A TEST FOR THE DIVERGENCE OF CERTAIN SERIES WITH POSITIVE TERMS

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ABSTRACT. In this short note we prove a simple divergent test for certain series with positive terms which are related in some way with the Theory of Dirichlet's series.

1. Introduction

While I was preparing some examples for ordinary Dirichlet series I encountered the following problem: I need to show that the series

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n}$$

is divergent. Since the method I used can be somewhat generalized it seems to me worthwhile to write down this short note.

2. The theorem

Theorem 1. *Let be*

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n}. \tag{1}$$

if

(1) $f : \mathbb{R} \to \mathbb{R}$ *is a bounded function.*
(2) *There exists a real number* c *and two bounded functions* $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ *such that*

$$0 < f(c) \leqslant f(n+1)g(n) + h(n)\, f(n) \quad \forall n \in \mathbb{N}.$$

then the series (1) *diverges.*

Proof. Since the functions f, g, h are bounded on $\mathbb{R}$ we can choose a positive constant M so that, for each $n \in \mathbb{N}$ it is
$|f(n)| < M$, $|g(n)| < M$,$|h(n)| < M$. From the triangular inequality we have
$0 < |f(c)| \leqslant |f(n+1)|\,|g(n)| + |f(n)|\,|h(n)| \leqslant M\left(|f(n+1)| + |f(n)|\right).$

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so that

$$(|f(n+1)| + |f(n)|) \geqslant \frac{M}{f(c)}.$$

Now, since

$$\frac{|f(n)|}{n} = \left(\frac{|f(n)| - |f(n+1)|}{2n}\right) + \left(\frac{|f(n)| + |f(n+1)|}{2n}\right).$$

it follows that

$$\frac{|f(n)|}{n} \geqslant \left(\frac{|f(n)| - |f(n+1)|}{2n}\right) + \frac{M}{2f(c)}\frac{1}{n}.$$

Since, for each $k \in \mathbb{N}$ it is

$$S_k = \sum_{n=1}^{k} (|f(n)| - |f(n+1)|) = |f(1)| - |f(k+1)|$$

we have that

$$|S_k| \leqslant |f(1)| + M < +\infty$$

and so the series

$$\sum_{n=1}^{+\infty} \frac{|f(n)| - |f(n+1)|}{2n}$$

converges by Dirichlet's convergence test. Thus

$$\lim_{m\to\infty} \sum_{n=1}^{m} \frac{|f(n)|}{n} \geqslant \lim_{m\to\infty} \sum_{n=1}^{m} \frac{|f(n)| - |f(n+1)|}{2n} + \lim_{m\to\infty} \frac{M}{2f(c)} \sum_{n=1}^{m} \frac{1}{n} = +\infty$$

and the given series is divergent. □

Corollary 1. *The series*

$$\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n}, \quad \sum_{n=1}^{\infty} \frac{|\cos(n)|}{n}$$

are divergent.

Proof. It is enough to observe that for the first series we have

$$0 < \sin(1) = \sin(n+1-n) = \sin(n+1)\cos n - \cos(n+1)\sin n$$

so that

- $c = 1$.
- $g(n) = \cos n$ and $h(n) = -\cos(n+1)$.

while, for the second, it is

$$0 < \cos\left(\frac{\pi}{2} - 1\right) = \sin(1) = \sin(n+1)\cos n - \cos(n+1)\sin n$$

so that

- $c = \frac{\pi}{2} - 1$.
- $g(n) = \sin(n+1)$, $h(n) = -\sin(n)$.

□

Corollary 2. *The Dirichlet's series*

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^s}, \quad \sum_{n=1}^{\infty} \frac{|\cos n|}{n^s}$$

have abscissa of convergence $\sigma_c = 1$.

References

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