Iterative methods for the saddle-point problem arising from the $\mathbf{H}_C/\mathbf{E}_I$ formulation of the eddy current problem.

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Abstract

This paper is concerned with the resolution of the linear system arising from a finite element approximation of the time-harmonic eddy current problem. We consider the $\mathbf{H}_C/\mathbf{E}_I$ formulation introduced and analyzed in [2], where an optimal error estimate for the finite element approximation using edge elements of the first order is proved. We reduce the linear system by eliminating the Lagrange multiplier introduced in the insulator region. Two different iterative procedures are proposed: a modified SOR method and an Uzawa-like method. The finite element scheme has been implemented in MATLAB and the two iterative procedures have been compared by solving four different test problems.

1 Introduction

To model the electromagnetic phenomena concerning alternating currents at low frequencies, it is often used the time-harmonic eddy currents model. The main equations of this model are Faraday's law

$$\mathbf{curl}\,\mathbf{H} = \sigma\mathbf{E} + \mathbf{J}_e \quad \text{in } \Omega\,, \tag{1}$$

and Ampère's law

$$\mathbf{curl}\,\mathbf{E} = -i\omega\mu\mathbf{H} \quad \text{in } \Omega\,, \tag{2}$$

where $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{J}_e$ denote the electric field, the magnetic field and a given generator current respectively. For the sake of simplicity the computational domain $\Omega \subset \mathbb{R}^3$ is assumed to be a simply connected Lipschitz polyhedron. The typical setting for an eddy current model distinguishes between a conducting region Ω_C, which we suppose strictly contained in Ω, and its complement $\Omega_I := \Omega \setminus \overline{\Omega}_C$. We shall assume that both Ω_C and Ω_I are Lipschitz polyhedron and that Ω_C is connected but not necessarily simply connected. The magnetic permeability μ is assumed to be a uniformly positive definite 3×3 tensor, whereas the electric conductivity σ is supposed to be a positive definite tensor in the conducting regions, and to be null in non-conducting regions. The real scalar constant $\omega \neq 0$ is a given angular frequency.

Since $\sigma \equiv 0$ in the non-conducting region, the generator current has to satisfy the compatibility conditions

$$\operatorname{div}\mathbf{J}_{e,I} = 0 \ \text{ in } \Omega_I \quad \text{and} \quad \int_\Gamma \mathbf{J}_{e,I}\cdot\mathbf{n}\,dS = 0\,, \tag{3}$$

where $\Gamma := \overline{\Omega}_C \cap \overline{\Omega}_I$ and $\mathbf{n}$ denotes the unit outward normal vector on Γ pointing towards Ω_I. Here and in the sequel, given any vector field $\mathbf{v}$ defined in Ω, we denote by $\mathbf{v}_L$ its restriction to Ω_L, $L = C, I$.

Equations (1)-(2) do not completely determine the electric field in Ω_I, and it is necessary to demand the gauge condition

$$\operatorname{div}(\epsilon\mathbf{E}_I) = 0 \ \text{ in } \Omega_I\,, \tag{4}$$

where ϵ is the dielectric permittivity, which is also assumed to be a uniformly positive definite symmetric tensor.

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In the boundary of the computational domain, suitable boundary conditions must be assigned. Most often, the tangential component of the electric field, $\mathbf{E} \times \mathbf{n}$, or the magnetic field, $\mathbf{H} \times \mathbf{n}$, are given (here $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$). Below we demand

$$\mathbf{H} \times \mathbf{n} = \mathbf{0} \ \text{ on } \partial\Omega \, . \tag{5}$$

This implies another compatibility condition for $\mathbf{J}_{e,I}$, namely

$$\mathbf{J}_{e,I} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \, . \tag{6}$$

Moreover an additional gauge condition for the electric field in the insulator of the type

$$\epsilon \mathbf{E} \cdot \mathbf{n} = 0 \ \text{ on } \partial\Omega \, , \tag{7}$$

must be added.

The complete eddy current model we consider is the system of equations (1)-(7) (see [1]). In this system it is possible to reduce the number of unknowns by eliminating either the electric field $\mathbf{E}$ or the magnetic field $\mathbf{H}$. In the so-called hybrid formulations the main unknowns are different fields in the conductor and in the insulator. This kind of formulation is particularly interesting for finite elements approximation, since it is possible to use unrelated (“*non-matching*”) meshes on Ω_C and Ω_I.

This paper concerns the hybrid formulation of the eddy current model which uses as main unknowns the magnetic field in the conductor and the electric field in the insulator: the $\mathbf{H}_C/\mathbf{E}_I$ formulation. It is obtained in the following way: from Ampère's law the electric field in the conductor is written $\mathbf{E}_C = \sigma^{-1}(\mathbf{curl}\, \mathbf{H}_C - \mathbf{J}_{e,C})$, and replacing it in Faraday's law one gets

$$\mathbf{curl}(\sigma^{-1}\, \mathbf{curl}\, \mathbf{H}_C) + i\omega\mu \mathbf{H}_C = \mathbf{curl}(\sigma^{-1} \mathbf{J}_{e,C}) \quad \text{in } \Omega_C \, .$$

On the other hand, from Faraday's law the magnetic field in the insulator is $\mathbf{H}_I = (-i\omega\mu)^{-1}\, \mathbf{curl}\, \mathbf{E}_I$, so replacing it in Ampère's law and multiplying the equation by $-i\omega$ one obtains

$$\mathbf{curl}(\mu^{-1}\, \mathbf{curl}\, \mathbf{E}_I) = -i\omega \mathbf{J}_{e,I} \quad \text{in } \Omega_I \, .$$

The tangential components of the electric and magnetic fields must be continuous on the interface Γ, so we have

$$\mathbf{H}_C \times \mathbf{n} = \mathbf{H}_I \times \mathbf{n} = (-i\omega\mu)^{-1}\, \mathbf{curl}\, \mathbf{E}_I \times \mathbf{n} \, ,$$

and

$$\mathbf{E}_I \times \mathbf{n} = \mathbf{E}_C \times \mathbf{n} = \sigma^{-1}(\mathbf{curl}\, \mathbf{H}_C - \mathbf{J}_{e,C}) \times \mathbf{n} \, .$$

Finally the boundary condition $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ can be written in terms of $\mathbf{E}$, since $\mathbf{H}_I \times \mathbf{n} = \mu^{-1}\, \mathbf{curl}\, \mathbf{E}_I \times \mathbf{n}$.

So we consider the system of equations

$$\begin{array}{ll}
\mathbf{curl}(\sigma^{-1}\, \mathbf{curl}\, \mathbf{H}_C) + i\omega\mu \mathbf{H}_C = \mathbf{curl}(\sigma^{-1} \mathbf{J}_{e,C}) & \text{in } \Omega_C \, , \\
\mathbf{curl}(\mu^{-1}\, \mathbf{curl}\, \mathbf{E}_I) = -i\omega \mathbf{J}_{e,I} & \text{in } \Omega_I \, , \\
\operatorname{div}(\epsilon \mathbf{E}_I) = 0 & \text{in } \Omega_I \, , \\
\mu^{-1}\, \mathbf{curl}\, \mathbf{E}_I \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \, , \\
\epsilon \mathbf{E}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \, , \\
\mathbf{H}_C \times \mathbf{n} = (-i\omega\mu)^{-1}\, \mathbf{curl}\, \mathbf{E}_I \times \mathbf{n} & \text{on } \Gamma \, , \\
\mathbf{E}_I \times \mathbf{n} = \sigma^{-1}(\mathbf{curl}\, \mathbf{H}_C - \mathbf{J}_{e,C}) \times \mathbf{n} & \text{on } \Gamma \, .
\end{array} \tag{8}$$

This formulation of the eddy current problem has been proposed and analyzed in [2]. It is motivated as a convenient approach for complicated geometrical configurations where the conductor Ω_C is not simply connected. In that paper the continuous and discrete variational formulations of (8) are discussed and an optimal error estimate for edge finite elements is proved.

The outline of this paper is as follows. In Section 2 we recall the weak formulation and the finite element approximation of the hybrid problem. In Section 3 we consider the system obtained from the finite element discretization. Since the Lagrange multiplier introduced in the insulator region is zero, we eliminate it reducing the number of unknowns. Then we introduce two iterative procedures to solve the reduced linear system: a modified SOR method and an Uzawa-like method. In Section 4 we compare this two algorithms by showing some numerical results corresponding to four different test cases: in the first test case the problem has known analytical solution; the second and third test cases are model problems where the conductor is not simply connected, and the last test problem is Problem 7 in the TEAM workshop.

2 Weak formulation and finite element approximation

We start this section introducing some notation that we shall use in the following. The space $\mathbf{H}(\mathbf{curl};\Omega)$ indicates the set of real or complex vector valued functions $\mathbf{v} \in (L^2(\Omega))^3$ such that $\mathbf{curl}\,\mathbf{v} \in (L^2(\Omega))^3$, and $\mathbf{H}^0(\mathbf{curl};\Omega)$ is the subspace of $\mathbf{H}(\mathbf{curl};\Omega)$ constituted by curl free functions. We recall the trace space for $\mathbf{H}(\mathbf{curl};\Omega)$:

$$\mathbf{H}^{1/2}(\mathrm{div}_\tau;\Gamma) := \{\mathbf{v}\times\mathbf{n} \,|\, \mathbf{v}\in\mathbf{H}(\mathbf{curl};\Omega)\}\,.$$

The electric field in the insulator $\mathbf{E}_I$ belongs to the space

$$\mathbf{Z}_I := \{\mathbf{z}_I \in \mathbf{H}(\mathbf{curl};\Omega_I) \mid \mathrm{div}(\epsilon\mathbf{z}_I) = 0 \text{ and } \epsilon\mathbf{z}_I\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\,.$$

The weak formulation of the $(\mathbf{H}_C,\mathbf{E}_I)$ hybrid model reads (see [2]):

$$\begin{cases} \text{Find } (\mathbf{H}_C,\mathbf{E}_I) \in \mathbf{H}(\mathbf{curl};\Omega_C)\times\mathbf{Z}_I\ : \\[1ex] \int_{\Omega_C}(\sigma^{-1}\,\mathbf{curl}\,\mathbf{H}_C\cdot\mathbf{curl}\,\overline{\mathbf{v}}_C + i\omega\mu\mathbf{H}_C\cdot\overline{\mathbf{v}}_C) \quad + \quad \int_\Gamma \overline{\mathbf{v}}_C\times\mathbf{n}\cdot\mathbf{E}_I \quad = \quad f_C(\mathbf{v}_C) \\[1ex] \int_\Gamma \mathbf{H}_C\times\mathbf{n}\cdot\overline{\mathbf{z}}_I \quad + \quad i\omega^{-1}\int_{\Omega_I}\mu^{-1}\,\mathbf{curl}\,\mathbf{E}_I\cdot\mathbf{curl}\,\overline{\mathbf{z}}_I \quad = \quad g_I(\mathbf{z}_I) \\[1ex] \text{for all } (\mathbf{v}_C,\mathbf{z}_I)\in\mathbf{H}(\mathbf{curl};\Omega_C)\times\mathbf{Z}_I \end{cases}$$

where

$$f_C(\mathbf{v}_C) := \int_{\Omega_C}\sigma^{-1}\mathbf{J}_{e,C}\cdot\mathbf{curl}\,\overline{\mathbf{v}}_C \quad \text{and} \quad g_I(\mathbf{z}_I) := \int_{\Omega_I}\mathbf{J}_{e,I}\cdot\overline{\mathbf{z}}_I\,.$$

It can be shown, via the standard theory for saddle-point problems, that this problem has a unique solution. The proof relies on the stability of the pairing $(\mathbf{u},\mathbf{w}) \mapsto \int_\Gamma(\mathbf{u}\times\mathbf{n})\cdot\overline{\mathbf{w}}$ on $\mathbf{H}^{1/2}(\mathrm{div}_\tau;\Gamma)\times\mathbf{H}^{1/2}(\mathrm{div}_\tau;\Gamma)$, and this stability is very hard to preserve in the discrete setting. The remedy proposed in [2] is to work in a smaller constrained space. The drawback is that now the solution in Ω_I is not the physical electric field $\mathbf{E}_I$, but a suitable magnetic vector potential $\widetilde{\mathbf{E}}_I$ such that $\mathbf{H}_I = -(i\omega\mu)^{-1}\,\mathbf{curl}\,\widetilde{\mathbf{E}}_I$. Assuming for simplicity that

$$\mathrm{Supp}\,\mathbf{J}_e\cap\Gamma = \emptyset\,, \tag{9}$$

it follows that $\mathrm{div}_\Gamma(\mathbf{H}_C\times\mathbf{n}) = \mathbf{curl}\,\mathbf{H}_I\cdot\mathbf{n} = 0$ on Γ. Hence we shall work on the spaces

$$\widetilde{\mathbf{X}}_C := \{\mathbf{v}_C\in\mathbf{H}(\mathbf{curl};\Omega_C) \mid \mathrm{div}_\Gamma(\mathbf{v}_C\times\mathbf{n}) = 0 \text{ on } \Gamma\}\,,$$

and

$$\widetilde{\mathbf{Z}}_I := \{\mathbf{z}_I\in\mathbf{H}(\mathbf{curl};\Omega_I) \mid \int_{\Omega_I}\mathbf{z}_I\cdot\nabla\overline{\psi}_I = 0 \text{ for all } \psi_I\in H^1(\Omega_I)\}\,.$$

We shall use below the following orthogonal decomposition of the space $\widetilde{\mathbf{Z}}_I$:

$$\widetilde{\mathbf{Z}}_I = (\mathbf{H}^0(\mathbf{curl};\Omega_I))^\perp \oplus \mathcal{H}_I\,,$$

where

$$\mathcal{H}_I := \{\mathbf{z}_I\in\mathbf{H}^0(\mathbf{curl};\Omega_I) \mid \mathrm{div}\,\mathbf{z}_I = 0 \text{ and } \mathbf{z}_I\cdot\mathbf{n} = 0 \text{ on } \partial\Omega_I\}\,.$$

In order to get rid of the constrained space $\widetilde{\mathbf{X}}_C\times\widetilde{\mathbf{Z}}_I$, we shall introduce two Lagrange multipliers. Let us define the space

$$\mathbf{X}_C^* := \{\mathbf{v}_C\in\mathbf{H}(\mathbf{curl};\Omega_C) \mid \mathrm{div}_\Gamma(\mathbf{v}_C\times\mathbf{n})\in L^2(\Gamma)\},$$

endowed with the graph norm. We deal with the uncostrained problem:

$$
\left\{
\begin{array}{llllll}
\multicolumn{6}{l}{\text{Find } (\mathbf{H}_C, \widetilde{\mathbf{E}}_I, q, \phi_I) \in \mathbf{X}_C^* \times \mathbf{H}(\mathbf{curl}; \Omega_I) \times L^2(\Gamma)\backslash\mathbb{C} \times H^1(\Omega_I)\backslash\mathbb{C} \ :} \\[2mm]
a_C(\mathbf{H}_C, \mathbf{v}_C) & + \ \overline{d(\mathbf{v}_C, \widetilde{\mathbf{E}}_I)} & + \ \int_\Gamma \mathrm{div}_\tau(\overline{\mathbf{v}}_C \times \mathbf{n}) q & & = & f_C(\mathbf{v}_C) \\[2mm]
d(\mathbf{H}_C, \mathbf{z}_I) & + \ a_I(\widetilde{\mathbf{E}}_I, \mathbf{z}_I) & & + \int_{\Omega_I} \overline{\mathbf{z}}_I \cdot \nabla \phi_I & = & g_I(\mathbf{z}_I) \\[2mm]
\int_\Gamma \mathrm{div}_\tau(\mathbf{H}_C \times \mathbf{n}) \overline{p} & & & & = & 0 \\[2mm]
& \int_{\Omega_I} \widetilde{\mathbf{E}}_I \cdot \nabla \overline{\psi}_I & & & = & 0 \\[2mm]
\multicolumn{6}{l}{\text{for all } (\mathbf{v}_C, \mathbf{z}_I, p, \psi_I) \in \mathbf{X}_C^* \times \mathbf{H}(\mathbf{curl}; \Omega_I) \times L^2(\Gamma)\backslash\mathbb{C} \times H^1(\Omega_I)\backslash\mathbb{C} \,,}
\end{array}
\right. \tag{10}
$$

where

$$
a_C(\mathbf{u}_C, \mathbf{v}_C) := \int_{\Omega_C} [\sigma^{-1} \, \mathbf{curl}\, \mathbf{u}_C \cdot \mathbf{curl}\, \overline{\mathbf{v}}_C + i\omega\mu \mathbf{u}_C \cdot \overline{\mathbf{v}}_C] \ (= s_C(\mathbf{u}_C, \mathbf{v}_C) + i\, m_C(\mathbf{u}_C, \mathbf{v}_C)) \,, \tag{11}
$$

$$
a_I(\mathbf{w}_I, \mathbf{z}_I) := i\omega^{-1} \int_{\Omega_I} \mu^{-1} \, \mathbf{curl}\, \mathbf{w}_I \cdot \mathbf{curl}\, \overline{\mathbf{z}}_I \ (= i\, s_I(\mathbf{w}_I, \mathbf{z}_I)) \tag{12}
$$

and

$$
d(\mathbf{v}_C, \mathbf{w}_I) := \int_\Gamma \mathbf{v}_C \times \mathbf{n} \cdot \overline{\mathbf{w}}_I \,. \tag{13}
$$

In [2] it has been proved that this problem has a unique solution.

We notice that the Lagrange multiplier $\phi_I \in H^1(\Omega_I)\backslash\mathbb{C}$ is equal to zero. In fact, taking as a test function in the second equation $\mathbf{z}_I = \nabla \phi_I \in \mathbf{H}(\mathbf{curl}; \Omega_I)$ we obtain, from the assumptions on $\mathbf{J}_{e,I}$ (3), (6) and (9),

$$
g_I(\nabla \phi_I) = \int_{\Omega_I} \mathbf{J}_{e,I} \cdot \nabla \overline{\phi}_I = -\int_{\Omega_I} \mathrm{div}\, \mathbf{J}_{e,I} \, \overline{\phi}_I + \int_{\partial\Omega} \mathbf{J}_{e,I} \cdot \mathbf{n} \, \overline{\phi}_I - \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n} \, \overline{\phi}_I = 0 \,.
$$

Moreover, $a_I(\widetilde{\mathbf{E}}_I, \nabla \phi_I) = 0$ and $d(\mathbf{H}_C, \nabla \phi_I) = -\int_\Gamma \mathrm{div}_\tau(\mathbf{H}_C \times \mathbf{n}) \overline{\phi}_I = 0$, hence

$$
\int_{\Omega_I} |\nabla \phi_I|^2 = 0 \,.
$$

Remark 2.1 A different hybrid formulation can be obtained by eliminating the magnetic field in the conductor and the electric field in the insulator: the $\mathbf{E}_C/\mathbf{H}_I$ formulation. In this formulation the magnetic field $\mathbf{H}_I$ belongs to the constrained space

$$
V_I^{\mathbf{J}_{e,I}} := \{ \mathbf{v}_I \in \mathbf{H}_{0,\partial\Omega}(\mathbf{curl}; \Omega_I) \mid \mathbf{curl}\, \mathbf{v}_I = \mathbf{J}_{e,I} \} \,.
$$

If we impose the constraint $\mathbf{curl}\, \mathbf{H}_I = \mathbf{J}_{e,I}$ by means of a Lagrange multiplier it must belong to a constrained functional space, namely

$$
\mathbf{U}_I := \{ \mathbf{u}_I \in (L^2(\Omega_I))^3 \mid \mathrm{div}(\epsilon \mathbf{u}_I) = 0 \text{ and } \epsilon \mathbf{u}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \} \,,
$$

which means that a further multiplier must be introduced. In fact, the Lagrange multiplier for the constraint $\mathbf{curl}\, \mathbf{H}_I = \mathbf{J}_{e,I}$ turns out to be the electric field in the insulator $\mathbf{E}_I$, so the numerical resolution of this hybrid formulation would be very costly.

Another way to deal with the constrained space $V_I^{\mathbf{J}_{e,I}}$ makes use of scalar magnetic potentials by representing $V_I^{\mathbf{0}} = \nabla H^1_{0,\partial\Omega}(\Omega_I) \oplus \mathcal{H}(\partial\Omega; \Gamma)$, where $\mathcal{H}(\partial\Omega; \Gamma)$ is the set of harmonic fields

$$
\mathcal{H}(\partial\Omega; \Gamma) := \{ \mathbf{v}_I \in (L^2(\Omega_I))^3 \mid \mathbf{curl}\, \mathbf{v}_I = \mathbf{0} \,,\ \mathrm{div}(\mu \mathbf{v}_I) = 0 \,,\ \mathbf{v}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega \,,\ \mu \mathbf{v}_I \cdot \mathbf{n} = 0 \text{ on } \Gamma \} \,.
$$

This approach requires the construction of a basis of $\mathcal{H}(\partial\Omega; \Gamma)$. Such a basis is readily available, once we know a collection of surfaces in Ω_I that cut any non-bounding cycle (see [5]). Finding these cuts for an arbitrary shape of Ω_C seems to be a challenging problem(see [7]). For this reason the hybrid $\mathbf{H}_C/\mathbf{E}_I$ formulation seems to be more convenient for complicated geometrical configurations. □

For the finite element approximation of (10) we consider two families of regular tetrahedral meshes $\mathcal{T}_{C,h}$ and $\mathcal{T}_{I,h}$ of Ω_C and Ω_I respectively. We employ the space of (complex valued) Nédélec curl-conforming edge elements of the lowest order $\mathbf{X}_{L,h}$ to approximate the functions in $\mathbf{H}(\mathbf{curl};\Omega_L)$, $L = C, I$. Let us denote by P_k the standard space of complex polynomial of total degree less than or equal to k. To approximate the functions in $L^2(\Gamma)$ we use the finite element space

$$Y_{\Gamma,h} := \{p_h \in L^2(\Gamma) \mid p_{h|T} \in P_0,\ \forall T \in \mathcal{T}_{\Gamma,h}\}\ ,$$

where $\mathcal{T}_{\Gamma,h}$ is the restriction to Γ of the mesh $\mathcal{T}_{C,h}$, and to approximate the functions in $H^1(\Omega_I)$ the space

$$H_{I,h} := \{\psi_{I,h} \in C^0(\Omega_I) \mid \psi_{I,h|K} \in P_1,\ \forall\, K \in \mathcal{T}_{I,h}\}\ .$$

The discrete problem that we consider reads:

$$\begin{cases} \text{Find } (\mathbf{H}_{C,h}, \widetilde{\mathbf{E}}_{I,h}, q_h, \phi_{I,h}) \in \mathbf{X}_{C,h} \times \mathbf{X}_{I,h} \times Y_{\Gamma,h}\backslash\mathbb{C} \times H_{I,h}\backslash\mathbb{C} : \\ a_C(\mathbf{H}_{C,h}, \mathbf{v}_{C,h}) + \overline{d(\mathbf{v}_{C,h}, \widetilde{\mathbf{E}}_{I,h})} + \overline{b_C(\mathbf{v}_{C,h}, q_h)} = f_C(\mathbf{v}_{C,h}) \\ d(\mathbf{H}_{C,h}, \mathbf{z}_{I,h}) + a_I(\widetilde{\mathbf{E}}_{I,h}, \mathbf{z}_{I,h}) + \overline{b_I(\mathbf{z}_{I,h}, \phi_{I,h})} = g_I(\mathbf{z}_{I,h}) \\ b_C(\mathbf{H}_{C,h}, p_h) = 0 \\ b_I(\widetilde{\mathbf{E}}_{I,h}, \psi_{I,h}) = 0 \\ \text{for all}\, (\mathbf{v}_{C,h}, \mathbf{z}_{I,h}, p_h, \psi_{I_h}) \in \mathbf{X}_{C,h} \times \mathbf{X}_{I,h} \times Y_{\Gamma,h}\backslash\mathbb{C} \times H_{I,h}\backslash\mathbb{C}\,, \end{cases} \tag{14}$$

where

$$b_C(\mathbf{v}_C, q) := \int_\Gamma \mathrm{div}_\tau(\mathbf{v}_C \times \mathbf{n})\, \overline{q}$$

and

$$b_I(\mathbf{z}_I, \psi_I) := \int_{\Omega_I} \mathbf{z}_I \cdot \nabla\overline{\psi}_I\ .$$

This problem has a unique solution and optimal error estimates can be proved (see [2]).

As in the continuous problem, the Lagrange multiplier $\phi_{I,h} \in H_{I,h}\backslash\mathbb{C}$ is equal to zero. In fact, taking $\nabla\phi_{I,h} \in \mathbf{X}_{I,h}$ as the test function in the second equation of (14) we have that $g_I(\nabla\phi_{I,h}) = 0$ and $a_I(\widetilde{\mathbf{E}}_{I,h}, \nabla\phi_{I,h}) = 0$. Moreover, since $\mathrm{div}_\tau(\mathbf{H}_{C,h} \times \mathbf{n}) \in Y_{\Gamma,h}$ the third equation in (14) implies that $\mathrm{div}_\tau(\mathbf{H}_{C,h} \times \mathbf{n}) = 0$, hence also $d(\mathbf{H}_{C,h}, \nabla\phi_{I,h}) = 0$ and it follows that $b_I(\nabla\phi_{I,h}, \phi_{I,h}) = \int_{\Omega_I} |\nabla\phi_{I,h}|^2 = 0$.

3 Solving the linear system

We shall use the following notation: let V and W be Hilbert spaces of complex valued functions and $r : V \times W \to \mathbb{C}$ a sesquilinear form. Let us consider finite-dimensional subspaces $V_h \subset V$ and $W_h \subset W$ with bases $\{v_l\}_{l=1}^{N_h}$ and $\{w_k\}_{k=1}^{M_h}$ respectively. We assume that v_l and w_k for $1 \le l \le N_h$, $1 \le k \le M_h$ are real valued functions. Then R denotes the $M_h \times N_h$ matrix with coefficients $R_{k,l} = r(v_l, w_k) \in \mathbb{C}$.

Choosing a basis for each finite dimensional space $\mathbf{X}_{C,h}$, $\mathbf{X}_{I,h}$, $Y_{\Gamma,h}\backslash\mathbb{C}$ and $H_{I,h}\backslash\mathbb{C}$, and using the notation stated above, system (14) can be written as:

$$\begin{bmatrix} A_C & D^T & B_C^T & \\ D & A_I & & B_I^T \\ B_C & & & \\ & B_I & & \end{bmatrix} \begin{bmatrix} H_C \\ \widetilde{E}_I \\ Q \\ \Phi_I \end{bmatrix} = \begin{bmatrix} F_C \\ G_I \\ 0 \\ 0 \end{bmatrix}, \tag{15}$$

with $A_C = S_C + iM_C$ as in (11), and $A_I = iS_I$ as in (12). The complex vectors H_C, $\widetilde{E}_I$, Q and Φ_I are the coefficients of $\mathbf{H}_{C,h}$, $\widetilde{\mathbf{E}}_{I,h}$, q_h and $\phi_{I,h}$ in the chosen bases of $\mathbf{X}_{C,h}$, $\mathbf{X}_{I,h}$, $Y_{\Gamma,h}\backslash\mathbb{C}$ and $H_{I,h}\backslash\mathbb{C}$, respectively. The complex vectors F_C and G_I are obtained applying the functionals f_C and g_I to the elements of the basis of $\mathbf{X}_{C,h}$

and $\mathbf{X}_{I,h}$, respectively. The matrices S_L for $L = C, I$, M_C, B_C, B_I and D are real. S_L with $L = C, I$ is symmetric and positive semidefinite and M_C is symmetric and positive definite.

Problem (15) is an indefinite system that arises from a saddle-point problem. It has the form

$$\begin{bmatrix} A & B^T \\ B & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

with A and B block structured matrices and A symmetric positive semidefinite. It can be solved using, for instance, the method presented in [9](see also [4] for a review of numerical methods for the solution of saddle point problems). However, to take advantage of the fact that our problem arises from an eddy current problem with two different subdomains, we rearrange system (15) in the following way:

$$\begin{bmatrix} A_C & B_C^T & D^T & \\ B_C & & & \\ D & & A_I & B_I^T \\ & & B_I & \end{bmatrix} \begin{bmatrix} H_C \\ Q \\ \widetilde{E}_I \\ \Phi_I \end{bmatrix} = \begin{bmatrix} F_C \\ 0 \\ G_I \\ 0 \end{bmatrix}. \tag{16}$$

Since $\Phi_I = 0$ and $B_I \widetilde{E}_I = 0$, it is possible to eliminate this unknown considering the reduced system

$$\begin{bmatrix} A_C & B_C^T & D^T \\ B_C & & \\ D & & A_I + i\gamma B_I^T B_I \end{bmatrix} \begin{bmatrix} H_C \\ Q \\ \widetilde{E}_I \end{bmatrix} = \begin{bmatrix} F_C \\ 0 \\ G_I \end{bmatrix}, \tag{17}$$

where the parameter γ is any positive real number.

Proposition 3.1 *System (16) and system (17) are equivalent.*

Proof. Since system (16) has a unique solution $(H_C, Q, \widetilde{E}_I, \Phi_I)$ with $\Phi_I = 0$, and in particular $B_I \widetilde{E}_I = 0$, it is clear that $(H_C, Q, \widetilde{E}_I)$ is solution of (17). Hence to show that both systems are equivalent it is enough to show that (17) has a unique solution. If there exists a non null solution of the homogeneous problem

$$\begin{bmatrix} A_C & B_C^T & D^T \\ B_C & & \\ D & & A_I + i\gamma B_I^T B_I \end{bmatrix} \begin{bmatrix} V_C \\ P \\ Z_I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

recalling that $A_I = iS_I$ we have

$$\begin{aligned} & V_C^*(S_C + iM_C)V_C + V_C^* B_C^T P + V_C^* D^T Z_I = 0\,, \\ & P^* B_C V_C = 0\,, \\ & Z_I^* D V_C = -i Z_I^* (S_I + \gamma B_I^T B_I) Z_I\,. \end{aligned} \tag{18}$$

Replacing $V_C^* B_C^T P$ and $V_C^* D^T Z_I$ in the first equation of (18) by the values given in the second and third equations we get

$$V_C^* S_C V_C + i(V_C^* M_C V_C + Z_I^* S_I Z_I + \gamma Z_I^* B_I^T B_I Z_I) = 0\,.$$

In particular, since M_C is symmetric positive definite and S_I is symmetric positive semidefinite it follows that $B_I Z_I = 0$. This means that

$$\begin{bmatrix} A_C & B_C^T & D^T & \\ B_C & & & \\ D & & A_I & B_I^T \\ & & B_I & \end{bmatrix} \begin{bmatrix} V_C \\ P \\ Z_I \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and hence, since the matrix in this system is non singular, the vectors V_C, P and Z_I are equal to zero. $\square$

Even if for any value of $\gamma > 0$ systems (16) and (17) are equivalent, the computed solutions for small values of γ could be different, because in the limit case $\gamma = 0$ the reduced system is singular. On the other hand for big values of γ the matrix of the reduced system is ill-conditioned. The convergence rate of the resolution algorithms depends on the choice of this parameter, more precisely γ should be chosen such that the matrices S_I and $\gamma B_I^T B_I$ are balanced.

Remark 3.1 It is worthy to note that the matrix $A_I + i\gamma B_I^T B_I = i(S_I + \gamma B_I^T B_I)$ is invertible if and only if Ω_I is simply connected. In fact, let us consider the space

$$\widetilde{\mathbf{Z}}_{I,h} := \{\mathbf{z}_{I,h} \in \mathbf{X}_{I,h} \, : \, \int_{\Omega_I} \mathbf{z}_{I,h} \cdot \nabla\phi_{I,h} = 0 \quad \forall\, \phi_{I,h} \in H_{I,h}\} \,,$$

that, analogously to its continuous counterpart, can be decomposed as the following direct sum

$$\widetilde{\mathbf{Z}}_{I,h} = [\mathbf{H}^0(\mathbf{curl}; \Omega_I) \cap \mathbf{X}_{I,h}]^\perp \oplus \mathcal{H}_{I,h} \,,$$

where

$$\mathcal{H}_{I,h} := \mathbf{H}^0(\mathbf{curl}; \Omega_I) \cap \widetilde{\mathbf{Z}}_{I,h} \,.$$

Given $Z_I \in \mathbb{C}^n$, let us denote $\mathbf{z}_{I,h}$ the function in $\mathbf{X}_{I,h}$ with coefficients Z_I. From the definitions of S_I and B_I, and since S_I is symmetric and positive semidefinite, it holds that $(S_I + \gamma B_I^T B_I)\, Z_I = 0$ if and only if $\mathbf{curl}\, \mathbf{z}_{I,h} = \mathbf{0}$ and $\mathbf{z}_{I,h} \in \widetilde{\mathbf{Z}}_{I,h}$ which means that $\mathbf{z}_{I,h} \in \mathcal{H}_{I,h}$. Moreover it can be proved that $\dim \mathcal{H}_{I,h} = \dim \mathcal{H}_I$, and it is well known (see [3]) that $\dim \mathcal{H}_I = p_I$, where $p_I = \beta_1(\Omega_I)$ stands for the first Betti number of Ω_I, that is zero if and only if Ω_I is simply connected.

On the other hand let us consider the perturbed matrix $S_I + \gamma B_I^T B_I + \varepsilon D\, D^T$. It is possible to prove that for each $\varepsilon > 0$ this matrix is not singular. In fact, if $(S_I + \gamma B_I^T B_I + \varepsilon D\, D^T)\, Z_I = 0$ we have in particular $(S_I + \gamma B_I^T B_I)\, Z_I = 0$ and $D^T\, Z_I = 0$, hence $\mathbf{z}_{I,h} \in \mathcal{H}_{I,h}$ and

$$\int_\Gamma \mathbf{v}_{C,h} \times \mathbf{n} \cdot \bar{\mathbf{z}}_{I,h} = 0 \,,$$

for all $\mathbf{v}_{C,h} \in \mathbf{X}_{C,h}$. From the discrete inf–sup condition proved in [2]

$$\exists \beta > 0 : \qquad \sup_{\substack{\mathbf{v}_{C,h} \in \mathbf{X}_{C,h} \\ \mathbf{v}_{C,h} \neq \mathbf{0}}} \frac{\left| \int_\Gamma \mathbf{v}_{C,h} \times \mathbf{n} \cdot \bar{\mathbf{z}}_{I,h} \right|}{\|\mathbf{v}_{C,h}\|_{\mathbf{H}(\mathbf{curl}; \Omega_C)}} \geq \beta \|\mathbf{z}_{I,h}\|_{(L^2(\Omega_I))^3} \quad \forall\, \mathbf{z}_{I,h} \in \mathcal{H}_{I,h} \,,$$

it follows that $\mathbf{z}_{I,h} = \mathbf{0}$, which implies $Z_I = 0$. □

Next we present two different algorithms to solve system (17), both of them taking advantage of the fact that the problem is formulated in two subdomains.

Modified SOR method

It is a block version of the SOR method. If the domain is not simply connected, in order to have non singular matrices on the diagonal of the block decomposition, the subproblem in the air region is modified by adding the term $i\varepsilon D\, D^T$ (see Remark 3.1) . In [10] a similar idea has been used to solve the problem formulated in terms of a magnetic vector potential in the whole domain Ω. In that paper the problem is perturbed adding a term of the form εM, where M is the matrix that corresponds to the bilinear form in $(L^2(\Omega))^3$, $m(\mathbf{w}, \mathbf{z}) := \int_\Omega \mathbf{w} \cdot \bar{\mathbf{z}}$. In our experience, the use of DD^T instead of the mass matrix in the insulator, improves the convergence of the method. The algorithm reads:

Algorithm 1 *Given H_C^0, Q^0 and $\widetilde{E}_I^0$ for $k \geq 0$ solve*

$$\begin{bmatrix} A_C & B_C^T \\ B_C & \end{bmatrix} \begin{bmatrix} H_C^{k+1/2} \\ Q^{k+1/2} \end{bmatrix} = \begin{bmatrix} F_C - D^T \widetilde{E}_I^k \\ 0 \end{bmatrix} ,$$

set

$$\begin{bmatrix} H_C^{k+1} \\ Q^{k+1} \end{bmatrix} = (1-\theta) \begin{bmatrix} H_C^k \\ Q^k \end{bmatrix} + \theta \begin{bmatrix} H_C^{k+1/2} \\ Q^{k+1/2} \end{bmatrix} ,$$

then solve

$$i(S_I + \gamma B_I^T B_I + \varepsilon D\, D^T) \widetilde{E}_I^{k+1/2} = G_I - D H_C^{k+1} + i\varepsilon D\, D^T \widetilde{E}_I^k \,,$$

and set

$$\widetilde{E}_I^{k+1} = (1-\theta) \widetilde{E}_I^k + \theta \widetilde{E}_I^{k+1/2} \,.$$

The real number θ is the relaxation parameter of the SOR method and must be chosen $0 < \theta < 2$. The parameter ε is taken equal to zero if the subdomain Ω_I is simply connected; in other case it must be greater than zero. The performance of the algorithm depends on the appropriate choice of both parameters (see Table 2).

At each iteration of the algorithm one needs to solve a linear system in each subdomain. To solve the subproblem in the insulator we use the preconditioned conjugate gradient method, taking as the preconditioner an incomplete Cholesky factorization of $S_I + \gamma B_I^T B_I + \varepsilon D\, D^T$. For the subproblem in the conductor we make use of its saddle-point structure, and solve it with an inexact Uzawa's algorithm with variable relaxation parameters (see [8]). For a system of the general form

$$\begin{bmatrix} A_C & B_C^T \\ B_C & \end{bmatrix} \begin{bmatrix} h \\ q \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

the algorithm reads: given h^0 and q^0, for $j \geq 0$ set

$$h^{j+1} = h^j + \omega_j \hat{A}_C^{-1} \left[f - (A_C h^j + B_C^T q^j) \right] ,$$

and

$$q^{j+1} = q^j + \tau_j \hat{P}_C^{-1} B_C h^{j+1},$$

where $\hat{A}_C$ is a preconditioner for A_C and $\hat{P}_C$ is a preconditioner for $B_C \hat{A}_C^{-1} B_C^T$. In particular we take as the preconditioner $\hat{A}_C$ an incomplete LU factorization (ILU) of A_C, and as $\hat{P}_C$ an ILU of $B_C \Lambda_C^{-1} B_C^T$, where Λ_C is a diagonal matrix with the elements of the main diagonal of A_C. The parameters ω_i and τ_i are computed dinamically at each iteration, as it is done in [8].

Uzawa-like method

This algorithm is a preconditioned Uzawa's method with variable relaxation parameter chosen as in [8], but adapted to the particular structure of our problem. The algorithm reads as follows:

Algorithm 2 *Given $\widetilde{E}_I^0$ for $k \geq 0$ solve*

$$\begin{bmatrix} A_C & B_C^T \\ B_C & \end{bmatrix} \begin{bmatrix} H_C^{k+1} \\ Q^{k+1} \end{bmatrix} = \begin{bmatrix} F_C - D^T \widetilde{E}_I^k \\ 0 \end{bmatrix},$$

then compute

$$r_k = D H_C^{k+1} + i(S_I + \gamma B_I^T B_I)\widetilde{E}_I^k - G_I \,,$$
$$d_k = \hat{N}^{-1} r_k \,,$$

and

$$\tau_k = \begin{cases} \frac{(r_k, d_k)}{(\hat{A}_C^{-1} D^T d_k, D^T d_k)} \,, & r_k \neq 0 \,, \\ 1 \,, & r_k = 0 \,, \end{cases}$$

and set

$$\widetilde{E}_I^{k+1} = \widetilde{E}_I^k + \theta_k \tau_k d_k.$$

Here θ_k is a parameter that, for an exact Uzawa's algorithm, can be chosen $0 < \theta_k \leq \frac{1}{2}$; the matrix $\hat{N}$ is a preconditioner for

$$P = \begin{bmatrix} D & 0 \end{bmatrix} \begin{bmatrix} A_C & B_C^T \\ B_C & 0 \end{bmatrix}^{-1} \begin{bmatrix} D^T \\ 0 \end{bmatrix} - i(S_I + \gamma B_I^T B_I) \,, \tag{19}$$

and $\hat{A}_C$ is a preconditioner for A_C.

In all the numerical tests presented in Section 4 we set θ_k constant and equal to $\frac{1}{2}$, and we take as preconditioners $\hat{N}$ and $\hat{A}_C$ the ILU of the matrices $S_I + \gamma B_I^T B_I$ and A_C, respectively.

We notice that this algorithm doesn't require to solve any linear system in the insulator, hence it is not necessary to modify the matrix of that subproblem, even in the case of Ω_I being non simply connected, i.e. when the matrix $S_I + \gamma B_I^T B_I$ is singular (see Remark 3.1). On the other hand in the conductor one needs to solve a linear system at each iteration, and it can be used, for instance, the inexact Uzawa's method already proposed in the modified SOR scheme.

Remark 3.2 The straightforward extension to our system of the method presented in [8] would require, for the computation of τ_k, to calculate the expression

$$\left(\widehat{\mathcal{A}}_C^{-1}\left[\begin{array}{c} D^T d_k \\ 0 \end{array}\right], \left[\begin{array}{c} D^T d_k \\ 0 \end{array}\right]\right),$$

$\widehat{\mathcal{A}}_C$ being a preconditioner of the block structured matrix corrsponding to the conductor. However, to take advantage of the null blocks, we are only considering the main part of the matrix, and computing $(\widehat{A}_C^{-1} D^T d_k, D^T d_k)$ where $\hat{A}_C$ is a preconditioner for A_C.

$\square$

4 Numerical results

The finite element method and the algorithms introduced in the previous section have been implemented in MATLAB. In the following we present some numerical tests illustrating how the algorithms perform. In the first set of numerical experiments we solve a problem with known analytical solution to validate the computer code and test the convergence properties of the methods. In the second and third numerical tests we consider a torus-shaped coil inducing eddy-currents in a non simply connected conductor, which is a torus in the second test problem and a trefoil knot in the third one. The last case concerns the benchmark Problem number 7 in the TEAM Workshop, which deals with an asymmetrical conductor with a hole (see [6], [10]). All the simulations have been run on a single processor Intel Xeon 5140 2.33GHz.

A problem with known analytical solution

In this set of tests the conductor Ω_C and the domain Ω are two cubes centered at the origin and with edge lengths 2 and 10, respectively. We shall construct an analytical solution $(\mathbf{H}_C, \widetilde{\mathbf{E}}_I)$ which will consist of two $\mathcal{C}^2$ functions with compact supports in Ω_C and Ω_I, respectively.

Let us suppose that ω, μ and $\sigma|_{\Omega_C}$ are positive constants equal to one, and that $\sigma|_{\Omega_I} \equiv 0$, as it was said before. Given a closed ball centered at $\mathbf{x}_0 \in \Omega$ and with radius r_0, we define the function p with support in this ball as follows:

$$p(x) = \begin{cases} q\left(\frac{|\mathbf{x}-\mathbf{x}_0|}{r_0}\right), & \text{if } |\mathbf{x}-\mathbf{x}_0| \leq r_0, \\ 0, & \text{if } |\mathbf{x}-\mathbf{x}_0| > r_0, \end{cases}$$

q being the unique eighth degree polynomial such that $q(0) = 1$, and the polynomial and its first three derivatives are null at the points 1 and -1. It is easily seen that q is given by the expression

$$q(x) = x^8 - 4x^6 + 6x^4 - 4x^2 + 1\,.$$

Now, let Θ_C be the closed ball centered at the origin with radius $r_0 = 0.9$, and Θ_I the ball with center at $\mathbf{x}_0 = (0, 3, 0)$ and radius $r_0 = 1.9$. Obviously, the two balls are disjoint and they are strictly contained in Ω_C and Ω_I, respectively. Let us denote by p_C and p_I the functions corresponding to the balls Θ_C and Θ_I, and define the electric field in the insulator as

$$\widetilde{\mathbf{E}}_I := \mathbf{curl}(0, 0, p_I) = \left(\frac{\partial p_I}{\partial y}, -\frac{\partial p_I}{\partial x}, 0\right),$$

and the magnetic field in the conductor as

$$\mathbf{H}_C := i\,\mathbf{curl}(\mathbf{curl}(0, 0, p_C)) = i\left(\frac{\partial^2 p_C}{\partial x \partial z}, \frac{\partial^2 p_C}{\partial y \partial z}, -\frac{\partial^2 p_C}{\partial x^2} - \frac{\partial^2 p_C}{\partial y^2}\right).$$

Now from first and second equations in (8) one can easily compute $\mathbf{J}_{e,C}$ and $\mathbf{J}_{e,I}$, and check that the excitation current density $\mathbf{J}_e$ satisfies the three compatibility conditions.

The program has been tested by solving this academic problem with four successively refined meshes, with grid sizes corresponding to h, $h/2$, $h/3$ and $h/4$ and setting the parameter γ equal to one in the four cases. In Table 1 we present the relative error between the computed and the exact solutions. More precisely, we set

$$e_{\mathbf{H}} = \frac{\|\mathbf{H}_C - \mathbf{H}_{C,h}\|_{\mathbf{H}(\mathbf{curl};\Omega_C)}}{\|\mathbf{H}_C\|_{\mathbf{H}(\mathbf{curl};\Omega_C)}}, \qquad e_{\mathbf{E}} = \frac{\|\widetilde{\mathbf{E}}_I - \widetilde{\mathbf{E}}_{I,h}\|_{\mathbf{H}(\mathbf{curl};\Omega_I)}}{\|\widetilde{\mathbf{E}}_I\|_{\mathbf{H}(\mathbf{curl};\Omega_I)}}.$$

Figure 1 shows the plots in a log–log scale of the relative errors of $e_{\mathbf{H}}$ and $e_{\mathbf{E}}$ versus the number of degrees of freedom. As can be seen, the error is reduced when the mesh is refined, even though the relative error for the finest mesh is still quite large. One of the reasons for these large errors is that the solution of our problem is a polynomial of seventh degree, and the support is concentrated in a small part of the domain.

Number of				SOR ($\theta = 0.5$)		Uzawa	
elements	d.o.f	$e_{\mathbf{H}}$	$e_{\mathbf{E}}$	iterations	time [s]	iterations	time [s]
1155	1721	0.9710	0.8128	20	3.9	16	2.0
9240	12245	0.6668	0.5163	20	78.9	31	60.2
31185	39656	0.5377	0.3399	20	576.9	203	699.2
73920	92039	0.4253	0.2607	20	3219.9	365	4248.4

Table 1: Results for problem with known analytical solution.

In Table 1 are also presented the number of iterations and computational times when solving system (16) using SOR method with relaxation parameter $\theta = 0.5$ and when solving with Uzawa's method. In this case, since Ω_I is simply connected, the perturbation parameter in the SOR method can be taken $\varepsilon = 0$ and the number of iterations is independent on the mesh size. The computational time includes the calculation of the preconditioners.

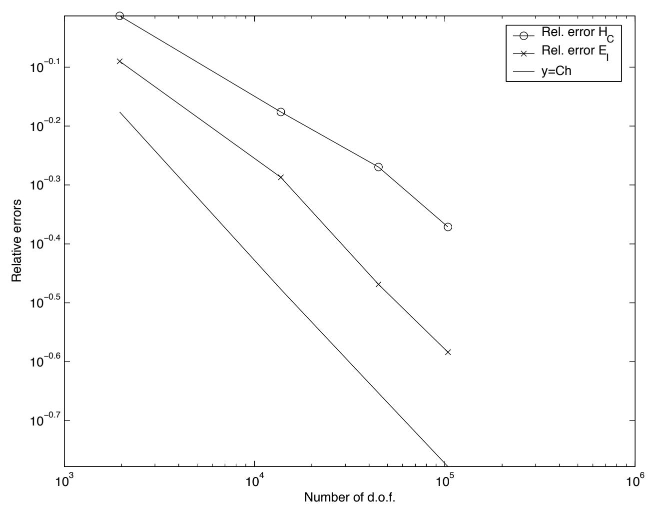

Figure 1: Relative error versus number of d.o.f.

Two tori

In the second test the computational domain is a cube with edge length 27 cm. We consider two coaxial tori of square section with edge length 1 cm. and radius 6.5 cm. (see Figure 2 where a similar torus is shown): the upper torus is a coil, that is part of the insulator region Ω_I, where we impose a clockwise current density $\mathbf{J}_{e,I}$ of magnitude 10^6 A/m^2, whereas the second torus is the conductor. For the physical magnitudes we are taking $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m, the magnetic permeability of the air, $\sigma = 10^7$ S/m and the angular frequency $\omega = 2\pi \times 50$ rad/s. The parameter γ is set equal to 10^6.

In this case, since the air region Ω_I is not simply connected, the parameter ε in the SOR method must be positive. We present in Table 2 the convergence results for different choices of ε and θ for a mesh with 27152 elements. Then we take the best values found for both parameters ($\theta = 0.5$, $\varepsilon = 10^5$) and use them in three different meshes to compare the behaviour of the modified SOR and the Uzawa's method. The results are summarized in Table 3.

	$\varepsilon = 10^5$		$\varepsilon = 10^6$		$\varepsilon = 10^7$	
	iterations	time [s]	iterations	time [s]	iterations	time [s]
$\theta = 0.25$	51	720.8	196	1392.2	1751	6372.1
$\theta = 0.5$	22	461.9	96	761.3	874	3247.5
$\theta = 0.75$	NC	NC	63	580.6	582	2251.1
$\theta = 1$	NC	NC	47	496.3	436	1760.4
$\theta = 1.25$	NC	NC	NC	NC	357	1505.0

Table 2: Two tori. Results for the SOR method with several values of θ and ε. (NC: not convergent)

Number of elements	d.o.f	SOR ($\theta = 0.5$, $\varepsilon = 10^5$)		Uzawa	
		iterations	time [s]	iterations	time [s]
3394	4763	24	32.8	336	55.6
27152	34795	22	461.9	239	751.3
91638	113853	25	2937.1	211	4356.1

Table 3: Two tori. Comparison of SOR and Uzawa's method.

The trefoil knot

We want to show how the $(\mathbf{H}_C, \widetilde{\mathbf{E}}_I)$ formulation performs in problems with complicated geometries, (see Remark 2.1). In particular we consider a problem in which the conductor is a socalled trefoil knot. It is well known (see e.g. [5], [7]) that there exists one "cutting" surface which cuts any non-bounding cycle in Ω_I, but its construction is a difficult task.

We suppose that Ω_C is a trefoil knot formed joining cubes of edge length 1 cm. Above the conductor there is placed a torus-shaped coil (see Figure 2). The sizes of the coil and the computational domain Ω are the same that in the previous test case. The physical magnitudes, the source current and the parameter γ are also taken as in the two tori test case.

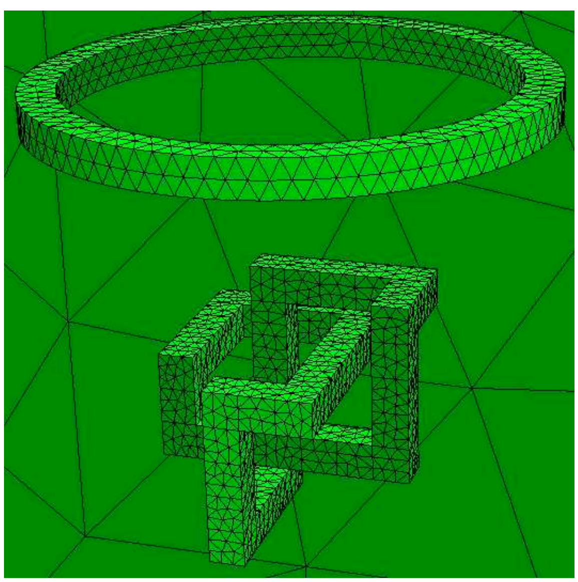

Figure 2: Detail of the mesh for the trefoil knot

In Table 4 we present the number of iterarions and the CPU time for the modified SOR method with relaxation parameter $\theta = 1$ and $\varepsilon = 10^6$, and for the Uzawa-like scheme. We represent, in Figure 3 and Figure 4 the real part and the imaginary part of the current density $\mathbf{J}_C = \mathbf{curl}\,\mathbf{H}_C$ on the surface of the knot.

Number of elements	d.o.f	SOR ($\theta = 1$, $\varepsilon = 10^6$)		Uzawa	
		iterations	time [s]	iterations	time [s]
37057	51665	402	4524.4	678	5673.4

Table 4: Results for the trefoil knot.

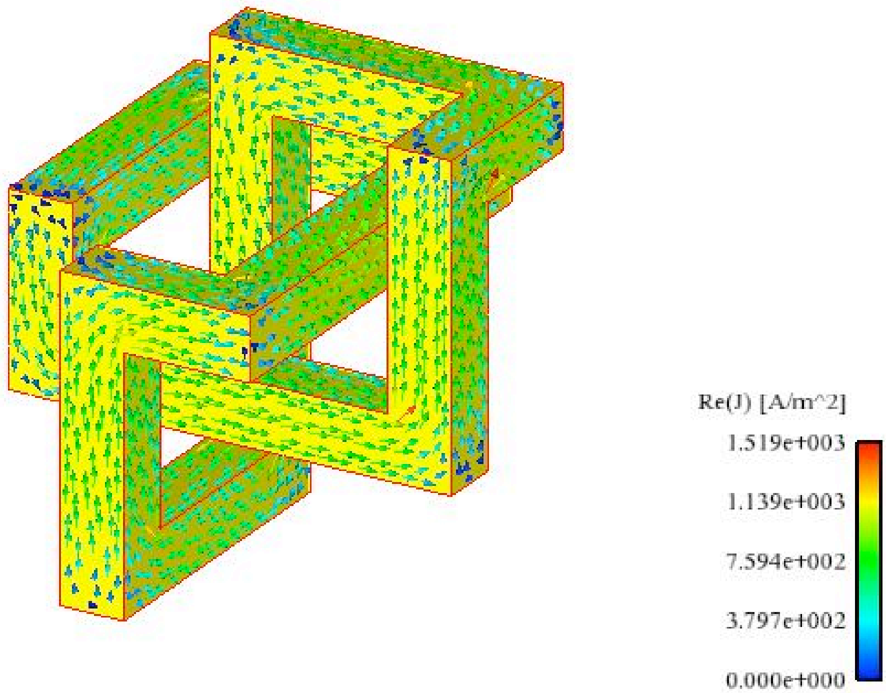

Figure 3: The current density in the trefoil knot: real part

Benchmark problem 7 in the TEAM Workshop

Our last test corresponds to benchmark problem number 7 in the TEAM workshop (see [6]). It consists of a thick aluminum plate with a hole eccentrically placed, and subjected to an asymmetric magnetic field. The field is produced by an exciting current traversing a coil above the plate (see Figure 5). The magnetic permeability is $\mu = 4\pi \times 10^{-7}$ H/m, the electrical conductivity is $\sigma = 3.526 \times 10^7$ S/m, the angular frequency is $\omega = 2\pi \times 50$ rad/s, and the absolute value of the real part (respectively, imaginary) of the excitation current density $\mathbf{J}_{e,I}$ is 1.0968×10^6 (respectively 0) A/m^2. The parameter γ is set equal to 5×10^6.

In Table 5 we present the number of iterations and CPU time for the perturbed-SOR method with relaxation parameter $\theta = 1$ and $\varepsilon = 10^6$, and for the Uzawa's scheme.

Number of elements	d.o.f	SOR ($\theta = 1$, $\varepsilon = 10^6$)		Uzawa	
		iterations	CPU time [s]	iterations	CPU time [s]
98857	131434	134	16868.6	191	17794.4

Table 5: Results for problem 7 in the TEAM Workshop.

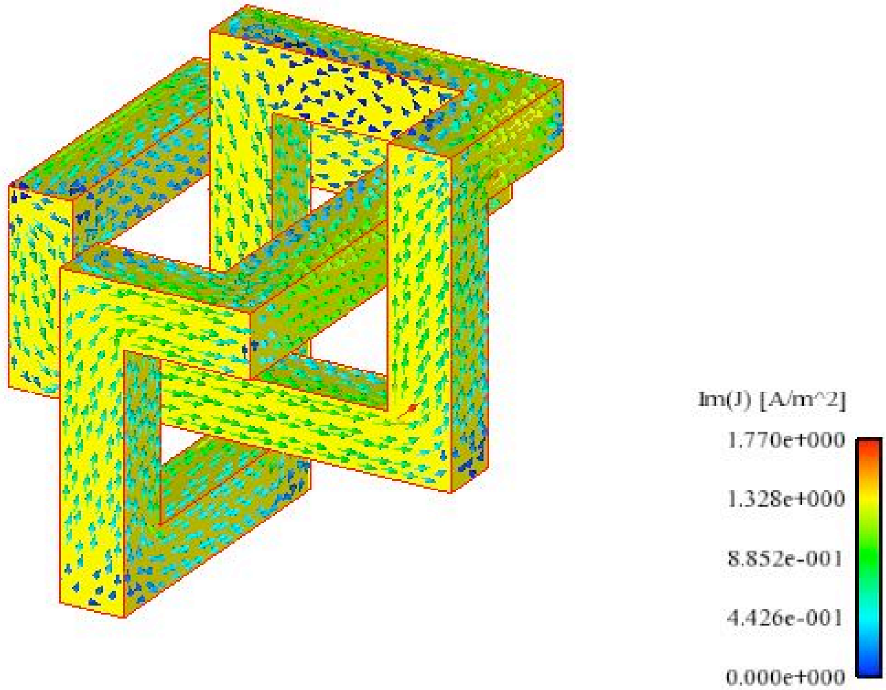

Figure 4: The current density in the trefoil knot: imaginary part

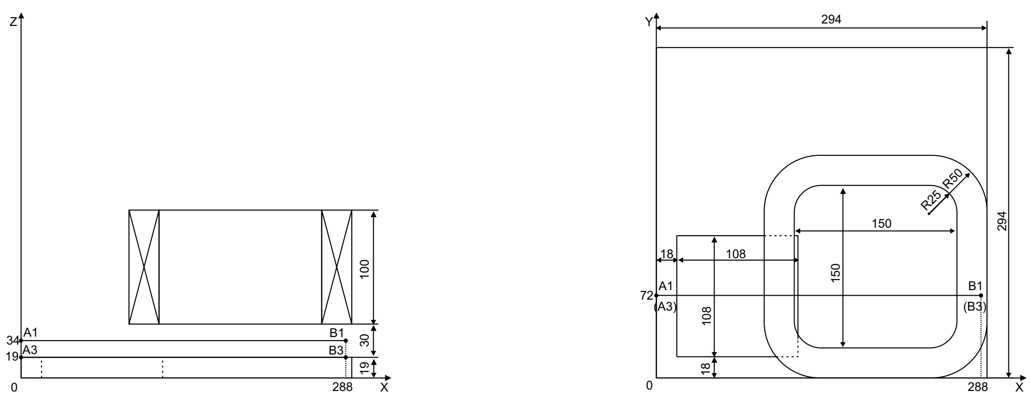

Figure 5: Geometry of the TEAM model

We also present a comparison of our numerical results with the experimental data given in [6]. In the left of Figure 6 is represented the z component of the magnetic induction $\mathbf{B}_I$ ($\equiv \frac{i}{\omega}\,\mathbf{curl}\,\widetilde{\mathbf{E}}_I$) along a straight line in the air region, with $y = 72\,\mathrm{mm}$. and $z = 34\,\mathrm{mm}$. In the right of Figure 6 is represented the y component of the current density $\mathbf{J}_C$ ($\equiv \mathbf{curl}\,\mathbf{H}_C$) along a straight line on the surface of the conductor, with $y = 72\,\mathrm{mm}$ and

$z = 19\,\mathrm{mm}$. We notice that the results for the magnetic induction in the air are not completely satisfactory. This is due to the fact that $\mathbf{B}_I$ is calculated from the curl of the computed $\widetilde{\mathbf{E}}_I$ and since we are using first order edge elements this curl is constant on each element. The oscillations are due to the relative position of the elements of the mesh where we compute the numerical values, with respect to the line where the field is measured.

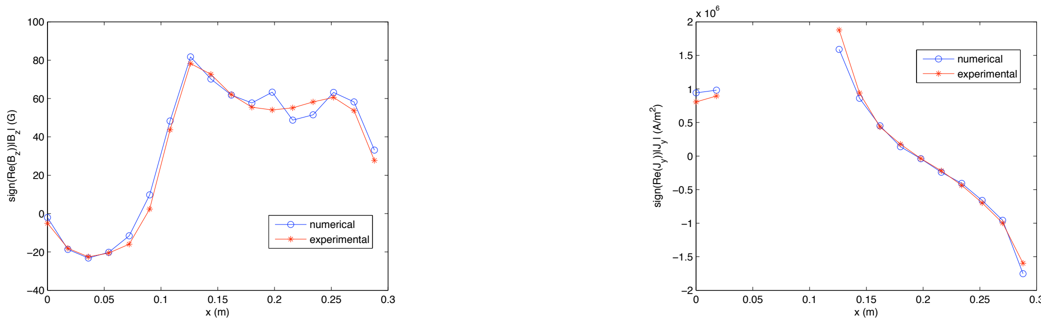

Figure 6: Left: z component of $\mathbf{B}_I$ along line A1–B1. Right: y component of $\mathbf{J}_C$ along line A3–B3.

5 Conclusions

We have implemented the finite element approximation of the $\mathbf{H}_C/\mathbf{E}_I$ formulation of the eddy-current problem studied in [2]. In particular we have considered a reduced linear system eliminating the Lagrange multiplier introduced in the insulator region by the penalty method. For the resolution of the resulting linear system we have proposed two different algorithms: a modified SOR method and an Uzawa-like scheme. Both algorithms take advantage of the coupled nature of the eddy current problem, which distinguishes between a conductor and an insulator.

In the light of the results presented above, the modified SOR method performs better in all the four test cases. The number of iterations by subdomains is smaller than for the Uzawa's method and also the CPU time is lower. In our opinion the performance of Uzawa's method can be improved with a better preconditioner for the matrix P defined in (19).

In the case of a simply connected conductor the penalization parameter of the modified SOR method is taken $\varepsilon = 0$ (so the method is actually the standard SOR), and the rate of convergence turns out to be independent of the mesh size. In other geometrical situations ε must be positive, and the performance of the modified SOR method depends on the choice of this ε and the relaxation parameter θ.

Morever it is worthy to note that the condition number of the reduced system is quite sensible to the choice of the penalization parameter γ.

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