# Why Three Measurements are not Enough for Trilateration-based Localisation

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***Abstract*—In this paper, we analyse the problem of simultaneous trilateration, i.e. when three ranging sensors retrieve their distance from a target, and we found that is ill-posed from an algebraic view-point. Then, we explain this fact using the concept of "delayed" trilateration, i.e. we consider three ranging sensors that measure their distance from the agent at three distinct time steps, while the agent has moved through the environment. We prove with a counterexample that, in a general case, there are multiple trajectories that are compliant with the motion of the target and with the ranging measurements collected by the anchor of known position. Therefore, we claim here that three measurements are not sufficient to localise a moving target in a given environment even in ideal conditions. In particular, we claim that the classic trilateration problem assumes an additional implicit information besides the three ranging measurements, that is that the three measurements are taken with respect to the same point in space or, in the most general and most probable case of a moving target, simultaneously.**

***Index Terms*—Positioning, Localisation, Range sensors, Trilateration**

## I. Introduction

Finding the position of a (moving) target in an indoor environment is a problem that has been deeply analysed in the past years. A natural choice for solving this *positioning* problem relies on ranging sensors, given the large number of sensors capable of measuring the distance between a number of fixed-frame points and the target. Ranging sensors that may be employed to this aim are LiDAR systems, Ultra-Wide Band (UWB) nodes and Wi-Fi nodes measuring the time elapsed between the signal emission and reception, usually dubbed *Time of Flight* (ToF), or measuring some features of the signal related to the sensed power [1], [2], [3], [4], [5]. A common approach to the positioning problem of a target through ranging sensors is the *trilateration*: collecting three or more ranging measurements from beacons of known locations, it is possible to uniquely determine the target position on the $\mathbb{R}^2$ plane or in the $\mathbb{R}^3$ space [6]. In the rather big technical literature on the subject, the fixed-frame sensors collecting the ranging measurements fall under different naming conventions, depending on their nature, physical phenomenon detected, limitations, etc. Since we are not interested in any particular type of sensor, but rather on the properties of the ranging system, in the following, we will refer to the fixed-frame ranging sensors as "anchors" (with an explicit reference to UWB systems), but they could be indifferently referred to as "beacons" or "landmarks", since they are considered as synonyms in this paper. Moreover, in the rest of the paper we will make reference to two classes of targets that can be considered in this analysis: mobile robots and human beings.

**Related work:** There is a large variety of technical literature that deals with the problem of uncertainty minimisation in trilateration systems, by leveraging different techniques. For instance, Yi et al. [7] extend the set of measurements including also the known distance between anchors, and use these measurements to adapt the range estimation to dynamic environment conditions. A similar approach, i.e. increasing the number of measurements, is adopted by Diao et al. [8] where each of the $N$ beacons retrieves a set of measurements, and only the three beacons with the lowest standard deviation in their measurement set are employed for the trilateration process. With a similar purpose, [9] and [10] define new frameworks, which are based on Neural Networks, that allow the system to reduce the position uncertainty. Moreover, Thomas et al. [11] propose a new framework, specifically conceived for mobile robots, suitable for a complete and deep analysis on the effect of noises and disturbances affecting the sensor readings. A different approach to reduce the positioning uncertainty is based on the so-called *multilateration*, where the number of ranging measurements is greater than 3. This technique can be obtained through a multi-channel approach [12], on an increased number of anchors [13] or explicitly using the filter characteristics [14]. This technique leverages the (nonlinear) least square solution, which may be directly applied to the equations of the retrieved distances [15], or on a different set of algebraic equations descending from the former ones [16]. The positioning problem has been deeply analysed also in the field of robotics, where the concept of positioning is strictly related to the concept of *localisation*. In fact many research works are not focusing only on reconstructing the position of a standing target in the environment, but rather on reconstructing its trajectory assuming the knowledge of its dynamics and of the sensor readings [17], [18]. In the same setting but with a reversed perspective, Han et al. [19] uses a moving robot as a mobile anchor with limited sensing range instead of a target with unknown position. They propose a path planning strategy that maximises the amount of space that is dynamically in sight of at least three anchors with limited sensing range, an approach similar to [20].

**Paper contributions:** We consider a target moving in an environment equipped with an infrastructure of three anchors of known position and measuring the distances to the target. Dictated by various actual applications, e.g., limited sensing range, limited bandwidth in the target-beacon communication or scalability issues [21], the measurements are retrieved at different time steps. Contrary to the intuition that a *delayed trilateration* should have the same properties of a *simultaneous trilateration* (i.e., three measurements are sufficient to localise the target), we show that in this setting we are not able to recover the target location, even if an ideal, perfect knowledge of the manoeuvres performed by the target is available. In the developments, we additionally prove that this result roots in an algebraically ill-posed solution of the trilateration.

The rest of the paper is organised as follows: in Section II, we define the model of the target and of the sensors, and we discuss the differences between simultaneous and delayed trilateration presenting the problem at hand. Section III presents well-known results on the simultaneous trilateration and then reports analogies and differences with the delayed trilateration. In Section IV, we present a numerical example that supports the conclusions drawn in the previous sections, while in Section V we derive the final considerations on this work and present future research directions.

## II. Models Description and Problem Formulation

In this section we will present the background knowledge and results that are fundamental to derive the problem we are tackling in this paper.

### A. Dynamical model – continuous-time dynamics

In a previous work [22], Farina et al. presented a dynamical model able to capture the relevant dynamics of the motion of a pedestrian. In the same spirit and for the sake of the problem at hand, we decide to abstract that dynamic model to a pair of integrators in the plane endowed with the orientation $\phi$. As depicted in Figure 1, we consider the target having two independent inputs $v_x$ and $v_y$ in the target reference frame $\langle B\rangle$, which leads to the following dynamics in the fixed inertial reference frame $\langle I\rangle$

$$ {}^{\langle I\rangle}\dot{x} = v_x\cos\phi - v_y\sin\phi, \qquad {}^{\langle I\rangle}\dot{y} = v_x\sin\phi + v_y\cos\phi. \quad (1)$$

To simplify the forthcoming analysis, we consider the system to be sampled at discrete time instants with sampling time $T_s$ (dictated by the hardware available), thus leading to

$$ {}^{\langle I\rangle}x_{k+1} = {}^{\langle I\rangle}x_k + \Delta x, \qquad {}^{\langle I\rangle}y_{k+1} = {}^{\langle I\rangle}y_k + \Delta y, \quad (2)$$

where ${}^{\langle I\rangle}x_k$ denotes the horizontal position of the target at the time instant $t = kT_s$ and $\Delta x$ and $\Delta y$ depend on the system inputs, i.e.

$$\begin{aligned} \Delta x &= (v_{x,k}\cos\phi - v_{y,k}\sin\phi)T_s, \\ \Delta y &= (v_{x,k}\sin\phi + v_{y,k}\cos\phi)T_s, \end{aligned} \quad (3)$$

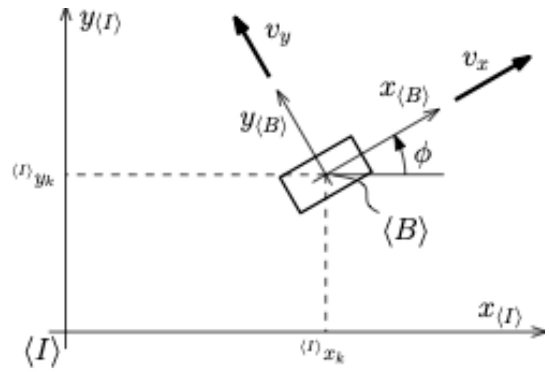


Fig. 1. Figure with the absolute and the relative reference frame.

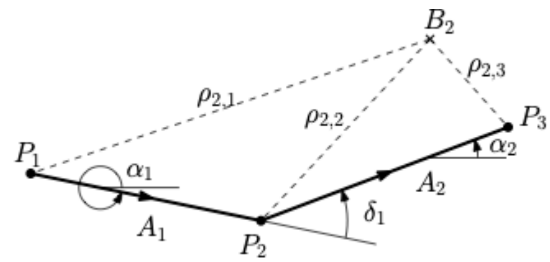


Fig. 2. Typical trajectories followed by the target. Through the velocity input sequence, we have an immediate description of the segment lengths $A_1$ and $A_2$ and their relative orientation $\delta_1$. As an example, the anchor $B_2$ is represented together with its distances from the target at three consecutive time instants.

where $v_{x,k} = v_x(kT_s)$ and $v_{y,k} = v_y(kT_s)$. In the following, we will denote the position of the target at time step $k$ as $P_k = [x_k, y_k]^\top$, and define $A_k$ as the length of the path travelled by the target between steps $k$ and $k+1$, i.e.

$$A_k = \|P_{k+1} - P_k\| = \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}.$$

Notice that we can compute the value of $A_k$ by using the discrete-time dynamics (2) of the target, thus yielding

$$A_k = \sqrt{\Delta x^2 + \Delta y^2}, \quad (4)$$

which only depends on the relative displacements $\Delta x$ and $\Delta y$ in (3).

Moreover, we can express the angular increment $\delta_k$ described by the segments connecting $P_k$ to $P_{k+1}$, and $P_{k+1}$ to $P_{k+2}$ as

$$\begin{aligned} \delta_k &= \alpha_{k+1} - \alpha_k \\ &= \arctan2(v_{y,k+1}, v_{x,k+1}) - \arctan2(v_{y,k}, v_{x,k}), \end{aligned} \quad (5)$$

where

$$\begin{aligned} \alpha_k &= \arctan2({}^{\langle B\rangle}y_{k+1} - {}^{\langle B\rangle}y_k, {}^{\langle B\rangle}x_{k+1} - {}^{\langle B\rangle}x_k) + \phi \\ &= \arctan2(v_{y,k}, v_{x,k}) + \phi. \end{aligned} \quad (6)$$

This way, the target trajectories can be represented by segments connecting $P_k$ and the successive points $P_{k+1}$ by using the length $A_k$ and the inclination $\alpha_k$, as represented in Figure 2.

**Remark 1.** *We assume in this paper that the inputs $v_{x,k}, v_{y,k}$ are known perfectly (i.e., no measurement uncertainty), and even in this setting the trilateration problem arises. Moreover, without the ranging measurements, since we are not aware of the initial position $P_0$ of the target and of its inclination $\phi$, given the history of the inputs $v_{x,k}, v_{y,k}$ in any discrete-time interval $[0, 1, \ldots, k_f]$, we are not able to reconstruct the absolute trajectory in the inertial reference frame $\langle I \rangle$, but we can only reconstruct the "relative geometrical shape" of the trajectory, i.e. the length of the segments $A_k$, $k \in [0, k_f]$ and their relative angle $\delta_k$, $k \in [0, k_f - 1]$. This is an immediate consequence of the knowledge of relative measurements.*

### *B. Sensor model*

We assume that the environment is equipped with a set of anchors, e.g., UWB anchors, $B_i = [X_i, Y_i]^\top$, $i = 1, \ldots, n$, retrieving the distance to the target, i.e. the measurement output of the systems at time $k$ are the distances $\rho_{i,k}$, such that

$$\rho_{i,k}^2 = ({}^{\langle I \rangle}x_k - X_i)^2 + ({}^{\langle I \rangle}y_k - Y_i)^2. \tag{7}$$

### *C. Problem formulation*

It is widely known that the problem of positioning a target on a plane, i.e., to retrieve its coordinates $x_k, y_k$ at a certain time $kT_s$, is solved by means of *trilateration*, i.e., at time $kT_s$ at least three ranging measurements from non collinear anchors are available [23]. With respect to (7), it amounts to collect $\rho_{i,k}$, for $i = 1, \ldots, 3$, i.e., all the measurements come at the same time instant. In this case, the positioning problem is *statically observable*. When, instead, the measurements from the three anchors come at different time instants, e.g., we have access to $\rho_{1,k}$, $\rho_{2,k+1}$ and $\rho_{3,k+2}$, the positioning problem turns to a *localisation* problem [24], which entails the concept of *dynamic observability*, or simply *observability*. The main idea is that the notion of the motion model compensates for a reduced amount of measurements at time $k$. In this paper, we will prove that this is counterintuitively: we analyse both the two different situations: the first is the traditional *simultaneous trilateration* problem where the three landmarks retrieve the distance measurements at the same time, and then we will analyse the problem of the *delayed trilateration*, where the measurements are retrieved at three different time steps. In the latter case, we will show that three measurements from three different anchors *are not sufficient*: in other words, the standard trilateration does not consider just three measurements, but *four*: the last one is the knowledge of the *simultaneous* measurements. In carrying out the analysis, we are not considering explicitly the role played by the measurement uncertainties. In fact, the results here obtained are applicable also in the *ideal* case, i.e., perfect measurements.

We would like here to stress that the problem we are dealing with is associated with the concept of *observability*, which depends only on the dynamics of the system, on the model of the sensors and on the trajectory followed by the system itself. Therefore, actuation uncertainty and measurement noise play no role at this level [24].

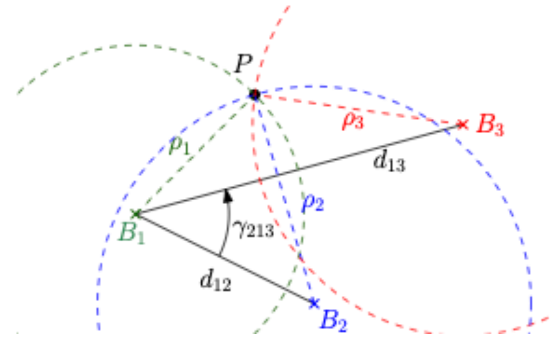


Fig. 3. Three range sensors measure their distance from the target: whenever the three anchors are not aligned (i.e. $\gamma_{213} \neq h\pi$), we have only one intersection among the three circles, i.e. we know where the target is.

## III. Trilateration

As aforementioned, the simultaneous trilateration involves three anchors retrieving the distance ideal measurements from the target at the same time. To compact the notation, in the following we will drop the subscript $k$ in (7). We introduce here the formal definition of trilateration and its proof.

**Proposition 1** (Simultaneous trilateration)**.** *Let $P = [x, y]^\top \in \mathbb{R}^2$ be the position of the target on the plane and let $B_i = [X_i, Y_i]^\top$, $i = 1, 2, 3$ be the positions of three anchors, each of them measuring their distance $\rho_i$ from $P$. Whenever the three anchors are not collinear, $P$ is the only point compliant with the three retrieved distances.*

*Proof.* By taking the differences $\rho_2^2 - \rho_1^2$ and $\rho_3^2 - \rho_1^2$, we come up with two linear equations in the unknown $x, y$, reading

$$M \begin{bmatrix} x \\ y \end{bmatrix} = h, \qquad \text{with } M = \begin{bmatrix} X_1 - X_2 & Y_1 - Y_2 \\ X_1 - X_3 & Y_1 - Y_3 \end{bmatrix} \tag{8}$$

which is invertible as soon as $M$ is nonsingular, i.e. $\det M \neq 0$. The determinant of $M$ can be obtained as the only nonzero element of the cross product between $B_2 - B_1$ and $B_3 - B_1$

$$\begin{bmatrix} 0 \\ 0 \\ \det M \end{bmatrix} = \begin{bmatrix} X_2 - X_1 \\ Y_2 - Y_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} X_3 - X_1 \\ Y_3 - Y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d_{12} d_{13} \cos \gamma_{213} \end{bmatrix}, \tag{9}$$

where $\cdot \times \cdot$ denotes the cross product between two vectors, $d_{12}$ and $d_{13}$ are the distances between the anchors $B_1$ and $B_2$, and $B_1$ and $B_3$, respectively, while $\gamma_{213}$ is the amplitude of the angle described by the three anchors, with vertex $B_1$, as represented in Figure 3. Whenever $\gamma_{213} = 0$, the three anchors are collinear and we are not able to uniquely identify the position $P$ of the target. □

The widely known geometric interpretation is the following: for each anchor $B_i$, we build a circle centred in the anchor itself, with *radius* equal to the retrieved distance $\rho_i$. The three circles have two intersection points as soon as the three centres are aligned, otherwise they only have one unique intersection.

**Remark 2.** *The proof of Proposition 1 is built upon the differences of the squares of the distances, which ensures that the solution will correspond to the actual target location.*

*However, when the distances are not collected from a real scenario, but the positions of the anchors and their ranges are fixed upfront, we can still find a point* $[x, y]^\top$ *by using* (8)*, but it will not be a solution to* (7).

Although Remark 2 seems to account for a situation that is never occurring, it turns out to be fundamental: indeed, a solution to the trilateration problem may be wrongly considered correct even if $M$ is invertible but the circles do not intersect in a single point. We will explicitly consider this situation in Section III-A, where we introduce the concept of delayed trilateration, i.e., the ranging measurements are collected at different time instants for a target that is moving, which may lead to the problem discussed in Remark 2.

**Remark 3.** *In the case of simultaneous trilateration, we are able to reconstruct the position $P$ of a still target independently on the orientation angle $\phi$, which has no effect on the measurements retrieved by the three landmarks, since it does not appear in the definition of the distances in* (7).

### A. Delayed trilateration

We address now the case of the delayed trilateration. In this scenario, the target is assumed to move according to (2) with unknown initial position $P_1 = [x_1, y_1]^\top$ and unknown orientation $\phi$. Assuming that the three measurements in (7) are given at time instants $k_1 \neq k_2 \neq k_3$ and by leveraging on our knowledge on the system inputs over time (see Remark 1), we will try to recover the unknown initial condition $P_1$ in order to reconstruct the entire trajectory (indeed, the system inputs are assumed to be perfectly known).

In the previous section, we have used the condition of *noncollinearity* among the three anchors $B_i$, in order to reconstruct the position $P$ of the target. Since in this scenario the target is moving, we will need a different *generalised noncollinearity* condition, as in the following definition.

**Definition 1** (Generalised noncollinearity)**.** *Given three consecutive positions $P_k$, $k = 1, 2, 3$ of the target and three landmarks $B_i$, $i = 1, 2, 3$, such that the $i$-th anchor distance to the target is retrieved at time $k_i$, the anchors are said noncollinear if the following holds:*

$$(\bar{B}_2 - \bar{B}_1) \times (\bar{B}_3 - \bar{B}_1) \neq 0,$$

*where the translated anchors $\bar{B}_i$ are defined as*

$$\bar{B}_1 = B_1, \quad \bar{B}_2 = B_2 - \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix}, \quad \bar{B}_3 = B_3 - \begin{bmatrix} \Delta x_1 + \Delta x_2 \\ \Delta y_1 + \Delta y_2 \end{bmatrix}.$$

From a geometric point of view, the condition expressed in Definition 1 may be interpreted as follows: we move the pair anchor–measurement (i.e. the pair $B_i$–$P_i$) such that all the measured points $P_i$ coincide with $P_1$, to recover a scenario similar to trilateration. The *generalised noncollinearity* holds if the three translated anchors $\bar{B}_i$ are not collinear.

For the sake of simplicity and without loss of generality, we assume that $k_1 = 1$, $k_2 = 2$ and $k_3 = 3$ in (7), while we are interested in the initial position $P_1 = [x_1, y_1]^\top$ of the target, together with the inclination $\phi$ (see Figure 4). In light of

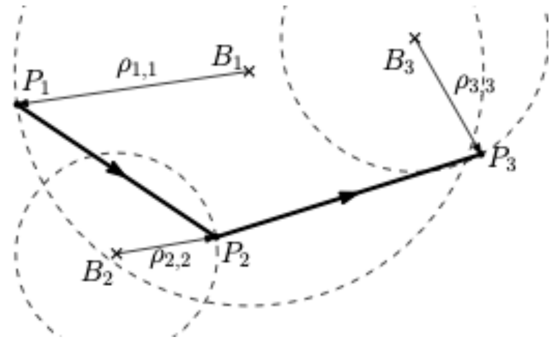


Fig. 4. Three anchors $B_i$ measure their distance $\rho_{i,i}$ from the target, each of them at time $k_i = i$.

Definition 1, we are ready to prove the following proposition.

**Proposition 2.** *Given the target dynamics* (2)*, the system inputs $v_{x,k}, v_{y,k}$, $k = 1, 2$, the sensor model* (7)*, the measurement outputs $\rho_{1,1}, \rho_{2,2}, \rho_{3,3}$ and the initial angle $\phi$, we can reconstruct the initial position $P_1$ of the target only if the generalised noncollinearity condition holds.*

*Proof.* For the proof of this proposition, we follow the same *rationale* as in the proof of Proposition 1, thus we build the differences $\rho^2_{2,2} - \rho^2_{1,1}$ and $\rho^2_{3,3} - \rho^2_{1,1}$. By leveraging on Definition 1, we compute the $i$-distance as

$$\rho_{i,i} = \|B_i - P_i\| = \|\bar{B}_i - P_1\|.$$

Being $P_1$ constant and common to all the measurement results, we recover the same structure as in the proof of Proposition 1

$$\bar{M} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \bar{h}, \qquad \text{with } \bar{M} = \begin{bmatrix} \bar{X}_1 - \bar{X}_2 & \bar{Y}_1 - \bar{Y}_2 \\ \bar{X}_1 - \bar{X}_3 & \bar{Y}_1 - \bar{Y}_3 \end{bmatrix}, \quad (10)$$

where $\bar{X}_i$ and $\bar{Y}_i$ are such that $\bar{B}_i = [\bar{X}_i, \bar{Y}_i]^\top$, $i = 1, 2, 3$.

Given the structure of $M$, we know that the matrix is invertible if $\bar{B}_1$, $\bar{B}_2$ and $\bar{B}_3$ makes the matrix $\bar{M}$ invertible, thus compliant with the condition of *generalised noncollinearity* of the three anchors $B_1, B_2, B_3$. □

Proposition 2 states that there exists *only one* trajectory compliant with the manoeuvres performed by the target, with its initial inclination $\phi$ and with the three measurement retrieved by the sensors. However ,we can draw a consideration that directly descends from Remark 2, which is discussed in the following remark and turns out to be fundamental.

**Remark 4.** *As in the case of simultaneous trilateration, the proof of Proposition 2 is based on the differences of the collected distances. However, in this situation, we use the measurement collected by the vehicle moving across the environment, but we fix an arbitrary value of $\phi$, which leads us to find an initial point $P_1$ according to* (10)*, but we have no guarantees that $P_1$ is also a solution to* (7)*. Thus solutions to* (7) *may be found only fixing some specific (unknown) values for the inclination angle $\phi$.*

The main difference with the case of simultaneous trilateration is the dependence on $\phi$ $M$ in (10) for the delayed trilateration. Therefore, there exist multiple solutions having the same sequence of manoeuvres and of measurements, but different values of $\phi$. As a consequence, based on Proposition 2, we can state that the knowledge of the system inputs, the model and the measurement is not sufficient to reconstruct $P_1$. Considering the Remark 2 and the Remark 4, we are now ready to introduce the main proposition of this paper.

**Proposition 3.** *Given a target moving accordingly to* (2) *with known velocity inputs* $v_{x,k}, v_{y,k}$, $k = 1, 2$, *and three fixed-frame anchors* $B_1, B_2, B_3$, *measuring their distance from the target at time* $k = 1, 2, 3$ *respectively, we cannot localise the target in the environment, i.e. we cannot reconstruct its initial position* $P_1$ *in the inertial reference frame.*

The main consequence of this proposition is that three ranging measurements are not sufficient for trilateration, but the three measurements should be collected simultaneously to solve the problem, even with the additional perfect knowledge of the model and the system inputs.

## IV. Simulation results

To support our claim, we provide a simulation of the scenario presented in Proposition 3, with a counterexample that shows that, despite the assumptions in Proposition 2 hold, there are many trajectories that are compliant with the manoeuvres performed by the target and with the range measurements retrieved by the anchors.

**Example 1.** We assume that the target moves with given inputs (thus we assume to know the relative displacement in (3) and the shape of the trajectory) and collects one measurement from each of the three anchors. We further assume the following configuration:

Sensor positions:

$$B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 9 \\ 6 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 14 \\ 3 \end{bmatrix},$$

Sensor readings:

$$\rho_{1,1} = 4, \quad \rho_{2,2} = 3, \quad \rho_{3,3} = 2,$$

Control inputs:

$$v_{x,1} = 5,\ v_{y,1} = 3, \quad v_{x,2} = 7,\ v_{y,2} = -4,$$

with sampling time $T_s = 1\,s$.

Figure 5 shows the results obtained in the simulation with the parameters above, i.e. a set of four trajectories that are compliant with both the manoeuvres performed by the target (see Remark 1) and with the readings of the three range sensors. As reported in Table I, where the results obtained with the four distinct solutions represented in Figure 5 are quantified, we can notice that: despite the generalised noncollinearity condition holds (for each trajectory, we can check the generalised noncollinearity condition by building the matrix $M$ as in (10) and computing its determinant $\det M$: the last row of Table I contains only non-zero values), all the four solutions are compliant with manoeuvres and measurements but are all different: one may simply check by using the obtained numerical results about the positions $P_1$, $P_2$ and $P_3$.

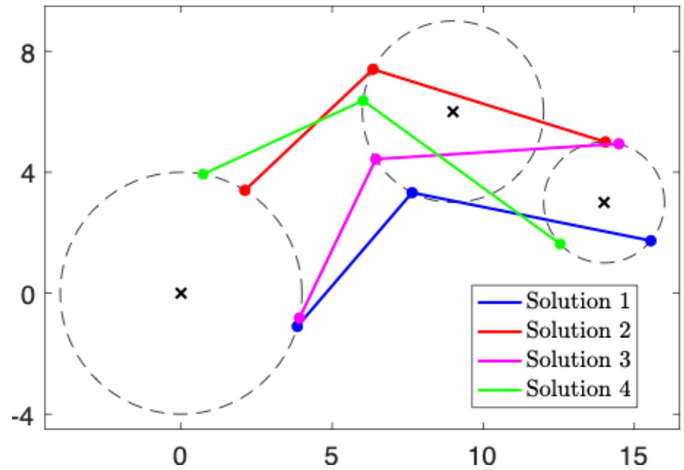


Fig. 5. Graphical representation of Example 1. In this case, despite the *generalised noncollinearity* condition holds, we have (at least) four trajectories that are compliant with the manoeuvres and the measurements retrieved by the three anchors, showing that three range measurements are not sufficient to uniquely identify the trajectory followed by the target.

| | Solution 1 | Solution 2 | Solution 3 | Solution 4 |
|---|---|---|---|---|
| $P_1$ | 3.85<br>$-1.10$ | 2.11<br>3.40 | 3.91<br>$-0.82$ | 0.73<br>3.93 |
| $P_2$ | 7.64<br>3.32 | 6.35<br>7.40 | 6.44<br>4.43 | 6.02<br>6.37 |
| $P_3$ | 15.55<br>1.74 | 14.04<br>5.00 | 14.49<br>4.94 | 12.54<br>1.63 |
| $\det M$ | $-2.76$ | 2.54 | $-20.43$ | 11.87 |

TABLE I

Numerical results of the simulation in Example 1 and depicted in Figure 5. For each solution, we report the position of the points $P_k$ reached by the target at time $k$ and whose distance is measured by the anchor $B_k$. In the last row we report the determinant of matrix $M$ built as in (10).

From the analysis carried out on the simultaneous and delayed trilateration, which are supported by the results obtained in the numerical example, we draw the following consideration: even though the intuition suggests that with three measurements we are able to reconstruct the position of the target on the plane $\mathbb{R}^2$, this is not sufficient whenever we add the dynamics of the system, i.e. whenever the target moves while the measurements are taken. The problem of finding the minimum number of measurements needed to find the position of the target with *three simultaneous* ranging measurements is exhaustively addressed in Proposition 1, while Example 1 and Proposition 3 state that the minimum number of anchors with *delayed measurements* to reconstruct the target location is still open. From a practical point of view, we are considering a target that is initially unaware of its position and orientation on the plane. These results imply that, when it collects 3 measurements with a delayed trilateration as in Section III-A, the target can build a set of positions where it could be located. Although the actual position of the vehicle is included in this set, the target cannot retrieve it with only 3 measurements.

With these considerations in mind, we are now ready to state the main claim of the paper with a clear statement.

**Claim 1.** *Whenever we consider a scenario of simultaneous trilateration as in Section III, we are collecting* ***three*** *measurements from the ranging sensors, but we are also relying on one* ***additional*** *information, which is the implicit assumption that the target is still.*

Notice that this assumption is not explicitly used in the computations and proofs (see proof to Proposition 1), but it allows us to find the position of the target on the plane. This claim is supported by Section III-A and by the numerical simulation in the Example 1, where the implicit assumption of simultaneous measurements is explicitly removed, i.e. the target is not still while the measurements are collected, thus leading to reconstruction failure.

## V. Conclusions

In this paper, we have revisited the well-known results on trilateration exposing the implicit time requirements, i.e., the readings should be collected simultaneously. Irrespective of the fact that the measurements are collected by a still agent or not, if the three anchors collecting the ranging measurements are not collinear, the positioning problem is solved. By considering for the first time the concept of *delayed trilateration*, where the target is moving and the anchors are collecting the measurements at different time instants, we have concluded that three measurements are not sufficient to uniquely determine the trajectory of the target and a larger number of measurements is needed. Given this mismatch in the two different scenarios, we have assumed that the simultaneous trilateration relies on four measurements: three ranging measurements and a fourth additional and implicit assumption that the target is still in the environment when the measurements are taken.

Therefore, we have underlined how three measurements are not sufficient to reconstruct the state of the agent, i.e., localise it in the environment, but we have given no hint on the minimum number of measurements and anchors needed to reconstruct the state, i.e., to attain the so-called *global observability*. Therefore, we plan to address this problem, by considering different dynamical models and a more general setup in the number of measurements retrieved by each anchor.

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