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Møller Maps for Dirac Fields in External Backgrounds

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Abstract
In this paper we study the foundations of the algebraic treatment of classical and quantum field theories for Dirac fermions under external backgrounds following the initial contributions already present in various places in the literature. The treatment is restricted to contractible spacetimes of globally hyperbolic nature in dimensions $d \geq 4$ and to external fields modelled with trivial principal bundles. In particular, we construct the classical Møller maps intertwining the configuration spaces of *charged* and *uncharged* fermions, and we show some of its properties in the case of a $U(1)$ gauge charge. In the last part, as a first step towards a quantization of the theory, we explore the combination of the classical Møller maps with Hadamard bidistributions and prove that they are involutive isomorphisms (algebraically and topologically) between suitable (formal) algebras of functionals (observables) over the configuration spaces of charged and uncharged Dirac fields.

Keywords pAQFT · Fermions · Classical Møller maps · Gauge charge

Mathematics Subject Classification 81T20 · 81T15 · 81R15 · 35A18 · 53C80

1 Introduction

The main aim of this paper is to contribute to a by now well established framework for the algebraic treatment of quantum systems made of fermions in arbitrary backgrounds (metrics, external fields etc.) which aims at the rigorous determination of physical effects (see, *e.g.*, [1–13]). It is a known fact that especially for strong space-time dependent external fields the mostly used theoretical frameworks suffer from several deficits. In general, the lack of space-time symmetries implies a missing preferred state

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(vacuum) with the related impossibility to use familiar tools as Fourier transformations and Fock spaces. All these limit terribly the ability of physicists to deduce observable effects in these quantum situations without making further drastic and simplifying assumptions. Indeed, there exists a very large literature on the subject which is full of interesting ideas, techniques and results (see, *e.g.*, [14, 15]). Tipically, however, the proposals are made without referring to general and deep basic concepts and hence with a lot of *ad hoc* assumptions. A possible way out of these difficulties is to refer to recent structural advances in quantum field theories using the algebraic approach. The new perspective refers to deep conceptual advancements, for instance local covariance [16], and technically, instead of Fourier transformations, uses its modern improvement named micro-local analysis [17], in particular wave front sets [18, 19]. There have been several recent interesting results that point towards the validity of this claim. For instance, Fröb and Zahn [10] have shown how to rigorously derive the trace anomaly for chiral fermions using in particular the rigorous method of Hadamard subtraction. At variance w.r.t. the literature in physics, this was done in Lorentzian spacetime, and invoking physical principles as invariance of the stress-energy tensor to show the cancellation of unwanted terms on which physicists debated for long.

Our main concern is to build up at first the classical tools that can be used later to develop the formalism towards the quantum aspects. In the present paper we concentrate basically only in the former aspects but make an initial step into the latter. We demonstrate that the classical Møller maps are involutive isomorphisms for the various classes of mathematical objects of pertinence for us. Indeed, at first we introduce various spinorial configurations spaces as sections of several bundles over semi-Riemannian spaces, and prove that the Møller maps are isomorphisms between the charged and uncharged spinor bundles. Then, we extend the structure to (nonlinear) *functionals* over such bundles forming (involutive) algebras. One should look at them as the (abelian) algebras of observables of the theory. A first step forward is done here by the construction of a *Poisson* algebra. This entails at first the selection of a "good" subset of functionals. The term good refers to the fact that in order to rigorously define a Poisson structure for such field theories one can make a covariant choice which is determined by the use of the Peierls' backets (see, *e.g.*, [20]). This implies the use of causal Green operators (propagators) whose kernels, seen as distributions, do not directly allow multiplications by generic functionals. It is here that microlocal analysis appears as the right tool. The good selection is indeed made out of the desire to define the product of functionals, and their derivatives, with the propagators. A sufficient criterion for those products to exist is the Hörmander's one based on wave front sets. Hence, the so called *microcausal functionals* make their crucial appearance here. These are functionals with prescribed singularities whose wave front sets combine well with the wave front set of the propagators as to satisfy Hörmander's criterion.

We then develop the formalism doing a first step into the quantum realm by extending the algebras to the formal algebras of deformation quantization, by changing the classical product with the use of the Hadamard prescription.

In the course of the paper we develop geometric and analytic descriptions by adopting a practical and precise view which has the merit of being rather explicit. The resulting formalism has advantages which we do hope compensate the heaviness of notations.

2 Geometric and Analytic Preliminaries

2.1 Basic Notions in Spin Geometry on Pseudo-riemannian Manifolds

We begin the exposition by recalling some definitions and results in order to fix the notation used throughout the paper. We refer the reader to [21], [22], [23] and [24] for more details.

We shall work on n-dimensional spacetimes, that is, couples (M, g) consisting of a connected, paracompact, orientable, time-orientable Hausdorff smooth manifold M and a non-degenerate, pseudo-riemannian metric g. We shall further suppose that two additional conditions hold:

- (i) we assume (M, g) is *globally hyperbolic*: this entails that given a normally hyperbolic operator, this admits retarded and advanced Green operators (see, *e.g.*, [25, 26]);
- (ii) we also assume that $\dim(M) \geq 4$: this entails that there exists a universal covering homomorphism $\xi_0 \colon \mathrm{Spin}^0_{r,1} \to \mathrm{SO}^0_{r,1}$ between the identity component of the Spin group $\mathrm{Spin}_{r,1}$ and the identity component of the signature $(r, 1)$ of the special orthogonal group [24, Proposition 12.1.41].

On the manifold M modelling our spacetime, we shall consider *fiber bundles*, *i.e.*, quadruples (B, M, π, F) where $\pi \colon B \to M$ is a smooth, surjective map and such that there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of the base manifold M and an associated collection $\{\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times F\}_{\alpha\in A}$ of diffeomorphisms, called *trivialization of the bundle*, such that

$$\pi_1 \circ \phi_\alpha = \pi|_{\pi^{-1}(U_\alpha)} \text{ for every } \alpha \in A\ . \tag{2.1}$$

Notice that given a point $p \in M$, the fiber $B_p \doteq \pi^{-1}(p)$ is diffeomorphic to F; this is the reason why F is called *typical fiber*. We can consider the *transition functions* of the bundle, that is, the maps

$$\begin{aligned} \phi_{\alpha\beta} \colon (U_\alpha \cap U_\beta) \times F &\to (U_\alpha \cap U_\beta) \times F \\ (p, f) \qquad &\mapsto \phi_\alpha\left(\phi_\beta^{-1}(p, f)\right)\ . \end{aligned}$$

By the condition (2.1), we have that $\phi_{\alpha\beta}(p, f) = (p, g_{\alpha\beta}(p)(f))$ for some $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Aut}(F)$; the collection $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$ is called *cocycle*.

In the context of quantum field theory, one usually deals with two kinds of fiber bundles:

- (i) *vector bundles*, that is, fiber bundles whose typical fiber is a vector space and whose cocycle is such that $g_{\alpha\beta}(p) \in \mathrm{GL}(F)$ for every $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$;
- (ii) *principal G-bundles*, that is, fiber bundles whose typical fiber is the group G, whose total space P_G is a right G-manifold with right action $r_g \colon P_G \to P_G$ and whose trivialization $\{\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times G\}$ consists of right G-equivariant maps. This entails that the cocycle $\{g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Aut}(G)\}$ consists of left-translations [21, Lemma 27.7].

Given a principal G-bundle $\pi: P_G \to M$ and a representation $\rho: G \to \mathrm{GL}(V)$ of the group on a finite-dimensional vector space V, we can construct the *associated vector bundle* $\pi: P_G \times_\rho V \to M$ with typical fiber V. [21, Theorem 31.9] shows that there exist linear isomorphisms

$$
\begin{aligned}
\cdot^\flat &: \Omega^q_\rho(P_G, V) \to \Gamma\left(\wedge^q T^*M \otimes P_G \times_\rho V\right) \\
\cdot^\sharp &: \Gamma\left(\wedge^q T^*M \otimes P_G \times_\rho V\right) \to \Omega^q_\rho(P_G, V)
\end{aligned}
\tag{2.2}
$$

between the spaces $\Omega^q_\rho(P_G, V)$ of V-valued, tensorial q-forms of type ρ on P_G and the space of q-forms on M with values in $P_G \times_\rho V$.

It is a well-known fact that gauge fields are represented by Ehresmann connections $\omega \in \Omega^1(P, \mathfrak{g})$ on a suitable principal G-bundle [27, 28]. These connection 1-forms can be used to induce an *exterior covariant differentiation* $D: \Omega^q_\rho(P_G, V) \to \Omega^{q+1}_\rho(P_G, V)$,

$$
D\varphi \doteq d\varphi + \omega \cdot_\rho \varphi \tag{2.3}
$$

where

$$
(\omega \cdot_\rho \varphi)_p(v_1, \ldots, v_{q+1}) = \frac{1}{q!} \sum_{\sigma \in S_{q+1}} \mathrm{sgn}(\sigma) \rho_*(\omega_p(v_{\sigma(1)})) \varphi_p(v_{\sigma(2)}, \ldots, v_{\sigma(q+1)}) \, .
$$

We can then use the exterior covariant derivative to endow the associated vector bundle $P_G \times_\rho V$ with a connection $\nabla: \mathfrak{X}(M) \times \Gamma(P_G \times_\rho V) \to \Gamma(P_G \times_\rho V)$,

$$
\nabla_v s \doteq (D(s^\sharp))^\flat(v) \, .
$$

This section of the associated vector bundle can be expressed locally as

$$
(\nabla_v s)(p) = \left[\sigma(p), v(p)\left(s^\sharp \circ \sigma\right) + \rho_*\Big(\omega_{\sigma(p)}\big(\sigma_{*p}(v_p)\big)\Big) s^\sharp(\sigma(p))\right] \tag{2.4}
$$

where $\sigma: U \to P_{G U}$ is a *local* section of the principal G-bundle P_G, usually deemed *local gauge choice*. The pullback $\sigma^*\omega \in \Omega^1(U, \mathfrak{g})$ is called *local gauge potential*.

An important construction which can be carried out in the case of complex vector bundles is that of the *conjugate bundle*. Let us recall that given a complex vector space V, the conjugate vector space $\overline{V}$ is a vector space which is real-isomorphic to V and whose complex structure is the conjugate complex structure. The antilinear isomorphism bewteen V and $\overline{V}$ shall be denoted with $C: \overline{V} \to V$, and it can be used to induce, starting from a linear map $L: V \to V$, a linear map $\overline{L}: \overline{V} \to \overline{V}$ by defining

$$
\overline{L} \doteq C^{-1} \circ L \circ C \, .
$$

Given a basis $\{\overline{e}_i\}_{i\in I}$ of $\overline{V}$, the components $\{\overline{L}^i_j\}_{i,j\in I}$ of $\overline{L}$ are the complex conjugates of the components of L in the basis $\{C\overline{e}_i\}_{i\in I}$, i.e.

$$\overline{L}^i_j = \overline{L^i_j}\,.$$

If V is endowed with a sesquilinear form, then one can show that $\overline{V} \simeq V^*$. These facts can be extended to the case of complex vector bundles in the following way: first of all, consider a vector bundle $E \xrightarrow{\pi} M$ with typical fiber V, whose cocycle is given by $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$; then one can construct another vector bundle with typical fiber $\overline{V}$ and whose cocycle is given by $\{\overline{g_{\alpha\beta}}\}_{\alpha,\beta\in A}$; this vector bundle is the conjugate vector bundle $\overline{E} \xrightarrow{\pi} M$. The conjugation map $C\colon \overline{V} \to V$ can be extended to a vector bundle anti-isomorphism $C\colon \overline{E} \to E$, and if E is endowed with a hermitian metric h, then

(i) h, which assigns to every point $p \in M$ a sesquilinear form h_p on E_p, can be understood as a map assigning to each $p \in M$ a bilinear map $h_p\colon E_p \times \overline{E}_p \to \mathbb{C}$;
(ii) we have an isomorphism $\overline{E} \simeq E^*$.

Let us now consider the tangent bundle $\pi\colon TM \to M$ on our spacetime; by the assumptions on (M, g) this is an oriented vector bundle endowed with a Lorentzian structure, and we can thus consider its oriented, time-oriented, pseudo-orthonormal frame bundle, that is, a principal $\mathrm{SO}^0_{r,1}$-bundle $\pi_P\colon P_{\mathrm{SO}^0_{r,1}}(M) \to M$ naturally associated to it. If we further assume that the second Stiefel–Whitney class $w_2(TM)$ vanishes, we can consider a *spin structure* on TM, that is, a principal $\mathrm{Spin}^0_{r,1}$-bundle $\pi_{P'}\colon P_{\mathrm{Spin}^0_{r,1}}(M) \to M$ coupled with a 2-sheeted covering

$$\xi\colon P_{\mathrm{Spin}^0_{r,1}} \to P_{\mathrm{SO}^0_{r,1}} \tag{2.5}$$

such that $\xi\left(r_g(p)\right) = r_{\xi_0(g)}(\xi(p))$ for every $p \in P_{\mathrm{Spin}^0_{r,1}}$, $g \in \mathrm{Spin}^0_{r,1}$, with $\xi_0\colon \mathrm{Spin}^0_{r,1} \to \mathrm{SO}^0_{r,1}$ being the universal covering map. This entails in particular that the cocycle $\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Aut}(\mathrm{SO}^0_{r,1})\}$ of the principal $\mathrm{SO}^0_{r,1}$-bundle is given by $\{\xi_0(s_{\alpha\beta})\}$, where $\{s_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Aut}(\mathrm{Spin}^0_{r,1})\}$ is a cocycle for the principal $\mathrm{Spin}^0_{r,1}$-bundle.
By the fundamental theorem of Riemannian geometry, there exists a unique torsion-free, metric-compatible connection $\omega \in \Omega^1(P_{\mathrm{SO}^0_{r,1}}, \mathfrak{so}^0_{r,1})$, the Levi–Civita connection; using the covering map (2.5), this connection can be pulled back to a connection 1-form $\omega^s \in \Omega^1(P_{\mathrm{Spin}^0_{r,1}}, \mathfrak{spin}^0_{r,1})$, usually called *spin connection*.

We can then consider the following: depending on the dimension $r+1$ of our spacetime M, there exists a complex representation of the Spin group $\Delta^{\mathbb{C}}_{r,1}\colon \mathrm{Spin}^0_{r,1} \to \mathrm{GL}(V, \mathbb{C})$ which is irreducible in the odd dimensional case and reducible in the even dimensional one [22, §I, Proposition 5.15]. In both cases, the representation descends from a complex representation $\rho\colon \mathscr{C}\ell_{r,1} \to \mathrm{End}(V, \mathbb{C})$ of the Clifford algebra $\mathscr{C}\ell_{r,1}$. In [22, §I, Theorem 5.7] it is proved that there exists a unique Clifford algebra representation if $r+1$ is even, while there exist two inequivalent representations if $r+1$ is odd; the induced Spin representation in the latter case doesn't depend on the chosen representation. These representations can be used to carry out the following construction:

(i) given a complex Spin representation $\Delta^{\mathbb{C}}_{r,1}\colon \mathrm{Spin}^0_{r,1} \to \mathrm{GL}(V,\mathbb{C})$, we can consider the *complex spinor bundle*

$$S(M) \doteq P_{\mathrm{Spin}^0_{r,1}} \times_{\Delta^{\mathbb{C}}_{r,1}} V \tag{2.6}$$

(ii) we can also consider the *Clifford algebra bundle*

$$\mathscr{C}\ell(M) \doteq P_{\mathrm{SO}^0_{r,1}} \times_{c\ell} \mathscr{C}\ell_{r,1} \tag{2.7}$$

where $c\ell\colon \mathrm{SO}^0_{r,1} \to \mathrm{Aut}(\mathscr{C}\ell_{r,1})$ denotes the unique extension to the Clifford algebra $\mathscr{C}\ell_{r,1}$ of the action of $\mathrm{SO}^0_{r,1}$ on $(\mathbb{R}^{r+1}, q_{r,1})$. It can be shown that the fiber of $\mathscr{C}\ell(M)$ at $p \in M$ is isomorphic to $\mathscr{C}\ell(T^*_pM, g_p)$.

These two bundles are related by means of the *Clifford module multiplication*

$$\mu\colon \mathscr{C}\ell(M) \otimes S(M) \to S(M)$$

which explicitly uses the fact that $\Delta^{\mathbb{C}}_{r,1}$ descends from an algebra representation of $\mathscr{C}\ell_{r,1}$.
Thanks to the spin structure, this map can then be extended to yield a Clifford module multiplication

$$\cdot_\Gamma\colon \mathfrak{X}(M) \otimes \Gamma(S(M)) \to \Gamma(S(M)) \tag{2.8}$$

between vector fields and sections of the spinor bundle.

We shall denote with

$$\nabla^s\colon \mathfrak{X}(M) \otimes \Gamma(S(M)) \to \Gamma(S(M)) \tag{2.9}$$

the covariant derivative induced as of (2.4) on the spinor bundle by the spin connection ω^s. This connection behaves well with respect to the Clifford multiplcation, in the sense that it is a module derivation: given $s \in \Gamma(S(M))$ and $v, u \in \mathfrak{X}(M)$ we have

$$\nabla^s_u(v \cdot_\Gamma s) = (\nabla_u v) \cdot_\Gamma s + v \cdot_\Gamma \nabla^s_u s\ . \tag{2.10}$$

The Clifford multiplication can be combined with the covariant derivative ∇^s and with the musical isomorphism induced by the metric to yield a partial differential operator acting on sections of the spinor bundle, the *Dirac operator*

$$\begin{aligned} \mathbf{D}\colon \Gamma(S(M)) &\to \Gamma(S(M)) \\ s \quad &\mapsto \quad \mathbf{D}(s) \doteq i\left(\cdot_\Gamma(\nabla^s(s))\right)\ . \end{aligned} \tag{2.11}$$

Given a local orthonormal frame $\{e_i\} \subset \Gamma(TM)$ and its dual $\{e^i\} \subset \Gamma(T^*M)$ and considering the local section $\sigma : U_\alpha \to P_{\mathrm{Spin}^0_{r,1}}(M)$, (2.11) can be written locally as

$$\mathbf{D}(s)(p) = ie^j(p) \cdot_\Gamma \left[\sigma(p), e_j(p)\left(s^\sharp \circ \sigma\right) + \Delta^{\mathbb{C}}_{r,1*}\left(\omega^s_{\sigma(p)}(\sigma_{*p} e_j(p))\right) s^\sharp(\sigma(p))\right] \tag{2.12}$$

where we use Einstein's convention.

Remark 2.1 The local expression above can be written even more explicitly as

$$\mathbf{D}(s)(p) = ie^j(p) \cdot_\Gamma \left[\sigma(p), e_j(p)\left(s^\sharp \circ \sigma\right) + \frac{1}{4}\sum_{k,l} \Gamma^l_{kj} e^k(p) \cdot_V e_l(p) \cdot_V s^\sharp(\sigma(p))\right]$$

where Γ^l_{jk} denote the Christoffel symbols of the Levi–Civita connection on M inducing the spin connection ω^s via pullback. In the following, we present the derivation of the explicit form of the Dirac operator in three different spacetimes.

(*a*) Let us consider (flat) Minkowski spacetime $(M, g) = (\mathbb{R}^4, \eta)$ and the embedding of $\mathfrak{X}(\mathbb{R}^4)$ into $\Gamma(\mathscr{C}\ell(\mathbb{R}^4))$ given by $e_j \mapsto \gamma_j, \gamma_j(p) \equiv \gamma_j$ where $\{\gamma_j\}_{1\leq j\leq 4}$ are the usual Gamma matrices in the Dirac representation. By choosing the global pseudo-orthonormal frame $\{\partial_i\}_{1\leq i\leq 4}$ one can write, for a function $s \in \Gamma(S(\mathbb{R}^4)) \simeq C^\infty(\mathbb{R}^4, \mathbb{C}^4)$

$$\mathbf{D}(s) = i\gamma^j \partial_j s$$

which is the usual expression of the (massless) Dirac operator.

(*b*) A more interesting example is that of the Dirac operator in Schwarzschild spacetime,

$$M = \mathbb{R} \times (r_g, +\infty) \times \mathbb{S}^1$$
$$g = -\left(1 - \frac{r_g}{r}\right) dt \otimes dt + \left(1 - \frac{r_g}{r}\right)^{-1} dr \otimes dr + r^2 g_{\mathbb{S}^2}$$

To carry out the computations, one considers a (in this case global) pseudo-orthonormal frame $\{e_j\}_{1\leq j\leq 4}$,

$$e_1 = \left(1 - \frac{r_g}{r}\right)^{-1/2} \partial_t \quad e_2 = \left(1 - \frac{r_g}{r}\right)^{1/2} \partial_r \quad e_3 = \frac{1}{r}\partial_\theta \quad e_4 = \frac{1}{r\sin(\theta)}\partial_\varphi$$

and the associated embedding $e_j \mapsto \gamma_j$ where, as before, $\{\gamma_j\}_{1\leq j\leq 4}$ are the Gamma matrices in the Dirac representation. In this non-holonomic, pseudo-orthornormal frame the Christoffel symbols are given by

$$\Gamma^i_{jk} = \frac{1}{2}\eta^{im}\left(c_{mjk} + c_{mkj} - c_{jkm}\right)$$

where η is the flat metric and

$$[e_i, e_j] = c^m_{jk} e_m, \quad c_{jkp} = \eta_{pm} c^m_{jk}$$

One can also exploit the properties of Christoffel symbols and of the Clifford module multiplication to write

$$\frac{1}{4} \sum_{k,l} \Gamma_{mkj} e^k(p) \cdot_V e^j(p) = \frac{1}{8} \sum_{k,l} \Gamma_{mkj} [\gamma^k, \gamma^j]$$

A lengthy but straightforward computation then leads to

$$\begin{aligned} \mathbf{D}(s) = i\gamma^1 \left(1 - \frac{r_g}{r}\right)^{-1/2} \partial_t s + i\gamma^2 \left(1 - \frac{r_g}{r}\right)^{1/2} \left(\partial_r s - \frac{r_g}{4r^2}\left(1 - \frac{r_g}{r}\right)^{-1} s - \frac{1}{r} s\right) \\ + i\gamma^3 \left(\frac{1}{r}\partial_\theta s - \frac{1}{2}\frac{\cot(\theta)}{r} s\right) + i\gamma^4 \frac{1}{r\sin(\theta)} \partial_\varphi s\ . \end{aligned}$$

(c) Another interesting example is that of Friedman–Lemaitre–Robertson–Walker spacetime,

$$M = I \times J \times \mathbb{S}^2 \qquad g = S(t)^2 \left(-dt \otimes dt + dr \otimes dr + f(r)^2 g_{\mathbb{S}^2}\right)\ .$$

Proceeding in the same way as in the Schwarzschild case, a simple computation leads to

$$\begin{aligned} \mathbf{D}(s) = i\gamma^1 \left(\frac{1}{S(t)}\partial_t s - \frac{3}{2}\frac{\dot{S}(t)}{S(t)^2} s\right) + i\gamma^2 \left(\frac{1}{S(t)}\partial_r s - \frac{f'(r)}{S(t) f(r)} s\right) \\ + i\gamma^3 \left(\frac{1}{S(t) f(r)}\partial_\theta s - \frac{\cot(\theta)}{S(t) f(r)} s\right) + i\gamma^4 \frac{1}{S(t) f(r)\sin(\theta)} \partial_\varphi s\ . \end{aligned}$$

As one can see from (2.12), the Dirac operator is a first-order partial differential operator whose principal symbol $\sigma(\mathbf{D})$ is given pointwise by

$$\sigma(\mathbf{D})\left(p, \xi_p\right)(s_p) = \mu|_p(\xi_p, s_p)\ .$$

Owing to the properties of Clifford multiplication and of the principal symbol, it is then evident that $\mathbf{D}^2$ is a normally hyperbolic, second-order partial differential operator. Therefore, $\mathbf{D}$ admits unique advanced and retarded Green operators

$$\not{S}_\pm \colon \Gamma_c(S(M)) \to \Gamma(S(M)) \tag{2.13}$$

i.e. $\mathbf{D} \circ \not{S}_\pm = \mathrm{id}_{\Gamma(S(M))}$, $\not{S}_\pm \circ \mathbf{D}|_{\Gamma_c(S(M))} = \mathrm{id}_{\Gamma_c(S(M))}$ and $\mathrm{supp}(\not{S}_\pm(s)) \subseteq J^\mp(\mathrm{supp}(s))$ [26], where $J^\pm(R)$ is, respectively, the causal future/past of the region $R \subset M$. These operators can be extended to continuous linear maps

$$\not{S}_\pm \colon \Gamma_{\mathrm{fc/pc}}(S(M)) \to \Gamma_{\mathrm{fc/pc}}(S(M))$$

where the subscripts fc and pc denote sections with *future-compact* and *past-compact* support respectively; we recall here that a subset $S \subseteq M$ is past-compact if $\overline{S} \cap J^-(p)$ is compact for every $p \in M$ (future-compact are defined in an analogous way). We remind, for later purposes, that we can also define the so-called *causal Green operator* (or, more simply, propagator) by posing $\mathcal{S} \doteq \mathcal{S}_- - \mathcal{S}_+$.

Remark 2.2 In the example case of Minkowski spacetime $(\mathbb{R}^4, \eta)$ previously examined, the advanced and retarded Green operators for the (massive) Dirac operator can be written explicitly by using the fact that

$$\mathcal{S}_\pm = (i\gamma^j \partial_j + m)\Delta_\pm$$

where $\Delta_\pm$ denote the advanced and retarded propagators of the Klein-Gordon operator, whose integral kernel can be written, in the sense of distributions, as

$$\Delta_\pm(x-y) = \frac{1}{(2\pi)^4} \lim_{\epsilon\to 0} \int_{\mathbb{R}^4} \frac{e^{ik_0(x^0-y^0)+i(\mathbf{k},\mathbf{x}-\mathbf{y})}}{-(k_0 \mp i\varepsilon)^2 + \|\mathbf{k}\|^2 + m^2} dk_0 d\mathbf{k} \,.$$

Then

$$\mathcal{S}_\pm(x-y) = \frac{1}{(2\pi)^4} \lim_{\epsilon\to 0} \int_{\mathbb{R}^4} \frac{(-\gamma^i k_i + m)e^{ik_0(x^0-y^0)+i(\mathbf{k},\mathbf{x}-\mathbf{y})}}{-(k_0 \mp i\varepsilon)^2 + \|\mathbf{k}\|^2 + m^2} dk_0 d\mathbf{k} \,.$$

Explicit expressions for the advanced and retarded propagators in the more interesting case of de Sitter spacetime can be found in [29].

By [24, Proposition 12.1.65], we know that $S(M)$ admits a Hermitian metric such that the Clifford multiplication by $\alpha \in \Gamma(T^*M)$ is an Hermitian automorphism of $S(M)$. We shall denote the Hermitian metric by $(\cdot, \cdot)$, and the associated two form by h.

2.2 The Geometric Description of Charged Spinors

Following [8], the coupling of a Dirac field with an external gauge field exploits the notion of *direct product bundle*, which we recall here:

Definition 2.3 Let $\pi_P : P \to M$ and $\pi_Q : Q \to M$ be two principal G- and H-bundles respectively. The product space $P \times Q$ is a principal $G \times H$-bundle over $M \times M$; by considering the diagonal $\Delta = \{(x, x) \in M \times M,\ x \in M\}$ and the inclusion map given by $i : \Delta \hookrightarrow M \times M$, we can then consider the pullback bundle

$$P + Q \doteq i^*(P \times Q) = \left\{(x, p, q) \in \Delta \times (P \times Q) \text{ s.t. } (x, x) = (\pi_P(p), \pi_Q(q))\right\} \,.$$

As Δ is diffeomorphic to M, we have that $P + Q$ is a principal $G \times H$-bundle over M, called the direct product bundle.

Notice that $P + Q$ can be naturally viewed as a subset of $P \times Q$; we can consider the restriction of the natural projections

$$f_P : P \times Q \to P \qquad f_Q : P \times Q \to Q$$

to $P + Q$. These are principal bundle homomorphisms, meaning that

$$f_P \left(r_{(g,h)}(p,q) \right) = r_g p \qquad f_Q \left(r_{(g,h)}(p,q) \right) = r_h q \ .$$

If the two principal bundles in Definition 2.3 are endowed with the connections $\omega^P \in \Omega^1(P, \mathfrak{g})$ and $\omega^Q \in \Omega^1(Q, \mathfrak{h})$ respectively, then by [30, Proposition 6.3] we have that the 1-form

$$\omega \doteq f_P^* \omega^P \oplus f_Q^* \omega^Q \tag{2.14}$$

is an Ehresmann connection on $P + Q$.
Let us now consider two representations

$$\rho_1 : G \to \mathrm{GL}(V) \qquad \rho_2 : H \to \mathrm{GL}(V)$$

such that $\rho_1(g)\rho_2(h) = \rho_2(h)\rho_1(g)$ for every $g \in G, h \in H$. Then one can construct the representation

$$\begin{aligned} \rho : G \times H &\to \mathrm{GL}(V) \\ (g,h) &\mapsto \rho\,(g,h) \doteq \rho_1(g)\rho_2(h) \end{aligned} \tag{2.15}$$

whose adjoint representation is given by

$$\begin{aligned} \rho_* : \mathfrak{g} \oplus \mathfrak{h} \to &\qquad \mathrm{End}(V) \\ (\mathbf{g}, \mathbf{h}) &\mapsto \rho_{1*}(\mathbf{g}) + \rho_{2*}(\mathbf{h}) \end{aligned}$$

and consider the associated bundle

$$(P + Q) \times_\rho V$$

which will then be endowed with a covariant derivative.

In our case of interest we will deal with a principal $\mathrm{Spin}^0_{r,1}$-bundle $\pi_s : P_{\mathrm{Spin}^0_{r,1}} \to M$ coupled with a principal G-bundle $\pi_g : P_G \to M$, where G is a compact Lie group which admits a representation $\rho_G : G \to \mathrm{GL}(V)$ such that

$$\Delta^{\mathbb{C}}_{r,1}(s)\rho_G(g) = \rho_G(g)\Delta^{\mathbb{C}}_{r,1}(s) \text{ for every } s \in \mathrm{Spin}^0_{r,1},\ g \in G \ .$$

By considering the direct product bundle $P_{\mathrm{Spin}^0_{r,1}} + P_G$ endowed with the connection given by (2.14) as well as the representation given by (2.15) we then can construct the associated vector bundle

$$S_G(M) \doteq (P_{\mathrm{Spin}^0_{r,1}} + P_G) \times_\rho V$$

which we shall call *charged spinor bundle*. This can be endowed with the covariant derivative

$$\nabla^{s,G} \colon \mathfrak{X}(M) \otimes \Gamma(S_G(M)) \to \Gamma(S_G(M)))$$

locally given by

$$\begin{aligned}(\nabla^{s,G}_v s)(p) &= \Big[\sigma(p), v(p)\left(s^{\sharp_G} \circ \sigma\right) + \rho_*\Big(\omega_{\sigma(p)}(\sigma_{*p} v(p))\Big) s^{\sharp_G}\,(\sigma(p))\Big] \\ &= \Bigg[\sigma(p), v(p)\left(s^{\sharp_G} \circ \sigma\right) + \Bigg(\Delta^{\mathbb{C}}_{r,1\,*}\left((f_{\mathrm{Spin}^0_{r,1}} \circ \sigma)^* \omega^s_p(v(p))\right) \\ &\quad + \rho_{G*}\Big((f_G \circ \sigma)^* \omega^G_p(v(p))\Big)\Bigg) s^{\sharp_G}\,(\sigma(p))\Bigg]\end{aligned} \tag{2.16}$$

where $\sigma \colon U \to P_{\mathrm{Spin}^0_{r,1}} + P_G$ is a local section of the direct product bundle and

$$\cdot^{\sharp_G} \colon \Gamma(\wedge^q T^* M \otimes S_G(M)) \to \Omega^q_\rho(P_{\mathrm{Spin}^0_{r,1}} + P_G, V)$$

denotes the isomorphism as of (2.2). Notice that as V is a $\mathscr{C}\ell_{r,1}$-module, even in this case we have a Clifford module multiplication

$$\cdot_{\Gamma,G} \colon \mathfrak{X}(M) \otimes \Gamma(S_G(M)) \to \Gamma(S_G(M))$$

which we can use to construct a Dirac operator $\mathbf{D}^G$ as in (2.11). Notice that the highest-order term is analogous to that of $\mathbf{D}$, therefore the principal symbols of these two operators coincide: it follows that $\mathbf{D}^G$ admits unique advanced and retarded Green operators

$$\mathcal{S}^G_\pm \colon \Gamma_c(S_G(M)) \to \Gamma(S_G(M))\ , \tag{2.17}$$

with the analogous properties of the uncharged operators as after (2.13). Again, we can define the *causal Green operator* by posing $\mathcal{S}^G \doteq \mathcal{S}^G_- - \mathcal{S}^G_+$.

Remark 2.4 Notice that the main difference between the local expressions (2.16) and (2.4) of the covariant derivatives, apart from the different spaces of section they are acting on, is given by the presence of the term

$$\Big[\sigma(p), \rho_{G*}\left((f_G \circ \sigma)^* \omega^G_p(v(p))\right) s^{\sharp_G}(\sigma(p))\Big]$$

which, when considering the Dirac operators $\mathbf{D}^G$ and $\mathbf{D}$, amounts to

$$ie^j(p)\cdot_{\Gamma,G}\left[\sigma(p),\rho_{G*}\left((f_G\circ\sigma)^*\omega^G_p(e_j(p))\right)s^{\sharp_G}(\sigma(p))\right]\,.$$

In the spirit of Remark 2.1, let us now exhibit an explicit expression in the simple case of Minkowski spacetime $(\mathbb{R}^4,\eta)$. First of all, let us define the quantities

$$A_j(p)\doteq\rho_{G*}\left((f_G\circ\sigma)^*\omega^G_p(e_j(p))\right)$$

which are maps $A_j\colon\mathbb{R}^4\to\mathfrak{gl}(\mathbb{C}^4)$. Then in the same way as before it follows that

$$\mathbf{D}^G(s)(p)=i\gamma^j(\partial_j s+A_j s)\,.$$

□

3 The Classical Møller Map on Field Configurations

3.1 The Entwining Maps Between Sections

In order to compare the uncharged theory and the charged one, we need a way to relate two different spaces of functions: on one hand we have the smooth sections of the spinor bundle $\Gamma(S(M))$, while on the other we have the sections of the charged spinor bundle $\Gamma(S_G(M))$. Notice that these spaces are isomorphic, thanks to the linear isomorphisms in (2.2), to the spaces

$$\begin{aligned}F&\doteq\left\{f\colon P_{\mathrm{Spin}^0_{r,1}}\to V,\ f\text{ right}-\mathrm{Spin}^0_{r,1}\text{ equivariant}\right\}\\ F_G&\doteq\left\{f\colon P_{\mathrm{Spin}^0_{r,1}}+P_G\to V,\ f\text{ right}-\mathrm{Spin}^0_{r,1}\times G\text{ equivariant}\right\}\end{aligned}$$

respectively. We recall that owing to [31, Sect. II] principal $\mathrm{Spin}^0_{r,1}$-bundles over non-compact spin spacetimes admit a global section $\sigma\colon M\to P_{\mathrm{Spin}^0_{r,1}}$, and are thus always trivial. To simplify the computations and the definition in this first case, we then make the following assumptions:

(i) *we assume that the principal G-bundle P_G is trivial*; this holds, for instance, if we consider a contractible spacetime, or if we consider principal bundles $P_{U(1)}$ whose first characteristic class $c_1(P)$ vanishes. On P_G, we assume that a fixed Ehresmann connection $\omega_G\in\Omega^1(P,\mathfrak{g})$ is given.
As P_G is trivial, there exists a global section $\sigma_G\colon M\to P_G$; we can use it to define the global section

$$M\ni p\overset{\tilde{\sigma}}{\mapsto}(p,\sigma(p),\sigma_G(p))$$

of the direct product bundle $P_{\mathrm{Spin}^0_{r,1}} + P_G$. It follows easily that $f_{\mathrm{Spin}^0_{r,1}} \circ \tilde{\sigma} = \sigma$, $f_G \circ \tilde{\sigma} = \sigma_G$. Moreover, notice that as the principal bundles $P_{\mathrm{Spin}^0_{r,1}}$ and $P_{\mathrm{Spin}^0_{r,1}} + P_G$ are trivial, the associated fiber bundles are also trivial, with diffeomorphisms given respectively by

$$\begin{aligned} \varphi^s \colon (M \times \mathrm{Spin}^0_{r,1}) \times_{\Delta^{\mathbb{C}}_{r,1}} V \to \quad & M \times V \\ [(p, s), v] \quad \mapsto & (p, \Delta^{\mathbb{C}}_{r,1}(s)v) \end{aligned}$$

and

$$\begin{aligned} \varphi^{s,G} \colon (M \times (\mathrm{Spin}^0_{r,1} \times G)) \times_\rho V \to \quad & M \times V \\ [(p, s, g), v] \quad \mapsto & (p, \rho(s, g)v) \end{aligned}.$$

As the vector bundles are trivial, the Hermitian metrics $(\cdot, \cdot)$ on $S(M)$ and $(\cdot, \cdot)_G$ on $S_G(M)$ can be induced by a Hermitian metric $(\cdot, \cdot)_V$ on V; notice that thanks to [24, Proposition 12.1.27] the Hermitian metric can be chosen to be such that Clifford multiplication satisfies

$$\rho(\phi)^* = -\rho(\phi^\dagger), \qquad \phi^\dagger = \tilde{\alpha}(\phi^t)$$

where $\cdot^t \colon \mathscr{C}\ell_{r,1} \to \mathscr{C}\ell_{r,1}$ denotes the transpose and $\tilde{\alpha}$ is the extension of $\mathbb{R}^{r,1} \ni \mathbf{v} \to -\mathbf{v}$ to $\mathscr{C}\ell_{r,1}$. We shall denote the Hermitian matrix associated with $(\cdot, \cdot)_V$ by (h_{ij}).

(ii) As a second core assumption, *we shall assume that the global gauge potential* computed using the section $\tilde{\sigma} \colon M \to P_{\mathrm{Spin}^0_{r,1}} + P_G$,

$$(f_G \circ \tilde{\sigma})^* \omega^G \in \Omega^1(M, \mathfrak{g})$$

has past-compact support. Clearly, this requirement depends on the gauge choice and is *not* gauge invariant; we shall expand on this in Remark 3.3.

(iii) *We will assume that the representation $\rho_G \colon G \to \mathrm{GL}(V)$ commutes with the Clifford algebra representation inducing $\Delta^{\mathbb{C}}_{r,1} \colon \mathrm{Spin}^0_{r,1} \to \mathrm{GL}(V)$*. Notice that this greatly reduces the freedom in the group G: indeed, it can be shown [22, Theorem 4.3] that

$$\begin{aligned} \mathbb{C}\ell_{r+1} \simeq M(2^{\lfloor (r+1)/2 \rfloor}, \mathbb{C}) \text{ if } r+1 = 0 \mod 2 \\ \mathbb{C}\ell_{r+1} \simeq M(2^{\lfloor (r+1)/2 \rfloor}, \mathbb{C}) \oplus M(2^{\lfloor (r+1)/2 \rfloor}, \mathbb{C}) \text{ if } r+1 = 1 \mod 2 \end{aligned}$$

and that an irreducible $\mathbb{C}$-module for $\mathscr{C}\ell_{r,1}$ (which descends from an irreducible module for $\mathbb{C}\ell_{r+1}$) has complex dimension $2^{\lfloor (r+1)/2 \rfloor}$. We are therefore assuming that the image of ρ_G lies in the centre of the algebra $M(2^{\lfloor (r+1)/2 \rfloor}, \mathbb{C})$, that is, $\rho_G(G) \subseteq \mathbb{C}\mathrm{id}_V$. Although it may seem rather restrictive, this case encompasses the interesting case of $G = U(1)$, that is, the case of electrically charged spinors.

The global section $\sigma_G\colon M \to P_G$ allows us to construct the following maps:

Definition 3.1 Let us define the maps $p\colon F_G \to F$,

$$F_G \ni f_g \mapsto (pf_g)(p,s) \doteq f_g(p,s,\sigma_G(p)) \tag{3.1}$$

and $i\colon F \to F_G$,

$$F \ni f \mapsto (if)(p,s,g) \doteq \rho_G(\tilde{g}^{-1})f(p,s) \tag{3.2}$$

where $\tilde{g} \in G$ is the unique group element such that $r_{\tilde{g}}\sigma_G(p) = (p,g)$, which exists as the action of G on P_G is free and transitive.

Notice that Definition 3.1 is well-posed, in the sense that given $f_g \in F_G$, pf_g is right $-\,\mathrm{Spin}^0_{r,1}$ equivariant, and given $f \in F$, if is right $-\,\mathrm{Spin}^0_{r,1} \times G$ equivariant: indeed,

$$\begin{aligned}\left(r^*_{\overline{s}}(pf_g)\right)(p,s) &= (pf_g)(p,r_{\overline{s}}s) = f_g(p,r_{\overline{s}}s,\sigma_G(p)) = (r^*_{(\overline{s},\mathrm{id}_G)}f_g)(p,s,\sigma_G(p))\\ &= \rho(\overline{s}^{-1},\mathrm{id}_G)f_g(p,s,\sigma_G(p)) = \Delta^{\mathbb{C}}_{r,1}(\overline{s}^{-1})(pf_g)(p,s)\end{aligned}$$

and

$$\begin{aligned}\left(r^*_{(\overline{s},\overline{g})}(if)\right)(p,s,g) &= (if)(p,r_{\overline{s}}s,r_{\overline{g}}g) = \rho_G(\overline{g}^{-1})\rho_G(\tilde{g}^{-1})f(p,r_{\overline{s}}s)\\ &= \rho_G(\overline{g}^{-1})\rho_G(\tilde{g}^{-1})r^*_{\overline{s}}f(p,s)\\ &= \rho_G(\overline{g}^{-1})\rho_G(\tilde{g}^{-1})\Delta^{\mathbb{C}}_{r,1}(\overline{s}^{-1})f(p,s)\\ &= \Delta^{\mathbb{C}}_{r,1}(\overline{s}^{-1})\rho_G(\overline{g}^{-1})\rho_G(\tilde{g}^{-1})f(p,s)\\ &= \rho(\overline{s}^{-1},\overline{g}^{-1})(if)(p,s,g)\,.\end{aligned}$$

We can combine the maps i and p defined in Definition 3.1 with the linear maps in (2.2) to yield maps

$$\begin{array}{ccccccc} i\colon \Gamma(S(M)) & \to & \Gamma(S_G(M)) & \quad & p\colon \Gamma(S_G(M)) & \to & \Gamma(S(M))\\ s & \mapsto & (is^{\sharp})^{\flat_G} & & s & \mapsto & (ps^{\sharp_G})^{\flat}\end{array} \tag{3.3}$$

which are denoted with the same symbol for the sake of notational simplicity. Let us now endow the spaces $\Gamma(S(M))$ and $\Gamma(S_G(M))$ with the Fréchet topology induced by the families of seminorms

$$\begin{aligned}\|s\|_{K,n} &\doteq \max_{0\le i\le n}\sup_{p\in K}\left\|\left(\nabla^{s\,i}s\right)(p)\right\|_{S(M)\otimes T^*M^{\otimes i}}\\ \|s\|^G_{K,n} &\doteq \max_{0\le i\le n}\sup_{p\in K}\left\|\left(\nabla^{s,G\,i}s\right)(p)\right\|_{S_G(M)\otimes T^*M^{\otimes i}}\,.\end{aligned} \tag{3.4}$$

We shall indicate these topological spaces as $\mathcal{E}(S(M))$ and $\mathcal{E}(S_G(M))$ respectively.

Lemma 3.2 *The maps $\imath$ and $\mathfrak{p}$ defined in (3.3) are continuous with respect to the Fréchet topologies on $\mathcal{E}(S(M))$ and $\mathcal{E}(S_G(M))$.*

Proof We need to exhibit, for each $K \subset\subset M$ and for each $n \in \mathbb{N}$, constants $k_{K,n}, k^G_{K,n} \in \mathbb{R}^+$ and families $\{(K_l, n_l)\}_{1\leq l\leq m}, \{(K^G_l, n^G_l)\}_{1\leq l\leq m^G}$ such that

$$\|\imath s\|^G_{K,n} \leq k_{K,n} \max_{1\leq l\leq m} \|s\|_{K_l,n_l} \qquad \|\mathfrak{p}s\|_{K,n} \leq k^G_{K,n} \max_{1\leq l\leq m^G} \|s\|^G_{K^G_l,n^G_l} \,.$$

The case $n = 0$ is easy: indeed,

$$\begin{aligned}
\|\imath s\|^G_{K,0} &= \sup_{p\in K} \|(\imath s)(p)\|_{S_G(M)} \\
&= \sup_{p\in K} \left\|\rho(\pi_2(\tilde{\sigma}(p)))(\imath s)^{\sharp_G}(\tilde{\sigma}(p))\right\|_V \\
&= \sup_{p\in K} \left\|\rho_G(\pi_2\circ\sigma_G(p))\Delta^{\mathbb{C}}_{r,1}(\pi_2\circ\sigma(p))s^{\sharp}(\sigma(p))\right\|_V \\
&\leq \sup_{p\in K} \|\rho_G\,(\pi_2\circ\sigma_G(p))\|_{\mathrm{End}(V)} \sup_{p\in K} \left\|\Delta^{\mathbb{C}}_{r,1}(\pi_2\circ\sigma(p))s^{\sharp}(\sigma(p))\right\|_V \\
&= k_{K,0}\,\|s\|_{K,0}
\end{aligned}$$

where we use that $K \subset\subset M$ and that ρ_G is a smooth representations onto a matrix algebra on V finite-dimensional vector space.
Consider now $n = 1$; in this case we compute $\nabla^{s,G}_{e_i}(\imath s)(p)$, where $\{e_i\}_{1\leq i\leq \dim(M)}$ is a local pesudo-orthonormal basis for TM. In particular, using a section $\widehat{\sigma}: M \to P_{\mathrm{Spin}^0_{r,1}} + P_G$ with $\widehat{\sigma} = r_g\tilde{\sigma}$,

$$\begin{aligned}
\nabla^{s,G}_{e_i}(\imath s)(p) &= \Big[\widehat{\sigma}(p), e_i(p)\Big(\rho_G\,(\pi_G\circ g(p))^{-1}\, s^{\sharp}(f_P\circ\widehat{\sigma}(p))\Big) \\
&\quad +\Big(\Delta^{\mathbb{C}}_{r,1*}\left(\left((f_P\circ\widehat{\sigma})^*\omega^s\right)_p(e_i(p))\right) \\
&\quad +\rho_{G*}\left(\left((f_G\circ\widehat{\sigma})^*\omega^G\right)_p(e_i(p))\right)(\imath s)^{\sharp_G}(\widehat{\sigma}(p))\Big] \\
&= \Big[\widehat{\sigma}(p), \rho_G\,(\pi_G\circ g(p))^{-1}\Big(e_i(p)\left(s^{\sharp}\circ f_P\circ\widehat{\sigma}\right) \\
&\quad +\Delta^{\mathbb{C}}_{r,1*}\left(\left((f_P\circ\widehat{\sigma})^*\omega^s\right)_p(e_i(p))\right)s^{\sharp}(\sigma(p))\Big)\Big] \\
&\quad +\Big[\widehat{\sigma}(p), e_i(p)\left(\rho_G\,(\pi_G\circ g(p))^{-1}\right)s^{\sharp}(f_P\circ\widehat{\sigma}(p)) \\
&\quad +\rho_{G*}\left(\left((f_G\circ\widehat{\sigma})^*\omega^G\right)_p(e_i(p))\right)(\imath s)^{\sharp_G}(\widehat{\sigma}(p))\Big].
\end{aligned}$$

Notice that the first term is equal to $\imath\circ\nabla^s_{e_i}s$, while the second one can be written as

$$\left[\widehat{\sigma}(p), \rho_{G*}\left(\left(-(\pi_G\circ g)^*\theta_G + (f_G\circ\widehat{\sigma})^*\omega^G\right)_p(e_i(p))\right)(\imath s)^{\sharp_G}(\widehat{\sigma}(p))\right] \quad (3.5)$$

where θ_G denots the Maurer–Cartan form of G. Using the notation $\varepsilon_i = g(e_i, e_i)$ we have

$$\begin{aligned}
\left\| \nabla^{s,G}(\imath s)(p) \right\|^2_{S_G(M)\otimes T^*M} &= \sum_{i=1}^{\dim M} \varepsilon_i \left\| \nabla^{s,G}_{e_i}(\imath s)(p) \right\|^2_{S_G(M)} \\
&\leq k \sum_{i=1}^{\dim M} \varepsilon_i \left\| \imath \circ \nabla^s_{e_i} s(p) \right\|^2_{S_G(M)} \\
&\quad + k \sum_{i=1}^{\dim M} \varepsilon_i \left\| \rho\,(\pi_2 \circ \widehat{\sigma}(p))\, \rho_{G*}\Big(\big(-(\pi_G \circ g)^* \theta_G \right\| \\
&\quad + (f_G \circ \widehat{\sigma})^* \omega^G \big)_p \,(e_i(p)) \Big) (\imath s)^{\sharp_G} (\sigma(p))^2_V \ .
\end{aligned}$$

Given a compact set $K \subset\subset M$, the first term can be bounded from above by

$$k_1 \sup_{p\in K} \left\| \nabla^s s \right\|^2_{S(M)\otimes T^*M} \leq k_1 \left\| s \right\|^2_{K,1}$$

while for the second term we have the upper bound

$$k_2 \sup_{p\in K} \left\| s \right\|^2_{S(M)} = k_2 \left\| s \right\|^2_{K,0} \leq k_2 \left\| s \right\|^2_{K,1} \ .$$

Therefore we can give the estimate

$$\left\| \imath s \right\|^G_{K,1} \leq k_{K,1} \left\| s \right\|_{K,1} \ .$$

We proceed by induction: suppose we have an inequality of the form $\left\| \imath s \right\|^G_{K,i} \leq k_{K,i} \left\| s \right\|_{K,i}$ for i up to $n-1 \in \mathbb{N}$. Then by the above discussion $\nabla^{s,G^n}(\imath s)$ can be written as $\imath \nabla^{s^n} s$, which can be bounded by $k_n \left\| s \right\|_{K,n}$, plus lower order terms in the covariant derivative of s and of the map $\rho_{G*}\Big(\big(-(\pi_G \circ g)^* \theta_G + (f_G \circ \widehat{\sigma})^* \omega^G\big)_p (e_i(p))\Big)$ which are bounded in every compact set $K \subset\subset M$ by the inductive hypothesis and the smoothness of the involved maps respectively.

An analogous proof holds for the map $p\colon \Gamma(S_G(M)) \to \Gamma(S(M))$: indeed,

$$\begin{aligned}
\left\| p s \right\|_{K,0} &= \sup_{p\in K} \left\| (p s)(p) \right\|_{S(M)} \\
&= \sup_{p\in K} \left\| \Delta^{\mathbb{C}}_{r,1}(\pi_2 \circ \sigma(p))(p s)^{\sharp}\,(\sigma(p)) \right\|_V \\
&= \sup_{p\in K} \left\| \Delta^{\mathbb{C}}_{r,1}(\pi_2 \circ \sigma(p)) s^{\sharp_G}\,(\tilde{\sigma}(p)) \right\|_V \\
&\leq \sup_{p\in K} \left\| \rho_G(\pi_2 \circ \sigma_G(p))^{-1} \right\|_{\mathrm{End}(V)}
\end{aligned}$$

$$\sup_{p\in K} \left\| \rho_G(\pi_2 \circ \sigma_G(p)) \Delta^{\mathbb{C}}_{r,1}(\pi_2 \circ \sigma(p)) s^{\sharp_G}(\tilde{\sigma}(p)) \right\|_V$$
$$= k^G_{K,0} \sup_{p\in K} \|s\|_{S^G(M)}$$

while for the case $n = 1$ we have

$$\begin{aligned}
\nabla^s_{e_i}(ps)(p) &= \Big[\sigma(p), e_i(p)\left(s^{\sharp_G}(\tilde{\sigma}(p))\right) + \Delta^{\mathbb{C}}_{r,1*}\Big((\sigma^*\omega^s)_p(e_i(p))\Big) s^{\sharp_G}(\tilde{\sigma}(p))\Big] \\
&= \Big[\sigma(p), e_i(p)\left(s^{\sharp_G}(\tilde{\sigma}(p))\right) + \Delta^{\mathbb{C}}_{r,1*}\Big((\sigma^*\omega^s)_p(e_i(p))\Big) s^{\sharp_G}(\tilde{\sigma}(p)) \\
&\quad + \rho_{G*}\Big((\sigma^*_G\omega^G)_p(e_i(p))\Big) s^{\sharp_G}(\tilde{\sigma}(p)) \\
&\quad - \rho_{G*}\Big((\sigma^*_G\omega^G)_p(e_i(p))\Big) s^{\sharp_G}(\tilde{\sigma}(p))\Big] .
\end{aligned}$$

Notice that

$$\Big[\sigma(p), e_i(p)(s^{\sharp_G} \circ \tilde{\sigma}) + \Big(\Delta^{\mathbb{C}}_{r,1*}\Big((\sigma^*\omega^s)_p(e_i(p))\Big) + \rho_{G*}\Big((\sigma^*_G\omega^G)_p(e_i(p))\Big)\Big) s^{\sharp_G}(\tilde{\sigma}(p))\Big] = p\left(\nabla^{s,G}_{e_i} s\right)(p) .$$

Then

$$\begin{aligned}
\left\|\nabla^s(ps)(p)\right\|^2_{S(M)\otimes T^*M} &= \sum_{i=1}^{\dim M} \varepsilon_i \left\|\nabla^s_{e_i}(ps)(p)\right\|^2_{S(M)} \\
&\leq k \sum_{i=1}^{\dim M} \varepsilon_i \left\| p\left(\nabla^{s,G}_{e_i} s\right)(p)\right\|^2_{S(M)} \\
&\quad + k \sum_{i=1}^{\dim M} \varepsilon_i \left\| \Delta^{\mathbb{C}}_{r,1}(\pi_2 \circ \sigma(p)) \right\| \\
&\quad \rho_{G*}\Big((\sigma^*_G\omega^G)_p(e_i(p)))\Big) s^{\sharp_G}(\tilde{\sigma}(p))^2_V .
\end{aligned}$$

As before, given a compact $K \subset\subset M$ the first term can be bounded from above by

$$k_1 \sup_{p\in K} \left\|\nabla^{s,G} s\right\|_{S_G(M)\otimes T^*M} \leq k_1 \left(\|s\|^G_{K,1}\right)^2$$

and the second term can be bounded from above by

$$k_2 \sup_{p\in K} \|s\|^2_{S_G(M)} \leq k_2 \left(\|s\|^G_{K,0}\right)^2 .$$

By reasoning as above by induction we have the desired result. □

Remark 3.3 Notice how a key object in the proof of Lemma 3.2 is given by (3.5), which we recall here for a section $s \in \Gamma(S_G(M))$:

$$\left[\widehat{\sigma}(p), \rho_{G*}\left(\left(-(\pi_G \circ g)^*\theta_G + (f_G \circ \widehat{\sigma})^*\omega^G\right)_p (e_i(p))\right) s^{\sharp_G}(\widehat{\sigma}(p))\right] .$$

In the above expression, $\widehat{\sigma}\colon M \to P_{\mathrm{Spin}^0_{r,1}} + P_G$ is a section which can be written as $\widehat{\sigma} = r_g\tilde{\sigma}$ in terms of the the global reference section and of a smooth function $g\colon M \to \mathrm{Spin}^0_{r,1} \times G$. The above expression is gauge independent: indeed, given another section $\overline{\sigma}\colon M \to P_{\mathrm{Spin}^0_{r,1}} + P_G$ we have that $\overline{\sigma} = r_{g'} \circ \widehat{\sigma}$ where $g'\colon M \to \mathrm{Spin}^0_{r,1} \times G$ and

$$f_P(r_{g'} \circ \widehat{\sigma}) = r_{\pi_{\mathrm{Spin}}(g')} f_P(\widehat{\sigma}) \qquad f_G(r_{g'} \circ \widehat{\sigma}) = r_{\pi_G(g')} f_G(\widehat{\sigma}) .$$

The following relations holds between the different pullbacks of the connection form ω^G and of the Maurer–Cartan form θ_G:

$$(f_G \circ \overline{\sigma})^*\omega^G = (r_{\pi_G(g')} \circ f_G \circ \widehat{\sigma})^*\omega^G = \mathrm{Ad}_{\pi_G(g')^{-1}} \circ (f_G \circ \widehat{\sigma})^*\omega^G + (\pi_G(g'))^*\theta_G ,$$
$$(\pi_G(gg'))^*\theta_G = (r_{\pi_G(g')} \circ \pi_G(g))^*\theta_G = \mathrm{Ad}_{\pi_G(g')^{-1}} \circ (\pi_G(g))^*\theta_G + (\pi_G(g'))^*\theta_G .$$

Therefore,

$$\begin{aligned}
&\left[\overline{\sigma}(p), \rho_{G*}\left(\left(-(\pi_G(gg'))^*\theta_G + (f_G \circ \overline{\sigma})^*\omega^G\right)_p (e_i(p))\right) s^{\sharp_G}(\overline{\sigma}(p))\right] \\
&= \Big[\widehat{\sigma}(p), \rho(g'(p)) \times \rho_{G*}\Big(\mathrm{Ad}_{\pi_G(g')^{-1}} \circ \Big((f_G \circ \widehat{\sigma})^*\omega^G \\
&\quad -(\pi_G(g))^*\theta_G\Big)\Big)_p (e_i(p)) \rho(g'^{-1}(p)) s^{\sharp_G}(\widehat{\sigma}(p))\Big] \\
&= \left[\widehat{\sigma}(p), \rho_{G*}\left(\left(-(\pi_G(g))^*\theta_G + (f_G \circ \widehat{\sigma})^*\omega^G\right)_p (e_i(p))\right) s^{\sharp_G}(\widehat{\sigma}(p))\right] .
\end{aligned}$$

Notice moreover that if the section $\widehat{\sigma}$ is such that $\widehat{\sigma} = \tilde{\sigma}$ the above expression simplifies to

$$\left[\tilde{\sigma}(p), \rho_{G*}\left(\left(\sigma_G^*\omega^G\right)_p (e_i(p))\right) s^{\sharp_G}(\tilde{\sigma}(p))\right]$$

as the map $g\colon M \to \mathrm{Spin}^0_{r,1} \times G$ is constantly the identity. Clearly, whereas the above expression is past-compactly supported by the assumption (ii), the same doesn't hold in general for

$$\left[\widehat{\sigma}(p), \rho_{G*}\left(\left(-(\pi_G \circ g)^*\theta_G + (f_G \circ \widehat{\sigma})^*\omega^G\right)_p (e_i(p))\right) (is)^{\sharp_G}(\widehat{\sigma}(p))\right] .$$

The compactness property holds for instance if $\pi_G \circ g\colon M \to G$ is constant outside of a past-compact region of M. As this property is fundamental in the construction of the (retarded) Møller map, *we restrict ourselves to considering gauge potentials induced by sections* $\widehat{\sigma}: M \to P_{\mathrm{Spin}^0_{r,1}} + P_G$ *which are such that* $\pi_G \circ g$ *is constant outside of a past-compact set, where* $g: M \to \mathrm{Spin}^0_{r,1} \times G$ *is such that* $\widehat{\sigma} = r_g \tilde{\sigma}$. Having made this choice, we define the map $A: \mathcal{E}(S_G(M)) \to \mathcal{E}(S_G(M))$ to act pointwise as

$$\begin{aligned}(As)(p) \doteq\; & i e^j(p) \cdot_{\Gamma,G} \Big[\widehat{\sigma}(p), \rho_{G*}\Big(\Big(-(\pi_G \circ g)^* \theta_G \\ & +(f_G \circ \widehat{\sigma})^* \omega^G\Big)_p (e_i(p))\Big) s^{\sharp_G}(\widehat{\sigma}(p))\Big] .\end{aligned} \tag{3.6}$$

We define

$$\mathrm{supp}(\mathscr{A}, g) \doteq \mathrm{supp}\left(-(\pi_G \circ g)^* \theta_G + (f_G \circ \widehat{\sigma})^* \omega^G\right) .$$

Notice that under the restriction on the possible gauge transformations, $\mathrm{supp}(\mathscr{A}, g)$ is always past-comapct. We now exhibit an important property of the map A defined in (3.6): to do so, we first show that $\imath: \mathcal{E}(S(M)) \to \mathcal{E}(S_G(M))$ commutes with the Clifford multiplication by sections of the cotangent bundle, that is, we show that

$$e^i \cdot_{\Gamma,G} \imath(u) = \imath\left(e^i \cdot_\Gamma u\right) .$$

Consider the section $\widehat{\sigma}: M \to P_{\mathrm{Spin}^0_{r,1}} + P_G$ inducing a section $f_P \circ \widehat{\sigma}: M \to P_{\mathrm{Spin}^0_{r,1}}$; locally $\imath(u(p)) = [\widehat{\sigma}(p), \rho_G((\pi_2 \circ h)(p))^{-1} u^\sharp(f_P \circ \widehat{\sigma}(p))]$ where $h: M \to \mathrm{Spin}^0_{r,1} \times G$ is such that $\widehat{\sigma} = r_h \tilde{\sigma}$. Then

$$\begin{aligned} e^i(p) \cdot_{\Gamma,G} \imath(u)(p) &= \left[\widehat{\sigma}(p), \left(g^{ij}(p) e_j(p)\right) \cdot_V \rho_G\left((\pi_2 \circ h)(p)\right)^{-1} u^\sharp(f_P \circ \widehat{\sigma}(p))\right] \\ &= \left[\widehat{\sigma}(p), \rho_G\left((\pi_2 \circ h)(p)\right)^{-1} \left(\left(g^{ij}(p) e_j(p)\right) \cdot_V u^\sharp(f_P \circ \widehat{\sigma}(p))\right)\right] \\ &= \imath(e^i \cdot_\Gamma u)(p) . \end{aligned}$$

Notice that assumption (iii) was used in the second equality.
Thus, as shown in the proof of Lemma 3.2 we know that

$$\mathbf{D}^G \circ \imath = \imath \circ \mathbf{D} + A \circ \imath . \tag{3.7}$$

□

3.2 The Explicit Construction of the Classical Møller Map

We now use these maps, in particular $\imath: \mathcal{E}(S(M)) \to \mathcal{E}(S_G(M))$ to give an explicit formula for the retarded classical Møller operator, whose definition in the examined case we recall:

Definition 3.4 Let us consider an admissible gauge potential (as of Remark 3.3), i.e. an admissible section $\widehat{\sigma}\colon M \to P_{\mathrm{Spin}^0_{r,1}} \times P_G$ such that $\widehat{\sigma} = r_g\tilde{\sigma}$. The *retarded classical Møller map on field configurations* is a map $R^-_A\colon \mathcal{E}(S(M)) \to \mathcal{E}(S_G(M))$ such that

(i) $\mathbf{D}^G \circ R^-_A = \imath \circ \mathbf{D}$,
(ii) $R^-_A(s)|_{M\setminus J^+(\mathrm{supp}(\mathscr{A},g))} = (\imath s)|_{M\setminus J^+(\mathrm{supp}(\mathscr{A},g))}$.

Remark 3.5 A similar construction can be made for the *advanced Møller map* $R^+_A\colon \mathcal{E}(S(M)) \to \mathcal{E}(S_G(M))$, which is defined analogously to Definition 3.4 except that requirement (ii) now reads

$$R^+_A(s)|_{M\setminus J^-(\mathrm{supp}(\mathscr{A},g))} = (\imath s)|_{M\setminus J^-(\mathrm{supp}(\mathscr{A},g))}$$

In the following, we shall focus on the retarded classical Møller map which shall be denote simply with $R_A\colon \mathcal{E}(S(M)) \to \mathcal{E}(S_G(M))$. The reader may refer to [32] for furhter details.

In this case, as anticipated, the classical Møller operator has an explicit form:

Theorem 3.6 *The unique solution to the requirements in Definition 3.4 is given by*

$$R_A = \imath - \not{S}^G_- \circ A \circ \imath \tag{3.8}$$

where $A\colon \mathcal{E}(S_G(M)) \to \mathcal{E}(S_G(M))$ is given by Eq. 3.6.

Proof First of all, notice that the composition $\not{S}^G_- \circ A \circ \imath$ is well-defined thanks to the fact that $\mathrm{supp}(\mathscr{A}, g)$ is past-compact: indeed, we know that

$$\not{S}^G_-\colon \mathcal{E}_{\mathrm{pc}}(S_G(M)) \to \mathcal{E}_{\mathrm{pc}}(S_G(M))$$

and as As vanishes outside of $\mathrm{supp}(\mathscr{A}, g)$ for every $s \in \mathcal{E}(S_G(M))$ we have that $(A \circ \imath)(s) \in \mathcal{E}_{\mathrm{pc}}(S_G(M))$ for every $s \in \mathcal{E}(S(M))$.
Thanks to the properties of the retarded propagator $\not{S}^G_-$, we thus have that

$$\mathrm{supp}\left((\not{S}^G_- \circ A \circ \imath)(s)\right) \subseteq J^+\left(\mathrm{supp}(A \circ \imath)(s)\right) \subseteq J^+\left(\mathrm{supp}(\mathscr{A}, g)\right)$$

and therefore the requirement (ii) in Definition 3.4 is fulfilled.

Using the property $\mathbf{D}^G \circ \not{S}^G_- = \mathrm{id}_{\mathcal{E}_{\mathrm{pc}}(S_G(M))}$ of the retarded propagator associated to $\mathbf{D}^G$ we have $\mathbf{D}^G \circ (\not{S}^G_- \circ A \circ \imath) = A \circ \imath$; therefore, by (3.7),

$$\mathbf{D}^G \circ (\imath - \not{S}^G_- \circ A \circ \imath) = \imath \circ \mathbf{D} .$$

Therefore, also the requirement (i) in Definition 3.4 is fulfilled. Thus we can say that

$$R_A = \imath - \not{S}^G_- \circ A \circ \imath .$$

As far as the uniqueness statement is concerned, we proceed as done in [33] and [34]: given $s \in \mathcal{E}(S(M))$, let us define $R_A s \doteq \psi \in \mathcal{E}(S_G(M))$; then we have that ψ satisfies the following:

$$\mathbf{D}^G \psi = (i \circ \mathbf{D})s \ , \quad \psi|_{M \setminus J^+(\mathrm{supp}(\mathscr{A}, g))} = (is)|_{M \setminus J^+(\mathrm{supp}(\mathscr{A}, g))} \ .$$

As M is globally hyperbolic, it is isometric to $\mathbb{R} \times \Sigma$, with $\{a\} \times \Sigma$ Cauchy surface for every $a \in \mathbb{R}$. In particular, we can assume that $\{0\} \times \Sigma \subseteq M \setminus J^+(\mathrm{supp}(\mathscr{A}, g))$. Let $\{K_n\}_{n\in\mathbb{N}}$ be an invading sequence of compact sets for $\{0\} \times \Sigma$, and define

$$\widehat{K}_n \doteq D(K_n) \cap [-n, n] \times \Sigma$$

where $D(K_n)$ denotes the Cauchy development of K_n. We then consider the family $\{\chi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}(M)$, with $\chi_n \equiv 1$ on $\widehat{K}_n$. Using these family, we consider

$$\begin{cases} (\mathbf{D}^G \circ \mathbf{D}^G)\phi_n = (i \circ \mathbf{D})(\chi_n s) & \text{on } M \\ \phi_n = (\mathcal{S}_-^G \circ i)(\chi_n s) & \text{on } \{0\} \times \Sigma \\ \nabla_\nu^{s,G} \phi_n = \nabla_\nu^{s,G} \big((\mathcal{S}_-^G \circ i)(\chi_n s)\big) & \text{on } \{0\} \times \Sigma \ . \end{cases}$$

Notice that $\mathbf{D}^G \circ \mathbf{D}^G$ is normally hyperbolic, and thus the above system admits a unique solution which depends continuously on the initial data [25]. Notice in particular that on $M \setminus J^+(\mathrm{supp}(\mathscr{A}, g))$ the map

$$(\mathcal{S}_-^G \circ i)(\chi_n s)$$

is a solution of the above problem, as there the part due to the gauge potential vanishes. Therefore by the uniqueness of the solution we have

$$\phi_n|_{M \setminus J^+(\mathrm{supp}(\mathscr{A}, g))} = (\mathcal{S}_-^G \circ i)(\chi_n s)|_{M \setminus J^+(\mathrm{supp}(\mathscr{A}, g))} \ .$$

By the properties of $\{\widehat{K}_n\}_{n\in\mathbb{N}}$ and $\{\chi_n\}_{n\in\mathbb{N}}$ and an analogous reasoning, we have that if $m > n$ then $\phi_m = \phi_n$ on $\widehat{K}_n$; therefore by defining

$$\phi(p) \doteq \phi_n(p) \text{ with } n \text{ such that } p \in \widehat{K}_n$$

and $\psi \doteq \mathbf{D}^G \phi$ we have that $\mathbf{D}^G \psi = (\mathbf{D}^G \circ \mathbf{D}^G)\phi = (i \circ \mathbf{D})s$ and

$$\psi|_{M \setminus J^+(\mathrm{supp}(\mathscr{A}, g))} = (\mathbf{D}^G \circ \phi)|_{M \setminus J^+(\mathrm{supp}(\mathscr{A}, g))} = (is)|_{M \setminus J^+(\mathrm{supp}(\mathscr{A}, g))} \ .$$

By the uniqueness, the ψ above is unique for every s, and therefore R_A is as well. □

Remark 3.7 Notice that $R_A \colon \mathcal{E}(S(M)) \to \mathcal{E}(S_G(M))$ admits both a right and left inverse. Let us consider the map

$$\widehat{R}_A \doteq p + \mathcal{S}_- \circ p \circ A \tag{3.9}$$

and consider

$$R_A \circ \widehat{R}_A = \imath \circ p + \imath \circ \not{S}_- \circ p \circ A - \not{S}_-^G \circ A \circ \imath \circ p - \not{S}_-^G \circ A \circ \imath \circ \not{S}_- \circ p \circ A \,.$$

It is easy to see that $\imath \circ p = \mathrm{id}$ and $p \circ \imath = \mathrm{id}$ on $\mathcal{E}(S_G(M))$ and $\mathcal{E}(S(M))$ respectively, while by the previous proof we know that $A \circ \imath = \mathbf{D}^G \circ \imath - \imath \circ \mathbf{D}$; combining these observations we conclude that

$$R_A \circ \widehat{R}_A = \mathrm{id} + \imath \circ \not{S}_- \circ p \circ A - \not{S}_-^G \circ A - \not{S}_-^G \circ \Big(\mathbf{D}^G \circ \imath - \imath \circ \mathbf{D}\Big) \circ \not{S}_- \circ p \circ A \,.$$

Due to the properties of the propagators and of the support of A, using the fact that M is globally hyperbolic and the fact that $\not{S}_-^G \circ \mathbf{D}^G = \mathrm{id}_{\mathcal{E}_{\mathrm{pc}}(S_G(M))}$ we then infer that

$$\begin{aligned} R_A \circ \widehat{R}_A &= \mathrm{id}_{\mathcal{E}(S_G(M)} + \imath \circ \not{S}_- \circ p \circ A - \not{S}_-^G \circ A - \imath \circ \not{S}_- \circ p \circ A + \not{S}_-^G \circ \imath \circ p \circ A \\ &= \mathrm{id}_{\mathcal{E}(S_G(M))} \,. \end{aligned}$$

In the same way, using the fact that

$$\begin{aligned} (p \circ A)(s) &= i e^j(p) \cdot_{\Gamma,G} \left[\sigma(p), \rho_{G*}\Big(\left(\sigma_G^* \omega^G\right)_p (e_j(p))\Big) s^{\sharp_G}(\tilde{\sigma}(p)) \right] \\ &= p \circ \mathbf{D}^G - \mathbf{D} \circ p \end{aligned}$$

one can show that

$$\widehat{R}_A \circ R_A = \mathrm{id}_{\mathcal{E}(S(M))} \,.$$

We shall thus write $\widehat{R}_A$ as R_A^{-1}.

4 The Behaviour of the Classical Møller Map in the Case of a $U(1)$ Gauge Charge

4.1 The Classical Møller Map and Green Operators

If the gauge group G is $U(1)$ the entwining maps $\imath$ and p defined in Sect. 3.1 and the classical Møller map on field configurations R_A, whose explicit expression is given in Theorem 3.6, enjoy some further properties which we now explore.

First of all, notice that if we consider the hermitian inner products on $S(M)$ and $S_G(M)$, then i and p are the adjoint of one another: indeed,

$$\begin{aligned}
((it), s)_G\,(p) &= \left(\rho(\pi_2 \circ \widehat{\sigma}(p))(it)^{\sharp_G}(\widehat{\sigma}(p)),\, \rho(\pi_2 \circ \widehat{\sigma}(p))s^{\sharp_G}(\widehat{\sigma}(p))\right)_V \\
&= \Big(\Delta^{\mathbb{C}}_{r,1}\,(f_P \circ \widehat{\sigma}(p))\,\rho_G\,(f_G \circ \widehat{\sigma}(p))\,(it)^{\sharp_G}(\widehat{\sigma}(p)), \\
&\quad \times\, \Delta^{\mathbb{C}}_{r,1}\,(f_P \circ \widehat{\sigma}(p))\,\rho_G(f_G \circ \widehat{\sigma}(p))s^{\sharp_G}\,(\widehat{\sigma}(p))\Big)_V \\
&= \Big(\Delta^{\mathbb{C}}_{r,1}\,(f_P \circ \widehat{\sigma}(p))\,\rho_G(g(p))^{-1}t^{\sharp}(\sigma(p)), \\
&\quad \times\, \Delta^{\mathbb{C}}_{r,1}\,(f_P \circ \widehat{\sigma}(p))\,\rho_G(g(p))^{-1}s^{\sharp_G}\,(\tilde{\sigma}(p))\Big)_V \\
&= \Big(\Delta^{\mathbb{C}}_{r,1}\,(\pi_2 \circ \sigma(p))\,t^{\sharp}(\sigma(p)),\, \Delta^{\mathbb{C}}_{r,1}\,(\pi_2 \circ \sigma(p))\,s^{\sharp_G}\,(\tilde{\sigma}(p))\Big)_V \\
&= \Big(\Delta^{\mathbb{C}}_{r,1}\,(\pi_2 \circ \sigma(p))\,t^{\sharp}(\sigma(p)),\, \Delta^{\mathbb{C}}_{r,1}\,(\pi_2 \circ \sigma(p))\,(ps)^{\sharp}\,(\sigma(p))\Big)_V \\
&= (t, (ps),)\,(p)
\end{aligned}$$

where we have used the fact that $G = U(1)$ and the G-equivariance of the sections.

We now move on to the behaviour of the map $A\colon \mathcal{E}(S_G(M)) \to \mathcal{E}(S_G(M))$ defined in (3.6). First of all, recall that $\mathfrak{u}(n)$ consists of skew-hermitian matrices, and thus $\mathfrak{u}(1)$ consists of purely imaginary complex numbers; therefore, if we define the quantity

$$A_k(p) \doteq \rho_{G*}\left(\left((-\pi_G \circ g)^*\theta_G + (f_G \circ \widehat{\sigma})^*\omega^G_p\right)(e_k(p))\right)$$

we have

$$\begin{aligned}
((At), s)_G\,(p) &= ((At)(p), s(p))_G \\
&= \left(i(g^{kj}(p)e_j(p)) \cdot_V A_k(p)t^{\sharp_G}(\widehat{\sigma}(p)),\, s^{\sharp_G}(\widehat{\sigma}(p))\right)_V \\
&= -\left(A_k(p)t^{\sharp_G}(\widehat{\sigma}(p)),\, i(g^{kj}(p)e_j(p)) \cdot_V s^{\sharp_G}(\widehat{\sigma}(p))\right)_V \\
&= \left(t^{\sharp_G}(\widehat{\sigma}(p)),\, iA_k(p)\left((g^{kj}(p)e_j(p)) \cdot_V s^{\sharp_G}(\widehat{\sigma}(p))\right)\right)_V \\
&= \left(t^{\sharp_G}(\widehat{\sigma}),\, i(g^{kj}(p)e_j(p)) \cdot_V \left(A_k(p)s^{\sharp_G}(\widehat{\sigma})\right)\right)_V \\
&= (t(p), (As)(p))_G = ((As), t)_G\,(p)\,.
\end{aligned}$$

Let us then consider the formal adjoint ${R_A}^*\colon \mathcal{E}(\overline{S_G(M)}) \to \mathcal{E}(\overline{S(M)})$ of the classical Møller map on field configurations, which has the following behaviour:

$$\int_M ((R_A t), s)_G\,(p)\,d\mu_g = \int_M \left(t, ({R_A}^* s)\right)\,d\mu_g \qquad \text{for every } t \in \mathcal{D}(S(M)),\, s \in \mathcal{D}(\overline{S_G(M)})\,.$$

Notice that as $R_A : \mathcal{E}(S(M)) \to \mathcal{E}(S_G(M))$ is linear, using the antilinear isomorphisms between a vector bundle and its conjugate bundle

$$C : \overline{S(M)} \to S(M) \qquad C_G : \overline{S_G(M)} \to S_G(M)$$

we can naturally induce a linear map $\overline{R}_A : \mathcal{E}(\overline{S(M)}) \to \mathcal{E}(\overline{S_G(M)})$, $\overline{R}_A \doteq C_G^{-1} \circ R_A \circ C$, whose formal adjoint is given by $\overline{R}_A{}^* : \mathcal{E}(S_G(M)) \to \mathcal{E}(S(M))$. In particular, it holds that $\overline{R}_A{}^* = \overline{R_A^*}$. In light of the previous equalities, we can write

$$\overline{R}_A{}^* = p - p \circ A \circ \mathbin{\$\!\!\!/}_+^G \qquad R_A^* = C^{-1} \circ \overline{R}_A{}^* \circ C_G \; .$$

Notice that the antilinear isomorphisms can be used to define

$$\overline{\imath} : \mathcal{E}(\overline{S(M)}) \to \mathcal{E}(\overline{S_G(M)}) \qquad \overline{p} : \mathcal{E}(\overline{S_G(M)}) \to \mathcal{E}(\overline{S(M)}) \; .$$

These maps, as well as the map $\overline{R}_A : \mathcal{E}(\overline{S(M)}) \to \mathcal{E}(\overline{S_G(M)})$ satisfy the same results as the ones previously proven.

Proposition 4.1 *The advanced and retarded propagators of $\mathbf{D}^G$ and $\mathbf{D}$ are related by*

$$\mathbin{\$\!\!\!/}_-^G = R_A \circ \mathbin{\$\!\!\!/}_- \circ p \qquad \mathbin{\$\!\!\!/}_+^G = \imath \circ \mathbin{\$\!\!\!/}_+ \circ \overline{R}_A{}^* \; . \tag{4.1}$$

where the composition of maps appearing on the right-hand sides are restricted to $\mathcal{D}(S_G(M))$.

Proof Let us denote with $\widehat{\mathbin{\$\!\!\!/}}_\pm^G$ the operators on the right-hand sides of the equalities in (4.1). It is easy to see that on $\mathcal{D}(S_G(M))$ it holds that

$$\mathbf{D}^G \circ \widehat{\mathbin{\$\!\!\!/}}_-^G = \mathbf{D}^G \circ R_A \circ \mathbin{\$\!\!\!/}_- \circ p \overset{\text{Def. } 3.4}{=} \imath \circ \mathbf{D} \circ \mathbin{\$\!\!\!/}_- \circ p = \imath \circ p = \mathrm{id}$$

and

$$\begin{aligned}\widehat{\mathbin{\$\!\!\!/}}_-^G \circ \mathbf{D}^G &= R_A \circ \mathbin{\$\!\!\!/}_- \circ p \circ \mathbf{D}^G = R_A \circ \mathbin{\$\!\!\!/}_- \circ \mathbf{D} \circ R_A^{-1} = R_A \circ R_A^{-1} \\ &= \mathrm{id} \; .\end{aligned}$$

The same relations for $\widehat{\mathbin{\$\!\!\!/}}_+^G$ can be obtained in the following way: we know that $\mathbf{D}^{G*} = C_G^{-1} \circ \mathbf{D}^G \circ C_G$ and that $\widehat{\mathbin{\$\!\!\!/}}_-^{G*} = C_G^{-1} \circ \widehat{\mathbin{\$\!\!\!/}}_+^G \circ C_G$; therefore given any $u \in \mathcal{D}(S_G(M))$, $v \in \mathcal{D}(\overline{S_G(M)})$

$$\begin{aligned}\int_M (u, v)_G \, d\mu_g &= \int_M \left(\mathbf{D}^G \widehat{\mathbin{\$\!\!\!/}}_-^G u, v\right)_G d\mu_g = \int_M \left(\widehat{\mathbin{\$\!\!\!/}}_-^G u, C_G^{-1} \circ \mathbf{D}^G \circ C_G v\right)_G d\mu_g \\ &= \int_M \left(u, C_G^{-1} \circ \widehat{\mathbin{\$\!\!\!/}}_+^G \circ \mathbf{D}^G \circ C_G v\right) d\mu_g \; .\end{aligned}$$

As the above equality holds for any functions $u \in \mathcal{D}(S_G(M))$, $v \in \mathcal{D}(\overline{S_G(M)})$, we infer that $\widehat{\slashed{S}}^G_+ \circ \mathbf{D}^G = \mathrm{id}_{\mathcal{D}(S_G(M))}$. The other equality can be obtained in a similar fashion.

To conclude the proof, it suffices to show that

$$\mathrm{supp}(\widehat{\slashed{S}}^G_\pm(u)) \subseteq J^\mp(\mathrm{supp}(u)) \text{ for every } u \in \mathcal{D}(S_G(M)) \,.$$

We proceed for the retarded propagator $\widehat{\slashed{S}}^G_-$, as the proof for the advanced one is entirely analogous. In particular, we shall prove that

$$M \setminus J^+(\mathrm{supp}(u)) \subseteq M \setminus \mathrm{supp}(\widehat{\slashed{S}}^G_-(u)) \,.$$

To this end, it suffices to show that $M \setminus \mathrm{supp}(\slashed{S}^G_-(u)) \subseteq M \setminus \mathrm{supp}(\widehat{\slashed{S}}^G_-(u))$; therefore, let $p \notin \mathrm{supp}(\slashed{S}^G_-(u))$, and let us consider two Cauchy surfaces $\Sigma_1, \Sigma_2 \subseteq M$ such that

(i) $\Sigma_2 \subseteq J^+(\Sigma_1)$;
(ii) $\Sigma_1 \cap \Sigma_2 = \varnothing$;
(iii) $(\mathrm{supp}(\mathscr{A}) \cup \{p\} \cup \mathrm{supp}(u)) \cap J^+(\Sigma_1) = \varnothing$.

We then consider a smooth function $\varphi \in \mathcal{E}(M)$ such that

$$\varphi \equiv 1 \text{ on } J^-(\Sigma_1), \qquad \varphi \equiv 0 \text{ on } J^+(\Sigma_2)$$

which we use to define the maps

$$\begin{aligned} \varphi \colon \mathcal{E}(S(M)) &\to \mathcal{E}(S(M)) & \qquad \varphi_G \colon \mathcal{E}(S_G(M)) &\to \mathcal{E}(S_G(M)) \\ s &\mapsto (\varphi s)(p) \doteq \varphi(p)s(p) & t &\mapsto (\varphi_G t)(p) \doteq \varphi(p)t(p) \end{aligned}$$

as well as the analogous maps $1-\varphi$ and $1-\varphi_G$. Notice that $i \circ \varphi = \varphi_G \circ i$ and $p \circ \varphi_G = \varphi \circ p$ and that $\mathrm{supp}(\varphi \circ \slashed{S}_- \circ p(u))$ is compact. Then

$$\begin{aligned} (\widehat{\slashed{S}}^G_- u) &= (R_A \circ S_- \circ p)\,(u) = \big(R_A \circ (1-\varphi+\varphi) \circ \slashed{S}_- \circ p\big)\,(u) \\ &= \big(R_A \circ \varphi \circ \slashed{S}_- \circ p\big)\,(u) + \big(R_A \circ (1-\varphi) \circ \slashed{S}_- \circ p\big)\,(u) \\ &= (i - \slashed{S}^G_- \circ A \circ i) \circ (\varphi \circ \slashed{S}_- \circ p)(u) + \big(R_A \circ (1-\varphi) \circ \slashed{S}_- \circ p\big)\,(u) \\ &= \Big(\slashed{S}^G_- \circ \mathbf{D}^G \circ i \circ \varphi \circ \slashed{S}_- \circ p\Big)\,(u) - \Big(\slashed{S}^G_- \circ A \circ i \circ \varphi \circ \slashed{S}_- \circ p\Big)\,(u) \\ &\quad + \big(R_A \circ (1-\varphi) \circ \slashed{S}_- \circ p\big)\,(u) \\ &= \Big(\slashed{S}^G_- \circ i \circ \mathbf{D} \circ \varphi \circ \slashed{S}_- \circ p\Big)\,(u) + \big(R_A \circ (1-\varphi) \circ \slashed{S}_- \circ p\big)\,(u) \\ &= \Big(\slashed{S}^G_- \circ i \circ \mathbf{D}(\varphi) \circ \slashed{S}_- \circ p\Big)\,(u) + \Big(\slashed{S}^G_- \circ i \circ \varphi \circ \mathbf{D} \circ \slashed{S}_- \circ p\Big)\,(u) \\ &\quad + \big(R_A \circ (1-\varphi) \circ \slashed{S}_- \circ p\big)\,(u) \\ &= \Big(\slashed{S}^G_- \circ i \circ \mathbf{D}(\varphi) \circ \slashed{S}_- \circ p\Big)\,(u) + \slashed{S}^G_-(u) + \big(R_A \circ (1-\varphi) \circ \slashed{S}_- \circ p\big)\,(u) \,. \end{aligned}$$

Now, notice that when evaluated at p, the above expression is zero: indeed, we know that $p \notin \operatorname{supp}(\mathcal{S}_-^G(u))$, and due to the properties of $\varphi \in \mathcal{E}(M)$ we also know that $\mathbf{D}(\varphi) \equiv 0$ on $J^-(\Sigma_1)$ and $J^+(\Sigma_2)$, and therefore $(i \circ \mathbf{D}(\varphi) \circ S_- \circ p)(u)$ is supported in $J^+(\Sigma_1) \cap J^-(\Sigma_2)$. But then $\mathcal{S}_-^G((i \circ \mathbf{D}(\varphi) \circ \mathcal{S}_- \circ p)u)$ is supported in the causal future of that set; as $p \in J^-(\Sigma_1)$ we then have that the first term vanishes at p. Let us then consider the last term: we have

$$\big(i \circ (1-\varphi) \circ \mathcal{S}_- \circ p\big)(u) = \big((1-\varphi_G) \circ i \circ \mathcal{S}_- \circ p\big)(u)$$

which is equal to zero when evaluated at p, and

$$-\Big(\mathcal{S}_-^G \circ A \circ i \circ (1-\varphi) \circ \mathcal{S}_- \circ p\Big)(u) = -\Big(\mathcal{S}_-^G \circ A \circ (1-\varphi_G) \circ i \circ \mathcal{S}_- \circ p\Big)(u)$$

which vanishes as $\operatorname{supp}(\mathscr{A}) \cap \operatorname{supp}(1-\varphi) = \varnothing$. Thus $p \notin \operatorname{supp}(\widehat{\mathcal{S}}_-^G(u))$, and we conclude. □

Corollary 4.2 *The causal propagators of $\mathbf{D}^G$ and $\mathbf{D}$ are related by*

$$\mathcal{S}^G = R_A \circ \mathcal{S} \circ \overline{R_A}^* \Big|_{\mathcal{D}(S_G(M))} . \tag{4.2}$$

Proof Let us define $o \doteq -\mathcal{S}_- \circ p \circ A$; then

$$\begin{aligned} R_A - i &= R_A - R_A \circ R_A^{-1} \circ i = R_A \circ \left(\mathrm{id} - R_A^{-1} \circ i\right) \\ &= R_A \circ \big(p \circ i - \big(p + \mathcal{S}_- \circ p \circ A\big) \circ i\big) \\ &= R_A \circ (-\mathcal{S}_- \circ p \circ A \circ i) = R_A \circ o \circ i \,. \end{aligned}$$

Analogously, $R_A - i = i \circ o \circ R_A$; moreover, we also have the same relations concerning the formal adjoints:

$$\overline{R_A}^* - p = \overline{R_A}^* \circ o^\dagger \circ p \qquad \overline{R_A}^* - p = p \circ o^\dagger \circ \overline{R_A}^*$$

where $o^\dagger \doteq -A \circ i \circ \mathcal{S}_+$. Now, using these equalities we have on $\mathcal{D}(S_G(M))$

$$\begin{aligned} R_A \circ (\mathcal{S}_- - \mathcal{S}_+) \circ \overline{R_A}^* \; &= \Big(R_A \circ \mathcal{S}_- \circ (p + p \circ o^\dagger \circ \overline{R_A}^*) \\ &\qquad -(i + R_A \circ o \circ i) \circ \mathcal{S}_+ \circ \overline{R_A}^*\Big) \,. \end{aligned}$$

Proposition 4.1 then entails that on $\mathcal{D}(S_G(M))$ we have that

$$\begin{aligned} R_A \circ \mathcal{S} \circ \overline{R_A}^* &= \mathcal{S}_-^G + R_A \circ \mathcal{S}_- \circ p \circ (-A \circ i \circ \mathcal{S}_+) \circ \overline{R_A}^* - \mathcal{S}_+^G \\ &\qquad - R_A \circ o \circ i \circ \mathcal{S}_+ \circ \overline{R_A}^* \\ &= \mathcal{S}^G + R_A \circ o \circ i \circ \mathcal{S}_+ \circ \overline{R_A}^* - R_A \circ o \circ i \circ \mathcal{S}_+ \circ \overline{R_A}^* = \mathcal{S}^G \,. \end{aligned}$$

□

4.2 The Classical Møller Map and Hadamard Bidistributions

Having assessed the properties of the classical Møller map on field configurations when coupled with the Green operators associated to the free and uncharged Dirac operators $\mathbf{D}$ and $\mathbf{D}^G$, we now examine the behaviour of Hadamard bidistributions when coupled with the Møller maps. This is of particular signficance because of the twofold importance of Hadamard bidistribution in the context of perturbative algebraic quantum field theory: indeed, on one hand these are the 2-point functions of the analogous of vacuum states in curved spacetimes, and on the other (as shall be shown in Sect. 5.2) Hadamard bidistributions are the right tools to construct topological $*$-algebras of observables containing meaningful interactions and to construct Wick powers (see for instance [35, 36])

First of all, let us recall that the algebras associated to the Dirac field need to account for both the spinor and cospinor field, that is, sections of both the vector bundle $S(M)$ and its conjugate bundle $\overline{S(M)}$. In order to do so, one considers the Whitney sum of the two vector bundles $S(M) \oplus \overline{S(M)}$ and $S_G(M) \oplus \overline{S_G(M)}$, which we shall denote with $S^{\oplus}(M)$ and $S^{\oplus}_G(M)$ respectively. A section u of $S^{\oplus}(M)$ can be then understood as a couple (u_1, u_2) with $u_1 \in \mathcal{E}(S(M))$ and $u_2 \in \mathcal{E}(\overline{S(M)})$; the same holds for sections of $S^{\oplus}_G(M)$.

The hermitian metric on $S(M)$, which can be understood as a bilinear map

$$\mathcal{E}(S(M)) \times \mathcal{E}(\overline{S(M)}) \ni u, v \mapsto (u, v) \in \mathcal{E}(M)$$

can be used to induce a symmetric and bilinear map (denoted with the same symbol)

$$\mathcal{E}(S^{\oplus}(M)) \times \mathcal{E}(S^{\oplus}(M)) \ni u, v \mapsto (u, v) \doteq (v_1, u_2) + (u_1, v_2) \in \mathcal{E}(M) .$$

One can also define an involution on the space of sections $\mathcal{E}(S^{\oplus}(M))$ by using the conjugation maps $C \colon \overline{S(M)} \to S(M)$ and $C^{-1} \colon S(M) \to \overline{S(M)}$:

$$\mathcal{E}(S^{\oplus}(M)) \ni u = (u_1, u_2) \mapsto u^* \doteq (Cu_2, C^{-1}u_1) \in \mathcal{E}(S^{\oplus}(M)) .$$

Using the Dirac operator $\mathbf{D} \colon \mathcal{E}(S(M)) \to \mathcal{E}(S(M))$ and its adjoint $\mathbf{D}^* = C^{-1} \circ \mathbf{D} \circ C$ as well as the causal propagators $\mathcal{S} \colon \mathcal{D}(S(M)) \to \mathcal{E}(S(M))$ and $\mathcal{S}^*$ we construct the operators

$$\mathbf{D}^{\oplus} \doteq \mathbf{D} \oplus -\mathbf{D}^* \qquad \mathcal{S}^{\oplus} \doteq \mathcal{S} \oplus -\mathcal{S}^* .$$

Notice that $\not{S}^{\oplus}$, which is the causal propagator for $\mathbf{D}^{\oplus}$, is formally self-adjoint: indeed, given $u, v \in \mathcal{D}(S^{\oplus}(M))$ we have

$$\begin{aligned}\int_M (\not{S}^{\oplus}u, v)\, d\mu_g &= \int_M (\not{S}u_1, v_2) - (v_1, \not{S}^* u_2)\, d\mu_g \\ &= \int_M -(u_1, \not{S}^* v_2) + (\not{S}v_1, u_2)\, d\mu_g \\ &= \int_M (u, \not{S}^{\oplus} v)\, d\mu_g \,.\end{aligned}$$

Therefore, the distribution $\not{S}^{\oplus} \in \mathcal{D}'(S^{\oplus}(M) \boxtimes S^{\oplus}(M))$ given by the Schwartz kernel theorem,

$$\not{S}^{\oplus}(u, v) \doteq \int_M (\not{S}^{\oplus}u, v)\, d\mu_g$$

is symmetric. Analogous extensions can be made in the case of the charged spinor bundle $S_G(M)$.

The Møller map $R_A : \mathcal{E}(S(M)) \to \mathcal{E}(S_G(M))$ as well as its conjugate $\overline{R}_A : \mathcal{E}(\overline{S(M)}) \to \mathcal{E}(\overline{S_G(M)})$, which satisfy Definition 3.4 and Theorem 3.6 (with suitable modifications), can be combined into one Møller map $\mathcal{R}_A : \mathcal{E}(S^{\oplus}(M)) \to \mathcal{E}(S^{\oplus}_G(M))$,

$$\mathcal{R}_A \doteq R_A \oplus \overline{R}_A = i^{\oplus} - \not{S}^{G\oplus}_{-} \circ A^{\oplus} \circ i^{\oplus} \,.$$

The same can be done with the formal adjoints of the Møller maps, yielding

$$\mathcal{R}_A{}^* = \overline{R}^*_A \oplus R^*_A \,.$$

We now recall the definition of Hadamard bidistribution for spinor fields.

Definition 4.3 A distribution $\zeta \in \mathcal{D}'(S^{\oplus}(M) \boxtimes S^{\oplus}(M))$ satisfies the Hadamard two-point condition if given any two sections $u, v \in \mathcal{D}(S^{\oplus}(M))$ we have:

(i) $\zeta((\mathbf{D}^{\oplus}u), v) = 0$;
(ii) $\zeta(u, v) + \zeta(v, u) = i\not{S}^{\oplus}(u, v)$;
(iii) we require that

$$\mathrm{WF}(\zeta) = \Big\{(x, y, \xi_x, -\xi_y) \in T^*M^2 \setminus z(M^2) \mid (x, \xi_x) \sim (y, \xi_y) \\ \text{or } x = y, \xi_x = \xi_y,\ \xi_x \rhd 0\Big\}$$

where $\xi_x \rhd 0$ means that ξ_x is future-directed and lightlike, and $(x, \xi_x) \sim (y, \xi_y)$ means that x can be connected to y by means of a future-directed lightlike geodesic γ such that ξ_x is the cotangent vector of γ at x and ξ_y is the cotangent vector of γ at y.

Notice that such distributions do exist; see for instance [8] and [12].

Proposition 4.4 *If* $\zeta \in \mathcal{D}'(S^{\oplus}(M) \boxtimes S^{\oplus}(M))$ *is a distribution satisfying the Hadamard two-point condition, then*

$$\zeta_G(\cdot, \cdot) \doteq \zeta(\mathcal{R}_A{}^* \cdot, \mathcal{R}_A{}^* \cdot) \tag{4.3}$$

is a distribution in $\mathcal{D}'(S_G^{\oplus}(M) \boxtimes S_G^{\oplus}(M))$ *satisfying the Hadamard two-point condition.*

Proof First of all, notice that ζ_G defined as in (4.3) is well-defined: indeed, $\overline{R}_A^*$ and $R_A{}^*$ are continuous with respect to the inductive limit topology of $\mathcal{D}(S_G(M))$ and $\mathcal{D}(\overline{S_G(M)})$, being linear maps which are sequentially continuous. Moreover, they map compactly supported smooth functions to compactly supported smooth functions: indeed, if we consider the expression of $\overline{R}_A^*$ when acting on $u \in \mathcal{D}(S_G(M))$ we have

$$\overline{R}_A^* u = pu - (p \circ A \circ \mathcal{S}_+^G)(u)$$

Recall $\operatorname{supp}(\mathcal{S}_+^G(u)) \subseteq J^-(\operatorname{supp}(u))$; therefore we have that

$$\operatorname{supp}((A \circ \mathcal{S}_+^G)(u)) \subseteq J^-(\operatorname{supp}(u)) \cap J^+(\Sigma)$$

where Σ is a Cauchy surface such that $\operatorname{supp}(\mathscr{A}, g) \subseteq J^+(\Sigma)$, the support being past-compact. But it holds [26, Corollary A.5.4] that

$$J^-(\operatorname{supp}(u)) \cap J^+(\Sigma) \subset\subset M$$

and so $\operatorname{supp}((A \circ \mathcal{S}_+^G)(u))$ is compact. This result directly translates to the map $\mathcal{R}_A{}^*$. Let us now prove that the three requirements are satisfied:

(i) it is easy to see that $\zeta_G(\mathbf{D}^{G\oplus} u, v) = 0$; indeed, by Definition 3.4 and Theorem 3.6 we have

$$R_A^* \circ \mathbf{D}^{G*} = (\mathbf{D}^G \circ R_A)^* = (i \circ \mathbf{D})^* = \mathbf{D}^* \circ i^*$$

and therefore $\mathcal{R}_A{}^* \circ \mathbf{D}^{G\oplus} = (\mathbf{D} \circ p) \oplus (-\mathbf{D}^* \circ i^*)$; this entails that

$$\zeta_G(\mathbf{D}^{G\oplus} u, v) = \zeta\left(\mathcal{R}_A{}^*(\mathbf{D}^{G\oplus} u), \mathcal{R}_A{}^* v\right) = \zeta(\mathbf{D}^{\oplus}((p \oplus i^*)u), \mathcal{R}_A{}^* v) = 0\,.$$

(ii) We know that $\zeta_G(u,v) = \zeta(\mathcal{R}_A{}^* u, \mathcal{R}_A{}^* v)$, and by using Corollary 4.2 we have

$$
\begin{aligned}
\zeta_G(u,v) + \zeta_G(v,u) &= \zeta(\mathcal{R}_A{}^* u, \mathcal{R}_A{}^* v) + \zeta(\mathcal{R}_A{}^* v, \mathcal{R}_A{}^* u) \\
&= i \mathcal{S}^{\oplus}(\mathcal{R}_A{}^* u, \mathcal{R}_A{}^* v) \\
&= i \int_M \left(\mathcal{S}^{\oplus}(\mathcal{R}_A{}^* u), \mathcal{R}_A{}^* v \right) d\mu_g \\
&= i \int_M \left(\mathcal{S} \circ \overline{R}^*_A u_1, R^*_A v_2 \right) - \left(\overline{R}^*_A v_1, \mathcal{S}^* \circ R^*_A u_2 \right) d\mu_g \\
&= i \int_M \left((R_A \circ \mathcal{S} \circ \overline{R}^*_A) u_1, v_2 \right) \\
&\quad - \left(v_1, (\overline{R}_A \circ \mathcal{S}^* \circ R^*_A) u_2 \right) d\mu_g \\
&= i \int_M \left(\mathcal{S}^{G\oplus} u, v \right)_G d\mu_g = i \mathcal{S}^{G\oplus}(u,v) \, .
\end{aligned}
$$

(iii) We know that ζ_G is a bisolution of $\mathbf{D}^{G\oplus}$, and that the principal symbol of $\mathbf{D}^{G\oplus}$ coincides with that of $\mathbf{D}^{\oplus}$; therefore by [37, Theorem 6.1.1] we know that the wavefront set of ζ_G coincides with that of ζ, being determined by the Hamiltonian flow associated to the principal symbol of $\mathbf{D}^{G\oplus}$.

□

5 The Classical Møller Map on the Observable Algebras

5.1 The Poisson ∗-Algebras of Observables

Our goal is to pass the classical Møller map on field configurations R_A, presented in Definition 3.4 and Theorem 3.6, to the algebras of observables of the charged and uncharged Dirac field. We briefly recall the construction of said topological algebras, as presented in [7, 8, 38] and in Sect. 2 of [13].

Let $E \xrightarrow{\pi} M$ be a vector bundle with typical fiber V, be it either $S(M) \oplus \overline{S(M)}$ or $S_G(M) \oplus \overline{S_G(M)}$, endowed with a symmetric bilinear metric h, and let us consider the exterior algebra of $\mathcal{E}(E)$, that is, the graded algebra

$$
\wedge^{\bullet} \mathcal{E}(E) = \bigoplus_{p \in \mathbb{N}} \wedge^p \mathcal{E}(E) \, .
$$

We can consider the spaces of homogeneous elements $\wedge^p \mathcal{E}(E)$ as embedded into $\Gamma(M^p, E^{\boxtimes p}) \simeq \overline{\Gamma(M,E)^{\otimes p}}$; then using the usual Fréchet topology (uniform convergence of all derivatives on compact sets) on $\Gamma(M^p, E^{\boxtimes p})$ we define the spaces of p-antisymmetric sections

$$
\mathcal{E}^a(M^p, E^{\boxtimes p}) \doteq \overline{\wedge^p \mathcal{E}(E)}
$$

as well as the configuration space

$$\mathcal{C}(E) \doteq \widehat{\bigoplus}_{p\in\mathbb{N}} \mathcal{E}^a(M^p, E^{\boxtimes p})$$

where $\widehat{\bigoplus}$ denotes the algebraic direct sum. Notice that the involution defined in Sect. 4.2 can be extended to an involution $\cdot^*\colon \mathcal{C}(E) \to \mathcal{C}(E)$ by requiring the behaviour

$$(u_1 \wedge \cdots \wedge u_p)^* \doteq u_p^* \wedge \cdots \wedge u_1^*$$

on homogeneous elements in $\wedge^p \mathcal{E}(E)$, by extending the above by continuity to $\mathcal{E}^a(M^p, E^{\boxtimes p})$ and by linearity to $\mathcal{C}(E)$.

We are interested in *antisymmetric functionals* on the space of sections $\mathcal{E}(E)$; these can be interpreted as a sequence $\{F_p\}_{p\in\mathbb{N}}$ of linear and continuous functionals on $\{\wedge^p\mathcal{E}(E)\}_{p\in\mathbb{N}}$, that is, a sequence of elements such that

$$F_p \in \mathcal{F}^p(E) \doteq \mathcal{E}^{a\prime}(M^p, E^{\boxtimes p}) \text{ for all } p \in \mathbb{N}\,,$$

where the $(\cdot)'$ means the strong topological dual.

Thus we define the space of *fermionic functionals* as

$$\mathcal{F}(E) \doteq \prod_{p\in\mathbb{N}} \mathcal{F}^p(E)\,.$$

There exists a duality pairing between $\mathcal{F}(E)$ and $\mathcal{C}(E)$ given by

$$\langle F, u\rangle \doteq \sum_{p\in\mathbb{N}} \langle F_p, u_p\rangle \text{ for all } F \in \mathcal{F}(E),\ u \in \mathcal{C}(E)\,.$$

Notice that the sum is finite and thus always well-defined, $\mathcal{C}(E)$ being an algebraic direct sum. We can endow $\mathcal{F}(E)$ with the weak topology τ_σ, that is, the topology given by the family of seminorms $\{p_u\}_{u\in\mathcal{C}(E)},\ p_u(F) = |F(u)|$, thus making it a locally convex topological vector space which happens to be nuclear and sequentially complete.

$\mathcal{F}(E)$ can be endowed with an antisymmetric, pointwise product initially defined on homogeneous elements in $\wedge^p\mathcal{E}(E)$ by

$$(F \wedge G)_p(u_1 \wedge \cdots \wedge u_p) \doteq \sum_{\sigma\in S_p} \operatorname{sgn}(\sigma) \sum_{k=0}^{p} \frac{1}{k!(p-k)!} F_k(u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(k)})$$
$$G_{p-k}(u_{\sigma(k+1)} \wedge \cdots \wedge u_{\sigma(p)})\,.$$

The object above is then extended by linearity and continuity to elements in $\mathcal{E}^a(M^p, E^{\boxtimes p})$, thus yielding a well-defined object in $\mathcal{F}(E)$. Moreover, said product is continuous with respect to the topology on $\mathcal{F}(E)$. $\mathcal{F}(E)$ is also naturally endowed with an involution $\cdot^*\colon \mathcal{F}(E) \to \mathcal{F}(E)$,

$$\{F_p\}_{p\in\mathbb{N}} \mapsto \left\{F_p^*\right\}_{p\in\mathbb{N}}, \qquad (F_p^*)(u_p) \doteq \overline{F_p(u_p^*)} \,. \tag{5.1}$$

One can then consider derivatives of fermionic functionals in the following way: given $F_p \in \mathcal{F}^p(E)$, $p \geq 1$, the left derivative of F_p in the direction $h \in \mathcal{E}(E)$ is defined on $\wedge^{p-1}\mathcal{E}(E)$ as

$$d_h F_p(u) \doteq F_p(h \wedge u)$$

and is then extended by continuity to $\mathcal{E}^a(M^{p-1}, E^{\boxtimes p-1})$, thus yielding a linear and continuous map $d_h F_p\colon \mathcal{E}^a(M^{p-1}, E^{\boxtimes p-1}) \to \mathbb{C}$, i.e. $d_h F_p \in \mathcal{F}^{p-1}(E)$. One can then extend the map $d_h\colon \mathcal{F}^p(E) \to \mathcal{F}^{p-1}(E)$ to the whole algebra $\mathcal{F}(E)$ by considering

$$d_h\colon F = \{F_p\}_{p\in\mathbb{N}} \mapsto d_h F \doteq \{d_h F_p\}_{p\in\mathbb{N}} \,.$$

It is easy to see that for any $h \in \mathcal{E}(E)$, d_h is a graded derivation. One can also consider higher order derivatives by iterating the left derivative: given $F_p \in \mathcal{F}^p(E)$, $k \leq p$ and $h_1, \ldots, h_k \in \mathcal{E}(E)$, we define for $u \in \wedge^{p-k}\mathcal{E}(E)$

$$d^k_{h_1,\ldots,h_k} F_p(u) = F_p(h_k \wedge \cdots \wedge h_1 \wedge u)$$

and then proceed by continuity as before, obtaining for each $F \in \mathcal{F}(E)$ a jointly continuous map

$$d^k F\colon \mathcal{E}(E)^k \times \mathcal{C}(E) \to \mathbb{C}$$

which is easily seen to be multilinear and alternating in the first k entries, that is, equivalently, a continuous map

$$F^{(k)}\colon \mathcal{E}^a(M^k, E^{\boxtimes k}) \times \mathcal{C}(E) \to \mathbb{C}$$

which is linear in the first entry. In particular, notice that these can be considered as an $\mathcal{F}(E)$-valued (compactly supported) distributional section of $E^{*\boxtimes k} \to M^k$, that is, an object of $\mathcal{D}'(M^k, E^{\boxtimes k})\widehat{\otimes}_\pi \mathcal{F}(E)$, where $\widehat{\otimes}_\pi$ denotes the completion of the tensor product in the projective topology[1].

In order to proceed with the quantization, one needs to restrict the $*$-algebra of fermionic functionals in order to endow it with a suitable $\star$-product. To do so, one needs to be able to control the wavefront set of derivatives of the relevant functionals;

[1] As both spaces are nuclear, the completion is independent on the chosen topology; we choose the projective topology just to fix one.

in particular, we consider the set of *microcausal* fermionic functionals $\mathcal{A}(E) \subseteq \mathcal{F}(E)$ consisting of those functionals $F \in \mathcal{F}(E)$ such that

$$\mathrm{WF}(F_u^{(n)}) \subseteq \Xi_n \doteq T^*\dot{M}^n \setminus \bigcup_{(p_1,\dots,p_n)\in M^n} \left(\overline{V_{p_1}^+} \times \cdots \times \overline{V_{p_n}^+}\right) \cup \left(\overline{V_{p_1}^-} \times \cdots \times \overline{V_{p_n}^-}\right)$$

for every $u \in \mathcal{C}(E)$, where with $\overline{V_p^\pm}$ we denote the closure of the future/past causal cone in T_p^*M. We endow this space with the initial locally convex topology induced by the family of linear maps $\{\ell_{k,u}\}$,

$$F \xmapsto{\ell_{k,u}} \begin{cases} \langle F, u\rangle \in \mathbb{C} & k = 0 \\ F_u^{(k)} \in \mathcal{E}^a_{\Xi_k}{}'(M^k, E^{\boxtimes k}) & k \geq 1 \end{cases}$$

indexed by an integer $k \in \mathbb{N}$ and a function in the configuration space $u \in \mathcal{C}(E)$, and where $\mathcal{E}^a_{\Xi_k}{}'(M^k, E^{\boxtimes k})$ denotes the inductive limit

$$\varinjlim \mathcal{E}^a_{\Gamma_{k,n}}{}'(M^k, E^{\boxtimes k})$$

with $\{\Gamma_{k,n}\}_{n\in\mathbb{N}}$ a sequence of closed cones in T^*M^k such that $\Gamma_{k,n} \subset \mathring{\Gamma}_{k,n+1}$ and $\cup_n \Gamma_{k,n} = \Xi_k$, and where $\mathcal{E}^a_{\Gamma_{k,n}}{}'(M^k, E^{\boxtimes k})$ is endowed with the usual Hörmander topology. Using the causal propagator $\mathcal{S} \in \mathcal{D}'(E^{\boxtimes 2}, M^2)$ one is then able to endow $\mathcal{A}(E)$ with a Peierls' bracket. First of all, given any homogeneous functional $F \in \mathcal{A}^p(E)$, we define the object $(\mathcal{S} * F^{(1)})$,

$$(\mathcal{S} * F^{(1)})(u) \doteq \int_M (\mathcal{S}(x, y), F_u^{(1)}(y))_G \, d\mu_g(y) \tag{5.2}$$

which is well-defined thanks to the wavefront set properties of both the causal propagator $\mathcal{S}$ and of $F_u^{(1)}$, as

$$\mathrm{WF}(\mathcal{S}) = \left\{(x, y, \xi_x, -\xi_y) \in T^*M^2 \setminus z(M^2) \mid (x, \xi_x) \sim (y, \xi_y)\right\} .$$

Moreover, notice that this object is actually a smooth function: indeed, using [17, Theorem 8.2.13] one obtains that

$$\mathrm{WF}((\mathcal{S} * F_u^{(1)})) \subseteq \mathrm{WF}_M(\mathcal{S}) \cup \mathrm{WF}'(\mathcal{S}) \circ \mathrm{WF}(F_u^{(1)}) = \varnothing \text{ for all } u \in \mathcal{E}^a(M^p, E^{\boxtimes p}) .$$

Therefore, we can consider it as an object in $\mathcal{E}(E^*) \simeq \mathcal{E}(E)$, where the isomorphism is due to the existence of the symmetric bilinear metric h. We can then compute, for any other homogeneous functional $G \in \mathcal{A}^q(E)$, the object

$$G^{(1)} \wedge (\mathcal{S} * F^{(1)}) \in \mathcal{A}^{p+q-2}(E)$$

which is defined on an element $u_1 \wedge \cdots \wedge u_{p+q-2}$

$$\begin{aligned}
&G^{(1)} \wedge (\not{\$} * F^{(1)})(u_1 \wedge \cdots \wedge u_{p+q-2}) \\
&\quad \doteq (-1)^{q+1} \sum_{\sigma \in S_{p+q-2}} \operatorname{sgn}(\sigma) G\left((\not{\$} * F^{(1)}_{u_{\sigma(q)} \wedge \cdots \wedge u_{\sigma(q+p-2)}}) \wedge u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(q-1)}\right)
\end{aligned}$$

and then extended to the whole of $\mathcal{E}^a(M^{p+q-2}, E^{\boxtimes p+q-2})$ by continuity. This procedure is then extended to non-homogeneous functionals by considering $G^{(1)} \wedge (\not{\$} * F^{(1)}) = \left\{\left(G^{(1)} \wedge (\not{\$} * F^{(1)})\right)_p\right\}_{p \in \mathbb{N}}$,

$$\left(G^{(1)} \wedge (\not{\$} * F^{(1)})\right)_p (u) \doteq \sum_{k=0}^{p} \frac{1}{k!(p-k)!} \left((G^{(1)})_k \wedge \left(\not{\$} * (F^{(1)})_{p-k}\right)\right)(u) \tag{5.3}$$

for $u \in \mathcal{E}^a(M^p, E^{\boxtimes p})$. Then the Peierls' bracket is given by $\{F, G\}_{\not{\$}} \doteq G^{(1)} \wedge (\not{\$} * F^{(1)})$; notice that on homogeneous elements $F \in \mathcal{A}^p(E)$, $G \in \mathcal{A}^q(E)$ we have the desired graded anticommutativity

$$\{G, F\}_{\not{\$}} = -(-1)^{qp} \{F, G\}_{\not{\$}}$$

as well as the graded Jacobi identity.

5.2 Deformation Quantization and the Classical Møller Maps

Starting from the Poisson algebra $(\mathcal{A}(E), \{\cdot, \cdot\}_{\not{\$}})$, the quantization proceeds in the following way: we consider the $*$-algebra $\mathcal{A}(E)[[\hbar]]$ of formal power series in $\hbar$ with coefficients in $\mathcal{A}(E)$, endowed with the product topology; clearly $\mathcal{A}(E) \subseteq \mathcal{A}(E)[[\hbar]]$. One is then able to introduce a $\star$-product on $\mathcal{A}(E)[[\hbar]]$, that is, a product such that given $F \in \mathcal{A}^p(E)$, $G \in \mathcal{A}^q(E)$

$$F \star G = F \wedge G + o(\hbar) \qquad F \star G - (-1)^{pq} G \star F = i\hbar \{F, G\}_{\not{\$}} + o(\hbar^2)\,.$$

Given a Hadamard bidistribution $\zeta \in \mathcal{D}'(M^2, E^{\boxtimes 2})$ (see Definition 4.3) and two functionals $F, G \in \mathcal{A}(E)$, we consider the fermionic functional

$$\Gamma^n_\zeta(G, F) \doteq \left(\frac{i}{2}\right)^n G^{(n)} \wedge \left(\zeta^{\boxtimes n} * F^{(n)}\right) \ \forall\, n \in \mathbb{N}\,.$$

where the quantity on the right is defined component-wise as in (5.3) and $\zeta^{\boxtimes n} * F^{(n)}$ is defined as in (5.2). Notice that due to the wavefront set properties of both $F^{(n)}$ and

$\zeta^{\boxtimes n}$ the object above is well-defined. Then we define

$$G \star F \doteq \sum_{n\in\mathbb{N}} \hbar^n \Gamma^n_\zeta(G, F) \in \mathscr{A}(E)[\![\hbar]\!] \,. \tag{5.4}$$

This product can then be easily extended to the whole topological $*$-algebra of formal power series. Now, let $\mathscr{A}(S)[\![\hbar]\!]$ and $\mathscr{A}(S_G)[\![\hbar]\!]$ be the topological $*$-algebras of microcausal fermionic functionals associated to the free and charged Dirac fields. Suppose that the $\star_G$-product on $\mathscr{A}(S_G)[\![\hbar]\!]$ is constructed using the Hadamard bidistribution induced by the one used to define $\star$-product on $\mathscr{A}(S)[\![\hbar]\!]$, as illustrated in Proposition 4.4. We define the *classical Møller map* $\mathscr{R}_A\colon \mathscr{A}(S_G)[\![\hbar]\!] \to \mathscr{A}(S)[\![\hbar]\!]$ by considering the pullbacks induced by $\mathcal{R}_A^{\wedge p}\colon \mathcal{E}^a(M^p, S^\oplus(M)^{\boxtimes p}) \to \mathcal{E}^a(M^p, S^\oplus_G(M)^{\boxtimes p})$, where $\mathcal{R}_A^{\wedge p}$ is the continuous extension of the map

$$\mathcal{R}_A^{\wedge p}(u_1 \wedge \cdots \wedge u_p) \doteq \mathcal{R}_A(u_1) \wedge \cdots \wedge \mathcal{R}_A(u_p)$$

to $\mathcal{E}^a(M^p, S^\oplus(M)^{\boxtimes p})$. Now,

Theorem 5.1 *$\mathscr{R}_A$ is a well-defined $*$-isomorphism, algebraically and topologically.*

Proof First of all, we need to show that given a functional $F \in \mathscr{A}(S_G) \subseteq \mathscr{A}(S_G)[\![\hbar]\!]$, $\mathscr{R}_A(F)$ is a well-defined functional in $\mathscr{A}(S)[\![\hbar]\!]$; that is, we need to show that

$$\mathrm{WF}\left((\mathscr{R}_A(F))^{(n)}_u\right) \subseteq \Xi_n \text{ for all } u \in \mathcal{C}(S^\oplus(M)) \text{ and } n \in \mathbb{N}. \tag{5.5}$$

To do so, we need to compute the wavefront set of the classical Møller map on field configurations $\mathcal{R}_A$, restricted to a continuous map $\mathcal{R}_A\colon \mathcal{D}(S^\oplus(M)) \to \mathcal{D}'(S^\oplus_G(M))$. This is defined as the wavefront set of the distribution r_A, where $r_A \in \mathcal{D}'(S^\oplus(M) \boxtimes S^\oplus_G(M))$ is obtained thanks to Schwartz's kernel theorem and satisfies

$$\int_M (\mathcal{R}_A u, v)_G \, d\mu_g = r_A(u, v) \text{ for every } u \in \mathcal{D}(S^\oplus(M)),\, v \in \mathcal{D}(S^\oplus_G(M)) \,.$$

To compute its wavefront set, we proceed locally, as presented in [17]. Namely, let $\{e_i\}_{1\le i\le \mathrm{rank}(S_G(M))}$ be a local[2] frame for $S_G(M)$ and $\{f_j\}_{1\le j\le \mathrm{rank}(S(M))}$ a local frame for $S(M)$; a local frame for $S^\oplus(M)$ and $S^\oplus_G(M)$ is then given by

$$\{f_1, \ldots, f_{\mathrm{rank}(S(M))}, \overline{f}_1, \ldots, \overline{f}_{\mathrm{rank}(S(M))}\} \quad \{e_1, \ldots, e_{\mathrm{rank}(S_G(M))}, \overline{e}_1, \ldots, \overline{e}_{\mathrm{rank}(S_G(M))}\} \tag{5.6}$$

respectively. Using these, we can write locally

$$r_A = r_{A\,ji} f^j \boxtimes e^i$$

[2] Notice that as we are supposing that the principal bundle P_G is trivial, there exist *global* frames for both $S(M)$ and $S_G(M)$.

with $r_{Aij} \in \mathcal{D}'(M^2)$ and where $\{e^i\}_i$ and $\{f^j\}_j$ denote the dual frames of $S^{\oplus}_G(M)^* \simeq \overline{S_G(M)} \oplus S_G(M)$ and $S^{\oplus}(M)^* \simeq \overline{S(M)} \oplus S(M)$ with respect to those in (5.6). Then

$$\mathrm{WF}(\mathcal{R}_A) \doteq \mathrm{WF}(r_A) = \bigcup_{\substack{1 \le i \le 2\mathrm{rank}(S_G(M)) \\ 1 \le j \le 2\mathrm{rank}(S(M))}} \mathrm{WF}\left(r_{A\,ji}\right)$$

where $\mathrm{WF}(r_{A\,ji})$ is given locally by

$$\widehat{\phi}_\alpha^{-1}(\mathrm{WF}(\phi_\alpha^* r_{A\,ji}))$$

with $\widehat{\phi}_\alpha : T^*M^2_{U_\alpha} \to U_\alpha^2 \times \mathbb{R}^{\dim(M^2)}$. Moreover, due to the properties of the wavefront set, we can study $\mathrm{WF}(\mathcal{R}_A)$ by studying separately the two terms $\iota^\oplus$ and $\not{S}^\oplus_- \circ A^\oplus \circ \iota^\oplus$. Let us thus first consider the map $\iota^\oplus : \mathcal{D}(S^\oplus(M)) \to \mathcal{D}'(S^\oplus_G(M))$; then we have that

$$\begin{aligned}
\int_M \left(\iota^\oplus u, v\right)_G \, d\mu_g &= \int_M (\iota u_1, v_2)_G + \left(v_1, \bar{\iota} u_2\right)_G \, d\mu_g \\
&= \int_M \left(u_1^\sharp(\sigma(p)), v_2^{\sharp_G}(\tilde{\sigma}(p))\right)_V \, d\mu_g \\
&\quad + \int_M \left(v_1^{\sharp_G}(\tilde{\sigma}(p)), u_2^\sharp(\sigma(p))\right)_V \, d\mu_g \\
&= \int_M u_1^{\sharp\,j}(\sigma(p)) h_{ji} v_2^{\sharp_G\,i}(\tilde{\sigma}(p)) \, d\mu_g \\
&\quad + \int_M v_1^{\sharp_G\,i}(\tilde{\sigma}(p)) h_{ij} u_2^{\sharp\,j}(\sigma(p)) \, d\mu_g
\end{aligned}$$

that is,

$$\iota_{ji} = \begin{cases} \delta_\Delta h_{ji} & j \le \mathrm{rank}(S(M)),\, i \le \mathrm{rank}(S_G(M)) \\ \delta_\Delta \overline{h}_{(j-\mathrm{rank}(S(M)))(i-\mathrm{rank}(S_G(M)))} & j > \mathrm{rank}(S(M)),\, i > \mathrm{rank}(S_G(M)) \end{cases} .$$

Therefore,

$$\mathrm{WF}(\iota^\oplus) = \left\{(x, x, k, -k) \in T^*M^2 \setminus z(M^2)\right\} .$$

As far as the map $\not{S}^\oplus_- \circ A^\oplus \circ \iota^\oplus$ is concerned, we just need to compute the wavefront set of $A^\oplus$. In particular, by considering a gauge choice $\widehat{\sigma} : M \to P_{\mathrm{Spin}_{r,1}+P_G}$ and defining

$$A_k(p) \doteq \rho_{G*}\left(\left((-\pi_G \circ g)^* \theta_G + (f_G \circ \widehat{\sigma})^* \omega_p^G\right)(e_k(p))\right)$$

we can proceed as done in the case of $\imath^{\oplus}$:

$$\begin{aligned}\int_M (A^{\oplus}u, v)_G \, d\mu_g &= \int_M \Big(i(g^{kj}(p)e_j(p)) \cdot_V A_k(p)u_1^{\sharp_G}(\widehat{\sigma}(p)), v_2^{\sharp_G}(\widehat{\sigma}(p))\Big)_V \, d\mu_g \\ &\quad + \int_M \Big(v_1^{\sharp_G}(\widehat{\sigma}(p)), \overline{i(g^{kj}(p)e_j(p))} \cdot_V \Big(\overline{A_k(p)}u_2^{\sharp_G}(\widehat{\sigma}(p))\Big)\Big)_V \, d\mu_g \\ &= \int_M \Big(i(g^{kj}(p)e_j(p)) \cdot_V A_k(p)\Big)^l_{\ j} u_1^{\sharp_G\, j}(\widehat{\sigma}(p))h_{li} v_2^{\sharp_G\, i}(\widehat{\sigma}(p)) \, d\mu_g \\ &\quad + \int_M \Big(\overline{i(g^{kj}(p)e_j(p)) \cdot_V A_k(p)}\Big)^l_{\ i} v_1^{\sharp_G\, i}(\widehat{\sigma}(p))u_2^{\sharp_G\, j}(\widehat{\sigma}(p))h_{lj} \; .\end{aligned}$$

Thus $A_{ji} = \delta_\Delta f_{ji}$, f_{ji} given by

$$f_{ji} = \begin{cases} \Big(i(g^{kj}(p)e_j(p)) \cdot_V A_k(p)\Big)^l_{\ j} h_{li} & j, i \leq d \\ \Big(\overline{i(g^{ij}(p)e_j(p)) \cdot_V A_k(p)}\Big)^l_{\ i-d} \overline{h}_{(j-d)l} & j, i > d \,, \end{cases}$$

with $d = \mathrm{rank}(S_G(M))$. Using the fact that $(f_G \circ \widehat{\sigma})^* \omega^G$ is assumed to be supported in $\mathrm{supp}(\mathscr{A}, g)$, we then have that

$$\begin{aligned}\mathrm{WF}(A^{\oplus}) &\subseteq \Big(\pi^{-1}_{T^*M^2}(\mathrm{supp}(\mathscr{A}, g) \times M) \setminus z(M^2)\Big) \cap \Big\{(x, x, k, -k) \in T^*M^2 \setminus z(M^2)\Big\} \\ &= \Big\{(x, x, k, -k) \in T^*M^2 \setminus z(M^2) \mid x \in \mathrm{supp}(\mathscr{A}, g)\Big\} \; .\end{aligned}$$

We are now in a position to compute $\mathrm{WF}(\mathscr{R}_A)$. First of all, notice that the composition $A^{\oplus} \circ \imath^{\oplus}$ is, as we expected, well-defined: indeed, as $\mathrm{supp}(\imath^{\oplus}) \subseteq \Delta$, we have that $\mathrm{supp}(\imath^{\oplus}) \ni (x, y) \mapsto y$ is proper, and moreover

$$\begin{aligned}\mathrm{WF}'(\imath^{\oplus})_M &= \big\{(y, \xi_y) \mid (x, y, 0, -\xi_y) \in \mathrm{WF}(\imath^{\oplus})\big\} = \varnothing \\ \mathrm{WF}(A^{\oplus})_M &= \big\{(x, \xi_x) \mid (x, y, \xi_x, 0) \in \mathrm{WF}(A^{\oplus})\big\} = \varnothing \; .\end{aligned}$$

Thus $\mathrm{WF}'(\imath^{\oplus})_M \cap \mathrm{WF}(A^{\oplus})_M = \varnothing$, and [17, Theorem 8.2.14] gives us the well-posedness of $A^{\oplus} \circ \imath^{\oplus}$ as well as

$$\mathrm{WF}(A^{\oplus} \circ \imath^{\oplus}) \subseteq \Big\{(x, x, k, -k) \in T^*M^2 \setminus z(M^2) \mid x \in \mathrm{supp}(\mathscr{A}, g)\Big\} \; .$$

Let us now consider $\mathcal{S}_-^{G\oplus} \circ A^{\oplus} \circ \imath^{\oplus}$; thanks to [39] we know that

$$\begin{aligned}\mathrm{WF}(\mathcal{S}_-^{G\oplus}) = \Big\{&(x, y, \xi_x, -\xi_y) \in T^*M^2 \setminus z(M^2) \mid x \in J^+(y), (x, \xi_x) \sim (y, \xi_y) \\ &\text{or } x = y, \xi_x = \xi_y\Big\} \; .\end{aligned}$$

As evidently $\mathrm{WF}'(A^{\oplus} \circ \imath^{\oplus})_M = \varnothing$ and $\mathrm{supp}(A^{\oplus} \circ \imath^{\oplus}) \ni (x, y) \mapsto y$ is proper, we can then apply again [17, Theorem 8.2.14] and we have that $\mathrm{WF}'(\mathcal{S}_-^{G\oplus} \circ A^{\oplus} \circ \imath^{\oplus})$ is contained in

$$\mathrm{WF}'(\slashed{S}_{-}^{G\oplus})\circ \mathrm{WF}'(A^{\oplus}\circ\iota^{\oplus})\cup\left(\mathrm{WF}(\slashed{S}_{-}^{G\oplus})_M\times M\times\{0\}\right)\cup\left(M\times\{0\}\times\mathrm{WF}'(A^{\oplus}\circ\iota^{\oplus})_M\right) .$$

The first set is given by

$$\left\{(x,z,\xi_x,\xi_z)\mid\exists(y,\xi_y)\text{ s.t. }(x,y,\xi_x,-\xi_y)\in\mathrm{WF}(\slashed{S}_{-}^{G\oplus}),\ (y,z,\xi_y,-\xi_z)\in\mathrm{WF}(A^{\oplus}\circ\iota^{\oplus})\right\} .$$

Using the properties of $\mathrm{WF}(A^{\oplus}\circ\iota^{\oplus})$ we conclude that the previous set is given by

$$\left\{(x,z,\xi_x,\xi_z)\mid(x,z,\xi_x,-\xi_z)\in\mathrm{WF}(\slashed{S}_{-}^{G}),\ z\in\mathrm{supp}(\mathscr{A},g)\right\} .$$

The second one and the third one are easily seen to be empty; therefore

$$\begin{aligned}\mathrm{WF}(\slashed{S}_{-}^{G\oplus}\circ A^{\oplus}\circ\iota^{\oplus})\ &\subseteq\{(x,z,\xi_x,-\xi_z)\mid(x,z,\xi_x,-\xi_z)\\&\qquad\in\ \mathrm{WF}(\slashed{S}_{-}^{G\oplus}),\ z\in\mathrm{supp}(\mathscr{A},g)\right\} .\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{WF}(\mathcal{R}_A)\subseteq&\left\{(x,z,\xi_x,-\xi_z)\mid(x,z,\xi_x,-\xi_z)\in\mathrm{WF}(\slashed{S}_{-}^{G\oplus}),\ z\in\mathrm{supp}(\mathscr{A},g)\right\}\\&\cup\left\{(x,x,k,-k)\in T^*M^2\setminus z(M^2)\right\} .\end{aligned}$$

As the map $(\mathscr{R}_A(F))_u^{(n)}$ involves the composition of $F_{\mathcal{R}_A u}^{(n)}$ with the map $\mathcal{R}_A{}^{\wedge n}$, the last step consists in computing $\mathrm{WF}(\mathcal{R}_A^{\wedge n})$. This, according to [17, Theorem 8.2.9], is a subset of

$$\begin{aligned}&\Big\{(x_1,y_1,x_2,y_2,\dots,x_n,y_n,\xi_1,\eta_1,\xi_2,\eta_2,\dots,\xi_n,\eta_n)\mid\exists I\subseteq\{1,\dots,n\},\ I\neq\varnothing\text{ s.t.}\\&\ (x_i,y_i,\xi_i,\eta_i)\in\mathrm{WF}(\mathcal{R}_A)\ \forall i\in I\text{ and }(x_j,y_j,\xi_j,\eta_j)=(x_j,y_j,0,0)\\&\ \text{with }(x_j,y_j)\in\mathrm{supp}(r_A)\ \forall j\in\{1,\dots,n\}\setminus I\Big\} .\end{aligned}$$

Finally, we are able to check whether $\mathscr{R}_A(F)$ is well-defined. First of all, notice that $(\mathscr{R}_A(F))_u^{(n)}=F_{\mathcal{R}_A u}^{(n)}\circ\mathcal{R}_A{}^{\wedge n}$ is well-defined: indeed, thanks to [17, Theorem 8.2.13] we know that the composition is well-defined if $\mathrm{WF}(F_{\mathcal{R}_A u}^{(n)})\cap\mathrm{WF}'(\mathcal{R}_A{}^{\wedge n})_{M^n}=\varnothing$; but

$$\left\{(x_1,\dots,x_n,\xi_1,\dots,\xi_n)\mid(x_1,y_1,\dots,x_n,y_n,-\xi_1,0,\dots,-\xi_n,0)\in\mathrm{WF}(\mathcal{R}_A{}^{\wedge n})\right\}=\varnothing .$$

We also infer that

$$\mathrm{WF}\left((\mathscr{R}_A(F))_u^{(n)}\right)\subseteq\mathrm{WF}(\mathcal{R}_A{}^{\wedge n})_{M^n}\cup\mathrm{WF}'(\mathcal{R}_A{}^{\wedge n})\circ\mathrm{WF}(F_{\mathcal{R}_A u}^{(n)})$$

i.e.

$$\mathrm{WF}\left((\mathscr{R}_A(F))_u^{(n)}\right) \subseteq \Big\{(x_1,\ldots,x_n,\xi_1,\ldots,\xi_n) \mid (x_1,y_1,\ldots,x_n,y_n,\xi_1,-\eta_1,\ldots,\xi_n,-\eta_n) \in \mathrm{WF}(\mathscr{R}_A^{\wedge n}) \text{ for some } (y_1,\ldots,y_n,\eta_1,\ldots,\eta_n) \in \mathrm{WF}(F^{(n)}_{\mathscr{R}_A u})\Big\}\ .$$

Using this, we shall prove (5.5) by contradiction. Assume then that $(x_1,\ldots,x_n,\xi_1,\ldots,\xi_n) \in \overline{V^+_{x_1}} \times \cdots \times \overline{V^+_{x_n}}$; then there exists $(y_1,\ldots,y_n,\eta_1,\ldots,\eta_n) \in \mathrm{WF}(F^{(n)}_{\mathscr{R}_A u})$ such that

$$(x_1,y_1,\ldots,x_n,y_n,\xi_1,-\eta_1,\ldots,\xi_n,-\eta_n) \in \mathrm{WF}({\mathscr{R}_A}^{\wedge n})\ .$$

If $i \in I$, then $(x_i,y_i,\xi_i,-\eta_i) \in \mathrm{WF}(\mathscr{R}_A)$, that is, either

$$(x_i,y_i,\xi_i,-\eta_i) \in \mathrm{WF}(\mathcal{S}_-^{G\oplus}) \text{ with } y_i \in \mathrm{supp}(\mathscr{A},g)$$

or

$$x_i = y_i \qquad \eta_i = \xi_i\ .$$

In the first case, we would then need to have $\eta_i \in \overline{V^+_{y_i}}$, as η_i is the cotangent vector to a future-directed lightlike geodesic, while in the second one the same result follows from the equality $\eta_i = \xi_i$. If $i \notin I$, then

$$(x_i,y_i,\xi_i,\eta_i) = (x_i,y_i,0,0)$$

i.e. $\eta_i = 0 \in \overline{V^+_{y_i}}$. Thus, we conclude that $(y_1,\ldots,y_n,\eta_1,\ldots,\eta_n) \in \overline{V^+_{y_1}} \times \cdots \overline{V^+_{y_n}}$; but we reached a contradiction, as this is not possible by the definition of F. The case $(x_1,\ldots,x_n,\xi_1,\ldots,\xi_n) \in \overline{V^-_{x_1}} \times \cdots \times \overline{V^-_{x_n}}$ leads to a similar conclusion: indeed, if $i \in I$ then $(x_i,y_i,\xi_i,-\eta_i) \in \mathrm{WF}(\mathscr{R}_A)$, which as before entails that either

$$(x_i,y_i,\xi_i,-\eta_i) \in \mathrm{WF}(\mathcal{S}_-^{G\oplus}) \text{ with } y_i \in \mathrm{supp}(\mathscr{A},g)$$

or

$$x_i = y_i \qquad \xi_i = \eta_i\ .$$

The first case is not possible, as we require $(x_i,\xi_i) \sim (y_i,\eta_i)$ which is not possible if ξ_i is past-directed; thus from the second case we infer that $(y_1,\ldots,y_n,\eta_1,\ldots,\eta_n) \in \overline{V^-_{y_1}} \times \cdots \overline{V^-_{y_n}}$, which is again a contradiction.

We thus have the well-posedness of the map $\mathscr{R}_A : \mathcal{A}(S_G)[\![\hbar]\!] \to \mathcal{A}(S)[\![\hbar]\!]$. Recall that R_A admits an inverse which is explicitly given by (3.9); it can be shown that by defining an analogous $\widehat{\mathscr{R}}_A : \mathcal{A}(S)[\![\hbar]\!] \to \mathcal{A}(S_G)[\![\hbar]\!]$ we reach the same conclusion, and that $\mathscr{R}_A \circ \widehat{\mathscr{R}}_A = \mathrm{id}_{\mathcal{A}(S)[[\hbar]]}$ and $\widehat{\mathscr{R}}_A \circ \mathscr{R}_A = \mathrm{id}_{\mathcal{A}(S_G)[[\hbar]]}$. Therefore, $\mathscr{R}_A$ is a vector space isomorphism.

The fact that $\mathscr{R}_A$ is an algebra homomorphism is due to the following fact: we know that the $\star$-product in $\mathcal{A}(S)[[\hbar]]$ is given by (5.4); given two homogeneous functionals $F \in \mathcal{A}^p(S)$ and $G \in \mathcal{A}^q(S)$, the functional $\Gamma^n_\zeta(G, F)$ appearing in the sum can be formally written, on an homogeneous element $u_1 \wedge \cdots \wedge u_{p+q-2n} \in \wedge^{p+q-2n}\mathcal{E}(S^\oplus(M))$ as

$$
\begin{aligned}
\Gamma^n_\zeta(G, F)(u_1 \wedge \cdots \wedge u_{p+q-2n}) = \left(\frac{i}{2}\right)^n \sum_{\sigma \in S_{p+q-2n}} \mathrm{sgn}(\sigma) \\
\times \int_{M^{2n}} d\mu_g(x_1) d\mu_g(y_1) \cdots d\mu_g(x_n) d\mu_g(y_n) \\
\left(F^{(n)}_{u_{\sigma(q-n+1)} \wedge \cdots \wedge u_{\sigma(p+q-2n)}}\right)_{f_1 \cdots f_n} (y_1, \ldots, y_n) \left(G^{(n)}_{u_{\sigma(1)} \wedge \cdots \wedge U_{\sigma(q-n)}}\right)_{g_1 \cdots g_n} \\
(x_1, \ldots, x_n) \\
\zeta_{s_1 t_1}(x_1, y_1) \cdots \zeta_{s_n t_n}(x_n, y_n) h^{f_1 s_1} h^{t_1 g_1} \ldots h^{f_n s_n} h^{t_n g_n} .
\end{aligned}
$$

We are interested in computing $\Gamma^n_\zeta(\mathscr{R}_A(G), \mathscr{R}_A(F))$ with $F \in \mathcal{A}^p(S_G)$ and $G \in \mathcal{A}^q(S_G)$; this amounts to substituting the formal integral above with

$$
\begin{aligned}
&\int_{M^{2n}} d\mu_g(\xi_1) d\mu_g(\eta_1) \cdots d\mu_g(\xi_n) d\mu_g(\eta_n) d\mu_g(x_1) d\mu_g(y_1) \cdots d\mu_g(x_n) d\mu_g(y_n) \\
&\quad \left(F^{(n)}_{\mathcal{R}_A{}^{\wedge p-n}(u_{\sigma(q-n+1)} \wedge \cdots \wedge u_{\sigma(p+q-2n)})}\right)_{l_1 \cdots l_n} (\xi_1, \ldots, \xi_n) \\
&\quad \left(G^{(n)}_{\mathcal{R}_A{}^{\wedge q-n}(u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(q-n)})}\right)_{k_1 \cdots k_n} (\eta_1, \ldots, \eta_n) \\
&\quad r_{A\, f_1 u_1}(x_1, \xi_1) r_{A\, g_1 v_1}(y_1, \eta_1) \cdots r_{A\, f_n u_n}(x_n, \xi_n) r_{A\, g_n v_n}(y_n, \eta_n) h^{l_1 u_1} h^{v_1 k_1} \ldots \\
&\quad h^{l_n u_n} h^{v_n k_n} \zeta_{s_1 t_1}(x_1, y_1) \cdots \zeta_{s_n t_n}(x_n, y_n) h^{f_1 s_1} h^{t_1 g_1} \ldots h^{f_n s_n} h^{t_n g_n} .
\end{aligned}
$$

By performing a simple computation one can notice that

$$
\begin{aligned}
&\int_{M^2} d\mu_g(x_i) d\mu_g(y_i) r_{A\, f_i u_i}(x_i, \xi_i) r_{A\, g_i v_i}(y_i, \eta_i) \zeta_{s_i t_i}(x_i, y_i) h^{f_i s_i} h^{t_i g_i} \\
&\quad = \int_{M^2} d\mu_g(x_i) d\mu_g(y_i) r_{A}{}^{*}_{u_i f_i}(\xi_i, x_i) r_{A}{}^{*}_{v_i g_i}(\eta_i, y_i) \zeta_{s_i t_i}(x_i, y_i) h^{f_i s_i} h^{t_i g_i} \\
&\quad = \zeta_{G\, u_i v_i}(\xi_i, \eta_i)
\end{aligned}
$$

and therefore we arrive at

$$\int_{M^{2n}} d\mu_g(\xi_1)d\mu_g(\eta_1)\cdots d\mu_g(\xi_n)d\mu_g(\eta_n)$$
$$\left(F^{(n)}_{\mathcal{R}_A{}^{\wedge p-n}(u_{\sigma(q-n+1)}\wedge\cdots\wedge u_{\sigma(p+q-2n)})}\right)_{l_1\cdots l_n}(\xi_1,\ldots,\xi_n)$$
$$\left(G^{(n)}_{\mathcal{R}_A{}^{\wedge q-n}(u_{\sigma(1)}\wedge\cdots\wedge u_{\sigma(q-n)})}\right)_{k_1\cdots k_n}(\eta_1,\ldots,\eta_n)$$
$$\zeta_{G\,u_1v_1}(\xi_1,\eta_1)\cdots\zeta_{G\,u_nv_n}(\xi_n,\eta_n)h^{l_1u_1}h^{v_1k_1}\ldots h^{l_nu_n}h^{v_nk_n}$$

which entails that

$$\Gamma^n_\zeta(\mathscr{R}_A(G),\mathscr{R}_A(F))(u_1\wedge\cdots\wedge u_{p+q-2n}) = \mathscr{R}_A(\Gamma^n_{\zeta_G}(G,F))(u_1\wedge\cdots\wedge u_{p+q-2n})\ .$$

The result can be extended to the whole $\mathcal{E}^a(M^{p+q-2n}, S^{\oplus}(M)^{\boxtimes p+q-2n})$ by continuity, as well as to the whole algebra $\mathcal{A}(S_G)$ and thus to the whole algebra of formal power series $\mathcal{A}(S_G)[\![\hbar]\!]$. Therefore,

$$\mathscr{R}_A(F)\star\mathscr{R}_A(H) = \mathscr{R}_A(F\star_G H)$$

and $\mathscr{R}_A$ is an algebra homomorphism as required.

As far as the behaviour of the classical Møller map with respect to conjugation is concerned, let us recall that on $\mathcal{A}(S)[\![\hbar]\!]$ and $\mathcal{A}(S_G)[\![\hbar]\!]$ the conjugation map is given by the natural extension of (5.1) (with the appropriate conjugation map, that is, with either $C\colon \mathcal{E}(\overline{S(M)}) \to \mathcal{E}(S(M))$ or $C_G\colon \mathcal{E}(\overline{S_G(M)}) \to \mathcal{E}(S_G(M))$) to formal power series.

Given $F \in \mathcal{A}(S_G) \subseteq \mathcal{A}(S_G)[\![\hbar]\!]$ and $u \in \mathcal{C}(S^{\oplus}(M))$ we thus have that

$$(\mathscr{R}_A(F))^*(u) = \overline{(\mathscr{R}_A(F))(u^*)} = \sum_{p\in\mathbb{N}} \overline{\langle F_p, \mathcal{R}_A{}^{\wedge p}u^*_p\rangle}\ .$$

Now, if $u_p = u_{i_1}\wedge\cdots\wedge u_{i_p}$, $u_{i_j} \in \mathcal{E}(S^{\oplus}(M))$ we have that

$$\begin{aligned}\mathcal{R}_A{}^{\wedge p}(u^*_p) &= \mathcal{R}_A u^*_{i_p}\wedge\cdots\wedge\mathcal{R}_A u^*_{i_1} = (\mathcal{R}_A u_{i_p})^*\wedge\cdots\wedge(\mathcal{R}_A u_{i_1})^* \\ &= \left(\mathcal{R}_A{}^{\wedge p}(u_1\wedge\cdots\wedge u_p)\right)^*\ .\end{aligned}$$

By continuity then it holds that

$$\sum_{p\in\mathbb{N}}\overline{\langle F_p, \mathcal{R}_A{}^{\wedge p}u^*_p\rangle} = \sum_{p\in\mathbb{N}}\overline{\langle F_p, (\mathcal{R}_A{}^{\wedge p}u_p)^*\rangle} = (\mathscr{R}_A(F^*))(u)\ .$$

Therefore, $\mathscr{R}_A\colon \mathcal{A}(S_G(M))[[\hbar]] \to \mathcal{A}(S(M))[[\hbar]]$ is a well-defined algebraic $*$-isomorphism. For what concerns the topological part, it is simple to understand that the sequential completeness of Hörmander topology is satisfied, since pointwise convergence holds term by term for the formal power series and that the wave front set condition is as well satisfied by the construction seen before. □

6 Conclusions and Outlook

Our classic treatment of fermions in external backgrounds has proceeded with the aim at establishing the most general framework possible for the passage to the quantum case, having in mind perturbation theory, hence in the language of formal power series. In doing so we have, however, taken shortcuts that is worth mentioning again. Two of them are particularly important: the first is that we have dealt with contractible spacetimes, which rules out topological effects (*i.e.* inequivalent spinor structures [40] and Aharonov–Bohm like effects, for which see, *e.g.*, [41, 42]), the second is our assumption about the past-compactness of the support of the gauge potentials, which rules out Coulomb potentials. Both assumptions can be relaxed, but to keep this paper into a reasonable length we postpone any further discussion. However, notice that for the first problem, the passage to Fredenhagen's universal algebra [43] may help to solve the issue and that, as far as the second is concerned, it is exactly due to the compactness of the *spatial* support of the potentials that one rules out secular effects in perturbation theory [44].

In this paper we have privileged the *pointwise* treatment of the geometric structures (sections, potentials *etc.*) to guarantee a detailed and unambiguous discussion of their features. In particular, our most interesting result has been to show how the *classical* Møller maps are algebraic and topological isomorphisms of the charged and uncharged microcausal fermionic algebras, as formal power series. Here, the use of wave front sets was essential. As for a possible next step, once the appropriate interactions are introduced, one would construct the respective *quantum* Møller maps (see, *e.g.*, [33]) and proceed to the explicit computation of physical effects. One of the most ambitious aims would be, for instance, to compute the Lamb Shift for hydrogenoid atoms [14] from first principles, avoiding *ad hoc* assumptions and rigorously controlling eventual approximations.

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Declarations

Conflict of interest The author has no conflict of interest to declare that are relevant to the content of this article.

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