4).

$\Longleftarrow$) For any $m \in \{1, ..., n\}$ consider the ordered Almansi decomposition (56) of f. Recall that $\partial \overline{x}_h / \partial x_k^c = 0$, $\forall h, k = 1, ..., n$ and $\partial / \partial x_{m+1}^c (\mathcal{S}_K^{[\![m]\!]}(f)) = 0$, for every $K \in \mathcal{P}(n)$, so applying [50, (73)] it holds

$$\frac{\partial f}{\partial x_{m+1}^c} = \frac{\partial}{\partial x_{m+1}^c} \left(\sum_{K \in \mathcal{P}(m)} (-1)^{|K^c|} (\overline{x})_{K^c} \mathcal{S}_K^{[\![m]\!]}(f) \right) = \sum_{K \in \mathcal{P}(m)} (-1)^{|K^c|} (\overline{x})_{K^c} \otimes \frac{\partial \mathcal{S}_K^{[\![m]\!]}(f)}{\partial x_{m+1}^c} = 0.$$

□

The previous characterization resembles the one given in Theorem 4.14, in which iterations of spherical values and spherical derivatives (also referred as truncated spherical derivatives $\mathcal{D}_\epsilon(f)$) have been used. It is easy to see that truncated spherical derivatives can be expressed as real combinations of the components $\mathcal{S}_K^{[\![m]\!]}(f)$ and viceversa, making the two characterizations equivalent. Next Lemma provides, indeed, a twofold relationship between truncated derivatives and the components of the ordered Almansi decomposition (Corollary 6.11).

Lemma 6.13. *Let $H \in \mathcal{P}(h)$, then it holds*

$$D_H^h(f) = \sum_{K \subset [\![h]\!] \setminus H} (-1)^{|K|} \mathcal{R}e(x_K) \mathcal{S}^{[\![h]\!]}_{([\![h]\!] \setminus H) \setminus K}(f). \tag{57}$$

Viceversa, $\forall K \in \mathcal{P}(h)$, we have

$$\mathcal{S}_K^{[\![h]\!]}(f) = \sum_{H \in \mathcal{P}(K)} \mathcal{R}e(x_H) D^h_{([\![h]\!] \setminus K) \cup H}(f). \tag{58}$$

Proof. Let's prove formula (57) by induction over $m = 1, ..., h$.
For $m = 1$, we have two possibilities: $H = \emptyset$ or $H = \{1\}$.

$H = \emptyset$:

$$\begin{aligned} D_\emptyset^1(f) &= (f)^\circ_{s,1} = (x_1 f)'_{s,1} - \mathcal{R}e(x_1) f'_{s,1} \\ &= (-1)^{|\emptyset|} \mathcal{R}e(x_\emptyset) \mathcal{S}^{[\![1]\!]}_{\{1\}}(f) + (-1)^{|\{1\}|} \mathcal{R}e(x_{\{1\}}) \mathcal{S}^{[\![1]\!]}_\emptyset(f), \end{aligned}$$

where we have used (27).

$H = \{1\}$:

$$D^1_{\{1\}}(f) = f'_{s,1} = (-1)^{|\emptyset|} \mathcal{R}e(x_\emptyset) \mathcal{S}^{[\![1]\!]}_\emptyset(f).$$

Assume now that (57) holds for $m-1$, namely that $\forall H' \in \mathcal{P}(m-1)$ it holds

$$D^{m-1}_{H'}(f) = \sum_{K \subset [\![m-1]\!] \setminus H')} (-1)^{|K|} \mathcal{R}e(x_K) \mathcal{S}^{[\![m-1]\!]}_{([\![m-1]\!] \setminus H') \setminus K}(f),$$

and let's prove the formula is true for every $H \in \mathcal{P}(m)$.
Suppose first $m \in H$, then $[\![m]\!] \setminus H = [\![m-1]\!] \setminus (H \setminus \{m\})$, thus define $H' := H \setminus \{m\} \in \mathcal{P}(m-1)$, so we have

$$\begin{aligned}
D^m_H(f) &= \left(D^{m-1}_{H'}(f)\right)'_{s,m} \\
&= \left(\sum_{K \subset [\![m-1]\!] \setminus H'} (-1)^{|K|} \mathcal{R}e(x_K) \mathcal{S}^{[\![m-1]\!]}_{([\![m-1]\!] \setminus H') \setminus K}(f) \right)'_{s,m} \\
&= \sum_{K \subset [\![m-1]\!] \setminus H'} (-1)^{|K|} \mathcal{R}e(x_K) \left(\mathcal{S}^{[\![m-1]\!]}_{([\![m-1]\!] \setminus H') \setminus K}(f) \right)'_{s,m} \\
&= \sum_{K \subset [\![m]\!] \setminus H)} (-1)^{|K|} \mathcal{R}e(x_K) \mathcal{S}^{[\![m]\!]}_{([\![m]\!] \setminus H) \setminus K}(f).
\end{aligned}$$

Conversely, suppose that $m \notin H$, then $H \in \mathcal{P}(m-1)$ too, so

$$\begin{aligned}
D^m_H(f) = \left(D^{m-1}_H(f)\right)^\circ_{s,m} &= \left(x_m D^{m-1}_H(f)\right)'_{s,m} - \mathcal{R}e(x_m) \left(D^{m-1}_H(f)\right)'_{s,m} \\
&= \left(x_m \sum_{K \subset [\![m-1]\!] \setminus H)} (-1)^{|K|} \mathcal{R}e(x_K) \mathcal{S}^{[\![m-1]\!]}_{([\![m-1]\!] \setminus H) \setminus K}(f) \right)'_{s,m} + \\
&- \mathcal{R}e(x_m) \left(\sum_{K \subset [\![m-1]\!] \setminus H)} (-1)^{|K|} \mathcal{R}e(x_K) \mathcal{S}^{[\![m-1]\!]}_{([\![m-1]\!] \setminus H) \setminus K}(f) \right)'_{s,m} \\
&= \sum_{K \subset [\![m-1]\!] \setminus H)} (-1)^{|K|} \mathcal{R}e(x_K) \left(x_m \mathcal{S}^{[\![m-1]\!]}_{([\![m-1]\!] \setminus H) \setminus K}(f) \right)'_{s,m} + \\
&- \mathcal{R}e(x_m) \sum_{K \subset [\![m-1]\!] \setminus H)} (-1)^{|K|} \mathcal{R}e(x_K) \left(\mathcal{S}^{[\![m-1]\!]}_{([\![m-1]\!] \setminus H) \setminus K}(f) \right)'_{s,m} \\
&= \sum_{K \subset [\![m-1]\!] \setminus H)} (-1)^{|K|} \mathcal{R}e(x_K) \mathcal{S}^{[\![m]\!]}_{([\![m]\!] \setminus H) \setminus H}(f) + \\
&+ \sum_{K \subset [\![m-1]\!] \setminus H)} (-1)^{|H \cup \{m\}|} \mathcal{R}e(x_{K \cup \{m\}}) \mathcal{S}^{[\![m]\!]}_{([\![m]\!] \setminus H) \setminus (K \cup \{m\})}(f) \\
&= \sum_{K \subset [\![m]\!] \setminus H)} (-1)^{|K|} \mathcal{R}e(x_K) \mathcal{S}^{[\![m]\!]}_{([\![m]\!] \setminus H) \setminus K}(f)
\end{aligned}$$

Let's prove, now, formula (58), again by induction.
For $m = 1$, $K \subset \{1\}) = \{\emptyset, \{1\}\}$:

$$\mathcal{S}^{[\![1]\!]}_{\emptyset}(f) = f'_{s,1} = D^1_{\{1\}}(f)$$

$$\mathcal{S}^{[\![1]\!]}_{\{1\}}(f) = (x_1 f)'_{s,1} = f^\circ_{s,1} + \mathcal{R}e(x_1) f'_{s,1} = D^1_{\emptyset}(f) + \mathcal{R}e(x_1) D^1_{\{1\}}(f).$$

Assume now that (58) holds for $m-1$, namely that $\forall K' \in \mathcal{P}(m-1)$ it is true that

$$\mathcal{S}^{[\![m-1]\!]}_{K'}(f) = \sum_{H \in \mathcal{P}(K')} \mathcal{R}e(x_H) D^{m-1}_{([\![m-1]\!] \setminus K') \cup H}(f),$$

let's prove that it holds for every $K \in \mathcal{P}(m)$.
Let $K \in \mathcal{P}(m)$, assume first $m \notin K$, then $K \in \mathcal{P}(m-1)$ too, therefore we have

$$\begin{aligned}
\mathcal{S}_K^{[\![m]\!]}(f) &= \left(\mathcal{S}_K^{[\![m-1]\!]}(f)\right)'_{s,m} = \left(\sum_{H\in\mathcal{P}(K)} \mathcal{R}e(x_H) D^{m-1}_{([\![m-1]\!]\setminus K)\cup H}(f)\right)'_{s,m} \\
&= \sum_{H\in\mathcal{P}(K)} \mathcal{R}e(x_H) \left(D^{m-1}_{([\![m-1]\!]\setminus K)\cup H}(f)\right)'_{s,m} \\
&= \sum_{H\in\mathcal{P}(K)} \mathcal{R}e(x_H) D^{m}_{([\![m]\!]\setminus K)\cup H}(f)
\end{aligned}$$

On the contrary, if $m \in K$, define $K' := K \setminus K \in \mathcal{P}(m-1)$, then

$$\begin{aligned}
\mathcal{S}_K^{[\![m]\!]}(f) &= \left(x_m \mathcal{S}_{K'}^{[\![m-1]\!]}(f)\right)'_{s,m} = \left(\mathcal{S}_{K'}^{[\![m-1]\!]}(f)\right)^{\circ}_{s,m} + \mathcal{R}e(x_m)\left(\mathcal{S}_{K'}^{[\![m-1]\!]}(f)\right)'_{s,m} \\
&= \left(\sum_{H\in\mathcal{P}(K')} \mathcal{R}e(x_H) D^{m-1}_{([\![m-1]\!]\setminus K')\cup H}(f)\right)^{\circ}_{s,m} + \\
&+ \mathcal{R}e(x_m)\left(\sum_{H\in\mathcal{P}(K')} \mathcal{R}e(x_H) D^{m-1}_{([\![m-1]\!]\setminus K')\cup H}(f)\right)'_{s,m} \\
&= \sum_{H\in\mathcal{P}(K')} \mathcal{R}e(x_H) D^{m}_{([\![m-1]\!]\setminus K')\cup H}(f) + \sum_{H\in\mathcal{P}(K')} \mathcal{R}e(x_{H\cup\{m\}}) D^{m}_{([\![m-1]\!]\setminus K')\cup(H\cup\{m\})}(f) \\
&= \sum_{H\in\mathcal{P}(K')} \mathcal{R}e(x_H) D^{m}_{([\![m]\!]\setminus K)\cup H}(f) + \sum_{H\in\mathcal{P}(K')} \mathcal{R}e(x_{H\cup\{m\}}) D^{m}_{([\![m]\!]\setminus K)\cup(H\cup\{m\})}(f) \\
&= \sum_{H\in\mathcal{P}(K)} \mathcal{R}e(x_H) D^{m}_{([\![m]\!]\setminus K)\cup H}(f)
\end{aligned}$$

□

Let us examine the case in which f is slice with respect to x_h.

Proposition 6.14. *Let $f \in \mathcal{S}_h(\Omega_D)$ for some $h \in \{1, ..., n\}$, then*

$$\mathcal{S}_K^{[\![h]\!]}(f) = 0, \qquad \forall K \in \mathcal{P}(h-1), K \neq \{1, ..., h-1\}. \tag{59}$$

In particular, the ordered decomposition of f of order h reduces to

$$f = \sum_{K\in\mathcal{P}(h-1)} (-1)^{|K^c|} \overline{x}_{K^c} \mathcal{S}^{[\![h]\!]}_{K\cup\{h\}}(f) - \overline{x}_h \mathcal{S}^{[\![h]\!]}_{[\![h-1]\!]}(f).$$

Proof. Assume $K \in \mathcal{P}(h-1)$, with $K \neq \{1, ..., h-1\}$, then there exists $m \in \{1, ..., h-1\} \setminus K$. Let $H := \{m, h\}$, then by (4) of Proposition 4.9 it holds $f'_{s,H} = 0$. Since $K \cap H = \emptyset$, $x_K \in \mathcal{S}_{c,H}(\Omega_D)$, then by (27) we have

$$\mathcal{S}_K^{[\![h]\!]}(f) = \left[(x_K \odot f)'_{s,H}\right]'_{s,[\![h]\!]\setminus H} = (x_K \odot f'_{s,H})'_{s,[\![h]\!]\setminus H} = 0.$$

This proves (59). By this and Corollary 6.11 follows

$$
\begin{aligned}
f &= \sum_{K\in\mathcal{P}(h)} (-1)^{|[\![h]\!]\setminus K|}\overline{x}_{[\![h]\!]\setminus K}\mathcal{S}_K^{[\![h]\!]}(f) = \sum_{K\in\mathcal{P}(h-1)} (-1)^{|[\![h]\!]\setminus K|}\overline{x}_{[\![h]\!]\setminus K}\mathcal{S}_K^{[\![h]\!]}(f)+\\
&+\sum_{K\in\mathcal{P}(h-1)} (-1)^{|[\![h]\!]\setminus (K\cup\{h\})|}\overline{x}_{[\![h]\!]\setminus (K\cup\{h\})}\mathcal{S}_{K\cup\{h\}}^{[\![h]\!]}(f)\\
&= -\overline{x}_h\mathcal{S}^{[\![h]\!]}_{[\![h-1]\!]}(f) + \sum_{K\in\mathcal{P}(h-1)} (-1)^{|[\![h-1]\!]\setminus K|}\overline{x}_{[\![h-1]\!]\setminus K}\mathcal{S}_{K\cup\{h\}}^{[\![h]\!]}(f).
\end{aligned}
$$

□

6.4.2 Applications in $\mathbb{H}^n$

The first application we give of Theorem 6.6 concerns quaternionic (ordered) polynomials with right coefficients, in which the components of the decomposition are given through zonal harmonics. Note that Theorem 6.6 can be fully applied, since every polynomial with right coefficients is a slice regular function [50, Proposition 3.14]. Before, we recall a result from [74, Corollary 6.7].

Lemma 6.15. *For every $m \geq 0$, consider the slice regular power $x^m : \mathbb{H} \to \mathbb{H}$. Then it holds*

$$(x^m)'_s = \tilde{\mathcal{Z}}_{m-1}(x),$$

where

$$\tilde{\mathcal{Z}}_k(x) := \begin{cases} \dfrac{\mathcal{Z}_k(x,1)}{k+1} & \text{if } k \geq 0 \\ 0 & \text{if } k=-1 \end{cases}$$

and $\mathcal{Z}_k(x,1)$ is the real valued zonal harmonic of $\mathbb{R}^4$ with pole 1 (see [7, Ch. 5]).

Proposition 6.16. *Let $P \in \mathbb{H}[X_1, ..., X_n]$ be any quaternionic polynomial with right coefficients*

$$P(x_1,...,x_n) = \sum_{k=1}^n \sum_{|\alpha|=k} x^\alpha a_\alpha = \sum_{k=1}^n \sum_{|\alpha|=k} x_1^{\alpha_1}...\, x_n^{\alpha_n} a_\alpha.$$

Then, for every $H \in \mathcal{P}(n)$ and $K \subset H$

$$\mathcal{S}_K^H(P)(x) = \sum_{k=1}^n \sum_{|\alpha|=k} \prod_{j\in H} \tilde{\mathcal{Z}}_{\alpha_j-1+\chi_K(j)}(x_j) \prod_{i\in H^c} x_i^{\alpha_i} a_\alpha, \tag{60}$$

where $\prod_i x_i^{\alpha_i}$ is an ordered product.

Proof. By linearity of the spherical derivative, we can assume without loss of generality $P(x_1,...,x_n) = x^\alpha = x_1^{\alpha_1}...\, x_n^{\alpha_n}$. We will proceed by induction over $|H|$. Suppose $H = \{h\}$, for some $h = 1,...,n$, then, since $x_1^{\alpha_1}...\, x_{h-1}^{\alpha_{h-1}} x_{h+1}^{\alpha_{h+1}}...\, x_n^{\alpha_n} \in \mathcal{S}_{c,h}(\Omega_D)$, we have

$$
\begin{aligned}
\mathcal{S}_\emptyset^{\{h\}}(P) &= (x^\alpha)'_{s,h} = (x_h^{\alpha_h} \odot x_1^{\alpha_1}...\, x_{h-1}^{\alpha_{h-1}} x_{h+1}^{\alpha_{h+1}}...\, x_n^{\alpha_n})'_{s,h}\\
&= (x_h^{\alpha_h})'_{s,h} \odot x_1^{\alpha_1}...\, x_{h-1}^{\alpha_{h-1}} x_{h+1}^{\alpha_{h+1}}...\, x_n^{\alpha_n} = \tilde{\mathcal{Z}}_{\alpha_h-1}(x_h) x_1^{\alpha_1}...\, x_{h-1}^{\alpha_{h-1}} x_{h+1}^{\alpha_{h+1}}...\, x_n^{\alpha_n},
\end{aligned}
$$

where we have used Lemma 6.15, (27) and that $_j$ is real valued. Similarly,

$$
\begin{aligned}
\mathcal{S}_{\{h\}}^{\{h\}}(P) &= (x_h \odot x^\alpha)'_{s,h} = (x_h^{\alpha_h+1} \odot x_1^{\alpha_1}...\, x_{h-1}^{\alpha_{h-1}} x_{h+1}^{\alpha_{h+1}}...\, x_n^{\alpha_n})'_{s,h}\\
&= \tilde{\mathcal{Z}}_{\alpha_h}(x_h) x_1^{\alpha_1}...\, x_{h-1}^{\alpha_{h-1}} x_{h+1}^{\alpha_{h+1}}...\, x_n^{\alpha_n}.
\end{aligned}
$$

Now, suppose that (60) holds for some $H \in \mathcal{P}(n)$ and let us prove it for $H' = H \cup \{m\}$, for any $m \notin H$. Suppose first $m \notin K$, then, as before

$$\begin{aligned}
\mathcal{S}_K^{H'}(P) = (\mathcal{S}_K^H(P))'_{s,m} &= \left(\prod_{j\in H} \tilde{\mathcal{Z}}_{\alpha_j-1+\chi_K(j)}(x_j) \prod_{i\in H^c} x_i^{\alpha_i}\right)'_{s,m} \\
&= \left(\prod_{j\in H} \tilde{\mathcal{Z}}_{\alpha_j-1+\chi_K(j)}(x_j) x_m^{\alpha_m} \odot \prod_{i\in (H')^c} x_i^{\alpha_i}\right)'_{s,m} \\
&= \prod_{j\in H} \tilde{\mathcal{Z}}_{\alpha_j-1+\chi_K(j)}(x_j) \tilde{\mathcal{Z}}_{\alpha_m-1}(x_m) \prod_{i\in (H')^c} x_i^{\alpha_i} = \prod_{j\in H'} \tilde{\mathcal{Z}}_{\alpha_j-1+\chi_K(j)}(x_j) \prod_{i\in (H')^c} x_i^{\alpha_i}.
\end{aligned}$$

If $m \in K$, let $K' = K \setminus \{m\}$, then

$$\begin{aligned}
\mathcal{S}_K^{H'}(P) = (x_m \odot \mathcal{S}_{K'}^H(P))'_{s,m} &= \left(\prod_{j\in H} \tilde{\mathcal{Z}}_{\alpha_j-1+\chi_{K'}(j)}(x_j) x_m^{\alpha_m+1} \odot \prod_{i\in (H')^c} x_i^{\alpha_i}\right)'_{s,m} \\
&= \prod_{j\in H} \tilde{\mathcal{Z}}_{\alpha_j-1+\chi_{K'}(j)}(x_j) \tilde{\mathcal{Z}}_{\alpha_m}(x_m) \prod_{i\in (H')^c} x_i^{\alpha_i} = \prod_{j\in H'} \tilde{\mathcal{Z}}_{\alpha_j-1+\chi_K(j)}(x_j) \prod_{i\in (H')^c} x_i^{\alpha_i}.
\end{aligned}$$

□

Example 9. Let $f \in \mathcal{SR}(\mathbb{H}^2)$, $f(x_1, x_2) := x_1 x_2$. We can give 2^2 decompositions of f for $H = \emptyset, \{1\}, \{1,2\}, \{2\}$: let $x = (\alpha_1 + J_1\beta_1, \alpha_2 + J_2\beta_2)$, then

$$\begin{aligned}
f(x) &= \mathcal{S}_\emptyset^\emptyset(f)(x) \\
&= \mathcal{S}_{\{1\}}^{[\![1]\!]}(f)(x) - \overline{x}_1 \mathcal{S}_\emptyset^{[\![1]\!]}(f)(x) = 2\alpha_1 x_2 - \overline{x}_1 x_2 \\
&= \mathcal{S}_{\{1,2\}}^{[\![2]\!]}(f)(x) - \overline{x}_1 \mathcal{S}_{\{2\}}^{[\![2]\!]}(f)(x) - \overline{x}_2 \mathcal{S}_{\{1\}}^{[\![2]\!]}(f)(x) + \overline{x}_1\overline{x}_2 \mathcal{S}_\emptyset^{[\![2]\!]}(f)(x) \\
&\quad = 4\alpha_1\alpha_2 - 2\alpha_2\overline{x}_1 - 2\alpha_1\overline{x}_1 + \overline{x}_1\overline{x}_2 \\
&= \mathcal{S}_{\{2\}}^{\{2\}}(f)(x) - \overline{x}_2 \odot \mathcal{S}_\emptyset^{\{2\}}(f)(x) = 2\alpha_2 x_1 - \overline{x}_2 \odot x_1 = 2\alpha_2 x_1 - x_1\overline{x}_2.
\end{aligned}$$

Note that in the first three decompositions the slice product is not needed. On the contrary, the last one, corresponding to $H = \{2\}$ needs the slice product. Moreover, $\mathcal{S}_\emptyset^\emptyset(f) = f \in \mathcal{SR}_1(\Omega_D)$ and $\mathcal{S}_\emptyset^{[\![1]\!]}(f), \mathcal{S}_{\{1\}}^{[\![1]\!]}(f) \in \mathcal{SR}_2(\Omega_D)$, as $f \in \mathcal{SR}(\Omega_D)$.

Example 10. Let $g \in \mathcal{SR}(\mathbb{H}^3)$, $g(x_1, x_2, x_3) = e^{x_1} x_2 x_3^3$. Now we have 2^3 decompositions for $H \in \mathcal{P}(3)$. Let $x = (\alpha_1 + J_1\beta_1, \alpha_2 + J_2\beta_2, \alpha_3 + J_3\beta_3)$, so aside from the trivial decomposition corresponding to $H=\emptyset$, we have the ordered decompositions for $H=\{1\},\{1,2\},\{1,2,3\}$

$$
\begin{aligned}
f(x) &= \mathcal{S}^{[\![1]\!]}_{\{1\}}(f)(x) - \overline{x}_1\mathcal{S}^{[\![1]\!]}_{\emptyset}(f)(x) = e^{\alpha_1}(\cos\beta_1+\alpha_1/\beta_1\sin\beta_1)x_2x_3^3 - \overline{x}_1 e^{\alpha_1}\sin(\beta_1)/\beta_1 x_2x_3^3\\
&= \mathcal{S}^{[\![2]\!]}_{\{1,2\}}(f)(x) - \overline{x}_1\mathcal{S}^{[\![2]\!]}_{\{2\}}(f)(x) - \overline{x}_2\mathcal{S}^{[\![2]\!]}_{\{1\}}(f)(x) + \overline{x}_1\overline{x}_2\mathcal{S}^{[\![2]\!]}_{\emptyset}(f)(x)\\
&\quad= e^{\alpha_1}(\cos\beta_1+\alpha_1/\beta_1\sin\beta_1)2\alpha_2x_3^3 - \overline{x}_1e^{\alpha_1}\sin(\beta_1)/\beta_1 2\alpha_2x_3^3+\\
&\quad - \overline{x}_2e^{\alpha_1}(\cos\beta_1+\alpha_1/\beta_1\sin\beta_1)x_3^3 + \overline{x}_1\overline{x}_2e^{\alpha_1}\sin(\beta_1)/\beta_1x_3^3\\
&= \mathcal{S}^{[\![3]\!]}_{\{1,2,3\}}(f)(x) - \overline{x}_1\mathcal{S}^{[\![3]\!]}_{\{2,3\}}(f)(x) - \overline{x}_2\mathcal{S}^{[\![3]\!]}_{\{1,3\}}(f)(x) - \overline{x}_3\mathcal{S}^{[\![3]\!]}_{\{1,2\}}(f)(x)+\\
&\quad + \overline{x}_1\overline{x}_2\mathcal{S}^{[\![3]\!]}_{\{3\}}(f)(x) + \overline{x}_1\overline{x}_3\mathcal{S}^{[\![3]\!]}_{\{2\}}(f)(x) + \overline{x}_2\overline{x}_3\mathcal{S}^{[\![3]\!]}_{\{1\}}(f)(x) - \overline{x}_1\overline{x}_2\overline{x}_3\mathcal{S}^{[\![3]\!]}_{\emptyset}(f)(x)\\
&\quad= e^{\alpha_1}(\cos\beta_1+\alpha_1/\beta_1\sin\beta_1)2\alpha_2 4\alpha_3(\alpha_3^2-\beta_3^2) - \overline{x}_1e^{\alpha_1}\sin(\beta_1)/\beta_1 2\alpha_2 4\alpha_3(\alpha_3^2-\beta_3^2)+\\
&\quad - \overline{x}_2e^{\alpha_1}(\cos\beta_1+\alpha_1/\beta_1\sin\beta_1)4\alpha_3(\alpha_3^2-\beta_3^2)+\\
&\quad - \overline{x}_3e^{\alpha_1}(\cos\beta_1+\alpha_1/\beta_1\sin\beta_1)2\alpha_2(3\alpha_3^2-\beta_3^2) + \overline{x}_1\overline{x}_2e^{\alpha_1}\sin(\beta_1)/\beta_1 4\alpha_3(\alpha_3^2-\beta_3^2)+\\
&\quad + \overline{x}_1\overline{x}_3e^{\alpha_1}\sin(\beta_1)/\beta_1 2\alpha_2(3\alpha_3^2-\beta_3^2) + \overline{x}_2\overline{x}_3e^{\alpha_1}(\cos\beta_1+\alpha_1/\beta_1\sin\beta_1)(3\alpha_3^2-\beta_3^2)+\\
&\quad - \overline{x}_1\overline{x}_2\overline{x}_3e^{\alpha_1}\sin(\beta_1)/\beta_1(3\alpha_3^2-\beta_3^2)
\end{aligned}
$$

and the remaining decompositions for $H=\{2\},\{3\},\{1,3\},\{2,3\}$

$$
\begin{aligned}
f(x) &= \mathcal{S}^{\{2\}}_{\{2\}}(f)(x) - \overline{x}_2\odot\mathcal{S}^{\{2\}}_{\emptyset}(f)(x) = e^{x_1}2\alpha_2x_3^3 - \overline{x}_2\odot e^{x_1}x_3^3\\
&= \mathcal{S}^{\{3\}}_{\{3\}}(f)(x) - \overline{x}_3\odot\mathcal{S}^{\{3\}}_{\emptyset}(f)(x) = e^{x_1}x_2 4\alpha_3(\alpha_3^2-\beta_3^2) - \overline{x}_3\odot e^{x_1}x_2(3\alpha_3^2-\beta_3^2)\\
&= \mathcal{S}^{\{1,3\}}_{\{1,3\}}(f)(x) - \overline{x}_1\odot\mathcal{S}^{\{1,3\}}_{\{3\}}(f)(x) - \overline{x}_3\odot\mathcal{S}^{\{1,3\}}_{\{1\}}(f)(x) + \overline{x}_1\overline{x}_3\odot\mathcal{S}^{\{1,3\}}_{\emptyset}(f)(x)\\
&\quad= e^{\alpha_1}(\cos\beta_1+\alpha_1/\beta_1\sin\beta_1)x_2 4\alpha_3(\alpha_3^2-\beta_3^2) - \overline{x}_1\odot e^{\alpha_1}\sin(\beta_1)/\beta_1 x_2 4\alpha_3(\alpha_3^2-\beta_3^2)+\\
&\quad - \overline{x}_3\odot e^{\alpha_1}(\cos\beta_1+\alpha_1/\beta_1\sin\beta_1)x_2(3\alpha_3^2-\beta_3^2) + \overline{x}_1\overline{x}_3\odot e^{\alpha_1}\sin(\beta_1)/\beta_1 x_2(3\alpha_3^2-\beta_3^2)\\
&= \mathcal{S}^{\{2,3\}}_{\{2,3\}}(f)(x) - \overline{x}_2\odot\mathcal{S}^{\{2,3\}}_{\{3\}}(f)(x) - \overline{x}_3\odot\mathcal{S}^{\{2,3\}}_{\{2\}}(f)(x) + \overline{x}_2\overline{x}_3\odot\mathcal{S}^{\{2,3\}}_{\emptyset}(f)(x)\\
&\quad= e^{x_1}2\alpha_2 4\alpha_3(\alpha_3^2-\beta_3^2) - \overline{x}_2\odot e^{x_1}4\alpha_3(\alpha_3^2-\beta_3^2) - \overline{x}_3\odot e^{x_1}2\alpha_2(3\alpha_3^2-\beta_3^2)+\\
&\quad + \overline{x}_2\overline{x}_3\odot e^{x_1}(3\alpha_3^2-\beta_3^2).
\end{aligned}
$$

6.4.3 Mean value and Poisson formulas

Exploiting the harmonic properties of spherical derivatives of slice regular functions in the quaternionic case, we are able to derive mean value and Poisson formulas. They first appeared in [11, 70] in the one variable case.

Let σ be the surface measure of $\mathbb{S}^3=\partial\mathbb{B}^4\subset\mathbb{H}\cong\mathbb{R}^4$ such that $\sigma\left(\mathbb{S}^3\right)=1$, namely $\sigma(y):=\mathcal{H}^3(y)/\omega_3$, where $\mathcal{H}^3$ denotes the three-dimensional Hausdorff measure of $\mathbb{R}^4$ and $\omega_3:=\mathcal{H}^3\left(\mathbb{S}^3\right)=2\pi^2$. Again, by σ^l we mean the l-th power of σ.

Important remark. Throughout the section we will always assume that $f\in\mathcal{SR}(\Omega_D)$ is a slice regular quaternionic valued function and for $a=(a_1,...,a_n)\in\Omega_D$ and $r_1,...,r_n\in\mathbb{R}^+\cup\{0\}$ it holds

$$\overline{B_{r_1}(a_1)}\times...\times\overline{B_{r_n}(a_n)}\subset\Omega_D.$$

Proposition 6.17. *Let $f\in\mathcal{SR}(\Omega_D)$, then for every $H=\{h_1,\dots,h_s\}\in\mathcal{P}(n)$ and $K\subset H$ it holds*

$$(\mathcal{S}^H_K(f))(a)=\int_{(\mathbb{S}^3)^{|H|}}\left(\mathcal{S}^H_K(f)\right)(a')\,d\sigma^{|H|}(\lambda),\tag{61}$$

with $a' = (a_1, \dots, a_{h_1} + r_{h_1}\lambda_{h_1}, \dots, a_{h_s} + r_{h_s}\lambda_{h_s}, \dots, a_n)$.

Proof. By Theorem 6.6, every $\mathcal{S}_K^H(f)$ is separately harmonic with respect to x_h, for every $h \in H$, thus, we can apply the classical mean value formula for harmonic functions for such variables

$$\left(\mathcal{S}_K^H(f)\right)(a) = \int_{\mathbb{S}^3} \left(\mathcal{S}_K^H(f)\right)(a')d\sigma(\lambda_h), \qquad \forall h \in H,$$

with $a' = (a_1, \dots, a_{h-1}, a_h + r_h\lambda_h, a_{h+1}, \dots, a_n)$. Thus, (61) follows applying the previous formula for any $h \in H$. □

Proposition 6.18 (First mean value formula)**.** *For any* $m = 1, ..., n$ *it holds*

$$f(a) = \sum_{K \in \mathcal{P}(m)} (-1)^{|K^c|}\overline{a}_{K^c} \int_{(\mathbb{S}^3)^m} \mathcal{S}_K^{[\![m]\!]}(f)(a')d\sigma^m(\lambda), \tag{62}$$

with $a' = (a_1 + r_1\lambda_1, \dots, a_m + r_m\lambda_m, a_{m+1}, \dots, a_n)$.

Proof. Apply (56) and (61) with $H = \{1, ..., m\}$

$$f(a) = \sum_{K \in \mathcal{P}(m)} (-1)^{|K^c|}\overline{a}_{K^c}\mathcal{S}_K^{[\![m]\!]}(f)(a) = \sum_{K \in \mathcal{P}(m)} (-1)^{|K^c|}\overline{a}_{K^c} \int_{(\mathbb{S}^3)^m} \mathcal{S}_K^{[\![m]\!]}(f)(a')d\sigma^m(\lambda).$$

□

We can give integral formulas through the general decomposition (50), but we must assume that the centers of the spheres are real.

Proposition 6.19. *With the notation of Proposition 6.17, let* $H \in \mathcal{P}(n)$ *and assume* $a_h \in \mathbb{R}$, *for any* $h \in H$. *Then*

$$f(a) = \sum_{K \subset H} (-1)^{|H \setminus K|} a_{H \setminus K} \int_{(\mathbb{S})^{|H|}} \mathcal{S}_K^H(f)(a')d\sigma^{|H|}(\lambda).$$

Proof. By (50) and (61) we have

$$\begin{aligned} f(a) &= \sum_{K \subset H} (-1)^{|H \setminus K|} \left[\overline{x}_{H \setminus K} \odot \mathcal{S}_K^H(f)\right](a) = \sum_{K \subset H} (-1)^{|H \setminus K|} a_{H \setminus K}\mathcal{S}_K^H(f)(a) = \\ &= \sum_{K \subset H} (-1)^{|H \setminus K|} a_{H \setminus K} \int_{(\mathbb{S}^3)^{|H|}} \left(\mathcal{S}_K^H(f)\right)(a')\, d\sigma^{|H|}(\lambda), \end{aligned}$$

where we have used that $a_h \in \mathbb{R}$, $\forall h \in H$. □

We give another integral formula through decomposition (56). For $m \geq 2$, it highly differs from (62), beacause of the components involved and the dimension of the domain of integration. On the contrary, they coincide if $m = 1$.

Proposition 6.20 (Second mean value formula)**.** *For any* $m = 1, ..., n$ *it holds*

$$\begin{aligned} f(a) = &\sum_{j=0}^{m-1} r_{[\![j]\!]} \int_{(\mathbb{S}^3)^{j+1}} \overline{\lambda}_{[\![j]\!]}\mathcal{S}_\emptyset^{[\![j]\!]}(f)(a^{\prime,j})d\sigma^{j+1}(\lambda) + \\ &+ r_{[\![m]\!]} \int_{(\mathbb{S}^3)^m} \overline{\lambda}_{[\![m]\!]}\mathcal{S}_\emptyset^{[\![m]\!]}(f)(a^{\prime,m})d\sigma^m(\lambda), \end{aligned} \tag{63}$$

where $a^{\prime,j} = (a_1 + r_1\lambda_1, \dots, a_j + r_j\lambda_j, a_{j+1}, \dots, a_n)$, *for every* $j = 0, \dots, m$ *and* $r_\emptyset = \overline{\lambda}_\emptyset = 1$.

Proof. We prove the identity by induction over m, using the corresponding one-variable formula [70, Proposition 2] that we can apply iteratively, since $\mathcal{S}_K^{[\![m]\!]}(f) \in \mathcal{SR}_{m+1}(\Omega_D)$. For $m=1$, (63) is precisely [70, Proposition 2], indeed

$$\begin{aligned} f(a) &= \int_{\mathbb{S}^3} f(a_1 + r_1\lambda_1, a_2, \dots, a_n) d\sigma(\lambda_1) + r_1 \int_{\mathbb{S}^3} \overline{\lambda}_1 \left(\mathcal{S}_\emptyset^{[\![1]\!]}(f)\right)(a_1 + r_1\lambda_1, a_2, \dots, a_n) d\sigma(\lambda_1) \\ &= \int_{\mathbb{S}^3} f(a_1 + r_1\lambda_1, a_2, \dots, a_n) d\sigma(\lambda_1) + r_1 \int_{\mathbb{S}^3} \overline{\lambda}_1 f'_{s,1}(a_1 + r_1\lambda_1, a_2, \dots, a_n) d\sigma(\lambda_1). \end{aligned}$$

Now, suppose that the formula holds for some m, then, apply [70, Proposition 2] to $\mathcal{S}_\emptyset^{[\![m]\!]}(f)_{m+1}^{a',m} \in \mathcal{SR}(\Omega_{D,m+1}(a'^{,m}))$:

$$\begin{aligned} f(a) = &\sum_{j=0}^{m-1} r_{[\![j]\!]} \int_{(\mathbb{S}^3)^{j+1}} \overline{\lambda}_{[\![j]\!]} \mathcal{S}_\emptyset^{[\![j]\!]}(f)(a'^{,j+1}) d\sigma^{j+1}(\lambda) + \\ &+ r_{[\![m]\!]} \int_{(\mathbb{S}^3)^m} \overline{\lambda}_{[\![m]\!]} \mathcal{S}_\emptyset^{[\![m]\!]}(f)(a'^{,m}) d\sigma^m(\lambda) \\ &= \sum_{j=0}^{m-1} r_{[\![j]\!]} \int_{(\mathbb{S}^3)^{j+1}} \overline{\lambda}_{[\![j]\!]} \mathcal{S}_\emptyset^{[\![j]\!]}(f)(a'^{,j+1}) d\sigma^{j+1}(\lambda) + \\ &+ r_{[\![m]\!]} \int_{(\mathbb{S}^3)^m} \overline{\lambda}_{[\![m]\!]} \left[\int_{\mathbb{S}^3} \mathcal{S}_\emptyset^{[\![m]\!]}(f)(a'^{,m+1}) d\sigma(\lambda_{m+1})\right] d\sigma^m(\lambda) + \\ &+ r_{[\![m]\!]} \int_{(\mathbb{S}^3)^m} \overline{\lambda}_{[\![m]\!]} \left[r_{m+1} \int_{\mathbb{S}^3} \overline{\lambda}_{m+1} \mathcal{S}_\emptyset^{[\![m+1]\!]}(f)(a'^{,m+1}) d\sigma(\lambda_{m+1})\right] d\sigma^m(\lambda) \\ &= \sum_{j=0}^{m-1} r_{[\![j]\!]} \int_{(\mathbb{S}^3)^{j+1}} \overline{\lambda}_{[\![j]\!]} \mathcal{S}_\emptyset^{[\![j]\!]}(f)(a'^{,j+1}) d\sigma^{j+1}(\lambda) + \\ &+ r_{[\![m]\!]} \int_{(\mathbb{S}^3)^{m+1}} \overline{\lambda}_{[\![m]\!]} \mathcal{S}_\emptyset^{[\![m]\!]}(f)(a'^{,m+1}) d\sigma^{m+1}(\lambda) + \\ &+ r_{[\![m+1]\!]} \int_{(\mathbb{S}^3)^{m+1}} \overline{\lambda}_{[\![m+1]\!]} \mathcal{S}_\emptyset^{[\![m+1]\!]}(f)(a'^{,m+1}) d\sigma^{m+1}(\lambda) \\ &= \sum_{j=0}^{m} r_{[\![j]\!]} \int_{(\mathbb{S}^3)^{j+1}} \overline{\lambda}_{[\![j]\!]} \mathcal{S}_\emptyset^{[\![j]\!]}(f)(a'^{,j+1}) d\sigma^{j+1}(\lambda) + \\ &+ r_{[\![m+1]\!]} \int_{(\mathbb{S}^3)^{m+1}} \overline{\lambda}_{[\![m+1]\!]} \mathcal{S}_\emptyset^{[\![m+1]\!]}(f)(a'^{,m+1}) d\sigma^{m+1}(\lambda). \end{aligned}$$

□

In the rest of the section we mimic what has been done so far, but with the Poisson kernel: first we find Poisson formulas for the components $\mathcal{S}_K^{[\![m]\!]}(f)$ and finally two types of formulas for f.

Proposition 6.21. *Let $x_1, ..., x_m \in \mathbb{B}_\mathbb{H}$, then it holds*

$$\mathcal{S}_K^{[\![m]\!]}(f)(a') = \int_{(\mathbb{S}^3)^m} \mathcal{S}_K^{[\![m]\!]}(f)(a") \prod_{j=1}^m P(x_j, \xi_j) d\sigma^m(\xi), \tag{64}$$

where $a' = (a_1+r_1x_1, \dots, a_m+r_mx_m, a_{m+1}, \dots, a_n)$, $a" = (a_1+r_1\xi_1, \dots, a_m+r_m\xi_m, a_{m+1}, \dots, a_n)$ *and* $P(x,\xi) := \dfrac{1-|x|^2}{|x-\xi|^4}$ *is the Poisson kernel of* $\mathbb{B} \subset \mathbb{R}^4$.

Proof. By Corollary 6.11, every $\mathcal{S}_K^{[\![m]\!]}(f)$ is harmonic with respect to $x_1, ..., x_m$, so by Poisson integral formula for harmonic functions it holds for any $k = 1, ..., m$

$$\mathcal{S}_K^{[\![m]\!]}(f)(a') = \int_{\mathbb{S}^3} \mathcal{S}_K^{[\![m]\!]}(f)(\tilde{a}) P(x_k, \xi_k) d\sigma(\xi_k),$$

with $\tilde{a} = (a_1+r_1x_1, \dots, a_{k-1}+r_{k-1}x_{k-1}, a_k+r_k\xi_k, a_{k+1}+r_{k+1}x_{k+1}, \dots, a_m+r_mx_m, a_{m+1}, \dots, a_n)$. Thus, (64) follows by applying the previous formula for $k = 1, ..., m$. ☐

Proposition 6.22 (First Poisson formula)**.** *With the notation of Proposition 6.21, let* $m = 1, ..., n$, *then it holds*

$$f(a') = \sum_{K \in \mathcal{P}(m)} (-1)^{|K^c|} (\overline{a} + r\overline{x})_{K^c} \int_{(\mathbb{S}^3)^m} \mathcal{S}_K^{[\![m]\!]}(f)(a") \prod_{j=1}^m P(x_j, \xi_j) d\sigma^m(\xi).$$

Proof. Apply (56) and (64) to every $\mathcal{S}_K^{[\![m]\!]}(f)$, to get

$$\begin{aligned} f(a') &= \sum_{K \in \mathcal{P}(m)} (-1)^{|K^c|} (\overline{a} + r\overline{x})_{K^c} \mathcal{S}_K^{[\![m]\!]}(a') \\ &= \sum_{K \in \mathcal{P}(m)} (-1)^{|K^c|} (\overline{a} + r\overline{x})_{K^c} \int_{(\mathbb{S}^3)^m} \mathcal{S}_K^{[\![m]\!]}(f)(a") \prod_{j=1}^m P(x_j, \xi_j) d\sigma^m(\xi). \end{aligned}$$

☐

Proposition 6.23 (Second Poisson formula)**.** *Let* $m = 1, ..., n$ *and* $x_1, ..., x_m \in \mathbb{B}_{\mathbb{H}}$, *then it holds*

$$\begin{aligned} f(a') = \sum_{j=1}^{m-1} r_{[\![j]\!]} \int_{(\mathbb{S}^3)^{j+1}} (\overline{\xi} - \overline{x})_{[\![j]\!]} (\mathcal{S}_\emptyset^{[\![j]\!]}(f))(a'^{,j+1}) \prod_{t=1}^{j+1} P(x_t, \xi_t) d\sigma^{j+1}(\xi) + \\ + r_{[\![m]\!]} \int_{(\mathbb{S}^3)^m} (\overline{\xi} - \overline{x})_{[\![m]\!]} \mathcal{S}_\emptyset^{[\![m]\!]}(f)(a'^{,m}) \prod_{t=1}^m P(x_t, \xi_t) d\sigma^m(\xi), \end{aligned}$$

where $a'^{,j} = (a_1 + r_1\xi, \dots, a_j + r_j\xi, a_{j+1}, \dots, a_n)$, *for every* $j = 1, \dots, m$.

Proof. The proof is analogue of the one of Proposition 6.20, but here apply [70, Proposition 3]. ☐

6.4.4 Applications in $(\mathbb{R}_m)^n$

Corollary 6.24. *Suppose that* $\Omega_D \subset (\mathbb{R}^{m+1})^n$ *is a star-like domain w.r.t any variable,* $f : \Omega_D \to \mathbb{R}_m$ *is a slice regular function and for* $H \in \mathcal{P}(n)$, *let*

$$f(x) = \sum_{K \subset H} (-1)^{|H \setminus K|} (\overline{x})_{H \setminus K} \odot \mathcal{S}_K^H(f)(x)$$

be the Almansi decomposition of f with respect to H. Then, for any $G \subset H$, we can further decompose

$$f(x) = \sum_{K \subset H} \sum_{\substack{T \in [\![0, \frac{m-3}{2}]\!]^{|G|}, \\ T=(t_1,\dots,t_{|G|})}} (-1)^{|H \setminus K|} |x_G|^{2T} \left(\overline{x}\right)_{H \setminus K} \odot \mathcal{E}^{H,G}_{K,T}(f)(x), \tag{65}$$

with $\mathcal{E}^{H,G}_{K,T}(f)(x) \in \ker \Delta_{m+1,G}$, where

$$|x_G|^{2T} := |x_{g_1}|^{2t_1} \cdot \dots \cdot |x_{g_s}|^{2t_s},$$

if $G = (g_1, \dots, g_s)$ and $T = (t_1, \dots, t_s)$.

Proof. Let us prove (65) by induction over $|G|$. Suppose firts that $G = \{g\} \subset H$, then, since $\mathcal{S}^H_K(f) \in \ker \Delta^{\frac{m-1}{2}}_{m+1,H}$, for any $K \subset H$ and since Ω_D is a star-like domain with respect to x_g, by classical Almansi decomposition (Theorem 6.1) there exist $\mathcal{E}^{H,\{g\}}_{K,0}, \dots, \mathcal{E}^{H,\{g\}}_{K,\frac{m-3}{2}} \in \ker \Delta_{m+1,g}$ such that, for any $K \subset H$,

$$\mathcal{S}^H_K(f)(x) = \sum_{j=0}^{\frac{m-3}{2}} |x_g|^{2j} \mathcal{E}^{H,\{g\}}_{K,j}(x)$$

and so

$$f(x) = \sum_{K \subset H} (-1)^{|H \setminus K|} \left(\overline{x}\right)_{H \setminus K} \odot \mathcal{S}^H_K(f)(x) = \sum_{K \subset H} \sum_{j=0}^{\frac{m-3}{2}} (-1)^{|H \setminus K|} |x_g|^{2j} \left(\overline{x}\right)_{H \setminus K} \odot \mathcal{E}^{H,\{g\}}_{K,j}(x).$$

Now, suppose that (65) holds for some $G \subset H$ and let us prove it for $\tilde{G} = G \cup \{g\}$, for some $g \in H \setminus G$. By induction, we have that

$$f(x) = \sum_{K \subset H} \sum_{\substack{T \in [\![0, \frac{m-3}{2}]\!]^{|G|}, \\ T=(t_1,\dots,t_{|G|})}} (-1)^{|H \setminus K|} |x_G|^{2T} \overline{x}_{H \setminus K} \odot \mathcal{E}^{H,G}_{K,T}(f)(x),$$

with $\mathcal{E}^{H,G}_{K,T}(f) \in \ker \Delta^{\frac{m-1}{2}}_{m+1,g}$, for every $K \subset H$ and $T \subset G$. Thus, by Theorem 6.1, for any $K \subset H$, $T \subset G$ there exist $\{\mathcal{E}^{H,G\cup\{g\}}_{K,T\cup t_g}(f)\}_{t_g=0}^{\frac{m-3}{2}} \in \ker \Delta_{m+1,g}$ such that

$$\mathcal{E}^{H,G}_{K,T}(f)(x) = \sum_{t_g=0}^{\frac{m-3}{2}} |x_g|^{2t_g} \mathcal{E}^{H,G\cup\{g\}}_{K,T\cup t_g}(f)(x)$$

and so

$$\begin{aligned} f(x) &= \sum_{K \subset H} \sum_{\substack{T \in [\![0, \frac{m-3}{2}]\!]^{|G|}, \\ T=(t_1,\dots,t_{|G|})}} (-1)^{|H \setminus K|} |x_G|^{2T} \overline{x}_{H \setminus K} \odot \sum_{t_g=0}^{\frac{m-3}{2}} |x_g|^{2t_g} \mathcal{E}^{H,G\cup\{g\}}_{K,T\cup t_g}(f)(x) \\ &= \sum_{K \subset H} \sum_{\substack{\tilde{T} \in [\![0, \frac{m-3}{2}]\!]^{|G|+1}, \\ \tilde{T}=T\cup t_g}} (-1)^{|H \setminus K|} |x_{G\cup\{g\}}|^{2\tilde{T}} \overline{x}_{H \setminus K} \odot \mathcal{E}^{H,G\cup\{g\}}_{K,\tilde{T}}(f)(x) \\ &\sum_{K \subset H} \sum_{\substack{\tilde{T} \in [\![0, \frac{m-3}{2}]\!]^{|G|+1}, \\ \tilde{T}=(t_1,\dots,t_{|G|+1})}} (-1)^{|H \setminus K|} |x_{G\cup\{g\}}|^{2\tilde{T}} \overline{x}_{H \setminus K} \odot \mathcal{E}^{H,G\cup\{g\}}_{K,\tilde{T}}(f)(x). \end{aligned}$$

□

Example 11. Let $f:(\mathbb{R}^6)^2 \to \mathbb{R}_5$, $f(x_1,x_2)=x_1^4x_2^7$. Then, choosing $H=\{1,2\}$, we can decompose f as

$$\begin{aligned}
f(x_1,x_2) &= \sum_{K\subset\{1,2\}} (-1)^{\{1,2\}\setminus K}\overline{x}_{\{1,2\}\setminus K}\mathcal{S}_K^{\{1,2\}}(f)(x_1,x_2)\\
&= \mathcal{S}_{1,2}^{\{1,2\}} - \overline{x}_1\mathcal{S}_2^{\{1,2\}} - \overline{x}_2\mathcal{S}_1^{\{1,2\}} + \overline{x}_1\overline{x}_2\mathcal{S}_\emptyset^{\{1,2\}}\\
&= (x_1^5)'_{s,1}(x_2^8)'_{s,2} - \overline{x}_1(x_1^4)'_{s,1}(x_2^8)'_{s,2} - \overline{x}_2(x_1^5)'_{s,1}(x_2^7)'_{s,2} + \overline{x}_1\overline{x}_2(x_1^4)'_{s,1}(x_2^7)'_{s,2}\\
&= (5\alpha_1^4 - 10\alpha_1^2\beta_1^2 + \beta_1^4)(8\alpha_2^7 - 56\alpha_2^5\beta_2^2 + 56\alpha_2^3\beta_2^4 - 8\alpha_2\beta_2^6)+\\
&\quad - (\alpha_1 - J_1\beta_1)(4\alpha_1^3 - 4\alpha_1\beta_1^2)(8\alpha_2^7 - 56\alpha_2^5\beta_2^2 + 56\alpha_2^3\beta_2^4 - 8\alpha_2\beta_2^6)+\\
&\quad - (\alpha_2 - J_2\beta_2)(5\alpha_1^4 - 10\alpha_1^2\beta_1^2 + \beta_1^4)(7\alpha_2^6 - 35\alpha_2^4\beta_2^2 + 21\alpha_2^2\beta_2^4 - \beta_2^6)+\\
&\quad + (\alpha_1 - J_1\beta_1)(\alpha_2 - J_2\beta_2)(4\alpha_1^3 - 4\alpha_1\beta_1^2)(7\alpha_2^6 - 35\alpha_2^4\beta_2^2 + 21\alpha_2^2\beta_2^4 - \beta_2^6).
\end{aligned}$$

Note that by Proposition 5.2, for any $K\subset\{1,2\}$, $\mathcal{S}_K^{\{1,2\}}(f)\in\ker\Delta_6^2$, hence we can further decompose

$$\mathcal{S}_K^{\{1,2\}}(f) = \sum_{T=(t_1,t_2)\in\{0,1\}^2} |x_1|^{2t_1}|x_2|^{2t_2}\mathcal{E}_{K,T}^{\{1,2\}}(f),$$

with $\mathcal{E}_{K,T}^{\{1,2\}}(f)\in\ker\Delta_6$. This correspond to the choice $G=\{1,2\}$ in Corollary 6.24. Explicitely, using the computation of Exalmples 7 and 8,

$$\begin{aligned}
\mathcal{S}_{1,2}^{\{1,2\}}(f) &= \mathcal{E}_{\{1,2\},(0,0)}^{\{1,2\}}(f) + |x_1|^2\mathcal{E}_{\{1,2\},(1,0)}^{\{1,2\}}(f) + |x_2|^2\mathcal{E}_{\{1,2\},(0,1)}^{\{1,2\}}(f) + |x_1|^2|x_2|^2\mathcal{E}_{\{1,2\},(1,1)}^{\{1,2\}}(f)\\
&= \left(7\alpha_1^4 - \frac{42}{5}\alpha_1^2\beta_1^2 + \frac{3}{5}\beta_1^4\right)\left(15\alpha_2^7 - 63\alpha_2^5\beta_2^2 + 45\alpha_2^3\beta_2^4 - 5\alpha_2\beta_2^6\right)+\\
&\quad + |x_1|^2\left(-2\alpha_1^2 + \frac{2}{5}\beta_1^2\right)(15\alpha_2^7 - 63\alpha_2^5\beta_2^2 + 45\alpha_2^3\beta_2^4 - 5\alpha_2\beta_2^6)+\\
&\quad + |x_2|^2\left(7\alpha_1^4 - \frac{42}{5}\alpha_1^2\beta_1^2 + \frac{3}{5}\beta_1^4\right)(-7\alpha_2^5 + 14\alpha_2^3\beta_2^2 - 3\alpha_2\beta_2^4)+\\
&\quad + |x_1|^2|x_2|^2\left(-2\alpha_1^2 + \frac{2}{5}\beta_1^2\right)(-7\alpha_2^5 + 14\alpha_2^3\beta_2^2 - 3\alpha_2\beta_2^4);\\
\mathcal{S}_2^{\{1,2\}}(f) &= \mathcal{E}_{\{2\},(0,0)}^{\{1,2\}}(f) + |x_1|^2\mathcal{E}_{\{2\},(1,0)}^{\{1,2\}}(f) + |x_2|^2\mathcal{E}_{\{2\},(0,1)}^{\{1,2\}}(f) + |x_1|^2|x_2|^2\mathcal{E}_{\{2\},(1,1)}^{\{1,2\}}(f)\\
&= \left(5\alpha_1^3 - 3\alpha_1\beta_1^2\right)\left(15\alpha_2^7 - 63\alpha_2^5\beta_2^2 + 45\alpha_2^3\beta_2^4 - 5\alpha_2\beta_2^6\right)+\\
&\quad + |x_1|^2(-\alpha_1)(15\alpha_2^7 - 63\alpha_2^5\beta_2^2 + 45\alpha_2^3\beta_2^4 - 5\alpha_2\beta_2^6)+\\
&\quad + |x_2|^2\left(5\alpha_1^3 - 3\alpha_1\beta_1^2\right)(-7\alpha_2^5 + 14\alpha_2^3\beta_2^2 - 3\alpha_2\beta_2^4)+\\
&\quad + |x_1|^2|x_2|^2(-\alpha_1)(-7\alpha_2^5 + 14\alpha_2^3\beta_2^2 - 3\alpha_2\beta_2^4);\\
\mathcal{S}_1^{\{1,2\}}(f) &= \mathcal{E}_{\{1\},(0,0)}^{\{1,2\}}(f) + |x_1|^2\mathcal{E}_{\{1\},(1,0)}^{\{1,2\}}(f) + |x_2|^2\mathcal{E}_{\{1\},(0,1)}^{\{1,2\}}(f) + |x_1|^2|x_2|^2\mathcal{E}_{\{1\},(1,1)}^{\{1,2\}}(f)\\
&= \left(7\alpha_1^4 - \frac{42}{5}\alpha_1^2\beta_1^2 + \frac{3}{5}\beta_1^4\right)\left(12\alpha_2^6 - 36\alpha_2^4\beta_2^2 + \frac{108}{7}\alpha_2^2\beta_2^4 - \frac{4}{7}\beta_2^6\right)+\\
&\quad + |x_1|^2\left(-2\alpha_1^2 + \frac{2}{5}\beta_1^2\right)\left(12\alpha_2^6 - 36\alpha_2^4\beta_2^2 + \frac{108}{7}\alpha_2^2\beta_2^4 - \frac{4}{7}\beta_2^6\right)+\\
&\quad + |x_2|^2\left(7\alpha_1^4 - \frac{42}{5}\alpha_1^2\beta_1^2 + \frac{3}{5}\beta_1^4\right)\left(-5\alpha_2^4 + 6\alpha_2^2\beta_2^2 - \frac{3}{7}\beta_2^4\right)+\\
&\quad + |x_1|^2|x_2|^2\left(-2\alpha_1^2 + \frac{2}{5}\beta_1^2\right)\left(-5\alpha_2^4 + 6\alpha_2^2\beta_2^2 - \frac{3}{7}\beta_2^4\right);
\end{aligned}$$

$$
\begin{aligned}
\mathcal{S}_{\emptyset}^{\{1,2\}}(f) &= \mathcal{E}_{\emptyset,(0,0)}^{\{1,2\}}(f) + |x_1|^2\mathcal{E}_{\emptyset,(1,0)}^{\{1,2\}}(f) + |x_2|^2\mathcal{E}_{\emptyset,(0,1)}^{\{1,2\}}(f) + |x_1|^2|x_2|^2\mathcal{E}_{\emptyset,(1,1)}^{\{1,2\}}(f) \\
&= \left(5\alpha_1^3 - 3\alpha_1\beta_1^2\right)\left(12\alpha_2^6 - 36\alpha_2^4\beta_2^2 + \frac{108}{7}\alpha_2^2\beta_2^4 - \frac{4}{7}\beta_2^6\right) + \\
&+ |x_1|^2\left(-\alpha_1\right)\left(12\alpha_2^6 - 36\alpha_2^4\beta_2^2 + \frac{108}{7}\alpha_2^2\beta_2^4 - \frac{4}{7}\beta_2^6\right) + \\
&+ |x_2|^2\left(5\alpha_1^3 - 3\alpha_1\beta_1^2\right)\left(-5\alpha_2^4 + 6\alpha_2^2\beta_2^2 - \frac{3}{7}\beta_2^4\right) + \\
&+ |x_1|^2|x_2|^2\left(-\alpha_1\right)\left(-5\alpha_2^4 + 6\alpha_2^2\beta_2^2 - \frac{3}{7}\beta_2^4\right).
\end{aligned}
$$

Thus, we can fully decompose f as

$$
\begin{aligned}
f = {} & \mathcal{E}_{\{1,2\},(0,0)}^{\{1,2\}}(f) + |x_1|^2\mathcal{E}_{\{1,2\},(1,0)}^{\{1,2\}}(f) + |x_2|^2\mathcal{E}_{\{1,2\},(0,1)}^{\{1,2\}}(f) + |x_1|^2|x_2|^2\mathcal{E}_{\{1,2\},(1,1)}^{\{1,2\}}(f) + \\
& - \overline{x}_1\mathcal{E}_{\{2\},(0,0)}^{\{1,2\}}(f) - \overline{x}_1|x_1|^2\mathcal{E}_{\{2\},(1,0)}^{\{1,2\}}(f) - \overline{x}_1|x_2|^2\mathcal{E}_{\{2\},(0,1)}^{\{1,2\}}(f) - \overline{x}_1|x_1|^2|x_2|^2\mathcal{E}_{\{2\},(1,1)}^{\{1,2\}}(f) + \\
& - \overline{x}_2\mathcal{E}_{\{1\},(0,0)}^{\{1,2\}}(f) - \overline{x}_2|x_1|^2\mathcal{E}_{\{1\},(1,0)}^{\{1,2\}}(f) - \overline{x}_2|x_2|^2\mathcal{E}_{\{1\},(0,1)}^{\{1,2\}}(f) - \overline{x}_2|x_1|^2|x_2|^2\mathcal{E}_{\{1\},(1,1)}^{\{1,2\}}(f) + \\
& + \overline{x}_1\overline{x}_2\mathcal{E}_{\emptyset,(0,0)}^{\{1,2\}}(f) + \overline{x}_1\overline{x}_2|x_1|^2\mathcal{E}_{\emptyset,(1,0)}^{\{1,2\}}(f) + \overline{x}_1\overline{x}_2|x_2|^2\mathcal{E}_{\emptyset,(0,1)}^{\{1,2\}}(f) + \overline{x}_1\overline{x}_2|x_1|^2|x_2|^2\mathcal{E}_{\emptyset,(1,1)}^{\{1,2\}}(f).
\end{aligned}
$$

7 Clifford Analysis and Fueter theorem in several variables

7.1 Fueter regular functions and Clifford Analysis

We recall two other operators on $\mathbb{H}$, and $\mathbb{R}_m$, known as Cauchy-Riemann-Fueter operators or Dirac operators.

Definition 7.1. Let $\Omega \subset \mathbb{H}$ be an open set and let

$$\partial_{CRF} := \frac{1}{2}\left(\frac{\partial}{\partial\alpha} - i\frac{\partial}{\partial\beta} - j\frac{\partial}{\partial\gamma} - k\frac{\partial}{\partial\delta}\right), \qquad \overline{\partial}_{CRF} := \frac{1}{2}\left(\frac{\partial}{\partial\alpha} + i\frac{\partial}{\partial\beta} + j\frac{\partial}{\partial\gamma} + k\frac{\partial}{\partial\delta}\right),$$

where α, β, γ and δ denotes the four real components of a quaternion $x = \alpha + i\beta + j\gamma + k\delta$. A function $f : \Omega \to \mathbb{H}$ in the kernel of $\overline{\partial}_{CRF}$ is called Fueter regular function.

Definition 7.2. Let $\Omega \subset \mathbb{R}^{m+1}$ be an open set and let

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x_0} - \sum_{i=1}^{k} e_i \frac{\partial}{\partial x_i}\right), \qquad \overline{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x_0} + \sum_{i=1}^{k} e_i \frac{\partial}{\partial x_i}\right). \tag{66}$$

A function $f : \Omega \to \mathbb{R}_m$ is called monogenic if $\overline{\partial} f = 0$. We will denote by $\mathcal{M}(\Omega)$ the set of monogenic functions with domain Ω. The symbol $\mathcal{AM}(\Omega_D) = \mathcal{S}(\Omega_D) \cap \mathcal{M}(\Omega_D)$ is used for monogenic functions that are also slice functions. They are known as axially monogenic functions.

The importance of these operators is evident as they factorize the Laplacian, indeed

$$4\partial_{CRF}\overline{\partial}_{CRF} = 4\overline{\partial}_{CRF}\partial_{CRF} = \Delta_4, \qquad 4\partial\overline{\partial} = 4\overline{\partial}\partial = \Delta_{m+1}. \tag{67}$$

Next definition allows us to treat uniformly the quaternionic and the Clifford algebras case. It was first introduced in [53].

Definition 7.3. A non empty subset S of A is called a genuine imaginary sphere of A if there exist a vector subspace M of A, with $\mathbb{R} \subset M \subset \mathcal{Q}_A$ such that $S = \mathbb{S}_A \cap M$. If such M exist it is unique and it holds

$$M = \bigcup_{I \in S} \mathbb{C}_I.$$

If $\dim(M) > 2$, we say that M is a hypercomplex subspace of A.

Remark 14. When $A = \mathbb{H}$, the whole algebra itself is a hypercomplex subspace of $\mathbb{H}$, with $S = \mathbb{S}_{\mathbb{H}}$. Moreover, also the reduced quaternions $\mathbb{H}_r = \{\alpha + i\beta + j\gamma \mid \alpha, \beta, \gamma \in \mathbb{R}\}$ form an hypercomplex subspace of $\mathbb{H}$, with genuine imaginary sphere $S = \{i\beta + j\gamma \in \mathbb{H} \mid \beta^2 + \gamma^2 = 1\}$.

More generally, when $A = \mathbb{R}_m$, the paravector suspace $\mathbb{R}^{m+1}$ is always contained in the quadratic cone $\mathcal{Q}_{\mathbb{R}_m}$ (Remark 1) and it is a hypercomplex subspace, indeed we can take as genuine imaginary sphere $S = \{x_1 e_i + \dots x_m e_m \mid x_1^2 + \dots x_m^2 = 1\}$.

7.2 Fueter and Fueter-Sce Theorem

Lemma 7.1 (Proposition 9, [72]). *Let M be a hypercomplex subspace and let $m+1 = \dim(M)$. Let $\Omega_D \subset M$, then for any slice regular function $f \in \mathcal{SR}(\Omega_D)$ it holds*

1. $\overline{\partial} f = \frac{1-m}{2} f_s'$;

2. $\Delta_{m+1} f = 2(1-m)\frac{\partial}{\partial x}(f_s') = 2(1-m)\partial(f_s')$.

Remark 15. When $A = \mathbb{H}$, we take $\mathbb{H}$ itself as hypercomplex subspace, for which $m = 3$. Thus, the previous relations become

$$\overline{\partial}_{CRF} f = -f'_s, \qquad \Delta_4 f = -4\frac{\partial}{\partial x}(f'_s) = -4\partial_{CRF}(f'_s). \tag{68}$$

Theorem 7.2 (Fueter theorem). *Let $\Omega_D \subset \mathbb{H}$ and let $f \in \mathcal{SR}(\Omega_D)$. Then $\Delta_4 f \in \mathcal{AM}(\Omega_D)$, namely*

$$\overline{\partial}_{CRF}\Delta_4 f = 0.$$

Proof. Since $f \in \mathcal{SR}(\Omega_D)$, by (68) and Proposition 5.1 it holds $f'_s = -\overline{\partial}_{CRF} f$ and $f'_s \in \ker \Delta_4$ and so

$$\overline{\partial}_{CRF}\Delta_4 f = \Delta_4 \overline{\partial}_{CRF} f = -\Delta_4 f'_s = 0.$$

□

Theorem 7.3 (Fueter-Sce theorem). *Let m be odd and let $\Omega_D \subset \mathbb{R}^{m+1}$. Then $\Delta_{m+1}^{\frac{m-1}{2}} f \in \mathcal{AM}(\Omega_D)$, namely*

$$\overline{\partial}\Delta_{m+1}^{\frac{m-1}{2}} f = 0.$$

Proof. Proceed as in the proof of Theorem 7.2, but it holds $\overline{\partial} f = \frac{1-m}{2} f'_s$ and $f'_s \in \ker \Delta_{m+1}^{\frac{m-1}{2}}$, by Proposition 5.2, so

$$\overline{\partial}\Delta_{m+1}^{\frac{m-1}{2}} f = \Delta_{m+1}^{\frac{m-1}{2}} \overline{\partial} f = \frac{1-m}{2}\Delta_{m+1}^{\frac{m-1}{2}} f'_s = 0.$$

□

Remark 16. In [77], the previous result was exteded to Clifford algebras with an even number of imaginary units, requiring techniques of fractional differential operators. We will not deal with them in these notes.

7.3 Fueter Theorem in several quaternionic variables

We can extend these operators to $\mathbb{H}^n$: for a slice function $f : \Omega_D \to \mathbb{H}$, we define, for any $h = 1, ..., n$, ∂_{x_h} and $\overline{\partial}_{x_h}$ as the Cauchy-Riemann-Fueter operators with respect to $x_h := \alpha_h + i\beta_h + j\gamma_h + k\delta_h$:

$$\partial_{x_h} := \frac{1}{2}\left(\frac{\partial}{\partial \alpha_h} - i\frac{\partial}{\partial \beta_h} - j\frac{\partial}{\partial \gamma_h} - k\frac{\partial}{\partial \delta_h}\right), \qquad \overline{\partial}_{x_h} := \frac{1}{2}\left(\frac{\partial}{\partial \alpha_h} + i\frac{\partial}{\partial \beta_h} + j\frac{\partial}{\partial \gamma_h} + k\frac{\partial}{\partial \delta_h}\right).$$

Then as before,

$$4\partial_{x_h}\overline{\partial}_{x_h} = 4\overline{\partial}_{x_h}\partial_{x_h} = \Delta_{m+1,h},$$

where $\Delta_h = \frac{\partial^2}{\partial \alpha_h^2} + \frac{\partial^2}{\partial \beta_h^2} + \frac{\partial^2}{\partial \gamma_h^2} + \frac{\partial^2}{\partial \delta_h^2}$. Finally, denote by $\mathcal{M}_h(\Omega) := \{f : \Omega \to \mathbb{H} : \overline{\partial}_{x_h} f = 0\}$ the set of monogenic functions w.r.t x_h and let $\mathcal{AM}_h(\Omega_D) := \mathcal{M}_h(\Omega_D) \cap \mathcal{S}^1(\Omega_D)$ be the set of axially monogenic functions with respect to x_h, i.e. the set of slice functions which are monogenic with respect to x_h. We extend from [74] properties of the spherical derivative of one-variable slice regular functions to several variables.

Lemma 7.4. *If $f \in \mathcal{SR}_h(\Omega_D)$, the following hold:*

1. $\overline{\partial}_{x_h} f = -f'_{s,h}$;

2. $\Delta_h f = -4\dfrac{\partial f'_{s,h}}{\partial x_h} = -4\partial_{x_h}(f'_{s,h})$.

Proof. 1. Note that $\forall y = (y_1, ..., y_n) \in \Omega_D$, $f_h^y \in \mathcal{SR}(\Omega_{D,h}(y))$, then we can apply (24) and [72, Proposition 9] to get

$$\overline{\partial}_{x_h} f(y) = \overline{\partial}_{CRF}(f_h^y)(y_h) = -(f_h^y)'_s(y_h) = -f'_{s,h}(y).$$

2. By (24), [72, Proposition 9] and [48, Theorem 2.2 (ii)] we have

$$\Delta_h f(y) = \Delta(f_h^y)(y_h) = -4\frac{\partial (f_h^y)'_s}{\partial x}(y_h) = -4\theta(f_h^y)'_s(y_h) = -4\partial_{CRF}(f_h^y)'_s(y_h) = -4\partial_{x_h} f'_{s,h}(y),$$

where $(\theta f)(x) = \frac{1}{2}\left(\frac{\partial f}{\partial \alpha}(x) + \frac{\mathrm{Im}(x)}{|\mathrm{Im}(x)|^2}(\beta\frac{\partial f}{\partial \beta}(x) + \gamma\frac{\partial f}{\partial \gamma}(x) + \delta\frac{\partial f}{\partial \delta}(x))\right)$ satisfies $\theta f = \frac{\partial f}{\partial x}$ and $2\theta f'_s = \partial_{CRF} f'_s$ for any slice function f.

□

Theorem 7.5 (Fueter theorem in several variables)**.** *Let $\Omega_D \subset \mathbb{H}^n$ be a circular set and let $f \in \mathcal{SR}_h(\Omega_D)$ be a slice function, which is slice regular with respect to x_h, for some $h = 1, ..., n$. Then $\Delta_h f$ is an axially monogenic function with respect to x_h, i.e.*

$$\Delta_h f \in \ker(\overline{\partial}_{x_h}).$$

In other words, the Fueter map extends to

$$\Delta_h : \mathcal{SR}_h(\Omega_D) \to \mathcal{AM}_h(\Omega_D).$$

Proof. Since $f \in \mathcal{SR}_h(\Omega_D)$, we can apply Lemma 7.4 1. and Corollary 5.9

$$\overline{\partial}_{x_h}\Delta_h f = \Delta_h\overline{\partial}_{x_h} f = -\Delta_h f'_{s,h} = 0.$$

□

We now find other relations with the theory of Fueter regular functions in several variables, thanks to Almansi decomposition. We also give another proof of Fueter theorem for several variables through Almansi decomposition.

Proposition 7.6. *Let $\Omega_D \subset \mathbb{H}^n$ and let $f \in \mathcal{SR}(\Omega_D)$. Then for every $m = 1, \ldots n$, the components of the ordered Almansi decomposition of f, $\mathcal{S}_K^{[\![m]\!]}(f)$ can be written as*

$$\mathcal{S}_K^{[\![m]\!]}(f) = (-1)^m\overline{\partial}_{x_m}(x_m^{\chi_K(m)} \ldots \overline{\partial}_{x_1}(x_1^{\chi_K(1)} f) \ldots). \tag{69}$$

Proof. Recall that if $f \in \mathcal{SR}(\Omega_D)$, by Proposition 4.9 (1), $f \in \mathcal{S}_1(\Omega_D)$ and $f'_{s,[\![j]\!]} \in \mathcal{SR}_{j+1}$, for any $j = 1, \ldots, n-1$. Then, we can iteratively apply 1. of Lemma 7.4 with $h = 1, \ldots, m$ to the definition of $\mathcal{S}_K^{[\![m]\!]}(f)$ to obtain (69). □

The components of the ordered decomposition provide examples of axially monogenic functions.

Proposition 7.7. *Let $f \in \mathcal{SR}(\Omega_D)$ and $m = 1, ..., n-1$. Then $\forall K \in \mathcal{P}(m)$, $\mathcal{S}_K^{[\![m]\!]}(f)$ satisfies*

1. $\partial_{x_m}(\mathcal{S}_K^{[\![m]\!]}(f)) \in \mathcal{AM}_m(\Omega_D)$;

2. $\Delta_{m+1}(\mathcal{S}_K^{[\![m]\!]}(f)) \in \mathcal{AM}_{m+1}(\Omega_D)$.

Proof. By Corollary 6.11, $\mathcal{S}_K^{[\![h]\!]}(f)$ is harmonic with respect to x_j, for any $j = 1, ..., h$, $\forall K \in \mathcal{P}(h)$, so

$$\overline{\partial}_{x_m}\partial_{x_m}\left(\mathcal{S}_K^{[\![m]\!]}(f)\right) = \frac{1}{4}\Delta_m\left(\mathcal{S}_K^{[\![m]\!]}(f)\right) = 0$$

and by Proposition 6.12

$$\overline{\partial}_{x_{m+1}}\Delta_{m+1}\left(\mathcal{S}_K^{[\![m]\!]}(f)\right) = \Delta_{m+1}\overline{\partial}_{x_{m+1}}\left(\mathcal{S}_K^{[\![m]\!]}(f)\right) = -\Delta_{m+1}\left(\mathcal{S}_K^{[\![m+1]\!]}(f)\right) = 0.$$

□

The following result highlights a difference between the one and several variables slice regular functions: in the first case the Laplacian of a slice regular function is always an axially monogenic functions (this is Fueter's Theorem [39]), in the latter this happens only for the first variable. But for any variable, we can at least write it as sum of axially monogenic functions.

Lemma 7.8. *Let $m = 1, ..., n$ and let $f \in \mathcal{S}^1(\Omega_D) \cap \ker(\partial/\partial x_m^c)$, then it holds*

$$\Delta_m f = -4\sum_{K\in\mathcal{P}(m-1)}(-1)^{|K^c|}\,(\overline{x})_{K^c}\,\partial_{x_m}\left(\mathcal{S}_K^{[\![m]\!]}(f)\right).$$

Proof. By Propositions 4.9 (1) and 4.10, it holds $\mathcal{S}_K^{[\![m-1]\!]}(f) = (x_K \odot f)'_{s,[\![m-1]\!]} \in \ker(\partial/\partial x_m^c) \cap \mathcal{S}_m(\Omega_D) = \mathcal{SR}_m(\Omega_D)$, $\forall K \in \mathcal{P}(m-1)$, then by (67) and Lemma 7.4 (1), it holds

$$\Delta_m\mathcal{S}_K^{[\![m-1]\!]}(f) = 4\partial_{x_m}\overline{\partial}_{x_m}\mathcal{S}_K^{[\![m-1]\!]}(f) = -4\partial_{x_m}\left[\left(\mathcal{S}_K^{[\![m-1]\!]}(f)\right)'_{s,m}\right] = -4\partial_{x_m}\left(\mathcal{S}_K^{[\![m]\!]}(f)\right),$$

with $\partial_{x_m}\left(\mathcal{S}_K^{[\![m]\!]}(f)\right) \in \mathcal{AM}_m(\Omega_D)$, by Proposition 7.7, 1. So, applying (56), we have

$$\begin{aligned}\Delta_m f &= \Delta_m\left(\sum_{K\in\mathcal{P}(m-1)}(-1)^{|K^c|}(\overline{x})_{K^c}\mathcal{S}_K^{[\![m-1]\!]}(f)\right) = \sum_{K\in\mathcal{P}(m-1)}(-1)^{|K^c|}(\overline{x})_{K^c}\Delta_m\mathcal{S}_K^{[\![m-1]\!]}(f)\\ &= -4\sum_{K\in\mathcal{P}(m-1)}(-1)^{|K^c|}(\overline{x})_{K^c}\partial_{x_m}\left(\mathcal{S}_K^{[\![m]\!]}(f)\right).\end{aligned}$$

□

The issue changes if we assume the function slice regular in that specific variable, as already proven in [9, Theorem 4.9], getting a generalization of Fueter's Theorem in several variables. We give another proof through the ordered decomposition of Corollary 6.11.

Proof of Theorem 7.5. By Proposition 7.7, Lemma 7.8 and (59) we get

$$\Delta_m f = -4\sum_{K\in\mathcal{P}(m-1)}(-1)^{|K^c|}(\overline{x})_{K^c}\partial_{x_m}\left(\mathcal{S}_K^{[\![m]\!]}(f)\right) = -4\partial_{x_m}\left(\mathcal{S}_{[\![m-1]\!]}^{[\![m]\!]}(f)\right) \in \mathcal{AM}_m(\Omega_D).$$

□

Corollary 7.9. *Every slice regular function is separately biharmonic in each variable.*

Proof. Let $f \in \mathcal{SR}(\Omega_D)$, then thanks to Corollary 6.11 2, Lemma 7.8 and (67) we have

$$\begin{aligned}\Delta_m^2 f &= \Delta_m \left(-4 \sum_{K \in \mathcal{P}(m-1)} (-1)^{|[\![m-1]\!] \setminus K|} \overline{x}_{[\![m-1]\!] \setminus K} \partial_{x_m} \left(\mathcal{S}_K^{[\![m]\!]}(f) \right) \right) \\ &= -4 \sum_{K \in \mathcal{P}(m-1)} (-1)^{|[\![m-1]\!] \setminus K|} \overline{x}_{[\![m-1]\!] \setminus K} \partial_{x_m} \left(\Delta_m \left(\mathcal{S}_K^{[\![m]\!]}(f) \right) \right) = 0.\end{aligned}$$

□

7.4 Fueter-Sce Theorem in several Clifford variables

Let $A = \mathbb{R}_m$ and let $\Omega_D \subset (\mathbb{R}^{m+1})^n$.

Lemma 7.10. *If $f \in \mathcal{SR}_h(\Omega_D)$, the following hold:*

1. $\overline{\partial}_{x_h} f = \frac{1-m}{2} f'_{s,h}$;
2. $\Delta_{m+1,h} f = 2(1-m) \dfrac{\partial f'_{s,h}}{\partial x_h} = 2(1-m) \partial_{x_h}(f'_{s,h})$.

Proof. The proof is equivalent to the one of Lemma 7.4. □

Theorem 7.11 (Futer-Sce theorem in several variables)**.** *Let $\Omega_D \subset (\mathbb{R}^{m+1})^n$ be a circular set and let $f \in \mathcal{SR}_h(\Omega_D)$ be a slice function, which is slice regular with respect to x_h, for some $h = 1, \dots, n$. Then $\Delta_{m+1,h}^{\frac{m-1}{2}} f$ is an axially monogenic function with respect to x_h, namely it holds*

$$\Delta_{m+1,h}^{\frac{m-1}{2}} f \in \ker(\overline{\partial}_{x_h}).$$

Thus, the Fueter-Sce map extends to

$$\Delta_{m+1,h}^{\frac{m-1}{2}} : \mathcal{SR}_h(\Omega_D) \to \mathcal{AM}_h(\Omega_D).$$

Proof. Since $f \in \mathcal{SR}_h(\Omega_D)$, we can apply Lemma 7.4 and Corollary 5.10

$$\overline{\partial}_{x_h} \Delta_{m+1,h}^{\frac{m-1}{2}} f = \Delta_{m+1,h}^{\frac{m-1}{2}} \overline{\partial}_{x_h} f = \frac{1-m}{2} \Delta_{m+1,h}^{\frac{m-1}{2}} f'_{s,h} = 0.$$

□

As before, we use Almansi decomposition for several Clifford variables to find new relations with the theory of axially monogenic functions.

Proposition 7.12. *Let $\Omega_D \subset (\mathbb{R}^{m+1})^n$ and let $f \in \mathcal{SR}(\Omega_D)$. Then for every $h = 1, \dots n$, the components of the ordered Almansi decomposition of f, $\mathcal{S}_K^{[\![m]\!]}(f)$ can be written as*

$$\mathcal{S}_K^{[\![m]\!]}(f) = \left(\frac{1-m}{2} \right)^m \overline{\partial}_{x_m} (x_m^{\chi_K(m)} \dots \overline{\partial}_{x_1} (x_1^{\chi_K(1)} f) \dots). \tag{70}$$

Proof. The proof is analogue to the one of Proposition 7.6, but applying 1. of Lemma 7.10. □

Corollary 7.13. *With the notation of Corollary 6.24, let*

$$f(x) = \sum_{K\subset H} \sum_{\substack{T\in[\![0,\frac{m-3}{2}]\!]^{|G|},\\ T=(t_1,\dots,t_{|G|})}} (-1)^{|H\setminus K|} |x_G|^{2T} \, (\overline{x})_{H\setminus K} \odot \mathcal{E}_{K,T}^{H,G}(f)(x)$$

be the decomposition (65)*, with harmonic components $\mathcal{E}_{K,T}^{H,G}(f)$. For any $T \in [\![0,\frac{m-3}{2}]\!]^{|G|}$, define*

$$\mathcal{G}_T^{H,G}(f) = \sum_{K\subset H} (-1)^{|H\setminus K|} \, (\overline{x})_{H\setminus K} \odot \mathcal{E}_{K,T}^{H,G}(f) \in \mathcal{S}(\Omega_D).$$

Then we can write

$$f(x) = \sum_{\substack{T\in[\![0,\frac{m-3}{2}]\!]^{|G|},\\ T=(t_1,\dots,t_{|G|})}} |x_G|^{2T} \mathcal{G}_T^{H,G}(f)(x), \tag{71}$$

with

$$\Delta_{m+1,g}^2 \mathcal{G}_T^{H,G}(f) = 0, \qquad \forall g \in G.$$

Proof. Decomposition (71) follows by (65) and the definition of $\mathcal{G}_T^{H,G}$, now let us prove that the components are biharmonic in every variable x_g, with $g \in G$. Following the proof of Corollary 6.5 we have

$$\begin{aligned}
\Delta_{m+1,g}\mathcal{G}_T^{H,G} &= \sum_{K\subset H, g\notin K} (-1)^{|H\setminus K|} \, (\overline{x})_{H\setminus K} \odot \Delta_{m+1,g}\left(\mathcal{E}_{K,T}^{H,G}(f)\right) + \\
&\quad + \sum_{K\subset H, g\notin K} (-1)^{|H\setminus (K\cup\{g\})|} \, (\overline{x})_{H\setminus K} \odot \Delta_{m+1,g}\left(\overline{x}_g \odot \mathcal{E}_{K\cup\{g\},T}^{H,G}(f)\right) \\
&= \sum_{K\subset H, g\notin K} (-1)^{|H\setminus (K\cup\{g\})|} \, (\overline{x})_{H\setminus K} \odot \partial_{x_g} \mathcal{E}_{K\cup\{g\},T}^{H,G}(f),
\end{aligned}$$

where we have used that $\Delta_{m+1,g}\mathcal{E}_{K,T}^{H,G} = 0$, for every $g \in G$ and that $\Delta_{m+1,g}\left(\overline{x}_g \odot \mathcal{E}_{K\cup\{g\},T}^{H,G}(f)\right) = \partial_{x_g}\mathcal{E}_{K\cup\{g\},T}^{H,G}(f)$. Finally

$$\begin{aligned}
\Delta_{m+1,g}^2\mathcal{G}_T^{H,G} &= \Delta_{m+1,g}\left(\sum_{K\subset H, g\notin K} (-1)^{|H\setminus (K\cup\{g\})|} \, (\overline{x})_{H\setminus K} \odot \partial_{x_g} \mathcal{E}_{K\cup\{g\},T}^{H,G}(f)\right) \\
&= \sum_{K\subset H, g\notin K} (-1)^{|H\setminus (K\cup\{g\})|} \, (\overline{x})_{H\setminus K} \odot \partial_{x_g} \Delta_{m+1,g}\left(\mathcal{E}_{K\cup\{g\},T}^{H,G}(f)\right) = 0.
\end{aligned}$$

□

Remark 17. Note that in the one variable case (Corollary 6.5) the components were more than biharmonic functions, namely they were in the kernel of the third-order differential operator $\overline{\partial}\Delta$. In several variables, the same happens for x_1, but in general it doesn't hold for the other variables.

Example 12. Let us resume Example 11, where we decomposed the function $f:(\mathbb{R}^6)^2 \to \mathbb{R}_5$, $f(x_1,x_2)=x_1^4x_2^7$ as

$$\begin{aligned} f = {} & \mathcal{E}^{\{1,2\}}_{\{1,2\},(0,0)}(f) + |x_1|^2\mathcal{E}^{\{1,2\}}_{\{1,2\},(1,0)}(f) + |x_2|^2\mathcal{E}^{\{1,2\}}_{\{1,2\},(0,1)}(f) + |x_1|^2|x_2|^2\mathcal{E}^{\{1,2\}}_{\{1,2\},(1,1)}(f) + \\ & - \overline{x}_1\mathcal{E}^{\{1,2\}}_{\{2\},(0,0)}(f) - \overline{x}_1|x_1|^2\mathcal{E}^{\{1,2\}}_{\{2\},(1,0)}(f) - \overline{x}_1|x_2|^2\mathcal{E}^{\{1,2\}}_{\{2\},(0,1)}(f) - \overline{x}_1|x_1|^2|x_2|^2\mathcal{E}^{\{1,2\}}_{\{2\},(1,1)}(f) + \\ & - \overline{x}_2\mathcal{E}^{\{1,2\}}_{\{1\},(0,0)}(f) - \overline{x}_2|x_1|^2\mathcal{E}^{\{1,2\}}_{\{1\},(1,0)}(f) - \overline{x}_2|x_2|^2\mathcal{E}^{\{1,2\}}_{\{1\},(0,1)}(f) - \overline{x}_2|x_1|^2|x_2|^2\mathcal{E}^{\{1,2\}}_{\{1\},(1,1)}(f) + \\ & + \overline{x}_1\overline{x}_2\mathcal{E}^{\{1,2\}}_{\emptyset,(0,0)}(f) + \overline{x}_1\overline{x}_2|x_1|^2\mathcal{E}^{\{1,2\}}_{\emptyset,(1,0)}(f) + \overline{x}_1\overline{x}_2|x_2|^2\mathcal{E}^{\{1,2\}}_{\emptyset,(0,1)}(f) + \overline{x}_1\overline{x}_2|x_1|^2|x_2|^2\mathcal{E}^{\{1,2\}}_{\emptyset,(1,1)}(f). \end{aligned}$$

Following Corollary 7.13, for every $T=(t_1,t_1)\in\{0,1\}^2$ define

$$\mathcal{G}_T = \sum_{K\in\mathcal{P}(2)} (-1)^{|\{1,2\}\setminus K|}\,(\overline{x})_{\{1,2\}\setminus K}\,\mathcal{E}^{\{1,2\}}_{K,T}(f),$$

explicitly, the four components are

$$\begin{aligned} \mathcal{G}_{(0,0)} &= \mathcal{E}^{\{1,2\}}_{\{1,2\},(0,0)}(f) - \overline{x}_1\mathcal{E}^{\{1,2\}}_{\{2\},(0,0)}(f) - \overline{x}_2\mathcal{E}^{\{1,2\}}_{\{1\},(0,0)}(f) + \overline{x}_1\overline{x}_2\mathcal{E}^{\{1,2\}}_{\emptyset,(0,0)}(f) \\ &= \left(7\alpha_1^4 - \frac{42}{5}\alpha_1^2\beta_1^2 + \frac{3}{5}\beta_1^4\right)\left(15\alpha_2^7 - 63\alpha_2^5\beta_2^2 + 45\alpha_2^3\beta_2^4 - 5\alpha_2\beta_2^6\right) + \\ &\quad - \overline{x}_1\left(5\alpha_1^3 - 3\alpha_1\beta_1^2\right)\left(15\alpha_2^7 - 63\alpha_2^5\beta_2^2 + 45\alpha_2^3\beta_2^4 - 5\alpha_2\beta_2^6\right) + \\ &\quad - \overline{x}_2\left(7\alpha_1^4 - \frac{42}{5}\alpha_1^2\beta_1^2 + \frac{3}{5}\beta_1^4\right)\left(12\alpha_2^6 - 36\alpha_2^4\beta_2^2 + \frac{108}{7}\alpha_2^2\beta_2^4 - \frac{4}{7}\beta_2^6\right) + \\ &\quad + \overline{x}_1\overline{x}_2\left(5\alpha_1^3 - 3\alpha_1\beta_1^2\right)\left(12\alpha_2^6 - 36\alpha_2^4\beta_2^2 + \frac{108}{7}\alpha_2^2\beta_2^4 - \frac{4}{7}\beta_2^6\right); \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{(1,0)} &= \mathcal{E}^{\{1,2\}}_{\{1,2\},(1,0)}(f) - \overline{x}_1\mathcal{E}^{\{1,2\}}_{\{2\},(1,0)}(f) - \overline{x}_2\mathcal{E}^{\{1,2\}}_{\{1\},(1,0)}(f) + \overline{x}_1\overline{x}_2\mathcal{E}^{\{1,2\}}_{\emptyset,(1,0)}(f) \\ &= \left(-2\alpha_1^2 + \frac{2}{5}\beta_1^2\right)(15\alpha_2^7 - 63\alpha_2^5\beta_2^2 + 45\alpha_2^3\beta_2^4 - 5\alpha_2\beta_2^6) + \\ &\quad - \overline{x}_1(-\alpha_1)(15\alpha_2^7 - 63\alpha_2^5\beta_2^2 + 45\alpha_2^3\beta_2^4 - 5\alpha_2\beta_2^6) + \\ &\quad - \overline{x}_2\left(-2\alpha_1^2 + \frac{2}{5}\beta_1^2\right)\left(12\alpha_2^6 - 36\alpha_2^4\beta_2^2 + \frac{108}{7}\alpha_2^2\beta_2^4 - \frac{4}{7}\beta_2^6\right) + \\ &\quad + \overline{x}_1\overline{x}_2(-\alpha_1)\left(12\alpha_2^6 - 36\alpha_2^4\beta_2^2 + \frac{108}{7}\alpha_2^2\beta_2^4 - \frac{4}{7}\beta_2^6\right); \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{(0,1)} &= \mathcal{E}^{\{1,2\}}_{\{1,2\},(0,1)}(f) - \overline{x}_1\mathcal{E}^{\{1,2\}}_{\{2\},(0,1)}(f) - \overline{x}_2\mathcal{E}^{\{1,2\}}_{\{1\},(0,1)}(f) + \overline{x}_1\overline{x}_2\mathcal{E}^{\{1,2\}}_{\emptyset,(0,1)}(f) \\ &= \left(7\alpha_1^4 - \frac{42}{5}\alpha_1^2\beta_1^2 + \frac{3}{5}\beta_1^4\right)(-7\alpha_2^5 + 14\alpha_2^3\beta_2^2 - 3\alpha_2\beta_2^4) + \\ &\quad - \overline{x}_1\left(5\alpha_1^3 - 3\alpha_1\beta_1^2\right)(-7\alpha_2^5 + 14\alpha_2^3\beta_2^2 - 3\alpha_2\beta_2^4) + \\ &\quad - \overline{x}_2\left(7\alpha_1^4 - \frac{42}{5}\alpha_1^2\beta_1^2 + \frac{3}{5}\beta_1^4\right)\left(-5\alpha_2^4 + 6\alpha_2^2\beta_2^2 - \frac{3}{7}\beta_2^4\right) + \\ &\quad + \overline{x}_1\overline{x}_2\left(5\alpha_1^3 - 3\alpha_1\beta_1^2\right)\left(-5\alpha_2^4 + 6\alpha_2^2\beta_2^2 - \frac{3}{7}\beta_2^4\right); \end{aligned}$$

$$
\begin{aligned}
\mathcal{G}_{(1,1)} &= \mathcal{E}^{\{1,2\}}_{\{1,2\},(1,1)}(f) - \overline{x}_1\mathcal{E}^{\{1,2\}}_{\{2\},(1,1)}(f) - \overline{x}_2\mathcal{E}^{\{1,2\}}_{\{1\},(1,1)}(f) + \overline{x}_1\overline{x}_2\mathcal{E}^{\{1,2\}}_{\emptyset,(1,1)}(f) \\
&= \left(-2\alpha_1^2 + \frac{2}{5}\beta_1^2\right)(-7\alpha_2^5 + 14\alpha_2^3\beta_2^2 - 3\alpha_2\beta_2^4) + \\
&- \overline{x}_1(-\alpha_1)(-7\alpha_2^5 + 14\alpha_2^3\beta_2^2 - 3\alpha_2\beta_2^4) + \\
&- \overline{x}_2\left(-2\alpha_1^2 + \frac{2}{5}\beta_1^2\right)\left(-5\alpha_2^4 + 6\alpha_2^2\beta_2^2 - \frac{3}{7}\beta_2^4\right) + \\
&+ \overline{x}_1\overline{x}_2(-\alpha_1)\left(-5\alpha_2^4 + 6\alpha_2^2\beta_2^2 - \frac{3}{7}\beta_2^4\right).
\end{aligned}
$$

With these functions, we have

$$
f(x_1, x_2) = \mathcal{G}_{(0,0)} + |x_1|^2\mathcal{G}_{(1,0)} + |x_2|^2\mathcal{G}_{(0,1)} + |x_1|^2|x_2|^2\mathcal{G}_{(1,1)},
$$

which corresponds to decomposition (71). Note that $\mathcal{G}_T \in \ker \Delta^2_{6,j}$, for every $T \in \{0,1\}^2$, $j = 1, 2$.

Proposition 7.14. *Let $\Omega_D \subset (\mathbb{R}^{\frac{m-1}{2}})^n$, $f \in \mathcal{SR}(\Omega_D)$ and let $h = 1, \ldots, n-1$. Then, for every $K \in \mathcal{P}(h)$, $\mathcal{S}^{[\![m]\!]}(f)$ satisfies the following:*

1. $\partial_{x_h}\left(\Delta^{\frac{m-3}{2}}_{m+1,h}\mathcal{S}^{[\![h]\!]}_K(f)\right) \in \mathcal{AM}_h(\Omega_D)$;
2. $\Delta^{\frac{m-1}{2}}_{m+1,h+1}\mathcal{S}^{[\![h]\!]}_K(f) \in \mathcal{AM}_{h+1}(\Omega_D)$.

Proof. 1. It holds

$$
\overline{\partial}_{x_h}\partial_{x_h}\Delta^{\frac{m-3}{2}}_{m+1,h}\mathcal{S}^{[\![h]\!]}_K(f) = \frac{1}{4}\Delta^{\frac{m-1}{2}}_{m+1,h}\mathcal{S}^{[\![h]\!]}_K(f) = 0,
$$

by Theorem 6.6.

2. Similarly,

$$
\begin{aligned}
\overline{\partial}_{x_{h+1}}\Delta^{\frac{m-1}{2}}_{m+1,h+1}\mathcal{S}^{[\![h]\!]}_K(f) = \Delta^{\frac{m-1}{2}}_{m+1,h+1}\overline{\partial}_{x_{h+1}}\mathcal{S}^{[\![h]\!]}_K(f) &= \frac{1-m}{2}\Delta^{\frac{m-1}{2}}_{m+1,h+1}\overline{\partial}_{x_{h+1}}\left(\mathcal{S}^{[\![h]\!]}_K(f)\right)'_{s,h+1} \\
&= \frac{1-m}{2}\Delta^{\frac{m-1}{2}}_{m+1,h+1}\overline{\partial}_{x_{h+1}}\mathcal{S}^{[\![h+1]\!]}_K(f) = 0,
\end{aligned}
$$

again by Theorem 6.6, Lemma 7.10 and Corollary 5.10.

□

Lemma 7.15. *Let $f \in \mathcal{S}^1(\Omega_D) \cap \ker(\partial/\partial x_h^c)$, then it holds*

$$
\Delta_{m+1,h}f = 2(1-m)\sum_{K\in\mathcal{P}(h-1)}(-1)^{|K^c|}(\overline{x})_{K^c}\,\partial_{x_h}\left(\mathcal{S}^{[\![h]\!]}_K(f)\right),
$$

with $K^c = \{1, \ldots, h-1\} \setminus K$.

Proof. By Theorem 6.6, we can decompose f as

$$
f = \sum_{K\in\mathcal{P}(h-1)}(-1)^{|K^c|}(\overline{x})_{K^c}\,\mathcal{S}^{[\![h-1]\!]}_K(f),
$$

then

$$\Delta_{m+1,h} f = \Delta_{m+1,h} \left(\sum_{K \in \mathcal{P}(h-1)} (-1)^{|K^c|} (\overline{x})_{K^c} \mathcal{S}_K^{[\![h-1]\!]}(f) \right)$$
$$= \sum_{K \in \mathcal{P}(h-1)} (-1)^{|K^c|} (\overline{x})_{K^c} \Delta_{m+1,h} \left(\mathcal{S}_K^{[\![h-1]\!]}(f) \right).$$

Now, by Lemma 7.10 (1) we have

$$\Delta_{m+1,h} \left(\mathcal{S}_K^{[\![h-1]\!]}(f) \right) = 4\partial_{x_h} \overline{\partial}_{x_h} \left(\mathcal{S}_K^{[\![h-1]\!]}(f) \right) = 2(1-m)\partial_{x_h} \left(\mathcal{S}_K^{[\![h]\!]}(f) \right)$$

and so

$$\Delta_{m+1,h} f = 2(1-m) \sum_{K \in \mathcal{P}(h-1)} (-1)^{|K^c|} (\overline{x})_{K^c} \partial_{x_h} \left(\mathcal{S}_K^{[\![h]\!]}(f) \right).$$

□

Proof of 7.11. By Lemma 7.10 (2) it holds

$$\Delta_{m+1,h} f = 2(1-m) \sum_{K \in \mathcal{P}(h-1)} (-1)^{|K^c|} (\overline{x})_{K^c} \partial_{x_h} \left(\mathcal{S}_K^{[\![h]\!]}(f) \right)$$

and recall that, since $f \in \mathcal{SR}_h(\Omega_D)$, $\mathcal{S}_K^{[\![h]\!]}(f) = 0$, for every $K \in \mathcal{P}(h-1) \setminus [\![h-1]\!]$ (Propositions 4.9 (1) and 4.10). Thus, the previous equation reduces to

$$\Delta_{m+1,h} f = 2(1-m)\partial_{x_h} \left(\mathcal{S}_{[\![h-1]\!]}^{[\![h]\!]}(f) \right),$$

so, again by Lemma 7.10 (2)

$$\Delta_{m+1,h}^{\frac{m-1}{2}} f = \Delta_{m+1,h}^{\frac{m-3}{2}} \Delta_{m+1,h}^{\frac{m-1}{2}} f = 2(1-m)\partial_{x_h} \left(\Delta_{m+1,h}^{\frac{m-3}{2}} \mathcal{S}_{[\![h-1]\!]}^{[\![h]\!]}(f) \right)$$

and we conclude with Corollary 5.10. □

Corollary 7.16. *Let $f \in \mathcal{SR}(\Omega_D)$, then $\Delta_{m+1,h}^{\frac{m+1}{2}} f = 0$, for every $h = 1, \dots, n$.*

Proof. By Lemma 7.10, it holds

$$\Delta_{m+1,h}^{\frac{m+1}{2}} f = \Delta_{m+1,h}^{\frac{m-1}{2}} \Delta_{m+1,h} f = 2(1-m) \sum_{K \in \mathcal{P}(h-1)} (-1)^{|K^c|} (\overline{x})_{K^c} \Delta_{m+1,h}^{\frac{m-1}{2}} \partial_{x_h} \left(\mathcal{S}_K^{[\![h]\!]}(f) \right)$$
$$= 2(1-m) \sum_{K \in \mathcal{P}(h-1)} (-1)^{|K^c|} (\overline{x})_{K^c} \partial_{x_h} \left(\Delta_{m+1,h}^{\frac{m-1}{2}} \mathcal{S}_K^{[\![h]\!]}(f) \right) = 0.$$

□

8 Slice regular Cliffordian holomorphic functions

Definition 8.1. Let $\Omega \subset \mathbb{R}^{m+1}$ be an open set. A function $f : \Omega \to \mathbb{R}_m$ of class $\mathcal{C}^{2k+1}$ is called holomorphic Cliffordian of order k if $\overline{\partial}\Delta^k f = 0$ or, equivalently, $\Delta^k f$ is monogenic. Holomorphic Cliffordian functions of order $\frac{m-1}{2}$ will by simply called holomorphic Cliffordian functions.

Note that monogenic functions are holomorphic Cliffordian of any order, since $\overline{\partial}\Delta^k f = \Delta^k \overline{\partial} f = 0$, for any $f \in \mathcal{M}(\Omega)$, while by Fueter-Sce theorem slice regular functions are holomorphic Cliffordian of order $k \geq \frac{m-1}{2}$.

From here on we assume m odd and we denote $\gamma_m := \frac{m-1}{2} \in \mathbb{N}$ the Sce exponent. For every $k < \gamma_m$, let

$$\mathfrak{F}_k := \overline{\partial}\Delta^k_{m+1}|_{\mathcal{SR}(\Omega_D)} \colon \mathcal{SR}(\Omega_D) \to \ker \Delta^{\gamma_m - k}.$$

Theorem 8.1. *Let $f \in \mathcal{SR}(\Omega_D)$, be a slice regular function on $\Omega_D \subset \mathbb{R}^{m+1}$ symmetric domain and let $k < \gamma_m = \frac{m-1}{2}$. Then f is holomorphic Cliffordian of order k if and only if it is a polynomial of degree at most $2k$. In other words*

$$\ker \mathfrak{F}_k = \mathbb{R}^{m+1}_{2k}[x], \qquad \forall k < \gamma_m.$$

Remark 18. Fueter-Sce theorem asserts that $\ker \mathfrak{F}_k = \mathcal{SR}(\Omega_D)$, for any $k \geq \gamma_m$. Hence, we can consider the following chain of inclusions, that ends with the Sce exponent $\gamma_m = \frac{m-1}{2}$:

$$\ker \mathfrak{F}_0 = \mathbb{R}_m \subset \ker \mathfrak{F}_1 = \mathbb{R}^{m+1}_2[x] \subset \cdots \subset \ker \mathfrak{F}_{\frac{m-3}{2}} = \mathbb{R}^{m+1}_{m-3}[x] \subset \ker \mathfrak{F}_{\frac{m-1}{2}} = \mathcal{SR}(\Omega_D).$$

In the trivial case $k = 0$ the results (already proven in [74, Corollary 3.3 (b)]) becomes

Corollary 8.2. *Let f be a slice regular and monogenic function. Then f is locally constant.*

Recall that we expressed the k^{th}-power of the Laplacian of the spherical derivative of a slice regular function in terms of lower order derivatives of its spherical derivative. The coefficients of the combination $a_j^{(k)}$ were defined in (31). Those coefficients are peculiar for producing a differential equation, whose solutions are polynomial with only even powers.

Lemma 8.3. *Let $I \subset \mathbb{R}$ be an open interval and let $y \colon I \to \mathbb{R}$ satisfy the following linear homogeneous differential equation of degree k*

$$\sum_{j=1}^{k} a_j^{(k)} x^{j-1} y^{(j)}(x) = 0, \qquad \forall x \in I, \tag{72}$$

with $a_j^{(k)}$ defined as in Proposition 5.5. Then y is the polynomial

$$y(x) = \sum_{j=0}^{k-1} c_j x^{2j},$$

for some $c_j \in \mathbb{R}$. Furthermore, if $I = I^+ \cup I^- \subset \mathbb{R}$, with $I^- = -I^+ = \{-x \mid x \in I^+\}$ and I^+ open interval of $[0, +\infty)$, then any even solution on I can be $\mathcal{C}^\infty$-extended to $\mathbb{R}$.

Proof. Let us start with the case I connected. In this case, the space of solution of (72) is a vector space of dimension k. Moreover, since $\{1, x^2, ..., x^{2(k-1)}\}$ is a set of linearly independent functions, it is enough to show that x^{2h} is solution of (72) for $h = 0, ..., k-1$. Let us compute x^{2h} in (72):

$$x^{2h}\sum_{j=1}^{k} a_j^{(k)}(2h)_j = 0 \iff \sum_{j=1}^{k}(-2)^j\frac{(2k-j-1)!}{(j-1)!(k-j)!}(2h)_j = 0.$$

It is easy to see that

$$\sum_{j=1}^{k}(-2)^j\frac{(2k-j-1)!}{(j-1)!(k-j)!}(2h)_j = -\frac{4h(2k-2)!}{(k-1)!}\,{}_2F_1(1-2h, 1-k, 2-2k; 2),$$

where ${}_2F_1$ is the hypergeometric function, defined by

$${}_2F_1(a, b, c; z) := \sum_{j=0}^{+\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}.$$

Finally, by [63, §8] we have that ${}_2F_1(-n, a, 2a; 2) = 0$ if n is odd. Since, $n = 2h-1$, we conclude.

Now, suppose that $I = I^+ \cup I^-$. If $0 \in I^+$, then I is connected and so $y(x) = \sum_{j=0}^{k-1} c_j x^{2j}$, for any $x \in \mathbb{R}$. Suppose that $0 \notin I^+$, so I is disconnected. I^+ and I^- are open intervals of $\mathbb{R}$, so there exist $c_j, d_j \in \mathbb{R}$ such that $y(x) = \sum_{j=0}^{k-1} c_j x^{2j}$ for any $x \in I^+$ and $y(x) = \sum_{j=0}^{k-1} d_j x^{2j}$ for any $x \in I^-$. Since y is an even function, we have

$$\sum_{j=0}^{k-1} d_j x^{2j} = y(x) = y(-x) = \sum_{j=0}^{k-1} c_j x^{2j}, \qquad \forall x \in I^-$$

and this holds if and only if $c_j = d_j$, for any $j = 0, ..., k-1$. Thus, $y(x) = \sum_{j=0}^{k-1} c_j x^{2j}$ for any $x \in \mathbb{R}$. $\square$

Remark 19. It is also possible to prove that for any $h \in \mathbb{N}$ it holds

$$\sum_{j=1}^{k}(-2)^j\frac{(2k-j-1)!}{(j-1)!(k-j)!}(2h)_j = (-4)^k\prod_{j=0}^{k-1}(h-j).$$

This immediately proves Lemma 8.3.

Corollary 8.4. *Let $m > 2k+1$ and let $f = \mathcal{I}(F)\colon \Omega_D \to \mathbb{R}_m$ be a slice regular function. Then*

$$\overline{\partial}\Delta_{m+1}^k f = 0 \iff f_s'(x) = \sum_{j=0}^{k-1} c_j(\mathrm{Re}(x))|\mathrm{Im}(x)|^{2j}, \tag{73}$$

for some functions c_j, $j = 0, \ldots, k-1$. In particular, f_s' can be extended to $\mathbb{R}$.

Proof. f is slice regular, then from (29) it holds

$$0 = \overline{\partial}\Delta_{m+1}^k f(x) = \Delta_{m+1}^k\overline{\partial} f(x) = \Delta_{m+1}^k f_s'(x) \iff \sum_{j=1}^{k} a_j^{(k)}|\mathrm{Im}(x)|^{j-2k}\partial_\beta^{(j)} f_s'(x).$$

Let us distinguish the case Ω_D slice domain or product domain. In the first case we immediately apply Lemma 8.3 to $y(\beta) = f_s'(\alpha, \beta)$ for any fixed α. Indeed, $f_s'(\alpha, \cdot)$ is defined over I_α, which is connected and it satisfies (72) and so $f_s'(\alpha, \beta) = \sum_{j=0}^{k-1} c_j(\alpha)\beta^{2j}$, for any $\beta \in \mathbb{R}$. On the contrary, if Ω_D is a product domain then $I_\alpha = I_\alpha^+ \cup I_\alpha^-$ and $I_\alpha^- = -I_\alpha^+$. Moreover, $f_s'(\alpha, \cdot)$ is an even function and so, thanks to Lemma 8.3, we conclude that $f_s'(\alpha, \beta) = \sum_{j=0}^{k-1} c_j(\alpha)\beta^{2j}$, for any $\beta \in \mathbb{R}$. $\square$

Now, we aim to reconstruct a slice regular function if its spherical derivative is of the form (73).

Lemma 8.5. *Let $v\colon \mathbb{R}^2 \to \mathbb{C}$ be the imaginary part of an entire function $F\colon \mathbb{C} \to \mathbb{C}$. Suppose that $v(\alpha,\beta)$ has the following form*

$$v(\alpha,\beta)=\sum_{j=0}^{k-1} c_j^{(k)}(\alpha)\beta^{2j+1},$$

for some functions $c_j^{(k)}(\alpha)$. Then, the function $c_j^{(k)}$ must be of the form

$$c_j^{(k)}(\alpha)=(-1)^j(2k+1)\cdots(2j+2)\sum_{\eta=0}^{2k-2j-1}\frac{\alpha^\eta}{\eta!}s_{\eta+2j}, \qquad j=0,...,k-1 \tag{74}$$

for some complex numbers $s_l \in \mathbb{C}$. In particular, F is the polynomial

$$F(z)=\sum_{j=0}^{2k}\frac{(2k+1)!}{j!}s_{j-1}z^j,$$

for some arbitrary real number s_{-1}.

Proof. Note that, in order for v to be harmonic, the functions $c_j^{(k)}$ satisfies the relation

$$\begin{cases} c_j^{''}(\alpha)=-(2j+2)(2j+3)c_{j+1}(\alpha), & j=0,...,k-2 \\ c_{k-1}^{''}=0. \end{cases} \tag{75}$$

Indeed

$$\begin{aligned}\Delta_2 v(\alpha,\beta)=\partial_\alpha^2 v+\partial_\beta^2 v &=\sum_{j=0}^{k-1}c_j^{''}(\alpha)\beta^{2j+1}+\sum_{j=1}^{k-1}(2j+1)(2j)c_j(\alpha)\beta^{2j-1}\\ &=\sum_{j=0}^{k-1}c_j^{''}(\alpha)\beta^{2j+1}+\sum_{j=0}^{k-2}(2j+3)(2j+2)c_{j+1}(\alpha)\beta^{2j+1}\\ &=\sum_{j=0}^{k-2}\beta^{2j+1}(c_j^{''}(\alpha)+(2j+2)(2j+3)c_{j+1}(\alpha))+c_{k-1}''(\alpha)\beta^{2k-1}.\end{aligned}$$

So

$$\Delta_2 v(\alpha,\beta)=0 \iff \begin{cases} c_j^{''}=-(2j+2)(2j+3)c_{j+1}, & j=1,...,k-2 \\ c_{k-1}^{''}=0. \end{cases}$$

Note that the functions $c_j^{(k)}$ defined in (74) satisfy (75): for every $j=0,...,k-2$

$$c_j'(\alpha)=(-1)^j(2k+1)\cdots(2j+2)\sum_{\eta=0}^{2k-2j-1}\frac{\alpha^{\eta-1}}{(\eta-1)!}s_{\eta+2j},$$

$$
\begin{aligned}
c_j''(\alpha) &= (-1)^j(2k+1)\cdots(2j+2)\sum_{\eta=0}^{2k-2j-1}\frac{\alpha^{\eta-2}}{(\eta-2)!}s_{\eta+2j}\\
&= -(2j+2)(2j+3)\left[(-1)^{j+1}(2k+1)\cdots(2(j+1)+2)\sum_{\eta=0}^{2k-2(j+1)-1}\frac{\alpha^{\eta}}{\eta!}s_{\eta+2(j+1)}\right]\\
&= -(2j+2)(2j+3)c_{j+1}(\alpha)
\end{aligned}
$$

and moreover $c_{k-1}(\alpha) = (-1)^{k-1}2k(2k+1)(s_{2k-2}+\alpha s_{2k-1})$ satisfies $c_{k-1}'' = 0$.

Now, let us prove that the imaginary part of $F(z) = \sum_{j=0}^{2k}\frac{(2k+1)!}{j!}s_{j-1}z^j$ is v. It holds

$$
\begin{aligned}
\mathrm{Im}(F(z)) &= \frac{1}{2i}(F(z)-F(\bar{z})) = \frac{1}{2i}\sum_{j=0}^{2k}\frac{(2k+1)!}{j!}(z^j-\bar{z}^j)\\
&= \sum_{j=0}^{2k}\frac{(2k+1)!}{j!}s_{j-1}\sum_{\eta=0}^{\lfloor\frac{j-1}{2}\rfloor}\binom{j}{2\eta+1}\alpha^{j-2\eta-1}(-1)^\eta\beta^{2\eta+1}\\
&= \sum_{j=1}^{k}\frac{(2k+1)!}{(2j)!}s_{2j-1}\sum_{\eta=0}^{j-1}\binom{2j}{2\eta+1}\alpha^{2j-2\eta-1}(-1)^\eta\beta^{2\eta+1}+\\
&+\sum_{j=0}^{k-1}\frac{(2k+1)!}{(2j+1)!}s_{2j}\sum_{\eta=0}^{j}\binom{2j+1}{2\eta+1}\alpha^{2j-2\eta}(-1)^\eta\beta^{2\eta+1}\\
&= (2k+1)!\sum_{j=1}^{k}\sum_{\eta=0}^{j-1}\frac{(-1)^\eta s_{2j-1}\alpha^{2j-2\eta-1}\beta^{2\eta+1}}{(2\eta+1)!(2j-2\eta-1)!}+\\
&+(2k+1)!\sum_{j=0}^{k-1}\sum_{\eta=0}^{j}\frac{(-1)^\eta s_{2j}\alpha^{2j-2\eta}\beta^{2\eta+1}}{(2\eta+1)!(2j-2\eta)!}.
\end{aligned}
$$

We can handle those two double sums for our purpose:

$$
\begin{aligned}
&\sum_{j=1}^{k}\sum_{\eta=0}^{j-1}\frac{(-1)^\eta s_{2j-1}\alpha^{2j-2\eta-1}\beta^{2\eta+1}}{(2\eta+1)!(2j-2\eta-1)!} = \sum_{j=0}^{k-1}\sum_{\eta=0}^{j}\frac{(-1)^\eta s_{2j+1}\alpha^{2j-2\eta+1}\beta^{2\eta+1}}{(2\eta+1)!(2j-2\eta+1)!}\\
&= \sum_{j=0}^{k-1}\sum_{\eta=0}^{k-1}\frac{(-1)^\eta s_{2j+1}\alpha^{2j-2\eta+1}\beta^{2\eta+1}}{(2\eta+1)!(2j-2\eta+1)!} = \sum_{\eta=0}^{k-1}\sum_{j=\eta}^{k-1}\frac{(-1)^\eta s_{2j+1}\alpha^{2j-2\eta+1}\beta^{2\eta+1}}{(2\eta+1)!(2j-2\eta+1)!}\\
&= \sum_{j=0}^{k-1}\sum_{\eta=j}^{k-1}\frac{(-1)^j s_{2\eta+1}\alpha^{2\eta-2j+1}\beta^{2j+1}}{(2j+1)!(2\eta-2j+1)!} = \sum_{j=0}^{k-1}\sum_{\eta=0}^{k-j-1}\frac{(-1)^j s_{2\eta+2j+1}\alpha^{2\eta+1}\beta^{2j+1}}{(2j+1)!(2\eta+1)!}
\end{aligned}
$$

and similarly

$$
\sum_{j=0}^{k-1}\sum_{\eta=0}^{j}\frac{(-1)^\eta s_{2j}\alpha^{2j-2\eta}\beta^{2\eta+1}}{(2\eta+1)!(2j-2\eta)!} = \sum_{j=0}^{k-1}\sum_{\eta=0}^{k-j-1}\frac{(-1)^j s_{2\eta+2j}\alpha^{2\eta}\beta^{2j+1}}{(2j+1)!(2\eta)!},
$$

so we get

$$
\begin{aligned}
\operatorname{Im}(F(z)) &= (2k+1)!\left[\sum_{j=0}^{k-1}\sum_{\eta=0}^{k-j-1}\frac{(-1)^j s_{2\eta+2j+1}\alpha^{2\eta+1}\beta^{2j+1}}{(2j+1)!(2\eta+1)!}+\sum_{j=0}^{k-1}\sum_{\eta=0}^{k-j-1}\frac{(-1)^j s_{2\eta+2j}\alpha^{2\eta}\beta^{2j+1}}{(2j+1)!(2\eta)!}\right]\\
&= \sum_{j=0}^{k-1}(-1)^j\frac{(2k+1)!}{(2j+1)!}\beta^{2j+1}\left[\sum_{\eta=0}^{k-j-1}\frac{s_{2\eta+2j+1}\alpha^{2\eta+1}}{(2\eta+1)!}+\sum_{\eta=0}^{k-j-1}\frac{s_{2\eta+2j}\alpha^{2\eta}}{(2\eta)!}\right]\\
&= \sum_{j=0}^{k-1}(-1)^j\frac{(2k+1)!}{(2j+1)!}\beta^{2j+1}\sum_{\eta=0}^{2k-2j-1}\frac{s_{\eta+2j}\alpha^{\eta}}{\eta!}=\sum_{j=0}^{k-1}c_j^{(k)}(\alpha)\beta^{2j+1}=v(\alpha,\beta).
\end{aligned}
$$

□

Proof of Theorem 8.1. The case $m \leq 2k+1$ is proved by Proposition 5.4, hence suppose $m > 2k+1$. From Corollary 8.4 we have

$$
\overline{\partial}\Delta_{m+1}^k f(x)=0 \iff f_s'(x)=\sum_{j=0}^{k-1}c_j(\operatorname{Re}(x))|\operatorname{Im}(x)|^{2j},
$$

thus

$$
F_1(\operatorname{Re}(x),|\operatorname{Im}(x)|)=|\operatorname{Im}(x)|f_s'(x)=\sum_{j=0}^{k-1}c_j(\operatorname{Re}(x))|\operatorname{Im}(x)|^{2j+1}.
$$

Finally, by Lemma 8.5, $F(z)=\sum_{j=0}^{2k}z^jc_j$ and so $f(x)=\mathcal{I}(F)=\sum_{j=0}^{2k}x^jc_j$. □

Example 13. Let us consider the slice regular function $f:\mathbb{R}^{5+1}\to\mathbb{R}_5$, $f(x)=x^5$. Note that $\gamma_5=2$, so we expect that $f\in\ker\mathfrak{F}_2=\mathcal{SR}(\mathbb{R}^{5+1})$, but since $\deg(f)=5>2k$, for any $k<\gamma_5$, $f\notin\ker\mathfrak{F}_k$, for $k=0,1$. Indeed, let us compute $\partial\Delta_{5+1}^k f=-4\Delta_{5+1}^k f_s'$, for $k=0,1,2$. Let $x=x_0+\sum_{j=1}^5 e_jx_j=\alpha+J\beta$, with $\alpha=\operatorname{Re}(x)=x_0$, $\beta=|\operatorname{Im}(x)|=\sqrt{\sum_{j=1}^5 x_i^2}$ and $J=\sum_{j=1}^5 x_i/\beta$, then

$$
f(x)=\alpha^5-10\alpha^3\beta^2+5\alpha\beta^4+J\beta(5\alpha^4-10\alpha^2\beta^2+\beta^4).
$$

Thus, we have

$$
\begin{aligned}
\mathfrak{F}_0(f)&=-4f_s'(x)=-20\alpha^4+40\alpha^2\beta^2-4\beta^4\neq 0;\\
\mathfrak{F}_1(f)&=-4\Delta_{5+1}f_s'(x)=160\alpha^2-32\beta^2\neq 0;\\
\mathfrak{F}_2(f)&=-4\Delta_{5+1}^2f_s'(x)=320-320=0.
\end{aligned}
$$

Example 14. Let us consider again the slice regular function $f(x)=x^5$, but with $f:\mathbb{R}^{9+1}\to\mathbb{R}_9$. Now, $\gamma_9=4$. We expect that $f\in\ker\mathfrak{F}_3=\mathbb{R}_6^{9+1}[x]\subsetneq\mathcal{SR}(\mathbb{R}^{9+1})=\ker\mathfrak{F}_4$, but $f\notin\ker\mathfrak{F}_k$, for $k=0,1,2$. Indeed, let us compute as before $\overline{\partial}\Delta_{9+1}^k f=-8\Delta_{9+1}^k f_s'$, for $k=0,1,2,3$.

$$
\begin{aligned}
\mathfrak{F}_0(f)&=-8f_s'(x)=-40\alpha^4+80\alpha^2\beta^2-8\beta^4\neq 0;\\
\mathfrak{F}_1(f)&=-8\Delta_{9+1}f_s'(x)=960\alpha^2-192\beta^2\neq 0;\\
\mathfrak{F}_2(f)&=-8\Delta_{9+1}^2f_s'(x)=1920-3456\neq 0;\\
\mathfrak{F}_3(f)&=-8\Delta_{9+1}^3f_s'(x)=0.
\end{aligned}
$$

9 Further directions

We can think of lines of research to carry forward the results found in the work.

1. Proceeding as the evolution of Fueter contruction, that has been extended first regarding Clifford algebras $\mathbb{R}_m$, generated by an odd number of imaginary units by Sce and then to all Clifford algebras by Qian, we shall provide Almansi decompositions for slice functions in several Clifford variables. In this work we did not deal with Clifford algebras $\mathbb{R}_m$ generated by even imaginary units. The problem is not in providing the decomposition, but in identifying the harmonic properties of the components. This requires applying the Fourier analysis methods exploiting by Qian in [77].

2. It would be appropriate to study an Almansi decomposition set in any hypercomplex space. This would lead to uniformity in the formulation of the theorem, as well as generalizing the result.

3. We can extend the new results we find in one variable to several variables, such as the computations with the powers of Laplacian or the study conducted in Section 8.

4. The Almansi decomposition is a special case of the Fischer decomposition [37], which concerns generic operators. One could produce different decompositions, following Fischer duality, for other operators such as the Dunkl-Dirac operator.

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