A FINITARY OUTER MEASURE LOGIC

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Abstract. We extend classical first order logic with a family of weak probability quantifiers, that we call submeasure quantifiers. Formulas are finitary, but infinitary deduction rules are needed. We consider first order structures which are equipped with a countable family of submeasures (hence the name of the new quantifiers). We prove that every consistent set of sentences in the resulting logic is satisfiable in some structure as above. Then we restrict the set of formulas by requiring that no submeasure quantifier occurs within the scope of some classical quantifier. By suitably extending the deduction rules, we prove that every consistent set of sentences from the restricted class of formulas is satisfiable in some structure whose submeasures are actually outer measures. To perform the last step, we apply nonstandard techniques *à la* A. Robinson.

1. Introduction

It is well known that, in general, measurability is not preserved under projections. Therefore many of the logics for probability that appear in the literature do discard the classical quantifiers and replace them with suitable probability quantifiers. As for the semantic side, the corresponding structures must satisfy necessary measurability conditions. See, for instance, [2] or [3].

In this paper we introduce a logic that extends classical first order logic by means of submeasure (or outer measure) quantifiers. We deal with finite submeasures only. Since submeasures are defined on all subsets of the domain under consideration, there is no problem with the definition of the satisfiability relation.

Ours is a logic with finitary formulas and some infinitary deduction rules. The latter are needed in order to get the equivalence of consistency and satisfiability. Throughout this paper, by *weak completeness* we mean the property that every consistent set of sentences is satisfiable in some first order structure which is equipped with a countable family of submeasures (see the description of an *outer measure structure* below).

Here is an outline of the paper: in Section 2 we introduce the setting. We describe the syntactic machinery and the class of structures that we consider. We call *submeasure structure* a first-order structure that is equipped with a countable family of $[0,1]$-valued submeasures, each defined on the

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power set of some finite power of the domain under consideration. An *outer measure structure* will be a submeasure structure with the property that each submeasure is actually an outer measure. The family of submeasures is required to satisfy some compatibility conditions, in particular a weak Fubini property. In Section 2, we also provide motivations for our approach and we relate it to the existing literature.

In Section 3, we get weak completeness (see above). Since we deal with finite formulas, we can perform a Henkin construction. We prove that every consistent set of sentences extends to a maximal consistent one which has the Henkin property in some suitably expanded language. Notice that, in presence of infinitary deduction rules, the proof that consistency is preserved by unions of ascending chains is not straightforward. Actually, one can easily verify that the infinitary rules make our logic non-compact.) After establishing the above mentioned extension result, we take inspiration from [2] to prove that every maximal consistent set of sentences with the Henkin property is satisfiable in some submeasure structure.

In Section 4, we restrict the set of formulas to those in which no submeasure quantifier occurs within the scope of a classical quantifier. We extend the logic by means of infinitary rules expressing suitable continuity properties. At the same time, we impose corresponding continuity properties on the class of submeasure structures, by restricting ourselves to the subclass of what we call *continuous submeasure structures*. Then we show that weak completeness holds with respect to the latter subclass. Finally, we apply nonstandard techniques *à la* A. Robinson and, adapting arguments from [2], we convert a continuous submeasure model of a set of sentences into a continuous outer measure model of the same set of sentences.

The extension of the latter result to the class of all formulas and the study of the model-theoretic properties of our logic will not be addresses in this work. We will briefly comment on those issues in Section 5.

2. Preliminaries

Throughout this paper, a *submeasure* on a set A is a function $\mu : P(A) \to [0,1]$ with the following properties:

(o1) $\mu(\varnothing) = 0$;
(o2) for all $B \subset C \in P(A)$, $\mu(B) \leq \mu(C)$;
(o3) for all $B, C \in P(A)$, $\mu(B \cup C) \leq \mu(B) + \mu(C)$.

An *outer measure* on A is a submeasure which also satisfies:

(o4) for every sequence $(B_i)_{i\in\omega}$ in $P(A)$, $\mu(\bigcup_{i\in\omega} B_i) \leq \sum_{i\in\omega} \mu(B_i)$.

We often refer to properties (o3) and (o4) as to *finite* and *countable subadditivity*, respectively.

We work in an extension of classical first order logic with equality.

In addition to the usual first order quantifiers and connectives, we have, for each finite sequence $\bar{x}$ of variables and each rational number $0 \leq r \leq 1$, the *submeasure quantifiers* $S\bar{x} \leq r$ and $S\bar{x} \geq r$.

We assume that the set of extralogical symbols (finitary function and relation symbols and constant symbols) is countable. As usual, we identify a language with the set of its extralogical. From now on L will denote some countable language. If f is a function symbol we shall write $f \in L$ meaning that f is a function symbol in L. We shall denote the arity of f by n_f. Same conventions hold for the predicate symbols.

The formation rules of terms are those of first order logic. The formula formation rules of first order logic are extended with the following clauses:

- $S\bar{x} \leq r\,\varphi$ is a formula whenever φ is a formula, $\bar{x}$ is a finite sequence of variables and $r \in \mathbb{Q} \cap [0,1]$;
- $S\bar{x} \geq r\,\varphi$ is a formula whenever φ is a formula, $\bar{x}$ is a finite sequence of variables and $r \in \mathbb{Q} \cap [0,1]$.

We write $S\bar{x} > r\varphi$ and $S\bar{x} < r\varphi$ as abbreviations for $\neg(S\bar{x} \leq r\varphi)$ and $\neg(S\bar{x} \geq r\varphi)$ respectively. The intended meaning of the formula $S\bar{x} \geq r\varphi(\bar{x})$ is that the submeasure of the set defined by the formula $\varphi(\bar{x})$ is at least r. Similarly with the other formulas.

Notice that, differently from [3], we have two kinds of primitive submeasure quantifiers. This is due to the fact that submeasures satisfy only a subadditivity property. We stress that the formulas are finitary.

We work in a natural deduction setting (this will simplify the formulation of some infinitary rules to be introduced below), enriched with axioms that govern the submeasure quantifiers or relate them to the classical quantifiers. Indeed, as the reader can easily verify, we may entirely dispose of those axioms and replace them with corresponding inference rules.

a1 $S\bar{x} \geq 0\,\varphi$

a2 $S\bar{x} \leq 1\,\varphi$

a3 $S\bar{x} \geq r\,\varphi \to S\bar{x} \geq s\,\varphi$ if $r > s$

a4 $S\bar{x} \leq s\,\varphi \to S\bar{x} \leq r\,\varphi$ if $r > s$

a5 $S\bar{x} > r\,\varphi \to S\bar{x} \geq r\,\varphi$

a6 $S\bar{x} > 0\,\varphi \to \exists\bar{x}\varphi$ (the empty set has measure zero)

a7 $\forall\bar{x}(\varphi \to \psi) \to (S\bar{x} \geq r\,\varphi \to S\bar{x} \geq r\,\psi)$ (monotonicity)

a8 $S\bar{x} \leq r\,\varphi \wedge S\bar{x} \leq s\,\psi \to S\bar{x} \leq r + s(\varphi \vee \psi)$ (finite subadditivity)

a9 $(S\bar{x} \leq r\,\varphi) \wedge (S\bar{y} \leq s\,\psi) \to S\bar{x}\bar{y} \leq rs(\varphi \wedge \psi)$ if $\bar{x}$ and $\bar{y}$ are disjoint sequences of variables.

a10 $S\bar{x} \leq s\,\varphi \to S\bar{x} < r\,\varphi$ if $r > s$.

a11 $S\bar{x} \geq r\,S\bar{y} \geq s\,\varphi(\bar{x},\bar{y}) \to S\bar{x}\bar{y} \geq rs\,\varphi(\bar{x},\bar{y})$ if $\bar{x}$ and $\bar{y}$ are disjoint sequences of variables.

a12 $S\bar{x} \leq r\,\varphi \to S\bar{y} \leq r\,\varphi$ if the sequence $\bar{y}$ is a permutation of the sequence $\bar{x}$.

a13 $S\bar{x} \geq r\,\varphi \to S\bar{y} \geq r\,\varphi$ if the sequence $\bar{y}$ is a permutation of the sequence $\bar{x}$.

An addition (subtraction) symbol appearing in some outer quantifier (see, for instance, the conclusion of axiom a8 above or rules r1, r2 below) always denotes truncated addition (subtraction) in the unit interval.

We point out that the restriction to finitary formulas prevents us from syntactically formulating the countable subadditivity or the archimedean property of [3] as axioms. We manage to recover the archimedean property by means of two infinitary rules:

r1
$$\frac{S\bar{x} \le (r + 1/n)\,\varphi \qquad \text{for all } n \in \omega_+}{S\bar{x} \le r\,\varphi}$$

r2
$$\frac{S\bar{x} \ge (r - 1/n)\,\varphi \qquad \text{for all } n \in \omega_+}{S\bar{x} \ge r\,\varphi}$$

According to the natural deduction practice, we regard derivations as rooted trees. The inductive definition of the set of derivations is standard.

The *depth* $d(\mathcal{D})$ of a derivation $\mathcal{D}$ is an ordinal. Atomic derivations (axioms or assumptions) have depth zero and, in case $\mathcal{D}$ ends, say, with an application of the infinitary rule r1, $d(\mathcal{D})$ is recursively defined as follows: let $\mathcal{D}_n$ be the derivation of the premiss $S\bar{x} \le (r + 1/n)\,\varphi$, then

$$d(\mathcal{D}) = \sup\{d(\mathcal{D}_n) + 1 : n \in \omega_+\}.$$

The other cases are formulated in a similar way.

Notions like provability, consistency, maximal consistency are standardly defined and the usual closure properties of maximal consistent sets can be easily proved. Due to the presence of infinitary rules, the extension of a consistent set of formulas to some maximal consistent one is not anymore a straightforward application of Zorn's Lemma.

Furthermore, the recursive definitions of the sets of free/bound variables in a formula and the operation of substitution of a term for a variable in a formula can be easily given, keeping in mind that both the sm-quantifiers $S\bar{x} \ge r$ and $S\bar{x} \le r$ bind all the variables occurring in the sequence $\bar{x}$. Also, the usual properties of substitutions can be easily proved.

When writing $\varphi(\bar{x})$ we mean that all the free variables occurring in the formula φ appear in the sequence $\bar{x}$. We adopt standard model-theoretic notation. We denote by $|\bar{x}|$ the length of a tuple $\bar{x}$.

We call *submeasure structure* (sm-structure, for short) a pair

$$\mathcal{M} = (\mathbf{M}, (\mu_n)_{n \ge 1}),$$

where $\mathbf{M}$ is an ordinary first-order L-structure and $(\mu_n : P(M^n) \to [0,1])_{n \ge 1}$ is a sequence of submeasures which satisfies the properties:

(o5) $\mu_k\{\bar{a} : \mu_n\{\bar{b} : (\bar{a}, \bar{b}) \in A\} \ge r\}\} \ge s \;\Rightarrow\; \mu_{k+n}(A) \ge rs$, for all $A \subseteq M^{k+n}$ and all $r, s \in \mathbb{Q} \cap [0,1]$.

(o6) $\mu_{k+n}(A \times B) \leq \mu_k(A)\mu_n(B)$ for all $A \subseteq M^k, B \subseteq M^n$.
(o7) $\mu_n(A) = \mu_n(\pi(A))$ for all $A \subset M^n$ and all coordinate permutations π.

Notice that axiom a11 and property (o5) state weak Fubini properties. These properties will play a crucial in role in establishing weak completeness.

An *outer measure structure* (om-structure, for short) is an sm-structure where each μ_n is an outer measure. In the current setting, the notions of sm- and om-structure correspond to those of *weak structure* and *graded probability structure* of [2] or [3], respectively.

A very simple example of om-structure is given by $(\mathbf{M}, (\mu_n)_{n\geq 1})$, where $\mathbf{M}$ is some finite first order structure and each $\mu_n : P(M^n) \to [0,1]$ is the counting measure.

The interpretation of a term and the truth value of a formula in an sm-structure $\mathcal{M}$ under an assignment of values in M to its free variables are defined as in first order logic, with the additional clause:

$$\mathcal{M} \vDash S\bar{x} \geq \varphi(\bar{x}, \bar{y})[\bar{b}] \iff \mu_{|\bar{x}|}(\{\bar{a} \in M^{|\bar{x}|} : \mathcal{M} \vDash \varphi[\bar{a}, \bar{b}]\}) \geq r$$

and with a similar clause for the sm-quantifier $S\bar{x} \leq r$. It can easily be proved that the truth value of a formula in $\mathcal{M}$ depends only on the assignment of values to the free variables in the formula.

3. Soundness and weak completeness

Theorem 1. *(Soundness Theorem) Every set of L-formulas which is satisfiable in some em-structure is consistent.*

Proof. It suffices to show that the axioms are valid and the deduction rules preserve validity. In particular, validity of a9 and a11 follows from (o6) and (o5) respectively.

Then, by induction on the depth of a derivation of a formula φ from a set Γ, one shows that $\Gamma \vdash \varphi$ implies $\Gamma \vDash \varphi$. The conclusion follows. □

Next we want to prove a converse of Theorem 1. For sake of simplicity we work with sets of L-sentences. We use a modified Henkin construction. Differently form [2], we do not make use of *consistency properties.*

We say that a set Γ of L-sentences has the *Henkin property in L* if, for every formula $\varphi(x)$ there exists some L-constant symbol c such that

$$\Gamma \vdash \exists x \varphi(x) \to \varphi(c/x).$$

Theorem 2. *Let Γ be a consistent set of L-sentences. Then there exist a countable language $L^* \supseteq L$ and a set $\Gamma^* \supseteq \Gamma$ of L^*-sentences such that Γ^* is maximal consistent and has the Henkin property in L^*.*

Proof. We recursively define increasing sequences $(L_n)_{n\in\omega}$ of languages and $(\Gamma_n)_{n\in\omega}$ of sentences with the properties that, for all $n \in \omega$,

(a) Γ_n is a consistent set of L_n-sentences;

(b) for each L_n-sentence φ, exactly one of $\varphi, \neg\varphi$ belongs to Γ_{n+1}.

We let $L_0 = L$ and $\Gamma_0 = \Gamma$. We define L_1 and Γ_1 (the same construction applies to get L_{n+1} and Γ_{n+1} from L_n and Γ_n, respectively).

Let $C = \{c_0, \dots, c_n, \dots\}$ be a countable set of new constant symbols. We let $L_1 = L \cup C$. We fix an enumeration

$$\varphi_0, \dots, \varphi_n, \dots$$

of the L-sentences with the property that each sentence occurs infinitely many times in the enumeration. We define a nondecreasing sequence

$$\Gamma = \Delta_0 \subseteq \dots \subseteq \Delta_n \subseteq \dots$$

of consistent sets of L_1-sentences as follows:

(1) If $\Delta_n \cup \{\varphi_n\}$ is inconsistent, we distinguish two subcases.

(1.1) φ_n is of the form $S\bar{x} \leq r\,\psi(\bar{x})$: we claim that there is some $l \in \omega_+$ such that $\Delta_n \cup \{S\bar{x} \geq (r + 1/l)\,\psi\}$ is consistent. For if not, by a5 and r1, we get $\Delta_n \vdash S\bar{x} \leq r\,\psi$, contradicting the consistency of Δ_n.

We point out that, with the current formulation of the infinitary rules, the previous argument may not work in a Hilbert style setting with MP, r1 and r2 as the only deduction rules, due to lack of a Deduction Theorem.

We let $\Delta_{n+1} = \Delta_n \cup \{S\bar{x} \geq (r + 1/k)\,\psi\}$, where k is the least l such that $\Delta_n \cup \{S\bar{x} \geq (r + 1/l)\,\psi\}$ is consistent (actually, any such l would do).

(1.2) φ_n is of the form $S\bar{x} \geq r\,\psi(\bar{x})$: we apply a dual argument to the above and we let $\Delta_{n+1} = \Delta_n \cup \{S\bar{x} \geq (r - 1/k)\,\psi\}$, for some $k \in \omega_+$ such that $\Delta_n \cup \{S\bar{x} \leq (r - 1/k)\,\psi\}$ is consistent.

(2) if $\Delta_n \cup \{\varphi_n\}$ is consistent, we put φ_n in Δ_{n+1}. Moreover, if φ_n is of the form $\exists x \psi(x)$, we pick the first constant c in C not occurring in Δ_n and we put $\psi(c/x)$ in Δ_{n+1}.

Next we notice that if $\Delta_n \cup \{\varphi_n, \psi(c/x)\}$ were inconsistent then, by a generalization theorem on constants which holds for our system and by standard first-order arguments, we would contradict the consistency of $\Delta_n \cup \{\exists x \psi(x)\}$. Hence Δ_{n+1} is consistent.

We let $\Gamma_1 = \bigcup_{n\in\omega} \Delta_n$. Since infinitary rules are present, the consistency of Γ_1 needs to be proved. For sake of contradiction, let us assume that it is inconsistent and let $\mathcal{D}$ be a derivation of a contradiction from Γ_1. We outline a procedure that removes from $\mathcal{D}$ all the applications of the infinitary rules. Since a deduction which is free from applications of the infinitary rules depends on finitely many assumptions, which are contained in some Δ_n, we will get a contradiction.

By the previous observation, the interesting case is when some application of the infinitary rules occurs in $\mathcal{D}$. Let α be the least ordinal such that there is a subderivation $\mathcal{E}$ of $\mathcal{D}$ of depth α ending with an application of some infinitary rule. Let us assume that $\mathcal{E}$ ends with an application of r1 and that its end formula is $S\bar{x} \leq r\,\varphi$. If the end formula is already in Γ_1, then $\mathcal{E}$ can be contracted to one node, thus eliminating the application of r1.

Hence we are left with the case when $(S\bar{x} \leq r\,\varphi) \notin \Gamma_1$. By construction, there is $k \in \omega_+$ such that $S\bar{x} \geq r + \frac{1}{k}\,\varphi$ is eventually in $(\Delta_n)_{n\in\omega}$. On the other hand, the minimality of α implies that all the proofs of the premisses of the infinitary rule are finitary ones. Hence there is a a sufficiently large l such that, at the same time, $\Delta_l \vdash S\bar{x} \leq r + \frac{1}{k+1}\,\varphi$ and $\Delta_l \vdash S\bar{x} \geq r + \frac{1}{k}\,\varphi$. By a10, Δ_l would be inconsistent. Therefore the case $(S\bar{x} \leq r\,\varphi) \notin \Gamma_1$ does not happen.

The case when the subderivation $\mathcal{E}$ ends with an application of r2 can be treated in the same way.

Proceeding along the ordinals, we remove all the applications of the infinitary rules, thus getting a finitary proof of a contradiction from Γ_1 and, consequently, a proof of a contradiction from some Δ_n. Since all Δ_n's are consistent, we conclude that the inconsistency of Γ_1 must be rejected.

Concerning property (b), if some L-sentence φ does not belong to Γ_1 then there is $m \in \omega$ such that $\Delta_n \cup \{\varphi\}$ is inconsistent for all $m \leq n$. Recall that $\neg\varphi$ appears infinitely often in the enumeration and pick $k \geq m$ such that $\varphi_k = \neg\varphi$. If $\Delta_k \cup \{\neg\varphi\}$ were inconsistent, then Δ_k would be inconsistent too: a contradiction. Therefore, by construction, $\neg\varphi \in \Delta_{k+1} \subseteq \Gamma_1$.

So far we have established the consistency of Γ_1. As already mentioned, the above argument also proves the consistency of Γ_{n+1}, under the assumption of consistency of Γ_n.

Finally, we let $L^* = \bigcup_{n\in\omega} L_n$ and $\Gamma^* = \bigcup_{n\in\omega} \Gamma_n$. Consistency of Γ^* can be established in the same way as that of Γ_1: assume that Γ^* is inconsistent and prove that there is a finitary proof of inconsistency of Γ^*, hence of Γ_n, for some n: a contradiction. We leave the details to the reader.

Maximal consistency of Γ^* follows from property (b). The Henkin property for Γ^* can be easily proved.

□

The next result extends the so-called Model Existence Theorem of classical first-order logic to our setting. We take inspiration from [2] for the definition of the submeasures involved.

Theorem 3. *Let Γ be a maximal consistent set of L-sentences with the Henkin property in L. Then there exists a sm-structure $\mathcal{M}$ such that $\mathcal{M} \vDash \Gamma$. Moreover $\mathcal{M}$ can be taken to be countable.*

Proof. We proceed to define $\mathcal{M} = (\mathbf{M}, (\mu_n)_{n\in\omega_+})$. Let T be the set of closed L-terms. We let $M = T/\sim$, where $\sim$ is the equivalence relation defined by

$$s \sim t \iff (s = t) \in \Gamma.$$

We denote by $t^{\sim}$ the equivalence class of term t. For every constant symbol c, function symbol f and predicate symbol P we let

(m1) $c^{\mathbf{M}} = c^{\sim}$;

(m2) for all $t_1, \dots, t_{n_f} \in T$, $\quad f^{\mathbf{M}}(t_1^{\sim}, \dots, t_{n_f}^{\sim}) = (f(t_1, \dots, t_{n_f}))^{\sim}$;

(m3) for all $t_1, \dots, t_{n_P} \in T$, $\quad (t_1^{\sim}, \dots, t_{n_P}^{\sim}) \in P^{\mathbf{M}} \iff P(t_1, \dots, t_{n_P}) \in \Gamma$.

It is an easy consequence of the axioms for equality and the closure properties of maximal consistent sets that the interpretations of functions and predicate symbols are well-defined. Moreover $t^{\mathbf{M}} = t^{\sim}$ holds for all $t \in T$.

If $\bar{t}$ is a tuple of terms, we denote by $\bar{t}^{\sim}$ the tuple of their equivalence classes.

For each $n \in \omega_+$ we define μ_n as follows:

$$\mu_n(A) = \inf\{r : \ (S\bar{x} \le r\,\varphi(\bar{x})) \in \Gamma \text{ and } A \subseteq \{\bar{t}^{\sim} : \ \varphi(\bar{t}/\bar{x}) \in \Gamma\}\},$$

for all $A \in P(M^n)$. We stress that, in the definition above, the infimum is taken over all rationals $r \in [0,1]$ such that there exists in Γ some formula of the form $S\bar{x} \le r\,\varphi(\bar{x})$ with the property that $A \subseteq \{\bar{t}^{\sim} : \ \varphi(\bar{t}/\bar{x}) \in \Gamma\}$.

It can be easily verified that each μ_n satisfies properties (o1) and (o2) of a submeasure. Concerning property (o3), the nontrivial case is when $A, B \subseteq M^n$ are such that $\mu_n(A) + \mu_n(B) < 1$. Let $0 < \epsilon$ and let $r, s \in \mathbb{Q}$ be such that $\mu_n(A) < r < \mu_n(A) + \epsilon$ and $\mu_n(B) < s < \mu_n(B) + \epsilon$. Then there exist $S\bar{x} \le r\,\varphi, S\bar{x} \le s\,\psi \in \Gamma$ such that $A \subseteq \{\bar{t}^{\sim} : \ \varphi(\bar{t}/\bar{x}) \in \Gamma\}$ and $B \subseteq \{\bar{t}^{\sim} : \ \psi(\bar{t}/\bar{x}) \in \Gamma\}$. By a8 it follows that $S\bar{x} \le r + s\,(\varphi \vee \psi) \in \Gamma$. Moreover $A \cup B \subseteq \{\bar{t}^{\sim} : \ (\varphi \vee \psi)(\bar{t}/\bar{x}) \in \Gamma\}$. Being ϵ arbitrarily chosen, we conclude that $\mu_n(A \cup B) \le \mu_n(A) + \mu_n(B)$.

The proof that μ_n satisfies (o6) above is similar to that of finite subadditivity, with the help of axiom a9 and the possible use of suitable alphabetic changes: just notice that, for arbitrary $0 < \epsilon < 1$ and for $r, s \in \mathbb{Q}$ such that $\mu_k(A) < r < \mu_k(A) + \epsilon$ and $\mu_n(B) < r < \mu_n(B) + \epsilon$, it holds that $\mu_k(A)\mu_n(B) < rs < \mu_k(A)\mu_n(B) + 3\epsilon$.

Property (o7) is easily verified. Therefore it remains to prove that $\mathcal{M}$ satisfies (o5): we will do this after proving that

(1) $$\text{for all } L\text{-sentences } \varphi \qquad \mathcal{M} \models \varphi \iff \varphi \in \Gamma.$$

In order to establish (1), we repeatedly use the fact that

(2) $$\text{for every formula } \psi(\bar{x}) \qquad \mathcal{M} \models \psi[\bar{t}^{\sim}] \iff \mathcal{M} \models \psi(\bar{t}/\bar{x})$$

and we proceed by induction on φ. The atomic and the propositional cases are easily verified. If φ is of the form $\exists y \psi(y)$, we use (2), the inductive assumption and, for the right-to-left implication, the Henkin property.

Finally, we prove (1) for φ of the form $S\bar{x} \le r\,\psi(\bar{x})$ or $S\bar{x} \ge r\,\psi(\bar{x})$. The proofs of the two cases are quite similar. For this reason we will give a detailed proof of one of the two cases and we will sketch the other. We write $\psi(\bar{x})^{\mathcal{M}}$ for the set defined by $\psi(\bar{x})$ in $\mathcal{M}$. More explicitly, we let $\psi(\bar{x})^{\mathcal{M}} = \{\bar{t}^{\sim} \in M^{|\bar{x}|} : \mathcal{M} \models \psi(\bar{t}^{\sim})\}$.

– Let φ be of the form $S\bar{x} \leq r\,\psi(\bar{x})$.

($\Rightarrow$) Suppose $\mathcal{M} \models \varphi$. Then

$$\inf(\{s \in \mathbb{Q} : (S\bar{x} \leq s\,\eta(\bar{x})) \in \Gamma \text{ and } \psi(\bar{x})^{\mathcal{M}} \subseteq \{\bar{t}^{\sim} : \eta(\bar{t}/\bar{x}) \in \Gamma\}) \leq r.$$

We notice that, by inductive assumption, the inclusion in the above inequality is equivalent to

$$\{\bar{t}^{\sim} : \psi(\bar{t}/\bar{x}) \in \Gamma\} \subseteq \{\bar{t}^{\sim} : \eta(\bar{t}/\bar{x}) \in \Gamma\}.$$

Hence, by the Henkin property, the latter inclusion is equivalent to $\forall\bar{x}(\psi(\bar{x}) \to \eta(\bar{x})) \in \Gamma$. It follows from a7 that

$$\inf\{s \in \mathbb{Q} : S\bar{x} \leq s\,\psi(\bar{x}) \in \Gamma\} \leq r. \tag{3}$$

Finally, we claim that $(S\bar{x} \leq r\,\psi(\bar{x})) \in \Gamma$. For, if not, by the infinitary rule r1, there is $n \in \omega_+$ such that $(S\bar{x} > r + \frac{1}{n}\,\psi) \in \Gamma$. On the other hand, by (3) and by a4, $(S\bar{x} \leq r + \frac{1}{n}\,\psi) \in \Gamma$, contradicting the consistency of Γ.

($\Leftarrow$) Assume $\varphi \in \Gamma$. By inductive assumption $\psi(\bar{x})^{\mathcal{M}} = \{\bar{t}^{\sim} : \psi(\bar{t}/\bar{x}) \in \Gamma\}$. Therefore $\mu_{|\bar{x}|}(\psi(\bar{x})^{\mathcal{M}}) \leq r$. Hence $\mathcal{M} \models \varphi$.

– Let φ be of the form $S\bar{x} \geq r\,\psi(\bar{x})$.

($\Rightarrow$) Assume $(S\bar{x} < r\,\psi(\bar{x})) \in \Gamma$. Then, by a5 and r2, there is some $n \in \omega_+$ such that $(S\bar{x} \leq r - \frac{1}{n}\,\psi(\bar{x})) \in \Gamma$. By using the inductive hypothesis, we get $\mu_{|\bar{x}|}(\psi(\bar{x})^{\mathcal{M}}) < r$. Therefore $\mathcal{M} \not\models S\bar{x} \geq r\,\psi(\bar{x})$.

($\Leftarrow$) Assume $\mathcal{M} \models S\bar{x} < r\,\psi(\bar{x})$. Then, by the same argument used in the left-to-right implication of the previous case, we get $s < r$ such that $(S\bar{x} \leq s\,\psi(\bar{x})) \in \Gamma$. Hence $(S\bar{x} \geq r\,\psi(\bar{x})) \notin \Gamma$.

This concludes the proof of (1).

Eventually we prove that (o5) holds in $\mathcal{M}$. Let $A \subseteq P(M^{k+n})$ be such that

$$\mu_k\{\bar{a} : \mu_n\{\bar{b} : (\bar{a}, \bar{b}) \in A\} \geq r\}\} \geq s \tag{4}$$

We notice that, by (1), the following equality holds:

$$\mu_{k+n}(A) = \inf\{r \in \mathbb{Q} :\ (S\bar{x} \leq r\,\psi(\bar{x})) \in \Gamma \text{ and } A \subseteq \psi(\bar{x})^{\mathcal{M}}\}.$$

Hence, for all $0 < \epsilon$, there exists a definable set A_ϵ such that $A \subseteq A_\epsilon$ and $\mu_{k+n}(A_\epsilon) - \mu_{k+n}(A) < \epsilon$. From (4) and from $A \subseteq A_\epsilon$, it follows by monotonicity that $\mu_k\{\bar{a} : \mu_n\{\bar{b} : (\bar{a}, \bar{b}) \in A_\epsilon\} \geq r\}\} \geq s$. Since $\mathcal{M}$ satisfies a11, definability of A_ϵ implies that $\mu_{k+n}(A_\epsilon) \geq rs$. From $\mu_{k+n}(A_\epsilon) - \mu_{k+n}(A) < \epsilon$, for all $0 < \epsilon$, we get $\mu_{k+n}(A) \geq rs$. □

With reference to the previous proof, we just point out that if $A \subseteq M^n$ is definable in $\mathcal{M}$ and $\psi(\bar{x})$ is any defining formula then, letting

$$A_\psi = \inf\{r \in \mathbb{Q} : (S\bar{x} \leq r\,\psi(\bar{x})) \in \Gamma\},$$

we get $\mu_n(A) = A_\psi$.

Corollary 4. *Let Γ be a consistent set of L-sentences. Then there exists an sm-structure $\mathcal{M}$ such that $\mathcal{M} \vDash \Gamma$.*

Proof. Straightforward from Theorems 2 and 3. □

4. Strengthening weak completeness

In order to strengthen the weak completeness theorem obtained in the previous section, we need to introduce two rules which are analogous to the continuity rule of [2] relative to the admissible fragments not containing ω:

r3

$$\frac{S\bar{x} \geq t\,(S\bar{y} < r\,\varphi \wedge S\bar{y} \geq (s - 1/n)\,\varphi) \qquad \text{for all } n \in \omega_+}{S\bar{x} \geq t\,(S\bar{y} < r\,\varphi \wedge S\bar{y} \geq s\,\varphi)}$$

r4

$$\frac{S\bar{x} \geq t\,(S\bar{y} > s\,\varphi \wedge S\bar{y} \leq (r + 1/n)\,\varphi) \qquad \text{for all } n \in \omega_+}{S\bar{x} \geq t\,(S\bar{y} > s\,\varphi \wedge S\bar{y} \leq r\,\varphi)}$$

In presence of r3 and r4, we must refine the Henkin construction performed in the proof of Theorem 2 above. Retaining the notation used therein, we start with a consistent set of L-sentences with respect to the deduction system including r3 and r4 and, in the construction of the nondecreasing sequence $(\Delta_n)_{n\in\omega}$, we consider two more subcases of the case when $\Delta_n \cup \{\varphi_n\}$ is inconsistent:

(1.3) φ_n is of the form $S\bar{x} \geq t\,(S\bar{y} < r\,\varphi \wedge S\bar{y} \geq s\,\varphi)$: we claim that there is some $l \in \omega_+$ such that $\Delta_n \cup \{S\bar{x} < t\,(S\bar{y} < r\,\varphi \wedge S\bar{y} \geq (s - 1/l)\,\varphi)\}$ is consistent. For if not, by r3, we get $\Delta_n \vdash S\bar{x} \geq t\,(S\bar{y} < r\,\varphi \wedge S\bar{y} \geq s\,\varphi)$, contradicting the consistency of Δ_n.

We let $\Delta_{n+1} = \Delta_n \cup \{S\bar{x} < t\,(S\bar{y} < r\,\varphi \wedge S\bar{y} \geq (s - 1/k)\,\varphi)\}$, where k is the least l such that $\Delta_n \cup \{S\bar{x} < t\,(S\bar{y} < r\,\varphi \wedge S\bar{y} \geq (s - 1/l)\,\varphi)\}$ is consistent (actually, any such l would do).

(1.4) φ_n is of the form $S\bar{x} \geq t\,(S\bar{y} > s\,\varphi \wedge S\bar{y} \leq r\,\varphi)$: repeating the argument in (1.3), this time using r4, we let

$$\Delta_{n+1} = \Delta_n \cup \{S\bar{x} < t\,(S\bar{y} > s\,\varphi \wedge S\bar{y} \leq (r + 1/k)\,\varphi)\},$$

where k is the least l such that $\Delta_n \cup \{S\bar{x} < t\,(S\bar{y} > s\,\varphi \wedge S\bar{y} \leq (r + 1/l)\,\varphi)\}$ is consistent.

Continuing the refinement of the Henkin construction in the proof Theorem 3 in presence of r3 and r4, we have to make sure that the set $\Gamma_1 = \bigcup_{n\in\omega} \Delta_n$ is consistent: as the reader can easily verify, the argument used in the proof of Theorem 3 can still be applied to establish the consistency of Γ_1, thanks to (1.3) and (1.4).

We also notice that the proof of Theorem 3 remains the same.

We call *continuous (outer) submeasure structure* an sm- (om-) structure $\mathcal{M}$ with the property that whenever it satisfies all the premisses of some instance of rule r3 (r4) then it also satisfies the conclusion of r3 (r4). We will use the abbreviations csm- and com-structure accordingly.

In light of the the previous arguments, we can strengthen Corollary 4 as follows:

Corollary 5. *Every consistent set of L-sentences is satisfiable in some csm-structure.*

Our aim is to prove that every consistent set of sentences in a suitable subclass (to be defined below) of the L-formulas is satisfiable in some om-structure, thus obtaining, in our setting, an equivalent of the Graded Completeness Theorem (see[3]) due to Hoover [2].

Actually, what is missing in the proof of Theorem 3 above is the countable subadditivity of the submeasures in $\mathcal{M}$, namely property (o4). We notice that the same problem occurs in Hoover's construction of a weak model (see [2]). As in [2], in order to establish (o4) we use a nonostandard construction. In [2], the author makes use of the Loeb measure. As pointed out in [3, §5], the Loeb measure does not preserve the truth value of formulas involving the universal quantifier. Our construction suffers from the same drawback. For this reason, later we will restrict the class of formulas. First, we outline the nonstandard construction.

Let $\mathcal{M} = (\mathbf{M}, (\mu_n)_{n\in\omega_+}))$ be an sm-structure. We consider a nonstandard extension of $\mathcal{M}$ in a suitable nonstandard universe. We refer the reader to [1] for the construction and the properties of nonstandard universes. From now on we assume to work in an ω_1-saturated nonstandard universe

$$^* : (V_\omega(U), \in) \to (V_\omega(^*U), \in),$$

where U is some set of urelements such that $L \cup M \cup \mathbb{R} \subseteq U$, where L here denotes the alphabet of the language and M is the universe of the structure $\mathcal{M}$. We form the nonstandard extension $^*\mathcal{M}$ of $\mathcal{M}$.

Let $n \in \omega_+$ and let $\bar{\mu}_n : {}^*P(M^n) \to [0,1]$ be the composition of the standard part map with the nonstandard extension ${}^*\mu_n : {}^*P(M^n) \to {}^*[0,1]$ of μ_n, namely $\bar{\mu}_n(A) = {}^\circ({}^*\mu_n(A))$, for all $A \in {}^*P(M^n)$. It turns out that $\bar{\mu}_n$ is a submeasure on ${}^*P(M^n)$.

Furthermore, by saturation, if $A \in {}^*P(M^n)$ and $(A_i)_{i\in\omega}$ is a sequence in ${}^*P(M^n)$ such that $A \subseteq \bigcup_{i\in\omega} A_i$ then there exists some $k \in \omega$ such that $A \subseteq \bigcup_{i\le k} A_i$. Hence $\bar{\mu}_n(A) \le \sum_{i\in\omega} \bar{\mu}_n(A_i)$. Therefore we can extend $\bar{\mu}_n$ to an outer measure $\hat{\mu}_n : P({}^*M^n) \to [0,1]$ as follows:

$$\hat{\mu}_n(A) = \inf\{\sum_{i\in\omega} \bar{\mu}_n(A_i) : A_i \in {}^*P(M^n) \text{ for all } i \in \omega \text{ and } A \subseteq \bigcup_{i\in\omega} A_i\}.$$

Next we use a straightforward saturation argument to simplify the definition of $\hat{\mu}_n$. We provide the details for sake of completeness. Let $A \in P({}^*M^n)$. We fix $0 < \epsilon \in \mathbb{R}$. Then there exists some nondecreasing sequence $(A_i)_{i\in\omega}$ in ${}^*P(M^n)$ such that $A \subseteq \bigcup_{n\in\omega} A_i$ and $\hat{\mu}_n(A) \le \sum_{i\in\omega} \bar{\mu}_n(A_i) \le \hat{\mu}_n(A) + \epsilon/2$.

For each $k \in \omega$ we consider the internal set F_k whose elements are the internal sequences $(B_i)_{i\in{}^*\mathbb{N}}$ with the properties that

(1) for all $0 \le i \le k$, $B_i = A_i$;
(2) for all $i \in {}^*\mathbb{N}$, $B_i \subseteq B_{i+1}$;
(3) for all $i \in {}^*\mathbb{N}$, ${}^*\mu_n(B_i) \le \hat{\mu}(A) + \epsilon$.

The family $\{F_k : k \in \omega\}$ has the finite intersection property. By saturation, there exists $(B_i)_{i\in{}^*\mathbb{N}} \in \bigcap\{F_k : k \in \omega\}$. Let $N \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$. From $A \subseteq \bigcup_{i\in\omega} A_i \subseteq B_N$, we get $\hat{\mu}_n(A) \le \bar{\mu}(B_N) \le \hat{\mu}_n(A) + \epsilon$. Since $0 < \epsilon$ was arbitrarily chosen, we conclude that

$$\hat{\mu}_n(A) = \inf\{\bar{\mu}_n(B) : B \in {}^*P(M^n) \text{ and } A \subseteq B\}. \tag{5}$$

In the following, when $A \in {}^*P(M^n)$, we will use without further mention the equality $\hat{\mu}_n(A) = \bar{\mu}_n(A)$.

We form the structure

$$\hat{\mathcal{M}} = ({}^*M, (f^{{}^*\mathbf{M}})_{f\in L}, (P^{{}^*\mathbf{M}})_{P\in L}, (c^{{}^*\mathbf{M}})_{c\in L}, (\hat{\mu}_n)_{n\in\omega_+}).$$

As in [2], we assume that the L-formulas are constructed set-theoretically. In particular, each L-formula is an element of $V_\omega(U)$ and, in $V_\omega(U)$, there are also relations expressing properties like a formula being the conjunction of two formulas or the negation of some other formula, etc.... Since formulas are finitary and since $L_a \subset U$, then ${}^*\varphi$ is just φ, for each L-formula φ.

In $V_\omega(U)$ there is also a relation (denoted by $\models$, as usual) between structures, representations of formulas and assignments of values to variables in M expressing the property that a formula is true in $\mathcal{M}$ under some assignment of values to variables. Hence, suppressing from now on the * in ${}^*\models$, if $\varphi(\bar{x})$ is an L-formula and $\bar{b} \in ({}^*M)^{|\bar{x}|}$, the relation ${}^*\mathcal{M} \models \varphi(\bar{b})$ is internal.

Moreover, by the Transfer Principle, we have

$$\mathcal{M} \models \varphi(\bar{a}) \iff {}^*\mathcal{M} \models \varphi(\bar{a}), \tag{6}$$

for all L-formulas $\varphi(\bar{x})$ and all $\bar{a} \in M^{|\bar{x}|}$.

Proposition 6. *For every sm structure $\mathcal{M}$, the structure $\hat{\mathcal{M}}$ defined above is an om-structure.*

Proof. Properties (o1)–(o4) are satisfied by construction of $\hat{\mathcal{M}}$. In order to establish (o5)–(o7), we use the Transfer Principle to show that they are inherited by $(\hat{\mu}_n)_{n\in\omega_+}$. We deal with (o5), the other cases being easier.

Let $A \in P({}^*M^{k+n})$. For $\bar{a} \in {}^*M^k$, we let $A_{\bar{a}} = \{\bar{b} \in {}^*M^n : (\bar{a}, \bar{b}) \in A\}$. We assume that $\hat{\mu}_k(\{\bar{a} \in {}^*M^k : \hat{\mu}_n(A_{\bar{a}}) \ge r\}) \ge s$ and we prove that $\hat{\mu}_{k+n}(A) \ge rs$. Let $B \in {}^*P(M^{k+n})$ be such that $A \subseteq B$. Since for all $\bar{a} \in {}^*M^k$, $A_{\bar{a}} \subseteq B_{\bar{a}}$, from $\hat{\mu}_n(B_{\bar{a}}) = \bar{\mu}_n(B_{\bar{a}})$ it follows that, for all $0 < \epsilon \in \mathbb{R}$,

$$s \le \hat{\mu}_k(\{\bar{a} \in {}^*M^k : {}^*\mu_n(B_{\bar{a}}) \ge r - \epsilon\}).$$

Since the set $\{\bar{a} \in {}^*M^k : {}^*\mu_n(B_{\bar{a}}) \ge r - \epsilon\}$ is internal, we get

$$s - \epsilon \le {}^*\mu_k(\{\bar{a} \in {}^*M^k : {}^*\mu_n(B_{\bar{a}}) \ge r - \epsilon\}).$$

Therefore, by Transfer of property (o5), ${}^*\mu_{k+n}(B) \geq (r-\epsilon)(s-\epsilon)$, for all $0 < \epsilon \in \mathbb{R}$. Hence $\bar{\mu}_n(B) \geq rs$. Since $A \subseteq B \in {}^*P(M^{k+n})$ was arbitrarily chosen, we conclude that $\hat{\mu}_{k+n}(A) \geq rs$.

□

From now on we work in a subclass of the language L-formulas that we call L^--formulas. A formula is in L^- if and only if no outer measure quantifier occurs in any of its universal subformulas. Otherwise said, the L^--formulas are those obtained by closing the set of formulas from classical first-order logic under applications of the propositional connectives and of the outer measure quantifiers.

Let $\mathcal{M}$ be any structure and $\varphi(\bar{x},\bar{y})$ an L-formula. For ease of notation we denote the set defined by φ in $\mathcal{M}$ by $\varphi^{\mathcal{M}}$. If $\bar{a} \in M^{|\bar{y}|}$, we let $\varphi^{\mathcal{M}}_{\bar{a}} = \{\bar{b} \in M^{|\bar{x}|} : \mathcal{M} \models \varphi(\bar{b},\bar{a})\}$.

We can now to prove the following:

Theorem 7. *Let $\mathcal{M}$ be a csm-structure. Then, for every L^--formula $\varphi(\bar{x},\bar{y})$ and every $\bar{a} \in M^{|\bar{y}|}$,*

$$\hat{\mu}_{|\bar{x}|}(\varphi^{\hat{\mathcal{M}}}_{\bar{a}} \triangle \varphi^{^*\mathcal{M}}_{\bar{a}}) = 0.$$

Proof. By induction on $\varphi(\bar{x},\bar{y})$. For notational simplicity, from now on we suppress the parameters from M and we let $l = |\bar{x}|$.

The case when $\varphi(\bar{x})$ is a first-order formula from classical logic is straightforward. If $\varphi(\bar{x})$ is of the form $\neg\psi(\bar{x})$ or $(\psi \wedge \eta)(\bar{x})$, we assume that the inductive assumption hold for ψ and η and we use the the set-theoretic identity $X \triangle Y = X^c \triangle Y^c$ and the set-theoretic inclusion $(X \cap X') \triangle (Y \cap Y') \subseteq (X \triangle Y) \cup (X' \triangle Y')$ respectively.

Next we consider the cases relative to the outer measure quantifiers.

– $\varphi(\bar{x})$ is of the form $S\bar{z} \geq r\,\psi(\bar{x},\bar{z})$. Let $k = |\bar{z}|$. We assume inductively that $\hat{\mu}_{l+k}(\psi^{\hat{\mathcal{M}}} \triangle \psi^{^*\mathcal{M}}) = 0$.

  By Proposition 6, $\hat{\mathcal{M}}$ satisfies property (o5). Therefore, for each positive rational t, $\hat{\mu}_l(\{\bar{b} \in {}^*M^l : \hat{\mu}_k(\psi^{\hat{\mathcal{M}}}_{\bar{b}} \triangle \psi^{^*\mathcal{M}}_{\bar{b}}) \geq t\}) = 0$. We have the following chain of inclusions:

$$\begin{aligned}\{\bar{b} \in {}^*M^l : \hat{\mu}_k(\psi^{\hat{\mathcal{M}}}_{\bar{b}}) \neq \hat{\mu}_k(\psi^{^*\mathcal{M}}_{\bar{b}})\} &\subseteq \{\bar{b} \in {}^*M^l : \hat{\mu}_k(\psi^{\hat{\mathcal{M}}}_{\bar{b}} \triangle \psi^{^*\mathcal{M}}_{\bar{b}}) \neq 0\} \\ &\subseteq \bigcup_{t \in]0,1]}\{\bar{b} \in {}^*M^l : \hat{\mu}_k(\psi^{\hat{\mathcal{M}}}_{\bar{b}} \triangle \psi^{^*\mathcal{M}}_{\bar{b}}) \geq t\},\end{aligned}$$

  where t ranges over the rationals. By countable subadditivity,

$$\hat{\mu}_l(\{\bar{b} \in {}^*M^l : \hat{\mu}_k(\psi^{\hat{\mathcal{M}}}_{\bar{b}}) \neq \hat{\mu}_k(\psi^{^*\mathcal{M}}_{\bar{b}})\}) = 0.$$

  A fortiori,

$$\hat{\mu}_l(\varphi^{\hat{\mathcal{M}}} \triangle \{\bar{b} \in {}^*M^l : \hat{\mu}_k(\psi^{^*\mathcal{M}}_{\bar{b}}) \geq r\}) = 0 \tag{7}$$

  Next we prove that

$$\hat{\mu}_l(\varphi^{^*\mathcal{M}} \triangle \{\bar{b} \in {}^*M^l : \hat{\mu}_k(\psi^{^*\mathcal{M}}_{\bar{b}}) \geq r\}) = 0 \tag{8}$$

For sake of contradiction, we assume that, for some $0 < t \in \mathbb{R}$, $\hat{\mu}_l(\varphi^{*\mathcal{M}} \triangle \{\bar{b} \in {}^*M^l : \hat{\mu}_k(\psi_{\bar{b}}^{*\mathcal{M}}) \geq r\}) > t$. Since

$$\{\bar{b} \in {}^*M^l : \hat{\mu}_k(\psi_{\bar{b}}^{*\mathcal{M}}) \geq r\} = \bigcap_{n \in \omega_+} \{\bar{b} \in {}^*M^l : {}^*\mu_k(\psi_{\bar{b}}^{*\mathcal{M}}) \geq r - 1/n\}$$

then ${}^*\mu_l((S\bar{z} \geq (r-1/n)\,\psi)^{*\mathcal{M}} \smallsetminus \varphi^{*\mathcal{M}}) \geq t$ and, by Transfer,

$$\mu_l((S\bar{z} \geq (r-1/n)\,\psi)^{\mathcal{M}} \smallsetminus \varphi^{\mathcal{M}}) \geq t \qquad \text{for all } n \in \omega_+.$$

Therefore, for all $n \in \omega_+$, $\mathcal{M} \vDash S\bar{x} \geq t\,(S\bar{z} < r\,\psi \wedge S\bar{z} \geq (r-1/n)\,\psi)$.

By assumption, the continuity rule r3 is valid in $\mathcal{M}$. Hence we get the contradiction $\mu_l(\{b \in M^l : r \leq \mu_k(\psi_{\bar{b}}^{\mathcal{M}}) < r\}) \geq t$.

Summing up: we have established (7) and (8). From the set theoretic identity $X \triangle Y \subseteq (X \triangle Z) \cup (Z \triangle Y)$, we finally get

$$\hat{\mu}_l(\varphi^{\hat{\mathcal{M}}} \triangle \varphi^{*\mathcal{M}}) = 0.$$

– $\varphi(\bar{x})$ is of the form $S\bar{z} \leq r\,\psi(\bar{x}, \bar{z})$. The structure of the proof is the same as in the previous case, with the obvious changes. This time, validity of rule r4 in $\mathcal{M}$ does play a role.

□

Corollary 8. *For every L^--formula $\varphi(\bar{x})$ and every $\bar{a} \in M^{|\bar{x}|}$*

$$\mathcal{M} \vDash \varphi(\bar{a}) \;\Leftrightarrow\; \hat{\mathcal{M}} \vDash \varphi(\bar{a}).$$

Proof. By induction on $\varphi(\bar{x})$. We present in detail the case when φ is of the form $S\bar{z} \geq r\,\psi(\bar{z}, \bar{x})$. Let $l = |\bar{z}|$.

($\Rightarrow$) $\mathcal{M} \vDash \varphi(\bar{a}) \Leftrightarrow {}^*\mathcal{M} \vDash \varphi(\bar{a}) \Leftrightarrow {}^*\mu_l(\psi_{\bar{a}}^{*\mathcal{M}}) \geq r \Rightarrow \hat{\mu}_l(\psi_{\bar{a}}^{*\mathcal{M}}) \geq r \overset{\bullet}{\Leftrightarrow} \hat{\mu}_l(\psi_{\bar{a}}^{\hat{\mathcal{M}}}) \geq r \Leftrightarrow \hat{\mathcal{M}} \vDash \varphi(\bar{a})$.

($\Leftarrow$) $\hat{\mathcal{M}} \vDash \varphi(\bar{a}) \Leftrightarrow \hat{\mu}_l(\psi_{\bar{a}}^{\hat{\mathcal{M}}}) \geq r \overset{\bullet}{\Leftrightarrow} \hat{\mu}_l(\psi_{\bar{a}}^{*\mathcal{M}}) \geq r \Rightarrow$ for all $n \in \omega_+$, ${}^*\mu_l(\psi_{\bar{a}}^{*\mathcal{M}}) \geq r - 1/n \Rightarrow$ for all $n \in \omega_+$, $\mu_l(\varphi_{\bar{a}}^{\mathcal{M}}) \geq r - 1/n \Leftrightarrow \mathcal{M} \vDash \varphi(\bar{a})$.

(The two equivalences marked with • hold by Theorem 7.) □

Corollary 9. *Every consistent set of L^--sentences is satisfiable in some com-structure.*

5. Concluding comments

We point out that, in a com-structure, the continuity properties formulated by requiring the validity of rules r3 and r4 do hold for the definable subsets only. It would be interesting to obtain structures where those continuity properties hold for all subsets. In this regard we notice that, in the proof of Theorem 3, we were able to extend property (o5) from the family of definable subsets to that of all subsets.

Even more important would be an extension of Corollary 9 to sets of L-sentences. As we already remarked, the outer measures that we have obtained by applying nonstandard techniques do not behave well with respect to the universal quantifier. It seems that a different approach is needed. We leave this issue, as well as the study of the model-theoretic properties of the proposed logic, to future work.

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