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MINKOWSKI INEQUALITY ON COMPLETE RIEMANNIAN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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We consider Riemannian manifolds of dimension at least 3, with nonnegative Ricci curvature and Euclidean volume growth. For every open bounded subset with smooth boundary we establish the validity of an optimal Minkowski inequality. We also characterise the equality case, provided the domain is strictly outward minimising and strictly mean convex. Along with the proof, we establish in full generality sharp monotonicity formulas, holding along the level sets of p-capacitary potentials in p-nonparabolic manifolds with nonnegative Ricci curvature.

1. Introduction

1A. ***Statements of the main results.*** Given an open bounded convex domain with smooth boundary $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, the classical Minkowski inequality, originally proven in [Minkowski 1903], gives a sharp lower bound for the average of the mean curvature H of $\partial\Omega$ in terms of the inverse of its surface radius, that is,

$$\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{1}{n-1}} \leq \fint_{\partial\Omega} \frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma,$$

with the equality satisfied if and only if Ω is a ball. It was clear to many authors that such inequality deserved to be further investigated. For example one would like to relax the convexity assumption on one hand, and to prove that the inequality holds on more general ambient manifolds on the other.

The first question has been positively answered using techniques based on geometric flows [Huisken 2009], optimal transport [Chang and Wang 2013; Castillon 2010], and recently also nonlinear potential theory [Fogagnolo et al. 2019; Agostiniani et al. 2022a]. The latter method actually provides the most general statement available so far, namely the extended Minkowski inequality

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|\mathrm{d}\sigma \tag{1-1}$$

holding for every open bounded domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary. Here Ω^* denotes the *strictly outward minimising hull* of Ω. The precise definition of Ω^* is reported in (4-12) below and analysed in full detail in [Fogagnolo and Mazzieri 2022]. However, in this preliminary discussion, we just point out that Ω^* minimises the perimeter among bounded subsets containing Ω.

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Many improvements can be found in the literature also concerning the question of extending the Minkowski inequality to more general settings. Firstly Gallego and Solanes [2005] established quermassintegral inequalities for convex domains in the hyperbolic space. Using the inverse mean curvature flow (IMCF for short), de Lima and Girão [2016] extended the result to star-shaped and strictly mean-convex domains lying in the same ambient manifold. The IMCF has been also employed to establish a Minkowski-type inequality for outward minimising sets sitting in the Schwarzschild manifold by Wei [2018], in the anti-de Sitter–Schwarzschild manifold by Brendle, Hung and Wang [Brendle et al. 2016], and on asymptotically flat static manifolds by McCormick [2018].

A natural context in which to test the validity of a Minkowski inequality is provided by complete noncompact Riemannian manifolds with nonnegative Ricci curvature. A very recent work [Brendle 2023] actually points in this direction. Indeed, choosing $f = 1$ in Corollary 1.5 of that work a nonsharp Minkowski inequality can be deduced for complete Riemannian manifolds with nonnegative sectional curvature and Euclidean volume growth. In the present paper, we prove the following theorem.

Theorem 1.1 (extended Minkowski inequality). *Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth. Let $\Omega \subseteq M$ be an open bounded set with smooth boundary. Then*

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma, \tag{1-2}$$

where $\mathrm{AVR}(g)$ is the asymptotic volume ratio of (M, g), H is the mean curvature of $\partial\Omega$ with respect to the outward normal unit vector and Ω^* is the strictly outward minimising hull of Ω.

In the case a strictly outward minimising $\Omega \subset M$ with strictly mean-convex boundary achieves the identity in (1-2), we show that $M \smallsetminus \Omega$ splits as a (truncated) cone.

Theorem 1.2 (rigidity for the Minkowski inequality). *A bounded strictly outward minimising subset $\Omega \subset M$ with smooth strictly mean-convex boundary satisfies*

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\frac{\mathrm{H}}{n-1} \mathrm{d}\sigma$$

if and only if $(M \smallsetminus \Omega, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial\Omega,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right), \quad \textit{where } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

Some comments are in order about the above statements. First, we recall for the reader's convenience that the asymptotic volume ratio of (M, g) is given by

$$\mathrm{AVR}(g) = \lim_{r \to +\infty} \frac{|B(o, r)|}{r^n |\mathbb{B}^n|}$$

for some $o \in M$. The fact that, on complete manifolds with nonnegative Ricci curvature, the above limit is well-defined and does not depend on the base point o, is a consequence of the classical Bishop–Gromov volume comparison theorem. Moreover, one has that $0 \leq \mathrm{AVR}(g) \leq 1$, with $\mathrm{AVR}(g) = 1$ if and only if

(M, g) is the standard n-dimensional Euclidean space. Beside the intrinsic fundamental role played by manifolds with nonnegative Ricci curvature with Euclidean volume growth in geometric analysis, this class includes a diversity of explicit manifolds naturally arising from different fields, such as asymptotically locally Euclidean spaces (ALE for short) *gravitational instantons*. These are noncompact hyperkähler Ricci flat 4-dimensional manifolds playing a role in the study of Euclidean quantum gravity theory, gauge theory and string theory (see [Hawking 1977; Eguchi and Hanson 1979; Kronheimer 1989a; 1989b; Minerbe 2009; 2010; 2011]).

It is worth noticing that inequality (1-2) is sharp and it provides the optimal Minkowski inequality on manifolds with nonnegative Ricci curvature for *outward minimising subsets*, see Corollary 4.6. These subsets are mean-convex and satisfy $|\partial\Omega^*| = |\partial\Omega|$, so that the Minkowski inequality reads

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \frac{\mathrm{H}}{n-1}\, d\sigma,$$

in this case. In addition to the Euclidean spaces, where it is immediately seen that balls achieve the identity in (1-2), the sharpness of this inequality is checked in far greater generality, as specified in Remark 4.7 below.

Combining Theorem 1.1 with the sharp isoperimetric inequality for manifolds with nonnegative Ricci curvature, first proved in dimension 3 in [Agostiniani et al. 2020, Theorem 1.4] and recently extended to any dimension in [Brendle 2023] (see also [Fogagnolo and Mazzieri 2022; Johne 2021; Balogh and Kristály 2023]), reading

$$\frac{|\mathbb{S}^{n-1}|^n}{|\mathbb{B}^n|^{n-1}} \mathrm{AVR}(g) \leq \frac{|\partial\Omega^*|^n}{|\Omega^*|^{n-1}},$$

we get the following sharp volumetric version of the Minkowski inequality.

Theorem 1.3 (volumetric Minkowski inequality). *Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth. Let $\Omega \subseteq M$ be an open bounded set with smooth boundary. Then*

$$\left(\frac{|\Omega|}{|\mathbb{B}^n|}\right)^{\frac{n-2}{n}} \mathrm{AVR}(g)^{\frac{2}{n}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| d\sigma, \tag{1-3}$$

where $\mathrm{AVR}(g)$ is the asymptotic volume ratio of (M, g), H is the mean curvature of $\partial\Omega$ with respect to the outward normal unit vector. Moreover, the equality is satisfied if and only if (M, g) is isometric to the flat Euclidean space and Ω is a ball.

As for the extended Minkowski inequality, (1-3) is easily recognised to be sharp, while the rigidity statement directly follows from the rigidity of the isoperimetric inequality. We finally point out that earlier contributions to the volumetric Minkowski inequality were given in [Chang and Wang 2011; Qiu 2015], holding in the flat Euclidean space and under stronger geometric assumptions on the boundary of Ω.

1B. ***Outline of the proof.*** We now describe the main features of our approach, which is in line with [Agostiniani and Mazzieri 2020; Agostiniani et al. 2020; 2022a; Fogagnolo et al. 2019]. Given (M, g) a Riemannian n-manifold, $n \geq 3$, with nonnegative Ricci curvature, and an open bounded subset $\Omega \subseteq M$

with smooth boundary we consider, for every $1 < p < n$, the p-capacitary potential associated to Ω. This is the solution u to the problem

$$\begin{cases} \Delta_g^{(p)} u = 0 & \text{on } M \smallsetminus \overline{\Omega}, \\ u = 1 & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } d_g(x, o) \to +\infty, \end{cases} \tag{1-4}$$

where $\Delta_g^{(p)}$ is the p-Laplace operator associated with the metric g, and $d_g(\,\cdot\,, o)$ is the distance induced by g to some fixed reference point o. Provided the manifold (M, g) is p-nonparabolic (see Definition 2.5 below, as well as [Holopainen 1990; 1999]), the solution to problem (1-4) exists and is unique. Such a solution is commonly referred to as the *p-capacitary potential* associated with Ω. It is worth specifying that manifolds with Euclidean volume growth (i.e., $\mathrm{AVR}(g) > 0$) do satisfy the p-nonparabolicity assumption for $1 < p < n$ by the characterisation given in [Holopainen 1999, Proposition 5.10]. As a crucial step in our method, we will establish families of *monotonicity formulas*, holding along the level sets of the p-capacitary potentials associated with Ω. More precisely, for every $t \in [1, +\infty)$, we set

$$F_p^\beta(t) = t^{\beta\frac{(n-1)(p-1)}{(n-p)}} \int_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)} \,\mathrm{d}\sigma, \tag{1-5}$$

and we show that for

$$\beta > \frac{n-p}{(p-1)(n-1)}$$

the above quantity admits a nonincreasing $\mathscr{C}^1(1, +\infty)$ representative.

Some remarks are mandatory at this stage. First of all, let us point out that the monotonicity statement provided here for the functions F_p^β holds in full generality and with no restriction on the geometry of Ω. As such, it is also new for domains sitting in $\mathbb{R}^n$, where the same conclusions were provided in [Fogagnolo et al. 2019] only for convex domains, and in fact for smooth level sets flows. In the general case, it is well known that the level sets flow of p-harmonic functions might present a much less regular behaviour since no general bound is available for the Hausdorff dimension of the critical set. To overcome these difficulties, the authors in [Agostiniani et al. 2022a] settled for the *effective inequalities*

$$\lim_{t\to+\infty} F_p^\beta(t) \le F_p^\beta(1) \quad \text{and} \quad (F_p^\beta)'(1) \le 0. \tag{1-6}$$

The derivation of these two bounds, however, heavily relied on the compactness of the critical set of u, that is a particular feature of spaces with finite topology, and as such it is not directly viable in our setting (see [Menguy 2000]). In contrast with this, the present treatment provides the desired extension to the nonlinear setting and to the general framework of nonnegatively Ricci curved p-nonparabolic manifolds of the monotonicity formulas discovered in [Colding 2012; Colding and Minicozzi 2014b; Agostiniani and Mazzieri 2020; Agostiniani et al. 2020] for harmonic functions. As a second remark, to let the reader appreciate the $\mathscr{C}^1$-regularity result, we observe that in principle even the fact that formula (1-5) yields a well-posed definition is not granted for free. The most serious difficulty here is that the set of singular values cannot be controlled through Sard's theorem, since p-harmonic functions only enjoy a mild — though optimal — $\mathscr{C}^{1,\beta}$-regularity. We managed to solve these problems also taking advantage of recent insights given in [Gigli and Violo 2023]. The full statement of the monotonicity theorem is found in Theorem 3.1 below.

Through the monotonicity of F_p^β, with $\beta = (p-1)^{-1}$, we arrive at the following L^p*-Minkowski inequality*

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \, \mathrm{AVR}(g)^{\frac{1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^p \mathrm{d}\sigma, \tag{1-7}$$

where $\mathrm{C}_p(\Omega)$ is the normalised p-capacity of Ω defined in (2-5) below. A major advantage we draw out of the full monotonicity of F_p^β is the bypassing of the computation of its limit as $t \to +\infty$ when reaching for (1-7). Indeed, this step is now replaced by a suitable contradiction argument that combines the full monotonicity of our quantities with the sharp *iso-p-capacitary inequality* (see Theorem 4.1 below)

$$\frac{\mathrm{C}_p(\mathbb{B}^n)^n}{|\mathbb{B}^n|^{n-p}} \, \mathrm{AVR}(g)^p \leq \frac{\mathrm{C}_p(\Omega)^n}{|\Omega|^{n-p}}. \tag{1-8}$$

Such a statement is of independent interest in our opinion and can be achieved by taking advantage of the already-mentioned sharp isoperimetric inequality in manifolds with nonnegative Ricci curvature and Euclidean volume growth, following rather classical arguments (see, e.g., [Jauregui 2012]).

With the L^p-Minkowski inequality (1-7) at hand, the extended Minkowski inequality (1-7) simply follows by letting $p \to 1^+$ since

$$\lim_{p \to 1^+} \mathrm{C}_p(\Omega) = \frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|},$$

as proven in [Fogagnolo and Mazzieri 2022, Theorem 1.2]. This particular feature of our approach, namely the fact that the Minkowski inequality is obtained as the limit of its L^p-versions, makes the rigidity statement a particularly nontrivial task, although we show that (1-7) holds with equality only on cones. This leads us to prove the rigidity statement, Theorem 1.2, through an argument involving the study of the IMCF starting at boundaries of domains that saturate the Minkowski inequality (1-1). More precisely, we first show that the flow is smooth and given by constantly mean-curved totally umbilical hypersurfaces for a short time. This crucially exploits the nonnegativity of the Ricci curvature (Lemma 4.8). Then, a splitting procedure along such flow, inspired by [Huisken and Ilmanen 2001], shows that an outer neighbourhood of $\partial\Omega$ is isometric to a truncated cone with the same volume ratio as $\mathrm{AVR}(g)$, and this allows us to conclude (Lemma 4.9).

1C. *Further monotonicity-rigidity results.* Beside the monotonicity-rigidity properties of F_p^β discussed above, we also establish analogous ones for the function

$$F_p^\infty(t) = t^{\frac{n-1}{n-p}} \sup_{\{u=1/t\}} |\mathrm{D}u|.$$

This is the content of Theorem 3.2, which is again proved in the general setting of p-nonparabolic manifolds with nonnegative Ricci curvature, extending [Fogagnolo et al. 2019, Theorem 1.3]. As geometric consequences of this statement, we provide a rigidity result under pinching conditions and a sphere theorem for smooth boundaries in manifolds with $\mathrm{Ric} \geq 0$ (see Theorems 4.11 and 4.12 below) and Euclidean volume growth. It is worth mentioning that the monotonicity of F_p^∞ also leads to a new insight on the critical set of the p-capacitary potential, which we believe deserves some further investigation. Namely, it turns out that every level set of u displays some nonempty relatively open region, where $\mathrm{D}u$ does not vanish, and where in particular u is smooth (see Corollary 3.3).

1D. ***Summary.*** In Section 2 we report, for the ease of the reader, some relevant facts from the theory of p-harmonic functions on Riemannian manifolds, focussing on the regularity theory as well as on the existence and uniqueness of solutions to (1-4). Some important — though already well known — estimates and identities are also recalled in this section. Section 3 is devoted to the proof of monotonicity-rigidity theorems (see Theorems 3.1 and 3.2). After having introduced a convenient conformally related setting, we restate them in this framework and we conclude the section with their proofs. In Section 4, after having provided (1-8), we make use of these tools to prove the L^p-Minkowski inequality (see Theorem 4.3), deduce the extended Minkowski inequality Theorem 1.1 and some rigidity results under pinching conditions as consequences of the monotonicity-rigidity theorems.

2. The p-capacitary potential in Riemannian manifolds

We have collected here, for the sake of future reference, some substantially well-known results that will be repeatedly applied in our arguments. Before considering the specific case of problem (1-4), we recall the definition of p-harmonic functions, as well as their regularity estimates. We then analyse the existence and uniqueness of the solution u_p to (1-4) on complete Riemannian manifolds. It turns out that these questions are intimately related to the notion of p-nonparabolicity, and p-nonparabolic manifolds will then constitute the natural setting for the monotonicity-rigidity theorems. We afterwards recall some global standard estimates on u_p and its gradient as well as a Kato-type identity for p-harmonic functions.

2A. ***p-harmonic functions and regularity.*** Given an open subset U of a complete Riemannian manifold (M, g), we say that $v \in W^{1,p}(U)$ is p-harmonic if

$$\int_U \big\langle |\mathrm{D}v|^{p-2}\mathrm{D}v \,\big|\, \mathrm{D}\psi \big\rangle \, \mathrm{d}\mu = 0 \tag{2-1}$$

for any test function $\psi \in \mathscr{C}_c^\infty(U)$. With $\langle \cdot | \cdot \rangle$ we denote as usual the scalar product induced by the underlying Riemannian metric g on the tangent space at each point. Regularity results for p-harmonic functions (see [Tolksdorf 1984; DiBenedetto 1983; Lieberman 1988]) ensure that v belongs to $\mathscr{C}^{1,\beta}_{\mathrm{loc}}(U)$ for some $\beta \in (0, 1)$ and is smooth around each point where $|\mathrm{D}v| > 0$.

Since the $\mathscr{C}^{1,\beta}$-regularity is not sufficient to employ Sard's theorem, we are going to heavily rely on the coarea formula. We report it here for ease of further references. The statement below follows from [Maggi 2012, Lemma 18.5 and Theorem 18.1] coupled with standard approximation results.

Proposition 2.1 (coarea formula). *Let (M, g) be a complete Riemannian manifold. Consider a locally Lipschitz function $v : U \to [0, +\infty)$ on some open subset $U \subseteq M$ such that $v^{-1}([a, b])$ is compact for every $[a, b] \subset (0, +\infty)$. Then the following hold:*

(1) $|\{v = t\} \cap \mathrm{Crit}(v)| = 0$ *for almost every* $t \in [0, +\infty)$.

(2) *For every measurable f such that $f|\mathrm{D}v| \in L^1_{\mathrm{loc}}(U)$ we have $f \in L^1(\{v = t\})$ for almost every $t \in (0, +\infty)$ and*

$$\int_U \psi(v) f |\mathrm{D}v| \, \mathrm{d}\mu = \int_0^{+\infty} \psi(t) \int_{\{v=t\}} f \, \mathrm{d}\sigma \, \mathrm{d}t \tag{2-2}$$

for every ψ bounded measurable function compactly supported in $(0, +\infty)$. In particular,

$$t \mapsto \int_{\{v=t\}} f \, \mathrm{d}\sigma \in L^1_{\mathrm{loc}}(0, +\infty),$$

and its equivalence class does not depend on the representative of f.

Remark 2.2. If $h \in L^1_{\mathrm{loc}}(U)$ and $h = 0$ almost everywhere on $\mathrm{Crit}(v)$, the function $f = h|\mathrm{D}v|^{-1}$, satisfies the assumptions of Proposition 2.1(2). Clearly, if $f \in L^1(U)$, then (2-2) holds for every ψ bounded measurable, even without compact support.

With the idea of applying the previous result for $f = |\mathrm{D}|\mathrm{D}v|^{p-1}|$, a higher integrability degree of p-harmonic functions is required. We refer the reader to [Lou 2008, Lemma 2.1] for a self-contained proof of the following lemma in the Euclidean case. The general case follows in the same way, as it is ultimately due to a careful integration of the Bochner identity. Indeed, computations are the same provided a lower bound on the Ricci tensor is in force, which is always true locally (see [Benatti 2022, Appendix C] for a complete proof).

Lemma 2.3. *Let (M, g) be a complete Riemannian manifold and $U \subseteq M$ be an open subset. Given $v \in W^{1,p}(U)$ a p-harmonic function, then $|\mathrm{D}v|^{p-1} \in W^{1,2}_{\mathrm{loc}}(U)$.*

Given $U \subseteq M$ with Lipschitz boundary, a p-harmonic function $u \in W^{1,p}(U)$ attains some Dirichlet data $g \in L^p(\partial U)$ if u coincides with g on ∂U in the sense of the trace operator.

2B. ***p-nonparabolic manifolds and the p-capacitary potential.*** Given a noncompact Riemannian manifold M, we consider the p-capacitary potential of a bounded set with smooth boundary $\Omega \subset M$, that is, a function $u \in W^{1,p}(M \smallsetminus \overline{\Omega})$ solving (1-4). The function u belongs to $\mathscr{C}^{1,\beta}(M \smallsetminus \Omega)$ (see [Lieberman 1988]) and it is smooth near the points where the gradient does not vanish. In particular, by the Hopf maximum principle in [Tolksdorf 1983, Proposition 3.2.1] the datum on $\partial\Omega$ is attained smoothly.

We now focus on some classical sufficient conditions to ensure the existence of the p-capacitary potential, which turns out to be related to the notion of p-Green's function we are going to recall.

Definition 2.4 (p-Green's function). Let (M, g) be a complete Riemannian manifold. Let $\mathrm{Diag}(M) = \{(x, x) \in M \times M \mid x \in M\}$. For $p \geq 1$, we say that $G_p : M \times M \smallsetminus \mathrm{Diag}(M) \to \mathbb{R}$ is a p-Green's function for M if it weakly satisfies $\Delta_p G(o, \cdot\,) = -\delta_o$ for any $o \in M$, where δ_o is the Dirac delta centred at o, that is, if it holds

$$\int_M \big\langle |\mathrm{D}G_p(o, \cdot)|^{p-2} \mathrm{D}G_p(o, \cdot) \,\big|\, \mathrm{D}\psi \big\rangle \, \mathrm{d}\mu = \psi(o)$$

for any $\psi \in \mathscr{C}^\infty_c(M)$.

The notion of p-Green's function calls for that of p-nonparabolic Riemannian manifold.

Definition 2.5 (p-nonparabolicity). We say that a complete noncompact Riemannian manifold (M, g) is p-nonparabolic if there exists a *positive* p-Green's function $G_p : M \times M \smallsetminus \mathrm{Diag}(M) \to \mathbb{R}$. With the expression *$p$-Green function* we are in fact referring to the positive minimal one.

The notion of p-nonparabolicity is intimately related to the existence of a solution to (1-4), in that if the positive p-Green's function of a p-nonparabolic Riemannian manifold vanishes at infinity, then such a solution exists for any open bounded subset $\Omega \subset M$ with smooth boundary. A complete and self-contained proof of this fact is provided in the Appendix of [Fogagnolo and Mazzieri 2022]. We report the statement of such basic thought fundamental result.

Theorem 2.6 (existence of the p-capacitary potential). *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Assume also that the p-Green's function G_p satisfies $G_p(o, x) \to 0$ as $d_g(o, x) \to +\infty$ for some $o \in M$. Then, there exists a unique solution u_p to (1-4).*

If (M, g) is a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth, then it is in fact p-nonparabolic for every $1 < p < n$ and the p-Green's function satisfies

$$G_p(o, x) \leq \mathrm{C}\, d_g(o, x)^{-\frac{n-p}{p-1}} \tag{2-3}$$

for some constant C. This is a direct consequence of [Holopainen 1999, Proposition 5.10].

We find convenient to recall here the definition of p-capacity of an open bounded subset $\Omega \subset M$ together with a normalised version of it which turns out to be more advantageous for our computations.

Definition 2.7 (p-capacity and normalised p-capacity). Let (M, g) be a complete noncompact Riemannian manifold, and let Ω be an open bounded subset of M. For $1 < p < n$, the p-capacity of Ω is defined as

$$\mathrm{Cap}_p(\Omega) = \inf\left\{ \int_M |\mathrm{D}v|^p \, \mathrm{d}\mu \;\middle|\; v \in \mathscr{C}^\infty_c(M),\ v \geq 1 \text{ on } \Omega \right\}. \tag{2-4}$$

On the other hand, the normalised p-capacity of Ω is defined as

$$\mathrm{C}_p(\Omega) = \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p} \right)^{p-1} \mathrm{Cap}_p(\Omega). \tag{2-5}$$

A function u solving (1-4) realises the p-capacity of the initial set Ω, and actually one can also characterise such quantity with a suitable integral on $\partial\Omega$. We resume these facts in the following statement.

Proposition 2.8. *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold for some $1 < p < n$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then the solution u_p to (1-4) realises*

$$\mathrm{C}_p(\Omega) = \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p} \right)^{p-1} \int_{M \smallsetminus \overline{\Omega}} |\mathrm{D}u_p|^p \, \mathrm{d}\mu. \tag{2-6}$$

Moreover, we have that

$$\mathrm{C}_p(\Omega) = \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p} \right)^{p-1} \int_{\{u_p = 1/t\}} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma \tag{2-7}$$

holds for almost every $t \in [1, +\infty)$, including any $1/t$ that is a regular value for u_p.

Proof. The function u_p can be approximated in $W^{1,p}(M \smallsetminus \bar{\Omega})$ by functions φ in $\mathscr{C}_c^\infty(M)$ which satisfy $\varphi \geq 1$ on Ω. Then

$$\mathrm{Cap}_p(\Omega) \leq \int_{M \smallsetminus \bar{\Omega}} |\mathrm{D}u_p|^p \, \mathrm{d}\mu.$$

On the other hand, the weak formulation in (2-1) can be relaxed in duality with functions in $W_0^{1,p}(M \smallsetminus \bar{\Omega})$. Hence, taking any competitor $\psi \in \mathscr{C}_c^\infty(M)$ with $\psi \geq 1$ on Ω, $u_p - \psi \in W_0^{1,p}(M \smallsetminus \bar{\Omega})$, we get that

$$\int_{M \smallsetminus \bar{\Omega}} |\mathrm{D}u_p|^p \, \mathrm{d}\mu = \int_{M \smallsetminus \bar{\Omega}} \langle |\mathrm{D}u_p|^{p-2} \mathrm{D}u_p, \mathrm{D}u_p \rangle \, \mathrm{d}\mu = \int_{M \smallsetminus \bar{\Omega}} \langle |\mathrm{D}u_p|^{p-2} \mathrm{D}u_p, \mathrm{D}\psi \rangle \, \mathrm{d}\mu.$$

Applying the Hölder inequality to the right-hand side, we are left with

$$\int_{M \smallsetminus \bar{\Omega}} |\mathrm{D}v|^p \, \mathrm{d}\mu \leq \int_{M \smallsetminus \bar{\Omega}} |\mathrm{D}\psi|^p \, \mathrm{d}\mu$$

for every competitor ψ in (2-4), proving (2-6). Since $|\mathrm{D}u_p| \in L^p(M \smallsetminus \bar{\Omega})$, applying the coarea formula (2-2) with $f = |\mathrm{D}u_p|^{p-1}$ to (2-6) (see Remark 2.2) one can obtain that

$$\mathrm{Cap}_p(\Omega) = \int_0^1 \int_{\{u_p = \tau\}} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma \, \mathrm{d}\tau. \tag{2-8}$$

Employing again the coarea formula (2-2) with $f = |\mathrm{D}u_p|^{p-1}$ and integration by parts we get

$$\begin{aligned}\int_0^1 \varphi'(\tau) \int_{\{u_p = \tau\}} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma \, \mathrm{d}\tau &= \int_{M \smallsetminus \bar{\Omega}} \varphi'(u_p) |\mathrm{D}u_p|^p \, \mathrm{d}\mu = -\int_{M \smallsetminus \bar{\Omega}} |\mathrm{D}u_p|^{p-2} \langle \mathrm{D}u_p, \mathrm{D}(\varphi(u_p)) \rangle \, \mathrm{d}\mu \\ &= \int_{M \smallsetminus \bar{\Omega}} \varphi(u_p) \, \mathrm{div}(|\mathrm{D}u_p|^{p-2} \mathrm{D}u_p) \, \mathrm{d}\mu = 0\end{aligned}$$

for every $\varphi \in \mathscr{C}_c^\infty(0, 1)$, which gives that

$$\tau \mapsto \int_{\{u_p = \tau\}} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma$$

admits a constant representative; that coupled with (2-8) yields (2-7), letting $t = 1/\tau$. □

In particular, evaluating (2-7) at $t = 1$, which is a regular value by the Hopf maximum principle [Tolksdorf 1983, Proposition 3.2.1], we have that

$$\mathrm{C}_p(\Omega) = \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p} \right)^{p-1} \int_{\partial\Omega} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma.$$

Moreover, one can actually relate the capacity of $\Omega_t = \{u > 1/t\} \cup \Omega$ to the capacity of Ω. The proof of the following lemma is contained in [Holopainen 1990, Lemma 3.8].

Proposition 2.9. *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold for some $1 < p < n$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then a solution u_p to (1-4) realises*

$$\mathrm{C}_p(\Omega_t) = t^{p-1} \mathrm{C}_p(\Omega) \tag{2-9}$$

for every $t \in [1, +\infty)$, where $\Omega_t = \{u > 1/t\} \cup \Omega$. In particular, the map $t \mapsto \mathrm{C}_p(\Omega_t)$ is smooth.

2C. *Li–Yau-type estimates.* We provide a sharp lower estimate for the p-Green's function, extending the well-known

$$d_g(o,x)^{2-n} \le G_2(o,x) \tag{2-10}$$

holding true for any couple of points o, x belonging to a 2-nonparabolic Riemannian manifolds with nonnegative Ricci curvature. The proof of (2-10) builds on the Laplacian comparison, which applies to showing that

$$\Delta\, d_g(o,\,\cdot\,)^{2-n} \ge 0$$

in the sense of distributions. This amounts to saying that

$$-\int_M \langle \mathrm{D} d_g(o,\cdot)^{2-n} \mid \mathrm{D}\psi\rangle \,\mathrm{d}\mu = \int_M d_g(o,\cdot)^{2-n}\Delta\psi \,\mathrm{d}\mu \ge 0 \tag{2-11}$$

for any test function $\psi \in \mathscr{C}^\infty_c(M)$. This leads to (2-10) substantially through the maximum principle. We refer the reader to [Agostiniani et al. 2020, Lemma 2.12] for details. The nonlinear version of (2-10), that, to our knowledge, has not been explicitly pointed out in literature yet, actually relies on (2-11) too.

Proposition 2.10 (sharp lower bound for the p-Green's function). *Let (M, g) be a complete p-nonparabolic Riemannian manifold, $1 < p < n$, with $\mathrm{Ric} \ge 0$. Let $o \in M$. Then, we have*

$$d_g(o,x)^{-\frac{n-p}{p-1}} \le G_p(o,x) \tag{2-12}$$

for any $x \in M \smallsetminus \{o\}$.

Proof. Fix for simplicity $o \in M$, and let $r(x) = d_g(o,x)$. We first show that $\Delta_p r^{-(n-p)/(p-1)} \ge 0$ holds in the weak sense, that is,

$$\int_M \langle |\mathrm{D} r^{-\frac{n-p}{p-1}}|^{p-2} \mathrm{D} r^{-\frac{n-p}{p-1}}, \mathrm{D}\psi\rangle \,\mathrm{d}\mu \le 0$$

for any $\psi \in \mathscr{C}^\infty_c(M)$. In fact, we have

$$\begin{aligned}\int_M \langle |\mathrm{D} r^{-\frac{n-p}{p-1}}|^{p-2} \mathrm{D} r^{-\frac{n-p}{p-1}}, \mathrm{D}\psi\rangle \,\mathrm{d}\mu &= -\left(\frac{n-p}{p-1}\right)^{p-1}\int_M r^{1-n}\langle \mathrm{D} r, \mathrm{D}\psi\rangle\,\mathrm{d}\mu \\ &= \frac{1}{n-2}\left(\frac{n-p}{p-1}\right)^{p-1}\int_M \langle \mathrm{D} r^{2-n}, \mathrm{D}\psi\rangle\,\mathrm{d}\mu \le 0,\end{aligned}$$

where the last inequality is the Laplacian comparison theorem (2-11).

Let now be $\delta > 0$. Since both $r^{-(n-p)/(p-1)}$ and G_p vanish at infinity, we have $r^{-(n-p)/(p-1)} \le G_p + \delta$ on $\partial B(o, R)$ for any $R > 0$ big enough. On the other hand, the general result [Serrin 1964, Theorem 12] ensures that $G_p(o, x)$ is asymptotic to $r(x)^{-(n-p)/(p-1)}$ as $d_g(o, x) \to 0^+$, and thus we also get $r^{-(n-p)/(p-1)} \le G_p + \delta$ on $\partial B(o, \varepsilon)$ for any $\varepsilon > 0$ small enough. Thus, applying the comparison principle to the subsolution $r^{-(n-p)/(p-1)}$ and to the solution $G_p + \delta$ (with respect to the p-Laplacian), in the annulus $B(o, R) \smallsetminus \overline{B(o, \varepsilon)}$, we get $r^{-(n-p)/(p-1)} \le G_p + \delta$ on such an annulus. Letting $\varepsilon \to 0^+$ and $R \to +\infty$, we deduce that the same holds on the whole $M \smallsetminus \{o\}$. Finally, letting $\delta \to 0^+$, we are left with (2-12). ☐

Coupling (2-3) and (2-12) with the comparison principle, we deduce the following important estimate for the p-capacitary potential.

Theorem 2.11. *Let (M, g) be a complete p-nonparabolic Riemannian manifold for some for some $1 < p < n$, with $\mathrm{Ric} \geq 0$. Let $\Omega \subset M$ be a bounded subset with smooth boundary, and let u_p be its p-capacitary potential. Then, there exists a positive constant C_1 such that*

$$\mathrm{C}_1\, d_g(o, x)^{-\frac{n-p}{p-1}} \leq u_p(x) \tag{2-13}$$

for any $x \in M \smallsetminus \Omega$. If in addition (M, g) has Euclidean volume growth, then there also exists another positive constant C_2 such that

$$u_p(x) \leq \mathrm{C}_2\, d_g(o, x)^{-\frac{n-p}{p-1}}. \tag{2-14}$$

Proof. In light of (2-12) and (2-3), this one holding true if (M, g) satisfies the additional Euclidean volume growth assumption, it suffices to show that there exist positive constants C_1 and C_2 such that $\mathrm{C}_1 G_p \leq u_p \leq \mathrm{C}_2 G_p$. Choose any $\mathrm{C}_1 < 1/\sup_{\partial\Omega} u_p$. Then, $\mathrm{C}_1 G_p < u_p$ on $\partial\Omega$. Moreover, since both u_p and G_p vanish at infinity, for any $\delta > 0$ we have $\mathrm{C}_1 G_p < u_p + \delta$ on $\partial B(o, R)$ for any R big enough. The comparison principle applied to the p-harmonic functions $u_p + \delta$ and G_p in $B(o, R) \smallsetminus \overline{\Omega}$ shows that $\mathrm{C}_1 G_p < u + \delta$ in the latter subset. The radius R being arbitrarily big, this implies that, by passing to the limit as $R \to +\infty$, that $\mathrm{C}_1 G_p < u_p + \delta$ in the whole $M \smallsetminus \Omega$. Letting $\delta \to 0^+$ leaves us with $\mathrm{C}_1 G_p \leq u_p$, and consequently with (2-13). The inequality $u_p \leq \mathrm{C}_2 G_p$, yielding (2-14), is shown the same way. □

We now couple (2-13) with the general Cheng–Yau-type inequality for p-harmonic functions on manifolds with nonnegative Ricci curvature provided in [Wang and Zhang 2011]. It asserts that a p-harmonic function v, with $1 < p < n$ defined in a ball $B(o, 2R) \subset M$, where M is endowed with a Riemannian metric such that $\mathrm{Ric} \geq 0$, satisfies the estimate

$$\sup_{B(o,R)} \frac{|\mathrm{D}v|}{v} \leq \frac{\mathrm{C}}{R} \tag{2-15}$$

for a constant C depending only on the dimension of the ambient manifold and p. With these tools we immediately obtain:

Proposition 2.12. *Let (M, g) be a p-nonparabolic Riemannian manifold for some $1 < p < n$, with $\mathrm{Ric} \geq 0$. Let $\Omega \subset M$ be a bounded subset with smooth boundary, and let u_p be its p-capacitary potential. Then, there exists a positive constant* C *such that*

$$|\mathrm{D}u_p| u_p^{-\frac{n-1}{n-p}} \leq \mathrm{C} \tag{2-16}$$

holds on the whole $M \smallsetminus \Omega$.

Proof. By the $\mathscr{C}^1$-regularity of u_p, it clearly suffices to show that (2-16) holds outside some compact set containing $\overline{\Omega}$. Let then $o \in \Omega$ and $R > 0$ be such that $\Omega \subset B(o, R)$, and let $x \in M \smallsetminus \overline{B(o, 4R)}$. With this choice, we have $B(x, 2d_g(o, x) - 2R) \subset M \smallsetminus \overline{B(o, 2R)}$. Thus, applying inequality (2-15) to the function u_p, on $B(x, d_g(o,x)-R)$, we get

$$\frac{|\mathrm{D}u_p|}{u_p^{\frac{n-1}{n-p}}}(x) \leq \mathrm{C}\frac{u_p(x)}{d_g(o,x)-R}u_p^{-\frac{n-1}{n-p}}(x) \leq 2\mathrm{C}\frac{u_p^{-\frac{p-1}{n-p}}(x)}{d_g(o,x)}$$

and the rightmost-hand side is bounded by means of (2-13). □

2D. ***Kato-type identity and a warped product splitting theorem.*** Finally, we give the statement of the refined Kato-type identity for p-harmonic functions obtained in [Fogagnolo et al. 2019, Proposition 4.4], which will be at the core of the monotonicity and rigidity of F_p^β.

Definition 2.13 (geometry of level sets and orthogonal decomposition). Let (M, g) be a Riemannian manifold and v be a smooth function on M. At any point where $|\mathrm{D}v| \neq 0$ we denote by h and H respectively the second fundamental form and the mean curvature of the level set of u with respect to the unit normal $\mathrm{D}v/|\mathrm{D}v|$ and $g^\top$ the metric induced by g on the level set of u. Finally, for a given differentiable function f, we denote by $\mathrm{D}^\top f$ the tangential part of the gradient, according to the orthogonal decomposition

$$\mathrm{D}^\perp f = \left\langle \mathrm{D}f, \frac{\mathrm{D}v}{|\mathrm{D}v|}\right\rangle \frac{\mathrm{D}v}{|\mathrm{D}v|} \quad \text{and} \quad \mathrm{D}^\top f = \mathrm{D}f - \mathrm{D}^\perp f.$$

In particular, the following formula holds:

$$|\mathrm{D}|\mathrm{D}f||^2 = |\mathrm{D}^\top|\mathrm{D}f||^2 + |\mathrm{D}^\perp|\mathrm{D}f||^2.$$

We are now ready to state the Kato-type identity for p-harmonic function.

Proposition 2.14 (Kato-type identity). *Let (M, g) be a Riemannian manifold and let v be a p-harmonic function on some subset of M, $p > 1$. Then, in an open neighbourhood of a point where $|\mathrm{D}v| \neq 0$, the following identity holds*:

$$|\mathrm{DD}v|^2 - \left(1+\frac{(p-1)^2}{n-1}\right)|\mathrm{D}|\mathrm{D}v||^2 = |\mathrm{D}v|^2\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}g^\top\right|^2 + \left(1-\frac{(p-1)^2}{n-1}\right)|\mathrm{D}^\top|\mathrm{D}v||^2,$$

according to the notation in Definition 2.13. Moreover, if, for some $t_0 \in \mathbb{R}$, $|\mathrm{D}v| > 0$ and

$$\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}g^\top\right|^2 = 0, \quad |\mathrm{D}^\top|\mathrm{D}v||^2 = 0$$

hold at each point of $\{v \geq t_0\}$, then the Riemannian manifold $(\{v \geq t_0\}, g)$ is isometric to the warped product $([t_0, +\infty) \times \{v = t_0\}, \mathrm{d}t \otimes \mathrm{d}t + \eta^2(t) g_{\{v=t_0\}})$, where the relation between v, η and t is given by

$$\eta(t) = \left(\frac{v'(t_0)}{v'(t)}\right)^{\frac{p-1}{n-1}}. \tag{2-17}$$

3. Monotonicity-rigidity theorems

In this section we are going to prove our *monotonicity formulas* in the p-nonparabolic setting. The results we present here are the natural extensions of the ones shown in [Agostiniani and Mazzieri 2020;

Agostiniani et al. 2020], as well as of the ones obtained in [Fogagnolo et al. 2019; Agostiniani et al. 2022a]. In the first two mentioned papers the authors established the monotonicity in the case of the harmonic potential, respectively in $\mathbb{R}^n$ and in a general 2-nonparabolic manifold with nonnegative Ricci curvature, whereas an analogous theory has been developed in the case of the p-capacitary potential in the Euclidean setting in the second two papers. More precisely, in [Fogagnolo et al. 2019], the authors worked out the smooth computations and took advantage of the fact that the p-capacitary potential associated with a convex domain is smooth and has no critical points (see [Colesanti et al. 2015; Lewis 1977]), whereas the main technical achievement in [Agostiniani et al. 2022a] is the treatment of the general case, when the critical points are present and even possibly arranged in sets of full measure. On the other hand, the approach presented in that work only produces *effective inequalities* (1-6), that are anyway sufficient to prove Theorem 1.1 in the flat setting, as mentioned in the Introduction. Here, we extend these results to the setting of p-nonparabolic manifolds and we improve them, establishing the full monotonicity of the integral quantities defined in (3-1) along the p-capacitary level sets flow.

As usual, the main difficulty amounts to ensuring that the monotonicity survives the singular values of u, that, as far as we know, could even form a set of positive measure. Inspired by the analysis in [Gigli and Violo 2023], where the authors were forced to face severe technical problems caused by the typical low regularity of the nonsmooth setting, we compute the derivative of our integral quantities (3-1) in the distributional sense, appealing to the full strength of the coarea formula in Proposition 2.1, and exploiting the integrability properties of the p-harmonic functions in Lemma 2.3.

From now on, except where it is necessary, we fix $1 < p < n$ and we drop the subscript p when we consider a solution u_p to the problem (1-4).

3A. *Statement of the monotonicity-rigidity theorems.* Let $u : M \smallsetminus \Omega \to \mathbb{R}$ be a solution of (1-4). For $\beta \in [0, +\infty)$ we consider the function

$$F_p^\beta(t) = t^{\beta\frac{(n-1)(p-1)}{(n-p)}} \int_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)} \, \mathrm{d}\sigma \tag{3-1}$$

defined for every $t \geq 1$ such that $|\{u = 1/t\} \cap \mathrm{Crit}(u)| = 0$, which is fulfilled for almost every $t \in [1, +\infty)$ by Proposition 2.1. We also set

$$F_p^\infty(t) = t^{\frac{n-1}{n-p}} \sup_{\{u=1/t\}} |\mathrm{D}u|, \tag{3-2}$$

which is defined on the whole $[1, +\infty)$. If $1/t$ is a regular value for u, then F_p^β is differentiable at t for every $\beta \in [0, +\infty)$ and its derivative is

$$(F_p^\beta)'(t) = -\beta t^{\beta\frac{(n-1)(p-1)}{(n-p)}-2} \int_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)-1} \left(\mathrm{H} - \frac{(n-1)(p-1)}{(n-p)} |\mathrm{D} \log u| \right) \mathrm{d}\sigma. \tag{3-3}$$

As said before, the aim of this section is to prove monotonicity-rigidity theorems for $t \mapsto F_p^\beta(t)$ and $t \mapsto F_p^\infty(t)$.

Theorem 3.1 (monotonicity-rigidity theorem for F_p^β). *Let (M, g) be a p-nonparabolic Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let $\Omega \subseteq M$ be a bounded open subset with smooth boundary. Let F_p^β be the* *function defined in (3-1) with*

$$\frac{n-p}{n-1)(p-1)} < \beta < +\infty.$$

Then F_p^β belongs to $W^{2,1}(1,+\infty)$ and the identity

$$\begin{aligned}(F_p^\beta)'(t) = -\beta\left(\frac{(n-2)(p-1)}{(n-p)}\right)^{(\beta+1)(p-1)} &\int_{\{u\le 1/t\}\setminus \mathrm{Crit}(u)} u^{2-\beta\frac{(p-1)(n-1)}{(n-p)}} |\mathrm{D}u|^{(\beta+1)(p-1)-1} \\ &\times\left\{\left[\beta-\frac{(n-p)}{(n-1)(p-1)}\right]\left[\mathrm{H}-\left[\frac{(n-1)(p-1)}{(n-p)}\right]|\mathrm{D}\log u|\right]^2\right. \\ &\left.+\left|\mathrm{h}-\frac{\mathrm{H}}{n-1}g^\top\right|^2+(p-1)\left[\beta+\frac{p-2}{p-1}\right]\frac{|\mathrm{D}^\top|\mathrm{D}u||^2}{|\mathrm{D}u|^2}+\mathrm{Ric}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|},\frac{\mathrm{D}u}{|\mathrm{D}u|}\right)\right\}\mathrm{d}\mu \end{aligned} \tag{3-4}$$

holds for every $t \in [1,+\infty)$ and

$$\begin{aligned}(F_p^\beta)''(t) = \beta\left(\frac{(n-2)(p-1)}{(n-p)}\right)^{(\beta+1)(p-1)} t^{\beta\frac{(n-1)(p-1)}{(n-p)}-2} &\int_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)-2} \\ &\times\left\{\left[\beta-\frac{(n-p)}{(n-1)(p-1)}\right]\left[\mathrm{H}-\left[\frac{(n-1)(p-1)}{(n-p)}\right]|\mathrm{D}\log u|\right]^2\right. \\ &\left.+\left|\mathrm{h}-\frac{\mathrm{H}}{n-1}g^\top\right|^2+(p-1)\left[\beta+\frac{p-2}{p-1}\right]\frac{|\mathrm{D}^\top|\mathrm{D}u||^2}{|\mathrm{D}u|^2}+\mathrm{Ric}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|},\frac{\mathrm{D}u}{|\mathrm{D}u|}\right)\right\}\mathrm{d}\mu \end{aligned} \tag{3-5}$$

holds for almost every $t \in [1,+\infty)$. In particular, F_p^β admits a convex and monotone nonincreasing $\mathscr{C}^1$ representative. Moreover, $(F_p^\beta)'(t_0) = 0$ at some $t_0 \ge 1$ such that $1/t_0$ is a regular value for u if and only if $(\{u \le 1/t_0\}, g)$ is isometric to

$$\left([\tau_0,+\infty)\times\{u=1/t_0\},\ \mathrm{d}\tau\otimes\mathrm{d}\tau+\left(\frac{\tau}{\tau_0}\right)^2 g_{\{u=1/t_0\}}\right), \quad \textit{where } \tau_0=\left(\frac{|\{u=1/t_0\}|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

In this case $\{u = 1/t_0\}$ is a connected totally umbilical hypersurface with constant mean curvature in $(M \setminus \Omega, g)$.

We also highlight that the rigidity statement is expressed in terms of the derivative. However, if $F_p^\beta(t) = F_p^\beta(T)$ for $1 \le t < T < +\infty$ such that $1/t$ and $1/T$ are regular values for u, the rigidity statement still triggers. Indeed, since the set of regular values is open, monotonicity ensures the existence of a decreasing sequence $(t_j)_{j\in\mathbb{N}}$ such that $t_j \to t$ as $j \to +\infty$, $1/t_j$ is regular for u and $(F_p^\beta)'(t_j) = 0$. Since $t \mapsto F_p^\beta(t)$ is smooth in a neighbourhood of t, this implies that $(F_p^\beta)'(t) = 0$; hence the splitting of $\{u \le 1/t\}$.

Theorem 3.2 (monotonicity-rigidity theorem for F_p^∞). *Let (M, g) be a p-nonparabolic Riemannian manifold with $\mathrm{Ric} \ge 0$. Let $\Omega \subseteq M$ be a bounded open subset with smooth boundary. Let F_p^∞ be the function defined in (3-2). Then F_p^∞ is a continuous monotone nonincreasing function. Furthermore, we* *have*

$$\left[\mathrm{H}_g - \frac{(n-1)(p-1)}{(n-p)}|\mathrm{D}\log u|_g\right](x_t) = -(p-1)\frac{\partial}{\partial \nu_t}\log\frac{|\mathrm{D}u|_g}{u^{\frac{n-1}{n-p}}}(x_t) \geq 0, \tag{3-6}$$

where $x_t \in \{u = 1/t\}$ *is the point where* $\sup_{\{u=1/t\}}|\mathrm{D}u|_g/u^{(n-1)/(n-p)}$ *is achieved and* $\nu_t = -\mathrm{D}u/|\mathrm{D}u|_g$ *is the unit normal to* $\{u = 1/t\}$. *Moreover,* $F_p^\infty(t_0) = F_p^\infty(T)$ *for some* $t_0 < T$ *or the equality holds in* (3-6) *for some* t_0 *such that* $1/T$ *and* $1/t_0$ *are regular for* u *if and only if* $(\{u \leq 1/t_0\}, g)$ *is isometric to*

$$\left([\tau_0, +\infty) \times \{u = 1/t_0\},\ \mathrm{d}\tau \otimes \mathrm{d}\tau + \left(\frac{\tau}{\tau_0}\right)^2 g_{\{u=1/t_0\}}\right), \quad \text{where } \tau_0 = \left(\frac{|\{u = 1/t_0\}|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

In this case $\{u = 1/t_0\}$ *is a connected totally umbilical hypersurface with constant mean curvature in* $(M \smallsetminus \Omega, g)$.

A direct consequence of the monotonicity of F_p^∞ is the following regularity theorem for the p-capacitary potential.

Corollary 3.3. *The function* F_p^∞ *is strictly positive. In particular, every level of* u *has at least one regular point.*

We want also to emphasise that these theorems can be applied in particular in $\mathbb{R}^n$ for every Ω open bounded with smooth boundary, where they naturally extend the monotonicity-rigidity theorems in [Fogagnolo et al. 2019; Agostiniani et al. 2022a].

We conclude this introduction by rewriting the functions F_p^β and F_p^∞ defined in (3-1) and (3-2) in a different formulation. We make use of this tool only to simplify computations, but as shown in [Agostiniani and Mazzieri 2020; Fogagnolo et al. 2019; Agostiniani et al. 2022b] monotonicity-rigidity theorems have their counterpart in this framework. Let (M, g) be a complete p-nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$ and $u : M \smallsetminus \Omega \to \mathbb{R}$ be the solution of the problem (1-4). We consider the conformally related Riemannian manifold $(M \smallsetminus \Omega, \tilde{g})$, where $\tilde{g}$ is given by

$$\tilde{g} = u^{2\left(\frac{p-1}{n-p}\right)} g. \tag{3-7}$$

It is also convenient to consider the new variable

$$\varphi = -\frac{(p-1)(n-2)}{(n-p)}\log u, \tag{3-8}$$

so that the metric $\tilde{g}$ can be equivalently rewritten as

$$\tilde{g} = \mathrm{e}^{-\frac{2\varphi}{n-2}} g.$$

With the same formal computation as in [Fogagnolo et al. 2019], one can prove that $\Delta_{\tilde{g}}^p \varphi = 0$ on $M \smallsetminus \overline{\Omega}$ where $\Delta_{\tilde{g}}^p$ is the p-Laplace operator with respect to the metric $\tilde{g}$.

From now on, given (M, g) *a* p*-nonparabolic manifold with* $\mathrm{Ric} \geq 0$ *and* u *a solution to* (1-4), φ *will be the function obtained by* u *through* (3-8), *whereas* $\tilde{g}$ *will indicate the metric on* $M \smallsetminus \Omega$ *obtained from* u *and* g *through* (3-7).

The gradient of φ is related to the one of u by

$$|\nabla\varphi|_{\tilde{g}} = \frac{(n-2)(p-1)}{(n-p)} \frac{|Du|_g}{u^{\frac{n-1}{n-p}}}, \tag{3-9}$$

where ∇ is the Levi-Civita connection associated to the metric $\tilde{g}$. We can observe that if t is a regular value for u then $s = -[(p-1)(n-2)/(n-p)]\log t$ is a regular value for φ, thanks to (3-8) and the previous relation. Moreover, we recognise from the above expression and the estimate (2-16) that the fundamental property of $|\nabla\varphi|_{\tilde{g}}$ is uniformly bounded, that is, there exists a constant C such that

$$|\nabla\varphi|_{\tilde{g}} \leq \mathrm{C} \tag{3-10}$$

on the whole $M \smallsetminus \Omega$.

Using (3-9), the family of functions $t \mapsto F_p^\beta(t)$ for $\beta \in [0, +\infty]$ defined in (3-1) and (3-2) can be rewritten in terms of $\tilde{g}$ and φ obtained through (3-8) and (3-7). For any $\beta \in [0, +\infty)$ we can now consider the function

$$\Phi_p^\beta(s) = \int_{\{\varphi=s\}} |\nabla\varphi|_{\tilde{g}}^{(\beta+1)(p-1)} \, d\sigma_{\tilde{g}}, \tag{3-11}$$

whenever $s \geq 0$ is such that $|\{\varphi = s\} \cap \mathrm{Crit}(\varphi)|$. Correspondingly we set

$$\Phi_p^\infty(s) = \sup_{\{\varphi=s\}} |\nabla\varphi|_{\tilde{g}}, \tag{3-12}$$

which is defined on the whole $[0, +\infty)$. The function Φ_p^β can be obtained from F_p^β through a change of variable, that is,

$$\Phi_p^\beta(s) = F_p^\beta\big(e^{\frac{(n-p)}{(p-1)(n-2)}s}\big).$$

For $\beta < +\infty$ it thus holds that

$$(\Phi_p^\beta)'(s) = \frac{(n-p)}{(p-1)(n-2)} e^{\frac{(n-p)}{(p-1)(n-2)}s} (F_p^\beta)'\big(e^{\frac{(n-p)}{(p-1)(n-2)}s}\big)$$

for almost every $s \in [0, +\infty)$. The previous relations reveal how proving the monotonicity results for F_p^β and F_p^∞, stated in Theorems 3.1 and 3.2, are equivalent to show the same one for Φ_p^β and Φ_p^∞. The same argument applies for the regularity of F_p^β.

3B. ***Proof of monotonicity-rigidity theorems.*** A basic property we will need is the essential uniform boundedness of Φ_p^β of Φ_p^∞ defined in (3-11) and (3-12).

Lemma 3.4. *Let be $1 < p < n$, and (M, g) be a p-nonparabolic Riemannian manifold. Let $\Omega \subset M$ be a open bounded subset with smooth boundary. For every $\beta \in [0, +\infty)$, Φ_p^β is essentially uniformly bounded, namely $\Phi_p^\beta(s) \leq C$ for almost every $s \in [0, +\infty)$, including any s that is regular for φ. Moreover, the function Φ_p^∞ is uniformly bounded.*

Proof. It suffices to write Φ_p^β as

$$\Phi_p^\beta(s) = \int_{\{\varphi=s\}} |\nabla\varphi|_{\tilde g}^{(\beta+1)(p-1)}\, d\sigma_{\tilde g} \le C^{\beta(p-1)} \int_{\{\varphi=s\}} |\nabla\varphi|_{\tilde g}^{p-1}\, d\sigma_{\tilde g} = C^{\beta(p-1)}\left[\frac{(n-2)(p-1)}{(n-p)}\right]^{p-1} \int_{\{u=1/t\}} |Du|^{p-1}\, d\sigma,$$

where C is the constant in (3-10), the last identity is due to (3-9) and (3-8) taking

$$s = -\left[\frac{(p-1)(n-2)}{n-p}\right] \log t.$$

By (2-7) we have that the integral on the rightmost-hand side coincides with $\mathrm{Cap}_p(\Omega)$ for almost any t, including any of those such that $1/t$ is a regular value for u. This settles the boundedness of Φ_p^β for finite β. On the other hand the uniform boundedness of Φ_p^∞ is a direct consequence of (3-10) alone. □

From now on, we will drop the subscript $\tilde g$ whenever it is clear to which metric we are referring.

Suppose by now that $\beta \in [0, +\infty)$ and consider the vector field

$$X = e^{-\frac{(n-p)}{(n-2)(p-1)}\varphi} |\nabla\varphi|^{p-2}\big(\nabla|\nabla\varphi|^{\beta(p-1)} + (p-2)\nabla^{\perp}|\nabla\varphi|^{\beta(p-1)}\big), \tag{3-13}$$

defined in a neighbourhood of each point such that $|\nabla\varphi| > 0$, where the function φ is actually smooth, being p-harmonic with respect to the metric $\tilde g$. This vector field is related to the derivative of Φ_p^β through the following identity.

Proposition 3.5. *Let (M, g) be a p-nonparabolic Riemannian manifold with* $\mathrm{Ric} \ge 0$. *For every $\beta \in [0, +\infty)$, the function $s \mapsto \Phi_p^\beta(s)$ defined in* (3-11) *belongs to $W^{1,1}_{\mathrm{loc}}(0, +\infty)$ and its derivative is given by*

$$e^{-\frac{(n-p)}{(n-2)(p-1)}s} (\Phi_p^\beta)'(s) = \frac{1}{p-1} \int_{\{\varphi=s\}} \left\langle X, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle d\sigma \tag{3-14}$$

for almost every $s \in [0, +\infty)$, where X is the vector field defined in (3-13).

Before starting the proof, observe that the quantity appearing in the left-hand side of (3-14) is actually well-defined for almost every $s \in (0, +\infty)$ even if X is a priori defined only where $|\nabla\varphi| > 0$. Indeed, by Proposition 2.1 $|\mathrm{Crit}\,\varphi \cap \{\varphi = s\}| = 0$ for almost every $s \in (0, +\infty)$.

Proof. By the definition of X, it is easy to check that

$$e^{-\frac{(n-p)}{(n-2)(p-1)}\varphi} \left\langle |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle = \frac{1}{p-1}\left\langle X, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle$$

holds around each point such that $|\nabla\varphi| \ne 0$. Hence, it remains only to prove that $\Phi_p^\beta(s) \in W^{1,1}_{\mathrm{loc}}(0 + \infty)$ and that

$$(\Phi_p^\beta)'(s) = \int_{\{\varphi=s\}} \left\langle |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle d\sigma$$

holds for almost any $s \in (0, \infty)$. Let $\eta \in \mathscr{C}^\infty_c(0, +\infty)$. Since $|\nabla\varphi| \leq C$ by (3-10), applying the coarea formula (2-2) with $f = |\nabla\varphi|^{(\beta+1)(p-1)}$ and the chain rule we obtain that

$$\begin{aligned}\int_0^{+\infty} \eta'(s)\Phi_p^\beta(s)\,\mathrm{d}s &= \int_0^{+\infty} \eta'(s) \int_{\{\varphi=s\}} |\nabla\varphi|^{(\beta+1)(p-1)}\,\mathrm{d}\sigma\,\mathrm{d}s \\ &= \int_{M\smallsetminus\bar{\Omega}} \eta'(s)\langle\nabla\varphi, \nabla\varphi\rangle|\nabla\varphi|^{(\beta+1)(p-1)-1}\,\mathrm{d}\mu \\ &= \int_{M\smallsetminus\bar{\Omega}} \langle\nabla(\eta(\varphi)), \nabla\varphi\rangle|\nabla\varphi|^{(\beta+1)(p-1)-1}\,\mathrm{d}\mu.\end{aligned}$$

Integrating by parts the right-hand side, $\Delta^{(p)}\varphi = 0$ yields

$$\int_0^{+\infty} \eta'(s)\Phi_p^\beta(s)\,\mathrm{d}s = -\int_{M\smallsetminus\bar{\Omega}} \eta(\varphi)\langle|\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}, \nabla\varphi\rangle\,\mathrm{d}\mu.$$

Thanks to (3-10) and Lemma 2.3, we are in position to apply the coarea formula in Proposition 2.1 with $f = \langle|\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}, \nabla\varphi\rangle/|\nabla\varphi|$ (see Remark 2.2) to get

$$\int_0^{+\infty} \eta'(s)\Phi_p^\beta(s)\,\mathrm{d}s = -\int_0^1 \eta(s) \int_{\{\varphi=s\}} \left\langle|\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle\mathrm{d}\sigma\,\mathrm{d}s,$$

which ensures both that $\Phi_p^\beta \in W^{1,1}_{\mathrm{loc}}(0, +\infty)$ and (3-14). □

The nonnegative divergence of X is what substantially rules the monotonicity of Φ_p^β, and this is true when β ranges in a suitable set of parameters.

Lemma 3.6 (divergence of X). *Let (M, g) be a p-nonparabolic manifold and X be the vector field defined in (3-13). Then*

$$\operatorname{div} X = \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi} Q$$

holds at any point such that $|\nabla\varphi| > 0$, with

$$\begin{aligned}Q = \beta(p-1)|\nabla\varphi|^{\beta(p-1)+p-2}\Bigg\{\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}\tilde{g}^\top\right|^2 + (p-1)\left[\beta + \frac{p-2}{p-1}\right]\frac{|\nabla^\top|\nabla\varphi||^2}{|\nabla\varphi|^2} \\ + (p-1)^2\left[\beta - \frac{(n-p)}{(p-1)(n-1)}\right]\frac{|\nabla^\perp|\nabla\varphi||^2}{|\nabla\varphi|^2} + \mathrm{Ric}_g\left(\frac{\nabla\varphi}{|\nabla\varphi|^2}, \frac{\nabla\varphi}{|\nabla\varphi|^2}\right)\Bigg\},\end{aligned} \tag{3-15}$$

where h *and* H *are respectively the second fundamental form and the mean curvature of the level sets of φ with respect to the unit normal $\nabla\varphi/|\nabla\varphi|$, $\nabla^\top$ is defined in Definition 2.13 and* Ric_g *denotes the Ricci tensor of the background metric. In particular,*

$$\operatorname{div}(X) \geq 0 \quad \text{for } \frac{n-p}{(n-1)(p-1)} \leq \beta < +\infty.$$

Proof. The proof follows the same lines of [Agostiniani et al. 2022a, Lemma 4.1], replacing accordingly the vector fields $W = |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}$ and $Z = (p-2)|\nabla\varphi|^{p-2}\nabla^\perp|\nabla\varphi|^{\beta(p-1)}$. The Ricci curvature term appears computing the divergence of W thanks to the Bochner identity for p-harmonic functions, as the reader can see following [Fogagnolo et al. 2019, Proposition 4.3]. □

Suppose that $|\nabla\varphi| \neq 0$ everywhere. We can apply the divergence theorem in the domain $\{s < \varphi < S\}$ to obtain

$$\int_{\{\varphi=S\}} \left\langle X, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma - \int_{\{\varphi=s\}} \left\langle X, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma = \int_{\{s<\varphi<S\}} \operatorname{div} X \,\mathrm{d}\mu \geq 0. \tag{3-16}$$

Using (3-14) we deduce that

$$\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s} (\Phi_p^\beta)'(s) \leq \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}S} (\Phi_p^\beta)'(S).$$

This almost concludes the proof of the monotonicity theorem for Φ_p^β with

$$\frac{n-p}{(n-1)(p-1)} < \beta < +\infty,$$

assuming the absence of critical points. Indeed, by integrating it, monotonicity will follow as in [Fogagnolo et al. 2019, Theorem 3.4]. This case lies in the same trail blazed in [Agostiniani and Mazzieri 2020] since if $|\nabla\varphi| \neq 0$, the p-Laplace operator is elliptic nondegenerate, and thus the techniques used for harmonic functions fit perfectly.

If we want to pursue the previous path, even when the critical set of φ is not empty, we are first committed to providing a version of (3-16) that holds even in presence of critical values. The main issue is that $\operatorname{div}(X)$ does not belong to L^1_{loc} a priori. Following the same lines of [Gigli and Violo 2023, Proposition 4.6], testing $s \mapsto \mathrm{e}^{-s(n-p)/((n-2)(p-1))}(\Phi_p^\beta)'(s)$ against nonnegative functions $\eta \in \mathscr{C}_c^\infty(0,+\infty)$ and using the coarea formula Proposition 2.1 for $f = \langle X, \nabla\varphi/|\nabla\varphi|\rangle(1-\chi_{\mathrm{Crit}\,\varphi})$, one gets

$$(p-1)\int_0^{+\infty} \eta'(s)\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s} (\Phi_p^\beta)'(s)\,\mathrm{d}s = \int_{M\smallsetminus\mathrm{Crit}(\varphi)} \langle X, \nabla[\eta(\varphi)]\rangle \,\mathrm{d}\mu.$$

We now would like to integrate by parts and use the nonnegativity of $\operatorname{div}(X)$ outside the critical set of φ. In doing this, we are hampered by the fact that $\operatorname{div}(\chi_{M\smallsetminus\mathrm{Crit}\,\varphi}X)$ is actually a measure that is possibly not absolutely continuous. Hence we can aim to prove that $s \mapsto \mathrm{e}^{-s(n-p)/((n-2)(p-1))}(\Phi_p^\beta)'(s)$ belongs to $\mathrm{BV}_{\mathrm{loc}}(0,+\infty)$, but not the absolute continuity. Differently from the nonsmooth case, we can here employ the higher regularity of φ outside its critical set to refine the result.

Proposition 3.7. *Let (M, g) be a p-nonparabolic Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. For every*

$$\frac{n-p}{(n-1)(p-1)} < \beta < +\infty,$$

the function $s \mapsto \mathrm{e}^{-s(n-p)/((n-2)(p-1))}(\Phi_p^\beta)'(s)$ defined in (3-14) belongs to $W^{1,1}_{\mathrm{loc}}(0,+\infty)$ and its derivative is given by

$$\left(\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s} (\Phi_p^\beta)'(s)\right)' = \frac{1}{p-1}\int_{\{\varphi=s\}} \frac{\operatorname{div} X}{|\nabla\varphi|}\,\mathrm{d}\sigma \tag{3-17}$$

for almost every $s \in [0,+\infty)$, where X is the vector field defined in (3-13).

We remark again that the quantity appearing in the left-hand side of (3-17) is actually well-defined for almost every $s \in (0,+\infty)$ even if X is a priori defined only where $|\nabla\varphi| > 0$. Indeed, by Proposition 2.1

$|\mathrm{Crit}\,\varphi \cap \{\varphi = s\}| = 0$ for almost every $s \in (0, +\infty)$. Moreover, since $\varphi \in \mathscr{C}^\infty$ around each point where $|\nabla\varphi| > 0$, the field X is smooth around such points; thus its divergence can be classically computed.

Proof. Proposition 3.7 follows if we prove that $\mathrm{div}(X)(1 - \chi_{\mathrm{Crit}(\varphi)})$ belongs to $L^1_{\mathrm{loc}}(M \smallsetminus \overline{\Omega})$ and

$$(p-1)\int_0^{+\infty} \eta'(s)\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)\,\mathrm{d}s = -\int_{M\smallsetminus \mathrm{Crit}\,\varphi} \eta(\varphi)\,\mathrm{div}\,X\,\mathrm{d}\mu \tag{3-18}$$

holds for every $\eta \in \mathscr{C}^\infty_c(0, +\infty)$. By the coarea formula in Proposition 2.1, with $f = \mathrm{div}(X)(1 - \chi_{\mathrm{Crit}(\varphi)})$, we would get

$$\int_{M\smallsetminus \mathrm{Crit}\,\varphi} \eta(\varphi)\,\mathrm{div}\,X\,\mathrm{d}\mu = \int_0^{+\infty} \eta(s)\int_{\{\varphi=s\}} \frac{\mathrm{div}\,X}{|\nabla\varphi|}\,\mathrm{d}\sigma\,\mathrm{d}t,$$

which implies both that $\mathrm{e}^{-(n-p)s/(n-2)(p-1)}(\Phi_p^\beta)' \in W^{1,1}_{\mathrm{loc}}(0, +\infty)$ and (3-17).

Step 1: proof for nonnegative η. Let $\eta \in \mathscr{C}^\infty_c(0, +\infty)$ be nonnegative. For any given $\varepsilon > 0$ consider the smooth nonnegative cut-off function $\chi_\varepsilon : [0, +\infty) \to \mathbb{R}$ defined as

$$\begin{cases} \chi_\varepsilon(t) = 0 & \text{in } t < \frac{1}{2}\varepsilon, \\ 0 < \chi'_\varepsilon(t) \le 2\varepsilon^{-1} & \text{in } \frac{1}{2}\varepsilon \le t \le \frac{3}{2}\varepsilon, \\ \chi_\varepsilon(t) = 1 & \text{in } t > \frac{3}{2}\varepsilon. \end{cases}$$

Consider accordingly the vector field $X_\varepsilon = \chi_\varepsilon(|\nabla\varphi|^{\beta(p-1)})X$, where X is the vector field given in (3-13). Let $\eta \in \mathscr{C}^\infty_c(0, +\infty)$ be nonnegative. We notice that $|\langle X_\varepsilon, \nabla\varphi\rangle| \le |\langle X, \nabla\varphi\rangle|$ which is in $L^2_{\mathrm{loc}}(M \smallsetminus \overline{\Omega})$ by (3-10) and Lemma 2.3. Hence (3-14), the coarea formula with $f = \eta'(\varphi)\langle X, \nabla\varphi/|\nabla\varphi|\rangle$ and the dominated convergence theorem imply

$$\int_0^{+\infty} \eta'(s)\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)\,\mathrm{d}s = \lim_{\varepsilon\to 0^+} \frac{1}{p-1}\int_M \eta'(\varphi)\langle X_\varepsilon, \nabla\varphi\rangle\,\mathrm{d}\mu.$$

Employing the coarea formula in (2-2) with $f = \langle X_\varepsilon, \nabla\varphi/|\nabla\varphi|\rangle$ and integration by parts, we obtain that

$$\begin{aligned}\int_0^{+\infty} \eta'(s)\int_{\{\varphi=s\}} \left\langle X_\varepsilon, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma\,\mathrm{d}s \\ &= \int_M \eta'(\varphi)\langle X_\varepsilon, \nabla\varphi\rangle\,\mathrm{d}\mu = -\int_M \mathrm{div}(X_\varepsilon)\eta(\varphi)\,\mathrm{d}\mu \\ &= -\int_{M\smallsetminus N_{\varepsilon/2}} \eta(\varphi)\chi_\varepsilon(|\nabla\varphi|^{\beta(p-1)})\,\mathrm{div}\,X\,\mathrm{d}\mu - \int_{N_{3\varepsilon/2}\smallsetminus N_{\varepsilon/2}} \eta(\varphi)\chi'_\varepsilon(|\nabla\varphi|^{\beta(p-1)})\langle X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle\,\mathrm{d}\mu,\end{aligned}$$

where $N_\delta = \{|\nabla\varphi|^{\beta(p-1)} < \delta\}$ for every $\delta > 0$. By the monotone convergence theorem, the first integral in the rightmost-hand side gives

$$\lim_{\varepsilon\to 0^+} \int_{M\smallsetminus N_{\varepsilon/2}} \eta(\varphi)\chi_\varepsilon(|\nabla\varphi|^{\beta(p-1)})\,\mathrm{div}\,X\,\mathrm{d}\mu = \int_{M\smallsetminus \mathrm{Crit}(\varphi)} \eta(\varphi)\,\mathrm{div}\,X\,\mathrm{d}\mu \ge 0.$$

To achieve Step 1, it thus remains to prove that the second integral vanishes as $\varepsilon \to 0^+$. Observe that the integral in question is always nonnegative, as $\langle X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle \ge 0$, $\eta \ge 0$ and $\chi'_\varepsilon \ge 0$. Hence, we only need to estimate it from above with a quantity that vanishes as $\varepsilon \to 0^+$. Since $|\nabla\varphi|^{\beta(p-1)} \geq \varepsilon/2$ on $N_{3\varepsilon/2} \smallsetminus N_{\varepsilon/2}$, φ is smooth in such a region. The coarea formula in Proposition 2.1 and $\chi'_\varepsilon \leq 2/\varepsilon$ would give

$$\int_{N_{3\varepsilon/2} \smallsetminus N_{\varepsilon/2}} \eta(\varphi)\chi'_\varepsilon(|\nabla\varphi|^{\beta(p-1)})\langle X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle \,\mathrm{d}\mu \leq \frac{2}{\varepsilon}\int_{\varepsilon/2}^{3\varepsilon/2}\int_{\partial N_s} \frac{\langle \eta(\varphi)X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma\,\mathrm{d}s. \quad (3\text{-}19)$$

However, to apply Proposition 2.1 without further specifications, the set $N_{3\varepsilon/2} \smallsetminus N_{\varepsilon/2}$ should be compactly contained in $M \smallsetminus \overline{\Omega}$ for every $\varepsilon > 0$ small enough. Since $|\nabla\varphi| > 0$ on $\partial\Omega$ and $\varphi \in \mathscr{C}^{1,\beta}_{\mathrm{loc}}(M \smallsetminus \Omega)$, it is clear that the set $N_{3\varepsilon/2} \smallsetminus N_{\varepsilon/2}$ does not touch $\partial\Omega$. Nonetheless, it could be unbounded. This is not a real issue since we are integrating $\eta(\varphi)$, which has compact support. More rigorously, choose $S > 0$ such that $\eta(s) = 0$ for every $s \geq S$. Let $\xi : \mathbb{R} \to [0, 1]$ be a smooth cut-off function such that $\xi = 1$ on $[0, S]$ and $\xi = 0$ on $[2S, +\infty)$. Observe that the function $\xi(\varphi)|\nabla\varphi|^{\beta(p-1)} + (1 - \xi(\varphi))$ is smooth outside $\operatorname{Crit}\varphi$, its sublevels $\widetilde{N}_\delta$ are compact for $\delta < 1$ and its gradient coincides with $\nabla|\nabla\varphi|^{\beta(p-1)}$ on the support of $\eta(\varphi)$. Moreover, one can replace N_δ with $\widetilde{N}_\delta$ in both sides of (3-19) without changing the value of the integrals. Indeed, such sets coincide on the support of $\eta(\varphi)$, where integrations are actually performed. Hence, (3-19) holds. Up to the end of this step, we will implicitly use this truncation argument when the coarea formula is applied.

Let $0 < R < 1$ be a regular value for $|\nabla\varphi|$. Define $\mathcal{H}$ as

$$\mathcal{H}(r) = \int_{\partial N_r} \frac{\langle \eta(\varphi)X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma \geq 0$$

for every $r \in (0, R)$ that is a regular value of $|\nabla\varphi|$, hence for almost every $r \in (0, R)$ thanks to Sard's theorem applied to the smooth function $|\nabla\varphi|$. We claim that $\mathcal{H}(r)$ vanishes as $r \to 0^+$. This is enough for Step 1, since it would give

$$\frac{2}{\varepsilon}\int_{\varepsilon/2}^{3\varepsilon/2}\int_{\partial N_s} \frac{\langle \eta(\varphi)X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma\,\mathrm{d}s \leq 2 \sup_{r\in[\frac{\varepsilon}{2}, \frac{3\varepsilon}{2}]} \mathcal{H}(r) \to 0$$

as $\varepsilon \to 0^+$.

Let $0 < t < r < R$ be two regular values for $|\nabla\varphi|$, applying the divergence theorem to the smooth vector field X on $N_r \smallsetminus N_t$ we get

$$\begin{aligned}\mathcal{H}(r) - \mathcal{H}(t) &= \int_{\partial N_r} \frac{\langle \eta(\varphi)X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma - \int_{\partial N_t} \frac{\langle \eta(\varphi)X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma \\ &= \int_{N_r \smallsetminus N_t} \operatorname{div}(\eta(\varphi)X)\,\mathrm{d}\mu = \int_{N_r \smallsetminus N_t} \eta(\varphi)\operatorname{div}(X)\,\mathrm{d}\mu + \int_{N_r \smallsetminus N_t} \langle X, \nabla\varphi\rangle \eta'(\varphi)\,\mathrm{d}\mu. \end{aligned} \quad (3\text{-}20)$$

Since $\operatorname{Ric} \geq 0$ and

$$|\nabla\varphi|^2 \left| \mathrm{h} - \frac{\mathrm{H}}{n-1} g^\top \right|^2 \geq 0,$$

by (3-15) we have that

$$\begin{aligned}\operatorname{div} X &\geq \beta(p-1)^2 \mathrm{e}^{-\frac{(n-p)}{(p-1)(n-2)}\varphi} |\nabla\varphi|^{\beta(p-1)+p-4} \\ &\qquad \times \left(\left[\beta + \frac{p-2}{p-1}\right] |\nabla^{\top}|\nabla\varphi||^2 + (p-1)\left[\beta - \frac{(n-p)}{(p-1)(n-1)}\right] |\nabla^{\perp}|\nabla\varphi||^2 \right) \\ &\geq \mathrm{C}\,\beta^2 p(p-1)^2 \mathrm{e}^{-\frac{(n-p)}{(p-1)(n-2)}\varphi} |\nabla\varphi|^{\beta(p-1)+p-4} (|\nabla^{\perp}|\nabla\varphi||^2 + |\nabla^{\top}|\nabla\varphi||^2) \\ &\geq \mathrm{C}\, p\, \mathrm{e}^{-\frac{(n-p)}{(p-1)(n-2)}\varphi} \frac{|\nabla\varphi|^{p-2} |\nabla|\nabla\varphi|^{\beta(p-1)}|^2}{|\nabla\varphi|^{\beta(p-1)}} \geq \mathrm{C} \frac{\langle X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla\varphi|^{\beta(p-1)}},\end{aligned}$$

where

$$\mathrm{C} = \frac{1}{p\beta} \min\left\{ \left[\beta + \frac{p-2}{p-1}\right], (p-1)\left[\beta - \frac{(n-p)}{(p-1)(n-1)}\right] \right\} > 0.$$

If we plug the above estimate into (3-20) and use the coarea formula in Proposition 2.1 with $f = |\nabla\varphi|^{-\beta(p-1)+p-2} \big|\nabla|\nabla\varphi|^{\beta(p-1)}\big|$, we get

$$\mathcal{H}(r) - \mathcal{H}(t) - \int_{N_r \smallsetminus N_t} \langle X, \nabla\varphi\rangle \eta'(\varphi)\, \mathrm{d}\mu \geq \mathrm{C} \int_t^r \frac{\mathcal{H}(s)}{s}\, \mathrm{d}s. \tag{3-21}$$

On the other hand, the map

$$t \mapsto \mathcal{G}(t) = \int_{N_t \smallsetminus \operatorname{Crit}\varphi} \langle X, \nabla\varphi\rangle \eta'(\varphi)\, \mathrm{d}\mu$$

is a well-defined bounded function in $\mathscr{C}^0([0, R])$. Indeed, $\eta'(\varphi)$ has compact support and

$$|\langle X, \nabla\varphi\rangle| \leq \beta(p-1)|\nabla\varphi|^{\beta(p-1)} |\nabla|\nabla\varphi|^{p-1}| \in L^2_{\mathrm{loc}}(N_R \smallsetminus \operatorname{Crit}\varphi)$$

by Lemma 2.3. Equation (3-21) states that $t \mapsto \mathcal{H}(t) - \mathcal{G}(t)$ is monotonically increasing, whereas $\mathcal{H}(s) \geq 0$ for almost every $s \in (0, r)$. Thus, $t \mapsto \mathcal{H}(t) = \mathcal{H}(t) - \mathcal{G}(t) + \mathcal{G}(t)$ admits a limit as $t \to 0^+$, being the sum of a monotone and a continuous function. Denote by $\mathcal{H}(0)$ such a limit. Since $\mathcal{G}(t) \to 0$ as $t \to 0^+$, by dominated convergence theorem and $\mathcal{H}(0) \geq 0$, we have

$$\mathcal{H}(R) - \mathcal{G}(R) \geq [\mathcal{H}(R) - \mathcal{G}(R)] - [\mathcal{H}(0) - \mathcal{G}(0)] \geq \mathrm{C} \int_0^R \frac{\mathcal{H}(s)}{s}\, \mathrm{d}s.$$

Hence $\mathcal{H}(s) \to 0$ as $s \to 0^+$; otherwise $\mathcal{H}(s)/s$ would not belong to $L^1(0, R)$, contradicting the boundedness of $\mathcal{H}(R) - \mathcal{G}(R)$.

Step 2: conclusions. In the previous step we proved (3-18) for every nonnegative function $\eta \in \mathscr{C}^\infty_c(0, +\infty)$. Let be $K \subset M \smallsetminus \overline{\Omega}$. Then, there exists an $\eta_K \in \mathscr{C}^\infty_c(0, +\infty)$, $\eta_K \geq 0$, such that $\eta_K(\varphi) \geq 1$ on K. In particular, since $\operatorname{div}(X) \geq 0$ outside $\operatorname{Crit}(\varphi)$ we have

$$\begin{aligned}\int_K \operatorname{div}(X)(1 - \chi_{\operatorname{Crit}(\varphi)})\, \mathrm{d}\mu &\leq \int_{M \smallsetminus \operatorname{Crit}(\varphi)} \eta_K(\varphi) \operatorname{div}(X)\, \mathrm{d}\mu \\ &= -(p-1) \int_0^{+\infty} \eta_K'(s) \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s} (\Phi^\beta_p)'(s)\, \mathrm{d}s,\end{aligned}$$

which is finite thanks to Proposition 3.5. This ensures that $\operatorname{div}(X)(1-\chi_{\operatorname{Crit}(\varphi)})$ belongs to $L^1_{\mathrm{loc}}(M \smallsetminus \overline{\Omega})$. Approximating the positive and the negative part of a general $\eta \in \mathscr{C}^\infty_c(0,+\infty)$, that are nonnegative Lipschitz with compact support, we can conclude. □

Proof of Theorem 3.1. We use an argument due to [Colding and Minicozzi 2014a]. By Propositions 3.7 and 3.5, Φ^β_p is $W^{2,1}_{\mathrm{loc}}(0,+\infty)$. By (3-17), $s \mapsto \mathrm{e}^{-s(n-p)/((n-2)(p-1))}(\Phi^\beta_p)'(s)$ is nondecreasing. Then for every $0 \le s < S < +\infty$ we have

$$\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}(S-s)}(\Phi^\beta_p)'(s) \le (\Phi^\beta_p)'(S).$$

Integrating the above inequality, we get

$$\frac{(n-2)(p-1)}{(n-p)}\big(\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}(S-s)}-1\big)(\Phi^\beta_p)'(s) \le \Phi^\beta_p(S) - \Phi^\beta_p(s) \tag{3-22}$$

for every $0 \le s < S < +\infty$. Suppose, by contradiction, that $(\Phi^\beta_p)'(s) > 0$ for some $s \in [0,+\infty)$. Passing to the limit as $S \to +\infty$ in (3-22) we would get that $\Phi^\beta_p(S) \to +\infty$ against the boundedness property ensured by Lemma 3.4. Hence, $(\Phi^\beta_p)'(s) \le 0$ and in particular $s \mapsto \Phi^\beta_p(s)$ is nonincreasing. Notice that Φ^β_p is a bounded, nonincreasing $\mathscr{C}^1(0,+\infty)$ function. Then $(\Phi^\beta_p)'(s) \to 0$ as $s \to +\infty$. Coupling this with the coarea formula in Proposition 2.1 for $f = \operatorname{div}(X)(1-\chi_{\operatorname{Crit}(\varphi)})/|\nabla\varphi|$ one gets that

$$\begin{aligned} \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s) &= \lim_{S\to+\infty} \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s) - \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}S}(\Phi^\beta_p)'(S) \\ &= \lim_{S\to+\infty} -\int_{\{s\le\varphi\le S\}\smallsetminus \operatorname{Crit}(\varphi)} \operatorname{div} X \,\mathrm{d}\mu = -\int_{\{\varphi\ge s\}\smallsetminus\operatorname{Crit}(\varphi)} \operatorname{div} X\,\mathrm{d}\mu \end{aligned} \tag{3-23}$$

for almost every $s \in [0,+\infty)$, which also ensures that $\operatorname{div} X \in L^1(M \smallsetminus (\overline{\Omega} \cup \operatorname{Crit}(\varphi)))$. We also observe that (3-23) holds actually for every $s \in [0,+\infty)$ and this is why (3-4) is in turn true for every $t \in [1,\infty)$. Indeed, the left-hand side is continuous by the statement. By the locality of the gradient, $\{\varphi = s\} \cap \operatorname{Crit}\varphi$ is negligible with respect to the volume measure μ, since φ is a $\mathscr{C}^{1,\beta}$ function. The integration in (3-23) can be thus performed on $\{\varphi > s\} \cap \operatorname{Crit}\varphi$. This shows that the right-hand side is right-continuous (hence continuous) by the monotone convergence theorem.

One can now obtain (3-4) rewriting (3-23) in terms of u. The proof proceeds through direct computations. The main ones are contained in [Fogagnolo et al. 2019, Section 3.3], the only difference is the Ricci term that can be computed as

$$\operatorname{Ric}(\nabla\varphi,\nabla\varphi) = \left[\frac{(p-1)(n-2)}{(n-p)}\right]^2 u^{-2\frac{n+p-2}{n-p}} \operatorname{Ric}(\mathrm{D}u,\mathrm{D}u).$$

Consequently, (3-5) follows by (3-4) and coarea formula.

For the rigidity statement, suppose that $(F^\beta_p)'(t_0) = 0$ for some $t_0 \in [1,+\infty)$ regular for u. Then by (3-4)

$$\left|\mathrm{h} - \frac{\mathrm{H}}{n-1} g^\top\right|_g = 0 \quad\text{and}\quad |\mathrm{D}^\top|\mathrm{D}u|_g|_g = 0$$

hold on $\{u \le 1/t_0\} \smallsetminus \operatorname{Crit}(u)$. By Proposition 2.14, $(\{u \le 1/t_0\}, g)$ splits to a warped product near the level set $\{u = 1/t_0\}$. In particular, the mean curvature H depends only on u. By (3-3) also $|\mathrm{D}u|$ depends only on u and

$$\frac{\partial}{\partial u}|\mathrm{D}u|_g = \frac{\mathrm{H}}{p-1} = \frac{n-1}{n-p}\frac{|\mathrm{D}u|_g}{u}.$$

Integrating it, we get that for some $A(t_0) > 0$ the identity

$$|\mathrm{D}u|_g = u^{\frac{n-1}{p-1}} A(t_0)$$

holds, which gives that $|\mathrm{D}u|_g$ never vanishes on $\{u \leq 1/t_0\}$ by the continuity of gradient. Recalling the relation between u, η and t in (2-17), we obtain that $\eta(t) = B(t_0)t_0 t + (1 - B(t_0))$ for some $B(t_0) > 0$. If we define the new coordinate as

$$\tau = t + \frac{1 - B(t_0)}{B(t_0)t_0} \quad \text{and} \quad \tau_0 = \frac{1}{t_0(B(t_0) - 1)},$$

we have that $\{\tau \geq \tau_0\} = \{u \leq 1/t_0\}$, $\eta(t) = \tau/\tau_0$ and $\mathrm{d}\tau = -\mathrm{d}t$. To sum up, we have proven that $(\{u \leq 1/t_0\}, g)$ is isometric to

$$\left([\tau_0, +\infty) \times \{u = 1/t_0\},\ \mathrm{d}\tau \otimes \mathrm{d}\tau + \left(\frac{\tau}{\tau_0}\right)^2 g_{\{u=1/t_0\}}\right),$$

leaving us only to characterise τ_0. Observe that, by the conical splitting, the measures of the level sets of τ satisfy

$$|\{\tau = R\}| = \left(\frac{R}{\tau_0}\right)^{n-1} |\{u = 1/t_0\}|.$$

One can easily prove that on a cone

$$1 = \lim_{R\to+\infty} \frac{|\{\tau \leq R\}|}{|B(o, R)|} = \lim_{R\to+\infty} \frac{|\{\tau = R\}|}{|\partial B(o, R)|},$$

which can be used to compute the claimed value of τ_0,

$$\mathrm{AVR}(g) = \lim_{R\to+\infty} \frac{|\{\tau = R\}|}{R^{n-1}|\mathbb{S}^{n-1}|} = \frac{|\{u = 1/t_0\}|}{\tau_0^{n-1}|\mathbb{S}^{n-1}|}. \qquad \square$$

We conclude this section by sketching the proof of the monotonicity-rigidity theorem for Φ_p^∞, which does not require much more effort than in $\mathbb{R}^n$ [Fogagnolo et al. 2019].

Proof of Theorem 3.2. Lemma 5.1 in [Fogagnolo et al. 2019] holds also in this setting. The only difference in proving that $|\nabla\varphi|^p$ is a subsolution of the nondegenerate uniformly elliptic operator

$$\mathscr{L}(f) = \Delta f + (p-2)\nabla\nabla f\left(\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|}\right) - \frac{n-p}{n-2}\langle\nabla f, \nabla\varphi\rangle,$$

acting on smooth f in a neighbourhood of points such that $|\nabla\varphi| > 0$, is that the curvature term that appears when the Bochner identity for p-harmonic functions is applied can be controlled by $\mathrm{Ric} \geq 0$. We claim that

$$|\nabla\varphi|(x) \leq \sup_{\{\varphi=s\}} |\nabla\varphi| \tag{3-24}$$

for every $s \in [0, +\infty)$ and $x \in \{\varphi \geq s\}$, which is the main ingredient in the proof of [Fogagnolo et al. 2019, Theorem 3.5]. Firstly suppose that $\Phi_p^\infty(s) > 0$ and let be $0 < \delta < \Phi_p^\infty(s)$. By (3-10), $|\nabla\varphi| \leq \mathrm{C}$ uniformly in $M \smallsetminus \Omega$. For some $S > s$ let

$$w = |\nabla\varphi|^p - \sup_{\{\varphi=s\}} |\nabla\varphi|^p - \mathrm{C}^p \mathrm{e}^{\frac{n-p}{(n-2)(p-1)}(\varphi-S)}$$

be defined on $\{s \leq \varphi \leq S\} \smallsetminus N_\delta$, where $N_\delta = \{|\nabla\varphi| < \delta\}$. Since $w \leq 0$ on the boundary of $\{s \leq \varphi \leq S\} \smallsetminus N_\delta$ and $\mathscr{L}(w) \geq 0$ in its interior, by the maximum principle we have that

$$|\nabla\varphi|^p \leq \sup_{\{\varphi=s\}} |\nabla\varphi|^p + \mathrm{C}^p \mathrm{e}^{\frac{n-p}{(n-2)(p-1)}(\varphi-S)} \tag{3-25}$$

on $\{s \leq \varphi \leq S\} \smallsetminus N_\delta$. Moreover, since $|\nabla\varphi| < \delta$ on N_δ, (3-25) is thus satisfied in the whole $\{s \leq \varphi \leq S\}$. Passing to the limit as $S \to +\infty$, (3-24) is proven for $s \in [0, +\infty)$ such that $\Phi_p^\infty(s) > 0$.

We now prove Corollary 3.3, namely that $\Phi_p^\beta(s) > 0$ for every $s \in [0, +\infty)$, which in particular yields (3-24) proving the monotonicity. Suppose by contradiction that $\Phi_p^\infty(s) = 0$ for some $s \in [0, +\infty)$. By Proposition 2.1 there exists a sequence of $(s_j)_{j\in\mathbb{N}}$, $s_j \to s$ as $j \to +\infty$ and $\Phi_p^\infty(s_j) > 0$. If, up to a subsequence, we can assume that $\Phi_p^\infty(s_j) \to 0$, then we can conclude. Indeed, $\Phi_p^\infty(s_j) \geq |\nabla\varphi|(x)$ for every $x \in \{\varphi \geq s\}$ and $\Phi_p^\infty(s_j) \to 0$ as $j \to +\infty$; hence $|\nabla\varphi| = 0$ on $\{\varphi \geq s\}$, contradicting the unboundedness of φ. Suppose now that every subsequence of $\Phi_p^\infty(s_j)$ does not vanish. Then there would be a $\delta > 0$ and $J \in \mathbb{N}$ such that $\Phi_p^\infty(s_j) > \delta$ for every $j \geq J$. Since level sets of φ are compact, $\Phi_p^\beta(s_j)$ is actually achieved at some point $x_{s_j} \in \{\varphi = s_j\}$. Moreover, $(x_{s_j})_{j\in\mathbb{N}}$ is bounded, since it is contained in $\{\varphi \leq s\}$. Hence, we can assume that there exists $x \in \{\varphi \leq s\}$ such that $x_{s_j} \to x$ as $j \to +\infty$. Since φ is $\mathscr{C}^1$, we obtain that $\varphi(x) = s$ and $|\nabla\varphi|(x) \geq \delta$, contradicting the fact that $\Phi_p^\infty(s) = 0$.

Using a similar argument we can infer that $s \mapsto \Phi_p^\beta(s)$ is left continuous. Indeed, by contradiction there would be a $\delta > 0$ such that $\Phi_p^\infty(s) \geq \Phi_p^\infty(s_0) + \delta$ for any $s < s_0$. Let $x_s \in \{\varphi = s\}$ such that $\Phi_p^\infty(s) = |\nabla\varphi|(x_s)$. By the compactness of $\{\varphi \leq s_0\}$, there exists a sequence $(s_j)_{j\in\mathbb{N}}$ and a point $x \in \{\varphi \leq s_0\}$ such that $s_j < s_0$, $s_j \to s_0$ and $x_{s_j} \to x$. Since $\varphi \in \mathscr{C}^1$, we have $\varphi(x) = s_0$ and $|\nabla\varphi|(x) \geq \Phi_p^\infty(s_0) + \delta$, contradicting the definition of Φ_p^∞. To prove the right continuity it is the enough to prove that $s \mapsto \Phi_p^\infty(s)$ is lower semicontinuous. Since $\Phi_p^\infty > 0$, the maximum of $|\nabla\varphi|$ on $\{\varphi = s\}$ is achieved at a regular point x. Let $(s_j)_{j\in\mathbb{N}}$ be a sequence such that $s_j \to s$ as $j \to +\infty$. Seeing as $|\nabla\varphi|$ is continuous, there exists a sequence of points $(x_{s_j})_{j\in\mathbb{N}}$ such that $x_{s_j} \in \{\varphi = s_j\}$ and $x_{s_j} \to x$ as $j \to +\infty$. Since $|\nabla\varphi|(x_{s_j}) \leq \Phi_p^\infty(s_j)$ for every $j \in \mathbb{N}$, we complete the proof.

We turn to prove the second part of Theorem 3.2. Since x_t is a point of maximum for the function $|\mathrm{D}u|_g / u^{(n-1)/(n-p)}$ on $\{u \leq 1/t\}$, its derivative with respect to the normal unit vector $\nu_t = -\mathrm{D}u/|\mathrm{D}u|_g$ is nonpositive. Hence (3-6) follows by direct computations. To conclude, both rigidity statements follow in the same way as in [Fogagnolo et al. 2019, Theorem 3.5], since $|\mathrm{D}u|_g^p / u^{p(n-1)/(n-p)}$ is also a subsolution of $\mathscr{L} f = 0$, thanks to (3-9). □

4. Geometric consequences of the monotonicity theorems

In this section, we prove the geometric implications of the monotonicity-rigidity theorems, which are the Minkowski inequalities, a rigidity result under a pinching condition and a sphere theorem. The proof of these theorems follows, along with the monotonicity already mentioned, by a contradiction argument that involves the iso-p-capacitary inequality, which we are going to state and prove immediately since we believe it to be of independent interest.

4A. ***Iso-p-capacitary inequality.*** We provide the sharp iso-p-capacitary inequality in complete noncompact Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growth. As for the standard iso-p-capacitary inequality in the Euclidean setting, the proof fully relies on the isoperimetric inequality combined with a Pólya–Szegő principle. In particular, the sharpness of the inequality that follows is a direct consequence of the sharp isoperimetric constant in this setting, which has been found first in dimension 3 in [Agostiniani et al. 2020] and later extended to all dimensions in [Brendle 2023]. See also [Fogagnolo and Mazzieri 2022; Balogh and Kristály 2023; Johne 2021] for related results. The proof below is classical, and it is inspired by [Jauregui 2012], where it is illustrated for the 2-capacity in $\mathbb{R}^n$.

Theorem 4.1 (iso-p-capacitary inequality). *Let (M, g) be a complete, noncompact Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth. Let be $\Omega \subseteq M$ open bounded subset with smooth boundary. Then*

$$\frac{\mathrm{Cap}_p(\mathbb{B}^n)^n}{|\mathbb{B}^n|^{n-p}}\,\mathrm{AVR}(g)^p \leq \frac{\mathrm{Cap}_p(\Omega)^n}{|\Omega|^{n-p}}. \tag{4-1}$$

Moreover, if the equality holds then (M, g) is isometric to the Euclidean space and Ω is a ball.

Proof. By (2-6) and the coarea formula in Proposition 2.1 we have that

$$\mathrm{Cap}_p(\Omega) = \int_{M\smallsetminus\overline{\Omega}} |\mathrm{D}u|^p \,\mathrm{d}\mu = \int_0^1 \int_{\{u=\tau\}} |\mathrm{D}u|^{p-1} \,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{4-2}$$

The Hölder inequality with exponents $a = p$ and $b = p/(p-1)$ gives

$$|\{u=\tau\}|^p \leq \left(\int_{\{u=\tau\}} |\mathrm{D}u|^{p-1}\,\mathrm{d}\sigma\right)\left(\int_{\{u=\tau\}} \frac{1}{|\mathrm{D}u|}\,\mathrm{d}\sigma\right)^{p-1} \tag{4-3}$$

for almost every $\tau \in (0, 1]$. Let $V' : (0, 1] \to \mathbb{R}$ be defined as

$$V'(\tau) = -\int_{\{u=\tau\}} \frac{1}{|\mathrm{D}u|}\,\mathrm{d}\sigma. \tag{4-4}$$

Moreover, let $V : (0, 1] \to \mathbb{R}$ be the primitive of $V'(\tau)$ chosen as

$$V(\tau) = |\Omega| - \int_\tau^1 V'(s)\,\mathrm{d}s = |\Omega_\tau \smallsetminus \mathrm{Crit}(u)|, \tag{4-5}$$

where the second identity is obtained coupling (4-4) with the coarea formula (2-2) applied with $f = (1-\chi_{\mathrm{Crit}(u)})|\mathrm{D}u|^{-1}$ (see Remark 2.2).

By the isoperimetric inequality in [Brendle 2023, Corollary 1.3] we have that

$$|\{u=\tau\}| \geq |\partial\Omega_\tau| \geq |\Omega_\tau|^{\frac{n-1}{n}}\,\mathrm{AVR}(g)^{\frac{1}{n}} n|\mathbb{B}^n|^{\frac{1}{n}} \geq V(\tau)^{\frac{n-1}{n}}\,\mathrm{AVR}(g)^{\frac{1}{n}} n|\mathbb{B}^n|^{\frac{1}{n}}. \tag{4-6}$$

Let $R(\tau)$ be the radius of the ball in $\mathbb{R}^n$ which has volume $V(\tau)$. Then $V(\tau)=|\mathbb{B}^n|R(\tau)^n$ and $V'(\tau)=|\mathbb{S}^{n-1}|R(\tau)^{n-1}R'(\tau)$. Coupling (4-6) with (4-2), (4-3) and (4-4) we obtain

$$\begin{aligned}\operatorname{Cap}_p(\Omega) &\geq \int_0^1 \frac{|\{u=\tau\}|^p}{[-V'(\tau)]^{p-1}}\,\mathrm{d}\tau \geq n^p(|\mathbb{B}^n|\operatorname{AVR}(g))^{\frac{p}{n}}\int_0^1 \frac{V(\tau)^{\frac{p(n-1)}{n}}}{[-V'(\tau)]^{p-1}}\,\mathrm{d}\tau \\ &= |\mathbb{S}^{n-1}|\operatorname{AVR}(g)^{\frac{p}{n}}\int_0^1 \frac{R(\tau)^{n-1}}{[-R'(\tau)]^{p-1}}\,\mathrm{d}\tau.\end{aligned}$$

Let now $v:\{|x|\geq R(1)\}\subset\mathbb{R}^n\to(0,1]$ be the function which is τ on $\{|x|=R(\tau)\}$. By (4-6) and (2-16) there exists a positive constant $\mathrm{C}=\mathrm{C}(p,n)$ such that

$$-V'(\tau)=\int_{\{u=\tau\}}\frac{1}{|\mathrm{D}u|}\,\mathrm{d}\sigma \geq \mathrm{C}|\Omega|^{\frac{n-1}{n}}\tau^{\frac{n-p}{p-1}}.$$

Seeing as

$$|\mathrm{D}v|=-\frac{1}{R'(\tau)}=-|\mathbb{S}^{n-1}|\frac{R^{n-1}(\tau)}{V'(\tau)},$$

the function v is locally Lipschitz. Since $|\mathbb{S}^{n-1}|R(\tau)^{n-1}=|\{|x|=R(\tau)\}|=|\{v=\tau\}|$, by the coarea formula (2-2) applied with $f=|\mathrm{D}v|^{p-1}$(see Remark 2.2) we have

$$\begin{aligned}|\mathbb{S}^{n-1}|\operatorname{AVR}(g)^{\frac{p}{n}}\int_0^1\frac{R(\tau)^{n-1}}{[-R'(\tau)]^{p-1}}\,\mathrm{d}\tau &= \operatorname{AVR}(g)^{\frac{p}{n}}\int_0^1\int_{\{v=\tau\}}|\mathrm{D}v|^{p-1}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= \operatorname{AVR}(g)^{\frac{p}{n}}\int_{\{|x|\geq R(1)\}}|\mathrm{D}v|^p\,\mathrm{d}x \geq \operatorname{AVR}(g)^{\frac{p}{n}}\operatorname{Cap}_p(\{|x|<R(1)\}),\end{aligned}$$

where the last one is by the definition of the p-capacity (2-4) in flat $\mathbb{R}^n$. Using (2-9) and the fact that $|\{|x|\leq R(1)\}|=V(1)=|\Omega|$, we finally obtain

$$\operatorname{AVR}(g)^{\frac{p}{n}}\operatorname{Cap}_p(\{|x|<R(1)\})=\operatorname{AVR}(g)^{\frac{p}{n}}\operatorname{Cap}_p(\mathbb{B}^n)R(1)^{n-p}=\operatorname{AVR}(g)^{\frac{p}{n}}\frac{\operatorname{Cap}_p(\mathbb{B}^n)}{|\mathbb{B}^n|^{\frac{n-p}{n}}}|\Omega|^{\frac{n-p}{n}},$$

and consequently (4-1).

Clearly, if the equality holds in (4-1) then also the equality holds in the use of the isoperimetric inequality, and [Brendle 2023, Theorem 1.2] forces the rigidity both of the ambient manifold and Ω. □

We conclude this subsection with the following remark, whose importance will be clarified in the very proof of the L^p-Minkowski inequality (Theorem 4.3 below), where a sharp lower bound for the p-capacity of the superlevel sets of the p-capacitary potential of Ω will be needed.

Remark 4.2. We observe that, replacing Ω and u with $\Omega_t=\{u>1/t\}\cup\Omega$ and $u_t=tu$ respectively and defining $V:(0,1]\to\mathbb{R}$ in (4-5) as

$$V(\tau)=|\Omega_t\cup\{u_t=1\}|+\int_\tau^1\int_{\{u_t=s\}}\frac{1}{|\mathrm{D}u_t|}\,\mathrm{d}\sigma\,\mathrm{d}s=|\Omega_{\tau/t}\smallsetminus(\operatorname{Crit}(u)\cap\{\tau<u_t<1\})|,$$

we obtain that

$$\frac{\operatorname{Cap}_p(\mathbb{B}^n)^n}{|\mathbb{B}^n|^{n-p}}\operatorname{AVR}(g)^p\leq\frac{\operatorname{Cap}_p(\Omega_t)^n}{|\Omega_t|^{n-p}}$$

holds for every $t\in[1,+\infty)$.

4B. *Minkowski inequality.* We are now ready to prove the L^p-Minkowski inequality in our setting. Let (M, g) be a noncompact, complete Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth. Consider the function $t \mapsto F_p(t)$ defined in (3-1) as F_p^β with $\beta = 1/(p-1)$. By (2-9) we can rewrite F_p in a more geometric fashion as

$$F_p(t) = t^{\frac{n-1}{n-p}} \int_{\{u=1/t\}} |\mathrm{D}u|^p \,\mathrm{d}\sigma = \left(\frac{\mathrm{C}_p(\Omega_t)}{\mathrm{C}_p(\Omega)}\right)^{-\frac{n-p-1}{n-p}} \int_{\{u_t=1\}} |\mathrm{D}u_t|^p \,\mathrm{d}\sigma, \tag{4-7}$$

where $u_t = tu$ and $\Omega_t = \{u > 1/t\} \cup \Omega$.

Theorem 4.3 (L^p-Minkowski inequality). *Let (M, g) be complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean volume growth. Let $\Omega \subseteq M$ be a open bounded subset with smooth boundary. Then, for every $1 < p < n$, the following inequality holds*:

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \,\mathrm{AVR}(g)^{\frac{1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^p \mathrm{d}\sigma. \tag{4-8}$$

Moreover, the equality holds in (4-8) if and only if $(M \smallsetminus \Omega, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial\Omega,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right), \quad \textit{where } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

Proof. We first show that

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \,\mathrm{AVR}(g)^{\frac{1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p}\right)^p \int_{\partial\Omega} |\mathrm{D}u|^p \,\mathrm{d}\sigma \tag{4-9}$$

holds for any open subset $\Omega \subseteq M$ with smooth boundary.

Let then $\theta < \mathrm{AVR}(g)$ and suppose by contradiction that there exists an open subset $\Omega \subseteq M$ with smooth boundary such that

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \theta^{\frac{1}{n-p}} \geq \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p}\right)^p \int_{\partial\Omega} |\mathrm{D}u|^p \,\mathrm{d}\sigma.$$

Define $\tau = 1/t \in (0, 1]$. By Theorem 3.1, the function $\tau \mapsto F_p(\tau)$ is nondecreasing for $\tau \in (0, 1]$. Exploiting this monotonicity as in (4-7) we have

$$\left(\frac{n-p}{p-1}\right)^p |\mathbb{S}^{n-1}| \theta^{\frac{1}{n-p}} \geq \mathrm{C}_p(\Omega)^{-\frac{n-p-1}{n-p}} \int_{\partial\Omega} |\mathrm{D}u|^p \,\mathrm{d}\sigma \geq \mathrm{C}_p(\Omega_\tau)^{-\frac{n-p-1}{n-p}} \int_{\{u=\tau\}} |\mathrm{D}u_\tau|^p \,\mathrm{d}\sigma, \tag{4-10}$$

where $u_\tau = u/\tau$. The Hölder inequality with conjugate exponents $a = (p+1)/p$ and $b = p+1$ yields

$$\mathrm{Cap}_p(\Omega_\tau)^{\frac{p+1}{p}} \leq \left(\int_{\{u=\tau\}} |\mathrm{D}u_\tau|^p \,\mathrm{d}\sigma\right) \left(\int_{\{u=\tau\}} \frac{1}{|\mathrm{D}u_\tau|} \,\mathrm{d}\sigma\right)^{\frac{1}{p}}.$$

Therefore, plugging it into (4-10), we get

$$|\mathbb{S}^{n-1}| \mathrm{C}_p(\Omega_\tau)^{\frac{n}{n-p}} \leq \left(\frac{n-p}{p-1}\right) \theta^{\frac{p}{n-p}} \int_{\{u=\tau\}} \frac{1}{|\mathrm{D}u_\tau|} \,\mathrm{d}\sigma.$$

Using (2-9) and integrating both sides we obtain

$$|\mathbb{S}^{n-1}|\mathrm{C}_p(\Omega)^{\frac{n}{n-p}}\int_\tau^1 s^{-\frac{n(p-1)}{n-p}-1}\,\mathrm{d}s \le \left(\frac{n-p}{p-1}\right)\theta^{\frac{p}{n-p}}\int_\tau^1\int_{\{u=s\}}\frac{1}{|\mathrm{D}u|}\,\mathrm{d}\sigma\,\mathrm{d}s,$$

which, together with the coarea formula (2-2) with $f=(1-\chi_{\mathrm{Crit}(u)})|\mathrm{D}u|^{-1}$ (see Remark 2.2), leaves us with

$$\frac{|\mathbb{S}^{n-1}|}{n}(\mathrm{C}_p(\Omega_\tau)^{\frac{n}{n-p}}-\mathrm{C}_p(\Omega)^{\frac{n}{n-p}})\le\theta^{\frac{p}{n-p}}|\Omega_\tau\smallsetminus(\Omega\cup\mathrm{Crit}(u))|$$

for every $\tau\in[0,1)$. Applying the sharp iso-p-capacitary inequality (4-1) to the left-hand side we obtain

$$\mathrm{AVR}(g)^{\frac{p}{n-p}}(|\Omega_\tau|-\mathrm{C}_p(\Omega)^{\frac{n}{n-p}})\le\theta^{\frac{p}{n-p}}|\Omega_\tau|.$$

Dividing both sides by $|\Omega_\tau|$ and passing to the limit as $\tau\to 0$, we get a contradiction with $\theta<\mathrm{AVR}(g)$, proving that for any $\theta<\mathrm{AVR}(g)$

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}}\theta^{\frac{1}{n-p}}<\frac{1}{|\mathbb{S}^{n-1}|}\left(\frac{p-1}{n-p}\right)^p\int_{\partial\Omega}|\mathrm{D}u|^p\,\mathrm{d}\sigma$$

holds for every any bounded open $\Omega\subset M$ with smooth boundary. Letting $\theta\to\mathrm{AVR}(g)^-$ yields (4-9).

To conclude observe that Theorem 3.1 implies $(F_p)'(1)\le 0$ and thus, thanks to (3-3), we have

$$\int_{\partial\Omega}\left(\frac{p-1}{n-p}\right)|\mathrm{D}u|^p\,\mathrm{d}\sigma\le\int_{\partial\Omega}|\mathrm{D}u|^{p-1}\frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma.$$

By the Hölder inequality with conjugate exponents $a=p/(p-1)$ and $b=p$, we get

$$\int_{\partial\Omega}|\mathrm{D}u|^p\,\mathrm{d}\sigma\le\left(\frac{n-p}{p-1}\right)^p\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|^p\mathrm{d}\sigma, \tag{4-11}$$

which coupled with (4-9) concludes the proof of (4-8).

If we now assume that the equality holds in (4-8), then the two sides of (4-11) are identical too. In particular, by (3-3), $F_p'(1)=0$ and the rigidity statement in Theorem 3.1 applies. □

Remark 4.4 (a sharp bound on F_p^β and other geometric inequalities). The previous proof combines a lower bound on $F_p(+\infty)$ with $F_p'(1)\le 0$. Such an argument can be generalised for every

$$\beta\ge\frac{n-p}{(n-1)(p-1)}.$$

In fact, with a similar reasoning one can get

$$\lim_{t\to+\infty}F_p^\beta(t)\ge\left(\frac{n-p}{p-1}\right)^{\beta(p-1)}\mathrm{C}_p(\partial\Omega)^{1-\beta\frac{p-1}{n-p}}\,\mathrm{AVR}(g)^{\beta\frac{p-1}{n-p}},$$

and couple it with $(F_p^\beta)'(1)\le 0$ to obtain the family of inequalities

$$\mathrm{C}_p(\partial\Omega)^{1-\beta\frac{p-1}{n-p}}\,\mathrm{AVR}(g)^{\beta\frac{p-1}{n-p}}\le\frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|^{(\beta+1)(p-1)}\mathrm{d}\sigma$$

depending on parameters

$$\beta \geq \frac{n-p}{(n-1)(p-1)} \quad \text{and} \quad 1 < p < n$$

(see [Benatti 2022, Theorem 4.2.1] and its proof for the details). Among them, we have the above-mentioned L^p-Minkowski inequality for $\beta = 1/(p-1)$ and the Willmore-type inequality proved in [Agostiniani et al. 2020, Theorem 1.1] for $\beta = (n-p)/(p-1)$.

In order to derive the extended Minkowski inequality we want to briefly recall the definition of outward minimising sets and the notion of strictly outward minimising hull in accordance to [Huisken and Ilmanen 2001] and some related properties that the interested reader can find in [Fogagnolo and Mazzieri 2022]. We are denoting with $\partial^* E$ the reduced boundary of a finite perimeter set E.

Definition 4.5 (outward minimising and strictly outward minimising sets). Let (M, g) be a complete Riemannian manifold. Let $E \subset M$ be a bounded measurable set with finite perimeter. E is *outward minimising* if for any $F \supseteq E$ we have $|\partial^* E| \leq |\partial^* F|$, where by $\partial^* F$ we denote the reduced boundary of a set F. We say E is *strictly outward minimising* if it is outward minimising and whenever $|\partial^* E| = |\partial^* F|$ for some $F \supseteq E$ we have that $|F \smallsetminus E| = 0$.

We can define the *strictly outward minimising hull* Ω^* of an open bounded subset Ω with smooth boundary as

$$\Omega^* = \operatorname{Int} E \quad \text{for some bounded } E \text{ containing } \Omega \text{ such that } |E| = \inf_{F \in \mathrm{SOMBE}(\Omega)} |F|, \tag{4-12}$$

where by $\mathrm{SOMBE}(\Omega)$ we denote the family of all bounded strictly outward minimising sets containing Ω and $\operatorname{Int} E$ is the measure-theoretic interior of E. As a consequence of [Fogagnolo and Mazzieri 2022, Theorem 1.1], if (M, g) is a manifold with nonnegative Ricci curvature and Euclidean volume growth, then Ω^* as defined above is unique and it is a maximal volume solution to the problem of area minimisation with obstacle Ω, that is,

$$|\partial^* \Omega^*| = \inf\{|\partial^* F| \mid F \text{ is bounded and } \Omega \subseteq F\}.$$

Outward minimising sets can be characterised as those satisfying

$$|\partial \Omega| = |\partial \Omega^*|. \tag{4-13}$$

The relation between the strictly outward minimising hull of a bounded set with smooth boundary Ω and its p-capacity in the family of manifolds we are working on is resumed in the limit

$$\lim_{p \to 1^+} \mathrm{C}_p(\Omega) = \frac{|\partial \Omega^*|}{|\mathbb{S}^{n-1}|}.$$

Such a result is contained in the far more general [Fogagnolo and Mazzieri 2022, Theorem 1.2], having in mind the relation between the p-capacity and the normalised p-capacity given in Definition 2.7. Letting $p \to 1^+$ in the L^p-Minkowski inequality (4-8) and employing the dominated convergence theorem complete the proof of the extended Minkowski inequality of Theorem 1.1,

$$\left(\frac{|\partial \Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial \Omega} \left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma. \tag{4-14}$$

Outward minimising sets are mean-convex, as a simple variational argument immediately shows, and satisfy (4-13). As a corollary, the Minkowski inequality can be simplified for this particular class of subsets as in the following statement.

Corollary 4.6 (Minkowski inequality for outward minimising sets). *Let (M, g) be complete Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth. Let $\Omega \subseteq M$ be a bounded outward minimising subset with smooth boundary, then*

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma. \tag{4-15}$$

Remark 4.7 (sharpness of the Minkowski inequality for outward minimising sets). The sharpness of the Minkowski inequality for outward minimising sets (4-15) is not difficult to check even in nonflat spaces. In fact, in a $\mathscr{C}^1$-asymptotically conical manifold, where the metric g approaches the cone metric $\mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2 g_L$ in the $\mathscr{C}^1$-topology, big level sets of ρ are outward minimising (see, e.g., [Benatti et al. 2024, Lemma 4.3]) and is straightforward to check that $\{\rho = R\}$ saturates (4-15) in the limit as $R \to +\infty$.

Going beyond asymptotically conical spaces, one can infer the sharpness of the Minkowski inequality for outward minimising sets in manifolds of nonnegative Ricci curvature and Euclidean volume growth of dimension $n \leq 7$. Indeed, the proof of [Fogagnolo and Mazzieri 2022, Theorem 1.3] can be readapted by exploiting (4-15) in place of the Willmore-type inequality [Agostiniani et al. 2022b, Theorem 1.1]. This would allow showing that the infimum among all outward minimising smooth sets of $|\partial\Omega|^{-(n-2)/(n-1)} \int_{\partial\Omega} \mathrm{H}$ is the lower bound given by (4-15), exactly in the same way [Fogagnolo and Mazzieri 2022, Theorem 1.3] provides the sharpness of the Willmore-type inequality.

4C. *Rigidity statement.* We finally characterise the subsets Ω that saturate the inequality (4-14). We are getting this rigidity result evolving $\partial\Omega$ by smooth IMCF, proving that, in an outer neighbourhood of $\partial\Omega$, the manifold is a truncated cone with the same volume ratio of (M, g). The conclusion then follows from a generalisation of the Bishop–Gromov theorem.

Going into more detail, since $\partial\Omega$ is strictly mean-convex, we can consider a sequence of sets Ω_t with $t \in [0, T)$ such that $\partial\Omega_t = F_t(\partial\Omega)$, where $F_t : \partial\Omega \to M$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t(\partial\Omega) = \frac{1}{\mathrm{H}_t}\nu_t, \tag{4-16}$$

where ν_t and H_t are respectively the outer unit normal and the mean curvature of $\partial\Omega_t$. The conical splitting we aim for is inspired by an argument contained in [Huisken and Ilmanen 2001, Section 8]. The first step consists in the following fundamental lemma.

Lemma 4.8. *Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$ and $\Sigma \subseteq M$ a totally umbilical closed hypersurface such that $\mathrm{Ric}(\nu, \nu) = 0$ where ν is the normal unit vector field to Σ. Then Σ has **constant** mean curvature.*

Proof. The (traced) Codazzi–Mainardi equations and the totally umbilicity yield

$$\mathrm{Ric}_{j\nu} = \mathrm{D}_i \mathrm{h}_{ij} - \mathrm{D}_j \mathrm{H} = -\frac{n-2}{n-1}\mathrm{D}_j \mathrm{H}$$

for any $j = 1, \dots, n-1$. Consider, at a fixed point on Σ, the vector $\eta_\lambda = \lambda\,\mathrm{D}^\top \mathrm{H} + \nu$, with $\lambda \in \mathbb{R}$. Since $\mathrm{Ric}(\nu, \nu) = 0$, we have

$$0 \le \mathrm{Ric}(\eta_\lambda, \eta_\lambda) = 2\,\mathrm{Ric}_{j\nu}\, \eta_\lambda^j \eta_\lambda^\nu + \mathrm{Ric}_{ij}\, \eta_\lambda^i \eta_\lambda^j = -2\lambda \frac{n-2}{n-1} |\mathrm{D}^\top \mathrm{H}|^2 + \lambda^2\, \mathrm{Ric}_{ij}\, \mathrm{D}^i \mathrm{H}\, \mathrm{D}^j \mathrm{H}$$

for every $\lambda \in \mathbb{R}$. This can happen only if $|\mathrm{D}^\top \mathrm{H}| = 0$, so that H is constant on Σ. □

The following straightforward but very important consequence of the Bishop–Gromov monotonicity ensures in particular that if an outer neighbourhood of a bounded open set with smooth boundary $\Omega \subset M$ is isometric to a truncated cone, then the whole complement of Ω is isometric to a truncated cone based at $\partial\Omega$.

Lemma 4.9. *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \ge 0$. Let $K \subset M$ be a bounded open set. Suppose there exists an outer neighbourhood $A \subset M \smallsetminus K$ of K such that (A, g) is isometric to*

$$\left([\rho_0, \rho_1] \times \partial K,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial K} \right)$$

for $0 < \rho_0 < \rho_1$. Then

$$|\partial K| \ge \rho_0^{n-1} |\mathbb{S}^{n-1}|\, \mathrm{AVR}(g), \tag{4-17}$$

and the equality holds if and only if $(M \smallsetminus K, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial K,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial K} \right).$$

Proof. Consider the cone $(C, \hat{g})$ given by

$$\left((0, \rho_1) \times \partial K,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial K} \right),$$

and the Riemannian manifold, with a conical singularity, obtained by gluing $(C, \hat{g})$ with $(M \smallsetminus (K \cup A), g)$ along $\{\rho = \rho_1\}$. By our assumptions, such a manifold is well-defined with nonnegative Ricci curvature outside of the tip o of C, and coincides with (M, g) in the complement of K. In C, the geodesic distance from o is given by ρ, and in particular, by Bishop–Gromov monotonicity,

$$\frac{|\{\rho = r\}|}{r^{n-1} |\mathbb{S}^{n-1}|} \ge \mathrm{AVR}(g)$$

for any $r \in (0, \rho_1)$. Since $|\{\rho = \rho_0\}| = |\partial K|$, setting $r = \rho_0$ proves (4-17). If equality holds, then, by the rigidity statement in the Bishop–Gromov theorem for manifolds with a conical singularity, the whole manifold we constructed is isometric to a cone, and in particular, $(M \smallsetminus K, g)$ splits as claimed. This well-known, slightly enhanced version of the Bishop–Gromov rigidity statement can be readily deduced from its classic proof, or seen as a very special case of its version for nonsmooth metric spaces [De Philippis and Gigli 2016]. □

We finally have at our disposal all the tools we need to work out the splitting argument leading to Theorem 1.2.

Proof of Theorem 1.2. Suppose that some strictly outward minimising $\Omega \subset M$ with strictly mean-convex boundary satisfies

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma. \tag{4-18}$$

Since $\partial\Omega$ is by assumption strictly mean-convex, we can evolve it by (smooth) IMCF $\partial\Omega_t$ defined in (4-16) for $t \in [0, T)$. By the [Huisken and Ilmanen 2001, Smooth Start Lemma 2.4], up to shortening the time interval, we can assume that Ω_t is strictly outward minimising for any $t \in [0, T)$. Indeed, since Ω is strictly outward minimising, the flow coincides for a short time with the weak notion of IMCF, which exists in our setting by [Mari et al. 2022, Theorem 1.7]. The sublevel sets of the weak IMCF being strictly outward minimising is a basic and fundamental property illustrated in [Huisken and Ilmanen 2001, Minimizing Hull Property 1.4]. Consider then the function $\mathcal{Q} : [0, T) \to \mathbb{R}$ defined by

$$\mathcal{Q}(t) = |\partial\Omega_t|^{-\frac{n-2}{n-1}} \int_{\partial\Omega_t} \mathrm{H}_t\,\mathrm{d}\sigma.$$

By evolution equations for curvature flows derived for example in [Huisken and Polden 1999, Theorem 3.2], a straightforward computation shows that

$$\mathcal{Q}'(t) = -|\partial\Omega_t|^{-\frac{n-2}{n-1}} \int_{\partial\Omega_t} \frac{|\mathring{\mathrm{h}}_t|^2 + \mathrm{Ric}(\nu_t, \nu_t)}{\mathrm{H}_t}\,\mathrm{d}\sigma \le 0,$$

where by $\mathring{\mathrm{h}}_t$ we denote the trace-free part of the second fundamental form h_t of $\partial\Omega_t$. On the other hand, the strict inequality for some $t \in [0, T)$ would result in a contradiction to the Minkowski inequality. Thus $\mathcal{Q}'(t)$ vanishes for any $t \in [0, T)$ and, in particular $\partial\Omega_t$ satisfies (4-18) for any $t \in [0, T)$. Hence, $\partial\Omega_t$ is totally umbilical and satisfies $\mathrm{Ric}(\nu_t, \nu_t) = 0$ for every $t \in [0, T)$. By Lemma 4.8 $\partial\Omega_t$ has constant mean curvature for every $t \in [0, T)$.

On $\{0 \le t < T\}$, the solution to the weak level set formulation of the IMCF w, which in our smooth case just means $\{w = t\} = \partial\Omega_t$, satisfies the relation

$$\mathrm{H}_t = \mathrm{div}\left(\frac{\mathrm{D}w}{|\mathrm{D}w|}\right)(x_t) = |\mathrm{D}w|(x_t)$$

at any $x_t \in \partial\Omega_t$. Hence, since $\mathrm{H}_t > 0$, a well-known extension of the Gauss' lemma yields

$$g = \frac{\mathrm{d}w \otimes \mathrm{d}w}{|\mathrm{D}w|^2} + g_{\partial\Omega_t} = \frac{\mathrm{d}t \otimes \mathrm{d}t}{\mathrm{H}_t^2} + g_{\partial\Omega_t}. \tag{4-19}$$

The evolution equation (see [Huisken and Polden 1999, Theorem 3.2(i)]) satisfied by $g_{\partial\Omega_t}$ is

$$\frac{\partial}{\partial t} g_{\partial\Omega_t} = 2\frac{\mathrm{h}_t}{\mathrm{H}_t} g_{\partial\Omega_t} = \frac{2}{n-1} g_{\partial\Omega_t},$$

where the last identity is due to the total umbilicity of $\partial\Omega_t$. Integrating such equation we deduce

$$g_{\partial\Omega_t} = \mathrm{e}^{\frac{2t}{(n-1)}} g_{\partial\Omega}. \tag{4-20}$$

On the other hand, the evolution equation for the mean curvature along the IMCF (see [Huisken and Polden 1999, Theorem 3.2(v)]) gives

$$\frac{\partial}{\partial t}\mathrm{H}_t = -\Delta_{\partial\Omega_t}\left(\frac{1}{\mathrm{H}_t}\right) - \frac{1}{\mathrm{H}_t}[|\mathrm{h}_t|^2 + \mathrm{Ric}(\nu_t, \nu_t)] = -\frac{\mathrm{H}_t}{n-1},$$

where the last identity is due to the fact that $\partial\Omega_t$ is totally umbilical, $\mathrm{Ric}(\nu_t, \nu_t) = 0$ and the mean curvature H_t of $\partial\Omega_t$ depends only on t. Integrating it we obtain that

$$\mathrm{H}_t = \mathrm{e}^{-\frac{t}{n-1}}\mathrm{H}_0, \tag{4-21}$$

where H_0 is the mean curvature of $\partial\Omega$.

Plugging (4-20) and (4-21) into (4-19), we deduce that $(\{0 \le t < T\}, g)$ is isometric to

$$\left([0, T) \times \partial\Omega,\ \mathrm{e}^{\frac{2t}{n-1}}\frac{\mathrm{d}t \otimes \mathrm{d}t}{\mathrm{H}_0^2} + \mathrm{e}^{\frac{2t}{n-1}} g_{\partial\Omega}\right).$$

Performing the change of variables

$$\rho = \frac{(n-1)}{\mathrm{H}_0}\mathrm{e}^{\frac{t}{(n-1)}},$$

the metric can be written as

$$\left([\rho_0, \rho(T)) \times \partial\Omega,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right), \quad \text{where } \rho_0 = \frac{(n-1)}{\mathrm{H}_0}.$$

On the other hand, since by assumption $\partial\Omega$ saturates the Minkowski inequality, that is, (4-18) holds, we immediately get

$$\rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}},$$

and we conclude by the rigidity statement in Lemma 4.9 that the whole $M \smallsetminus \Omega$ is isometric to a truncated cone. □

In the following remark, we briefly discuss how the assumptions for the rigidity can be relaxed in small dimensions.

Remark 4.10. In dimension $3 \le n \le 7$, an open bounded subset Ω with smooth strictly mean-convex boundary satisfying

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma$$

is a priori strictly outward minimising, and thus, in this case, such an assumption can be dropped. Indeed, by approximating Ω via mean curvature flow with smooth strictly outward minimising domains, as described in [Huisken and Ilmanen 2001, Lemma 5.6], we deduce that (4-14) holds also for $\mathscr{C}^{1,1}$-hypersurfaces. In particular, the Minkowski inequality holds also for the strictly outward minimising hull of Ω (see the regularity results recalled in [Huisken and Ilmanen 2001, Regularity Theorem 1.3] and [Fogagnolo and Mazzieri 2022, Theorem 2.18]) for every Ω with smooth boundary, provided the dimensional bound holds. We can then argue by contradiction. Suppose that Ω^* does not coincide with Ω. Then

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma > \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega^*} \frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma,$$

where the last inequality is due to the fact that $\mathrm{H}=0$ on $\partial\Omega^* \smallsetminus \partial\Omega$. But this contradicts the Minkowski inequality for Ω^*; hence $\Omega = \Omega^*$.

4D. ***A pinching condition and a sphere theorem.*** In this subsection, we exploit the monotonicity of the function $t \mapsto F_p^\infty(t)$ defined in (3-2) to prove a couple of rigidity statements involving a pinching condition on the mean curvature of $\partial\Omega$ and an a priori bound on the gradient of the p-capacitary potential associated to Ω. These results without any convexity assumption are also new in $\mathbb{R}^n$, and they constitute the complete nonlinear generalisation of [Borghini et al. 2019, Corollary 1.4 and 1.9]. For convex subsets of the Euclidean space they are the content of [Fogagnolo et al. 2019, Corollary 2.16 and 2.17].

Theorem 4.11. *Let* (M, g) *be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean volume growth. If there exists an open bounded subset* $\Omega \subseteq M$ *with smooth boundary satisfying*

$$-\left[\frac{\mathrm{AVR}(g)}{\mathrm{C}_p(\Omega)}\right]^{\frac{1}{n-p}} \leq \frac{\mathrm{H}}{n-1} \leq \left[\frac{\mathrm{AVR}(g)}{\mathrm{C}_p(\Omega)}\right]^{\frac{1}{n-p}} \tag{4-22}$$

on every point of $\partial\Omega$, *then* $(M \smallsetminus \Omega, g)$ *is isometric to*

$$\left([\rho_0, +\infty) \times \partial\Omega,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right), \quad \text{where } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

In this case $\partial\Omega$ *is a connected totally umbilical hypersurface with constant mean curvature in* $(M \smallsetminus \Omega, g)$.

Proof. We can argue by contradiction as in Theorem 4.3 to prove that

$$\left(\frac{n-p}{p-1}\right)\left[\frac{\mathrm{AVR}(g)}{\mathrm{C}_p(\Omega)}\right]^{\frac{1}{n-p}} \leq \sup_{\partial\Omega}|\mathrm{D}u|.$$

Indeed, we can follow the same lines replacing the consequence of the monotonicity of F_p with the corresponding of F_p^∞, which thanks to (2-9) can be rewritten as

$$F_p^\infty(t) = t^{\frac{n-1}{n-p}} \sup_{\{u=1/t\}} |\mathrm{D}u| = \left(\frac{\mathrm{C}_p(\Omega_t)}{\mathrm{C}_p(\Omega)}\right)^{\frac{1}{n-p}} \sup_{\{u_t=1\}} |\mathrm{D}u_t|,$$

where $u_t = tu$ and $\Omega_t = \{u > 1/t\} \cup \Omega$. Accordingly, we employ the Hölder inequality with conjugate exponents $a = +\infty$ and $b = 1$, that is,

$$\mathrm{Cap}_p(\Omega_t)^{\frac{1}{p}} \leq \sup_{\{u=1/t\}} |\mathrm{D}u_t| \left(\int_{\{u=1/t\}} \frac{1}{|\mathrm{D}u_t|}\,\mathrm{d}\sigma\right)^{\frac{1}{p}}.$$

In the end, by Theorem 3.2 we get

$$\sup_{\partial\Omega}|\mathrm{D}u| \leq \frac{(n-p)}{(p-1)(n-1)} \sup_{\partial\Omega}|\mathrm{H}|$$

and the equality holds if and only if $(M \smallsetminus \Omega, g)$ splits as in the statement. Condition (4-22) easily implies the equality. □

The above result is a rigidity theorem under a pinching condition on the mean curvature of $\partial\Omega$ with respect to its p-capacity. From the proof above we can also get that

$$\frac{1}{p-1}\left[\frac{\operatorname{AVR}(g)}{\mathrm{C}_p(\Omega)}\right]^{\frac{1}{n-p}} \leq \sup_{\partial\Omega}\left|\frac{\mathrm{D}u}{n-p}\right| \tag{4-23}$$

and the equality is satisfied only on metric cones. The previous inequality gives a lower bound on the gradient of u on $\partial\Omega$ in terms of the p-capacity of Ω that, when attained, forces (M, g) to be (isometric to) $\mathbb{R}^n$ with Ω a Euclidean ball.

Theorem 4.12. *Let (M, g) be a complete Riemannian manifold with $\operatorname{Ric} \geq 0$ curvature and Euclidean volume growth. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary, u the p-capacitary potential associated to Ω and assume that*

$$\sup_{\partial\Omega}\left|\frac{\mathrm{D}u}{n-p}\right| \leq \frac{1}{p-1}\operatorname{AVR}(g)^{\frac{1}{p-1}}\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{1}{n-1}}. \tag{4-24}$$

Then (M, g) is isometric to $\mathbb{R}^n$ with the Euclidean metric and Ω is a ball.

Proof. Under the assumption (4-24), we get

$$\mathrm{C}_p(\Omega) = \left(\frac{p-1}{n-p}\right)^{p-1}\frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}|\mathrm{D}u|^{p-1}\,\mathrm{d}\sigma \leq \operatorname{AVR}(g)\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{-\frac{n-p}{n-1}},$$

which yields

$$\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{n-p}{n-1}} \leq \frac{\operatorname{AVR}(g)}{\mathrm{C}_p(\Omega)} \leq (p-1)^{n-p}\sup_{\partial\Omega}\left|\frac{\mathrm{D}u}{n-p}\right|^{n-p} \leq \operatorname{AVR}(g)^{\frac{n-p}{p-1}}\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{n-p}{n-1}}, \tag{4-25}$$

where we used (4-23) together with the condition (4-24). Thus, we obtain that $\operatorname{AVR}(g) = 1$, and hence, by the Bishop–Gromov theorem, that (M, g) is isometric to $\mathbb{R}^n$ with the Euclidean metric. Since all inequalities in (4-25) become equalities, by the second one we can apply the rigidity statement in Theorem 3.2 which ensures that $\partial\Omega$ is a compact connected and totally umbilical hypersurface of $\mathbb{R}^n$, that is, Ω is a ball. □

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