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Graded Lie algebras of maximal class in positive characteristic, generated by two elements of different weights

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Contents

Introduction

The notion of maximal class for p-groups was introduced in 1958 by Blackburn [**Bla58**]. Let $G, \gamma_2(G), \gamma_3(G), \ldots$ be the terms of the lower central series of a group G with p^n elements. It is well known that all finite p-groups are nilpotent, namely there exists an integer k such that $\gamma_k(G) = 1$. If k is the smallest among these integers, G is said to be of nilpotency class $k-1$. The nilpotency class for G is at most $n-1$ and, if this upper bound is reached, we say that G is of maximal class.

Finite abelian p-groups are easily classified. When one moves to class two groups, one finds that there are simply too many of them, and a classification is impossible. However, the situation is different if one looks at the coclass instead.

If G has order p^n and (nilpotency) class c, then its coclass is

$$cc(G) = n - c.$$

The notion of coclass can be extended to pro-p-groups. Let G be a pro-p-group. We denote by $G, \gamma_2(G), \ldots$ the terms of the lower central series and by $G_i = G/\gamma_i(G)$. The coclass of G is defined as

$$cc(G) = \lim_{i\to\infty} cc(G_i).$$

Aiming at classifying p-groups and pro-p groups, given p and the coclass r, in 1980 Leedham-Green and Newman formulated five conjectures. The proof of these conjectures has involved many people and nowadays they are all proved. Details about these results can be found in [**LGM02**].

The notions of maximal class and coclass can be defined also for Lie algebras. Let L be a residually nilpotent Lie algebra and L^i the terms of the lower central series. Suppose that $\dim(L^i/L^{i+1})$ are all finite and $\dim(L^i/L^{i+1}) \leq 1$ for sufficiently large i. Then we say that L has finite coclass and define

$$cc(L) = \sum_{\substack{i\geq 1\\ L^i\neq 0}} (\dim(L^i/L^{i+1}) - 1).$$

When $cc(L) = 1$ we say that L is of maximal class. Of special interest from a group-theoretic point of view are $\mathbf{N}$-graded Lie algebras of maximal class, $L = \bigoplus_{i\geq 1} L_i$, generated by L_1 and satisfying $\dim(L_1) = 2$ and $\dim(L_i) = 1$, for $i > 1$, since graded Lie algebras arising from pro-p-groups of maximal class are of this type.

Shalev and Zelmanov [**SZ97**] developed a coclass theory for Lie algebras of characteristic zero in analogy with the theory established by Leedham-Green and Newman for groups. Shalev and Zelmanov first dealt with the $\mathbf{N}$-graded Lie algebras of finite coclass that are generated, as it happens for groups, by their first homogeneous component. They proved that there is only one just infinite algebra, namely

$$a \quad = \quad \langle x, y : [yx^iy] = 0, \text{for all } i \geq 1\rangle.$$

Indeed such an algebra is of maximal class and metabelian.

In positive characteristic, there is exactly one infinite-dimensional metabelian Lie algebra of maximal class. Shalev [**Sha94**] showed that there are countable many infinite-dimensional graded Lie algebras of maximal class over a field of positive characteristic p generated by the first homogeneous component.We will refer to theses algebras as AFS-algebras.

In 1997, Caranti, Mattarei and Newman [**CMN97**] proved that, for each prime p, there are $2^{\aleph_0}$ isomorphism types of $\mathbf{N}$-graded Lie algebras of maximal class over $\mathbf{F}_p$, generated in weight 1. Such algebras were constructed using the algebras in [**Sha94**], by means of a technical process called inflation.

In 1999, Caranti and Newman [**CN00**] proved that all $\mathbf{N}$-graded Lie algebras of maximal class in characteristic $p > 2$ generated in weight 1 are obtained via possibly infinitely many inflation steps from some AFS-algebra.

Later the case of $\mathbf{N}$-graded Lie algebras of maximal class in characteristic $p = 2$ generated in weight 1 has been dealt by Jurman [**Jur05**]. The author proved that, in addition to algebras obtained by inflations of AFS-algebras, there is also a further family of infinite-dimensional graded Lie algebras of maximal class.

There are other possibilities for a graded Lie algebra to be of maximal class. One of them is that each homogeneous component of L has dimension one and that $[L_1L_i] = L_{i+1}$, for $i > 1$. We will refer to such algebras as algebras of type 2, since they are generated by one element of weight 1 and one of weight 2. Shalev and Zelmanov [**SZ97**] proved that over a field of characteristic zero there are three infinite-dimensional algebras of this type, namely

$$m \quad = \quad \langle e_1, e_2 : [e_2e_1^ie_2] = 0, \text{for all } i \geq 1\rangle,$$

$$\begin{aligned} m_2 = \langle e_i \quad : \quad & [e_ie_1] = e_{i+1}, \text{for all } i \geq 2, \\ & [e_ie_2] = e_{i+2}, \text{for all } i \geq 3, \\ & [e_i, e_j] = 0, \text{for all } i, j \geq 3\rangle. \end{aligned}$$

and

$$W \quad = \quad \langle e_i, i \geq 1 : [e_ie_j] = (i-j)e_{i+j}\rangle.$$

In characteristic $p > 2$, the algebras m and m_2 are still graded Lie algebras of maximal class, but the same does not hold for W. The other algebras of type 2 are the algebras obtainable as subalgebra of an algebra generated in weight one, one further family of soluble algebras and, for $p = 3$, one additional family of soluble algebras [**CVL00**].

Caranti and Vaughan-Lee [**CVL03**] have studied algebras of type 2 also in characteristic two. This case is more uniform than the odd characteristic case.

The aim of this thesis is to begin the study of graded Lie algebras over a field of odd characteristic

$$L = L_1 \oplus \bigoplus_{i=n}^{\infty} L_i$$

where $\dim(L_1) = 1$, $\dim(L_i) = 1$, for $i \geq n$, and $[L_1 L_i] = L_{i+1}$, for each integer $i \geq n$. From now on, when we will write Lie algebra of maximal class, we will refer to an algebra satisfying the conditions above.

In analogy with [**CMN97**] and [**CVL00**] in Chapter 2 the notion of constituent is introduced for the algebras we study in our work. We denote by $e_k = [e_n, e_1^{k-n}]$ the generator of the homogeneous component of weight k of an algebra of maximal class and say that the first constituent has length k if

$$\begin{aligned} [e_i, e_n] &= 0 \quad \text{for } i < k \\ [e_k, e_n] &\neq 0. \end{aligned}$$

In Chapter 2 it will be proved that there are some restrictions for the length l of the first constituent of a graded Lie algebra of maximal class. In particular, provided that $p > 2n$, there is no graded Lie algebra of maximal class such that $p^h + n \leq l \leq p^{h+1} - n + 1$, where p is the characteristic of the underlying field and h a positive integer. The question about the existence of such algebras for $p^h \leq l \leq p^h + n - 1$ and $l \leq p - n + 1$ is open. We suppose that there are no algebras of maximal class with such constituent length, except for $l = n + 1$. Computations performed in GAP suggest that, for $p > 2n$, and an arbitrary choice of $\lambda \in \mathbf{F}$ there is exactly one graded Lie algebra of maximal class such that $[[e_n e_1] e_n] = \lambda e_{2n+1}$.

The problem of the existence of graded Lie algebras of maximal class with first constituent length l in the range $p^h - n + 2 \leq l \leq p^h + 1$ or $l = 2p^h - n + 1$ is addressed in Chapter 3. In the latter case, it will be proved that such algebras are obtainable as subalgebras of a graded Lie algebra of maximal class generated by two elements of weight one. In the case $p^h - n + 2 \leq l \leq p^h + 1$, it will be proved that there is an upper bound for the number of graded Lie algebras for any value l in the range.

In Chapter 4 it will be shown that such bound is actually reached. Moreover an explicit construction of such algebras will be provided.

We wish to mention that in a work not reported here we have considered the case of graded Lie algebras of the form

$$L = L_1 \oplus \bigoplus_{i=q}^{\infty} L_i,$$

where $\dim(L_1) = 1$ and $\dim(L_i) = 1$, for $i \geq q$, and $[L_1 L_i] = L_{i+1}$, where $q = p^h$, for some prime p and positive integer h, over a field of characteristic p. We think that, in analogy with the case of algebras of type 2 in characteristic two, such case is more uniform than that of algebras generated in weight 1 and n.

CHAPTER 1

Preliminaries

1.1. Lie algebras

We begin this introductory section recalling the definition of Lie algebra.

DEFINITION 1.1.1. A Lie algebra L over a field $\mathbf{F}$ is a vector space over $\mathbf{F}$ endowed with a binary operation

$$\begin{array}{rcl} L \times L & \to & L \\ (x, y) & \mapsto & [x, y] \end{array}$$

called the bracket or commutator of x and y such that

(1) $[x, x] = 0$, for any $x \in L$;
(2) for any $x, y, z \in L$, the Jacobi identity holds:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Remark. As an immediate consequence of the definition we have that

$$[x, y] = -[y, x], \quad \text{for any } x, y \in L,$$

namely the bracket operation is anticommutative.

Herein we point out that, given $x, y, z \in L$, we will often write $[xyz]$ in place of $[[x, y], z]$.

As a consequence of the Jacobi identity we have that

$$[x, [yz^k]] = \sum_{i=0}^{k} (-1)^i \binom{k}{i} [xz^i yz^{k-i}].$$

1.1.1. Free Lie algebras. In this section we recall the notion of free algebra and free Lie algebra. We follow the approach of Bourbaki.

Let X be a set. We can construct the free magma $M(X)$ on X. We define inductively the sets X_n, for $n \geq 1$. For $n = 1$ we define $X_1 = X$. Once we have defined $X_1, X_2, \ldots, X_{n-1}$ we say that X_n is the union of the sets $X_i \times X_{n-i}$, for $1 \leq i \leq n-1$. We denote the union of the sets X_n just defined by $M(X)$. Consider two elements $w_i, w_j \in M(X)$ and suppose that $w_i \in X_i, w_j \in X_j$ respectively. Denote $n = i + j$ and consider the injection

$$\begin{array}{rcl} \iota : X_n \times X_{n-i} & \to & X_n \\ (w_i, w_j) & \mapsto & \iota(w_i, w_j). \end{array}$$

We denote $\iota(w_i, w_j)$ by $w_i * w_j$ and say that this is the product of w_i and w_j.

The algebra of the magma $M(X)$ with the coefficients in the ring K is denoted by $Lib_K(X)$, or $Lib(X)$ when there is no ambiguity.

We are in a position to give the definition of free Lie algebra.

DEFINITION 1.1.2. The free Lie algebra over the set X is the quotient algebra

$$L(X) = Lib(X)/\mathfrak{a},$$

where $\mathfrak{a}$ is the ideal of $Lib(X)$ generated by the elements of one of the forms:

$$\begin{array}{cl} a * a, & \text{for } a \in Lib(X) \\ a * (b * c) + b * (c * a) + c * (a * b), & \text{for } a, b, c \in Lib(X). \end{array}$$

1.1.2. Presentation of a Lie algebra. Let $\mathfrak{g}$ be a Lie algebra and $\boldsymbol{a} = (a_i)_{i \in I}$ a family of elements of $\mathfrak{g}$. Consider the homomorphism

$$\begin{array}{rcl} f_{\boldsymbol{a}} : L(I) & \to & \mathfrak{g} \\ i & \mapsto & a_i, \quad \text{for } i \in I. \end{array}$$

The elements of $\ker(f_{\boldsymbol{a}})$ are called the *relators* of the family $\mathfrak{a}$. If $f_{\boldsymbol{a}}$ is surjective we say that $\boldsymbol{a}$ is generating.

Given a Lie algebra $\mathfrak{g}$, we say that the ordered pair $(\boldsymbol{a}, \boldsymbol{r})$ is a *presentation* of $\mathfrak{g}$ if the family $\boldsymbol{a} = (a_i)_{i \in I}$ generates $\mathfrak{g}$ and $\boldsymbol{r} = (r_j)_{j \in J}$ generates the kernel of the homomorphism $f_{\boldsymbol{a}}$, between $L(I)$ and $\boldsymbol{g}$, defined above.

In general, if we take a set I, we can construct the free Lie algebra over I. If $\boldsymbol{r} = (r_j)_{j \in J}$ is a subset of $L(I)$ and $\mathfrak{a}_r$ the ideal generated by $\boldsymbol{r}$, then the quotient algebra $L(I)/\mathfrak{a}_r$ is defined by the presentation $(I, \boldsymbol{r})$.

1.1.3. Derivations of Lie algebras. Let L be a Lie algebra. We say that a linear map $D : L \to L$ is a derivation if

$$D[a, b] = [Da, b] + [a, Db], \quad \text{for any } a, b \in L.$$

If we denote by juxtaposition the composition of two derivations, we can define the commutator of two derivations D, E as

$$[D, E] = DE - ED.$$

Then, $\mathrm{Der}(L)$ is a Lie algebra. For any $x \in L$ it is possible to define the endomorphism

$$\begin{array}{rcl} \mathrm{ad}(x) : L & \to & L \\ y & \mapsto & [x, y]. \end{array}$$

As a consequence of the Jacobi identity, it is easily seen that $\mathrm{ad}(x)$ is a derivation. In fact $[x, [y, z] = [[x, y], z] + [y, [x, z]]$.

A subspace I of a Lie algebra L is an *ideal* of L if, for any $x \in L$ and $y \in I$, the bracket $[x, y] \in I$. An important example of ideal of L is the *derived algebra* of L, denoted by $[L, L]$, which consists of the all linear combinations of commutators $[x, y]$, for $x, y \in L$.

We remind the following result on vector spaces.

LEMMA 1.1.3. *Let U, V be vector spaces and W a vector subspace of U. Let $f : U \to V$ be a linear map, whose kernel contains W, and $\pi : U \to U/W$ the* *projection onto U/W. Then there exists a unique linear map $\bar{f} : U/W \to V$ such that the following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\pi"] & V \\ U/W \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Consider now a free Lie algebra $L(X)$ over a set X. The following holds.

LEMMA 1.1.4. *Every mapping of X into $L(X)$ can be extended uniquely to a derivation of $L(X)$.*

The following result holds.

LEMMA 1.1.5. *Let $D : L(X) \to L(X)$ be a derivation and I an ideal of $L(X)$ such that $I \subseteq \ker(D)$. There exists and is unique a derivation $\bar{D} : L(X)/I \to L(X)/I$ such that the following diagram commutes:*

$$\begin{tikzcd} L(X) \arrow[r, "\pi\circ D"] \arrow[d, "\pi"] & L(X)/I \\ L(X)/I \arrow[ur, "\bar{D}"'] & \end{tikzcd}$$

PROOF. Since D and $\pi \circ D$ are a linear maps, as a consequence of Lemma 1.1.3 we have that there exists a unique linear map $\bar{D} : L(X)/I \to L(X)/I$ making the diagram commute. The map $\bar{D}$ takes an element $x + I$ to $D(x) + I$. It is now easy to verify that $\bar{D}$ is a derivation too. In fact,

$$\begin{aligned} \bar{D}([x+I, y+I]) &= \bar{D}([x,y]+I) \\ &= D([x,y]) + I = [Dx, y] + [x, Dy] + I \\ &= [\bar{D}(x+I), y+I] + [x, \bar{D}(y+I)]. \end{aligned}$$

□

1.1.4. Solvable, nilpotent and residually nilpotent Lie algebras. For any Lie algebra L it is possible to define the *derived series* by $L^{(1)} = L, L^{(2)} = [L^1, L^1]$ and more generally $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$. We say that L is *solvable*, if $L^{(n)} = 0$, for some n. For example, if $[x, y] = 0$ for any $x, y \in L$ (namely, L is *abelian*), then L is solvable.

For any Lie algebra L it is also possible to define another sequence of ideals, the *lower central series*, by $L^1 = L, L^2 = [L, L]$ and, in general, $L^i = [LL^{i-1}]$. We say that L is nilpotent if $L^n = 0$, for some n. As before, an abelian Lie algebra is nilpotent and all nilpotent algebras are solvable.

Finally, we say that a Lie algebra L is *residually nilpotent* if $\cap_i L^i = \emptyset$.

1.2. Graded Lie algebras of maximal class

We remind firstly what a graded vector space is. Denote by $\mathbf{N}^+$ the set of positive integers and let V be a vector space. We say that V is *graded* if it can be decomposed as a direct sum

$$V = \bigoplus_{n \geq 1} V_n,$$

where each V_n is a vector space.

A *graded Lie algebra* L is a Lie algebra endowed with a gradation compatible with the Lie bracket. Therefore there exists a family $\{L_n\}_{n \in \mathbf{N}^+}$ of Lie algebras such that L can be decomposed as a direct sum of vector spaces

$$L = \bigoplus_{n \geq 1} L_n$$

and, for each $e_i \in L_i$ and $e_j \in L_j$,

$$[e_i, e_j] \in L_{i+j}.$$

It is always possible to grade a Lie algebra L. In fact, we can define, for $i \in \mathbf{N}^+$,

$$L_i = L^i / L^{i+1}$$

and then consider the algebra

$$\bigoplus_{i=1}^{\infty} L_i.$$

Now, consider a graded Lie algebra

$$L = \bigoplus_{i=1}^{\infty} L_i.$$

If $\dim(L^i/L^{i+1}) < \infty$, for each i, and $\dim(L^i/L^{i+1}) \leq 1$, for all sufficiently large i, then L has *finite coclass*. If this is the case, we define

$$cc(L) = \sum_{\substack{i \geq 1 \\ L^i \neq 0}} (\dim(L^i/L^{i+1}) - 1).$$

If $cc(L) = 1$, then L is of *maximal class*.

In [**CMN97**], [**CN00**] and [**CVL00**] the authors studied the following:

(1) infinite dimensional graded Lie algebras over a field of odd characteristic, generated by two elements x, y of weight one, such that $\dim(L_1) = 2$, $\dim(L_i) = 1$, for any $i \geq 2$, and $[L_i, L_1] = L_{i+1}$;
(2) infinite dimensional graded Lie algebras over a field of odd characteristic, generated by two elements x, y of weights respectively 1 and 2, such that $\dim(L_i) = 1$, for any i, and $[L_i, L_1] = L_{i+1}$.

Of course, these are not the only possibilities for an infinite dimensional graded Lie algebra over a field of odd characteristic to be of maximal class.

We focused on another case. The object of study of our thesis has been the infinite dimensional graded Lie algebras of maximal class over a field $\mathbf{F}$ of odd characteristic p of the form

$$L = L_1 \oplus \bigoplus_{i=n}^{\infty} L_i,$$

where each L_i has dimension one, $L_1 = \langle e_1 \rangle$, $L_n = \langle e_n \rangle$ and, for $i > n$, $L_i = \langle e_i \rangle$, having defined inductively $e_i = [e_{i-1}, e_1]$.

From now on, when we refer to a graded Lie algebra of maximal class without further specifications, we mean a graded Lie algebra as defined above.

1.3. Binomial identities and determinants

In the following we will use without further mention some identities for binomial coefficients. Here we write down some of them.

Assume that m, n, r, s are integers.

$$\begin{aligned} \binom{n}{k} &= \binom{n}{n-k} \\ \sum_{k \in \mathbf{Z}} \binom{r}{m+k}\binom{s}{n+k} &= \binom{r+s}{r-m+n} \\ \binom{r}{m}\binom{m}{k} &= \binom{r}{k}\binom{r-k}{m-k}. \end{aligned}$$

We will often rely upon the following Theorem, named after Lucas.

THEOREM 1.3.1 (Lucas' Theorem). *Let p be a prime and a, b two positive integers. If*

$$\begin{aligned} a &= a_h p^h + a_{h-1} p^{h-1} + \cdots + a_1 p + a_0, \\ b &= b_h p^h + b_{h-1} p^{h-1} + \cdots + b_1 p + b_0, \end{aligned}$$

where h is a non-negative integer and $0 \le a_i, b_i < p$ for $0 \le i \le h$, then

$$\binom{a}{b} \equiv \binom{a_h}{b_h} \cdot \binom{a_{h-1}}{b_{h-1}} \cdots \binom{a_0}{b_0} \pmod{p}. \tag{1.3.1}$$

As a consequence of Lucas' theorem we deduce the following.

COROLLARY 1.3.2. *If $q = p^h$, for some prime p and positive integer h, and $0 \le a, b < q$, then*

$$\binom{a}{q-1-b} \equiv (-1)^{a+b} \binom{b}{q-1-a} \pmod{p}. \tag{1.3.2}$$

PROOF. Suppose firstly that $h = 1$. Therefore $0 \le a, b < p$. Consider

$$\begin{aligned} (p-1)! &= (p-1)(p-2)\dots(p-a)\cdot(p-a-1)! \equiv (-1)^a a!(p-a-1)! \pmod{p} \\ (p-1)! &= (p-1)(p-2)\dots(p-b)\cdot(p-b-1)! \equiv (-1)^b b!(p-b-1)! \pmod{p}. \end{aligned}$$

Then,

$$\begin{aligned}
\binom{a}{p-1-b} &= \frac{a!}{(p-1-b)!(a-p+1+b)!} = \frac{a!(p-1)\dots(p-b)}{(p-1)!(a-p+1+b)!} \\
&\equiv \frac{a!(-1)^b b!}{(-1)^a a!(p-1-a)!(a-p+1+b)!} \\
&\equiv (-1)^{a+b}\frac{b!}{(p-1-a)!(a-p+1+b)!} \\
&\equiv (-1)^{a+b}\binom{b}{p-1-a} \pmod p.
\end{aligned}$$

Now consider the case of h greater than 1. Let

$$\begin{aligned}
a &= a_{h-1}p^{h-1} + \dots + a_1 p + a_0; \\
b &= b_{h-1}p^{h-1} + \dots + b_1 p + b_0; \\
q-1-a &= (p-1-a_{h-1})p^{h-1} + \dots + (p-1-a_1)p + (p-1-a_0); \\
q-1-b &= (p-1-b_{h-1})p^{h-1} + \dots + (p-1-b_1)p + (p-1-b_0).
\end{aligned}$$

By means of Lucas' theorem we get:

$$\begin{aligned}
\binom{a}{q-1-b} &\equiv \prod_{i=0}^{h-1}\binom{a_i}{p-1-b_i} \equiv \prod_{i=0}^{h-1}(-1)^{a_i+b_i}\binom{b_i}{p-1-a_i} \\
&\equiv \prod_{i=0}^{h-1}(-1)^{p^i(a_i+b_i)}\binom{b_i}{p-1-a_i} \equiv (-1)^{a+b}\binom{b}{q-1-a} \pmod p.
\end{aligned}$$

□

In some of our proofs it will be useful to evaluate the determinant of matrices whose entries are binomials. The determinant evaluations we are going to write down are taken by [**Kra99**].

The following holds.

Theorem 1.3.3. *Let n be a positive integer, and let $L_1, L_2, \dots, L_n$ and A, B be indeterminates. Then there holds*

$$\det_{1\le i,j\le n}\left(\binom{BL_i+A}{L_i+j}\right) =$$
$$= \frac{\prod_{1\le i<j\le n}(L_i-L_j)}{\prod_{i=1}^{n}(L_i+n)!}\prod_{i=1}^{n}\frac{(BL_i+A)!}{((B-1)L_i+A-1)!}\prod_{i=1}^{n}(A-Bi+1)_{i-1},$$

where $(A-Bi+1)_{i-1} = (A-Bi+1)\cdot(A-Bi+2)\dots(A-Bi+i-1)$, if $i>1$, or $(A-Bi+1)_{i-1} = 1$, if $i=1$.

If we set

$$\begin{aligned} L_i &= x - i \\ B &= -1 \\ A &= 2x + y \end{aligned}$$

we deduce immediately the following

Corollary 1.3.4. *Let x, y be integers. Then,*

$$\det_{1\le i,j\le n}\left(\binom{x+y+i}{x-i+j}\right) = $$
$$= \frac{\prod_{1\le i<j\le n}(j-i)}{\prod_{i=1}^{n}(x-i+n)!}\prod_{i=1}^{n}\frac{(x+y+i)!}{(y+2i-1)!}\prod_{i=1}^{n}(2x+y+i+1)_{i-1}.$$

Another useful determinant evaluation is the following.

Lemma 1.3.5. *Let a, b be nonnegative integers and n a positive integer. Then,*

$$\det_{1\le i,j\le n}\left(\binom{a+b}{a-i+j}\right) = \prod_{i=1}^{n}\prod_{j=1}^{a}\prod_{k=1}^{b}\frac{i+j+k-1}{i+j+k-2}.$$

CHAPTER 2

The length of the first constituent

Through this section L will denote a graded Lie algebra of maximal class, generated by two elements of weights 1 and n respectively, over a field $\mathbf{F}$ of characteristic $p > 2n$.

We define the length and the type of a constituent in a graded Lie algebra of maximal class $M = \bigoplus_i M_i$ generated by two elements of weight 1 and n such that $[M_{i-1}M_1] = M_i$ in analogy with [**CVL00**].

Let e_1 and e_n be the generators of the homogeneous components of weight respectively 1 and n. We define inductively $e_{i+1} = [e_i, e_1]$, for $i \geq n$. Suppose that $[e_{k-1}, e_n] = 0$, but $[e_k, e_n] = \lambda e_{k+n} \neq 0$, for some integer k and non-zero $\lambda \in \mathbf{F}$. Let m, r be positive integers and suppose that $[e_{k+m}, e_n] = \cdots = [e_{k+m+r-1}, e_n] = 0$. Finally suppose that $[e_{k+m+r}, e_n] \neq 0$. Let us denote $[e_i, e_n] = \lambda_i e_{i+n}$ for $k \leq i \leq k + m - 1$.

Definition 2.0.6. We call the pattern

$$\begin{array}{rcll}
[e_{k-1}, e_n] & = & 0 & \\
[e_i, e_n] & = & \lambda_i \cdot e_{i+n} & \text{for } k \leq i \leq k+m-1 \\
[e_i, e_n] & = & 0 & \text{for } k+m \leq i \leq k+m+r-1 \\
[e_{k+m+r}, e_n] & \neq & 0 &
\end{array}$$

a constituent of length $m + r$ and type $(\lambda_k, \ldots, \lambda_{k+m-1})$ or $\{\lambda_i\}_{i=k}^{k+m-1}$.

We define separately the first constituent.

Definition 2.0.7. Let

$$\begin{array}{rcll}
[e_i, e_n] & = & 0 & \text{for } i \leq n < k, \\
[e_k, e_n] & \neq & 0, &
\end{array}$$

for some integer k. We say that the length of the first constituent is k.

Denote by l the length of the first constituent of L. The following holds.

Lemma 2.0.8. *The length l is even if and only if n is odd.*

PROOF. Take a positive integer n. If n is even, suppose that l is even too, otherwise suppose that l is odd. In both cases $l \pm n$ is even. Then,

$$\begin{aligned} 0 &= [e_{\frac{l+n}{2}}, e_n e_1^{\frac{l-n}{2}}] \\ &= \left(\sum_{i=0}^{\frac{l-n}{2}} (-1)^i \binom{\frac{l-n}{2}}{i} \lambda_{i+\frac{l+n}{2}} \right) e_{l+n} \\ &= (-1)^{\frac{l-n}{2}} \lambda_l e_{l+n}, \end{aligned}$$

in contradiction with the hypothesis that l is the length of the first constituent.

Hence, l is even if and only if n is odd. $\square$

We begin proving a Lemma, we will use repeatedly through this and the following sections.

LEMMA 2.0.9. *Suppose that the length of the first constituent of L is $l \geq n+3$, namely $[e_i, e_n] = 0$, for $n \leq i < l$, and $[e_l, e_n] = \lambda_l e_{n+l}$, for some nonzero element $\lambda_l \in \mathbf{F}$.*

Then

$$[e_{l+n+i}, e_n] = 0, \quad \text{for } 0 \leq i \leq \frac{l-n-3}{2}.$$

More generally, if $\{\lambda_i\}_{i=k}^{k+m-1}$ is a constituent of length $s+m$ and $s \geq \dfrac{l-n-1}{2}$,

$$[e_{k+m+s+n+i}, e_n] = 0, \quad \text{for } 0 \leq i \leq \frac{l-n-3}{2}.$$

PROOF. Since the length of the first constituent is l,

$$[e_n e_1^i, e_n e_1^{i+1}] = 0, \quad \text{for } 0 \leq i \leq \frac{l-n-3}{2}.$$

Now we prove the thesis, in the general case, by induction on $0 \leq i \leq \dfrac{l-n-3}{2}$. The particular case follows taking $k = n, m = 0, s = l - n$.

When $i = 0$,

$$\begin{aligned} 0 &= [e_{k+m+s-1}, [e_n, e_n e_1]] \\ &= \lambda_{k+m+s}[e_{k+m+s+n}, e_n] = \lambda_{k+m+s} \cdot \lambda_{k+m+s+n} \cdot e_{k+m+s+2n}, \end{aligned}$$

hence $\lambda_{k+m+s+n} = 0$.

Suppose that $\lambda_{k+m+s+n+i} = 0$, for all integers i smaller or equal to r, where $0 \leq r < \frac{l-n-3}{2}$. Then,

$$\begin{aligned} 0 &= [e_{k+m+s-r-2}, [e_n e_1^{r+1}, e_n e_1^{r+2}]] \\ &= (-1)^{r+1} \lambda_{k+m+s} \left(\sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i \lambda_{k+m+s+n+i} \right) e_{k+m+s+2n+r+1} \\ &= \lambda_{k+m+s} \cdot \lambda_{k+m+s+n+r+1} e_{k+m+s+2n+r+1}. \end{aligned}$$

Hence, $\lambda_{k+m+s+n+r+1} = 0$ and we are done. $\square$

Given an integer r, we define the parameter

$$\varepsilon_r = \begin{cases} 0 & \text{if } r \text{ is even} \\ 1 & \text{if } r \text{ is odd.} \end{cases}$$

We want to prove that the length of the first constituent takes the values

- $n+1$
- $p^h + \varepsilon_n, p^h + \varepsilon_n - 2, \ldots, p^h - (n-2)$, for some positive integer h
- $2p^h - (n-1)$, for some positive integer h.

Firstly we prove the following.

LEMMA 2.0.10. *There is no graded Lie algebra of maximal class with the length of the first constituent l, such that*

$$rp^h + n \leq l \leq (r+1)p^h - n, \quad \textit{for } 1 \leq r \leq p-1$$

PROOF. According to Lemma 2.0.9,

$$\lambda_{l+n+j} = 0 \quad \text{for } 0 \leq j \leq \frac{l-n-3}{2}. \tag{2.0.3}$$

Now we can prove, by induction on i, that

$$[e_{l+n-i}, e_n] = 0 \quad \text{for } 0 \leq i \leq n. \tag{2.0.4}$$

The base step is trivially true. We write $l = (r+1)p^h - n - k$, for some non-negative integer k. Suppose that (2.0.4) is true for all indices i smaller than a given positive integer $j < n$. Consider

$$\begin{aligned} &[e_{l+n-j}, e_n e_1^{p^h-n-k-j+1}] = \\ &= \sum_{i=0}^{p^h-n-k-j+1} (-1)^i \binom{p^h-n-k-j+1}{i} \lambda_{l+n-j+i} e_{l+n-k-2j+1+p^h}. \end{aligned}$$

We note that

$$\begin{aligned} \frac{l-n-3}{2} &= \frac{(r+1)p^h - 2n - 3 - k}{2} \\ &\geq p^h - n - \frac{3+k-\varepsilon_{3+k}}{2}. \end{aligned}$$

We know that $\lambda_{l+n+i} = 0$, for $0 \leq i \leq \dfrac{l-n-3}{2}$. At the same time, $\lambda_{l+n-i} = 0$, for $0 \leq i < j$.

Supposing that $0 \leq i \leq p^h - n - k - j + 1$ and $j \geq 1$,

$$\begin{aligned} l+n-j+i &\leq l+n-1+p^h-n-k-j+1 \leq l+n+(p^h-n-k-1) \\ &\leq l+n+(p^h-n-\frac{k+3-\varepsilon_{k+3}}{2}). \end{aligned}$$

Hence,

$$\begin{aligned}
&[e_{l+n-j}, e_n e_1^{p^h-n-k-j+1}] = \\
&\qquad \lambda_{l+n-j} e_{l+n-k-2j+1+p^h} \\
&= -[e_{p^h-k-j+1}, e_n e_1^{(r+1)p^h-k-j-n}] \\
&= -\sum_{i=0}^{(r+1)p^h-k-j-n} (-1)^i \binom{(r+1)p^h-k-j-n}{i} \lambda_{p^h-k-j+1+i} e_{l+n-k-2j+1+p^h} \\
&= -\sum_{h=0}^{r} (-1)^h \binom{r}{h} \sum_{i=0}^{p^h-k-j-n} (-1)^i \binom{p^h-k-j-n}{i} \lambda_{(h+1)p^h-k-j+1+i} e_{l+n-k-2j+1+p^h} \\
&= 0.
\end{aligned}$$

Therefore, also $\lambda_{l+n-j} = 0$. We conclude that all $\lambda_{l+n-j} = 0$, for $0 \leq j \leq n$. In particular $\lambda_l = 0$, in contradiction with the fact that the length of the first constituent was l. $\square$

Consider the following.

LEMMA 2.0.11. *There is no graded Lie algebra of maximal class with the length of the first constituent l, such that*

$$\begin{aligned}
2p^h - n + 2 &\leq l \leq 2p^h, \\
rp^h - n &\leq l \leq rp^h, \quad \text{when } 3 \leq r \leq p-1.
\end{aligned}$$

PROOF. In Lemma 2.0.9 we proved that, if $\lambda_i = 0$, for $i < l$, and $\lambda_l \neq 0$, then $\lambda_i = 0$, for $l + n \leq i \leq l + n + \frac{l-n-3}{2}$.

Consider

$$\begin{aligned}
[e_n e_1^{l+(1-r)p^h-n}, e_n e_1^{rp^h}] &= (-1)^{r-1} \binom{rp^h}{(r-1)p^h} [e_l, e_n e_1^{p^h}] = (-1)^{r-1} r \lambda_l \cdot e_{l+p^h+n} \\
&= -[e_{rp^h+n}, e_n e_1^{l+(1-r)p^h-n}] \\
&= \sum_{i=0}^{l+(1-r)p^h-n} (-1)^{i+1} \binom{l+(1-r)p^h-n}{i} \lambda_{i+rp^h+n} \cdot e_{l+p^h+n}. \qquad (2.0.5)
\end{aligned}$$

We have that, for $0 \leq i \leq l + (1-r)p^h - n$,

$$l + n \leq rp^h + n + i \leq l + p^h.$$

If $r = 2$,

$$n + \frac{l-n-3}{2} \geq \frac{2p^h - 1}{2},$$

namely $\lambda_k = 0$, for $l + n \leq k \leq l + p^h$, hence the sum in (2.0.5) is 0. This implies that $\lambda_l = 0$ and we are in contradiction.

If $3 \leq r < p - 1$,

$$n + \frac{l-n-3}{2} \geq \frac{rp^h - 3}{2} > p^h,$$

hence the sum in (2.0.5) is 0 is zero too and we are in contradiction with the assumption $\lambda_l \neq 0$. $\square$

We proceed proving the following.

LEMMA 2.0.12. *There is no graded Lie algebra of maximal class with the length of the first constituent l, such that*

$$rp^h \quad < \quad l < rp^h + n, \quad \textit{for } 2 \leq r < p.$$

PROOF. With the usual notation, suppose that $\lambda_i = 0$, for $i < l$ and $\lambda_l \neq 0$, where l is an integer greater than rp^h and smaller than $rp^h + n$ for some $2 \leq r < p$. Then $\lambda_i = 0$, for $l + n \leq i \leq l + n + \frac{l-n-3}{2}$. We can write $l = rp^h + k$, for some integer $1 \leq k < n$. Let $q = p^h$.

Consider

$$\left[e_n e_1^{rq+k}, e_n e_1^{q-n}\right] \quad = \quad \sum_{i=0}^{q-n} (-1)^i \binom{q-n}{i} \lambda_{rq+k+n+i} e_{n+(r+1)q+k} = 0.$$

The sum just written is zero, since the coefficients $\lambda_{rq+k+n+i}$ are zero. In fact,

$$l + n \leq rq + k + n + i \leq (r+1)q + k$$

and

$$\begin{aligned} l + n + \frac{l-n-3}{2} \quad &= \quad \frac{3l+n-3}{2} = rq + k + \frac{rq+k+n-3}{2} \\ &\geq \quad rq + k + q + \frac{k+n-3}{2} \geq rq + k + q, \end{aligned}$$

since $n \geq 2$ and $k \geq 1$.

At the same time,

$$\begin{aligned} &[e_n e_1^{rq+k}, e_n e_1^{q-n}] = -[e_q, e_n e_1^{rq+k}] = \\ \equiv \quad &\sum_{h=0}^{r} (-1)^h \binom{r}{h} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \lambda_{(h+1)q+i} \\ = \quad &(-1)^{r-1} r (-1)^k \lambda_{rq+k}. \end{aligned}$$

Hence, $\lambda_l = \lambda_{rq+k} = 0$ and we are in contradiction with the initial hypothesis that $\lambda_l \neq 0$. $\square$

The cases $l = q + \varepsilon_n - 2i$, for q a power of p and $0 \leq i \leq \dfrac{n-2+\varepsilon_n}{2}$, and $l = 2q - n + 1$ have not been excluded. In fact, there exist Lie algebras of maximal class for such values of l.

So far, we have not dealt with Lie algebras having first constituent length l smaller than $p - (n-2)$ or such that $p^h < l < p^h + n$, for a positive integer h. As regards the latter case, we have not yet proved the non-existence of Lie algebras of maximal class with such first constituent length. Computations performed in GAP suggest that there are no Lie algebras with such first constituent length. The case of Lie algebras with l smaller than $p - (n-2)$ is still unsolved, but based on a conjecture that we formulate in the subsection below.

2.0.1. Length smaller than p-(n-2). Experimental results seem to support the following

CONJECTURE 2.0.13. *Let L be a graded Lie algebra of maximal class generated by two elements of weights 1 and n over a field $\mathbf{F}$ of characteristic $p > 2n$. If $[e_n e_1 e_n] = \lambda e_{2n+1}$, for some non-zero $\lambda \in \mathbf{F}$, then*

$$[e_k, e_n] = \lambda e_{k+n}, \quad \textit{for all } k > n.$$

Indeed, the conjecture is true for $n = 2$ (see [**CVL00**] for the details).

Suppose that there exists a graded Lie algebra of maximal class L with length of the first constituent l such that $n+1 < l < p-(n-2)$. As usual, let us denote the generators of the homogeneous components of weights 1 and n by e_1 and e_n and in general $e_s = [e_n, e_1^{s-n}]$, for $s > n$. Define $k = l-(n+1)$. We note that k is even, so we can take the integer $m = n + \frac{k}{2}$. Consider now the subalgebra M of L generated by the elements e_1 and $e_m = [e_n e_1^{k/2}]$. Such an algebra is of maximal class with first constituent length equal to $m+1$. In fact

$$\begin{aligned}
[e_{m+1}, e_m] &= [e_{m+1}, e_n e_1^{k/2}] = \sum_{i=0}^{k/2} (-1)^i \binom{k/2}{i} [e_{m+1+i} e_n e_1^{k/2-i}] \\
&= [e_l, e_n] = (-1)^{k/2} \lambda_l e_{l+n},
\end{aligned}$$

since $[e_{m+i+1}, e_n] = 0$, whenever $m+i+1 < l$.

Moreover, $p > 2m$. In fact,

$$2m = 2n + k = 2n + l - (n+1) = n + l - 1 < n + p - (n-2) - 1 = p + 1.$$

Since $2m$ is even and $2m < p+1$, it follows that $2m \leq p-1$.

What we have just proved implies that M is a graded Lie algebra of maximal class generated in weights 1 and m and such that the length of the first constituent is $m+1$, where $p > 2m$. Then,

$$[e_k, e_m] = (-1)^{k/2} \lambda_l e_{k+m}, \quad \text{for } k > m.$$

This fact implies that in L the following holds

$$[e_{l+i}, e_n] = (-1)^i \binom{p-k/2-1}{i} \lambda_l e_{l+n+i} \quad \text{for } 0 \leq i \leq p-k/2-1. \tag{2.0.6}$$

This can be proved by induction. The base step is obvious, since

$$[e_l, e_n] = \lambda_l e_{l+n}.$$

Now we prove the inductive step. To do that, suppose to have proved (2.0.6) for $0 \leq i \leq r-1$. Then,

$$\begin{aligned}
(-1)^{k/2}\lambda_l e_{l+r+n} &= [e_{m+1+r}, e_n e_1^{k/2}] \\
&= \sum_{j=0}^{k/2-1} (-1)^j \binom{k/2}{j} [e_{m+1+r+j} e_n e_1^{k/2-j}] + (-1)^{k/2}[e_{l+r}, e_n] \\
&= (-1)^{r-k/2}\lambda_l \sum_{j=0}^{k/2-1} \binom{k/2}{j} \binom{p-k/2-1}{j-(k/2-r)} e_{l+r+n} + (-1)^{k/2}[e_{l+r}, e_n] \\
&= (-1)^{r-k/2}\lambda_l \left(\binom{p-1}{r} - \binom{p-k/2-1}{r} \right) e_{l+r+n} + (-1)^{k/2}[e_{l+r}, e_n] \\
&= \lambda_l \left((-1)^{k/2} - (-1)^{r-k/2} \binom{p-k/2-1}{r} \right) e_{l+r+n} + (-1)^{k/2}[e_{l+r}, e_n].
\end{aligned}$$

Hence,

$$[e_{l+r}, e_n] = (-1)^r \binom{p-k/2-1}{r} \lambda_l e_{l+r+n}.$$

We remind that $n+1 < l < p-(n-2)$, so $[e_n e_1 e_n] = 0$. It follows that $[e_{l+n}, e_n] = 0$. Moreover,

$$n + \frac{k}{2} + 1 = m + 1 < l < p,$$

namely $n < p - (\frac{k}{2}+1)$, so we are in contradiction with the fact that

$$[e_{l+n}, e_n] = (-1)^n \binom{p-k/2-1}{n} \lambda_l e_{l+2n} \neq 0.$$

CHAPTER 3

Lie algebras with given first constituent lengths

We will denote by L a graded Lie algebra of maximal class, generated in weights 1 and n, over a field $\mathbf{F}$ of characteristic p.

3.1. First constituent length $2q - n + 1$

In all this section we assume that q is a power of p, a prime integer greater than $2n$. In the following we will characterize the subalgebra, generated by the elements of weights 1 and n, of a graded Lie algebra of maximal class generated by two elements of weight one. The arguments used go back to [**CVL00**]. The reader can find the relevant definitions and the classification of graded Lie algebras of maximal class generated by two elements of weight one in [**CMN97**] and [**CN00**].

Let L be a graded Lie algebra of maximal class generated by two elements x and y of weight 1. Without loss of generality we assume that y generates the first two-step centralizer, namely $C_2 = \langle y \rangle$. Suppose that in L we have the following sequence of two-step centralizers,

$$C_2 = C_{r-2} = C_{r-1}, C_r = \langle y - \lambda x \rangle \neq C_2, C_{r+1} = \cdots = C_{r+n} = C_2.$$

Note in passing that $C_{r+1} = \cdots = C_{r+p-1} = C_2$ by the properties of the two-step centralizers proved in [**CMN97**]. By our assumption $p > 2n$ it follows that $C_{r+1} = \cdots = C_{r+n} = C_2$.

Let $e_n = [yx^{n-1}]$ and in general $e_i = [yx^{i-1}]$, for $i > 1$.

We have that

$$\begin{aligned} [e_{r-n+1}, e_n] &= [e_{r-n+1}, yx^{n-1}] = \\ &= (-1)^{n-1}([e_r, y - \lambda x] + [e_r, \lambda x]) \\ &= (-1)^{n-1} \lambda e_{r+1}. \end{aligned}$$

More in general, for $0 \leq i \leq n-1$,

$$[e_{r-i}, e_n] = (-1)^i \binom{n-1}{i} \lambda e_{r+n-i}.$$

We have just proved that the subalgebra generated by the elements of weight one and n of a graded Lie algebra of maximal class generated by two elements of weight one has all constituents of type

$$\left\{ (-1)^{n-1+i} \binom{n-1}{i} \lambda \right\}_{i=0}^{n-1},$$

for some $\lambda \neq 0$.

Under certain hypotheses the converse holds too. Firstly we note that, as a consequence of Lemma 2.0.9, the following holds.

LEMMA 3.1.1. *Let L be a graded Lie algebra of maximal class generated by two elements e_1, e_n of weight 1 and n with the length of the first constituent equal to $2q-(n-1)$. In the multiplication table, denote $[e_i, e_n] = \lambda_i e_{i+n}$. Suppose that, for some integer $k \geq 2q-n+1$, the coefficient $\lambda_k \neq 0$ and that $\lambda_{k-n-1} = \cdots = \lambda_{k-1} = 0$. Then $\lambda_{k+n} = \cdots = k+2n = 0$.*

LEMMA 3.1.2. *Let $\mathfrak{g}$ be a graded Lie algebra of maximal class generated by two elements of weight 1 and n with the length of the first constituent equal to $2q-(n-1)$. Suppose that all the constituents are of type*

$$\{\lambda_{j+i}\}_{i=0}^{n-1} = \left\{(-1)^i \binom{n-1}{i} \lambda_j \right\}_{i=0}^{n-1} \quad \text{for } j \in J \subset \mathbf{N} \text{ and } \lambda_j \in \mathbf{F}.$$

Then $\mathfrak{g}$ is isomorphic to a subalgebra of a graded Lie algebra of maximal class generated by two elements of weight one.

PROOF. For $n=2$ the result has been already proved in [**CVL00**]. Take $n>2$ and suppose that $\mathfrak{g}$ is a graded Lie algebra of maximal class generated by two elements e_1, e_n of weights respectively 1 and n. Such generators are related by the relators $[e_{j+i}, e_n] + (-1)^{i+1}\binom{n-1}{i}\lambda_j e_{n+j+i}$, when $j \in J$ and $0 \leq i \leq n-1$, and $[e_k, e_n]$ if k is not one of the integers $j+i$, for some $j \in J$ and $0 \leq i \leq n-1$. We want to extend $\mathfrak{g}$ by a derivation of weight $n-1$ on $\mathfrak{g}$. In such a way we get a graded Lie algebra of maximal class generated by two elements of weight 1 and $n-1$, which, by inductive hypothesis, is isomorphic to a subalgebra of a graded Lie algebra of maximal class generated by two elements of weight 1. Let $X = \{e_1, e_n\}$ and $L(X)$ the free Lie algebra over X. Define

$$\begin{array}{rcl} D : X & \to & L(X) \\ e_1 & \mapsto & e_n \\ e_n & \mapsto & 0 \end{array}$$

We can extend D uniquely to a derivation of $L(X)$ according to Lemma 1.1.4. Consider in $L(X)$ the ideal I generated by the relators introduced above. The quotient algebra $L(X)/I$ is then isomorphic to $\mathfrak{g}$.

We can order the elements of J and say that $j_1 < j_2 < \ldots$. Now we prove that $D(e_k) = 0$ for the integers $k \leq j_1$. Of course the assertion is true for $k=n$. Suppose to have proved the assertion for an integer $k-1 < j_1$. Then, $D(e_k) = D([e_{k-1}, e_1]) = [De_{k-1}, e_1] + [e_{k-1}, De_1] = [e_{k-1}, e_n] = 0$.

Suppose now to have reached the beginning of a constituent. That means that we have $k=j$, for some $j \in J$. We have that $[e_{j-1}, e_n] = 0$ and $[e_j, e_n] = e_{j+n}$. Moreover, $D(e_j) = 0$. Therefore $D(e_{j+1}) = D([e_j, e_1]) = [De_j, e_1] + [e_j, De_1] = [e_j, e_n] = \lambda_j e_{j+n}$. By inductive hypothesis, suppose that $D(e_{j+i}) =$

$(-1)^{i-1}\binom{n-2}{i-1}\lambda_j e_{j+i+n}$, for some i such that $0 \leq i < n-2$. Then,

$$\begin{aligned} D(e_{j+i+1}) &= D([e_{j+i}, e_1]) = [De_{j+i}, e_1] + [e_{j+i}, De_1] \\ &= \left((-1)^{i-1}\binom{n-2}{i-1} + (-1)^i\binom{n-1}{i}\right)\lambda_j e_{j+i+n+1} \\ &= (-1)^i\binom{n-2}{i}\lambda_j e_{j+i+n+1}. \end{aligned}$$

This proves that

$$De_{j+i} = (-1)^{i-1}\binom{n-2}{i-1}\lambda_j e_{i+j+n} \quad \text{for } 0 \leq i \leq n-1.$$

Now consider $j_i + n \leq k \leq j_{i+1}$. If $k = j_i + n$, then $D(e_k) = D([e_{j_i+n-1}, e_1]) = (-1)^{n-2} + (-1)^{n-1}$. If we have already proved that $D(e_k) = 0$, for some $j_i + n \leq k < j_{i+1}$, then $D(e_{k+1}) = [De_k, e_1] + [e_k, e_n] = 0$.

Now we check that $I \subseteq \ker(D)$. We do that considering the generators of I, namely the relators defined above. If $k \neq j, j+1, \ldots, j+(n-1)$ for some $j \in J$, it is easily verified that $D([e_k, e_n]) = [De_k, e_n] + [e_k, De_n] = 0$. Conversely, suppose that $k = j+i$, for some $j \in J$ and $i = 0, 1, \ldots, n-1$. Then,

$$D([e_{j+i}, e_n]) = [De_{j+i}, e_n] = (-1)^{i-1}\binom{n-2}{i-1}[e_{j+i+n-1}, e_n] = 0.$$

Hence we have extended $\mathfrak{g}$ to a graded Lie algebra of maximal class generated by one element of weight 1 and $n-1$, namely e_1 and D. □

Using the previous result we will prove that, if a graded Lie algebra of maximal class L has the length of the first constituent equal to $2q-n+1$, then it is obtainable from a graded Lie algebra of maximal class generated by two elements of weight one.

Let L be generated by the elements e_1 and e_n of weight respectively 1 and n. Suppose that the first constituent has length $l = 2q-(n-1)$, where $q = p^h$, for some positive integer h. With the usual notation it means that in the multiplication table $\lambda_i = 0$ for $i < l$, while $\lambda_l \neq 0$. Up to scaling e_n we can suppose that $\lambda_l = 1$. We remind that $\lambda_{l+n+i} = 0$, for $0 \leq i \leq n$. Consider the following relations:

$$\left[e_{q+h}, e_n e_1^{q+h-n}\right] = 0 \quad \text{for } 1 \leq h \leq n-1.$$

For a fixed h we have then

$$\sum_{i=0}^{q+h-n} (-1)^i\binom{q-(n-h)}{i}\lambda_{q+h+i}e_{2q+2h} = 0.$$

We note that each integer $q+h+i$ is not greater than $2q+n-2$. Since $\lambda_{q+h+i} \neq 0$ if and only if $2q-(n-1) \leq q+h+i \leq 2q$, we can consider the sum above over a smaller range of indices, namely for $q-h-(n-1) \leq i \leq q-h$. Fix an integer h. Then $i = q-(n-1)-h+j$, for some $0 \leq j \leq n-1$. By Lucas' theorem

$$\binom{q-(n-h)}{i} \equiv \binom{p-(n-h)}{p-(n-1)-h+j} \pmod{p}.$$

We can rewrite the previous sums as

$$\sum_{j=0}^{n-1}(-1)^j\binom{p-(n-h)}{p-(n-1)-h+j}\lambda_{2q-(n-1)+j}e_{2q+2h}=0 \quad \text{for } 1\le h\le n-1.$$

Taking into consideration the fact that $\lambda_{2q-n+1}=1$, such equalities can be reformulated in the following non-homogenous linear system

$$\begin{pmatrix} \binom{p-n+1}{p-n+1} & -\binom{p-n+1}{p-n+2} & \dots & (-1)^n\binom{p-n+1}{p-1} \\ \binom{p-n+2}{p-n} & -\binom{p-n+2}{p-n+1} & \dots & (-1)^n\binom{p-n+2}{p-2} \\ \vdots & \vdots & \vdots & \\ \binom{p-1}{p-2n+3} & -\binom{p-1}{p-2n+4} & \dots & (-1)^n\binom{p-1}{p-n+1} \end{pmatrix}\begin{pmatrix}\lambda_{2q-(n-2)}\\ \lambda_{2q-(n-3)}\\ \vdots\\ \lambda_{2q}\end{pmatrix}=\begin{pmatrix}1\\1\\ \vdots\\ 1\end{pmatrix}$$

We aim at evaluating the determinant of the matrix associated to this system. By basic properties of the determinant of a matrix, it is equal to

$$\pm\det\left(\binom{p-n+i}{p-n+1-i+j}\right)_{i,j=1,\dots,n-1}. \tag{3.1.1}$$

If we prove that determinant 3.1.1 is not congruent to zero $\pmod p$, the $(n-1)$-dimensional square matrix has full rank, hence the solution of the non-homogeneous system is unique. We rely upon Corollary 1.3.4 in order to evaluate determinant 3.1.1. According to the notation of the corollary we set $x=p-n+1$ and $y=-1$. Hence,

$$\begin{aligned}\det\left(\binom{p-n+i}{p-n+1-i+j}\right)_{i,j=1,\dots,n-1} &= \\ = \frac{\prod_{1\le i<j\le n-1}(j-i)}{\prod_{i=1}^{n-1}(p-i)!}\prod_{i=1}^{n-1}\frac{(p-n+i)!}{(2i-2)!}\prod_{i=1}^{n-1}(2p-2n+2+i)_{i-1}.\end{aligned}$$

In this expansion no factor is divisible by p. In fact, $0<j-i<n-2$, whichever values of i,j we take in the range $1,\dots,n-1$. As regards the factors $(p-n+i)!$, they are the product of integers not greater than $p-1$. We must take care at the factors

$$(2p-2n+2+i)_{i-1}=(2p-2n+2+i)\cdot(2p-2n+2+i+1)\dots(2p-2n+2i).$$

It is easily seen that each of the factors is greater than p, being $p>2n$. Moreover, each factor is smaller or equal to

$$2p-2n+2i\le 2p-2n+2n-2=2p-2.$$

We can conclude that our non-homogeneous system admits exactly one solution. By means of elementary properties of binomial coefficients, we can determine the solution. Indeed, the vector

$$(\lambda_{2q-(n-2)},\dots,\lambda_{2q})=\left(-\binom{n-1}{1},\binom{n-1}{2},\dots,(-1)^{n-1}\binom{n-1}{n-1}\right)$$

gets the job done.

Our next goal is proving that all the constituents are of the form

$$\left\{(-1)^i\binom{n-1}{i}\lambda_j\right\}_{i=0}^{n-1} \quad \text{for some } \lambda_j \neq 0 \in \mathbf{F} \tag{3.1.2}$$

and with length at least q and not exceeding $2q$. We suppose to have proved the assertion up to a certain constituent ending with $\lambda_{k-1}=0$. Moreover suppose to have proved that

$$\lambda_{k+i}=(-1)^i\binom{n-1}{i}\lambda_k \quad \text{for } 0\le i\le n-1 \text{ and } \lambda_k\neq 0\in\mathbf{F}.$$

We want to prove that $\{\lambda_{k+1}\}_{i=0,\dots,n-1}$ is the beginning of a constituent of length at least q and not greater than $2q$. We begin proving that $\lambda_{k+n}=\cdots=\lambda_{k+q-1}=0$. By Lemma (3.1.1) the coefficients $\lambda_{k+n}=\cdots=\lambda_{k+2n}=0$. Suppose to have proved that $\lambda_{k+2n}=\cdots=\lambda_{k+r-1}=0$ for a certain r such that $2n<r<q=p^h$. We prove that also $\lambda_{k+r}=0$. We note that

$$0=\left[e_{k+(n-2)},[e_ne_1^{r-(2n-2)}e_n]\right],$$

which implies that

$$0=\left(\lambda_{k+(n-2)}-(r-(2n-2))\lambda_{k+(n-1)}-(-1)^{r-(2n-2)}\lambda_{k+(n-2)}\right)\lambda_r.$$

If r is even we get $(r-(2n-2))\lambda_{k+(n-1)}\lambda_r=0$, hence $\lambda_r=0$.

We deal now with the case r odd. We get then

$$0=\left(2\lambda_{k+(n-2)}-(r-(2n-2))\lambda_{k+(n-1)}\right)\lambda_r.$$

If $r\not\equiv 0 \pmod p$, we have that $\left(2\lambda_{k+(n-2)}-(r-(2n-2))\lambda_{k+(n-1)}\right)\not\equiv 0 \pmod p$, hence $\lambda_r\equiv 0 \pmod p$.

Suppose now that $r\equiv 0 \pmod p$. We can write $r=\beta p^t$, for some integer $\beta\not\equiv 0 \pmod p$. Moreover $r<q$, hence $p^t<p^h=q$. Since $r+p^t-n=(\beta+1)p^t-n\le q-n$, the following holds:

$$0=\left[e_{k-p^t},[e_ne_1^{r+p^t-n}e_n]\right].$$

Since we have already proved that all the previous constituents are of type (3.1.2) and have length at least q, this relation implies

$$0=\left(\sum_{i=0}^{n-1}\binom{r+p^t-n}{p^t+i}\binom{n-1}{i}\lambda_k\right)\lambda_{k+r}.$$

Let us concentrate for a moment on the binomials which appear in the previous sum:

$$\binom{r+p^t-n}{p^t+i}\equiv\binom{\beta}{1}\binom{p-n}{i} \pmod p.$$

Hence we can rewrite the previous sum as

$$\begin{aligned} 0 &= \left(\sum_{i=0}^{n-1}\binom{p-n}{i}\binom{n-1}{i}\lambda_k\right)\lambda_{k+r} \\ &= \binom{p-1}{p-n}\lambda_k\lambda_{k+r}. \end{aligned}$$

Since neither $\binom{p-1}{p-n}$ nor λ_k are equivalent to zero $\pmod p$, we conclude that $\lambda_{k+r} \equiv 0 \pmod p$.

Now we prove that the length of the new constituent is at most $2q$. Suppose, on the converse, that $\lambda_{k+n} = \lambda_{k+n+1} = \cdots = \lambda_{k+2q} = 0$. Then,

$$0 = [e_{k+n-1}, e_n e_1^{2q-2n+1} e_n - e_n e_1^{2q-n+1}]$$

implies that $\lambda_{k+n-1} = 0$ and this is in contradiction with our hypotesis. Hence, we have proved that the constituent has length not shorter than q and not longer than $2q$. Let r be the length of the constituent. If $r < 2q-1$,

$$0 = [e_{k+(n-2)}, [e_n e_1^{r-(2n-2)} e_n]].$$

With the same argument used above we deduce that r must be divisible by p. If this is the case, $r = \beta p^t$, for some positive integer t and $\beta \not\equiv 0 \pmod p$. Consider

$$0 \quad = \quad [e_{k-(2q-\beta p^t-n)}, e_n e_1^{2q-2n+1} e_n - e_n e_1^{2q-n+1}].$$

Firstly,

$$\begin{aligned}
&[e_{k-(2q-\beta p^t-n)}, e_n e_1^{2q-2n+1} e_n] = \\
&= \quad (-1)^{\beta p^t+n} \left(\sum_{i=0}^{n-1} (-1)^i \binom{2q-2n+1}{2q-\beta p^t-n+i} \lambda_{k+i} \right) \lambda_{k+r} e_{k+r+n} \\
&\equiv \quad (-1)^{\beta p^t+n} \binom{2p^{h-1}-1}{2p^{h-1}-\beta p^{t-1}-1} \left(\sum_{i=0}^{n-1} (-1)^i \binom{p-2n+1}{p-n+i} \lambda_{k+i} \right) \lambda_{k+r} e_{k+r+n} \\
&\equiv \quad 0 \pmod p.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&[e_{k-(2q-\beta p^t-n)}, e_n e_1^{2q-n+1}] \\
&= \quad (-1)^{\beta p^t+n} \left(\sum_{i=0}^{n-1} (-1)^i \binom{2q-n+1}{2q-\beta p^t-n+i} \lambda_{k+i} \right) e_{k+r+n+1} \\
&\quad +(-1)^n \left(\sum_{i=0}^{1} (-1)^i \binom{2q-n+1}{2q-n+i} \lambda_{k+r+i} \right) e_{k+r+n+1} \\
&\equiv \quad (-1)^{\beta p^t+n} \binom{2p^{h-1}-1}{2p^{h-1}-\beta p^{t-1}-1} \left(\sum_{i=0}^{n-1} \binom{p-n+1}{p-n+i} \binom{n-1}{i} \lambda_k \right) e_{k+r+n+1} \\
&\quad +(-1)^n \left(\sum_{i=0}^{1} (-1)^i \binom{2q-n+1}{2q-n+i} \lambda_{k+r+i} \right) e_{k+r+n+1} \pmod p \\
&\equiv \quad (-1)^n \left(\sum_{i=0}^{1} (-1)^i \binom{2q-n+1}{2q-n+i} \lambda_{k+r+i} \right) e_{k+r+n+1} \pmod p
\end{aligned}$$

Summing all up, we have that

$$\begin{aligned} 0 &= [e_{k-(2q-\beta p^t-n)}, e_n e_1^{2q-2n+1} e_n - e_n e_1^{2q-n+1}] \\ &\equiv \left(\sum_{i=0}^{1} (-1)^i \binom{p-n+1}{p-n+i} \lambda_{k+r+i}\right) e_{k+r+n+1} \pmod p \end{aligned}$$

Hence $\lambda_{k+r+1} = -\binom{n-1}{1}\lambda_{k+r}$.

Suppose now to have proved that $\lambda_{k+r+i} = (-1)^i\binom{n-1}{i}\lambda_{k+r}$, for $i = 0, 1, \ldots, j-1 < n-1$. With the same argument used before, we consider the relation

$$0 = [e_{k-(2q-\beta p^t-n)+j-1}, e_n e_1^{2q-2n+1} e_n - e_n e_1^{2q-n+1}].$$

We observe that

$$\begin{aligned} &[e_{k-(2q-\beta p^t-n)+j-1}, e_n e_1^{2q-2n+1} e_n] \equiv \\ &\equiv \binom{2p^{h-1}-1}{2p^{h-1}-\beta p^{t-1}-1} \left(\sum_{i=0}^{n-1} (-1)^i \binom{p-2n+1}{p-n-j+1+i} \lambda_{k+i}\right) \lambda_{k+r+j} e_{k+r+j+n} \\ &\equiv 0 \pmod p. \end{aligned}$$

Moreover,

$$\begin{aligned} &[e_{k-(2q-\beta p^t-n)+j-1}, e_n e_1^{2q-n+1}] \\ &= (-1)^{\beta p^t-n-j+1} \left(\sum_{i=0}^{n-1} (-1)^i \binom{2q-n+1}{2q-\beta p^t-n-j+1+i} \lambda_{k+i}\right) e_{k+r+n+j} \\ &\quad +(-1)^{n+j-1} \left(\sum_{i=0}^{n-1} (-1)^i \binom{2q-n+1}{2q-n-j+1+i} \lambda_{k+r+i}\right) e_{k+r+n+j} \\ &\equiv (-1)^{\beta p^t-n-j+1} \binom{2p^{h-1}-1}{2p^{h-1}-\beta p^{t-1}-1} \left(\sum_{i=0}^{n-1} \binom{p-n+1}{p-n-j+1+i} \binom{n-1}{i} \lambda_k\right) e_{k+r+n+j} \\ &\quad +(-1)^{n+j-1} \left(\sum_{i=0}^{n-1} (-1)^i \binom{2q-n+1}{2q-n-j+1+i} \lambda_{k+r+i}\right) e_{k+r+n+j} \pmod p \\ &\equiv +(-1)^{n+j-1} \left(\sum_{i=0}^{n-1} (-1)^i \binom{2q-n+1}{2q-n-j+1+i} \lambda_{k+r+i}\right) e_{k+r+n+j} \pmod p \end{aligned}$$

Summing all up, we have that

$$\begin{aligned} 0 &= [e_{k-(2q-\beta p^t-n)+j-1}, e_n e_1^{2q-2n+1} e_n - e_n e_1^{2q-n+1}] \\ &\equiv \left(\sum_{i=0}^{j} (-1)^i \binom{p-n+1}{p-n-j+1+i} \lambda_{k+r+i}\right) e_{k+r+n+j} \pmod p \end{aligned}$$

Hence $\lambda_{k+r+j} = (-1)^j\binom{n-1}{j}\lambda_{k+r}$.

We deal now with the cases of a constituent of length $2q-1$ or $2q$. In both cases, $\lambda_{k+n} = \cdots = \lambda_{k+2q-2} = 0$. If the length of the constituent is $2q-1$ consider the relations

$$\left[e_{k+2n-2+i}, e_n e_1^{2q-2n+1} e_n - e_n e_1^{2q-n+1}\right] = 0, \quad 0 \le i \le n-2,$$

while, if the length is $2q$, consider

$$\left[e_{k+2n-1+i}, e_n e_1^{2q-2n+1} e_n - e_n e_1^{2q-n+1}\right] = 0, \quad 0 \le i \le n-2.$$

By Lemma (3.1.1), in the case of length $2q-1$, the coefficients $\lambda_{k+2q-1+n} = \cdots = \lambda_{k+2q-1+2n} = 0$, while in the other case the coefficients $\lambda_{k+2q+n} = \cdots = \lambda_{k+2q+2n} = 0$. As a consequence we have that in the former case $\left[e_{k+2n-2+i}, e_n e_1^{2q-2n+1} e_n\right] = 0$, while in the latter $\left[e_{k+2n-1+i}, e_n e_1^{2q-2n+1} e_n\right] = 0$. Hence we have in the two cases respectively

$$\begin{aligned} \left[e_{k+2n-2+i}, e_n e_1^{2q-n+1}\right] &= 0, \quad 0 \le i \le n-2, \\ \left[e_{k+2n-1+i}, e_n e_1^{2q-n+1}\right] &= 0, \quad 0 \le i \le n-2. \end{aligned}$$

As a consequence the following equalities hold respectively

$$\begin{aligned} 0 &= \sum_{j=0}^{n-1} (-1)^j \binom{2q-n+1}{2q-1-2n+2-i+j} \lambda_{k+2q-1+j} \\ &\equiv \sum_{j=0}^{n-1} (-1)^j \binom{p-n+1}{p-2n+1-i+j} \lambda_{k+2q-1+j} \pmod{p}, \\ 0 &\equiv \sum_{j=0}^{n-1} (-1)^j \binom{p-n+1}{p-2n+1-i+j} \lambda_{k+2q+j} \pmod{p}. \end{aligned}$$

Let us denote $v_1 = (\lambda_{k+2q-1}, \lambda_{2q}, \ldots, \lambda_{2q+n-2})$ and $v_2 = (\lambda_{k+2q}, \lambda_{2q+1}, \ldots, \lambda_{2q+n-1})$. Moreover, let $\mu_1 = \lambda_{2q-1}$ and $\mu_2 = \lambda_{2q}$ and

$$c_r = -\mu_r \cdot \left(\binom{p-n+1}{p-2n+1}, \binom{p-n+1}{p-2n}, \ldots, \binom{p-n+1}{p-2n-n+3} \right),$$

where r takes values 1 or 2.

The relations just written give rise to the following linear systems

$$\begin{pmatrix} -\binom{p-n+1}{p-2n+1+1} & \binom{p-n+1}{p-2n+1+2} & \cdots & (-1)^{n-1}\binom{p-n+1}{p-2n+1+(n-1)} \\ -\binom{p-n+1}{p-2n+1} & \binom{p-n+1}{p-2n+2} & \cdots & (-1)^{n-1}\binom{p-n+1}{p-2n+(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ -\binom{p-n+1}{p-2n-n+3+1} & \binom{p-n+1}{p-2n-n+3+2} & \cdots & (-1)^{n-1}\binom{p-n+1}{p-2n-n+3+(n-1)} \end{pmatrix} \cdot v_r^T = c_r^T$$

Denote by A the matrix of the coefficients of the system above. We want to prove that A has full rank, namely $n-1$. This will be done evaluating the determinant of the matrix B such defined

$$B=\left(\binom{p-n+1}{p-2n+2-i+j}_{ij}\right),\quad \text{for } i,j=1,\dots n-1.$$

Since $\det(B)=\pm\det(A)$, the determinant of A is non zero if and only if $\det(B)$ is different from zero. We rely upon Lemma (1.3.5) in order to evaluate $\det(B)$. Using the notation of the Lemma, we set $a=p-2n+2$ and $b=n-1$. Then,

$$\det(B)=\prod_{i=1}^{n-1}\prod_{j=1}^{p-2n+2}\prod_{k=1}^{n-1}\frac{i+j+k-1}{i+j+k-2}.$$

We note that

$$1<i+j+k-1\leq(n-1)+(p-2n+2)+(n-1)-1=p-1.$$

Hence $\det(B)\not\equiv 0 \pmod p$. So we are done. Now it is an easy matter to determine the solution of the nonhomogeneous system, namely

$$v_r=\left\{(-1)^j\binom{n-1}{j}\mu_r\right\}\quad \text{for } 0\leq j\leq n-1.$$

Hence we have proved that all the constituents of a graded Lie algebra, with length of the first constituent equal to $2q-(n-1)$, are of type $\{(-1)^j\binom{n-1}{j}\mu\}_{0\leq j\leq n-1}$. By Lemma (3.1.2) such an algebra is obtainable by a graded Lie algebra generated by two elements of weight one.

3.2. First constituent length $(q-n+4,q+1)$

When $p>4n$, the following result holds.

LEMMA 3.2.1. *If $p>4n$ and the length of the first constituent of L is $q+\varepsilon_n-2i$, where $0\leq i\leq\frac{n+\varepsilon_n-2}{2}$, then*

$$\lambda_{q+\varepsilon_n-2i+n+k}\ =\ 0\quad \textit{for } 0\leq k\leq\frac{q-1-2n}{2}$$

and at least one among the coefficients λ_{q+k}, with $1\leq k\leq n$, is different from zero.

PROOF. The first part of the Lemma is a direct consequence of Lemma 2.0.9. In fact, being $q+\varepsilon_n-2i\geq q-n+2$, we have that

$$\frac{l-n-3}{2}\geq\frac{q-2n+2-3}{2}.$$

Hence, $\lambda_{q+\varepsilon_n-2i+n+k}=0$ for $0\leq k\leq\frac{q-1-2n}{2}$.

As regards the second part of the Lemma, suppose that $\lambda_{q+k}=0$, for $1\leq k\leq n$. Take into consideration the following relations:

$$\left[e_ne_1^{\frac{q-2n+1}{2}+i},e_ne_1^{\frac{q-2n+1}{2}+i}\right]=0,\text{for } 1\leq i\leq n-1$$

Such $n-1$ relations give rise to a homogeneous system of $n-1$ linear equations in the indeterminates λ_i, for $q-n+2 \leq i \leq q$. The $n-1$ linear equations are

$$\sum_{j=1}^{2n-2} (-1)^j \binom{\frac{q-2n+1}{2}+i}{\frac{q-2n+1}{2}-i+j} \lambda_{q-n+1+j} = 0, \quad \text{for } 1 \leq i \leq n-1. \tag{3.2.1}$$

We note that

$$\begin{aligned} q-2n+1+2i &= (p-1+2i-2n+2) + \sum_{l=1}^{h-1} p^l(p-1) \\ q-2n+1-2i+2j &= (p-1+2-2n-2i+2j) + \sum_{l=1}^{h-1} p^l(p-1). \end{aligned}$$

Hence, by Lucas' theorem,

$$\binom{\frac{q-2n+1}{2}+i}{\frac{q-2n+1}{2}-i+j} \equiv \binom{\frac{p-1}{2}-(-i+n-1)}{\frac{p-1}{2}-(-j+i+n-1)} \pmod{p}.$$

Since $\lambda_{q+\varepsilon_n-2i+n+k} = 0$ for $0 \leq k \leq \frac{q-1-2n}{2}$, the coefficients a_{ij} of the matrix A associated with the equations (3.2.1) are

$$a_{ij} = (-1)^j \binom{(\frac{p-1}{2}-n+1)+i}{(\frac{p-1}{2}-n+1)-i+j}, \quad \text{for } 1 \leq i,j \leq n-1.$$

We want to prove that the determinant of A is different from zero. We can multiply each column of A by -1 and get the matrix B, having the same determinant of A up to the sign. For the sake of the clarity, we write $c = \frac{p-1}{2}-n+1$. So the entries of the matrix B are

$$b_{ij} = \binom{c+i}{c-i+j}, \quad \text{for } 1 \leq i,j \leq n-1.$$

It is possible to evaluate the determinant of B using the result given in Corollary (1.3.4). The evaluation of the determinant of B is

$$\det_{1\leq i,j\leq n-1}\left(\binom{c+i}{c-i+j}\right) = \frac{\prod_{1\leq i<j\leq n}(j-i)}{\prod_{i=1}^{n-1}(c-i+n)!} \prod_{i=1}^{n-1} \frac{(c+i)!}{(2i-1)!} \prod_{i=1}^{n-1} (2c+i+1)_{i-1}$$

The factors $(j-i)$ are not divisible by p, whichever the values of i and j are. As regards the factors $(c+i)!$, they are product of integers not greater than $(p-1)/2$. Consider finally the terms $(2c+i+1)_{i-1}$, for i not greater than $n-1$. They are defined as

$$(2c+i+1)_{i-1} = (2c+i+1)\cdot(2c+i+2)\dots(2c+2i-1).$$

We note that

$$0 < 2c+i+1 \leq \cdots \leq 2c+2i-1 \leq p-2.$$

Hence none of the $(2c+i+1)_{i-1}$ is divisible by p. We conclude that $\det(B)$ is not divisible by p and the same holds for $\det(A)$. This implies that the $n-1$ equations in (3.2.1) are linearly independent over $\mathbf{F}$, hence zero is the only element in $\mathbf{F}^n$

satisfying them. But this implies that the coefficient $\lambda_l = 0$, in contradiction with the fact that l is the length of the first constituent. $\square$

The following holds

LEMMA 3.2.2. *Let l be the length of the first constituent. Suppose that $q-(n-2) \leq l \leq q+\varepsilon_n$. Let k be a nonnegative integer.*

If k is odd suppose that at least one among $\lambda_{(k+1)q}, \dots, \lambda_{(k+1)q+n}$ is nonzero.

If k is even suppose that at least one among $\lambda_{(k+1)q+1}, \dots, \lambda_{(k+1)q+n}$ is nonzero.

If $k > 0$, suppose that one of the following holds:

(1) *for $0 \leq h < k$,*

$$\begin{aligned} &\lambda_{l+hq} \neq 0 \\ &\lambda_{l+hq+n} = \cdots = \lambda_{l+(h+1)q-1} = 0 \\ &\lambda_{l+(h+1)q} \neq 0; \end{aligned}$$

(2) *for $0 < h < k$,*

$$\begin{aligned} &\lambda_{l+hq-1} \neq 0 \\ &\lambda_{l+hq-1+n} = \cdots = \lambda_{l+(h+1)q-2} = 0 \end{aligned}$$

and

$$\begin{aligned} \lambda_{l+n} = \cdots = \lambda_{l+q-2} &= 0 \\ \lambda_{l+kq-1} &\neq 0. \end{aligned}$$

Then, in case (1) or $k=0$,

$$\lambda_{l+kq+n} = \cdots = \lambda_{l+(k+1)q-2} = 0,$$

while, in case (2),

$$\lambda_{l+kq-1+n} = \cdots = \lambda_{l+(k+1)q-3} = 0$$

PROOF. Set $c = l + kq$, in case (1) or $k = 0$, or $c = l + kq - 1$ in case (2). From Lemma 2.0.9 it follows that

$$\lambda_{c+n} = \lambda_{c+n+1} = \cdots = \lambda_{c+\frac{l-n-3}{2}+n} = 0.$$

Define $r = \max\{(k+1)q \leq t \leq (k+1)q+n : \lambda_t \neq 0\}$. We want to prove by induction on i that

$$\lambda_{c+\frac{l-n-3}{2}+n+i} = 0, \quad \text{for any } 0 \leq i \leq q-2-\frac{l-n-3}{2}-n$$

The base case has been already proved. Suppose to have proved that $\lambda_{c+\frac{l-n-3}{2}+n+t} = 0$, for some $0 \leq t < q-2-\frac{l-n-3}{2}-n$. Define $s = c + \frac{l-n-3}{2} - r$. We note that $s+t+1 < l-1-n$, hence

$$\left[e_r, e_n e_1^{s+t+1} e_n\right] = 0.$$

If $s+t$ is even,

$$\begin{aligned} 0 &= \left[e_r, e_n e_1^{s+t+1} e_n\right] \\ &= 2\lambda_r \lambda_{r+n+s+t+1} e_{2n+r+s+t+1} \end{aligned}$$

and we get $\lambda_{r+n+s+t+1} = 0$.

If $s+t$ is odd and r even,

$$\begin{aligned} -\lambda_{n+r+s+t+1}e_{2n+r+s+t+1} &= \\ &= \left[e_n, e_n e_1^{r+s+t+1}\right] \\ &= (\lambda_{n+r+s+t+1})e_{2n+r+s+t+1}, \end{aligned}$$

hence $\lambda_{n+r+s+t+1} = 0$ and we are done. Here we used the fact that

$$r+s+t+1 < (k+1)q + (l-2)$$

If $s+t$ is odd and r odd,

$$\begin{aligned} -\lambda_{n+r+s+t+2}e_{2n+r+s+t+2} &= \\ &= \left[e_n, e_n e_1^{r+s+t+2}\right] \\ &= (-(r+s+t+2)\lambda_{n+r+s+t+1} + \lambda_{n+r+s+t+2})e_{2n+r+s+t+2}, \end{aligned}$$

At the same time,

$$\begin{aligned} 0 &= \left[e_r, e_n e_1^{s+t+2} e_n\right] \\ &= \lambda_r(-(s+t+2)\lambda_{n+r+s+t+1} + 2\lambda_{r+n+s+t+2} - (s+t+2))e_{2n+r+s+t+2} \end{aligned}$$

Hence,

$$\begin{bmatrix} \text{-(r+s+t+2)} & 2 \\ \text{-(s+t+2)} & 2 \end{bmatrix} \begin{bmatrix} \lambda_{n+r+s+t+1} \\ \lambda_{n+r+s+t+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We remind that $(k+1)q \leq r \leq (k+1)q + n$ and r can assume the value $(k+1)q$ if and only if k is odd, namely $r = (k+1)q$ is even. In our case, r is odd, hence $r \not\equiv 0 \pmod p$. Then, the matrix of coefficients of the system above has rank 2, hence the unique solution of the system is $\lambda_{n+r+s+t+1} = \lambda_{n+r+s+t+2} = 0$.

$\square$

Let $q = p^h$ for some positive integer h and $q-(n-2) \leq l \leq q+1$. Denote by $g(n,l)$ the number of graded Lie algebras of maximal class, up to a rescaling of e_n, generated by two elements of weight 1 and n respectively and with the length of the first constituent equal to l. The purpose of all the forthcoming results is to prove that, under the assumption $p > 4n$:

$$g(n,l) \leq \begin{cases} 1, & \text{if } l = q+1 \text{ or } q-(n-2) \\ 2, & \text{if } n > 3 \text{ and } l = q \text{ or } l = q + \varepsilon_n - 2i \text{ for some } 1 \leq i \leq \frac{n-4+\varepsilon_n}{2} \end{cases}$$

For the time being we have proved that, if the length l of the first constituent is such that $q-(n-2) \leq l \leq q+1$, then $\lambda_{l+n} = \cdots = \lambda_{l+q-2} = 0$. Consider now the following:

$$\begin{aligned} -\lambda_{l+q}e_{l+q} &= [e_n, e_n e_1^{l+q-n}] \equiv \\ &\equiv \left(-\binom{l-n}{l-n}\lambda_l - \binom{l-n}{l-n-1}\lambda_{l+q-1} + \lambda_{l+q}\right) e_{l+q} \pmod p. \end{aligned}$$

Hence, $0 \equiv -\lambda_l - (l-n)\lambda_{l+q-1} + 2\lambda_{l+q} \pmod p$. Since $\lambda_l \not\equiv 0 \pmod p$ it is not possible that both λ_{l+q-1} and λ_{l+q} are congruent to zero modulo p. Then we have two possible cases:

(1) $\lambda_l \neq 0, \lambda_{l+n} = \cdots = \lambda_{l+q-2} = 0$ and $\lambda_{l+q-1} \not\equiv 0 \pmod p$)
(2) $\lambda_l \neq 0, \lambda_{l+n} = \cdots = \lambda_{l+q-1} = 0$ and $\lambda_{l+q} \not\equiv 0 \pmod p$.

We note that in case 1 also $\lambda_{l+(n-1)} = 0$. In fact,

$$\begin{aligned} 0 &= [e_{l+(n-1)}, e_n e_1^{q-2n} e_n] \\ &= 2\lambda_{l+(n-1)}\lambda_{l+q-1} e_{l+q-1+n}. \end{aligned}$$

Let $c = l+q-1$ in case 1 and $c = l+q$ in case 2. Then,

$$\lambda_{c+n} = \cdots = \lambda_{c+n+\frac{l-n-3}{2}} = 0.$$

Suppose that

$$\lambda_{2q} = \lambda_{2q+1} = \cdots = \lambda_c = \cdots = \lambda_{c+n+\frac{l-n-3}{2}} = 0. \tag{3.2.2}$$

Define $d = l-1$ in case 1 and $d = l$ in case 2.

Suppose firstly that $2i - \varepsilon_n = -1$. This means that $i = 0$ and $\varepsilon_n = 1$, namely $l = q+1$. In case 1 this implies $\lambda_{2q} \neq 0$ and we are in contradiction with (3.2.2). In case 2, we get $\lambda_{2q} = 0$ and $\lambda_{2q+1} \neq 0$ and we are still in contradiction with (3.2.2).

Suppose now that $2i - \varepsilon_n = 0$. This means that $l = q$. In case 1, we have that $\lambda_{2q-1} \neq 0$. At the same time, the following holds:

$$0 = [e_{q+(n-1)}, e_n e_1^{q-1}],$$

which implies that

$$\begin{aligned} 0 &= \left(\sum_{j=0}^{q-1} (-1)^j \binom{q-1}{j} \lambda_{q+(n-1)+j}\right) e_{2q+2n-2} \\ &= \binom{q-1}{q-1-(n-1)} \lambda_{2q-1} e_{2q+2n-2}. \end{aligned}$$

Hence $\lambda_{2q-1} = 0$ and we are still in contradiction. In case 2, we have that $\lambda_{2q} \neq 0$ and this is an absurd.

If $2i - \varepsilon_n > 0$ consider the following relations:

$$\begin{aligned} [e_{d+n}, e_n e_1^d] &= 0 \\ \vdots \quad & \vdots \quad \vdots \\ [e_{d+n+2i-\varepsilon_n-1}, e_n e_1^{d+2i-\varepsilon_n-1}] &= 0 \end{aligned} \tag{3.2.3}$$

and, if $d = l-1$, also

$$[e_{d+n+2i-\varepsilon_n}, e_n e_1^{d+2i-\varepsilon_n}] = 0.$$

More explicitly, $c = 2q + \varepsilon_n - 2i - 1$ in case 1 and $2q + \varepsilon_n - 2i$ in case 2.

The relations above can be expanded as follows,

$$\left(\sum_{j=0}^{d+r} (-1)^j \binom{d+r}{j} \lambda_{d+n+r+j}\right) e_{2d+2r+n} \quad \text{for } 0 \le r \le 2i - \varepsilon_n - 1$$

and, for $d = l - 1$ and $r = 2i - \varepsilon_n$

$$\left(\sum_{j=0}^{d+r} (-1)^j \binom{d+r}{j} \lambda_{d+n+r+j} \right) e_{2d+2r+n}.$$

In case $d = l - 1$ the relations (3.2.3) give rise to the following linear system:

$$\begin{pmatrix} \binom{q+\varepsilon_n-2i-1}{q-n} & -\binom{q+\varepsilon_n-2i-1}{q-n+1} & \dots & (-1)^{\varepsilon_n}\binom{q+\varepsilon_n-2i-1}{q-n-\varepsilon_n+2i} \\ \binom{q+\varepsilon_n-2i}{q-n-1} & -\binom{q+\varepsilon_n-2i}{q-n} & \dots & (-1)^{\varepsilon_n}\binom{q+\varepsilon_n-2i}{q-n-\varepsilon_n+2i-1} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{q-1}{q-n-2i+\varepsilon_n} & -\binom{q-1}{q-n-2i+\varepsilon_n+1} & \dots & (-1)^{\varepsilon_n}\binom{q-1}{q-n} \end{pmatrix} \cdot v_1^T = 0,$$

where $v_1 = (\lambda_{l+q-1}, \dots, \lambda_{2q-1})$. Define A the matrix of coefficients of the system. Then $\det(A) \equiv \pm\det(B)$, where

$$B = \left(\binom{p+\varepsilon_n - 2i - 2 + k}{p - n - k + j} \right)_{kj} \quad \text{for } k, j = 1, \dots, 2i - \varepsilon_n + 1. \tag{3.2.4}$$

We evaluate $\det(B)$ by means of Corollary (1.3.4) as usual. For the sake of clarity we denote by $m = 2i - \varepsilon_n + 1$. Following the notation of the Corollary, set $x = p - n$ and $y = n + \varepsilon_n - 2i - 2$. Then,

$$\det(B) = \frac{\prod_{1 \le k < j \le m}(j-k)}{\prod_{k=1}^m (x - i + m)!} \prod_{k=1}^m \frac{(x+y+k)!}{(y+2k-1)!} \prod_{k=1}^m (2x+y+k+1)_{k-1}.$$

Of course, the factors $(j-k)$ are not divisible by p. As regards the terms $(x+y+k)!$, we have that

$$(x+y+k)! = (p + \varepsilon_n - 2i - 2 + k)!$$

Such factorials are not divisible by p, since $p + \varepsilon_n - 2i - 2 + k \le p - 1$. Let us analyse the terms $(2x+y+k+1)_{k-1}$. We have that

$$\begin{aligned} (2x+y+k+1)_{k-1} &= (2p - n + \varepsilon_n - 2i - 2 + k + 1)_{k-1} \\ &= (2p - n + \varepsilon_n - 2i - 1 + k) \dots (2p - n + \varepsilon_n - 2i - 3 + 2k). \end{aligned}$$

We note that $(2p - n + \varepsilon_n - 2i - 1 + k) < \dots < (2p - n + \varepsilon_n - 2i - 3 + 2k)$. Now,

$$(2p - n + \varepsilon_n - 2i - 1 + k) \ \ge \ 2p - n - (n-1) = 2p - 2n + 1 > p.$$

Hence each of the factors above is greater than p. Now we prove that they are also smaller than $2p$. In fact,

$$\begin{aligned} (2p - n + \varepsilon_n - 2i - 3 + 2k) &\le 2p - n + \varepsilon_n - 2i - 3 + 4i - 2\varepsilon_n + 2 \\ &= 2p - n + 2i - \varepsilon_n - 1 \le 2p - n - 1 + n - 2 \\ &= 2p - 3. \end{aligned}$$

Hence, $\det(B)$ is not divisible by p and the same holds for $\det(A)$. That means that $v_1 = 0$ and this is absurd.

Consider now the case $d = l$. The relations (3.2.3) give rise to the system:

$$\begin{pmatrix} \binom{q+\varepsilon_n-2i}{q-n} & -\binom{q+\varepsilon_n-2i}{q-n+1} & \dots & (-1)^{\varepsilon_n-1}\binom{q+\varepsilon_n-2i}{q-n-\varepsilon_n+2i-1} \\ \binom{q+\varepsilon_n-2i+1}{q-n-1} & -\binom{q+\varepsilon_n-2i+1}{q-n} & \dots & (-1)^{\varepsilon_n-1}\binom{q+\varepsilon_n-2i+1}{q-n-\varepsilon_n+2i-2} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{q-1}{q-n-2i+\varepsilon_n+1} & -\binom{q-1}{q-n-2i+\varepsilon_n+2} & \dots & (-1)^{\varepsilon_n-1}\binom{q-1}{q-n} \end{pmatrix} \cdot v_2^T = 0,$$

where $v_2 = (\lambda_{l+q}, \dots, \lambda_{2q-1})$. With the same argument as before we get that $\det(A)$ is not divisible by p, hence $v_2 = 0$ and this is an absurd.

All considered, we deduce that it is not possible that

$$\lambda_{2q} = \lambda_{2q+1} = \dots = \lambda_c = \dots = \lambda_{c+n+\frac{l-n-3}{2}} = 0.$$

This means that at least one among $\lambda_{2q}, \lambda_{2q+1}, \dots, \lambda_{2q+n}$ is different from zero. Then, we can apply Lemma (3.2.2) and deduce that, in case 1,

$$\lambda_{l+q-1+n} = \dots = \lambda_{l+2q-3} = 0,$$

while, in case 2,

$$\lambda_{l+q+n} = \dots = \lambda_{l+2q-2} = 0.$$

In case 1, consider

$$\begin{aligned} -\lambda_{2q+l-1}e_{2q+l+n-1} &= \\ &= [e_n, e_n e_1^{2q+l-1-n}] \\ &= \left(\sum_{j=0}^{2q+l-1-n} (-1)^j \binom{2q+l-1-n}{j} \lambda_{n+j} \right) e_{2q+l+n-1} \\ &\equiv \left(-\binom{2q+l-1-n}{q+l-1-n} \lambda_{q+l-1} - \binom{2q+l-1-n}{2q+l-2-n} \lambda_{2q+l-2} + \lambda_{2q+l-1} \right) e_{2q+l-1+n}. \end{aligned}$$

Hence,

$$-2\lambda_{q+l-1} - (l-1-n)\lambda_{2q+l-2} + 2\lambda_{2q+l-1} \equiv 0 \pmod{p}. \tag{3.2.5}$$

Since $\lambda_{q+l-1} \not\equiv 0 \pmod{p}$, the coefficients λ_{2q+l-2} and λ_{2q+l-1} cannot be simultaneously zero modulo p.

Summing all up there are three cases, we are going to deal with.

(1) *Case (l+q-1, l+2q-1).*

$$\begin{aligned} \lambda_{l+n-1} = \dots = \lambda_{l+q-2} &= 0 \\ \lambda_{l+q-1} &\neq 0 \\ \lambda_{l+q-1+n} = \dots = \lambda_{l+2q-2} &= 0 \\ \lambda_{l+2q-1} &\neq 0. \end{aligned}$$

(2) *Case (l+q-1, l+2q-2).*

$$\begin{aligned} \lambda_{l+n-1} = \dots = \lambda_{l+q-2} &= 0 \\ \lambda_{l+q-1} &\neq 0 \\ \lambda_{l+q-1+n} = \dots = \lambda_{l+2q-3} &= 0 \\ \lambda_{l+2q-2} &\neq 0. \end{aligned}$$

(3) *Case (l+q, l+2q).*

$$\begin{aligned}\lambda_{l+n} = \cdots = \lambda_{l+q-1} &= 0\\ \lambda_{l+q} &\neq 0\\ \lambda_{l+q+n} = \cdots = \lambda_{l+2q-2} &= 0\end{aligned}$$

3.2.1. Case (l+q-1, l+2q-1). From (3.2.5) we deduce that $\lambda_{l+q-1} = \lambda_{l+2q-1}$. Now we prove by induction on $0 \leq j \leq l + (n-2) - q - 1$ that

$$\lambda_{l+n-2-j} = \lambda_{l+q+n-2-j}.$$

Let $j = 0$. Then,

$$\begin{aligned}0 &= \left[e_{l+n-2}, e_n e_1^{2q-2n+1} e_n\right] =\\ &= \left((\lambda_{l+n-2} - \lambda_{l+q+n-2})\lambda_{l+2q-1} - \lambda_{l+n-2}(\lambda_{l+q-1} - \lambda_{l+2q-1})\right) e_{l+2q-1+n}\\ &= (\lambda_{l+n-2} - \lambda_{l+q+n-2}) e_{l+2q-1+n},\end{aligned}$$

since $\lambda_{l+2q-1} \neq 0$ and $\lambda_{l+q-1} = \lambda_{l+2q-1}$.

Suppose now to have proved, for a fixed integer j greater than zero, that $\lambda_{l+n-2-i} = \lambda_{l+q+n-2-i}$, for $0 \leq i \leq j-1 < l+n-2-q-1$. Then,

$$\begin{aligned}0 &= \left[e_{l+n-2-j}, e_n e_1^{2q-2n+j+1} e_n\right] =\\ &= \left(\sum_{k=0}^{q-2n+j+1} (-1)^k \binom{q-2n+j+1}{k} (\lambda_{l+n-2-j+k} - \lambda_{l+q+n-2-j+k})\right) \lambda_{l+2q-1} e_{l+2q+n-1}\\ &\quad \pm \lambda_{l+n-2-j}(\lambda_{l+q-1} - \lambda_{l+2q-1}) e_{l+2q+n-1}\\ &= \left(\sum_{k=0}^{q-2n+j+1} (-1)^k \binom{q-2n+j+1}{k} (\lambda_{l+n-2-j+k} - \lambda_{l+q+n-2-j+k})\right) \lambda_{l+2q-1} e_{l+2q+n-1}.\end{aligned}$$

Since $\lambda_{l+n-2-j+k} = \lambda_{l+q+n-2-j+k}$, for $k \geq 1$, we get $\lambda_{l+n-2-j} = \lambda_{l+q+n-2-j}$.

Hence, we have proved that

$$\lambda_{q+1} = \lambda_{2q+1}, \lambda_{q+2} = \lambda_{2q+2}, \ldots, \lambda_{l+(n-2)} = \lambda_{l+(n-2)+q}. \tag{3.2.6}$$

We remind that the length of the first constituent l is odd if and only if n is even and that $q - (n-2) \leq l \leq q+1$. Since q is always odd, $q - n$ is odd if and only if n is even. All considered, it is easily seen that the length of the first constituent can be expressed as

$$l = q + s + \varepsilon_s - n, \quad \text{for some } 1 \leq s \leq n.$$

We note also that $l + n - 2 = q + \varepsilon_s + s - 2 \geq q + \frac{s+\varepsilon_s}{2}$, unless $s + \varepsilon_s = 2$, in which case $l = q - (n-2)$. We will deal in a separate section with this case.

In addition to (3.2.6), we remind that

$$\begin{aligned}\lambda_{l+(n-1)} &= \cdots = \lambda_{l+q-2} = 0\\ \lambda_{l+q-1+n} &= \cdots = \lambda_{l+2q-2} = 0\end{aligned} \tag{3.2.7}$$

Moreover, with he notation adopted,

$$\begin{aligned}
\lambda_{l+(n-2)} &= \lambda_{q+s+\varepsilon_s-2}\\
\lambda_{l+q-1} &= \lambda_{2q+s+\varepsilon_s-n-1}
\end{aligned}$$

Consider now the following relations:

$$\begin{aligned}
0 &= [e_{q+1}, e_n e_1^{q+1-n}]\\
0 &= [e_{q+2}, e_n e_1^{q+2-n}]\\
\vdots &\ \vdots\ \vdots\\
0 &= [e_{q+\frac{s+\varepsilon_s}{2}}, e_n e_1^{q+\frac{s+\varepsilon_s}{2}-n}]\\
\vdots &\ \vdots\ \vdots\\
0 &= [e_{q+n-1}, e_n e_1^{q-1}]
\end{aligned} \tag{3.2.8}$$

They can be expanded as follows:

$$\begin{aligned}
0 &= \sum_{j=0}^{s+\varepsilon_s-3} (-1)^j \binom{q+1-n}{j} \lambda_{q+1+j} e_{2q+2}\\
0 &= \sum_{j=0}^{s+\varepsilon_s-4} (-1)^j \binom{q+2-n}{j} \lambda_{q+2+j} e_{2q+4}\\
\vdots &\ \vdots\ \vdots\\
0 &= \sum_{j=0}^{q+\frac{s+\varepsilon_s}{2}-n} (-1)^j \binom{q+\frac{s+\varepsilon_s}{2}-n}{j} \lambda_{q+\frac{s+\varepsilon_s}{2}+j} e_{2q+s+\varepsilon_s}\\
\vdots &\ \vdots\ \vdots\\
0 &= \sum_{j=0}^{q-1} (-1)^j \binom{q-1}{j} \lambda_{q+n-1+j} e_{2q-2+n}
\end{aligned} \tag{3.2.9}$$

Taking into consideration the fact that $\lambda_{l+q-1} \neq 0$, up to scaling e_n we can assume that $\lambda_{l+q-1} = 1$. The relations (3.2.8) give rise to a non-homogeneous linear system, whose matrix of coefficients is square and its dimension is $n-1$. We note that, due to (3.2.7), the binomial coefficients really involved in (3.2.9) are of the form

(1) $\binom{q+k-n}{j}$, for $1 \leq k \leq n-1$ and $0 \leq j \leq n-2$
(2) $\binom{q+k-n}{j}$, for $1 \leq k \leq n-1$ and $q+s+\varepsilon_s-2n \leq j \leq q-1$.

By Lucas' theorem, in case (1) we have

$$\binom{q+k-n}{j} \equiv \binom{p+k-n}{j} \equiv \binom{2p+k-n}{p+j} \pmod{p}$$

As regards the binomial coefficients of the second form, we note that we can rewrite each j as

$$j = q - 2n + \tilde{j} \quad \text{where } 2 \leq \tilde{j} \leq 2n - 1$$

and then each binomial

$$\binom{q+k-n}{j} \equiv \binom{p+k-n}{k+n-\tilde{j}} \equiv \binom{2p+k-n}{p+k+n-\tilde{j}} \pmod{p}.$$

Let us define the vectors

$$\begin{aligned} u_1 &= [\lambda_{2q+s+\varepsilon_s-n}, \dots, \lambda_{2q}], \\ u_2 &= [\lambda_{q+1}, \dots, \lambda_{q+s+\varepsilon_s-2}], \\ v &= [u1 \mid u2]. \end{aligned}$$

We note that v is a vector of length $n-1$. We define also a vector w of length $n-1$, such that

$$w_i = \binom{2p-n+i}{p+s+\varepsilon_s-n-2-i}, \quad \text{for } 1 \leq i \leq n-1.$$

We are now in a position to translate the relations (3.2.9) in the non-homogeneous linear system

$$Av^T = w^T,$$

where the matrix A is defined as

$$\text{(3.2.10)} \qquad A = \left((-1)^j \binom{2p-n+i}{p+s+\varepsilon_s-n-1-i+j} \right)_{ij}, \quad \text{for } 1 \leq i,j \leq n-1.$$

We want to prove that $\det(A)$ is nonzero. To do that we consider the matrix

$$B = \left(\binom{2p-n+i}{p+s+\varepsilon_s-n-1-i+j} \right)_{ij}, \quad \text{for } 1 \leq i,j \leq n-1$$

having $\det(B) = \pm\det(A)$. In order to unravel a little the notation, we set

$$\begin{aligned} x &= p+s+\varepsilon_s-n-1 \\ y &= p-s-\varepsilon_s+1. \end{aligned}$$

Then,

$$\det(B) = \frac{\prod_{1\leq i<j\leq n-1}(j-i)}{\prod_{i=1}^{n-1}(x-i+n-1)!} \prod_{i=1}^{n-1} \frac{(x+y+i)!}{(y+2i-1)!} \prod_{i=1}^{n-1} (2x+y+i+1)_{i-1}.$$

We analyse the factors involved in the evaluation of $\det(B)$.

- $\prod_{1\leq i<j\leq n-1}(j-i)$ is clearly not divisible by p.
- $\prod_{i=1}^{n-1}(x-i+n-1)!$ can be divisible by p. In particular we expand each $(x-i+n-1)!$ We have that

$$(x-i+n-1)! = (p+s+\varepsilon_s-i-2)!$$

which is divisible by p if and only if $i \leq s+\varepsilon_s-2$. We can conclude that

$$\prod_{i=1}^{n-1}(x-i+n-1)! = p^{s+\varepsilon_s-2}\cdot m_1,$$

for some integer m_1 not divisible by p.

- As regards $\prod_{i=1}^{n-1}(x+y+i)!$, we have that each

$$(x+y+i)! = (2p-n+i)! \quad \text{for some } 1\leq i\leq n-1$$

is divisible by p (but not by higher powers of p). Hence,

$$\prod_{i=1}^{n-1}(x+y+i)! = p^{n-1}\cdot m_2,$$

for some integer m_2 not divisible by p.

- $\prod_{i=1}^{n-1}(2x+y+i+1)_{i-1}$ can be studied looking at each of the

$$(2x+y+i+1)_{i-1} = (3p+s+\varepsilon_s-2n+i)\cdot(3p+s+\varepsilon_s-2n+i+1)\dots(3p+s+\varepsilon_s-2n+2i-2).$$

Being

$$2p < (3p+s+\varepsilon_s-2n+i) < \cdots < (3p+s+\varepsilon_s-2n+2i-2) < 4p,$$

we have that each $(2x+y+i+1)_{i-1}$ is divisible by p if and only if $s+\varepsilon_s-2n+2i-2\geq 0$, or equivalently, $i\geq n+1-\frac{s+\varepsilon_s}{2}$. We conclude that

$$\prod_{i=1}^{n-1}(2x+y+i+1)_{i-1} = p^{\frac{s+\varepsilon_s}{2}-1}\cdot m_3,$$

for some integer m_3 not divisible by p.

- Finally we focus on $\prod_{i=1}^{n-1}(y+2i-1)!$. We have that

$$(y+2i-1)! = (p-s-\varepsilon_s+2i)!$$

is divisible by p if and only if $i\geq\frac{s+\varepsilon_s}{2}$. Hence,

$$\prod_{i=1}^{n-1}(y+2i-1)! = p^{n-\frac{s+\varepsilon_s}{2}}m_4,$$

for some integer m_4 not divisible by p.

Summing all up,

$$\det(B) = \frac{\prod_{1\leq i<j\leq n-1}(j-i)}{p^{s+\varepsilon_s-2}\cdot m_1}\frac{p^{n-1}\cdot m_2}{p^{n-\frac{s+\varepsilon_s}{2}}m_4}p^{\frac{s+\varepsilon_s}{2}-1}\cdot m_3.$$

This infers that $\det(B)$ is not divisible by p and the same holds for $\det(A)$. Then, the solution of our non-homogeneous system is

$$v_j = (-1)^j\binom{n-1}{j} \quad \text{for } 1\leq j\leq n-1.$$

In fact,

$$\begin{aligned}
&\sum_{j=0}^{n-1} \binom{n-1}{j} \binom{2p-n+i}{p+s+\varepsilon_s-n-2-i+j} = \\
&= \binom{2p+i-1}{p+s+\varepsilon_s-n-2-i} \equiv \binom{i-1}{p+s+\varepsilon_s-n-2-i} \equiv 0 \pmod p
\end{aligned}$$

since $i-1 \leq n-2$ and $n < p-2n+1 \leq p+s+\varepsilon_s-n-2-i < p$.

3.2.2. Case (l+q-1, l+2q-2). We will prove that

$$\lambda_{l+n-1-i} = \lambda_{l+q+n-2-i}, \quad \text{for } 0 \leq i \leq l-q+(n-2). \tag{3.2.11}$$

Firstly we consider the following relations, only for i even such that $1 \leq i \leq l-q+n-1$:

$$\begin{aligned}
0 &= [e_{l+q+(n-2)-i}, e_n e_1^{q-2n+i} e_n] \\
&= \left(2\lambda_{l+q+(n-2)-i} + \sum_{k\geq 1} (-1)^k \binom{q-2n+i}{k} \lambda_{l+q+(n-2)-i+k}\right) e_{l+2q-2+n}
\end{aligned} \tag{3.2.12}$$

and

$$\begin{aligned}
0 &= [e_{l+(n-1)-i}, e_n e_1^{q-2n+i} e_n] \\
&= \left(2\lambda_{l+(n-1)-i} + \sum_{k\geq 1} (-1)^k \binom{q-2n+i}{k} \lambda_{l+(n-1)-i+k}\right) e_{l+q-1+n}
\end{aligned} \tag{3.2.13}$$

Consider now

$$\begin{aligned}
0 &= [e_{l+n-2}, e_n e_1^{2q-2n} e_n] \\
&= \left(2n\lambda_{l+(n-1)} - \lambda_{l+q+n-2}\right) \lambda_{l+2q-2} e_{l+2q-2+n}.
\end{aligned}$$

Since $\lambda_{l+(n-1)} = 0$, also $\lambda_{l+q+n-2} = 0$.

Take $1 \le i \le l-q+(n-2)$ and suppose to have proved (3.2.11) for indices smaller than i. If i is odd,

$$\begin{aligned}
0 &= [e_{l+(n-1)-(i+1)}, e_n e_1^{2q-2n+i} e_n] \\
&= \Big(2\lambda_{l+(n-1)-(i+1)} + (2n-i)\lambda_{l+(n-1)-i} + \\
&\quad + \sum_{k\ge 2} (-1)^k \binom{q-2n+i}{k} \lambda_{l+(n-1)-(i+1)+k}\Big) e_{l+2q-2+n} \\
&\quad - \Big(\lambda_{l+q+(n-1)-(i+1)} + \sum_{k\ge 1} (-1)^k \binom{q-2n+i}{k} \lambda_{l+q+(n-1)-(i+1)+k}\Big) e_{l+2q-2+n} \\
&= \big(2\lambda_{l+(n-1)-(i+1)} + (2n-i)\lambda_{l+(n-1)-i} - \lambda_{l+q+(n-1)-(i+1)}\big) e_{l+2q-2+n} \\
&\quad + \Big(\sum_{k\ge 2} (-1)^k \Big(\binom{q-2n+i}{k} + \binom{q-2n+i}{k-1}\Big) \lambda_{l+(n-1)-(i+1)+k}\Big) e_{l+2q-2+n} \\
&= \big(2\lambda_{l+(n-1)-(i+1)} + (2n-i)\lambda_{l+(n-1)-i} - \lambda_{l+q+(n-1)-(i+1)}\big) e_{l+2q-2+n} \\
&\quad + \Big(\sum_{k\ge 2} (-1)^k \binom{q-2n+i+1}{k} \lambda_{l+(n-1)-(i+1)+k}\Big) e_{l+2q-2+n} \\
&= (\lambda_{l+(n-1)-i} - \lambda_{l+q+(n-1)-(i+1)}) e_{l+2q-2+n}.
\end{aligned}$$

Hence $\lambda_{l+(n-1)-i} = \lambda_{l+q+(n-2)-i}$.

If i is even, we get from (3.2.12)

$$0 = 2\lambda_{l+q+(n-2)-i} + \sum_{k\ge 1} (-1)^k \binom{q-2n+i}{k} \lambda_{l+q+(n-2)-i+k}.$$

Since $\lambda_{l+q+(n-2)-i+k} = \lambda_{l+(n-1)-i+k}$, for $1 \le k \le q-2n+i$,

$$0 = 2\lambda_{l+q+(n-2)-i} + \sum_{k\ge 1} (-1)^k \binom{q-2n+i}{k} \lambda_{l+(n-1)-i+k}.$$

At the same time,

$$0 = 2\lambda_{l+(n-1)-i} + \sum_{k\ge 1} (-1)^k \binom{q-2n+i}{k} \lambda_{l+(n-1)-i+k}.$$

Hence, $\lambda_{l+q+(n-2)-i} = \lambda_{l+(n-1)-i}$ and we are done.

Suppose now that $\lambda_{l+q+(n-3)} \ne 0$. Up to scaling e_n, there is no loss of generality in assuming that $\lambda_{l+q+(n-3)} = 1$. We remind that $l > q-(n-2)$, since it will be useful in the following. Then,

$$\begin{aligned}
0 &= [e_{l+q+(n-3)}, e_n e_1^{q-2n+2} e_n] \\
&= \lambda_{l+q+(n-3)} \cdot (2\lambda_{l+2q-1} + (2n-2)\lambda_{l+2q-2}) e_{l+2q-1+n}.
\end{aligned}$$

Since $\lambda_{l+q+(n-3)} \ne 0$, it follows that $\lambda_{l+2q-1} = -(n-1)\lambda_{l+2q-2}$.

Now we prove by induction on

$$0 \le j \le l+(n-3)-q$$

that

$$\lambda_{l+q+(n-3)-j} = (-1)^j \binom{n-1}{j}. \tag{3.2.14}$$

By hypothesis, the assertion is true for $j = 0$. Suppose that (3.2.14) is true for all the integers j such that $0 \leq j < k \leq l + (n-3) - q$.

If k is odd,

$$\begin{aligned} 0 &= [e_{l+q+(n-3)-k}, e_n e_1^{q+k-(2n-3)-2} e_n] \\ &= \left(2\lambda_{l+q+(n-3)-k} - \sum_{j\geq 1} \binom{q+k-(2n-3)-2}{j} \binom{n-1}{k-j} \right) \lambda_{l+2q-2} e_{l+2q-2+n} \end{aligned}$$

and, being $\lambda_{l+2q-2} \neq 0$, it is easily seen that $\lambda_{l+q+(n-3)-k}$ must assume the value $-\binom{n-1}{k}$. In fact,

$$\begin{aligned} &\lambda_{l+q+(n-3)-k} - \sum_{j\geq 1} \binom{q+k-(2n-3)-2}{j} \binom{n-1}{k-j} \\ &= -\sum_{j\geq 0} \binom{q+k-(2n-3)-2}{j} \binom{n-1}{k-j} \\ &= -\binom{q+k-n}{k} = -\binom{q+k-n}{q-n} \equiv \binom{n-1}{n-k-1} \\ &\equiv \binom{n-1}{k} = -\lambda_{l+q+(n-3)-k}. \end{aligned}$$

If conversely k is even, consider the relation

$$[e_{l+q+(n-3)-k}, e_n e_1^{q+k-(2n-3)-1} e_n] = 0.$$

It follows that

$$\begin{aligned} &\left(\lambda_{l+q+(n-3)-k} + \sum_{j\geq 1} \binom{q+k-(2n-3)-1}{j} \binom{n-1}{k-j} \right) (-(n-1))\lambda_{l+2q-2} \\ &= \lambda_{l+q+(n-3)-k}(q+k-(2n-3)-1+(n-1))\lambda_{l+2q-2}. \end{aligned} \tag{3.2.15}$$

Since $\lambda_{l+2q-2} \neq 0$, we can cancel λ_{l+2q-2} from both sides of (3.2.15) and deduce that $\lambda_{l+q+(n-3)-k}$ assume the value $\binom{n-1}{k}$. In fact, the left hand side of (3.2.15), after the simplification just mentioned, becomes

$$-\binom{q+k-n+1}{k}(n-1) \equiv -\frac{(k-n+1)(k-n)\dots(2-n)}{k!}(n-1).$$

The right hand side can be rewritten as follows:

$$\begin{aligned}
\binom{n-1}{k}(q+k-n+1) &\equiv \frac{(n-1)(n-2)\dots(n-k)}{k!}(k-n+1)\\
&\equiv (-1)^{k-1}\frac{(n-1)(2-n)\dots(k-n)}{k!}(k-n+1)\\
&\equiv -\frac{(k-n+1)(k-n)\dots(2-n)}{k!}(n-1),
\end{aligned}$$

being k even.

Hence, we have proved that

$$\begin{aligned}
\lambda_{2q} &= (-1)^{l+(n-3)-q}\binom{n-1}{l+(n-3)-q} = (-1)^{l+(n-3)-q}\binom{n-1}{q-l+2}\\
\lambda_{2q+1} &= (-1)^{l+(n-4)-q}\binom{n-1}{l+(n-4)-q} = (-1)^{l+(n-4)-q}\binom{n-1}{q-l+3}\\
&\vdots \quad \vdots \quad \vdots\\
\lambda_{l+q-1+(n-2)} &= \binom{n-1}{0} = \binom{n-1}{n-1}.
\end{aligned} \tag{3.2.16}$$

Our next step is to determine the coefficients $\lambda_{l+q-1}, \dots, \lambda_{2q-1}$. This will be done employing partially an argument already seen before. We remind that the length of the first constituent l can be expressed as

$$l = q+s+\varepsilon_s-n, \quad \text{for some } 1 \le s \le n.$$

Moreover, with this notation,

$$\begin{aligned}
\lambda_{l+(n-1)} &= \lambda_{q+s+\varepsilon_s-1},\\
\lambda_{l+q-1} &= \lambda_{2q+s+\varepsilon_s-n-1}.
\end{aligned}$$

Consider the relations

$$\begin{aligned}
0 &= \left[e_{q+s+\varepsilon_s-1}, e_n e_1^{q+s+\varepsilon_s-1-n}\right]\\
0 &= \left[e_{q+s+\varepsilon_s}, e_n e_1^{q+s+\varepsilon_s-n}\right]\\
\vdots \quad & \vdots \quad \vdots\\
0 &= \left[e_{q+n-1}, e_n e_1^{q-1}\right].
\end{aligned}$$

We can expand these relations as follows

$$
\begin{aligned}
0 &= \left(\sum_{j\geq 0}(-1)^j\binom{q+s+\varepsilon_s-1-n}{j}\lambda_{q+s+\varepsilon_s-1+j}\right)e_{2q+2s+2\varepsilon_s-2}\\
0 &= \left(\sum_{j\geq 0}(-1)^j\binom{q+s+\varepsilon_s-n}{j}\lambda_{q+s+\varepsilon_s+j}\right)e_{2q+2s+2\varepsilon_s}\\
\vdots\ &\ \vdots\ \ \vdots\\
0 &= \left(\sum_{j\geq 0}(-1)^j\binom{q-1}{j}\lambda_{q+n-1+j}\right)e_{2q-2+2n}.
\end{aligned}
$$

As a consequence the following non-homogeneous linear system holds.

$$
\begin{aligned}
0 &= \sum_{j=q-n-1}^{q-s-\varepsilon_s}(-1)^j\binom{q+s+\varepsilon_s-1-n}{j}\lambda_{q+s+\varepsilon_s-1+j}\\
&\quad+\sum_{j\geq q-s-\varepsilon_s+1}(-1)^j\binom{q+s+\varepsilon_s-1-n}{j}\lambda_{q+s+\varepsilon_s-1+j}\\
0 &= \sum_{j=q-n-2}^{q-s-\varepsilon_s-1}(-1)^j\binom{q+s+\varepsilon_s-n}{j}\lambda_{q+s+\varepsilon_s+j}\\
&\quad+\sum_{j\geq q-s-\varepsilon_s}(-1)^j\binom{q+s+\varepsilon_s-n}{j}\lambda_{q+s+\varepsilon_s+j}\\
\vdots\ &\ \vdots\ \ \vdots\\
0 &= \sum_{j=q-2n+s+\varepsilon_s-1}^{q-n}(-1)^j\binom{q-1}{j}\lambda_{q+n-1+j}\\
&\quad+\sum_{j\geq q-n+1}(-1)^j\binom{q-1}{j}\lambda_{q+n-1+j}.
\end{aligned}
\tag{3.2.17}
$$

Such a system is formed by $n-(s+\varepsilon_s)+1$ equations involving $n+2-s-\varepsilon_s$ indeterminates, namely $\lambda_{l+q-2},\lambda_{l+q-1},\dots,\lambda_{2q-1}$. We have already determined the values of $\lambda_{2q},\dots,\lambda_{l+q+(n-3)}$.

The matrix of the coefficients associated with this system is

$$
A=\left((-1)^{q-n-k-1+j}\binom{q+s+\varepsilon_s-2-n+k}{q-n-1-k+j}\right)_{\substack{k=1,\dots,n-(s+\varepsilon_s)+1\\ j=1,\dots,n-(s+\varepsilon_s)+2}}.
$$

We consider the matrix obtained eliminating the first column, having determinant equal to

$$
\det\left(\binom{q+s+\varepsilon_s-1-n+k}{q-n-1-k+j}\right)_{\substack{k=1,\dots,n-(s+\varepsilon_s)+1\\ j=1,\dots,n-(s+\varepsilon_s)+1}}
\tag{3.2.18}
$$

up to the sign. Such a determinant has been already evaluated (see (3.2.4)) and is not equivalent to zero modulo p.

One solution of the non-homogeneous system (3.2.17) is

$$\lambda_{l+q-2+j} = (-1)^{n-1+j}\binom{n-1}{j}, \quad \text{for } 0 \le j \le n-1.$$

Now we determine the general solution of the associated homogeneous system, whose matrix of coefficients is A. Shifting the index j, A can be rewritten as follows:

$$A = \left((-1)^{q-n-k+j}\binom{q+s+\varepsilon_s-2-n+k}{q-n-k+j}\right)_{\substack{k=1,\dots,n-(s+\varepsilon_s)+1\\ j=0,\dots,n-(s+\varepsilon_s)+1}}.$$

The general solution of the homogeneous system associated with A is

$$\lambda_{l+q-2+r} = (-1)^r\binom{n-(s+\varepsilon_s)+1}{r}\mu \quad \text{for } 0 \le r \le n-(s+\varepsilon_s)+1,$$

where $\mu \in \mathbf{F}$ can be arbitrarily chosen. The general solution of the non-homogeneous system can be expressed as follows:

$$\lambda_{l+q-2+r} = (-1)^{n-1+r}\binom{n-1}{r} + (-1)^r\binom{n-(s+\varepsilon_s)+1}{r}\mu, \quad \text{for } 0 \le r \le n-1.$$

We remind that in the algebra, we are dealing with, $\lambda_{l+q-2} = 0$. This constraint forces μ to assume the value $(-1)^n$. In particular,

$$\begin{aligned} \lambda_{l+q-1} &= (-1)^n(n-1-(n-(s+\varepsilon_s)+1)) \\ &= (-1)^n(s+\varepsilon_s-2), \end{aligned} \tag{3.2.19}$$

where $4 \le s+\varepsilon \le n+1$.

Now we remind that the coefficients $\lambda_{q+1} = \lambda_{2q}, \lambda_{q+2} = \lambda_{2q+1}, \dots, \lambda_{l+(n-2)} = \lambda_{l+q+(n-1)}$ have been explicitly determined (see (3.2.16)). Moreover $\lambda_{l+(n-1)} = \cdots = \lambda_{l+q-2} = 0$. We note that, being $\lambda_{q+1}, \dots, \lambda_{l+q-2}$ assigned, also the coefficients $\lambda_l, \lambda_{l+1}, \dots, \lambda_q$ are determined, since they can be obtained from the following relations

$$\begin{aligned} 0 &= \left[e_q, e_n e_1^{q-n}\right] \\ 0 &= \left[e_{q-1}, e_n e_1^{q-1-n}\right] \\ \vdots &\ \ \vdots \quad \vdots \\ 0 &= \left[e_l, e_n e_1^{l-n}\right]. \end{aligned}$$

In Chapter 4 we will construct an algebra M generated by two elements e_1 and e_n, whose length of the first constituent is equal to l, with the same values for $\lambda_{q+1}, \dots, \lambda_{l+q-2}$. If n is odd $\lambda_l = -\binom{n-1-(q-l+1)}{1}$, otherwise $\lambda_l = \binom{n-1-(q-l+1)}{1}$.

Now we note that $\lambda_l, \lambda_{l+q-1}, \lambda_{l+q}$ are related. In fact,

$$\begin{aligned} -\lambda_{l+q}e_{l+q+n} &= [e_n, e_1^{l+q-n}] \\ &= (-\lambda_l - (l+q-n)\lambda_{l+q-1} + \lambda_{l+q})e_{l+q+n} \end{aligned}$$

and

$$\begin{aligned} 0 &= [e_{l+n-2}, e_n e_1^{q-2n+2} e_n] \\ &= (\lambda_{l+q} + (n-1)\lambda_{l+q-1})\lambda_{l+n-2} e_{l+q+n}. \end{aligned}$$

Taking into consideration the fact that $\lambda_{l+n-2} \neq 0$, we have that

$$\begin{aligned} 0 &= -\lambda_l - (l+q-n)\lambda_{l+q-1} + 2\lambda_{l+q} \\ 0 &= \lambda_{l+q} + (n-1)\lambda_{l+q-1}. \end{aligned}$$

Hence, if n is odd, $\lambda_{l+q-1} = 1$, otherwise $\lambda_{l+q-1} = -1$. In any case we are in contradiction with (3.2.19).

We remind that we got such a contradiction assuming that $\lambda_{l+q+(n-3)} \neq 0$. Suppose now that $\lambda_{l+q+(n-3)} = 0$. This implies that $\lambda_{l+(n-2)} = 0$. We know that at least one among $\lambda_{q+1}, \dots, \lambda_{q+(n-1)}$ must be nonzero. Let $l + (n-k)$ be the greatest integer $r \in \{q+1, \dots, q+(n-1)\}$ such that $\lambda_r \neq 0$. Since $\lambda_{l+q-1} \neq 0$ it is a routine check to verify that k must be even. Moreover, being $1 \leq k \leq l-q+n-1$, we have that $\left[e_n e_1^{q-2n+k} e_n\right] = 0$.

Consider

$$\begin{aligned} 0 &= [e_{l+(n-k)}, e_n e_1^{q-2n+k} e_n] \\ &= \lambda_{l+(n-k)}(2\lambda_{l+q} + (2n-k)\lambda_{l+q-1}) e_{l+q+n}. \end{aligned}$$

We deduce that

$$\lambda_{l+q} = -(n-k/2)\lambda_{l+q-1}, \quad \text{for some even integer } k > 2.$$

Since $\lambda_{l+(n-k)} \neq 0$, up to scaling e_n there is no loss of generality in supposing that $\lambda_{l+(n-k)} = 1$. We are going to prove that

$$\lambda_{l+(n-k)-i} = (-1)^i \binom{n-k/2}{i}, \quad \text{for } 0 \leq i \leq l+(n-k)-q-1. \tag{3.2.20}$$

Of course the assertion is true if $i = 0$. Suppose then to have proved 3.2.20 for all the indices i such that $0 \leq i < j \leq l+(n-k)-q-1$.

If j is odd,

$$0 = \left[e_{l+(n-k)-j}, e_n e_1^{q-(2n-k)+j-1} e_n\right]$$

implies that

$$0 = 2\lambda_{l+(n-k)-j} - \sum_{r \geq 1} \binom{q-2n+j+k-1}{r} \binom{n-k/2}{j-r},$$

hence the value of $\lambda_{l+(n-k)-j}$ is uniquely determined, namely $\lambda_{l+(n-k)-j} = -\binom{n-k/2}{j}$. In fact,

$$\begin{aligned}
0 &= \lambda_{l+(n-k)-j} - \sum_{r\geq 1} \binom{q-2n+j+k-1}{r}\binom{n-k/2}{j-r} \\
&= -\sum_{r\geq 0} \binom{q-2n+j+k-1}{r}\binom{n-k/2}{j-r} \\
&= -\binom{q-n+j+k/2-1}{j} = -\binom{q-n+j+k/2-1}{q-n+k/2-1} \\
&\equiv \binom{n-k/2}{j}.
\end{aligned}$$

Consider now

$$\begin{aligned}
0 &= [e_{q+1}, e_{q+1}] = \left[e_{q+1}, e_n e_1^{q+1-n}\right] \\
&= \sum_{i\geq 0} \binom{q+1-n}{i}\binom{n-k/2}{l+(n-k)-q-1-i} \\
&= \binom{q+1-k/2}{l-k} \equiv \binom{p-k-2+1}{p+\varepsilon_n-2i-k} \not\equiv 0,
\end{aligned}$$

being $l = q+\varepsilon_n - 2i$, for some $0 \leq i \leq \frac{n+\varepsilon_n-2}{2}$. Hence we are in contradiction.

We can conclude that it is not possible that $\lambda_{l+q-1} \neq 0$ and $\lambda_{l+2q-2} \neq 0$.

3.2.3. Case (l+q, l+2q).

We begin proving that also $\lambda_{l+2q-1} = 0$. To do that, consider

$$\begin{aligned}
-\lambda_{l+2q-1} e_{l+2q-1+n} &= [e_n, e_n e_1^{l+2q-1-n}] \\
&= \sum_{i=0}^{l+2q-1-n} \binom{l+2q-1-n}{i} \lambda_{n+i} e_{l+2q-1+n} \\
&= \lambda_{l+2q-1} e_{l+2q-1+n}.
\end{aligned}$$

We deduce that $\lambda_{l+2q-1} = 0$.

At the same time, $\lambda_{l+2q} \neq 0$. Suppose on the contrary that $\lambda_{l+2q} = 0$. Since at least one among $\lambda_{q+1}, \ldots, \lambda_{q+(n-1)}$ is non-zero, we have that, for some $k \in \{1, \ldots, n-1\}$ and $\lambda_{q+k} \in \mathbf{F}$ different from zero,

$$\left[e_n e_1^{q+k-n} e_n - \lambda_{q+k} e_n e_1^{q+k}\right] = 0.$$

We have that

$$\begin{aligned}
0 &= \left[e_{l+q-k}, e_n e_1^{q+k-n} e_n - \lambda_{q+k} e_n e_1^{q+k}\right] \\
&= [e_{l+q-k}, [e_n e_1^{q+k-n}], e_n] - [[e_{l+q-k}, e_n], [e_n e_1^{q+k-n}]] - [e_{l+q-k}, \lambda_{q+k} e_n e_1^{q+k}] \\
&= \left(\sum_{i\geq 0} (-1)^i \binom{q+k-n}{i} \lambda_{l+q-k+i}\right) \lambda_{l+2q} e_{l+2q+n} - (-1)^k \lambda_{l+q} \lambda_{q+k} e_{l+2q+n} \\
&= (-1)^k \lambda_{l+q} \lambda_{q+k} e_{l+2q-n},
\end{aligned}$$

being $\lambda_{l+2q} = 0$. Hence $\lambda_{l+q} = 0$ and we are in contradiction.

We know that at least one among $\lambda_{q+1}, \lambda_{q+2}, \ldots, \lambda_{q+n}$ must be non-zero. We want to discard the possibility that

$$\begin{aligned}
0 &= \lambda_{q+2} = \cdots = \lambda_{l+q-1}, \\
0 &\neq \lambda_{q+1}.
\end{aligned}$$

If this is the case, consider

$$0 = [e_{q+1}, e_n e_1^{q-n+1}]. \tag{3.2.21}$$

Since the length of the first constituent is greater than $q-(n-2)$, we deduce that $l \geq q-n+4$. Expanding (3.2.21) we obtain that

$$0 = \left(\sum_{i\geq 0} (-1)^i \binom{q-n+1}{i} \lambda_{q+1+i}\right) e_{2q+2}.$$

We note that

$$q+1+i \leq 2q-n+2 < 2q-n+4 \leq l+q,$$

hence all $\lambda_{q+1+i} = 0$, for i greater than 0. But we get an absurd, because $\lambda_{q+1} \neq 0$.

Therefore for some $1 \leq k \leq l+n-(q+2)$ we have that

$$\begin{aligned}
0 &= \lambda_{l+n} = \cdots = \lambda_{l+n-k+1} \\
0 &= \lambda_{l+q+n} = \cdots = \lambda_{l+q+n-k+1}
\end{aligned}$$

and one of the following holds:

(1) $\lambda_{l+n-k} = 0, \lambda_{l+q+n-k} \neq 0$;

(2) $\lambda_{l+n-k} \neq 0, \lambda_{l+q+n-k} \neq 0$;

(3) $\lambda_{l+n-k} \neq 0, \lambda_{l+q+n-k} = 0$.

In any case we note that k must be odd. Suppose, on the contrary, that k is even.

Consider, in case (1)

$$\begin{aligned}
0 &= [e_{l+q+n-k}, e_n e_1^{q-2n+k} e_n] \\
&= 2 \cdot \lambda_{l+q+n-k} \cdot \lambda_{l+2q} e_{l+2q+n}.
\end{aligned}$$

We deduce that $\lambda_{l+q+n-k}$ or $\lambda_{l+2q} = 0$ and we are in contradiction.

Similarly, consider in cases (2), (3)

$$\begin{aligned} 0 &= [e_{l+n-k}, e_n e_1^{q-2n+k} e_n] \\ &= 2 \cdot \lambda_{l+n-k} \cdot \lambda_{l+q} e_{l+q+n}. \end{aligned}$$

We deduce that λ_{l+n-k} or $\lambda_{l+q} = 0$ and we are still in contradiction.

Now we analyse separately the three cases. We begin showing that case (1) is impossible.

To do that, consider

$$\begin{aligned} 0 &= [e_{l+(n-k)}, e_n e_1^{2q-2n+k} e_n] \\ &= \big((\lambda_{l+n-k} - \lambda_{l+q+(n-k)})\lambda_{l+2q} - \lambda_{l+(n-k)}(\lambda_{l+q} - \lambda_{l+2q})\big)\, e_{l+2q+n}. \end{aligned}$$

Being $\lambda_{l+n-k} = 0$, we deduce that $\lambda_{l+q+(n-k)}\lambda_{l+2q} = 0$ and we get a contradiction.

Before dealing with cases (2) and (3) we prove two technical results.

Lemma 3.2.3. *Suppose that there exists an odd integer k such that $1 \leq k \leq l+n-(q+2)$ and that*

$$\begin{aligned} \lambda_{l+n} = \cdots = \lambda_{l+n-k+1} = 0 \\ \lambda_{l+q+n} = \cdots = \lambda_{l+q+n-k+1} = 0. \end{aligned}$$

Finally, suppose that $\lambda_{l+n-k}, \lambda_{l+q+n-k} \neq 0$.

Then, $\lambda_{q+2} = \lambda_{2q+2}, \ldots, \lambda_{l+(n-1)} = \lambda_{l+q+(n-1)}$.

Lemma 3.2.4. *Suppose that there exists an odd integer k such that $1 \leq k \leq l+n-(q+2)$ and that*

$$\begin{aligned} \lambda_{l+n} = \cdots = \lambda_{l+n-k+1} = 0 \\ \lambda_{l+q+n} = \cdots = \lambda_{l+q+n-k+1} = 0. \end{aligned}$$

Finally, suppose that $\lambda_{l+n-k} \neq 0$, while $\lambda_{l+q+n-k} = 0$.

Then, $\lambda_{2q+1} = \lambda_{2q+2} = \cdots = \lambda_{l+q+(n-1)} = 0$.

Proof. (Lemma (3.2.3)) From

$$0 = \left[e_{\frac{l+2q+1+n}{2}}, e_n e_1^{\frac{l+2q+1-n}{2}} \right]$$

we deduce that

$$0 = \frac{l+1-n}{2}\lambda_{l+q} - \lambda_{l+q+1} - \frac{l+1-n}{2}\lambda_{l+2q} + \lambda_{l+2q+1}. \tag{3.2.22}$$

Moreover, from

$$0 = [e_{l+(n-k)}, e_n e_1^{q-2n+k+1} e_n] \tag{3.2.23}$$

$$0 = [e_{l+q+(n-1)}, e_n e_1^{q-2n+k+1} e_n] \tag{3.2.24}$$

we deduce that

$$0 = \lambda_{l+q-1} + (n-(k+1)/2)\lambda_{l+q}, \tag{3.2.25}$$

$$0 = \lambda_{l+2q-1} + (n-(k+1)/2)\lambda_{l+2q}. \tag{3.2.26}$$

Combining (3.2.22) - (3.2.26) we have that

$$\lambda_{l+q} = \lambda_{l+2q} \quad \text{and} \quad \lambda_{l+q+1} = \lambda_{l+2q+1}. \tag{3.2.27}$$

The next step is to prove that

$$\lambda_{l+(n-k)-j} = \lambda_{l+q+(n-k)-j}, \quad \text{for } 0 \le j \le l+(n-k)-(q+2). \tag{3.2.28}$$

Before proceeding we note that

$$[e_n e_1^{q-2n+r+2} e_n] = 0, \text{for } 0 \le r \le l+(n-1)-(q+2),$$

because $q-n+r+2 \le l-1$.

We begin proving that $\lambda_{l+(n-k)} = \lambda_{l+q+(n-k)}$. Since

$$\begin{aligned} 0 &= [e_{l+(n-k)}, e_n e_1^{2q-2n+k} e_n] \\ &= \big((\lambda_{l+(n-k)} - \lambda_{l+q+(n-k)})\lambda_{l+2q} - \lambda_{l+(n-k)}(\lambda_{l+q} - \lambda_{l+2q})\big)\, e_{l+2q+n} \end{aligned}$$

and $\lambda_{l+q} = \lambda_{l+2q}$ we deduce that $\lambda_{l+(n-k)} = \lambda_{l+q+(n-k)}$.

Suppose now to have proved (3.2.28) for all the indices $j < r$, where r is a positive integer not greater than $l+(n-1)-(q+2)$. We distinguish two cases, namely r odd or r even.

If r is odd,

$$\begin{aligned} 0 &= [e_{l+(n-1)-r}, e_n e_1^{q-2n+r+1} e_n] \\ &= \left(2\lambda_{l+(n-1)-r} + \sum_{i \ge 1} (-1)^i \binom{q-2n+r+1}{i} \lambda_{l+(n-1)-r+i}\right) \lambda_{l+q} e_{l+q+n} \end{aligned}$$

and

$$\begin{aligned} 0 &= [e_{l+q+(n-1)-r}, e_n e_1^{q-2n+r+1} e_n] \\ &= \left(2\lambda_{l+q+(n-1)-r} + \sum_{i \ge 1} (-1)^i \binom{q-2n+r+1}{i} \lambda_{l+q+(n-1)-r+i}\right) \lambda_{l+2q} e_{l+q+n}. \end{aligned}$$

Hence, $\lambda_{l+(n-1)-r} = \lambda_{l+q+(n-1)-r}$.

If r is even, from

$$\begin{aligned} 0 &= [e_{l+(n-1)-r}, e_n e_1^{q-2n+r+2} e_n], \\ 0 &= [e_{l+q+(n-1)-r}, e_n e_1^{q-2n+r+2} e_n] \end{aligned}$$

we get that

$$\begin{aligned} 0 &= (2\lambda_{l+q+1} + (2n-2-r)\lambda_{l+q})\lambda_{l+(n-1)-r} \\ &\quad + \left(\sum_{i \ge 1} (-1)^i \binom{q-2n+r+2}{i} \lambda_{l+(n-1)-r+i}\right) \lambda_{l+q+1}, \\ 0 &= (2\lambda_{l+2q+1} + (2n-2-r)\lambda_{l+2q})\lambda_{l+q+(n-1)-r} \\ &\quad + \left(\sum_{i \ge 1} (-1)^i \binom{q-2n+r+2}{i} \lambda_{l+q+(n-1)-r+i}\right) \lambda_{l+2q+1}. \end{aligned}$$

Taking into consideration (3.2.25) - (3.2.28), the fact that $k \not\equiv 0 \pmod p$ and the inductive hypothesis, we deduce that also in this case $\lambda_{l+(n-k)-r} = \lambda_{l+q+(n-k)-r}$. $\square$

PROOF. (Lemma (3.2.4)) Since

$$\begin{aligned} 0 &= [e_{l+n-k}, e_n e_1^{2q-2n+k} e_n] \\ &= ((\lambda_{l+n-k} - \lambda_{l+q+n-k})\lambda_{l+2q} - \lambda_{l+n-k}(\lambda_{l+q} - \lambda_{l+2q}))\, e_{l+2q+n} \\ &= \lambda_{l+n-k}(2\lambda_{l+2q} - \lambda_{l+q}) e_{l+q+n} \end{aligned}$$

necessarily $\lambda_{l+q} = 2\lambda_{l+2q}$.

In order to prove the thesis, we must prove that

$$\lambda_{l+q+n-k-j} = 0 \quad 0 \le j \le l+n-(q+k+2).$$

We proceed by induction. The base case is trivially true. Suppose to have proved the assertion for all the indices j smaller than r, where r is a positive integer not greater than $l+n-(q+k+2)$. As usual we distinguish the case r odd from r even.

To start with, suppose that r is odd. Then,

$$\begin{aligned} 0 &= [e_{l+q+n-k-r}, e_n e_1^{q-2n+k+r} e_n] \\ &= 2\lambda_{l+q+n-k-r}\lambda_{l+2q} e_{l+2q+n}. \end{aligned}$$

In the case r is even, we begin considering that

$$\begin{aligned} 0 &= [e_{l+n-k-r}, e_n e_1^{q-2n+k+r} e_n] \\ &= \left(\sum_{i\ge 1} (-1)^i \binom{q-2n+k+r}{i} \lambda_{l+n-k-r+i} \right) \lambda_{l+q} e_{l+2q+n}. \end{aligned}$$

implies that

$$0 = \sum_{i\ge 1} (-1)^i \binom{q-2n+k+r}{i} \lambda_{l+n-k-r+i}. \tag{3.2.29}$$

Then,

$$\begin{aligned} 0 &= [e_{l+n-k-r}, e_n e_1^{2q-2n+k+r} e_n] \\ &= \left(\lambda_{l+n-k-r} + \sum_{i\ge 1} (-1)^i \binom{q-2n+k+r}{i} \lambda_{l+n-k-r+i} \right) \lambda_{l+2q} e_{l+2q+n} \\ &\quad -\lambda_{l+n+q-k-r}\lambda_{l+2q} e_{l+2q+n} \\ &\quad -\lambda_{l+n-k-r}(\lambda_{l+q} - \lambda_{l+2q}) e_{l+2q+n} \\ &= (\lambda_{l+n-k-r} - \lambda_{l+n+q-k-r} - \lambda_{l+n-k-r})\lambda_{l+2q} e_{l+2q+n}. \end{aligned}$$

Being $\lambda_{l+2q} \neq 0$ we deduce that $\lambda_{l+n+q-k-r} = 0$ and we are done. $\square$

Now we proceed proving cases (2) and (3).

Case (2). In this case the hypotheses of Lemma (3.2.3) are satisfied. We remind that the length l of the first constituent is of the form

$$l = q + s + \varepsilon_s - n, \quad \text{for some } 1 \le s \le n,$$

where $3 \leq s \leq n$.

Suppose firstly that $k = 1$. Then $\lambda_{q+2} = \lambda_{2q+2}, \ldots, \lambda_{l+(n-1)} = \lambda_{l+q+(n-1)}$.

We will use an argument employed previously to prove that, if $\lambda_{l+q} = \mu$, then

$$\lambda_{l+q+i} = (-1)^i \binom{n-1}{i} \mu, \quad \text{for } 0 \leq i \leq n-1. \tag{3.2.30}$$

Then we consider the following $n-2$ relations:

$$\begin{array}{rcl} 0 & = & [e_{q+2}, e_n e_1^{q+2-n}] \\ \vdots & \vdots & \vdots \\ 0 & = & [e_{q+\frac{s+\varepsilon_s}{2}}, e_n e_1^{q+\frac{s+\varepsilon_s}{2}-n}] \\ \vdots & \vdots & \vdots \\ 0 & = & [e_{q+n-1}, e_n e_1^{q-1}] \end{array} \tag{3.2.31}$$

We remind also that $\lambda_{l+q+1} = -(n-1)\lambda_{l+q}$.

Let us define the vectors

$$\begin{array}{rcl} u_1 & = & [\lambda_{2q+s+\varepsilon_s-n+2}, \ldots, \lambda_{2q+1}], \\ u_2 & = & [\lambda_{q+2}, \ldots, \lambda_{q+s+\varepsilon_s-1}], \\ v & = & [u_1 \mid u_2] \end{array}$$

and the vectors w_i such defined:

$$w_i = \binom{2p-n+1+i}{p+s+\varepsilon_s-n-1-i} \lambda_{l+q} - \binom{2p-n+1+i}{p+s+\varepsilon_s-n-i} \lambda_{l+q+1},$$

for $1 \leq i \leq n-2$.

Relations (3.2.31) give rise to the non-homogeneous linear system

$$Av^T = w^T,$$

where

$$A = \left((-1)^j \binom{2p-n+1+i}{p+s+\varepsilon_s-n-i+j} \right)_{ij}, \quad \text{for } 1 \leq i, j \leq n-2. \tag{3.2.32}$$

We note that such a matrix is obtained from (3.2.10) eliminating the first row and the first column. In the same way we proved that the determinant of the matrix (3.2.10) is different from zero, it can be seen that also in this case $\det(A)$ is non-zero.

We conclude that the solution of the non-homogeneous system is unique and is

$$v_j = (-1)^{j+1} \binom{n-1}{j+1} \mu \quad \text{for } 1 \leq j \leq n-2.$$

Hence,

$$\lambda_{l+q+i} = (-1)^i \binom{n-1}{i} \mu \quad \text{for } 0 \leq 1 \leq n-1$$

and $\lambda_{q+2} = \lambda_{2q+2}, \ldots, \lambda_{l+(n-1)} = \lambda_{l+q+(n-1)}$.

Then we can easily obtain the values of $\lambda_{q+1}, \dots, \lambda_l$ by means of

$$\begin{array}{ccc} 0 & = & [e_{q+1}, e_n e_1^{q+1-n}], \\ \vdots & \vdots & \vdots \\ 0 & = & [e_l, e_n e_1^{l-n}]. \end{array}$$

Now we consider the case $k \geq 3$. Suppose that $l + n - k < q + \frac{s+\varepsilon_s}{2}$. We can now consider the following relations

$$\begin{array}{ccc} 0 & = & [e_{q+2}, e_n e_1^{q+2-n}], \\ \vdots & \vdots & \vdots \\ 0 & = & [e_{l+(n-k)}, e_n e_1^{l+(n-k)-n}]. \end{array}$$

Such relations give rise to a homogeneous linear system formed by $l+(n-k)-q-1$ equations. Since $\lambda_{l+(n-k)+1} = \cdots = \lambda_{l+q-1} = 0$, such equations involve only the $l+(n-k)-q-1$ parameters $\lambda_{q+2}, \dots, \lambda_{l+(n-k)}$. Moreover, the equations are linearly independent, being the system triangular. Hence, $\lambda_{q+2} = \cdots = \lambda_{l+n-k} = 0$ and we are in contradiction.

Suppose now that $l + (n-k) \geq q + \frac{s+\varepsilon_s}{2}$ and consider the relations

$$\begin{array}{ccc} 0 & = & [e_{q+2}, e_n e_1^{q+2-n}] \\ \vdots & \vdots & \vdots \\ 0 & = & [e_{q+\frac{s+\varepsilon_s}{2}}, e_n e_1^{q+\frac{s+\varepsilon_s}{2}-n}] \\ \vdots & \vdots & \vdots \\ 0 & = & [e_{q+(n-k)+2}, e_n e_1^{q+(n-k)+2-n}]. \end{array} \tag{3.2.33}$$

Define the vectors

$$\begin{array}{ccc} u_1 & = & [\lambda_{2q+s+\varepsilon_s-n}, \dots, \lambda_{2q+1}] \\ u_2 & = & [\lambda_{q+2}, \dots, \lambda_{l+(n-k)}] \\ v & = & [u_1 \mid u_2]. \end{array}$$

Relations (3.2.33) give rise to the homogeneous linear system

$$Av^T = 0,$$

where

$$A = \left((-1)^j \binom{2p-n+1+i}{p+s+\varepsilon_s-n-2-i+j} \right)_{ij}, \quad \text{for } 1 \leq i, j \leq n-k+1. \tag{3.2.34}$$

Such a determinant is easily seen to be different from zero, hence $v = 0$. But this is absurd, since $\lambda_{l+(n-k)} \neq 0$.

Case (3) Since the hypotheses of Lemma (3.2.4) are satisfied, we have that $\lambda_{2q+1} = \lambda_{2q+2} = \cdots = \lambda_{l+q+(n-1)} = 0$. We remind that, with the notation previously adopted, $l + (n-k) = q + s + \varepsilon_s - k$ where $3 \leq s + \varepsilon_s \leq n+1$.

Firstly we deal with the case $s+\varepsilon_s = n+1$ or n, which means $l = q+1$ or $l = q$. We have that

$$\begin{aligned} 0 &= [e_{l+(n-k)}, e_n e_1^{q-2n+k+1} e_n] \\ &= \lambda_{l+(n-k)} \left(\lambda_{l+q+1} + (n-(k+1)/2)\lambda_{l+q}\right) e_{l+q+1+n}. \end{aligned}$$

If $l = q+1$ it follows that $\lambda_{l+(n-k)} = 0$ and we are in contradiction. If $l = q$, we get

$$\lambda_{l+(n-k)}\lambda_{l+q} = 0$$

and we are in contradiction, because neither $\lambda_{l+(n-k)} = 0$ nor $\lambda_{l+q} = 0$.

We deal now with the case $4 \le s+\varepsilon_s \le n-1$. Consider the relations

$$\begin{aligned} 0 &= [e_{q+s+\varepsilon_s}, e_n e_1^{q+s+\varepsilon_s-n}] \\ \vdots &\;\;\vdots\;\; \vdots \\ 0 &= [e_{q+n-1}, e_n e_1^{q-1}], \end{aligned} \tag{3.2.35}$$

the vector

$$v = [\lambda_{2q+s+\varepsilon_s-n}, \dots, \lambda_{2q}]$$

and the associated homogeneous linear system

$$Av^T = 0,$$

where the matrix A is such defined:

$$A = \left((-1)^j \binom{2p+s+\varepsilon_s-n-1+i}{p-n-i+j} \right) \qquad \begin{array}{c} 1 \le i \le n-s-\varepsilon_s, \\ 1 \le j \le n-s-\varepsilon_s+1 \end{array} \tag{3.2.36}$$

The rank of A is $n-s-\varepsilon_s$. If we denote $\lambda_{l+q} = \mu$, for some non-zero $\mu \in \mathbf{F}$,

$$v_k = (-1)^k \binom{n-s-\varepsilon_s}{k} \mu \quad \text{for } k = 0, \dots, n-s-\varepsilon_s.$$

In particular,

$$\lambda_{l+q+1} = -(n-s-\varepsilon_s)\lambda_{l+q}.$$

At the same time, being

$$0 = [e_{l+(n-k)}, e_n e_1^{q-2n+k+1} e_n]$$

we have that $\lambda_{l+q+1} = -(n-(k+1)/2)\lambda_{l+q}$, where $1 \le k \le s+\varepsilon_s-2$ and we are in contradiction.

For the time being we have proved the following intermediate result.

LEMMA 3.2.5. *Let L be a graded Lie algebra of maximal class generated by two elements of weight 1 and n over a field $\mathbf{F}$ of characteristic $p > 4n$. Suppose that the length of the first constituent is*

$$l = p^h + \varepsilon_n - 2i, \quad \text{for } 0 \le i \le \frac{n+\varepsilon_n-4}{2}$$

and some positive integer h. Then, up to a rescaling of e_n, there are two possibilities for L.

- Case 1:

$$\begin{aligned} 0 &= \lambda_{l+(n-1)} = \cdots = \lambda_{l+q-2}, \\ 0 &= \lambda_{l+q-1+n} = \cdots = \lambda_{l+2q-2} \end{aligned}$$

  and

$$\lambda_{l+q-1+i} = (-1)^i \binom{n-1}{i}, \quad \textit{for } 0 \le i \le n-1.$$

- Case 2:

$$\begin{aligned} 0 &= \lambda_{l+n} = \cdots = \lambda_{l+q-1}, \\ 0 &= \lambda_{l+q+n} = \cdots = \lambda_{l+2q-1}, \end{aligned}$$

  and

$$\lambda_{l+q+i} = (-1)^i \binom{n-1}{i}, \quad \textit{for } 0 \le i \le n-1.$$

Moreover, in both cases, $\lambda_{q+1} = \lambda_{2q+1}, \ldots, \lambda_{q+n} = \lambda_{2q+n}$.

Corollary 3.2.6. *Suppose that n is odd and that $l = q+1$, where $q = p^h$, for some prime p and positive integer h. Let L be a graded Lie algebra of maximal class with first constituent length l. Then, $\lambda_{q+n} = 0$ and L belongs to case 1.*

Proof. Since $[e_n e_1 e_n] = 0$, we have that $[e_q, [e_n e_1 e_n]] = 0$. This implies that $\lambda_{q+1+n} = 0$. Now, consider

$$\begin{aligned} -\lambda_{q+1+n} e_{q+2n+1} &= [e_n, e_n e_1^{q+1}] \\ &= (-\lambda_{q+n} + \lambda_{q+n+1}) e_{q+2n+1}. \end{aligned}$$

Hence, $\lambda_{q+n} = 0$ and we are done. $\square$

Using Lemma 3.2.5 we prove the following.

Lemma 3.2.7. *In the same hypotheses of Lemma 3.2.5 the following holds, respectively for case 1 or 2.*

- Case 1. *For any positive integer h,*

$$0 = \lambda_{l+hq-1+n} = \cdots = \lambda_{l+(h+1)q-2}$$

  and

$$\lambda_{l+hq-1+i} = (-1)^i \binom{n-1}{i}, \quad \textit{for } 0 \le i \le n-1.$$

- Case 2. *For any positive integer h,*

$$0 = \lambda_{l+hq+n} = \cdots = \lambda_{l+(h+1)q-1}$$

  and

$$\lambda_{l+hq+i} = (-1)^i \binom{n-1}{i}, \quad \textit{for } 0 \le i \le n-1.$$

Proof. We prove this Lemma by induction on h. Of course the assertions are true for $h = 0$ by Lemma 3.2.5. Suppose then that thesis is true for all integers h smaller than a positive integer k. We set $s = l + (k-1)q - 2$ in case 1, while $s = l + (k-1)q - 1$ in case 2. In both cases, we have that $\lambda_{q+1}, \dots, \lambda_{q+(l-q+(n-2))}$ are non-zero. In case 2 also $\lambda_{l+(n-1)}$ is non-zero. Such a fact means that

$$\begin{array}{rcl} 0 & = & [e_n e_1^{q+1-n} e_n] - \lambda_{q+1}[e_n e_1^{q+1}], \\ \vdots & \vdots & \vdots \\ 0 & = & [e_n e_1^{l-2} e_n] - \lambda_{l+(n-2)}[e_n e_1^{l+n-2}] \end{array}$$

and, in case 2, also $0 = [e_n e_1^{l-1} e_n] - \lambda_{l+(n-1)}[e_n e_1^{l+n-1}]$.

We prove that

$$\lambda_{s+i} = \lambda_{s+q+i}, \quad \text{for } 1 \leq i \leq l - q + (n-2) \tag{3.2.37}$$

and, in case 2, also $\lambda_{s+l-q+(n-1)} = \lambda_{s+l+(n-1)}$.

We have that, taken i such that $1 \leq i \leq l - q + (n-1)$,

$$0 = [e_s, [e_n e_1^{q+i-n} e_n]].$$

In fact,

$$\begin{array}{rcl} [e_s, [e_n e_1^{q+i-n} e_n]] & = & [e_s, e_n e_1^{q+i-n}, e_n] - [e_s, e_n, e_n e_1^{q+i-n}] \\ & = & -\left(\displaystyle\sum_{j=0}^{n-1} \binom{q+i-n}{j+1} \binom{n-1}{j} \right) \lambda_{s+q+i} e_{s+q+i+n} - 0 \\ & = & -\displaystyle\binom{q+i-1}{n} \lambda_{s+q+i} e_{s+q+i+n} \\ & \equiv & -\displaystyle\binom{i-1}{n} \lambda_{s+q+i} e_{s+q+i+n} \pmod p \\ & \equiv & 0 \pmod p, \end{array}$$

being $i \leq n$.

We proceed proving (3.2.37). Consider firstly $i = 1$. We have that

$$\begin{array}{rcl} 0 & = & [e_s, [e_n e_1^{q+1-n} e_n] - \lambda_{q+1}[e_n e_1^{q+1}]] \\ & = & -\lambda_{q+1}(-\lambda_{s+1} + \lambda_{s+q+1}) e_{s+q+1+n}. \end{array}$$

Being $\lambda_{q+1} \neq 0$ we deduce that $\lambda_{s+1} = \lambda_{s+q+1}$.

Suppose now to have proved (3.2.37) for all integers i such that $1 \leq i < j \leq l - q + (n-2)$ in case 1 or such that $1 \leq i < j \leq l - q + (n-1)$ in case 2. Then,

$$\begin{array}{rcl} 0 & = & [e_s, [e_n e_1^{q+j-n} e_n] - \lambda_{q+j}[e_n e_1^{q+j}]] \\ & = & -\lambda_{q+j} \left(\displaystyle\sum_{i=1}^{j} (-1)^i (\lambda_{s+i} - \lambda_{s+q+i}) \right) e_{s+q+j+n} \end{array}$$

and, being $\lambda_{q+j} \neq 0$, we have that $\lambda_{s+j} = \lambda_{s+q+j}$.

We remind that in both cases $\lambda_{l+q}, \dots, \lambda_{2q-1}$ are non-zero and, in case 1, also $\lambda_{l+q-1} \neq 0$.

In case 1 consider the relations

$$\begin{array}{rcl} 0 & = & [e_{l+(k-2)q+(n-1)}, e_n e_1^{l+q-n-1} e_n - \lambda_{l+q-1} e_n e_1^{l+q-1}], \\ \vdots & \vdots & \vdots \\ 0 & = & [e_{l+(k-2)q+(n-1)}, e_n e_1^{2q-1-n} e_n - \lambda_{2q-1} e_n e_1^{2q-1}]. \end{array}$$

We have that $[e_{l+(k-2)q+(n-1)}, e_n e_1^{q+i} e_n] = 0$, for $l-n-1 \leq i \leq q-n-1$. In fact,

$$\begin{array}{l} [e_{l+(k-2)q+(n-1)}, e_n e_1^{q+i} e_n] = \\ \quad = \; [e_{l+(k-2)q+(n-1)}, [e_n e_1^{q+i}], e_n] - [e_{l+(k-2)q+(n-1)}, e_n, e_n e_1^{q+i}] \\ \quad = \; \left(\sum_{j=0}^{i} (-1)^j \binom{i}{j} (\lambda_{l+(k-2)q+(n-1)+j} - \lambda_{l+(k-1)q+(n-1)+j}) \right) \lambda_{l+(k-1)q+(2n-1)+i} e_{l+(k-1)q+3n-1+i} \end{array}$$

and we note that all the λ coefficients involved in the sum are zero by inductive hypotheses. Hence,

$$0 = [e_{l+(k-2)q+(n-1)}, e_n e_1^{q+i}], \tag{3.2.38}$$

for $l-1 \leq i \leq q-1$. Expanding each of these relations we get:

$$\begin{array}{rcl} 0 & = & (-1)^{q-n} \left(\sum_{j=q-n}^{i} \binom{i}{j} \binom{n-1}{j-q+n} \right) - \left(\sum_{j=q-n}^{i} (-1)^j \binom{i}{j} \lambda_{l+(k-1)q+n-1+j} \right) \\ & = & (-1)^{n+1} \binom{i+n-1}{i+n-q} - \left(\sum_{j=q-n}^{i} (-1)^j \binom{i}{j} \lambda_{l+(k-1)q+n-1+j} \right) \\ & \equiv & 0 - \left(\sum_{j=q-n}^{i} (-1)^j \binom{i}{j} \lambda_{l+(k-1)q+n-1+j} \right) \quad \pmod{p}. \end{array}$$

Our goal is to determine the values of $\lambda_{l+kq-1+j}$, for $0 \leq j \leq n-1$, or, with the notation adopted,

$$\lambda_{s+q+j}, \quad \text{for } 1 \leq j \leq n.$$

Indeed we have already determined the values of λ_{s+q+j}, for $1 \leq j \leq l-q+(n-2)$. If $l = q+1$ all λ_{s+q+j} are determined, except λ_{s+q+n}, which, for the moment, we leave undetermined.

Suppose then that $l < q+1$. We remind that l is of the form $q + \varepsilon_n - 2f$, where $0 \leq f \leq \frac{n-4+\varepsilon_n}{2}$. For the sake of clarity let us denote $t = l - q + (n-3)$.

Relations (3.2.38) can be expanded as follows:

$$\begin{aligned} 0 &= \sum_{j=0}^{t}\binom{p+\varepsilon_n-2f-1}{p-n+j}\binom{n-1}{j}+\sum_{j=t+1}^{n-1}(-1)^j\binom{p+\varepsilon_n-2f-1}{p-n+j}\lambda_{s+q+1+j},\\ \vdots\ &\ \vdots\ \ \vdots\\ 0 &= \sum_{j=0}^{t}\binom{p-1}{p-n+j}\binom{n-1}{j}+\sum_{j=t+1}^{n-1}(-1)^j\binom{p-1}{p-n+j}\lambda_{s+q+1+j}.\end{aligned}$$

If we denote $v=[\lambda_{s+q+2+t},\dots,\lambda_{s+q+n}]$, such relations give rise to the non-homogeneous linear system

$$Av^T=-w^T,$$

where

$$w=\left(\sum_{j=0}^{t}\binom{p+\varepsilon_n-2f-1}{p-n+j}\binom{n-1}{j},\dots,\sum_{j=0}^{t}\binom{p-1}{p-n+j}\binom{n-1}{j}\right)$$

and

$$A=\left((-1)^j\binom{p+\varepsilon_n-2f-1+i}{p-n+j}\right)_{ij},$$

for $0\le i\le 2f-\varepsilon_n$ and $j=t+1,\dots,n-1$. Being A in row echelon form, its rank is $2f-\varepsilon_n+1$, while the length of v is $m=2f-\varepsilon_n+2$.

One solution of the non-homogeneous system is

$$\left((-1)^{t+1}\binom{n-1}{t+1},\dots,(-1)^{n-1}\binom{n-1}{n-1}\right).$$

The null space of the matrix A has dimension 1 and is spanned by

$$c=\left(\binom{n-t-2}{0},-\binom{n-t-2}{1},\dots,(-1)^{n-t}\binom{n-t-2}{n-t-2}\right).$$

In fact, the generic row of A, for $0\le i\le 2f-\varepsilon_n$, is

$$\binom{p+\varepsilon_n-2f-1+i}{p+\varepsilon_n-2f-2+j},\quad \text{for } 0\le j\le n-t-2$$

and

$$\begin{aligned} Ac^T &= \sum_{j\ge 0}\binom{p+\varepsilon_n-2f-1+i}{p+\varepsilon_n-2f-2+j}\binom{2f-\varepsilon_n+1}{j}\\ &= \binom{p+i}{i+1}\equiv 0 \pmod p.\end{aligned}$$

Therefore the general solution of the non-homogeneous linear system can be expressed as

$$\lambda_{l+kq-1+j}=(-1)^j\left(\binom{n-1}{j}+\mu\binom{n-t-2}{j-t-1}\right),\quad \text{for } 0\le j\le n-1,$$

for some $\mu\in\mathbf{F}$. Herein we note that, if $\mu\ne -1$, then $\lambda_{l+kq-1+(n-1)}\ne 0$, otherwise $\lambda_{l+kq-1+(n-1)}=0$, while $\lambda_{l+kq-1+(n-2)}\ne 0$.

In case 2 one can consider the relations

$$\begin{array}{rcl} 0 & = & [e_{l+(k-2)q+n}, e_n e_1^{l+q-n} e_n - \lambda_{l+q} e_n e_1^{l+q}], \\ \vdots & \vdots & \vdots \\ 0 & = & [e_{l+(k-2)q+n}, e_n e_1^{2q-1-n} e_n - \lambda_{2q-1} e_n e_1^{2q-1}]. \end{array}$$

Similarly to the case 1, we have that $[e_{l+(k-2)q+n}, e_n e_1^{q+i} e_n] = 0$, for $l-n \le i \le q-n-1$.

Hence,

$$0 = [e_{l+(k-2)q+n}, e_n e_1^{q+i}], \tag{3.2.39}$$

for $l \le i \le q-1$. Expanding (3.2.39) for any i we get:

$$0 \equiv \left(\sum_{j=q-n}^{i} (-1)^j \binom{i}{j} \lambda_{l+(k-1)q+n+j} \right) \pmod p.$$

Even in this case we want to determine the values of λ_{l+kq+j}, for $0 \le j \le n-1$, or, with the notation adopted,

$$\lambda_{s+q+j}, \quad \text{for } 1 \le j \le n.$$

Indeed we have already determined the values of λ_{s+q+j}, for $1 \le j \le l-q+(n-1)$. If $l = q+1$ all λ_{s+q+j} are determined.

Suppose then that $l < q+1$. We remind that l is of the form $q + \varepsilon_n - 2f$, where $0 \le f \le \frac{n-4+\varepsilon_n}{2}$. Denote $t = l - q + (n-2)$.

Relations (3.2.39) can be expanded as follows:

$$\begin{array}{rcl} 0 & = & \displaystyle\sum_{j=0}^{t} \binom{p+\varepsilon_n-2f}{p-n+j} \binom{n-1}{j} + \sum_{j=t+1}^{n-1} (-1)^j \binom{p+\varepsilon_n-2f}{p-n+j} \lambda_{s+q+1+j}, \\ \vdots & \vdots & \vdots \\ 0 & = & \displaystyle\sum_{j=0}^{t} \binom{p-1}{p-n+j} \binom{n-1}{j} + \sum_{j=t+1}^{n-1} (-1)^j \binom{p-1}{p-n+j} \lambda_{s+q+1+j}. \end{array}$$

If we denote $v = [\lambda_{s+q+2+t}, \ldots, \lambda_{s+q+n}]$, such relations give rise to the non-homogeneous linear system

$$Av^T = -w^T,$$

where

$$w = \left(\sum_{j=0}^{t} \binom{p+\varepsilon_n-2f}{p-n+j} \binom{n-1}{j}, \ldots, \sum_{j=0}^{t} \binom{p-1}{p-n+j} \binom{n-1}{j} \right)$$

and

$$A = \left((-1)^j \binom{p+\varepsilon_n-2f+i}{p-n+j} \right)_{ij},$$

for $0 \le i \le 2f - \varepsilon_n - 1$ and $j = t+1, \ldots, n-1$. The matrix A is in row echelon form and has rank $2f - \varepsilon_n$, while the length of v is $m = 2f - \varepsilon_n + 1$.

One solution of the non-homogeneous system is

$$\left((-1)^{t+1}\binom{n-1}{t+1},\dots,(-1)^{n-1}\binom{n-1}{n-1}\right).$$

As in case 1 the null space of the matrix A has dimension 1 and is spanned by

$$c=\left(\binom{n-t-2}{0},-\binom{n-t-2}{1},\dots,(-1)^{n-t}\binom{n-t-2}{n-t-2}\right).$$

Hence, the general solution of the non-homogeneous linear system can be expressed as

$$\lambda_{l+kq+j}=(-1)^j\left(\binom{n-1}{j}+\mu\binom{n-t-2}{j-t-1}\right),\quad \text{for } 0\le j\le n-1,$$

where $\mu\in\mathbf{F}$. In particular, if $\mu\neq -1$, then $\lambda_{l+kq+(n-1)}\neq 0$, otherwise $\lambda_{l+kq+(n-1)}=0$, while $\lambda_{l+kq+(n-2)}\neq 0$.

We have that in both cases at least one among $\lambda_{l+kq+(n-3)},\dots,\lambda_{l+kq+(n-1)}$ is non-zero. Since

$$l+kq+(n-1)>l+kq+(n-3)\ge (k+1)q+\varepsilon_n-2f+(n-3)\ge (k+1)q+1,$$

hypotheses of Lemma 3.2.2 are satisfied and we get that, in case 1

$$\lambda_{l+kq+(n-1)}=\dots=\lambda_{l+(k+1)q-3}=0,$$

while in case 2

$$\lambda_{l+kq+n}=\dots=\lambda_{l+(k+1)q-2}=0.$$

We want to prove that, in case 1, also $\lambda_{l+(k+1)q-2}=0$. Suppose, on the contrary, that $\lambda_{l+(k+1)q-2}\neq 0$. Since

$$\begin{aligned}0&=[e_{l+kq+(n-2)},e_ne_1^{q-2n}e_n]\\&=2\lambda_{l+kq+(n-2)}\lambda_{l+(k+1)q-2}e_{l+(k+1)q-2+n},\end{aligned}$$

then $\lambda_{l+kq+(n-2)}=0$. That implies, with the notation adopted above,

$$\lambda_{l+kq+(n-3)}=(-1)^{n-2}(t+1).$$

Consider now the relation $[e_ne_1^{2q-2n+1}e_n]=0$. Then,

$$\begin{aligned}0&=[e_{l+(k-1)q+(n-3)},e_ne_1^{2q-2n+1}e_n]\\&=(\lambda_{l+(k-1)q+(n-3)}+(2n-1)\lambda_{l+(k-1)q+(n-2)}-\lambda_{l+kq+(n-3)})\lambda_{l+(k+1)q-2}e_{l+(k+1)q-2+n}\\&\quad+\lambda_{l+(k-1)q+(n-3)}\lambda_{l+(k+1)q-2}e_{l+(k+1)q-2+n}\\&=(2\lambda_{l+(k-1)q+(n-3)}+(2n-1)\lambda_{l+(k-1)q+(n-2)}-\lambda_{l+kq+(n-3)})\lambda_{l+(k+1)q-2}e_{l+(k+1)q-2+n}\end{aligned}$$

Since $2\lambda_{l+(k-1)q+(n-3)}+(2n-2)\lambda_{l+(k-1)q+(n-2)}=0$, we get that

$$\lambda_{l+(k-1)q+(n-2)}=\lambda_{l+kq+(n-3)}, \tag{3.2.40}$$

namely $(-1)^{n-1}=(-1)^{n-2}(t+1)$. Since $t=l-q+(n-3)\ge (n-3)-(n-4)=1$, we get a contradiction.

Hence, $\lambda_{k+(k+1)q-2}=0$.

In case 2, also $\lambda_{l+(k+1)q-1}=0$. Suppose, on the contrary, that $\lambda_{l+(k+1)q-1}\neq 0$. Since

$$\begin{aligned} 0 &= [e_{l+kq+(n-1)}, e_n e_1^{q-2n} e_n] \\ &= 2\lambda_{l+kq+(n-1)}\lambda_{l+(k+1)q-1} e_{l+(k+1)q-1+n}, \end{aligned}$$

then $\lambda_{l+kq+(n-1)}=0$. That implies, with the notation adopted above,

$$\lambda_{l+kq+(n-2)}=(-1)^{n-2}(t+1).$$

Consider now the relation $[e_n e_1^{2q-2n+1} e_n]=0$. Then,

$$\begin{aligned} 0 &= [e_{l+(k-1)q+(n-2)}, e_n e_1^{2q-2n+1} e_n] \\ &= (2\lambda_{l+(k-1)q+(n-2)}+(2n-1)\lambda_{l+(k-1)q+(n-1)}-\lambda_{l+kq+(n-2)})\lambda_{l+(k+1)q-1} e_{l+(k+1)q-1+n} \end{aligned}$$

Since $2\lambda_{l+(k-1)q+(n-2)}+(2n-2)\lambda_{l+(k-1)q+(n-1)}=0$, we get that

$$\lambda_{l+(k-1)q+(n-1)}=\lambda_{l+kq+(n-2)}, \tag{3.2.41}$$

namely $(-1)^{n-1}=(-1)^{n-2}(t+1)$ and, as before, we get a contradiction.

Hence, $\lambda_{l+(k+1)q-1}=0$. $\square$

All the results proved in this section can be summarized by the following.

LEMMA 3.2.8. *Let $g(n,l)$ denote the number of graded Lie algebras of maximal class generated by two elements of weights 1 and n over a field $\mathbf{F}$ of positive characteristic $p>4n$ and with the length of the first constituent l. Let $q=p^h$, for some positive integer h. Then,*

$$g(n,l)\leq \begin{cases} 1, & \text{if } l=q+1 \\ 2, & \text{if } n>3 \text{ and } l=q \text{ or } l=q+\varepsilon_n-2i \text{ for some } 1\leq i\leq \frac{n-4+\varepsilon_n}{2} \end{cases}$$

3.3. First constituent length $q-(n-2)$

In this section we deal with one special case, namely the case of a graded Lie algebra of maximal class, whose length of the first constituent is $q-(n-2)$. It will be proved that, with the usual hypothesis that $p>4n$, there exists and is unique a graded Lie algebra of maximal class with such first constituent length.

According to Lemma (3.2.2), we have that

$$\lambda_{q+2}=\cdots=\lambda_{2q-n}=0.$$

Since at least one among $\lambda_{q+1},\ldots,\lambda_{q+n}$ must be non-zero, in this particular case $\lambda_{q+1}\neq 0$. This implies that $\lambda_{2q-n+1}=0$, because

$$\begin{aligned} 0 &= [e_{q+1}, e_n e_1^{q-2n} e_n] \\ &= 2\lambda_{q+1}\lambda_{2q-n+1} e_{2q+1} \end{aligned}$$

and $\lambda_{q+1}\neq 0$.

Moreover,

$$0=[e_{q+1}, e_n e_1^{q+1-n}]$$

implies that $\lambda_{q+1}=(-1)^{n-1}\lambda_{2q+2-n}$.

Up to scaling e_n, we can suppose that $[e_{q-n+2}, e_n] = 2e_{q+2}$, namely $\lambda_{q-n+2} = 2$. That implies that $\lambda_{2q-n+2} = 1$. In fact,

$$\begin{aligned} -\lambda_{2q-n+2} e_{2q+2} &= [e_n, e_n e_1^{2q-2n+2}] \\ &= (-\lambda_{q-n+2} + \lambda_{2q-n+2}) e_{2q+2}. \end{aligned}$$

We prove the following.

LEMMA 3.3.1. *Let $k \geq 2$. Suppose that*

$$\lambda_{(k-1)q+2} = \cdots = \lambda_{kq-n+1} = 0.$$

Then,

$$\lambda_{kq-n+2+j} = (-1)^j \binom{n-1}{j} \mu_1 + (-1)^j \binom{n-2}{j-1} \mu_2, \quad \text{for } 0 \leq j \leq n-1 \tag{3.3.1}$$

and μ_1, μ_2 arbitrarily chosen in $\mathbf{F}$.

Moreover,

(1) $(n-2)\lambda_{kq-n+2} + \lambda_{kq-n+3} + (-1)^{n-1}\lambda_{kq+1} = 0$;

(2) $(-1)^{n-1}\lambda_{kq-n+2} + \lambda_{kq} + (n-2)\lambda_{kq+1} = 0$.

PROOF. Consider the following $n-2$ relations

$$\begin{array}{rcl} 0 &=& [e_{(k-1)q+2}, e_n e_1^{q-2n+2} e_n - 2 e_n e_1^{q-n+2}] \\ \vdots & \vdots & \vdots \\ 0 &=& [e_{(k-1)q+n-1}, e_n e_1^{q-2n+2} e_n - 2 e_n e_1^{q-n+2}]. \end{array} \tag{3.3.2}$$

Since $\lambda_{(k-1)q+2} = \cdots = \lambda_{kq-n+1} = 0$,

$$\begin{array}{rcl} 0 &=& [e_{(k-1)q+2}, e_n e_1^{q-2n+2} e_n] \\ \vdots & \vdots & \vdots \\ 0 &=& [e_{(k-1)q+n-1}, e_n e_1^{q-2n+2} e_n]. \end{array}$$

Define the vector

$$v = [\lambda_{kq-n+2}, \ldots, \lambda_{kq+1}]. \tag{3.3.3}$$

Relations (3.3.2) yield the homogeneous linear system

$$A v^T = 0,$$

where

$$A = \left((-1)^j \binom{p+2-n}{p-n-i+j} \right), \quad i = 1, \ldots n-2; j = 1, \ldots, n. \tag{3.3.4}$$

In order to prove that A has rank $n-2$, we compute the minor

$$\det \left((-1)^j \binom{p+2-n}{p-n-i+j} \right), \quad i, j = 1, \ldots, n-2,$$

which, up to the sign, is equal to the determinant of the matrix

$$B = \left(\binom{p+2-n}{p-n-i+j} \right), \quad i, j = 1, \dots, n-2. \tag{3.3.5}$$

We can rely upon Lemma (1.3.5) and evaluate $\det(B)$. Using the notation of the Lemma, we set $a = p - n$ and $b = 2$. Then,

$$\det(B) = \prod_{i=1}^{n-2} \prod_{j=1}^{p-n} \prod_{k=1}^{2} \frac{i+j+k-1}{i+j+k-2}. \tag{3.3.6}$$

Checking that such a determinant is non-zero is an easy matter, since

$$1 \le i+j+k-2 \le i+j+k-1 \le (n-2)+(p-n)+2-1 = p-1.$$

Then we have proved that the null space of A has dimension 2. The general solution can be expressed as

$$v_j = \mu_1 u_j + \mu_2 w_j, \quad \text{for } 0 \le j \le n-1 \text{ and } \mu_1, \mu_2 \in \mathbf{F},$$

where

$$u_j = (-1)^j \binom{n-1}{j} \quad \text{for } 0 \le j \le n-1,$$

while

$$w_j = (-1)^j \binom{n-2}{j-1} \text{for } 0 \le j \le n-1.$$

From (3.3.1) the last two equalities follow immediately. □

We have already proved that $\lambda_{q+2} = \cdots = \lambda_{2q-n+1} = 0$. That means that, setting $k = 2$, hypotheses of Lemma 3.3.1 are satisfied. Since $\lambda_{2q-n+2} = 1$, also $\mu_1 = 1$, with the notation of Lemma. Hence,

$$\lambda_{2q-n+2+j} = (-1)^j \binom{n-1}{j} + (-1)^j \binom{n-2}{j-1} \mu, \quad \text{for } 0 \le j \le n-1$$

and $\mu \in \mathbf{F}$.

Now we proceed proving the following.

LEMMA 3.3.2. *Let k be an even integer greater than zero. Suppose that, for $2 \le h < k$,*

$$\lambda_{hq-n+2+j} = (-1)^j \binom{n-1}{j}, \quad \textit{for } 0 \le j \le n-1$$

and that $\lambda_{kq-n+2} = 1$. Suppose also that, for $1 \le h < k$,

$$\lambda_{hq+2} = \cdots = \lambda_{(h+1)q-n+1} = 0.$$

Then,

$$\lambda_{kq-n+2+j} = (-1)^j \binom{n-1}{j} + (-1)^j \binom{n-2}{j-1} \mu, \quad \textit{for } 0 \le j \le n-1 \textit{ and } \mu \in \mathbf{F}. \tag{3.3.7}$$

Moreover

$$\lambda_{kq+2} = \cdots = \lambda_{(k+1)q-n+1} = 0.$$

PROOF. Since hypotheses of Lemma 3.3.1 are satisfied, we have that (3.3.7) holds.

If $\lambda_{kq+1} = 0$ then $\mu = -1$ and $\lambda_{kq} = (-1)^n$. Hence, whatever the value of λ_{kq+1} is, hypotheses of Lemma 3.2.2 are satisfied. It follows that

$$\lambda_{kq+2} = \cdots = \lambda_{(k+1)q-n} = 0.$$

In order to prove that also $\lambda_{(k+1)q-n+1} = 0$, consider

$$\begin{aligned}
-\lambda_{(k+1)q-n+1}e_{(k+1)q+1} &= [e_n, e_1^{(k+1)q-2n+1}] \\
&= \left(\sum_{i=0}^{(k+1)q-2n+1} (-1)^i \binom{(k+1)q-2n+1}{i} \lambda_{n+i}\right) e_{(k+1)q+1} \\
&= \lambda_{(k+1)q-n+1}e_{(k+1)q+1}.
\end{aligned}$$

□

We prove now the following.

LEMMA 3.3.3. *Let f be a positive integer and $k = 2f$. Suppose that the hypotheses of Lemma 3.3.2 are satisfied. Then,*

$$\begin{aligned}
\lambda_{kq-n+3} &= -(n-1) - \mu \\
\lambda_{kq} &= (-1)^n((n-1) + (n-2)\mu) \\
\lambda_{kq+1} &= (-1)^{n-1}(1+\mu)
\end{aligned}$$

for some $\mu \in \mathbf{F}$ such that $\mu \neq 1$. Then, $\mu = 0$ or, if $f \not\equiv 0 \pmod p$, $\mu = \frac{2f-3}{2f}$.

Moreover,

$$\begin{aligned}
\lambda_{(k+1)q-n+2} &= \frac{1}{1-\mu}, \\
\lambda_{(k+1)q-n+3} &= 1 - \frac{n}{1-\mu}.
\end{aligned}$$

PROOF. Since hypotheses of Lemma 3.3.2 are satisfied we obtain $\lambda_{kq-n+3} = -(n-1) - \mu$, $\lambda_{kq} = (-1)^n((n-1) + (n-2)\mu)$, $\lambda_{kq+1} = (-1)^{n-1}(1+\mu)$. Moreover,

$$\lambda_{kq+2} = \cdots = \lambda_{(k+1)q-n+1} = 0.$$

Consider the following:

$$[e_{fq+\frac{q+3}{2}}, e_n e_1^{fq+\frac{q+3}{2}-n}] = 0. \tag{3.3.8}$$

Expanding (3.3.8) we get:

$$0 = \sum_{i=0}^{f} (-1)^i \binom{f}{i} [e_{fq+\frac{q+3}{2}+iq}, e_n e_1^{\frac{q+3}{2}-n}, e_1^{fq-iq}]. \tag{3.3.9}$$

Moreover,

$$\begin{aligned}
&[e_{fq+\frac{q+3}{2}+iq}, e_n e_1^{\frac{q+3}{2}-n}] = \\
&= (-1)^{\frac{q+3}{2}-n-1} \left(\left(\frac{1}{2} - (n-1) \right) \lambda_{(f+1+i)q-n+2} - \lambda_{(f+i+1)q-n+3} \right) e_{(f+i+1)q+3}.
\end{aligned}$$

We remind that $\lambda_{(f+1+i)q-n+2} = 1$, for $0 \leq i < f$, and that $\lambda_{(f+1+i)q-n+3} = -(n-1)$, for $0 \leq i < f-1$. Hence, (3.3.8) can be rewritten as:

$$\begin{aligned}
0 = & \left(\sum_{i=0}^{f-2}\binom{f}{i}[e_{fq+\frac{q+3}{2}+iq}, e_n e_1^{\frac{q+3}{2}-n}, e_1^{fq-iq}]\right) + \left(\sum_{i=f-1}^{f}[e_{fq+\frac{q+3}{2}+iq}, e_n e_1^{\frac{q+3}{2}-n}, e_1^{fq-iq}]\right) \\
= & \left((-1)^{f+\frac{q+3}{2}-n-1}\frac{1}{2}(f-1)\right) e_{(k+1)q+3} \\
& +(-1)^{f-1+\frac{q+3}{2}-n-1} f\left(\frac{1}{2}-(n-1)+(n-1)+\mu\right) e_{(k+1)q+3} \\
& +(-1)^{f+\frac{q+3}{2}-n-1}\left(\left(\frac{1}{2}-(n-1)\right)\lambda_{l+(k+1)q-n+2}-\lambda_{l+(k+1)q-n+3}\right) e_{(k+1)q+3}.
\end{aligned}$$

For the sake of clarity we will denote $a = \lambda_{l+(k+1)q-n+2}$ and $b = \lambda_{l+(k+1)q-n+3}$. With this new notation, what we just got is:

$$-\frac{1}{2} - f\mu + \left(\frac{1}{2}-(n-1)\right)a - b = 0. \tag{3.3.10}$$

Consider now the following relations:

$$0 = [e_{kq}, e_n e_1^{q-2n+2} e_n - 2e_n e_1^{q-n+2}], \tag{3.3.11}$$

$$0 = [e_{kq+1}, e_n e_1^{q-2n+2} e_n - 2e_n e_1^{q-n+2}]. \tag{3.3.12}$$

Expanding (3.3.11) we get

$$\begin{aligned}
0 &= ((\lambda_{kq}+(n-1)\lambda_{kq+1}+(-1)^n)a - \lambda_{kq}+(-n+2)\lambda_{kq+1})e_{(k+1)q+2} \\
(-1)^n e_{(k+1)q+2} &= ((-1)^{n+1}\mu+(-1)^n)a \cdot e_{(k+1)q+2}.
\end{aligned}$$

Hence, $1-\mu \neq 0$, and we get

$$a = \frac{1}{1-\mu}. \tag{3.3.13}$$

Now, consider (3.3.12):

$$0 = 2(\lambda_{kq+1}b - \lambda_{kq+1}(-n+1)a - \lambda_{kq+1} - (-1)^n(-n+2)a - (-1)^{n+1}b)e_{(k+1)q+3}.$$

We deduce that

$$\begin{aligned}
0 &= (-1)^{n-1}(1+\mu)b + (-1)^n(-n+1)(1+\mu)a + (-1)^n(1+\mu) \\
&\quad +(-1)^{n+1}(-n+2)a + (-1)^n b \\
&= -\mu b + (-n+1)\mu a - a + (1+\mu) \\
&= -\mu b + (-n+1)\mu\frac{1}{1-\mu} - \frac{1}{1-\mu} + (1+\mu) \\
&= -\mu b + \mu\frac{-n+1-\mu}{1-\mu}.
\end{aligned}$$

Hence, provided that $\mu \neq 0$,

$$b = 1 - \frac{n}{1-\mu}. \tag{3.3.14}$$

Substituting (3.3.13) and (3.3.14) in (3.3.10) we obtain:

$$\begin{aligned} 0 &= -\frac{1}{2} - f\mu + \left(\frac{1}{2} - (n-1)\right)\frac{1}{1-\mu} - 1 + \frac{n}{1-\mu} \\ &= -\frac{3}{2} - f\mu + \frac{3}{2}\frac{1}{1-\mu} \\ &= \frac{\mu(3-2f+2f\mu)}{1-\mu}. \end{aligned}$$

Hence, provided that $f \not\equiv 0 \pmod p$,

$$\mu \in \left\{0, \frac{2f-3}{2f}\right\}. \tag{3.3.15}$$

□

Now we prove the following.

LEMMA 3.3.4. *Let f be a positive integer and $k = 2f$. Suppose that hypotheses of Lemma 3.3.2 are satisfied.*

Let r be a positive integer and suppose that, for $0 \le h < k + r$,

$$\lambda_{hq+2} = \cdots = \lambda_{(h+1)q-n+1} = 0.$$

For $0 \le i < r$, denote

$$\begin{aligned} a_i &= \lambda_{(k+i)q}, & b_i &= \lambda_{(k+i)q+1}, \\ c_i &= \lambda_{(k+1+i)q-n+2}, & d_i &= \lambda_{(k+1+i)q-n+3}. \end{aligned}$$

Then, provided that $a_i + (n-1)b_i + (-1)^n \neq 0$ and $b_i + (-1)^n \neq 0$,

$$\begin{aligned} c_i &= \frac{a_i + (n-2)b_i}{a_i + (n-1)b_i + (-1)^n}, \\ d_i &= \frac{-(n-2)a_i - ((n-2)^2 - 1)b_i}{a_i + (n-1)b_i + (-1)^n}. \end{aligned}$$

PROOF. Consider the relation

$$0 = [e_{(k+i)q}, e_n e_1^{q-2n+2} e_n - 2e_n e_1^{q-n+2}]. \tag{3.3.16}$$

As a consequence we have that

$$(a_i + (n-1)b_i + (-1)^n)c_i = a_i + (n-2)b_i,$$

hence

$$c_i = \frac{a_i + (n-2)b_i}{a_i + (n-1)b_i + (-1)^n}. \tag{3.3.17}$$

Consider now the relation

$$0 = [e_{(k+1+i)q}, e_n e_1^{q-2n+2} e_n - 2e_n e_1^{q-n+2}]. \tag{3.3.18}$$

We deduce that

$$b_i d_i + b_i c_i(n-1) - b_i + (-1)^{n-1}(-n+2)c_i + (-1)^n d_i = 0.$$

Hence,

$$\begin{aligned} d_i &= \frac{b_i-(n-1)b_ic_i+(-1)^n(-n+2)c_i}{b_i+(-1)^n} \\ &= \frac{(a_i+(n-1)b_i+(-1)^n)b_i-(n-1)a_ib_i-(n-2)(n-1)b_i^2+}{(a_i+(n-1)b_i+(-1)^n)(b_i+(-1)^n)} \\ &\quad + \frac{(-1)^n(-n+2)a_i+(-1)^{n+1}(n-2)^2b_i}{(a_i+(n-1)b_i+(-1)^n)(b_i+(-1)^n)} \\ &\quad = \frac{-(n-2)a_i(b_i+(-1)^n)-((n-2)^2-1)b_i(b_i+(-1)^n)}{(a_i+(n-1)b_i+(-1)^n)(b_i+(-1)^n)}. \end{aligned}$$

Finally we get

$$d_i = \frac{-(n-2)a_i-((n-2)^2-1)b_i}{a_i+(n-1)b_i+(-1)^n}. \tag{3.3.19}$$

□

Our next step is proving the following.

Lemma 3.3.5. *Suppose that hypotheses of Lemma 3.3.2 and 3.3.4 are satisfied. With the notation of Lemma 3.3.4 suppose that*

- $a_i+(n-1)b_i+(-1)^n \neq 0$,
- $b_i+(-1)^n \neq 0$

for $0 \leq i < r$.

If $b_0 \neq 0$, *then*

(1) $c_i \neq 0$, *for* $0 \leq i < r$;
(2) $b_i = (-1)^n \dfrac{b_{i-1}}{a_{i-1}+(n-1)b_{i-1}+(-1)^n}$, *for* $0 < i \leq r$;
(3) $b_i = (-1)^n \dfrac{b_0}{ib_0+(-1)^n(i+1)}$, *for* $0 \leq i < r$;
(4) $c_i = (-1)^n \dfrac{1}{(i+1)b_0+(-1)^n(i+2)}$, *for* $0 \leq i < r$.

Proof. (1) We proceed by induction on i. Let $i=0$. Since $\lambda_{kq-n+2}=1$, we deduce from Lemma 3.3.1 that

$$a_0+(n-2)b_0+(-1)^{n-1}=0.$$

Being

$$c_0 = \frac{a_0+(n-2)b_0}{a_0+(n-1)b_0+(-1)^n},$$

we have that $c_0 \neq 0$.

Suppose to have proved that $c_i \neq 0$, for $0 \leq i < j$, where j is an integer smaller than r. From Lemma 3.3.1 we deduce that

$$a_j+(n-2)b_j+(-1)^{n-1}c_{j-1}=0.$$

Hence, being

$$c_j = \frac{a_j + (n-2)b_j}{a_j + (n-1)b_j + (-1)^n} = (-1)^n \frac{c_{j-1}}{a_j + (n-1)b_j + (-1)^n},$$

also in this case $c_j \neq 0$.

(2) From Lemma 3.3.1 we have that $(n-2)c_{i-1} + d_{i-1} + (-1)^{n-1}b_i = 0$. Using formulas obtained for c_i and d_i in Lemma 3.3.4 we have that

$$\frac{(n-2)a_{i-1} + (n-2)^2 b_{i-1} - (n-2)a_{i-1} - ((n-2)^2 - 1)b_{i-1}}{a_{i-1} + (n-1)b_{i-1} + (-1)^n} + (-1)^{n-1}b_i = 0.$$

It follows that

$$b_i = (-1)^n \frac{b_{i-1}}{a_{i-1} + (n-1)b_{i-1} + (-1)^n}. \tag{3.3.20}$$

(3)-(4) We prove both formulas for b_i and c_i by induction on i.

If $i = 0$, there is nothing to prove for b_0. As regards c_0, from Lemma 3.3.4 we have that

$$c_0 = \frac{a_0 + (n-2)b_0}{a_0 + (n-1)b_0 + (-1)^n} = \frac{(-1)^n}{b_0 + 2(-1)^n}$$

and we are done.

If $i = 1$, we have that

$$\begin{aligned} b_1 &= (-1)^n \frac{b_0}{a_0 + (n-1)b_0 + (-1)^n} \\ &= (-1)^n \frac{b_0}{b_0 + (-1)^n 2}. \end{aligned}$$

From Lemma 3.3.4,

$$\begin{aligned} c_1 &= \frac{a_1 + (n-2)b_1}{a_1 + (n-1)b_1 + (-1)^n} = \frac{(-1)^n c_0}{b_1 + (-1)^n c_0 + (-1)^n} \\ &= (-1)^n \frac{(-1)^n}{b_0 + (-1)^n 2} \cdot \frac{b_0 + (-1)^n 2}{(-1)^n b_0 + 1 + (-1)^n b_0 + 2} \\ &= (-1)^n \frac{1}{2b_0 + (-1)^n 3} \end{aligned}$$

Suppose to have proved the assertions for b_i, c_i for all i not greater than j , for some $1 < j < r$. Then, consider

$$\begin{aligned} & a_{j-1} + (n-1)b_{j-1} + (-1)^n = \\ &= b_{j-1} + (-1)^n c_{j-2} + (-1)^n \\ &= (-1)^n \frac{b_0}{(j-1)b_0 + (-1)^n j} + \frac{1}{(j-1)b_0 + (-1)^n j} + (-1)^n \\ &= \frac{(-1)^n j b_0 + (j+1)}{(j-1)b_0 + (-1)^n j}. \end{aligned}$$

Now, using (2) and the inductive hypotheses

$$\begin{aligned} b_j &= (-1)^n \frac{b_{j-1}}{a_{j-1}+(n-1)b_{j-1}+(-1)^n} \\ &= \frac{b_0}{(j-1)b_0+(-1)^n j} \frac{(j-1)b_0+(-1)^n j}{(-1)^n j b_0+(j+1)} \\ &= (-1)^n \frac{b_0}{jb_0+(-1)^n(j+1)}. \end{aligned}$$

As regards c_j, from Lemma 3.3.4 and inductive hypothesis

$$\begin{aligned} c_j &= \frac{a_j+(n-2)b_j}{a_j+(n-1)b_j+(-1)^n} \\ &= (-1)^n c_{j-1} \cdot (-1)^n \frac{jb_0+(-1)^n(j+1)}{(j+1)b_0+(j+2)} \\ &= (-1)^n \frac{1}{jb_0+(-1)^n(j+1)} \cdot \frac{jb_0+(-1)^n(j+1)}{(j+1)b_0+(j+2)} \\ &= (-1)^n \frac{1}{(j+1)b_0+(j+2)}. \end{aligned}$$

□

Finally, consider the following technical Lemma.

LEMMA 3.3.6. *Let k be an integer greater or equal to 2 and suppose that, for $1 \le h < k$,*

$$\begin{aligned} \lambda_{q-n+2}, \lambda_{q+1} &\neq 0, \\ \lambda_{hq+2} = \dots \lambda_{(h+1)q-n+1} &= 0, \\ \lambda_{(h+1)q-n+2}, \lambda_{(h+1)q+1} &\neq 0. \end{aligned}$$

Then,

$$\lambda_{kq+2} = \cdots = \lambda_{(k+1)q-n+1} = 0.$$

PROOF. Indeed, this result follows easily from Lemma 3.2.2. In fact, being the hypotheses of Lemma 3.2.2 satisfied with $l = q-n+2$, we have that

$$\lambda_{kq+2} = \cdots = \lambda_{(k+1)q-n} = 0.$$

Then, from

$$0 = [e_{kq+1}, e_n e_1^{q-2n} e_n]$$

it follows that $\lambda_{kq+1}\lambda_{(k+1)q-n+1} = 0$, hence $\lambda_{(k+1)q-n+1} = 0$. □

We remind that in this section we are dealing with a graded Lie algebra of maximal class, whose length of the first constituent is $q-n+2$, where $q = p^h$, for some prime $p > 4n$ and $h > 0$. Up to scaling e_n we can suppose that $\lambda_{q-n+2} = 2$. We have already proved, at the beginning of the section, that

$$\lambda_{q+2} = \cdots = \lambda_{2q-n+1} = 0$$

and that $\lambda_{2q-n+2} = 1$.

Suppose now to have proved that, for some positive integer f and $k = 2f$,

$$\lambda_{hq-n+2+j} = (-1)^j \binom{n-1}{j}, \quad \text{for } 2 \le h < k \text{ and } 0 \le j \le n-1 \tag{3.3.21}$$

and that, for $1 \le h < k$,

$$\lambda_{hq+2} = \cdots = \lambda_{(h+1)q-n+1} = 0.$$

Then, according to Lemma 3.3.2,

$$\lambda_{kq-n+2+j} = (-1)^j \binom{n-1}{j} + (-1)^j \binom{n-2}{j-1} \mu, \quad \text{for } 0 \le j \le n-1 \tag{3.3.22}$$

and some $\mu \in F$. Indeed, being hypotheses of Lemma 3.3.2 and 3.3.3 satisfied, there are at most two possible values for μ, namely

(1) *Case 1:* $\mu = 0$.
(2) *Case 2:* $\mu = \frac{2f-3}{2f}$, provided that $f \not\equiv 0 \pmod p$.

We will deal separately with these two cases.

Case 1: $\mu = 0$. Being $\mu = 0$,

$$\lambda_{kq-n+2+j} = (-1)^j \binom{n-1}{j}, \quad \text{for } 0 \le j \le n-1. \tag{3.3.23}$$

In particular $\lambda_{kq+1} \neq 0$. From Lemma 3.2.2 it follows that

$$\lambda_{kq+2} = \cdots = \lambda_{(k+1)q-n} = 0.$$

Then, being $[e_{kq+1}, e_n e_1^{q-2n} e_n] = 0$, it follows that $\lambda_{kq+1}\lambda_{(k+1)q-n+1} = 0$, hence $\lambda_{(k+1)q-n+1} = 0$.

Since $[e_{kq}, e_n e_1^{q-2n+2} e_n - 2 e_n e_1^{q-n+2}] = 0$, we have that

$$\lambda_{(k+1)q-n+2} = \frac{\lambda_{kq} + (n-2)\lambda_{kq+1}}{\lambda_{kq} + (n-1)\lambda_{kq+1} + (-1)^n} = \frac{(-1)^{n-2}}{(-1)^n} = 1.$$

According to Lemma 2.0.9,

$$\lambda_{kq+2} = \cdots = \lambda_{kq+2+n} = 0.$$

Now, consider the following $n-1$ relations:

$$\begin{array}{rcl}
0 & = & [e_{\frac{(k+1)q+3}{2}}, e_n e_1^{\frac{(k+1)q+3}{2} - n}] \\
\vdots & \vdots & \vdots \\
0 & = & [e_{\frac{(k+1)q+3}{2}+i}, e_n e_1^{\frac{(k+1)q+3}{2} - (n-i)}] \\
\vdots & \vdots & \vdots \\
0 & = & [e_{\frac{(k+1)q+3}{2}+(n-2)}, e_n e_1^{\frac{(k+1)q+3}{2} - 2}].
\end{array}$$

Expanding the i-th relation, for some $0 \leq i \leq n-2$, we get

$$\begin{aligned} 0 &= \left(\sum_{h=0}^{f}(-1)^h\binom{f}{h}\sum_{j=0}^{2i+1}(-1)^j\binom{\frac{q-1}{2}+2-n+i}{\frac{q-1}{2}+2-n-1-i+j}\lambda_{(f+1+h)q-n+2+j}\right) \\ &\equiv \left(\sum_{h=0}^{f}(-1)^h\binom{f}{h}\sum_{j=0}^{2i+1}(-1)^j\binom{\frac{p-1}{2}+2-n+i}{\frac{p-1}{2}+2-n-1-i+j}\lambda_{(f+1+h)q-n+2+j}\right), \end{aligned}$$

applying Lucas' theorem and taking into consideration the fact that $p > 4n$.

Since, for $0 \leq h < f$,

$$\lambda_{(f+1+h)q-n+2+j} = (-1)^j\binom{n-1}{j},$$

the expansion above reduces to:

$$\begin{aligned} &\sum_{j=0}^{2i+1}(-1)^j\binom{\frac{p-1}{2}+2-n+i}{\frac{p-1}{2}+2-n-1-i+j}\lambda_{(k+1)q-n+2+j} \\ = &\sum_{j=0}^{2i+1}\binom{\frac{p-1}{2}+2-n+i}{\frac{p-1}{2}+2-n-1-i+j}\binom{n-1}{j}. \end{aligned}$$

We remind that $\lambda_{(k+1)q-n+2} = 1$. In order to unravel a little the notation, denote $x_j = \lambda_{(k+1)q-n+2+j}$, for $1 \leq j \leq n-1$, and $x = [x_1, \ldots, x_{n-1}]$. Finally, denote

$$y_i = \sum_{j=1}^{2i+1}\binom{\frac{p-1}{2}+2-n+i}{\frac{p-1}{2}+2-n-1-i+j}\binom{n-1}{j}.$$

We can construct the non-homogeneous linear system $Ax^T = y$, where

$$A = \left((-1)^j\binom{\frac{p-1}{2}+1-n+i}{\frac{p-1}{2}+2-n-i+j}\right)_{ij}, \quad \text{for } 1 \leq i,j \leq n-1.$$

Since we are interested in the determinant evaluation of A, we consider the matrix

$$B = \left(\binom{\frac{p-1}{2}+1-n+i}{\frac{p-1}{2}+2-n-i+j}\right)_{ij}, \quad \text{for } 1 \leq i,j \leq n-1, \tag{3.3.24}$$

having the same determinant of A up to the sign.

In order to evaluate $\det(B)$ we rely upon Corollary 1.3.4. We have that

$$\det(B) = \frac{\prod_{1\leq i<j\leq n-1}(j-i)}{\prod_{i=1}^{n-1}\left(\frac{p-1}{2}+1-i\right)} \cdot \prod_{i=1}^{n-1}\frac{\left(\frac{p-1}{2}-n+i\right)!}{(2i-2)!}\prod_{i=1}^{n-1}(p+3-2n+i)_{i-1}.$$

It is easily verified that $\det(B) \not\equiv 0 \pmod p$. In fact, for each $1 \leq i < j \leq n-1$, the difference $1 < j-i < n-1 < p$. Then, since $p > 4n$,

$$n+1 \leq \frac{p-1}{2} - n + i \leq \frac{p-1}{2}.$$

Hence,

$$\prod_{i=1}^{n-1}\left(\frac{p-1}{2}-n+i\right)!$$

is not divisible by p.

Finally, consider

$$(p+3-2n+i)_{i-1}=(p+3-2n+i)\dots(p+1-2n+2i).$$

Since,

$$1<(p+3-2n+i)<\dots<(p+1-2n+2i)\leq(p-1),$$

we have that also

$$\prod_{i=1}^{n-1}(p+3-2n+i)_{i-1}\not\equiv 0 \pmod{p}.$$

We can conclude that $\det(B)\neq 0$ and the same holds for $\det(A)$. Hence, the non-homogeneous system $Ax^T=y$ has exactly one solution. Summing all up, we have that

$$\lambda_{(k+1)q-n+2+j}=(-1)^j\binom{n-1}{j},\quad \text{for } 0\leq j\leq n-1.$$

In particular, being $\lambda_{(k+1)q+1}\neq 0$, we can apply Lemma 3.3.6 and get that

$$\lambda_{(k+1)q+2}=\dots=\lambda_{(k+2)q-n+1}=0.$$

Finally, with the same argument used before, we get that $\lambda_{(k+1)q-n+2}=1$.

Case 2: $\mu=\dfrac{2f-3}{2f}$**.** We suppose that $1+\mu\neq 0$. We will deal with this case later. Set $a_0=\lambda_{kq}$ and $b_0=\lambda_{kq+1}$. Since $1+\mu\neq 0$, we have that $b_0\neq 0$.

Set $i=0$ and consider the following iterative procedure.

Step 1. Being hypotheses of Lemma 3.3.6 satisfied, $\lambda_{(k+i)q}=\dots=\lambda_{(k+1+i)q-n+1}=0$.

- If $a_i+(n-1)b_i+(-1)^n\neq 0$ and $b_i+(-1)^n\neq 0$, according to Lemma 3.3.4, set

$$\begin{aligned} c_i &= \frac{a_i+(n-2)b_i}{a_i+(n-1)b_i+(-1)^n} \\ d_i &= \frac{-(n-2)a_i-((n-2)^2-1)b_i}{a_i+(n-1)b_i+(-1)^n}. \end{aligned}$$

- Otherwise break the iteration.

Step 2. Set $i=i+1$. Since $(-1)^{n-1}b_i+(n-2)c_{i-1}+d_{i-1}=0$, we have that

$$(-1)^n b_i=\frac{b_{i-1}}{a_{i-1}+(n-1)b_{i-1}+(-1)^n}\neq 0.$$

Hence, $b_i=\lambda_{(k+i)q+1}\neq 0$. Go to step 1.

A priori, such a procedure could run indefinitely. We want to prove that after some iterations it will be interrupted because one of the two conditions in Step 1 has not been verified.

Suppose that, for some positive integer j,

$$\begin{aligned} a_i+(n-1)b_i+(-1)^n &\neq 0 \quad \text{for } 0\leq i<j,\\ b_i+(-1)^n &\neq 0 \quad \text{for } 0\leq i<j,\\ b_j+(-1)^n &= 0. \end{aligned}$$

Since

$$b_j=(-1)^n\frac{b_0}{jb_0+(-1)^n(j+1)}$$

and $b_j=(-1)^{n-1}$, we have that

$$(j+1)b_0=(-1)^{n-1}(j+1).$$

That implies that either $j\equiv -1 \pmod p$ or $b_0=(-1)^{n-1}$. The second case is impossible, since we have supposed at the beginning that $b_0\neq(-1)^{n-1}$. Hence $j\equiv -1 \pmod p$. That means in particular that $b_i+(-1)^n\neq 0$ for $i=0,1,\dots,p-2$. We prove now that, for some $i<p-1$,

$$a_i+(n-1)b_i+(-1)^n=0,$$

in contradiction with our initial hypotheses. To do that, consider

$$\begin{aligned} a_i+(n-1)b_i+(-1)^n &= b_i+(-1)^n+(-1)^nc_{i-1}\\ &= \frac{(-1)^n(i+1)b_0+(i+2)}{ib_0+(-1)^n(i+1)}. \end{aligned}$$

We have that $a_i+(n-1)b_i+(-1)^n=0$ if and only if $(-1)^n(i+1)b_0+(i+2)=0$, namely

$$i=-\frac{(-1)^nb_0+2}{(-1)^nb_0+1}=-1-\frac{1}{(-1)^nb_0+1}\not\equiv -1 \pmod p.$$

Hence, for some $0\leq i\leq p-2$, we have that $a_i+(n-1)b_i+(-1)^n=0$, in contradiction with our initial hypotheses.

At the same time, we have proved that $b_i+(-1)^n\neq 0$, for all integers i such that $0\leq i\leq p-2$, while, for some $0\leq j\leq p-2$, the relation $a_j+(n-1)b_j+(-1)^n=0$. But also in this case we have a contradiction. In fact, with the usual notation,

$$(a_j+(n-1)b_j+(-1)^n)c_j=a_j+(n-2)b_j. \tag{3.3.25}$$

Hence, $a_j+(n-2)b_j=0$. But, $a_j+(n-2)b_j=(-1)^nc_{j-1}\neq 0$.

We conclude that, provided that $1+\mu\neq 0$, we cannot have an infinite dimensional graded Lie algebra of maximal class.

Consider now the possibility that $1+\mu=0$. Firstly we note that $f\neq 1$. In fact, if $f=1$, then $\mu=-1/2\not\equiv -1 \pmod p$.

Then,

$$\lambda_{kq-n+2+i}=(-1)^i\binom{n-2}{i}, \quad \text{for } 0\leq i\leq n-2,$$

while $\lambda_{kq+1} = 0$. According to Lemma 3.2.2,

$$\lambda_{kq+2} = \cdots = \lambda_{(k+1)q-n} = 0.$$

It is easily seen that also $\lambda_{(k+1)q-n+1} = 0$. In fact,

$$\begin{aligned}
-\lambda_{(k+1)q-n+1}e_{(k+1)q+1} &= [e_n, e_n e_1^{(k+1)q-2n+1}] \\
&= \sum_{h=0}^{k}(-1)^h\binom{k}{h}\left(\sum_{i=0}^{q-2n+1}(-1)^i\binom{q-2n+1}{i}\lambda_{hq+n+i}\right)e_{(k+1)q+1} \\
&= \lambda_{(k+1)q-n+1}e_{(k+1)q+1}.
\end{aligned}$$

Therefore, $\lambda_{(k+1)q-n+1} = 0$.

Now, consider the relation $[e_{kq}, e_n e_1^{q-2n+2} e_n - 2e_n e_1^{q-n+2}] = 0$. As a consequence,

$$(\lambda_{kq} + (-1)^n)\lambda_{(k+1)q-n+2} = \lambda_{kq}.$$

Being $\lambda_{kq} = (-1)^n$, we deduce that $\lambda_{(k+1)q-n+2} = 1/2$. Since $[e_{(k+1)q-n+1}, e_n e_1 e_n] = 0$, it follows that $\lambda_{(k+1)q+2} = 0$. Moreover, from $[e_{kq+1}, e_n e_1^{q-n+1} e_n + (-1)^n e_n e_1^{q+1}] = 0$, we deduce that

$$\begin{aligned}
0 &= [e_{kq+1}, [e_n e_1^{q-n+1}], e_n] - [e_{kq+1}, e_n, [e_n e_1^{q-n+1}]] + (-1)^n [e_{kq+1}, e_n e_1^{q+1}] \\
&= (-1)^{n+q}\lambda_{(k+1)q+1}e_{(k+1)q+2+n}.
\end{aligned}$$

Hence, $\lambda_{(k+1)q+1} = 0$. According to Lemma 3.3.1,

$$\lambda_{(k+1)q-n+2+i} = (-1)^i \cdot \frac{1}{2}\binom{n-2}{i}, \quad \text{for } 0 \le i \le n-2.$$

Since $[e_n e_1^{hq-n+1} e_n + (-1)^n e_n e_1^{hq+1}] = 0$, for $1 \le h < k$ and $k \ge 4$, in particular we have that $[e_n e_1^{2q-n+1} e_n + (-1)^n e_n e_1^{2q+1}] = 0$. Consider the following:

$$\begin{aligned}
0 &= [e_{(k-1)q+1}, [e_n e_1^{2q-n+1} e_n + (-1)^n e_n e_1^{2q+1}]] \\
&= [e_{(k-1)q+1}, e_n e_1^{2q-n+1}, e_n] - [e_{(k-1)q+1}, e_n, e_n e_1^{2q-n+1}] + (-1)^n [e_{(k-1)q+1}, e_n e_1^{2q+1}] \\
&= \left(\sum_{i=0}^{2q-n+1}(-1)^i\binom{2q-n+1}{i}\lambda_{(k-1)q+1+i}\right)\lambda_{(k+1)q+2}e_{(k+1)q+n+2} \\
&\quad -\lambda_{(k-1)q+1}\left(\sum_{i=0}^{n-1}\binom{q-n+1}{q-2n+1+i}\binom{n-2}{i}\right)e_{(k+1)q+n+2} \\
&\quad +\lambda_{(k-1)q+1}\sum_{i=0}^{n-1}\binom{q-n+1}{q-2n+1+i}\cdot\frac{1}{2}\cdot\binom{n-2}{i}e_{(k+1)q+n+2} + (-1)^n\lambda_{(k-1)q+1}e_{(k+1)q+n+2} \\
&= (-(-1)^{n-1}\binom{q-1}{n} + (-1)^{n-1}\frac{1}{2}\binom{q-1}{n} - 1)e_{(k+1)q+n+2} = -\frac{1}{2}e_{(k+1)q+n+2}.
\end{aligned}$$

Therefore also in this case we have a contradiction.

All what we have proved through this section can be summarized in the following.

LEMMA 3.3.7. *Let L be a graded Lie algebra of maximal class generated by two elements of weight 1 and n over a field $\mathbf{F}$ of characteristic $p > 4n$, whose first constituent length is $l = q - n + 2$. Suppose, up to scaling e_n, that $[e_l, e_n] = 2e_{l+n}$. For $k \geq n$, denote $[e_k, e_n] = \lambda_k e_{k+n}$. Then,*

$$\begin{aligned} \lambda_k &= 0 && \text{for } n \leq k < l, \\ \lambda_{hq-n+2+j} &= (-1)^j \tbinom{n-1}{j} && \text{for } h \geq 2 \text{ and } 0 \leq j \leq n-1, \\ \lambda_{hq+2} = \cdots = \lambda_{(h+1)q-n+1} &= 0 && \text{for } h \geq 1 \end{aligned}$$

and $\lambda_l, \dots, \lambda_{q+1}$ are uniquely determined by means of the relations

$$0 = [e_k, e_k], \quad \text{for } l \leq k \leq q+1.$$

PROOF. The proof of this Lemma is straightforward using all the facts proved through this section. Supposing that $\lambda_l = 2$, we have proved that $\lambda_{q+2}, \dots, \lambda_{2q-n+1} = 0$ and that $\lambda_{2q-n+2} = 1$. Then, by means of the relations $[e_k, e_k] = 0$, for $l \leq k \leq q+1$, the coefficients λ_k are uniquely determined.

Now we prove the assertions about the coefficients $\lambda_{hq-n+2+j}$ and $\lambda_{hq+2}, \dots, \lambda_{(h+1)q-n+1}$, for $h \geq 2$ and $0 \leq j \leq n-1$. We proceed by induction on $h = 2f$. Take $f = 1$. Since hypotheses of Lemma 3.3.2 hold for $k = 2$,

$$\lambda_{2q-n+2+j} = (-1)^j \binom{n-1}{j} + (-1)^j \binom{n-2}{j-1} \mu \quad \text{for } 0 \leq j \leq n-1,$$

where $\mu = 0$ or $\mu = \dfrac{2f-3}{2f}$, if $f \not\equiv 0 \pmod p$ (see Lemma 3.3.3). If $\mu = 0$, we proved that

$$\begin{aligned} \lambda_{hq-n+2+j} = \lambda_{(h+1)q-n+2+j} &= \binom{n-1}{j} \quad \text{for } 0 \leq j \leq n-1, \\ \lambda_{hq+2} = \cdots = \lambda_{(h+1)q-n+1} &= 0, \\ \lambda_{(h+1)q+2} = \cdots = \lambda_{(h+2)q-n+1} &= 0, \\ \lambda_{(h+2)q-n+2} &= 1. \end{aligned}$$

If $\mu = \dfrac{2f-3}{2f}$, we come to a contradiction.

As regards the inductive step, let $f > 1$ and suppose to have proved the assertions for h smaller than $2f$. Then, setting $h = 2f$, it is possible to repeat verbatim the proof above replacing $f = 1$ with the current value of f. □

Hence, in the initial hypotheses of this section, we proved that there exists at most one graded Lie algebra of maximal class.

CHAPTER 4

Construction of the Lie algebras

In Chapter 3 we defined $g(n, l)$ as the number of graded Lie algebras of maximal class over a field $\mathbf{F}$ of odd characteristic $p > 4n$ with first constituent length equal to l, up to a rescaling of e_n. What we got is that

$$g(n,l) \leq \begin{cases} 1, & \text{if } l = q+1 \text{ or } q-(n-2) \\ 2, & \text{if } n > 3 \text{ and } l = q \text{ or } l = q+\varepsilon_n - 2i \text{ for some } 1 \leq i \leq \frac{n-4}{2}, \end{cases}$$

where $q = p^h$ for some positive integer h and

$$\varepsilon_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

In the following we show that the upper bounds for $g(n, l)$ are reached for any choice of l contemplated above.

In the following Section of this Chapter, the reader can find the explicit construction of the following:

(1) $\mathfrak{g}_1(p, h, n)$, a graded Lie algebra of maximal class over a field of odd characteristic $p > 2n$ and odd n with first constituent length equal to $p^h + 1$, for some positive integer h;
(2) $\mathfrak{g}_2(p, h, n)$, a graded Lie algebra of maximal class over a field of odd characteristic $p > 2n$ and even n with first constituent length equal to $q = p^h$, for some positive integer h.

If $n = 2$ we have only to consider the case $l = p^h-(n-2) = p^h$. Since $\mathfrak{g}_2(p, h, n)$ is a graded Lie algebra of maximal class with $l = p^h$, we have that $g(n, l) = 1$.

If $n = 3$ we have only to consider the cases $l = p^h + 1$ or $l = p^h - (n - 2)$. For the moment consider the case $l = p^h + 1$. We have that $\mathfrak{g}_1(p, h, n)$ is a graded Lie algebra of maximal class with $l = p^h+1$, hence $g(n, l) = 1$. The case $l = p^h-(n-2)$ will be dealt with at the end of this section.

Consider now an odd integer $n \geq 5$. Even in this case $g(n, p^h+1) = 1$, for each positive integer h. Let $1 \leq i \leq \dfrac{n + \varepsilon_n - 4}{2}$ and $l = q + 1 - 2i$, where $q = p^h$. We want to prove that $g(n, l) = 2$. We consider $\mathfrak{g}_1(p, h, m)$, where $m = n - 2i$, and, in it, the subalgebra generated by the elements $e_1, e_n = [e_m, e_1^{2i}]$. This is a graded Lie algebra of maximal class, generated in weights 1 and n, with first constituent length equal to l. Looking at the multiplication table of $\mathfrak{g}_1$, we have that

$$[e_{q-2i+n}, e_n] = [e_{q+(n-2i)}, e_m e_1^{2i}] = 0,$$

being $[e_k, e_m] = 0$, for $q + (n - 2i) \leq k \leq n$.

We can also consider $\mathfrak{g}_2(p, h, m)$, where $m = n+1-2i$, and, in it, the subalgebra generated by the elements $e_1, e_n = [e_m, e_1^{2i-1}]$. This is a graded Lie algebra of maximal class, generated in weights 1 and n, with first constituent length equal to l. As before, we look at the multiplication table of $\mathfrak{g}_2$. Here we see that

$$[e_{q-2i+n}, e_n] = [e_{q+(n-2i)}, e_m e_1^{2i-1}] \neq 0,$$

because $[e_{q+(n-2i)}, e_m] \neq 0$, while $[e_k, e_m] = 0$, for $q - 2i + n + 1 \leq k \leq q + n - 1$.

Consider now an even integer $n \geq 4$. Let $0 \leq i \leq \dfrac{n-4}{2}$ and $l = q - 2i$, where $q = p^h$ for some positive integer h. We want to prove that $g(n, l) = 2$.

Take $\mathfrak{g}_1(p, h, m)$, where $m = n - 2i - 1$, and, in it, the subalgebra generated by $e_1, e_n = [e_m, e_1^{2i+1}]$. This is a graded Lie algebra of maximal class, generated in weights 1 and n, with first constituent length equal to l. We have that

$$[e_{q+(n-2i-1)}, e_m e_1^{2i+1}] = 0,$$

because $[e_k, e_m] = 0$, for $q + (n - 2i - 1) \leq k \leq q + n$.

Now consider $\mathfrak{g}_2(p, h, m)$, where $m = n-2i$, and, in it, the subalgebra generated by $e_1, e_n = [e_m, e_1^{2i}]$. This is a graded Lie algebra of maximal class, generated in weights 1 and n, with first constituent length equal to l. We have that

$$[e_{q+(n-2i-1)}, e_m e_1^{2i}] \neq 0,$$

because $[e_{q+(n-2i-1)}, e_m] \neq 0$, while $[e_k, e_m] = 0$, for $q + (n - 2i) \leq k \leq q + n - 1$.

Finally, consider the case $l = q - (n - 2)$, where $q = p^h$, for some positive integer h. We want to prove that $g(n, l) = 1$. Take into consideration the algebra $\mathfrak{g}_2(p, h, 2)$ and in it the subalgebra generated by the two elements $e_1, e_n = [e_2, e_1^{n-2}]$. Such an algebra is of maximal class and has first constituent length equal to $q - (n - 2)$, hence gets the job done.

4.1. Lie algebra with first constituent length q or $q + 1$

Let $q = p^h$, for some positive integer h. In this section we give a construction of the Lie algebra generated by two elements of weight 1 and n with the first constituent of length q, if n is even, or $q + 1$, if n is odd. In both cases $p > 2n$.

Let V be a vector space of dimension q over the field $\mathbf{F}_p(t)$ of rational functions over the field $\mathbf{F}_p$ with p elements. We grade V over the cyclic group of order q,

$$V = \langle v_0 \rangle \oplus \langle v_1 \rangle \oplus \cdots \oplus \langle v_{q-1} \rangle.$$

Let D and E be endomorphisms of V defined as follows:

$$E = \begin{cases} v_i \mapsto v_{i+1} & \text{if } i \neq q-1 \\ v_{q-1} \mapsto t v_0. & \end{cases}$$

$$D = \begin{cases} v_i \mapsto (-1)^{i+1}\binom{n-1}{i+1} v_{i+n} & \text{if } 0 \leq i \leq n-2 \\ v_i \mapsto 0 & \text{if } n-1 \leq i \leq q-2 \\ v_{q-1} \mapsto t v_{n-1} & \end{cases}$$

Consider the Lie algebra M spanned by D and E in the endomorphism algebra of V. In M, the endomorphism D and E have weights respectively n and 1.

Now we prove that, for every $0 \leq j < q$,

$$v_j[DE^m D] = 0, \quad \text{for } 0 \leq m \leq q - n - 1. \tag{4.1.1}$$

Suppose first that $0 \le j \le n-1$ and that $m+n+j < q-1$. In this case, $[DE^mD]$ is a linear combination of monomials of the form $E^\alpha DE^\beta D$ and $DE^\alpha DE^\beta$, being $\alpha+\beta = m$. Then, $v_jE^\alpha DE^\beta = c \cdot v_{j+m+n}$ and $v_jDE^\alpha = d \cdot v_{j+n+\alpha}$, for some $c, d \in \mathbf{F}_p$. Hence we conclude that $v_j[DE^mD] = 0$.

Now we deal with the case $q-1 \le m+n+j (< 2q)$.

$$
\begin{aligned}
v_j[DE^mD] &= \\
&= \left(\left(\sum_{k=j}^{j+m}(-1)^{j+1}\binom{n-1}{k+1}\binom{m}{k-j}\right)\cdot(-1)^{j+m+n}\cdot\binom{n-1}{m+n+j+1-q}\right.\\
&\quad \left. -(-1)^{j+1}\binom{n-1}{j+1}\cdot\sum_{k=j+n}^{j+n+m}(-1)^{j+n-q+1}\binom{n-1}{k-q+1}\binom{m}{k-(j+n)}\right)tv_{j+m+2n}\\
&= (-1)^{m+(n+1)}\left(\left(\sum_{k=0}^{m}\binom{n-1}{k+1+j}\binom{m}{k}\right)\cdot\binom{n-1}{m+n+j+1-q}\right.\\
&\quad \left. -(-1)^{n+1}\binom{n-1}{j+1}\cdot\sum_{k=0}^{m}\binom{n-1}{j+n+k-q+1}\binom{m}{k}\right)tv_{j+m+2n}\\
&= \left((-1)^{m+(n-1)}\binom{n+m-1}{m+j+1}\binom{n-1}{q-1-(j+m+1)}\right.\\
&\quad \left. -(-1)^{n+1}\binom{n-1}{j+1}\binom{m+n-1}{q-1-(j+1)}\right)tv_{j+m+2n}\\
&= \left((-1)^{j+1}\binom{n+m-1}{m+j+1}\binom{m+j+1}{q-1-(n-1)}\right.\\
&\quad \left. -(-1)^{m+(j+1)}\binom{n-1}{j+1}\binom{j+1}{q-1-(m+n-1)}\right)tv_{j+m+2n}\\
&= (-1)^{j+1}\left(\binom{n+m-1}{q-1-(n-1)}\binom{2n+m-1-q}{m+j+1-q+n}\right.\\
&\quad \left. +(-1)^{m+1}\binom{n-1}{q-1-(m+n-1)}\binom{2n+m-1-q}{m+j+1-q+n}\right)tv_{j+m+2n}\\
&= (-1)^{j+1}\binom{2n+m-1-q}{m+j+1-q+n}\binom{n+m-1}{q-1-(n-1)}\left(1+(-1)^{m+1}(-1)^m\right).\\
&= 0.
\end{aligned}
$$

This proves (4.1.1) for v_j such that $0 \le j \le n-1$.

To complete our proof we consider the case $n \le j \le q-1$. For the moment we exclude the case $j = q-1$. As considered before, $[DE^mD]$ is a linear combination of monomials of the form $E^\alpha DE^\beta D$ and $DE^\alpha DE^\beta$, being $\alpha+\beta = m$. Since $n \le j \le q-2$, we have that $v_jD = 0$ and $v_jDE^\alpha DE^\beta = 0$.

Consider now the monomials $E^\alpha DE^\beta D$. If $v_jE^\alpha D = 0$ we are done. So, suppose that $v_jE^\alpha D \ne 0$. Hence $v_jE^\alpha D = c \cdot tv_k$ for some $c \in \mathbf{F}_p$ and $k \ge n-1$. Therefore $v_jE^\alpha DE^\beta = d \cdot v_k$, where $d \in \mathbf{F}_p$ and $k = j+n+m-q$. Now we observe that $n-1 \leq j+n+m-q \leq q-2+n+q-n-1-q \leq q-3$. That means that $v_k D = 0$ and this proves our assertion.

As regards $j = q-1$, we have that $v_j E^\alpha D E^\beta = c \cdot t v_{m+n-1}$, for some $c \in \mathbf{F}_p$. As before, $n-1 \leq m+n-1 \leq q-2$, hence $v_j E^\alpha D E^\beta D = 0$, whenever $\alpha+\beta = m$ and $0 \leq m \leq q-n-1$. Similarly we conclude that $v_j D E^\alpha D E^\beta = 0$.

Now we proceed proving that

$$[DE^{q-n}] = (-1)^{n-1} t \cdot 1.$$

We distinguish two cases. Supposing that $0 \leq j \leq n-2$,

$$\begin{aligned} v_j[DE^{q-n}] &= \left(\sum_{k=j}^{n-1} (-1)^{k+1} \binom{n-1}{k+1} (-1)^{k-j} \binom{q-n}{k-j} \right) t v_j \\ &= (-1)^{j+1} \binom{q-1}{n-j-2} t v_j \equiv (-1)^{n-1} t v_j \pmod{p}. \end{aligned}$$

Finally, suppose that $n-1 \leq j \leq q-1$. Then,

$$\begin{aligned} v_j[DE^{q-n}] &= \left(\sum_{k=j}^{j+q-n} (-1)^k \binom{n-1}{k-(q-1)} (-1)^{k-j} \binom{q-n}{k-j} \right) t v_j \\ &= (-1)^j \left(\sum_{k=0}^{q-n} \binom{n-1}{k+j-(q-1)} \binom{q-n}{k} \right) t v_j \\ &= (-1)^j \binom{q-1}{j-n+1} t v_j \equiv (-1)^{n-1} t v_j \pmod{p}. \end{aligned}$$

Hence, all the $[DE^i]$ are non-zero, for $0 \leq i \leq q-n$ and are linearly independent over $\mathbf{F}_p$, because have distinct weights $n, \dots, q$. Moreover, $[DE^m D] = 0$, for $0 \leq m \leq q-n-1$. Therefore,

$$M = \langle E, [DE^i] : 0 \leq i \leq q-n \rangle$$

is a $(q-n+2)$-dimensional Lie algebra.

Now consider the semidirect product $V + \mathrm{End}(V)$ and in it the Lie Algebra L over $\mathbf{F}_p$ generated by

$$e_1 = E, \quad e_n = \frac{v_{n-1}}{t} + D.$$

For $n < i \leq q$ let

$$e_i = [e_{i-1}, e_1] = \frac{v_{i-1}}{t} + [DE^{i-n}].$$

If we take $i = q$ we have that

$$e_q = \frac{v_{q-1}}{t} + [DE^{q-n}] = \frac{v_{q-1}}{t} + (-1)^{n-1} t \cdot 1.$$

Moreover, $e_{q+1} = v_0$ and, for $0 \leq r \leq q-1$ and $k \geq 1$

$$e_{kq+r+1} = t^{k-1} v_r.$$

Since $[DE^m D] = 0$, for $0 \leq m \leq q-n-1$, we conclude that

$$[e_i, e_n] = 0, \quad \text{for } n \leq i < q.$$

Finally, consider the case $i = q$. Then,

$$[e_q, e_n] = \frac{v_{q-1}}{t} D + (-1)^n t \frac{v_{n-1}}{t} = (1 + (-1)^n) e_{q+n}.$$

Then,

$$[e_q, e_n] = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2e_{q+n} & \text{otherwise.} \end{cases}$$

In general, if $0 \leq r \leq n-2$ and $k \geq 1$,

$$\begin{aligned} [e_{kq+r+1}, e_n] &= \left[t^{k-1} v_r, \frac{v_{n-1}}{t} + D\right] = (-1)^{r+1} \binom{n-1}{r+1} t^{k-1} v_{r+n} \\ &= (-1)^{r+1} \binom{n-1}{r+1} e_{kq+r+n+1}, \end{aligned}$$

while, for $n-1 \leq r < q-1$ and $k \geq 1$,

$$[e_{kq+r+1}, e_n] = 0.$$

Finally, if $r = q-1$ and $k \geq 1$,

$$[e_{(k+1)q}, e_n] = t^k v_{n-1} = e_{(k+1)q+n}.$$

We have defined two graded Lie algebras of maximal class, respectively for n even and odd:

(1) If n is even,

$$[e_i, e_n] = \begin{cases} 0 & \text{for } i = n \ldots q-1 \\ 2e_{i+n} & \text{for } i = q \\ (-1)^r \binom{n-1}{r} e_{i+n} & \text{if } i = kq + r, \text{ for } r = 1, \ldots, n-1 \text{ and } k \geq 1 \\ 0 & \text{if } i = kq + r,\ r = n, \ldots, q-1 \text{ and } k \geq 1 \\ 1 & \text{if } i = kq \text{ and } k \geq 2. \end{cases}$$

(2) If n is odd,

$$[e_i, e_n] = \begin{cases} 0 & \text{for } i = n \ldots q \\ (-1)^r \binom{n-1}{r} e_{i+n} & \text{if } i = kq + r, \text{ for } r = 1, \ldots, n-1 \text{ and } k \geq 1 \\ 0 & \text{if } i = kq + r,\ r = n, \ldots, q-1 \text{ and } k \geq 1 \\ 1 & \text{if } i = kq \text{ and } k \geq 2. \end{cases}$$

Note in passing that both the algebras above are ultimately periodic. Therefore, they are also soluble, namely $[e_r, e_s] = 0$, for $r, s > q + n$. In fact, write $s - n = aq + b$ for some $0 \leq b < q$ and positive integer a. Then,

$$\begin{aligned} [e_r, e_s] &= [e_r, e_n e_1^{aq+b}] \\ &= \sum_{h=0}^{a} (-1)^h \binom{a}{h} \sum_{j=0}^{b} (-1)^j \binom{b}{j} [e_{r+hq+j}, e_n, e_1^{(a-h)q+b-j}] = 0, \end{aligned}$$

since $[e_{r+j}, e_n] = [e_{r+hq+j}, e_n]$, for any $0 \leq h \leq a$ and $0 \leq j \leq b$.

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