Preprint UTM 674 - July 2004

Bose-Einstein condensation and Spontaneous Breaking of Conformal Symmetry on Killing Horizons

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Abstract. Local scalar QFT (in Weyl algebraic approach) is constructed on degenerate semi-Riemannian manifolds corresponding to Killing horizons in spacetime. Covariance properties of the C^*-algebra of observables with respect to the conformal group $PSL(2,\mathbb{R})$ are studied. It is shown that, in addition to the state studied by Guido, Longo, Roberts and Verch for bifurcated Killing horizons, which is conformally invariant and KMS at Hawking temperature with respect to the Killing flow and defines a conformal net of von Neumann algebras, there is a further wide class of algebraic states representing spontaneous breaking of $PSL(2,\mathbb{R})$ symmetry. This class is labeled by functions in a suitable Hilbert space and their GNS representations enjoy remarkable properties. The states are non equivalent extremal KMS states at Hawking temperature with respect to the residual one-parameter subgroup of $PSL(2,\mathbb{R})$ associated with the Killing flow. The KMS property is valid for the two local sub algebras of observables uniquely determined by covariance and invariance under the residual symmetry unitarily represented. These algebras rely on the physical region of the manifold corresponding to a Killing horizon cleaned up by removing the unphysical points at infinity (necessary to describe the whole $PSL(2,\mathbb{R})$ action). Each of the found states can be interpreted as a different thermodynamic phase, containing Bose-Einstein condensate, for the considered quantum field. It is finally suggested that the quantum field could be interpreted as a noncommutative coordinate on the horizon since its mean value on any of the states introduced above defines, in fact, a (commutative) coordinate on the horizon.

1 Introduction.

In a remarkable paper [1] Guido, Longo, Roberts and Verch showed that, under suitable hypotheses the local algebra of observables in a spacetime containing a bifurcate Killing horizon induces a local algebra of observables on the Killing horizon which is covariant with respect to a unitary (projective) representation of $PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\{\pm I\}$. This was done by considering a net of von Neumann algebras in the representation of a certain state which is assumed to exist and satisfy the following requirement. Its restriction to the subnet of observables which are localized at the horizon, must be KMS at Hawking temperature for the Killing flow. The result uses

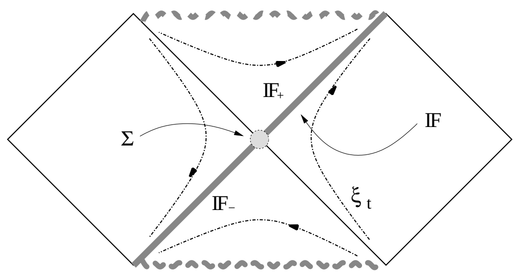
general theorems due to Wiesbrock [2, 3] (which shows the existence of $SL(2,\mathbb{R})$ representations related to modular operators of von Neumann algebras, and thus to the KMS condition) and has also some interplay with several "holographic" ideas (including LightFront Holography) in QFT [4, 5, 6, 7]. In the framework of *conformal nets* (see for instance [8, 9, 10, 11] and references therein) employed in [1], the full covariance of local observables with respect to Möbius group $PSL(2,\mathbb{R})$ is described extending the Killing horizon by adding points at infinity obtaining a manifold $\mathbb{S}^1 \times \Sigma$, Σ being the transverse manifold at the bifurcation of horizons. $\mathbb{S}^1$ represents the history $\mathbb{R}$ of a particle of light living on the horizon compactified into a circle by the addition of a point at infinity. This is necessary since the Möbius group acts as a subgroup of the diffeomorphisms of the circle $\mathbb{S}^1$. In particular there is a one-parameter subgroup of $PSL(2,\mathbb{R})$ which describes arbitrary angular displacements on $\mathbb{S}^1$ realized as $[-\pi,\pi]$ with the identification of the extremal points. The action of this subgroup has no physical meaning since it shifts in the physical region the point at infinity. However the covariance machinery contemplate also those unphysical transformations in principle. In spite of this drawback, the theory shows the existence of interplay of Killing horizons, thermal state at the correct physical temperature, and conformal symmetry the vacuum state used to built up the representation of the C^*-algebra of observables is $PSL(2,\mathbb{R})$-invariant. This fact is strongly remarkable in its own right.
In this paper, first of all, we give an explicit procedure to built up a local algebra of observables localized on a degenerate semi-Riemannian manifold $\mathbb{M} := \mathbb{S}^1 \times \Sigma$ (Killing horizon in particular) without referring to external algebra and states. We use Weyl quantization procedure for a real scalar field finding, in fact, a conformal net of observables relying on a $PSL(2,\mathbb{R})$-invariant vacuum λ which is KMS in suitable regions $\mathbb{F}_\pm$ of $\mathbb{M}$ corresponding to the physical part of the Killing horizon without the bifurcation point: At algebraic level there is a representation α of Möbius group $PSL(2,\mathbb{R})$ made of *-automorphisms of the Weyl algebra $\mathcal{W}(\mathbb{M})$ and there is a state λ on $\mathcal{W}(\mathbb{M})$ which is invariant under α and, in the GNS representation of λ, α is implemented unitarily and covariantly by a representation U of $PSL(2,\mathbb{R})$. Afterwards we show that it is possible to get rid of the drawback concerning the unphysical action of $PSL(2,\mathbb{R})$ and single out the physical part of the horizon at *quantum, i.e. Hilbert space, level* through a sort of *spontaneous breaking of $PSL(2,\mathbb{R})$ symmetry* referring to a new state $\lambda_\zeta \neq \lambda$ which *preserves* the relevant thermal properties: We establish the existence of other, unitarily inequivalent, GNS representations of $\mathcal{W}(\mathbb{M})$ based on new states λ_ζ – the functions ζ being in $L^2(\Sigma)$ – which are no longer invariant under the whole α, but such that, the residual symmetry still is covariantly and unitarily implementable and singles out the algebras $\mathcal{A}(\mathbb{F}_+)$ and $\mathcal{A}(\mathbb{F}_-)$ as unique invariant algebras. We show also that every λ_ζ enjoys the same thermal (KMS) properties as λ and it represents a different *thermodynamical phases* with respect to λ (this is because the states λ_ζ are extremal KMS states). The difference, in the case of ζ real, is related with the appearance of a Bose-Einstein condensate localized at the horizon. Finally we suggest that the bosonic field ϕ generating the Weyl representations could represent a noncommutative coordinate in the physical regions $\mathbb{F}_\pm$. Several comments concerning the representation of the whole group $Diff^+(\mathbb{S}^1)$ and in particular its Lie algebra in the presence of the transverse manifold Σ, are spread throughout the work.

In this work, concerning KMS states we adopt the definition 5.3.1 in [12] (see also chapter

V of [13] where the σ-weak topology used in the definition above in the case of a von Neumann algebra is called *weak *-topology* also known as *ultraweak topology*). The symbol $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \ldots\}$, whereas $\mathbb{N}'$ means $\mathbb{N} \setminus \{0\}$.

2 Scalar free QFT on degenerate semi-Riemannian manifolds.

2.1. *General geometrical extent.* In this section we want to generalize the construction of the local QFT for a scalar field on $\mathbb{S}^1$ [7] to other metric-degenerate semi-Riemannian manifolds of the form $\mathbb{M} := \mathbb{S}^1 \times \Sigma$ where Σ is a connected oriented d-dimensional *Riemannian* manifold, i.e. equipped with a positive metric Φ_Σ. $\mathbb{S}^1$ is supposed to be oriented and equipped with the null metric. The limit case is given by $\mathbb{M} := \mathbb{S}^1$ alone, omitting the transverse manifold Σ. That was the case treated in [7]. As a simple example consider the time-oriented Kruskal manifold $\mathbb{K}$ [14]. A basis of Killing vector fields of $\mathbb{K}$ is made of three fields: two generating the $\mathbb{S}^2$ symmetry and ∂_t generating time evolution in the two static open wedges of $\mathbb{K}$ where ∂_t is timelike (see the figure where ∂_t is indicated by ξ_t). The region where $(\partial_t, \partial_t) = 0$ is made of the union of two hypersurfaces $\mathbb{P}$ and $\mathbb{F}$ which are, respectively the past and the future event horizon of the manifold. $\mathbb{P} \cap \mathbb{F}$ is the *bifurcation surface* where $\partial_t = 0$, this is a spacelike two-dimensional hypersurface given by $\mathbb{S}^2$ equipped with the Euclidean standard metric of a 2-sphere with radius given by Schwarzschild one r_s. We assume that the bifurcation manifold is oriented. Let us focus attention on $\mathbb{F}$. It is isometric to the metrically degenerate manifold $\mathbb{R} \times \mathbb{S}^2$, $\mathbb{R}$ contains the orbits of the null Killing vector ∂_t restricted to $\mathbb{F}$ – taken as future-oriented in the positive half-line of $\mathbb{R}$ whose origin coincides with the bifurcation point – and $\mathbb{S}^2$ being the bifurcation surface equipped with the Euclidean above-mentioned metric. Notice that, by construction, the metric is invariant under $\mathbb{R}$-displacements. A degenerate manifold $\mathbb{M}$ of the form $\mathbb{S}^1 \times \Sigma$ can, obviously, be obtained from $\mathbb{F}$ by adding a point at infinity to $\mathbb{R}$ obtaining $\mathbb{S}^1$. In this case $\mathbb{M} = \mathbb{S}^1 \times \mathbb{S}^2$. Orientation of $\mathbb{S}^1$ is induced by time-orientation of $\mathbb{R}$. Both orientation of $\mathbb{S}^1$ and that of $\mathbb{S}^2$ induces an orientation on $\mathbb{M} = \mathbb{S}^1 \times \mathbb{S}^2$. Other examples arises from topological black-holes [15, 16] where Σ is replaced by a compact two-dimensional manifold of arbitrary nonnegative genus.



In the following we shall define the QFT for a bosonic field living on a degenerate manifold $\mathbb{M} = \mathbb{S}^1 \times \Sigma$. In [6, 7] we have seen that, at least in the limit case $\mathbb{M} = \mathbb{S}^1$ such a theory can be "induced" on $\mathbb{M}$ by means of an holographic procedure, from standard linear QFT in the bulk manifold ($2D$ Rindler wedge) admitting $\mathbb{M} \setminus \{\infty\}$ as part of its boundary made of a bifurcate Killing horizon. The holographic procedure could be generalized to more complicated manifolds (Kruskal manifold in particular) and this issue will be investigated elsewhere. Here we think the bosonic field as living on the degenerate manifold obtained by an event horizon as said above.

2.2. *Weyl/symplectic approach.* The formulation of real scalar QFT on $\mathbb{M}$ we present here is a straightforward adaptation of the theory of fields obeying linear field equations in globally hyperbolic spacetimes [12, 17, 14][1]. We start with a technical definition. Let $d\Sigma$ be the volume form on Σ induced by the metric Φ_Σ. If $\psi \in C_c^\infty(\mathbb{M};\mathbb{C})$ (the space of compactly-supported smooth real-valued function on $\mathbb{M}$), we define the $dim\, \mathbb{M}$ -form ω_ψ **canonically associated with** ψ by

$$\epsilon_\psi := \frac{\partial \psi}{\partial \alpha} d\alpha \wedge d\Sigma$$

[1]In this paper, barring few differences, we make use of conventions and notation of [14].

in local coordinates of the product atlas on $\mathbb{S}^1 \times \Sigma$, α being a (local) positive-oriented coordinate on $\mathbb{S}^1$. If $\mathbb{M} = \mathbb{S}^1$ ϵ_ψ is exact. The starting point of QFT is the real vector space of **wavefunctions** $\mathcal{S}(\mathbb{M}) := C_c^\infty(\mathbb{M};\mathbb{R})/\sim$ – where $C_c^\infty(\mathbb{M};\mathbb{R})$ is the space of compactly-supported real-valued smooth functions on $\mathbb{M}$ and $\psi \sim \psi'$ iff $\epsilon_{\psi-\psi'} = 0$ on $\mathbb{M}$ – equipped with the symplectic (i.e. bilinear and antisymmetric) form Ω

$$\Omega([\psi],[\psi']) := \int_{\mathbb{M}} \psi' \,\epsilon_\psi - \psi \,\epsilon_{\psi'} \,. \tag{1}$$

One can easily prove that Ω is well-defined, invariant under orientation-preserving diffeomorphisms of $\mathbb{S}^1$ and *nondegenerate* on $\mathcal{S}(\mathbb{M};\mathbb{R})$, i.e. $\Omega([\psi],[\psi']) = 0$ for all $[\psi] \in \mathcal{S}(\mathbb{M})$ implies $[\psi] = 0$. $\mathcal{S}(\mathbb{M})$ has been defined as a quotient just due to degenerateness of Ω in $C_c^\infty(\mathbb{M};\mathbb{R})$. The use of forms in (1) solves the problem of the absence of a nondegenerate measure associated with the degenerate metric on $\mathbb{M}$ (usually employed to define the symplectic form in spacetimes). From now on we indicate a wavefunction $[\psi]$ by ψ whenever the notation is free from misunderstanding. It is worth noticing that "wavefunction" is an improper term here, because we have no motion equation on $\mathbb{M}$. However, the "wavefunctions" introduced here play the same rôle as that played by real smooth solutions of Klein-Gordon equation in QFT on a globally hyperbolic spacetime. As $\mathcal{S}(\mathbb{M})$ is a real vector space equipped with a *nondegenerate* symplectic form Ω, there exist a unique, up to (isometric) *-isomorphisms, complex C^*-algebra $\mathcal{W}(\mathbb{M})$ (theorem 5.2.8 in [12]), called **Weyl algebra**, generated by elements, $W(\psi)$ with $\psi \in \mathcal{S}(\mathbb{M})$, called **symplectically-smeared (abstract) Weyl operators**, such that, for all $\psi, \psi' \in \mathcal{S}(\mathbb{M})$

$$\text{(W1)} \qquad W(-\psi) = W(\psi)^*, \qquad \text{(W2)} \qquad W(\psi)W(\psi') = e^{i\Omega(\psi,\psi')/2}W(\psi+\psi')\,.$$

As a consequence the algebra admits unit $I = W(0)$, each $W(\psi)$ is unitary and, from the nondegenerateness of Ω, $W(\psi) = W(\psi')$ if and only if $\psi = \psi'$. A general procedure to construct Weyl algebras is presented in [18]. The formal interpretation of elements $W(\psi)$ is $W(\psi) \equiv e^{i\Omega(\psi,\hat{\phi})}$ where $\Omega(\psi,\hat{\phi})$ are **symplectically-smeared scalar fields**. That interpretation makes mathematical sense in GNS representations associated with *regular states* [12].

2.3. *Implementing locality: fields smeared with forms.* It is impossible to assign a unique support to a class of equivalence $[\psi]$ and thus implementation of locality is not very straightforward in the symplectic approach. An improvement is obtained by giving an equivalent definition of $\mathcal{W}(\mathbb{M})$ using field operators smeared with forms instead of classes of equivalence. With this formal improvement a notion of locality becomes implementable straightforwardly. In usual spacetimes the *local* smearing is done employing functions instead of forms [14]. However, since there is no natural measure on $\mathbb{M}$ (because $\mathbb{S}^1$ is metrically degenerate) it is convenient to replace the functions with ($dim\mathbb{M}$)-forms because these do not need a measure to be integrated. Let us indicate by $\mathcal{D}(\mathbb{M})$ the space of real n-forms canonically associated with functions in $C_c^\infty(\mathbb{M};\mathbb{R})$. In a globally hyperbolic spacetime (see [14]), the relation between wavefunctions satisfying a suitable motion equation (now classes in $\mathcal{S}(\mathbb{M})$ which do *not* satisfy any motion equation) and smooth compactly supported functions (now elements of $\mathcal{D}(\mathbb{M})$ used to smear fields locally, is implemented by the *causal propagator*, $E : \mathcal{D}(\mathbb{M}) \to \mathcal{S}(\mathbb{M})$, [14] which is a $\mathbb{R}$-linear surjective map which associates a smooth function with a wavefunctions (supported in the causal set generated by the support of the smooth function) and satisfy several properties. Barring causal properties which depend on the existence of a linear field equation, the crucial property describing the interplay with Ω reads

$$\Omega(E\omega, E\omega') = E(\omega, \omega') \quad \text{for all } \omega, \omega' \in \mathcal{D}(\mathbb{M}) , \tag{2}$$

with $E(\omega, \omega') := \int_{\mathbb{M}} E(\omega)\omega'$. Actually, on $\mathbb{M}$, (2) together with surjectivity determine E uniquely.

Proposition 2.1. *There is a unique surjective $\mathbb{R}$-linear map $E : \mathcal{D}(\mathbb{M}) \to \mathcal{S}(\mathbb{M})$ satisfying (2). Moreover the following facts hold.*
(a) *If $\mathbb{S}^1$ is viewed as the segment $-\pi \leq \theta \leq \pi$ with $-\pi \equiv \pi$, $\omega \in \mathcal{D}(\mathbb{M})$ is realized as a 2π-periodic form in the positive-oriented coordinate $\theta \in \mathbb{R}$ and $s \in \Sigma$, E admits the representation*

$$(E(\omega))(\theta, s) = \left[\frac{1}{4}\int_{\theta' \in [-\pi,\pi]\, s' \in \Sigma} \left(sign(\theta') - \frac{\theta'}{\pi}\right) \delta(s, s')\omega(\theta - \theta', s')\right] . \tag{3}$$

(b) *E is* bijective *and in particular, for $\psi \in \mathcal{S}(\mathbb{M})$, $\omega \in \mathcal{D}(\mathbb{M})$, one has*

$$E(\omega) = \psi \quad \text{if and only if} \quad \omega = 2\epsilon_\psi \ . \tag{4}$$

Thus $(\omega, \omega') \mapsto E(\omega, \omega') := \int_{\mathbb{M}} E(\omega)\omega'$ is a nondegenerate symplectic form on $\mathcal{D}(\mathbb{M})$.

Proof. The fact that E defined in (3) satisfies (2) can be proved straightforwardly by direct computation. Direct computation shows also the validity of (4) proving injectivity and surjectivity. Any linear surjective map E satisfying (2) fulfills also $\Omega(\psi, E\omega') = \int_{\mathbb{M}} \psi\omega'$ for every $\psi \in \mathcal{S}(\mathbb{M})$ and $\omega' \in \mathcal{D}(\mathbb{M})$. If E, E' are surjective linear maps satisfying (2), one has $\Omega(\psi, E\omega - E'\omega) = \int_{\mathbb{M}} \psi(\omega - \omega) = 0$. for every $\psi \in \mathcal{S}(\mathbb{M})$. Ω is non degenerate and thus $E\omega - E'\omega = 0$ for every $\omega \in \mathcal{D}(\mathbb{M})$. Hence $E = E'$. The final statement is now obvious. $\Box$

Motivated by the theory in spacetimes we shall call **causal propagator** the map E in (3), also if this name is not very appropriate due to the lack of field equations, whose existence is responsible, in spacetimes, for the failure of the injectivity of the causal propagator. The presence of the Dirac delta in (3) concerning the part of causal propagator on Σ has an evident interpretation if $\mathbb{M} = (\mathbb{S}^1 \setminus \{\infty\}) \times \Sigma$ is thought as the future Kruskal event horizon and E is interpreted as the limit case of a properly defined causal propagator: As the boundary of a causal sets $J(S)$, for $S \subset \Sigma$, is made of portions of the factor $\mathbb{S}^1$, causal separation of sets $S, S' \subset \Sigma$ assigned at different "times" of $\mathbb{S}^1 \setminus \{\infty\}$, is equivalent to $S \cap S' = \emptyset$.
As in spacetimes, if $\omega \in \mathcal{D}(\mathbb{M})$, the **form-smeared (abstract) Weyl field** is defined as

$$V(\omega) := W(E\omega). \tag{5}$$

With this definition one immediately gets Weyl relations once again: For all $\omega, \eta \in \mathcal{D}(\mathbb{M})$,

$$\text{(V1)} \qquad V(-\omega) = V(\omega)^*, \qquad \text{(V2)} \qquad V(\omega)V(\eta) = e^{iE(\omega,\eta)/2}V(\omega+\eta)\,.$$

Since E is injective, differently from the extent in a spacetime, $V(\omega) = V(\omega')$ if and only if $\omega = \omega'$. A notion of *locality* on $\mathbb{M}$ (in a straightforward extension of original idea due to Sewell [19]) can be introduced at this point by the following proposition.

Proposition 2.2. $[V(\omega), V(\omega')] = 0$ *for* $\omega, \omega' \in \mathcal{D}(\mathbb{M})$ *if one of the conditions is fulfilled:*
(a) *there are two open disjoint segments* $I, I' \subset \mathbb{S}^1$ *with* $supp\,\omega \subset I \times \Sigma$ *and* $supp\,\omega' \subset I' \times \Sigma$,
(b) *there are two open disjoint sets* $S, S' \subset \Sigma$ *with* $supp\,\omega \subset \mathbb{S}^1 \times S$ *and* $supp\,\omega' \subset \mathbb{S}^1 \times S'$.

See the appendix for the proof. The *-algebra $\mathcal{W}(\mathbb{M})$ is *local* in the sense stated in the thesis of Proposition 2.2. Notice that $supp\,\omega \cap supp\,\omega' = \emptyset$ does not imply commutativity of $W(\omega)$ and $W(\omega')$ in general.

2.4. *Fock representations.* A Fock representation of $\mathcal{W}(\mathbb{M})$ can be introduced as follows generalizing part of the construction presented in 2.4 of [20] and in [7]. From a physical point of view, the procedure resembles quantization with respect to Killing time in a static spacetime. Fix a global lightlike Killing field $\partial_\theta := \frac{\partial}{\partial\theta}$ on $\mathbb{M} := \mathbb{S}^1 \times \Sigma$, such that $\mathbb{S}^1$ is realized as the segment $-\pi < \theta \leq \pi$ with $\pi \equiv +\pi$ and the coordinate θ is positive oriented in $\mathbb{S}^1$. Any representative ψ of $[\psi] \in \mathbb{S}(\mathbb{M})$ can be expanded in Fourier series in the parameter θ, where $\mathbb{N}' := \mathbb{N} \setminus \{0\}$,

$$\psi(\theta, s) \sim \sum_{n\in\mathbb{N}'} \frac{e^{-in\theta}\widetilde{\psi(s,n)}_+}{\sqrt{4\pi n}} + \sum_{n\in\mathbb{N}'} \frac{e^{in\theta}\overline{\widetilde{\psi(s,n)}_+}}{\sqrt{4\pi n}} = \psi_+(\theta, s) + \overline{\psi_+(\theta, s)}\,. \tag{6}$$

ψ_+ is the ∂_θ-**positive frequency part** of ψ. The term with $n = 0$ was discarded due to the equivalence relation used defining $\mathbb{S}(\mathbb{M})$, the remaining terms depend on $[\psi]$ only. $\Sigma \ni s \mapsto \widetilde{\psi(s,n)}_+$ is smooth, supported in a compact set of Σ independent from n and, using integration by parts, for any $\alpha > 0$, there is $C_\alpha \geq 0$ with $||\widetilde{\psi(\cdot,n)}_+||_\infty \leq C_\alpha n^{-\alpha}$ for $n \in \mathbb{N}'$ so that the series in (6) converges uniformly and θ-derivative operators can be interchanged with the symbol of summation. The found estimation and Fubini's theorem entail that the sesquilinear form

$$\langle \psi'_+, \psi_+ \rangle := -i\Omega(\overline{\psi'_+}, \psi_+) \tag{7}$$

on the space of complex linear combinations of ∂_θ-positive frequency parts satisfies

$$\langle \psi'_+, \psi_+ \rangle = \sum_{n=1}^{\infty} \int_\Sigma \overline{\widetilde{\psi'(s,n)}_+}\, \widetilde{\psi(s,n)}_+ d\Sigma(s) = \int_\Sigma \sum_{n=1}^{\infty} \overline{\widetilde{\psi'(s,n)}_+}\, \widetilde{\psi(s,n)}_+ d\Sigma(s). \tag{8}$$

Thus it is positive and defines a Hermitian scalar product. The **one-particle space** $\mathcal{H}$ is now defined as the completion w.r.t $\langle\cdot,\cdot\rangle$ of the space of positive ∂_θ-frequency parts ψ_+ of wavefunctions. Due to (8), $\mathcal{H}$ is isomorphic to $\ell^2(\mathbb{N}) \otimes L^2(\Sigma, d\Sigma)$ [2]. $\mathfrak{F}_+(\mathcal{H})$ is the symmetrized Fock space with vacuum state Ψ and one-particle space $\mathcal{H}$. The **symplectically-smeared field operator** and the **form-smeared field operator** are respectively the $\mathbb{R}$-linear maps, for $\psi \in \mathcal{S}(\mathbb{M})$ and $\omega \in \mathcal{D}(\mathbb{M})$,

$$\psi \mapsto \Omega(\psi, \hat{\phi}) := i\alpha(\overline{\psi_+}) - i\alpha^\dagger(\psi_+)\,, \quad \omega \mapsto \hat{\phi}(\omega) := \Omega(E\omega, \hat{\phi}) \tag{9}$$

where the operators $\alpha^\dagger(\psi_+)$ and $\alpha(\overline{\psi_+})$ ($\mathbb{C}$-linear in $\overline{\psi_+}$) respectively create and annihilate the state ψ_+. The common invariant domain of all the involved operators is the dense linear manifold $F(\mathcal{H})$ spanned by the vectors with finite number of particle. $\Omega(\psi, \hat{\phi})$ and $\hat{\phi}(\omega)$ are essentially self-adjoint on $F(\mathcal{H})$ (they are symmetric and $F(\mathcal{H})$ is dense and made of analytic vectors) and satisfy bosonic commutation relations, i.e. $[\Omega(\psi, \hat{\phi}), \Omega(\psi', \hat{\phi})] = -i\Omega(\psi, \psi')I$ and $[\hat{\phi}(\omega), \hat{\phi}(\omega')] = -iE(\omega, \omega')I$. Finally the unitary operators

$$\hat{W}(\psi) := e^{i\overline{\Omega(\psi,\hat{\phi})}} \quad \text{and, equivalently,} \qquad \hat{V}(\omega) := \hat{W}(E\omega) = e^{i\overline{\hat{\phi}(\omega)}} \tag{10}$$

enjoy properties (W1), (W2) and, respectively (V1), (V2), so that they define a unitary representation $\hat{\mathcal{W}}(\mathbb{M})$ of $\mathcal{W}(\mathbb{M})$ which is also irreducible. The proof of these properties follows from propositions 5.2.3 and 5.2.4 in [12][3]. If $\Pi : \mathcal{W}(\mathbb{M}) \to \hat{\mathcal{W}}(\mathbb{M})$ denotes the unique (Ω being nondegenerate) C^*-algebra isomorphism between those two Weyl representations, $(\mathfrak{F}_+(\mathcal{H}), \Pi, \Psi)$ coincides, up to unitary transformations, with the GNS triple associated with the algebraic pure state λ on $\mathcal{W}(\mathbb{M})$ uniquely defined by the requirement (see the appendix)

$$\lambda(W(\psi)) := e^{-\langle \psi_+, \psi_+ \rangle /2}\,. \tag{11}$$

We stress that the construction of the Fock representation $(\mathfrak{F}_+(\mathcal{H}), \hat{\mathcal{W}}(\mathbb{M}))$ relies upon the choice of the preferred global Killing field ∂_θ. There are as many choices as many the orientation-preserving diffeomorphisms of $\mathbb{S}^1$ are. Each gives a (generally different) Weyl representation.

3 Conformal nets on degenerate semi-Riemannian manifolds.

3.1. *$Diff^+(\mathbb{S}^1)$, $PSL(2,\mathbb{R})$ and associated *-automorphisms on* $\mathbb{M}$. The group of diffeomorphisms of $\mathbb{S}^1$, the subgroup $PSL(2,\mathbb{R})$ called **Möbius group** and their Lie algebras play a relevant rôle in QFT on $\mathbb{M}$. We recall here some basic notions in this areas. Let $Vect(\mathbb{S}^1)$ be the infinite-dimensional Lie algebra of the infinite-dimensional Lie group (see Milnor [21]) of orientation-preserving smooth diffeomorphisms of the circle $Diff^+(\mathbb{S}^1)$. $Vect(\mathbb{S}^1)$ is the real linear space of smooth vector fields on $\mathbb{S}^1$ whose associated one-parameter diffeomorphisms preserve the orientation of $\mathbb{S}^1$. If $Vect^{\mathbb{C}}(\mathbb{S}^1)$ denotes the complex Lie algebra $Vect(\mathbb{S}^1) \oplus iVect(\mathbb{S}^1)$ with

[2] The construction of $\mathcal{H}$ is equivalent to that performed in the approach of [14] (see also [17]) using the real scalar product on $\mathcal{S}(\mathbb{M})$, $\mu(\psi, \psi') := -Im\, \Omega(\overline{\psi_+}, \psi'_+)$ and the map $K : \mathcal{S}(\mathbb{M}) \ni \psi \mapsto \psi_+ \in \mathcal{H}$.

[3] There the symplectic form is $\sigma = -2\Omega$ and the field operator $\Phi(\psi_+)$ of prop. 5.2.3 of [12] is $\Phi(\psi_+) = 2^{-1/2}\Omega(J\psi, \hat{\phi})$ where $J\psi = -i\psi_+ + i\overline{\psi_+}$ if $\psi = \psi_+ + \overline{\psi_+}$. Notice that $J(\mathcal{S}(\mathbb{M})) \subset \mathcal{S}(\mathbb{M})$: That is false in general with other definitions of $\mathcal{S}(\mathbb{M})$!

usual Lie brackets $\{\cdot,\cdot\}$ and involution $\omega : X \mapsto -\overline{X}$ for $X \in Vect^{\mathbb{C}}(\mathbb{S}^1)$, so that $\omega(\{X,Y\}) = \{\omega(Y), \omega(X)\}$, $Vect(\mathbb{S}^1)$ is the (real) sub-Lie-algebra of $Vect^{\mathbb{C}}(\mathbb{S}^1)$ of anti-Hermitean elements with respect to ω. Further mathematical tools can be introduced fixing a preferred field ∂_θ on $\mathbb{S}^1$ as specified at the beginning of 2.4. d denotes the Lie subalgebra of $Vect^{\mathbb{C}}(\mathbb{S}^1)$ whose elements have a finite number of Fourier component with respect to ∂_θ. A basis for d is made of fields

$$\mathcal{L}_n := ie^{in\theta}\partial_\theta \,, \qquad \text{with } n \in \mathbb{Z}. \tag{12}$$

They enjoy the so-called *Hermiticity condition*, $\omega(\mathcal{L}_n) = \mathcal{L}_{-n}$ and the well-known *Virasoro commutation rules* with vanishing central charge, $[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m}$. To go on, we recall the group isomorphism $SL(2,\mathbb{R}) \ni h \mapsto g \in SU(1,1)$ where:

$$g := \begin{pmatrix} \zeta & \overline{\eta} \\ \eta & \overline{\zeta} \end{pmatrix}, h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \zeta := \frac{a+d+i(b-c)}{2}\,, \quad \eta := \frac{d-a-i(b+c)}{2}\,.$$

$Diff^+(\mathbb{S}^1)$ includes $PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\{\pm I\}$ as a finite-dimensional subgroup: Thinking $\mathbb{S}^1$ as the unit complex circle, an element $g \in SU(1,1)/\{\pm I\} \equiv PSL(2,\mathbb{R})$ turns out to be injectively associated with a diffeomorphism $g \in Diff^+(\mathbb{S})$, called **Möbius diffeomorphism**, such that

$$g : e^{i\theta} \mapsto \frac{\zeta e^{i\theta} + \overline{\eta}}{\eta e^{i\theta} + \overline{\zeta}}, \quad \text{with} \quad \theta \in [-\pi, \pi]\,. \tag{13}$$

The corresponding inclusion of Lie algebras is illustrated by the fact that the three ω-anti-Hermitean linearly-independent elements of d

$$\mathcal{K} := i\mathcal{L}_0 = -\partial_\theta\,, \quad \mathcal{S} := i\frac{\mathcal{L}_1 + \mathcal{L}_{-1}}{2} = -\cos\theta\partial_\theta\,, \quad \mathcal{D} := i\frac{\mathcal{L}_1 - \mathcal{L}_{-1}}{2} = -\sin\theta\partial_\theta \tag{14}$$

enjoy the commutation rules of the elements k, s, d of the basis of the Lie algebra $sl(2,\mathbb{R})$ with

$$k = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; s = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; d = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{15}$$

$Diff^+(\mathbb{S}^1)$ acts naturally as a group of *isometries* on the semi-Riemannian manifold $\mathbb{M} = \mathbb{S}^1 \times \Sigma$. If $g \in Diff^+(\mathbb{S}^1)$, we shall use the same symbol to indicate the associated diffeomorphism of $\mathbb{M}$. Since $g \in Diff^+(\mathbb{S}^1)$ depends on the variable in $\mathbb{S}^1$ only, if $[\psi] \in \mathcal{S}(\mathbb{M})$, the element $[\psi]^{(g)} := [\psi \circ g] \in \mathcal{S}(\mathbb{M})$ is well defined. It is also defined the usual pull-back action on forms $\omega \in \mathcal{D}(\mathbb{M})$, $\omega^{(g)} := g^*\omega$. Notice that g^* leaves $\mathcal{D}(\mathbb{M})$ fixed: Using (4), it results that if $\psi = E\omega$ with $\omega \in \mathcal{D}(\mathbb{M})$ then $\omega^{(g)} = 2\epsilon_{\psi^{(g)}} \in \mathcal{D}(\mathbb{M})$. With these definitions one gets straightforwardly, for all $\psi, \phi \in \mathcal{S}(\mathbb{M})$, $g \in Diff^+(\mathbb{S}^1)$ and $\omega, \eta \in \mathcal{D}(\mathbb{M})$,

$$\Omega(\psi,\phi) = \Omega(\psi^{(g)}, \phi^{(g)}) \quad \text{and} \quad E(\omega,\eta) = E(\omega^{(g)}, \eta^{(g)})\,. \tag{16}$$

Therefore, as a consequence of general results ((4) in theorem 5.2.8 of [12]), $Diff^+(\mathbb{M})$ admits a representation $A : g \mapsto A_g$ made of *-automorphisms of the algebra $\mathcal{W}(\mathbb{M})$ induced by

$$A_g(V(\omega)) := V(\omega^{(g^{-1})})\,. \tag{17}$$

Throughout we shall employ the subsequent representation α of Möbius group defined by $PSL(2,\mathbb{R}) \ni g \mapsto \alpha_g := A_g$ in terms of *-automorphisms of $\mathcal{W}(\mathbb{M})$.

3.2. *Virasoro representations and Conformal nets.* Let us investigate on the existence of representations of Virasoro algebra and the real sub algebra $sl(2,\mathbb{R})$ in the Fock space $\mathfrak{F}_+(\mathcal{H})$ introduced above focusing, in particular, on the relationship with the algebra $\hat{\mathcal{W}}(\mathbb{M})$. From now on we assume to single out a preferred vector field ∂_θ as said at the beginning of 2.4 and use that vector to build up the Fock space and the Weyl representation. It is possible to introduce in $\mathfrak{F}_+(\mathcal{H})$ a new class of operators which generalizes chiral currents straightforwardly. If $\mathbb{N}' := \{1,2,3,\ldots\}$ and $\{u_j\}_{j\in\mathbb{N}'}$ is a Hilbert basis of $L^2(\Sigma, d\Sigma)$ the vectors

$$Z_{jn}(\theta,s) := \frac{u_j(s)e^{-in\theta}}{\sqrt{4\pi n}}$$

define a Hilbert basis of the one-particle space $\mathcal{H}$. We can always reduce to the case of *real* vectors u_j and we assume that[4] henceforth. The functions field operator $\mathcal{D}(\mathbb{M}) \ni \omega \mapsto \alpha(E\omega)$ and $\mathcal{D}(\mathbb{M}) \ni \omega \mapsto \alpha^\dagger(E\omega)$, where the operators work on the domain $F(\mathcal{H})$, can be proved to be distributions using the strong-operator topology (to show it essentially use (1) in prop. 5.2.3 in [12]) and the usual test-function topology on $\mathcal{D}(\mathbb{M})$ induced by families of seminorms referred to derivatives (of any order) in coordinates of components of forms ω (see 2.8 in [22]). $\mathcal{D}(\mathbb{M}) \ni \omega \mapsto \hat{\phi}(\omega)$ admits the distributional kernel

$$\hat{\phi}(\theta,s) = \frac{1}{i\sqrt{4\pi}} \sum_{(n,j)\in\mathbb{Z}\times\mathbb{N}'} \frac{u_j(s)e^{-in\theta}}{n} J_n^{(j)}\,, \tag{18}$$

where the **(generalized) chiral currents** $J_n^{(j)} : F(\mathcal{H}) \to F(\mathcal{H})$ are defined as follows

$$J_0^{(j)} = 0\,, \quad J_n^{(j)} = i\sqrt{n}\alpha(\overline{Z_{jn}}) \quad \text{if } n>0 \text{ and } \quad J_n^{(j)} = -i\sqrt{-n}\alpha^\dagger(Z_{j,-n}) \quad \text{if } n<0\,.$$

They satisfy on $F(\mathcal{H})$ both the Hermiticity condition $J_n^{(j)\dagger}\restriction_{F(\mathcal{H})} = J_{-n}^{(j)}$ and the oscillator commutation relations $[J_n^{(j)}, J_m^{(i)}] = n\delta^{ij}\delta_{n,-m}I$. Introducing the usual normal order prescription $:\cdot\cdot:$ "operators $J_p^{(j)}$ with negative index p must precede those with positive index p", one can try to define the linearly-independent operators, with $c \in \mathbb{N}' \cup \{\infty\}$

$$L_k^{(c)} := \frac{1}{2} \sum_{n\in\mathbb{Z}, n\leq c} :J_n^{(j)} J_{k-n}^{(j)}:\,, \qquad L_k := L_k^{(\infty)} \tag{19}$$

[4] $L^2(\Sigma, d\Sigma)$ is separable since the Borel measure $d\Sigma$ is σ-finite and the Borel σ-algebra of Σ is countably generated (the topology of Σ being second countable by definition of manifold). If $\{u_j\}$ is a Hilbert basis $\{\overline{u_j}\}$ is such. Orthonormalization procedure of a maximal set of linearly independent generators in the set of all $u_j + \overline{u_j}$, $i(u_j - \overline{u_j})$ yields a real Hilbert basis.

on some domain in $\mathfrak{F}_+(\mathcal{H})$. We shall denote the complex infinite-dimensional algebra spanned by $L_k^{(c)}$ by $\hat{d}_c$. One can formally show that $L_k^{(c)}$ and L_k have two equivalent geometric expression

$$L_k^{(c)} = \frac{1}{2i} :\Omega(\hat{\phi}^{(c)}, \mathcal{L}_k(\hat{\phi}^{(c)})): \quad \text{and} \quad L_k = \frac{1}{2i} :\Omega(\hat{\phi}, \mathcal{L}_k(\hat{\phi})): \tag{20}$$

$$L_k^{(c)} = \int_{\mathbb{M}} :\partial_\theta \hat{\phi}^{(c)}(\theta, s) \partial_\theta \hat{\phi}^{(c)}(\theta, s): e^{ik\theta} d\theta d\Sigma \quad \text{and} \quad L_k = \int_{\mathbb{M}} :\partial_\theta \hat{\phi}(\theta, s) \partial_\theta \hat{\phi}(\theta, s): e^{ik\theta} d\theta d\Sigma \tag{21}$$

where $\hat{\phi}^{(c)}$ is the right-hand side of (18) with the sum over j restricted to the set $\{1, 2, \cdots, c\}$ and $\mathcal{L}_k(\hat{\phi}^{(c)})$ is the "scalar field" obtained by the action of the differential operator $\mathcal{L}_k$ (naturally extended from $\mathbb{S}^1$ to the product $\mathbb{M} = \mathbb{S}^1 \times \Sigma$) on the "scalar field" $\phi^{(c)}$. If c is finite the following proposition can be proved by direct inspection.

Proposition 3.1. *Fix a vector field ∂_θ on $\mathbb{M} = \mathbb{S}^1 \times \Sigma$ as said at the beginning of 2.4, take $c \in \mathbb{N}'$ and consider the operators $L_n^{(c)}$ in (19). They satisfy the following.*
(a) *They are well defined on $F(\mathcal{H})$ which is a dense invariant space of analytic vectors for all them. Moreover the following holds concerning operators $L_n^{(c)}$.*
(b) $(\hat{d}_c, [\cdot,\cdot], \cdot^\dagger\restriction_{F(\mathcal{H})})$ *is a central representation, with central charge c, of the algebra $(d, \{\cdot,\cdot\}, \omega)$ (that is a unitarizable Virasoro representation) since the following relations hold:*

$$L^{(c)}{}_{-n} = L_n^{(c)\dagger}\restriction_{F(\mathcal{H})}, \tag{22}$$

$$[L_n^{(c)}, L_m^{(c)}] = (n-m) L^{(c)}{}_{n+m} + \frac{(n^3-n)c}{12} \delta_{n+m,0} I . \tag{23}$$

(c) *The representation is* positive energy, *i.e. $L_0^{(j)}$ is non-negative.*
(d) *They do not depend on the real base $\{u_j\}_{j \leq c}$ but only on the finite dimensional subspace spanned by those vectors.*

Notice that the found Virasoro representations are strongly reducible [23]. Once they are decomposed into unitarizable irreducible highest-weight representations [23], they can be exponentiated ([24, 25, 11]) obtaining unitary strongly continuous representations of $Diff^+(\mathbb{S}^1)$.
In general there is no general physical way to select a Hilbert basis $\{u_j\}$ or equivalently a sequence $\ldots \mathcal{H}_k \subset \mathcal{H}_{k+1} \ldots$ of finite dimensional subspace of $L^2(\Sigma, d\Sigma)$. In the presence of particular symmetries for Σ a class of finite dimensional subspaces can be picked out referring to the invariant subspaces with respect to a unitary representation on $L^2(\Sigma, d\Sigma)$ of the symmetry group. For instance, think to $\Sigma = \mathbb{S}^2$, in that case one may decompose $\psi \in L^2(\mathbb{S}^2)$ using (real and imaginary parts of) spherical harmonics Y_m^l. Hence a suitable class of finite dimensional subspaces are those with fixed angular momentum $l = 0, 1, 2, \ldots$. The sphere $\mathbb{S}^2$ is reconstructed as a sequence of *fuzzy spheres* ([26]) with greater and greater angular momentum l. The associated Virasoro representations have central charges $c_l = 2l + 1$.

In the absence of symmetries only the case $c = \infty$ seems to be physically interesting. Let us turn attention on this case. Serious problems arises when trying to give a rigorous meaning to all the operators L_n. First of all (23) becomes meaningless due to $c = \infty$ in the right-hand side. Furthermore, by direct inspection one finds that, if $n < -1$, the domain of L_n cannot include any vector of $F(\mathcal{H})$ due to an evident divergence (this drawback would arise also for $|n| = 1$ if $J_0^{(j)} = 0$ were false). However, by direct inspection, one finds that L_n with $n \geq -1$ are well defined on $F(\mathcal{H})$ which is, in fact a common invariant dense domain made of analytic vectors, moreover $L_n\Psi = 0$. The central charge does not appear considering commutators of those operators. The complex space (finitely) spanned by those vectors is closed with respect to the commutator but, unfortunately, it is *not* with respect to the Hermitean conjugation so that they cannot represent a Lie algebra of observables. However, restricting to the case $|n| \leq 1$ everything goes right and one gets a Lie algebra closed with respect to the Hermitean conjugation. Anti-Hermitean linearly-independent operators generating that Lie algebra are

$$iK := iL_0\,, \quad iS := i\frac{L_1 + L_{-1}}{2}\,, \quad iD := \frac{L_1 - L_{-1}}{2}\,. \tag{24}$$

They enjoy the commutation rules of the elements k, s, d of the basis of the Lie algebra $sl(2, \mathbb{R})$ (15). As a consequence a representation $R : sl(2, \mathbb{R}) \to \mathcal{L}(F(\mathcal{H}))$ can be realized by assuming $iK = R(k), iS = R(s), iD = R(d)$ and $R : ak + bs + cd \mapsto aiK + biS + ciD$ for all $a, b, c \in \mathbb{R}$. One expects that this representation is associated, via exponentiation, with a strongly continuous (projective) unitary representation of the universal covering of $SL(2, \mathbb{R})$, $\widetilde{SL(2, \mathbb{R})}$. Let us prove that such a representation does exists and enjoys remarkable properties.

Theorem 3.1. *Fix a vector field ∂_θ on $\mathbb{M} = \mathbb{S}^1 \times \Sigma$ as at the beginning of 2.4, consider the GNS (Fock) realization of $\mathcal{W}(\mathbb{M})$ associated with the state λ in (11) and the representation R. The Hermitean operators $iR(x)$, with $x \in sl(2, \mathbb{R})$, are essentially selfadjoint on $F(\mathcal{H})$ and there is a unique strongly-continuous representation $PSL(2, \mathbb{R}) \ni g \mapsto U(g) : \mathfrak{F}_+(\mathcal{H}) \to \mathfrak{F}_+(\mathcal{H})$ such that*

$$U(\exp(tx)) = e^{t\overline{R(x)}}\,, \quad \text{for all } x \in sl(2, \mathbb{R}) \text{ and } t \in \mathbb{R}. \tag{25}$$

The following further facts hold.
(a) *U is a* positive-energy *representation of $PSL(2, \mathbb{R})$ (that is the self-adjoint generator $\overline{K}$ of the the representation, called* conformal Hamiltonian*, associated with the one-parameter subgroup $\mathbb{S}^1$, i.e. $\{exp(th)\}_{t\in\mathbb{R}} \subset SL(2, \mathbb{R})$, has nonnegative spectrum) and $\sigma(\overline{K}) = \{0, 1, 2, \ldots\}$.*
(b) *U and its generators do not depend on the choice of the basis $\{u_j\}_{j\in\mathbb{Z}} \subset L^2(\Sigma, d\Sigma)$.*
In particular, U is the tensorialization of $U\restriction_{\mathcal{H}}$ and it turns out that $U\restriction_{\mathcal{H}} = V \otimes I$, where the decomposition $\mathcal{H} = \ell^2(\mathbb{C}) \otimes L^2(\Sigma, d\Sigma)$ holds and V_{∂_θ} is the restriction to the one-particle space of the representation U in the case $\mathbb{M} = \mathbb{S}^1$ (without transverse manifold).
(c) *Each subspace of $\mathfrak{F}_+(\mathcal{H})$ with finite number of particles is invariant under U.*
(d) *The GNS representative of λ, the vacuum state Ψ, is invariant under U and is the only unit vector of $\mathfrak{F}_+(\mathcal{H})$ invariant under $\{e^{it\overline{D}}\}_{t\in\mathbb{R}}$ (and thus U itself) up to phases.*

The proof of the theorem is given in the appendix. The following further theorem states that $\hat{\mathcal{W}}(\mathbb{M})$ transform covariantly under this representation with respect to the action of the diffeomorphisms of $PSL(2,\mathbb{R}) \subset Diff^+(\mathbb{S}^1)$ seen in 3.1.

Theorem 3.2. *With hypotheses and notation of theorem 3.1, the following holds.*
(a) *U is Möbius covariant. In other words it implements unitarily the representation α of Möbius group $PSL(2,\mathbb{R})$ (see 3.1): For all $g \in PSL(2,\mathbb{R})$,*

$$U(g)\, w\, U(g)^\dagger = \alpha_g(w)\,, \quad \textit{for all } w \in \hat{\mathcal{W}}(\mathbb{M}) \tag{26}$$

(b) *The one-parameter group of *-automorphisms associated with the one-parameter group of diffeomorphisms respectively generated by vector fields $\mathcal{K}, \mathcal{S}, \mathcal{D}$ correspond, trough (26), to the one-parameter unitary subgroups of U respectively generated by iK, iS, iD[5].*
(c) *λ is invariant under the *-automorphism representation α of the Möbius group $PSL(2,\mathbb{R})$.*

The proof of the theorem is given in the appendix. Theorems 3.1 and 3.2 has a remarkable consequence concerning the existence of a conformal net on $\mathbb{S}^1$ associated with the the algebra $\hat{W}(\mathbb{M})$. This fact has a wide spectrum of relevant consequences in physics and in mathematics, see for instance [8, 9, 10, 11] and references therein.

Definition 3.1. *Let $\mathcal{I}$ be the set of non empty, nondense, open intervals of $\mathbb{S}^1$. A* **conformal net on** $\mathbb{S}^1$ *is a triple $(\mathcal{A}, \Psi, U)$ where $\mathcal{A}$ is a family $\{\mathcal{A}(I) \mid I \in \mathcal{I}\}$ of von Neumann algebras on an infinite-dimensional separable complex Hilbert space $\mathcal{H}_\mathcal{A}$, and the following properties hold.*
(C1) **Isotony**. *$\mathcal{A}(I) \subset \mathcal{A}(J)$, if $I \subset J$ with $I, J \in \mathcal{I}$.*
(C2) **Locality**. *$\mathcal{A}(I) \subset \mathcal{A}(J)'$, if $I \cap J = \emptyset$ with $I, J \in \mathcal{I}$.*
(C3) **Möbius covariance**. *$U(g)\mathcal{A}(I)U(g)^\dagger = \mathcal{A}(gI)$, $I \in \mathcal{I}$, $g \in PSL(2,\mathbb{R})$, where U is a strongly continuous unitary representation of $PSL(2,\mathbb{R})$ in $\mathcal{H}_\mathcal{A}$ and g denotes the Möbius transformation (13) with respect to a fixed representation of $\mathbb{S}^1$ as the segment $[-\pi, +\pi]$ with $-\pi \equiv \pi$.*
(C4) **Positivity of the energy**. *The representation U is a positive-energy representation.*
(C5) **U-invariance and uniqueness of the vacuum**. *$\Psi \in \mathcal{H}_\mathcal{A}$ is the unique (up to phases) unit vector invariant under U.*
(C6) **Cyclicity of the vacuum**. *Ψ is cyclic for the algebra $\mathcal{A}(\mathbb{S}^1) := \bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.*

Theorem 3.3. *Fix a vector field ∂_θ on $\mathbb{M} = \mathbb{S}^1 \times \Sigma$ as said at the beginning of 2.4 and define the associated Weyl algebra $\hat{\mathcal{W}}(\mathbb{M})$ in the Fock space $\mathfrak{F}_+(\mathcal{H})$ with vacuum state Ψ and the representation of $PSL(2,\mathbb{R})$, U of Theorems 3.1 and 3.2. With those hypotheses the family*

$$\mathcal{A} = \{\mathcal{A}(I) \mid I \in \mathcal{I}\}\,, \quad \textit{with} \quad \mathcal{A}(I) = \{\hat{V}(\omega) \mid supp\, \omega \subset I \times \Sigma\}'' \tag{27}$$

together with Ψ and U form a conformal net on $\mathbb{S}^1$ such that $\hat{\mathcal{W}}(\mathbb{M}) \subset \mathcal{A}(\mathbb{S}^1)$.

[5] Sign conventions should be clear, anyway to fix them notice that formally $[iK, \hat{\phi}(\theta, s)] = -\partial_\theta \hat{\phi}(\theta, s)$.

Proof. (C1), (C2) and (C3) are straightforward consequences of the definition (27) using the fact that (von Neumann's density theorem) $\mathcal{A}(K)$ is the closure with respect the strong operator topology of the *-algebra generated by the elements in $\{\hat{W}(\omega) \mid supp\, \omega \subset K \times \Sigma\}$, employing proposition 2.2 concerning (C2) and theorem 3.2 concerning (C3). (C4) and (C5) are part of theorem (3.1). (C6) is a consequence of the fact that Ψ is cyclic with respect to $\hat{\mathcal{W}}(\mathbb{M})$ (see the appendix) and $\hat{\mathcal{W}}(\mathbb{M}) \subset \mathcal{A}(\mathbb{S}^1)$. This inclusion is a consequence of the fact that, if $I, J \in \mathcal{I}$ and $\mathbb{S}^1 = I \cup J$, then, due to (W2), each element of $\hat{\mathcal{W}}(\mathbb{M})$ has the form $c\hat{W}(\omega)\hat{W}(\omega')$ where $supp\, \omega \subset I \times \Sigma$, $supp\, \omega' \subset J \times \Sigma$ and $|c| = 1$, so that $\hat{\mathcal{W}}(\mathbb{M}) \subset \mathcal{A}(I) \vee \mathcal{A}(J) \subset \mathcal{A}(\mathbb{S}^1)$. □

Remarks.
(1) Our construction of a conformal net for, in particular, a bifurcate Killing horizon, is explicit in particular in giving the effective form of of the unitary representation of $PSL(2,\mathbb{R})$ and the relationship with the whole Virasoro algebra. It does not requires any assumption on the existence of any algebra of observables in the spacetime where $(\mathbb{S}^1 \setminus \{\infty\}) \times \Sigma$ can be viewed to be embedded or any KMS state on that algebra. A different approach was presented in [1] where it is shown that a conformal net can be obtained by restriction to the horizon of a local algebra in the spacetime realized using a GNS representation with cyclic vector which satisfies the KMS condition with respect to the Killing time flow. The unitary representation of $PSL(2,\mathbb{R})$ was obtained there making use of relevant results by Weisbrock et al [3, 2, 20] on the interplay of modular theory and conformal theory. It seems plausible that our construction can be recovered also using the approach of [1] defining a bulk algebra of observables and a KMS state appropriately. This topic will be investigated elsewhere.
(2) Conformal nets enjoy relevant properties [8, 9, 10, 11]:
Reeh-Schlieder property. Ψ is cyclic and separating for every $\mathcal{A}(I)$.
Bisognano-Wichmann property. The modular operator Δ_I associated with every $\mathcal{A}(I)$ satisfies $\Delta_I^{it} = U(\exp(2\pi\mathcal{D}_I))$ for every $t \in \mathbb{R}$, $\{\exp(tD_I)\}_{t\in\mathbb{R}} \subset PSL(2,\mathbb{R})$ being the one-parameter subgroup which leaves I invariant (with $\mathcal{D}_I$ defined as in remark (2) after theorem 4.1 below) so that Ψ is a KMS state for $\mathcal{A}(I)$ at inverse temperature 2π w.r.to $-\mathcal{D}_I$ for $\mathcal{A}((0,\pi))$.
Haag duality. $\mathcal{A}(I)' = \mathcal{A}(Int(\mathbb{S}^1 \setminus I))$ for every $\mathcal{A}(I)$.
Irreducibility. $\mathcal{A}(\mathbb{S}^1)$ includes all of bounded operators on $\mathcal{H}_{\mathcal{A}}$.
Factoriality. Each $\mathcal{A}(I)$ is a type *III*$_1$ factor.
Additivity. For every $\mathcal{A}(I)$, it holds $\mathcal{A}(I) \subset \vee_{J\in S}\mathcal{A}(J)$ if $\cup_{J\in S} J \supset I$.
(3) Theorems 3.1, 3.2, 3.3 are still valid with some changes if one considers operators $L_n^{(c)}$ with $c < \infty$, $|n| \leq 1$ and the real basis u_j is made of smooth functions (that is the case, for instance, if $\Sigma = \mathbb{S}^2$ and the basis is that of real and imaginary parts of spherical harmonics).
One starts to replacing the space $\mathcal{S}(\mathbb{M})$ for its subspace $\mathcal{S}^{(c)}(\mathbb{M})$ of the real linear combinations of functions $\mathbb{M} \ni (\theta, s) \mapsto f(\theta)u_j(s)$ with $f \in C^\infty(\mathbb{S}^1)$ and $j \leq c < \infty$. In the following $\mathcal{H}_\Sigma^{(c)} := \mathcal{S}^{(c)}(\mathbb{M}) + i\mathcal{S}^{(c)}(\mathbb{M})$. $\mathcal{D}^{(c)}(\mathbb{M})$ is constructed exactly as $\mathcal{D}(\mathbb{M})$ but using the space $\mathcal{S}^{(c)}(\mathbb{M})$ as starting point. The construction of the algebras $\mathcal{W}^{(c)}(\mathbb{M})$ and $\mathcal{F}^{(c)}(\mathbb{M})$ goes on exactly as in $c = \infty$ case (the causal propagator is defined as the restriction to $\mathcal{D}^{(c)}(\mathbb{M})$ of E) obtaining sub algebras of $\mathcal{W}(\mathbb{M})$ and $\mathcal{F}(\mathbb{M})$. Re-defining the one-particle space $\mathcal{H}^{(c)} = \ell^2(\mathbb{C}) \otimes \mathcal{H}_\Sigma^{(c)}$ using the

same procedure as for $\mathcal{H}$ but starting from $\mathcal{S}^{(c)}(\mathbb{M})$, one gets Fock representations of the algebras $\mathcal{W}^{(c)}(\mathbb{M})$ and $\mathcal{F}^{(c)}(\mathbb{M})$ in $\mathfrak{F}_+(\mathcal{H}^{(c)})$. Finally, referring to operators

$$iK^{(c)} := iL_0^{(c)}\ , \quad iS^{(c)} := i\frac{L_1^{(c)} + L_{-1}^{(c)}}{2}\ , \quad iD^{(c)} := \frac{L_1^{(c)} - L_{-1}^{(c)}}{2}\ ,$$

theorems 3.1, 3.2, 3.3 hold true provided one replaces $L^2(\Sigma, d\Sigma)$ for $\mathcal{H}^{(c)}$, $\mathcal{H}$ for $\mathcal{H}^{(c)}$, $j \in \mathbb{Z}$ for $j \leq c$ and $\hat{\mathcal{W}}(\mathbb{M})$, $\hat{\mathcal{F}}(\mathbb{M})$ for, respectively, $\hat{\mathcal{W}}^{(c)}(\mathbb{M})$, $\hat{\mathcal{F}}^{(c)}(\mathbb{M})$ everywhere.

4 Spontaneous breaking of $SL(2,\mathbb{R})$ symmetry and thermal states.

4.1. *Back to Physics.* Consider the degenerate manifold $\mathbb{M}_{\mathbb{F}} = \mathbb{S}^1 \times \mathbb{S}^2$ obtained by the future event horizon $\mathbb{F} \equiv \mathbb{R} \times \mathbb{S}^2$ of the Kruskal manifold as discussed in section 2.1. We may arrange a coordinate θ on $\mathbb{S}^1 \equiv [-\pi, \pi]$ with the identification $\pi \equiv -\pi$ in order that, using the same notation as in section 2.1, the point ∞ added to $\mathbb{R}$ to get $\mathbb{S}^1$ corresponds to $\theta = \pi$ and $\theta = 0$ corresponds to the bifurcation point on $\mathbb{F}$ (see section 2.1). The orientation of $\mathbb{S}^1$ is fixed by that coordinate with the requirement that θ increases toward the future in $(-\pi, \pi)$. In this way a bosonic QFT can be built up on $\mathbb{M}_{\mathbb{F}}$ together with a Möbius-covariant representation of $PSL(2,\mathbb{R})$ everything associated with the preferred choice for the Killing vector ∂_θ. In the following we want to try to develop the idea that a bosonic field living on the event horizon can be related to some quantum properties of the black hole or of the spacetime in general. The natural GNS representation for that bosonic field seems to be the Fock one presented previously. In particular the vacuum state is a thermal state – in the KMS sense – with respect to the Killing vector $-\mathcal{D}$. Moreover it is in relation [1] with the analogous result valid in the Kruskal manifold where the natural reference state, the so called Hartle-Hawking state, is a KMS state with respect to the Killing field ∂_t that tends to $-\mathcal{D}$ approaching the future event horizon if the coordinate θ is fixed appropriately. However, from a physical point of view the choice of that representation does not seem to be very satisfactory. The whole $PSL(2,\mathbb{R})$ unitary representation, referred to the Fock space $\mathfrak{F}_+(\mathcal{H})$, includes arbitrary translations in the coordinate θ and, in fact, all the points in $\mathbb{S}^1 = \mathbb{R} \cup \{\infty\}$ are metrically equivalent. However, the circle $\mathbb{S}^1 = \mathbb{R} \cup \{\infty\}$ admits two physically distinguishable points: The point at infinity, which cannot be reached physically because it corresponds to a surface which does not belong to the Kruskal manifold, and another point, which correspond to the bifurcation manifold where ∂_t vanishes. (In any cases, $\mathbb{S}^1 \times \Sigma$ with the the degenerate semi-Riemannian metric considered in this work cannot represent a portion of spacetime because of the presence of closed causal curves lying in $\mathbb{S}^1$: At least a point of $\mathbb{S}^1$ must be removed to make contact with physics.) The remaining points of $\mathbb{S}^1$ are physically equivalent barring the fact that they are either in the past or in the future of $\theta = 0$. This determines two regions $\mathbb{F}_- \equiv (-\pi, 0) \times \mathbb{S}^2$ and $\mathbb{F}_+ \equiv (0, \pi) \times \mathbb{S}^2$ in the physical part of the manifold $\mathbb{R} \times \mathbb{S}^2$, corresponding to respectively the future and past part (with respect to the bifurcation point) of the future event horizon of $\mathbb{K}$. In turn, once a reference state ω is fixed on $\mathcal{W}(\mathbb{M}_{\mathbb{F}})$, these regions correspond to von Neumann algebras $\mathcal{A}(\mathbb{F}_+)$ and

$\mathcal{A}(\mathbb{F}_-)$ (based upon the GNS representation of ω) representing the observables in those regions. (A different – in a sense weaker – notion of observables associated with regions $\mathbb{F}_\pm$ consists of the Weyl algebras generated by forms supported in $\mathbb{F}_\pm$, we shall use that notion later.) In the following we show that it is possible to single out those physical regions at *quantum, i.e. Hilbert space, level* through a sort of *spontaneous breaking of* $SL(2,\mathbb{R})$ *symmetry* referring to a new state $\lambda_\zeta \neq \lambda$ which *preserves* the relevant thermal properties. We mean the following. At algebraic level there is a representation α of Möbius group $PSL(2,\mathbb{R})$ made of *-automorphisms of the Weyl algebra $\mathcal{W}(\mathbb{M}_\mathbb{F})$. Moreover, we have seen in theorems 3.1 and 3.2 that there is a state λ on $\mathcal{W}(\mathbb{M}_\mathbb{F})$ which is invariant under α and, in the GNS representation of λ, α is implemented unitarily and covariantly by a representation U of $PSL(2,\mathbb{R})$. We show below that there are other, unitarily inequivalent, GNS representations of $\mathcal{W}(\mathbb{M}_\mathbb{F})$ based on new states λ_ζ which are no longer invariant under the whole α, but such that, the residual symmetry is still covariantly and unitarily implementable and singles out the algebras $\mathcal{A}(\mathbb{F}_+)$ and $\mathcal{A}(\mathbb{F}_-)$ as unique invariant algebras. We show also that every λ_ζ enjoys the same thermal (KMS) properties as λ and it represents a different thermodynamical phase with respect to λ.

4.2. *Symmetry breaking.* We need some technical results and definitions to go on. Coming back to the general case where Σ is a generic Riemannian manifold, fix ∂_θ as at the beginning of 2.4. Consider $\omega \in \mathcal{D}(\mathbb{M})$. By proposition 2.1 it holds $\omega = 2\epsilon_\psi$ for a unique $\psi \in \mathcal{S}(\mathbb{M})$. This allow one to define the **∂_θ-positive-frequency part** of ω: $\omega_+ := 2\epsilon_{\psi_+}$. Obviously, it turns out that $\omega := \omega_+ + \overline{\omega_+}$. Moreover the following important technical lemma holds.

Lemma 4.1. *Fix ∂_θ on $\mathbb{M} := \mathbb{S}^1 \times \Sigma$ as in 2.4. If $g \in PSL(2,\mathbb{R})$ and $\omega \in \mathcal{D}(\mathbb{M})$,*

$$g^*(\omega_+) = (g^*\omega)_+ \,. \tag{28}$$

Proof. From *Remark* on p. 271 of [7] one finds that $(\psi^{(g)})_+ = (\psi_+ \circ g)$ for all $g \in PSL(2,\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{M})$. Now the result can be extended to forms $\omega \in \mathcal{D}(\mathbb{M})$ simply using the definition given above. $\Box$

We stress that the same result does *not* hold for generic diffeomorphisms $g \in Diff^+(\mathbb{S}^1)$.
The one-parameter subgroup of Möbius transformations $\mathbb{R} \ni t \to \exp(t\mathcal{D})$ is generated by the field $\mathcal{D} := -\sin\theta \frac{\partial}{\partial\theta}$ in $\mathbb{M}$. It admits 0 and π as unique fixed points and, on the other hand, it is simply proved that (up to nonvanishing factors) $\mathcal{D}$ is the unique nonzero vector field in the representation of $sl(2,\mathbb{R})$ which vanishes at 0 and π. As a consequence that subgroup is the unique (up to rescaling of the parameter) nontrivial one-parameter Möbius subgroup which admits $(0,\pi)$ and $(-\pi,0)$ as invariant segments. The parameter v of the integral curves of $-\mathcal{D}$ satisfies

$$v = \Gamma(\theta) := \ln\left|\tan\frac{\theta}{2}\right| \,, \tag{29}$$

where v ranges monotonically in $\mathbb{R}$ with $v' > 0$ for $\theta \in (0, \pi)$, whereas it ranges monotonically in $\mathbb{R}$ with $v' < 0$ for $\theta \in (-\pi, 0)$. In spite of its singularity at $\theta = 0$, the function Γ in (29) is locally integrable. Thus for any fixed function $\zeta \in L^2(\Sigma, d\Sigma)$, $\Lambda_\zeta(V(\omega)) := \lambda(V(\omega))e^{i\int_{\mathbb{M}}\Gamma(\zeta\omega_+ + \overline{\zeta\omega_+})}$ is well defined if $\omega \in \mathcal{D}(\mathbb{M})$. Let us show that Λ_ζ extends to a state on $\mathcal{W}(\mathbb{M})$. It holds $\Lambda_\zeta(V(0)) = 1$. Using (V1), (V2) and imposing linearity, Λ_ζ defines a linear functional on the *-algebra generated by all of objects $V(\omega)$. As λ is positive, Λ_ζ turns out to be positive too, finally $\mathbb{R} \ni t \mapsto \Lambda_\zeta(V(\omega))$ is continuous. For known theorems [27] there is a unique extension λ_ζ of Λ_ζ to a state on $\mathcal{W}(\mathbb{M})$: If the real function $\zeta \in L^1_{loc}(\Sigma, d\Sigma)$ is fixed, it is the unique state satisfying,

$$\lambda_\zeta(V(\omega)) = \lambda(V(\omega))\, e^{i\int_{\mathbb{M}}\Gamma(\zeta\omega_+ + \overline{\zeta\omega_+})} \tag{30}$$

for all $\omega \in \mathcal{D}(\mathbb{M})$. λ_ζ and its GNS triple $(\mathfrak{H}_\zeta, \Pi_\zeta, \Psi_\zeta)$ enjoy the remarkable properties stated in the theorem below whose proof is in the appendix.

Theorem 4.1. *Take ∂_θ on $\mathbb{M} = \mathbb{S}^1 \times \Sigma$ as in 2.4, define $\mathcal{D}$ as in (14) and the group of *-automorphisms α as in 3.1, $\{\alpha^{(\mathcal{X})}_t\}_{t\in\mathbb{R}}$ being any one-parameter subgroup associated with the vector field $\mathcal{X}$. If $\zeta \in L^2(\Sigma, d\Sigma)$ and λ_ζ is the state defined in (30) with GNS triple $(\mathfrak{H}_\zeta, \Pi_\zeta, \Psi_\zeta)$, the following holds:*

(a) *The map $V(\omega) \mapsto V(\omega)\, e^{i\int_{\mathbb{M}}\Gamma(\zeta\omega_+ + \overline{\zeta\omega_+})}$, $\omega \in \mathcal{D}(\mathbb{M})$, uniquely extends to a *-automorphism γ_ζ on $\mathcal{W}(\mathbb{M})$ and*

$$\lambda_\zeta(w) = \lambda(\gamma_\zeta w)\,, \quad \text{for all } w \in \mathcal{W}(\mathbb{M})\,, \tag{31}$$

$$\gamma_\zeta \circ \alpha^{(\mathcal{D})}_t = \alpha^{(\mathcal{D})}_t \circ \gamma_\zeta\,, \quad \text{for all } t \in \mathbb{R}, \tag{32}$$

(b) *(i) λ_ζ is pure, (ii) if $\zeta \neq \zeta'$ a.e., λ_ζ and $\lambda_{\zeta'}$ are not quasiequivalent, (iii) λ_ζ is invariant under $\{\alpha^{(\mathcal{D})}_t\}_{t\in\mathbb{R}}$, but it is not so under any other one-parameter subgroup of α (barring those associated with $c\mathcal{D}$ for $c \in \mathbb{R}$ constant) when $\zeta \neq 0$ almost everywhere.*

(c) *$\mathfrak{H}_\zeta$ identifies with a Fock space $\mathfrak{F}_+(\mathcal{H}_\zeta)$ with vacuum vector Ψ_ζ and, for all $\omega \in \mathcal{D}(\mathbb{M})$,*

$$\Pi_\zeta : V(\omega) \mapsto \hat{V}_\zeta(\omega) := e^{\overline{i\hat{\phi}_\zeta(\omega)}}\,, \quad \text{where} \quad \hat{\phi}_\zeta(\omega) := \hat{\phi}_0(\omega) + \left\{\int_{\mathbb{M}}\Gamma(\zeta\omega_+ + \overline{\zeta\omega_+})\right\} I, \tag{33}$$

$\hat{\phi}_0(\omega)$ being here the standard field operator in the Fock space $\mathfrak{F}_+(\mathcal{H}_\zeta)$ as in 2.4.

(d) *There is a strongly continuous one-parameter group of unitary operators $\{U^{(\mathcal{D})}_\zeta(t)\}_{t\in\mathbb{R}}$ with*

$$\alpha^{(\mathcal{D})}_t(w) = U^{(\mathcal{D})}_\zeta(t)\, w\, U^{(\mathcal{D})\dagger}_\zeta(t) \quad \text{for all } t \in \mathbb{R} \text{ and } w \in \hat{W}_\zeta(\mathbb{M}) := \Pi_\zeta(\mathcal{W}_\zeta(\mathbb{M})). \tag{34}$$

Moreover (the derivative is performed in the strong sense where it exists)

$$\frac{d}{dt}|_{t=0} U^{(\mathcal{D})}_\zeta(t) = \frac{-i}{2}\overline{:\Omega(\hat{\phi}_0, \mathcal{D}\hat{\phi}_0):}\,. \tag{35}$$

The following corollary remarks that the states λ_ζ share several properties with λ, KMS property in particular.

Corollary. *In the hypotheses of theorem 4.1 the following holds for net of von Neumann algebras*

$$\mathcal{A}_\zeta = \{\mathcal{A}_\zeta(I) \mid I \in \mathfrak{I}\} , \quad \text{with} \quad \mathcal{A}_\zeta(I) = \{\hat{V}_\zeta(\omega) \mid supp\, \omega \subset I \times \Sigma\}'' . \tag{36}$$

(a) $\mathcal{A}_\zeta \supset \hat{W}_\zeta(\mathbb{M})$ *and it enjoys the following properties: (i) isotony, (ii) locality, (iii)* $\{\exp(t\mathcal{D})\}_{t\in\mathbb{R}}$*-covariance, (iv)* $U_\zeta^{(\mathcal{D})}$*-invariance and uniqueness of the vacuum* Ψ_ζ*, (v) cyclicity of the vacuum* Ψ_ζ*, (vi) Reeh-Schlieder, (vii) Haag duality, (viii) factoriality, (iix) irreducibility, (ix) additivity.*
(b) *If* $\zeta \neq 0$ *a.e.,* $\mathcal{A}_\zeta(\mathbb{F}_+) := \mathcal{A}_\zeta((0,\pi))$ *and* $\mathcal{A}_\zeta(\mathbb{F}_-) := \mathcal{A}_\zeta((-\pi,0))$ *are the unique* $\{U_t^{(\mathcal{D})}\}_{t\in\mathbb{R}}$*-invariant algebras in* $\mathcal{A}_\zeta$*.*
(c) *If* Δ *is the modular operator associated with* $\mathcal{A}_\zeta(\mathbb{F}_+)$ *then*

$$\Delta^{it} = U_\zeta^{(\mathcal{D})}(2\pi t) , \quad \text{for all } t \in \mathbb{R} . \tag{37}$$

Thus λ_ζ *is a KMS state on* $\mathcal{A}_\zeta(\mathbb{F}_+)$ *with temperature* $T = 1/2\pi$*, with respect to* $\{\alpha_t^{(-\mathcal{D})}\}_{t\in\mathbb{R}}$ *(extended to* σ*-weak one-parameter group of* **-automorphisms of* $\mathcal{A}_\zeta(\mathbb{F}_+)$ *through (34)).*

Proof. (a) and (c) Since the difference between $\hat{V}_\zeta(\omega)$ and $e^{i\overline{\hat{\phi}_0(\omega)}}$ amounts to a phase only, each algebra $\mathcal{A}_\zeta(I)$ of $\mathcal{A}_\zeta$ coincides with the analog constructed starting from operators $e^{i\overline{\hat{\phi}_0(\omega)}}$ and using the same $I \in \mathfrak{I}$. Hence theorem 3.3 and subsequent remark 2 hold using the field $\hat{\phi}_0$, replacing Ψ with Ψ_ζ and employing the representation U of $PSL(2,\mathbb{R})$ which leaves Ψ_ζ unchanged. Notice that U does not implement α! In this way all the properties cited in the thesis turn out to be automatically proved with the exception of (iii) and (iv). However using (32), (35) and (d) of theorem 3.1 also those properties can be immediately proved. The proof of (c) is straightforward. $\mathcal{A}_\zeta((0,\pi))$ coincides with the analog constructed starting from operators $e^{i\overline{\hat{\phi}_0(\omega)}}$. In that case the thesis holds with respect to the subgroup of U, $e^{t\overline{:\Omega(\hat{\phi}_0,\mathcal{D}(\hat{\phi}_0)):/2}}$ (remark (2) after theorem 3.3). Now (35) implies the validity of the thesis in our case.
(b) Since $\mathcal{D}$ admits the only zeros at $\theta = 0$ and $\theta = \pi \equiv -\pi$, the only open nonempty and nondense intervals of $\mathbb{S}^1$ which are invariant under the one-parameter group $\{g_t^{(\mathcal{D})}\}_{t\in\mathbb{R}}$ generated by $\mathcal{D}$ are $(0,\pi)$ and $(-\pi,0)$. $\mathcal{D}$-covariance reads $U_\zeta^{(\mathcal{D})}(t)\mathcal{A}_\zeta(I)U_\zeta^{(\mathcal{D})\dagger}(t) = \mathcal{A}_\zeta(g_t^{(\mathcal{D})}(I))$ and thus $\mathcal{A}_\zeta((0,\pi))$ and $\mathcal{A}_\zeta((-\pi,0))$ are invariant under $\{U_\zeta^{(\mathcal{D})}(t)\}_{t\in\mathbb{R}}$. Let us prove their uniqueness. Consider the case of $I = (a,b)$ with $0 \leq a < b < \pi$. There are $t' > 0$ and $a' > 0$, with $a' < b$ and such that $g_{t'}^{(\mathcal{D})}(a',b) \cap (a,b) = \emptyset$. Therefore, by locality it holds $[U_\zeta^{(\mathcal{D})}(t')\mathcal{A}_\zeta((a',b))U_\zeta^{(\mathcal{D})\dagger}(t'), \mathcal{A}_\zeta((a,b))] = 0$, i.e. $[\mathcal{A}_\zeta((a',b)), U_\zeta^{(\mathcal{D})}(-t')\mathcal{A}_\zeta((a,b))U_\zeta^{(\mathcal{D})\dagger}(-t')] = 0$. If $\mathcal{A}_\zeta((a,b))$ were invariant under $\{U_\zeta^{(\mathcal{D})}(t)\}_{t\in\mathbb{R}}$, the latter identity above would imply that $[\mathcal{A}_\zeta((a',b)), \mathcal{A}_\zeta((a,b))] = 0$, and thus in particular $\mathcal{A}_\zeta((a',b)) \subset \mathcal{A}_\zeta((a',b))'$ which is trivially false because elements $\hat{V}_\zeta(\omega) \in \mathcal{A}_\zeta((a',b))$ generally do not commute. All the remaining cases can be reduced to that studied above with obvious adaptations. □

Remarks.
(1) (c) in the corollary is valid also replacing $\mathbb{F}_+$ for $\mathbb{F}_-$ and $\mathcal{D}$ for $-\mathcal{D}$ as well. Theorem 4.1 and the corollary hold in particular for $\Sigma = \mathbb{S}^2$ and $\mathbb{M} = \mathbb{M}_{\mathbb{F}}$. In that case one finds easily that: *λ_ζ is invariant under the group of *-automorphisms induced by the action of $SO(3)$ as isometry group on $\mathbb{S}^2$ if and only if ζ is constant a.e. on $\mathbb{S}^2$.*
Generic Σ do not admit $SO(3)$ as group of isometries, in that case λ_ζ is invariant under the relevant isometry group of Σ provided ζ is so. Finally we notice that the hypotheses $\zeta \in L^2(\Sigma, d\Sigma)$ can be relaxed in $\zeta \in L^1_{loc}(\Sigma, d\Sigma)$ (the space of locally integrable functions on Σ with respect to $d\Sigma$) both in the theorem and in the corollary, the only result that could fail to hold is (ii) in (b) of the theorem.
(2) The theorem and the corollary refer to the pair of segments $(0,\pi)$ and $(-\pi,0)$ in the circle realized as the segment $[-\pi,\pi]$ with $-\pi \equiv \pi$. From a physical point of view there is no way to distinguish between the pair of regions $(0,\pi)$, $(-\pi,0)$ and any other pair of open nonempty segments $I, J \subset \mathbb{S}^1$ such that $J = int(\mathbb{S}^1 \setminus I)$. This is because there is no way to measure segments on $\mathbb{S}^1$ as the metric is degenerate therein. In fact the theorem can be stated for any pair of such segments. To prove it we notice that there exists a Möbius diffeomorphism $g : \mathbb{S}^1 \to \mathbb{S}^1$ with $I = g((0,\pi))$ and $J = g((-\pi,0))$[6]. Hence, theorem 4.1 and its corollary can be re-stated replacing $(0,\pi)$ and $(-\pi,0)$ for, respectively I and J, replacing the state (30) for the state and assuming to have fixed some $\zeta \in L^2(\Sigma, d\Sigma)$,

$$\lambda_I(V(\omega)) := \lambda(V(\omega))e^{i\Gamma_I(\omega)}, \quad \text{with } \Gamma_I(\omega) := \int_{\mathbb{M}_{\mathbb{F}}} \Gamma(\zeta\, g^*\omega_+ + \overline{\zeta\, g^*\omega_+})$$

and replacing $\mathcal{D}$ for the generator $\mathcal{D}_I$ of the one-parameter Möbius subgroup $\mathbb{R} \ni t \mapsto \exp(t\mathcal{D}_I) := g \circ \exp(t\mathcal{D}) \circ g^{-1}$ which leaves invariant I and J ($\mathcal{D}_I$ does not depend on the choice of g).
Notice also that if I, J is a pair of segments as said above and h is any Möbius transformation, $h(I), h(J)$ still is a pair of open nonempty segments with $h(J) = int(\mathbb{S}^1 \setminus h(I))$ and it holds (using also lemma 4.1)

$$\lambda_{h(I)}(V(\omega)) = \lambda_I(V(h^*\omega))\,.$$

This fact means that the $PSL(2,\mathbb{R})$ symmetry, broken at Hilbert-space level, is restored at algebraic level by considering the whole class of states λ_I.

4.3. *Residual Virasoro representation after breaking $PSL(2,\mathbb{R})$ symmetry.* The complex Lie algebra $(d, \{\cdot,\cdot\}, \omega)$ of vector field on $\mathbb{S}^1$ (see discussion in 3.1) is made of vector fields on $\mathbb{S}^1$ whose diffeomorphism groups, generated by their real and imaginary parts, do not admit (in general) $\mathbb{F}_\pm$ as invariant regions, when extended to $\mathbb{M} = \mathbb{S}^1 \times \Sigma$. This happens in particular for generators $\mathcal{L}_n = ie^{in\theta}\partial_\theta$. However, it is possible to rearrange that basis in order to partially overcome the problem. Consider the equivalent basis of d made of the following real vector fields $-i\mathcal{L}_0$, $\mathcal{E}_n := (1-\cos((2n)\theta))\partial_\theta$, $\mathcal{O}_n := (1+\cos((2n+1)\theta))\partial_\theta$, $\mathcal{G}_n := -\sin(n\theta)\partial_\theta$ with $n = 1, 2, \ldots$. Barring $-i\mathcal{L}_0$ and $\mathcal{O}_n$, the other fields admit $\mathbb{F}_\pm$ as invariant regions. Moreover the

[6] Assume that, in coordinates θ, I has length equal or shorter than J. The diffeomorphism g^{-1} is the composition of a rigid rotation generated by $\mathcal{K}$ which maps the center of I in 0, a dilatation generated by $\mathcal{D}$ which enlarges the transformed I up to $(-\pi/2, \pi/2)$ and another anti-clockwise rigid rotation of $\pi/2$.

fields $\mathcal{G}_n$ define a Lie algebra with respect to the usual Lie bracket whereas $\mathcal{E}_n$, or $\mathcal{E}_n$ together $\mathcal{G}_n$, do not so. However allowing infinite linear combinations of vector fields – using for instance L^2-convergence for the components of vector fields with respect to ∂_θ (the same result hold anyway using stronger notions of convergence as uniform convergence of functions and their derivatives up to some order) – one sees that each $\mathcal{E}_n$ can be expanded as an infinite linear combination of $\mathcal{G}_n$. From these considerations one might expect, at least, that fields $\mathcal{E}_n$, but *not the vectors* $\mathcal{L}_n$ *and* $\mathcal{O}_n$, admit some operator representation in $\mathfrak{H}_\zeta$ in terms of the field operator $\hat{\phi}_\zeta$. In fact this is the case if ζ is a real function in $L^2(\Sigma, d\Sigma)$. If one tries to define operators $L^{(c)}_{\zeta n}$ as in (20) with $\hat{\phi}^{(c)}$ replaced for $\hat{\phi}^{(c)}_\zeta := \hat{\phi}^{(c)} + 2\zeta\Gamma$, one immediately faces ill-definiteness of those operators due to infinite additive terms and the same problem arises for formal operators $O^{(c)}_n := L^{(c)}_{\zeta 0} + (L^{(c)}_{\zeta 2n+1} + L^{(c)}_{\zeta\, -2n-1})/2$ and also for $E^{(c)}_n := L^{(c)}_{\zeta 0} - (L^{(c)}_{\zeta 2n} + L^{(c)}_{\zeta\, -2n})/2$. However these terms cancel out if considering the operators $G^{(c)}_n := (L^{(c)}_{\zeta\, -n} - L^{(c)}_{\zeta n})/(2i)$ with $n = 1, 2, \ldots$, which are well defined and essentially selfadjoint on $F(\mathcal{H}_\zeta)$. Moreover, the operators $G^{(c)}_n$ define a Lie algebra with respect to the commutator. (Direct inspection shows that if $c = \infty$ none of the considered operators is well-defined on $F(H_\zeta)$.) It is plausible that operators $\overline{G^{(c)}_n}$ define one-parameter groups which implement covariance with respect to analogous groups of diffeomorphisms generated by associated vector fields $\mathcal{G}_n$, and that the exponentiation of the algebra of $G^{(c)}_n$ produces a unitary representation of a (perhaps the) subgroup of $Diff^+(\mathbb{S}^1)$ of the diffeomorphisms which leaves $\mathbb{F}_\pm$ invariant. However, it is worth stressing that, barring the case $\overline{G^{(c)}_1}$ which generates just $U^{(\mathcal{D})}_\zeta(t)$, Ψ_ζ is not invariant under the remaining unitary groups.

5 Towards physical interpretation and conclusions.

5.1. *Extremal KMS states and thermodynamical phases Bose-Einstein condensate.* What about the meaning of the state λ_ζ with $\zeta \neq 0$? Notice that, by construction λ_ζ are KMS states on the C^*-algebra $\mathcal{W}(\mathbb{F}_+)$, the Weyl algebra generated by Weyl operators $V(\omega)$ with $supp\, \omega \subset \mathbb{F}_+$ which is contained in $\mathcal{A}_\zeta(\mathbb{F}_+)$. As states on $\mathcal{W}(\mathbb{F}_+)$, λ_ζ and $\lambda_{\zeta'}$ can be compared also if $\zeta \neq \zeta'$ (they do not belong to a common folium if (ii) in (b) of theorem 4.1 holds, so they cannot be compared on a common von Neumann algebra of observables in that case). The next proposition shows that $\{\lambda_\zeta\}_{\zeta \in L^2(\Sigma, d\Sigma)}$ is a family of extremal states in the convex space of KMS states over $\mathcal{W}(\mathbb{F}_+)$ at inverse temperature 2π with respect to $-\mathcal{D}$. The natural interpretation of this fact is that the states λ_ζ, restricted to the observables in the physical region $\mathbb{F}_+$, are nothing but *different thermodynamical phases* of the same system at the same temperature (see V.1.5 in [13]).

Proposition 5.1. *With the same hypotheses as in theorem 4.1 the following holds.*
(a) *Any state* λ_ζ *(with* $\zeta \in L^2(\Sigma, d\Sigma)$*) defines an extremal states in the convex set of KMS states on the* C^**-algebra* $\mathcal{W}(\mathbb{F}_+)$ *at inverse temperature* 2π *with respect to* $\{\alpha^{(-\mathcal{D})}_t\}_{t\in\mathbb{R}}$.
(b) *Different choices of* ζ *individuate different states on* $\mathcal{W}(\mathbb{F}_+)$ *which are not unitarily equivalent as well.*

Proof. Let $(\mathfrak{H}_\zeta, \Pi_\zeta, \Psi_\zeta)$ be the GNS representations of λ_ζ. The GNS representations of $\lambda_\zeta \restriction_{\mathcal{W}(\mathbb{F}_+)}$ must be (up to unitary equivalences) $(\mathfrak{H}_\zeta, \Pi_\zeta \restriction_{\mathcal{W}(\mathbb{F}_+)}, \Psi_\zeta)$ due to Reeh-Schlieder property ((a) in Corollary) of $\mathcal{A}_\zeta(\mathbb{F}_+)$. Since $\mathcal{A}_\zeta(\mathbb{F}_+) = \Pi_\zeta \restriction''_{\mathcal{W}(\mathbb{F}_+)}$ is a (type III_1) factor, the state $\lambda_\zeta \restriction_{\mathcal{W}(\mathbb{F}_+)}$ – namely $\Pi_\zeta \restriction_{\mathcal{W}(\mathbb{F}_+)}$ – is primary (see III.2.2 in [13]). As a consequence, by theorem 1.5.1 in [13], the KMS state $\lambda_\zeta \restriction_{\mathcal{W}(\mathbb{F}_+)}$ is extremal in the space of KMS states on $\mathcal{W}(\mathbb{F}_+)$ with respect to $\alpha^{(-\mathcal{D})}_t$ at the temperature of $\lambda_\zeta \restriction_{\mathcal{W}(\mathbb{F}_+)}$ itself. Obviously $\lambda_\zeta \restriction_{\mathcal{W}(\mathbb{F}_+)} \neq \lambda_{\zeta'} \restriction_{\mathcal{W}(\mathbb{F}_+)}$ because, if $\zeta - \zeta'$ is not zero almost everywhere, the integrals in the exponentials defining λ_ζ and $\lambda_{\zeta'}$ produce different results when applied to $V(\omega)$ with $supp\, \omega \subset \mathbb{F}_+$ with a suitable choice of ω. The proof of non equivalence is the same as done (see the appendix) for the states defined in the whole von Neumann algebras. □

Henceforth we consider the case of ζ real and perform more or less rigorous manipulations on the mathematical objects in order to grasp some physical meaning. Let us examine the generators $\hat{\phi}_\zeta$ of the Weyl representation associated with λ_ζ. Consider $\omega \in \mathcal{D}(\mathbb{M})$ such that $supp\, \omega \subset \mathbb{F}_+$ and such that $\omega(v, s)$ can be rewritten as $\frac{\partial \psi(v,s)}{\partial v} dv \wedge d\Sigma$ where ψ is smooth and compactly supported in $\mathbb{F}$. Similar "wavefunctions" ψ have been considered in [6] building up scalar QFT on a Killing horizon ($\mathbb{F}_+$ in our case). Using (29) we can write the formal expansion

$$\hat{\phi}_\zeta(\omega) = \int_{\mathbb{F}_+} \hat{\phi}_0(\theta_+(v))\omega(v, s) + \int_{\mathbb{F}_+} 2\zeta(s)\, v\omega(v, s)\,. \tag{38}$$

In terms of wavefunctions, where $\Omega_\mathbb{F}$ is the restriction of the right-hand side of (1) to real smooth functions compactly supported in $\mathbb{F}$ ($\Omega_\mathbb{F}$ is *nondegenerate* on that space)

$$\Omega(\psi, \hat{\phi}_\zeta) = \Omega_\mathbb{F}(\psi, \hat{\phi}_0) - \int_\Sigma d\Sigma(s) 2\zeta(s) \int_{-\infty}^{+\infty} \psi(v, s) dv\,. \tag{39}$$

The group of elements $e^{itH_\zeta} := U^{(-\mathcal{D})}_\zeta(t)$, $t \in \mathbb{R}$ generates displacements $v \mapsto v - t$ in the variable v in the argument of the wavefunctions ψ, since v is just the parameter of the integral curves of $-\mathcal{D}$ which takes the form $\frac{\partial}{\partial v}$ in $\mathbb{F}_+$. Using Fourier transformation with respect to v we can write down

$$\psi(v, s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} dE\ \widetilde{\psi_+(s, E)} e^{-iEv} + \overline{\widetilde{\psi_+(s, E)}} e^{iEv}\,. \tag{40}$$

In heuristic sense H_ζ acts on the wavefunctions ψ as the multiplicative operator $\widetilde{\psi_+(s, E)} \mapsto E\widetilde{\psi_+(s, E)}$. Physically speaking, thermal properties of λ_ζ are referred just to the energy notion associated with that Hamiltonian. Actually, as is well-known, this interpretation must be handled with great care: the interpretation of $\widetilde{\psi_+}$ as a representative of a one-particle quantum state can be done in a Fock space whose vacuum state does not coincide with the KMS state λ_ζ (see V.1.4 and the discussion in p. 219 of [13].) Using (40), (39) can be re-written as

$$\Omega(\psi, \hat{\phi}_\zeta) = \Omega_\mathbb{F}(\psi, \hat{\phi}_0) - \sqrt{8\pi} \int_\Sigma d\Sigma(s)\zeta(s)\widetilde{\psi(0, s)}_+\,. \tag{41}$$

From (41) it is apparent that $\hat{\phi}_\zeta$ gets contributions from *zero-energy modes* ($E = 0$) as it happens in Bose-Einstein condensation. To this end see chapter 6 of [28] and 5.2.5 of [12], especially p. 72, where in the decomposition of the KMS state ω (after the thermodynamical limit) in $\mathbb{R}^\nu$-ergodic states, the mathematical structure of the latter states resemble that of the states λ_ζ. Another interesting comment concerning this interpretation of the states λ_ζ as states describing the presence of a condensate is obtained by studying the explicit expression of the generator of $U_\zeta^{(-\mathcal{D})}(t) = e^{itH_\zeta}$. By (d) of theorem 4.1 we have (both sides are supposed to be restricted to the core $F(\mathcal{H}_\zeta)$)

$$H_\zeta = \int_{\mathbb{M}} \sin(\theta) :\frac{\partial\hat{\phi}_0(\theta,s)}{\partial\theta}\frac{\partial\hat{\phi}_0(\theta,s)}{\partial\theta}: d\theta d\Sigma(s) \,.$$

Passing from coordinates (θ, s) to coordinates (v, s) (see 4.2 for $\theta_\pm$) and employing the field $\hat{\phi}_\zeta$ one finds

$$H_\zeta = \lim_{N\to+\infty}\left\{\int_{\mathbb{F}_+}\chi_N(v) :\frac{\hat{\phi}_\zeta(\theta_+(v),s)}{\partial v}\frac{\hat{\phi}_\zeta(\theta_+(v),s)}{\partial v}: dvd\Sigma(s) - ||\zeta||^2\int_{\mathbb{R}}\chi_N(v)dv\right\}$$

$$-\lim_{N\to+\infty}\left\{\int_{\mathbb{F}_-}\chi_N(v) :\frac{\hat{\phi}_\zeta(\theta_-(v),s)}{\partial v}\frac{\hat{\phi}_\zeta(\theta_-(v),s)}{\partial v}: dvd\Sigma(s) - ||\zeta||^2\int_{\mathbb{R}}\chi_N(v)dv\right\} \,,$$

where the function χ_N is smooth with compact support in $[-N, N]$ and becomes the constant function 1 for $N \to +\infty$. The normal ordering prescription used in the integrals is defined by subtracting the distribution of $w(\theta', s', \theta, s) := \left(\Psi, \hat{\phi}(\theta', s')\hat{\phi}(\theta, s)\Psi\right)$ before applying derivatives and then smoothing with a product of delta in θ, θ' and s, s'. We do not enter into mathematical details here which are quite standard procedures of applied microlocal analysis similar to that used in Hadamard regularization [29, 30, 31]. What we want to stress concerning the expansion of the generator H_ζ of v displacements written above is that it has the usual form in terms of stress-energy tensor defined with the generators of the Weyl representation $\hat{\phi}_\zeta$ except for a further volume divergence. That divergence is due to the term

$$E_\zeta := ||\zeta||^2\int_{\mathbb{R}} dv \,.$$

This can be interpreted as the energy of the BE condensate whose density is *finite* and amounts to $||\zeta||^2$. As a final comment we notice that, up to now, we have used units with no physical dimension. If one takes seriously the presence of the thermal state associated with the field ϕ on the event horizon of a Schwarzschild black hole, one expects that the state is in thermal equilibrium with Hartle-Hawking state in the bulk and thus the temperature $1/\beta$ must coincide with Hawking's one: $\kappa/(2\pi)$, $\kappa = \hbar/(GM)$ being the surface gravity of the black hole of mass M. This select a preferred unit for the coordinate v. That coordinate has to be thought as the limit (towards the portion of future Killing horizon in the future of the bifurcation surface) of the analogous null coordinate defined in the right Schwarzschild wedge using the radial coordinate and the Schwarzschild time t (which approaches Minkowski time asymptotically at spatial infinity). In this case the coordinate v must be defined as

$$v(\theta) := \frac{1}{\kappa} \ln \left| \tan \frac{\theta}{2} \right| ,$$

and the Hamiltonian H_ζ must be rescaled correspondingly.

5.2. *Non commutative coordinates and some final comments.* Another interesting point is the following. Still assuming ζ real, one has for the field $\hat{\phi}_\zeta$

$$\lambda_\zeta(\hat{\phi}_\zeta(\theta, s)) = 2\zeta(s)v(\theta) .$$

Considering the simplest case $2\zeta = c$ constant (this is allowed for $\zeta \in L^2(\Sigma, d\Sigma)$ if Σ is compact as for the Kruskal manifold) one has, if $x := cv$

$$\lambda_\zeta(\hat{\phi}_\zeta(\theta, s)) = x(\theta) .$$

And thus the mean value of $\hat{\phi}_\zeta$ with respect to λ_ζ defines a coordinate on $\mathbb{F}_+$ along light rays. Obviously $\lambda_\zeta(\hat{\phi}_\zeta(\theta, s))$ is not well defined and it must be thought as the weak limit of a sequence $\lambda_\zeta(\hat{\phi}_\zeta(\omega_n))$ where the forms ω_n regularize Dirac' s delta centered in $(\theta, s) \in \mathbb{F}$. Actually all that is not completely correct because one has to take into account that the allowable forms have the shape $\omega_n(\theta, s) = \frac{\partial f_n(\theta,s)}{\partial \theta} d\theta \wedge d\Sigma$ were f_n is *periodic* in θ. It is not possible to produce a regularization sequence for $\delta(s, s')\frac{\partial \delta(\theta'-\theta)}{\partial \theta} d\theta \wedge d\Sigma$ in this way due to periodic constraint. The drawback can easily be skipped by fixing an origin x_0 (corresponding to some θ_0) for the coordinate x. In other words one considers a sequence of forms $\omega_n^{(\theta,s)}$ induced by smooth θ-periodic functions $f_n^{(\theta,s)}(\theta') = \delta_n(s' - s) \left[\Theta_n(\theta - \theta') + \Theta_n(\theta' - \theta_0)\right]$, where $\{\delta_n(s')\}$ regularize $\delta(s')$ and $\{\Theta_n(\theta')\}$ regularize the step distribution whose derivative is just $\delta(\theta')$. In weak sense

$$\lim_{n\to+\infty} \lambda_\zeta(\hat{\phi}_\zeta(\omega_n^{(\theta,s)})) = x(\theta) - x_0 .$$

So, up to the choice of the origin, the mean value of the field $\hat{\phi}_\zeta$ define a *classical coordinate* in the physical region $\mathbb{F}_+$. This suggest to interpret $\hat{\phi}_\zeta$, or more precisely the abstract field operator ϕ, as a *noncommutative coordinate* on $\mathbb{F}_+$. Noncommutativity arises from canonical commutation relations $[\phi(\theta, s), \phi(\theta', s')] = iE(\theta, s, \theta', s')$ enjoyed by the *-algebra generated by the field operator ϕ. Commutativity is restored under the choice of an appropriate state on that *-algebra. A recent remarkable application of some ideas of noncommutative geometry to conformal net theory and black holes appears in [32].

With a pair of fields ϕ one defined on $\mathbb{F}$ and the other defined on the past event horizon $\mathbb{P}$ we may define, through the outlined way, global null coordinates in the complete r, t section of right Schwarzschild wedge. A subsequent questions arises: In addition to the null coordinates in the plane r, t, is it possible to give a quantum interpretation to the transverse coordinate and the whole metric of the manifold using the QFT on $\mathbb{F}$ (and $\mathbb{P}$)?

The presented results could lead to an interesting scenario which deserves future investigation. The Kruskal spacetime could be a classical object arising by spontaneous breaking of $SL(2,\mathbb{R})$ symmetry as well as Bose-Einstein condensation due to a state of a local QFT defined on a certain conformal net.

A Appendix

A.1. Fock representation and GNS theorem. The interplay of the Fock representation presented in Section 3 and GNS theorem [13, 33] is simply sketched. Using notation introduced therein, if $\Pi : \mathcal{W}(\mathbb{M}) \to \hat{\mathcal{W}}(\mathbb{M})$ denotes the unique (Ω being nondegenerate) C^*-algebra isomorphism between those two Weyl representations, it turns out that $(\mathfrak{F}_+(\mathcal{H}), \Pi, \Psi)$ is the GNS triple associated with a particular pure algebraic state λ (*quasifree* [33, 17] and invariant under the automorphism group associated with ∂_θ) on $\mathcal{W}(\mathbb{M})$ we go to introduce. Define

$$\lambda(W(\psi)) := e^{-\langle\psi_+,\psi_+\rangle/2}$$

then extend λ to the *-algebra finitely generated by all the elements $W(\psi)$ with $\psi \in \mathcal{S}(\mathbb{M})$, by linearity and using (W1), (W2). It is simply proved that, $\lambda(\mathbb{I}) = 1$ and $\lambda(a^*a) \geq 0$ for every element a of that *-algebra so that λ is a state. As the map $\mathbb{R} \ni t \mapsto \lambda(W(t\psi))$ is continuous, known theorems [27] imply that λ extends uniquely to a state λ on the complete Weyl algebra $\mathcal{W}(\mathbb{M})$. On the other hand, by direct computation, one finds that $\lambda(W(\psi)) = \left\langle \Psi, \hat{W}(\psi)\Psi \right\rangle$. Since a state on a C^* algebra is continuous, this relation can be extended to the whole algebras by linearity and continuity and using (W1), (W2) so that a general GNS relation is verified:

$$\lambda(a) = \langle\Psi, \Pi(a)\Psi\rangle \quad \text{for all } a \in \mathcal{W}(\mathbb{M})\,. \tag{42}$$

To conclude, it is sufficient to show that Ψ is cyclic with respect to Π. Let us show it. If $\hat{\mathcal{F}}(\mathbb{M})$ denotes the *-algebra generated by field operators $\Omega(\psi, \hat{\phi})$, $\psi \in \mathcal{S}(\mathbb{M})$, defined on $F(\mathcal{H})$, $\hat{\mathcal{F}}(\mathbb{M})\Psi$ is dense in the Fock space (see proposition 5.2.3 in [12]). Let $\Phi \in \mathfrak{F}_+(\mathcal{H})$ be a vector orthogonal to both Ψ and to all the vectors $\hat{W}(t_1\psi_1)\cdots\hat{W}(t_n\psi_n)\Psi$ for $n = 1, 2, \ldots$ and $t_i \in \mathbb{R}$ and $\psi_i \in \mathcal{S}(\mathbb{M})$. Using Stone theorem to differentiate in t_i for $t_i = 0$, starting from $i = n$ and proceeding backwardly up to $i = 1$, one finds that Φ must also be orthogonal to all of the vectors $\Omega(\psi_1, \hat{\phi})\cdots\Omega(\psi_n, \hat{\phi})\Psi$ and thus vanishes because $\hat{\mathcal{F}}(\mathbb{M})\Psi$ is dense. This result means that $\Pi(\mathcal{W}(\mathbb{M}))\Psi$ is dense in the Fock space too, i.e. Ψ is cyclic with respect to Π. Since Ψ satisfies also (42), the uniqueness of the GNS triple proves that the triple $(\mathfrak{F}_+(\mathcal{H}), \Pi, \Psi)$ is just (up to unitary transformations) the GNS triple associated with λ. Since the Fock representation is irreducible, λ is pure.

A.2. Proofs of some theorems.
Proof of Proposition 2.2. Let us use the angular coordinate $\theta \in (-\pi, \pi]$ to describe $\mathbb{S}^1$. Assume the condition (a) holds. We can write $\omega = \epsilon_f$ and $\omega' = \epsilon_{f'}$ for some functions $f, f' \in \mathbb{C}^\infty(\mathbb{S}^1 \times \Sigma; \mathbb{C})$. By hypotheses f' is constant in the variable θ in $I \times \Sigma$ since $\frac{\partial f'(\theta,s)}{\partial\theta} = 0$ therein and $I \times \Sigma$

is connected by paths with s constant. Moreover, if t, t' are the extreme points of I, it must hold $f(t,s) = f(t',s)$ for every $s \in \Sigma$. Indeed $\frac{\partial f(\theta,s)}{\partial\theta} = 0$ vanishes outside $I \times \Sigma$ – and thus f is constant in θ in that set as before – and f is periodic in θ at s fixed by hypotheses. To use these facts we notice that, in the general case, it holds $\Omega(f, f') = E(\epsilon_f, \epsilon_{f'})$, so that, integrating by parts the expression of Ω (1),

$$\Omega(f, f') = 2\int_\Sigma d\Sigma(s) \int_{\mathbb{S}^1} f'(\theta,s)\frac{\partial f}{\partial\theta}(\theta,s)d\theta \;= 2\int_\Sigma d\Sigma(s) \int_I f'(\theta,s)\frac{\partial f}{\partial\theta}(\theta,s)d\theta\,.$$

f' is constant in θ in $I \times \Sigma$ and $f(t',s) = f(t,s)$, t, t' being the extreme points of I, so that

$$\frac{1}{2}\Omega(f, f') = \int_I f'(\theta,s)\frac{\partial f}{\partial\theta}(\theta,s)d\theta = f'(s)\int_I \frac{\partial f}{\partial\theta}(\theta,s)d\theta = f'(s)(f(t',s) - f(t,s)) = 0\,.$$

Now suppose that (b) holds true. In this case one has

$$i\Omega(f, f') = 2\int_\Sigma d\Sigma(s) \int_{\mathbb{S}^1} f'(\theta,s)\frac{\partial f}{\partial\theta}(\theta,s)d\theta \;= 2\int_S d\Sigma(s) \int_{\mathbb{S}^1} f'(\theta,s)\frac{\partial f}{\partial\theta}(\theta,s)d\theta\,.$$

Since $\frac{\partial f'(\theta,s)}{\partial\theta} = 0$ in the set $\mathbb{S}^1 \times S$ which is connected by paths with s constant, f' does not depend on θ in that set and thus

$$\frac{1}{2}\Omega(f, f') = 2\int_S d\Sigma(s) f'(s) \int_{\mathbb{S}^1} \frac{\partial f}{\partial\theta}(\theta,s)d\theta = 2\int_S d\Sigma(s) f'(s) = 0\,.$$

Finally (W2) or equivalently (V2) entails the thesis. $\Box$

Proof of Theorem 3.1. The operator $L := K^2 + S^2 + D^2$ is essentially selfadjoint on $F(\mathcal{H})$ since the dense invariant space $F(\mathcal{H})$ is made of analytic vectors. The proof is straightforward by direct estimation of $||L^n\Psi||$ with $\Psi \in F(\mathcal{H})$ (there is a constant $C_\Psi \geq 0$ with $||L^n\Psi|| \leq C^n_\Psi$). As a consequence of some results by Nelson (Theo. 5.2, Cor. 9.1, Lem. 9.1 and Lem. 5.1 in [34]) the Hermitean operators $iR(x)$ with $x \in sl(2,\mathbb{R})$ are essentially selfadjoint on $F(\mathcal{H})$ and there is a unique strongly-continuous representation $\widetilde{SL(2,\mathbb{R})} \ni g \mapsto U(g) : \mathfrak{F}_+(\mathcal{H}) \to \mathfrak{F}_+(\mathcal{H})$ such that (25) holds true.
(a) k generates the one-parameter subgroup $\mathbb{S}^1$ in $SL(2,\mathbb{R})$ – that is $\mathbb{R} \ni t \mapsto \exp(tk)$ with period 4π – as well as the one-parameter subgroup $\mathbb{R} \ni t \mapsto l(t)$ isomorphic to $\mathbb{R}$ in $\widetilde{SL(2,\mathbb{R})}$. From the general theory of $\widetilde{SL(2,\mathbb{R})}$ representations, a representation $\widetilde{SL(2,\mathbb{R})} \ni g \mapsto V(g)$ is in fact a representation of $SL(2,\mathbb{R})$ if $t \mapsto V(l(t))$ has period $4\pi/k$ for some integer $k \neq 0$. It is simply proved that the operator K is the tensorialization of the operator defined on $\ell^2(\mathbb{C}) \otimes L^2(\Sigma, d\Sigma)$ by extending

$$\{C_n\}_{n=1,2,\cdots} \otimes u_j \mapsto \{nC_n\}_{n=1,2,\cdots} \otimes u_j$$

by linearity. As a consequence the spectrum of $\overline{K}$ is the set $\sigma(\overline{K}) = \{0, 1, 2, \ldots\}$ where the eigenspace with eigenvalue 0 is one-dimensional and it is generated by the vacuum state Ψ.

This implies that $\mathbb{R} \ni t \mapsto e^{it\overline{K}} = U(l(t))$ has period 2π. As a first consequence U is a proper representation of $SL(2,\mathbb{R})$. Furthermore, since $\sigma(\overline{K})$ is nonnegative, the representation is a positive-energy representation. Finally, notice that $-I = e^{2\pi k}$ and thus $U(-I) = e^{i2\pi\overline{K}} = I$ and so U is a representation of $PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\pm I$.
(b) and (c). From direct inspection one sees that the operators K, S, D are tensorializations of the respective operators $K\restriction_{\mathcal{H}}, S\restriction_{\mathcal{H}}, D\restriction_{\mathcal{H}}$, in particular their restriction to the space generated by the vacuum vector coincide with the operator 0. Moreover, decomposing $\mathcal{H} = \ell^2(\mathbb{C})\otimes L^2(\Sigma, d\Sigma)$, one finds

$$K\restriction_{\mathcal{H}} = K_0 \otimes 0\,, \quad S\restriction_{\mathcal{H}} = S_0 \otimes 0\,, \quad D\restriction_{\mathcal{H}} = D_0 \otimes 0\,,$$

where K_0, S_0, D_0 are obtained by restricting to the one-particle space the operators K, S, D defined in the case $\mathbb{M} = \mathbb{S}^1$ (without transverse manifold). Using again Nelson results these operators give rise to a representation $\widetilde{SL(2,\mathbb{R})} \ni g \mapsto V(g)\otimes I$ in $\mathcal{H}$. (This representation is, in fact, an irreducible representation of $SL(2,\mathbb{R})$, see [7].) By tensorialization this representation extends to a representation U' in the whole Fock space. By construction, the generators iK', iS', iD' of this representation ad associated with k, s, d respectively coincides with iK, iS, iD on $F(\mathcal{H})$ respectively. Nelson's uniqueness property implies that $U' = U$. By construction U $(= U')$ admits every space with finite number of particles as invariant space, including the space with zero particles spanned by the vacuum state.
(d) First of all, as said above, U leaves invariant the space generated by the vacuum vector Ψ so that it is an invariant vector up to a phase. Let us show that this is the only unit vector with this property. By (b), the operator $\overline{D}$ is the tensorialization of $\overline{D_0}\oplus I = \overline{D_0 \oplus I}$ where the generator of V, $\overline{D_0}$, is defined on the one-particle space in the case of the absence of Σ, $\ell^2(\mathbb{C})$, and I acts on $L^2(\Sigma, d\Sigma)$. In [6, 7] the representation V has been studied, realized, under a suitable Hilbert space isomorphism, in the space $L^2(\mathbb{R}^+, dE)$. In that space $\overline{D_0}$ is the closure of the essentially-selfadjoint operator $-i(Ed/dE + 1/2)$. The original dense, invariant domain of $-i(Ed/dE+1/2)$ is a core for D_0 made of smooth functions on $(0,+\infty)$ (see [6] for details) of the form $\sqrt{E}e^{-\beta E}P(E)$ with $\beta > 0$ a constant not depending on the considered function and P any polynomial. Under the unitary transformation U, which takes the form $(U\psi)(x) := (2\pi)^{-1/2}\int_0^{+\infty} e^{-ix\ln E}\psi(E)/\sqrt{E}dE$ on the domain of $-i(Ed/dE+1/2)$, this operator becomes the operator position X (i.e $(X\psi)(x) = x\psi(x)$) on $L^2(\mathbb{R}, dx)$ restricted to a core contained in the Schwartz space. As a consequence $\sigma(\overline{D_0}) = \sigma_c(\overline{D_0}) = \sigma(X) = \mathbb{R}$ and, similarly, $\sigma(\overline{D_0\oplus I}) = \sigma_c(\overline{D_0 \oplus I}) = \mathbb{R}$. Therefore, passing to the tensorialization, $\sigma(\overline{D}) = \mathbb{R}$ and $\sigma_p(\overline{D}) = \{0\}$ with, up to phases, unique eigenvector given by the vacuum vector Ψ. If Φ is a unit vector which is up-to-phases invariant under U, it must be in particular $e^{it\overline{X}}\Phi = u_X(t)\Phi$ where X is any real linear combination of K, S, D and $|u_X| = 1$. As the domain of X is dense, it contains a vector Φ' with $\langle\Phi', \Phi\rangle \neq 0$ and thus $u_X(t) = \langle e^{-it\overline{X}}\Phi', \Phi\rangle/\langle\Phi', \Phi\rangle$ is differentiable at $t = 0$ by Stones' theorem. As a consequence, the left-hand side $e^{it\overline{X}}\Phi = u_X(t)\Phi$ must be differentiable at $t = 0$. By Stone theorem Φ belongs to the domain of $\overline{X}$ and it holds $\overline{X}\Phi = \lambda_X\Phi$ where $\lambda_X = -idu_X/dt|_{t=0}$. Specializing the identity to $X = D$, from the spectral structure of $\overline{D}$, one concludes that it must be $\lambda_D = 0$ and, up to phases, $\Phi = \Psi$. □

Proof of Theorem 3.2. (a) and (b). To establish (26) it is sufficient to prove those identities for $w = \hat{V}(\omega)$ with $\omega \in \mathcal{D}(\mathbb{M})$ and $g \in PSL(2,\mathbb{R})$. Actually, with the said choices for w

$$U(g)\, a\, U^\dagger(g) \quad = \quad \alpha'_g(a)\,, \quad \text{for all } a \in \hat{\mathcal{F}}(\mathbb{M})\,. \tag{43}$$

implies (26). For if (43) holds, taking the adjoint twice for both sides one gets the relations for selfadjoint field operators $U(g)\,\overline{\hat{\phi}(\omega)}\,U^\dagger(g) = \overline{\hat{\phi}(\omega^{(g^{-1})})}$. Then (10) implies (26) for $w = \hat{V}(\omega)$ via standard spectral theory. To conclude the proof of (a) it is now sufficient to show the validity of (43) with $a = \hat{\phi}(\omega)$ or of the equivalent statement

$$U(g)\,\Omega(\psi,\hat{\phi})\,U^\dagger(g) = \Omega(\psi^{(g^{-1})},\hat{\phi})\,, \quad \text{for all } \psi \in \mathcal{S}(\mathbb{M}) \text{ and } g \in PSL(2,\mathbb{R})\,. \tag{44}$$

In turn, using the fact that U preserve the vacuum vector and is the tensorialization of $U\restriction_{\mathcal{H}}$ (theorem 3.1) as well as (9) one sees that (44) is equivalent to

$$\psi^{(g)} = U(g^{-1})\restriction_{\mathcal{H}} \psi_+ + \overline{U(g^{-1})\restriction_{\mathcal{H}} \psi_+}\,, \quad \text{for all } \psi \in \mathcal{S}(\mathbb{M}) \text{ and } g \in PSL(2,\mathbb{R})\,. \tag{45}$$

Let us prove (45). If $\psi \in \mathcal{S}(\mathbb{M})$ and $g \in Diff^+(\mathbb{S}^1)$ the map $\psi \mapsto \psi^{(g)}$ induces a $\mathbb{R}$-linear map from the space of ∂_θ-positive frequency parts ψ_+ to the same space given by

$$\psi_+ \mapsto S(g)\psi_+ := ((\psi_+ + \overline{\psi_+})^{(g^{-1})})_+.$$

In this way the action of g on the wavefunction ψ is equivalent to the action of $S(g)$ on its positive frequency part ψ_+:

$$\psi^{(g^{-1})} = S(g)\psi_+ + \overline{S(g)\psi_+}\,. \tag{46}$$

However, in general, $S(g)$ is not $\mathbb{C}$-linear (and thus cannot be seen as a map $\mathcal{H} \to \mathcal{H}$) since, using $\chi_+ := i\psi_+$ above, one gets $S(g)(i\psi_+) = ((i\psi_+ - i\overline{\psi_+})^{(g^{-1})})_+ = i((\psi_+ - \overline{\psi_+})^{(g^{-1})})_+ \neq i((\psi_+ + \overline{\psi_+})^{(g^{-1})})_+ = iS(g)\psi_+$. Actually, if $g \in PSL(2,\mathbb{R})$, it turns out that $(\overline{\psi_+} \circ g^{-1})_+ = 0$ so that $S(g)\psi_+ = (\psi_+ \circ g^{-1})_+$ and S is $\mathbb{C}$-linear. This nontrivial result was proved in the remark in page 271 of [7] (the presence of the transverse manifold does not affect that proof). To conclude the proof it is sufficient to show that $S(g) = U(g)\restriction_{\mathcal{H}}$ for all $g \in PSL(2,\mathbb{R})$. Tho establish such an identity we first notice that $S(g) : \mathcal{H} \to \mathcal{H}$ is a unitary representation of $PSL(2,\mathbb{R})$. The only fact non self-evident is that $S(g)$ preserve the scalar product but it is true because, if $\chi := i\psi_+ - i\overline{\psi_+}$, it holds

$$\langle \psi_+, \psi'_+ \rangle = -i\Omega(\overline{\psi_+}, \psi'_+) = \frac{-i}{2}\left(\Omega(\psi,\psi') + i\Omega(\chi,\psi')\right)$$

now, due to (46) we can replace the arguments ψ_+, ψ'_+ by respectively $S(g)\psi_+, S(g)\psi'_+$ and the arguments ψ, ψ', χ by $\psi^{(g^{-1})}, \psi'^{(g^{-1})}, \chi^{(g^{-1})}$ respectively, obtaining a similar identity; finally, since the action of positive-oriented diffeomorphisms of $\mathbb{S}^1$ preserves the symplectic form, one has $\Omega(\psi^{(g^{-1})}, \psi'^{(g^{-1})}) + i\Omega(\chi^{(g^{-1})}, \psi'^{(g^{-1})}) = \Omega(\psi,\psi') + i\Omega(\chi,\psi')$ and thus $\langle S(g)\psi_+, S(g)\psi'_+ \rangle =$

$\langle\psi_+,\psi'_+\rangle$. To conclude the proof it is sufficient to notice that, by direct inspection making use of Stone theorem one finds [7] that, if $\psi_{np} = \{\delta_{np}\}_{p=1,2,\ldots}\otimes u_j \in \ell^2(\mathbb{C})\otimes L^2(\Sigma, d\Sigma) = \mathcal{H}$

$$iX\psi_{np} = \frac{d}{dt}S(exp(tx))\psi_{np}$$

where $X = K, S, D$ and, respectively, $x = k, s, d$ (k, d, s being the basis of $sl(2,\mathbb{R})$ introduced above). On the other hand the same result holds, by construction, for the representation $U\restriction_{\mathcal{H}}$

$$iX\psi_{np} = \frac{d}{dt}U(exp(tx))\psi_{np}\,.$$

Since the elements ψ_{np} span a dense space of analytic vectors for $K\restriction^2_{\mathcal{H}} + S\restriction^2_{\mathcal{H}} + D\restriction^2_{\mathcal{H}}$, by the results by Nelson cited in the proof of theorem 3.1, $S = U\restriction_{\mathcal{H}}$. Now (46) implies (45) and this concludes the proof.
(c) It is an immediate consequence of (a) and the fact that the GNS representative of λ, Ψ, is invariant under U as stated in (d) of theorem 3.1. □

Proof of Theorem 4.1. (a) Consider the closure $\mathcal{W}_\zeta(\mathbb{M})$ of the *-algebra of in $\mathcal{W}(\mathbb{M})$ spanned elements $V_\zeta(\omega) := V(\omega)e^{i\int_{\mathbb{M}}\Gamma(\zeta\omega_+ + \overline{\zeta\omega_+})}$ with $\omega \in \mathcal{D}(\mathbb{M})$. Obviously the obtained C^*-algebra coincides with $\mathcal{W}(\mathbb{M})$ itself. On the other hand its generators $V_\zeta(\omega)$ satisfy (V1) and (V2) and thus, by theorem 5.8.8 in [12] there is a unique *-isomorphism $\gamma_\zeta : \mathcal{W}(\mathbb{M}) \to \mathcal{W}_\zeta(\mathbb{M}) = \mathcal{W}(\mathbb{M})$ with $\gamma_\zeta(V(\omega)) = V(\omega)e^{i\int_{\mathbb{M}}\Gamma(\zeta\omega_+ + \overline{\zeta\omega_+})}$. Finally, by construction $\lambda(\gamma_\zeta(V(\omega))) = \lambda_\zeta(V(\omega))$ and thus, linearity and continuity imply (30). Let us proof (32). Due to linearity and continuity, it is sufficient to show the validity of the relation when restricting to elements $V_\zeta(\omega)$. In turn, since $V(\omega)$ is invariant under $g_t := \exp(t\mathcal{D})$ and using lemma 4.1, the validity of (32) for those elements is a consequence of the invariance of the integral $\int_{\mathbb{M}}\zeta\Gamma\omega_+$ under the action of g_t^* on the argument ω_+ which we go to prove. If $\mathcal{D}(\mathbb{M}) \ni \omega = \frac{\partial f(\theta,s)}{\partial\theta}d\theta\wedge d\Sigma(s)$ and defining $\theta_\pm(v) = \pm 2\tan^{-1}(e^v)$, direct computation yields:

$$\int_{\mathbb{M}}\zeta\Gamma\,\omega_+ = -\lim_{N\to+\infty}\int_{-N}^{N}dv\int_\Sigma d\Sigma(s)\zeta(s)\left[f_+(\theta_+(v),s) - f_+(\theta_-(v),s)\right] + \text{boundary terms}\,.$$

Using periodicity of f_+ in θ, boundary terms can be re-arranged into a term

$$\lim_{\Theta\nearrow\pi}\left[(\Theta-\pi)\ln\left(\left|\tan\frac{\Theta}{2}\right|\right)\int_\Sigma\zeta(s)\frac{f_+(\Theta,s) - f_+(\pi,s)}{\Theta-\pi}\,d\Sigma\right]$$

and three other similar terms where $-\pi$ or 0 replaces π. The last integral can be bounded uniformly in Θ using Lagrange theorem since $\frac{\partial f_+}{\partial\theta}$ is continuous and compactly supported. As a consequence the limit vanishes and the boundary terms can be dropped. Finally, using the fact

[7] details are very similar to those in the corresponding part of Theorem 2.4 in [7]

that v is the parameter of the integral curves of $\mathcal{D}$ one has,

$$\begin{aligned}\int_{\mathbb{M}} \zeta\Gamma\, g_t^*\omega_+ &= -\lim_{N\to+\infty}\int_{-N}^{N} dv \int_{\Sigma} d\Sigma(s)\zeta(s)\left[f_+(\theta_+(v-t),s) - f_+(\theta_-(v-t),s)\right] \\ &= -\lim_{N\to+\infty}\int_{-N+t}^{N+t} dv \int_{\Sigma} d\Sigma(s)\zeta(s)\left[f_+(\theta_+(v),s) - f_+(\theta_-(v),s)\right] = \int_{\mathbb{M}} \zeta\Gamma\,\omega_+\,,\end{aligned}$$

so that the invariance of the integral functional under $\exp(t\mathcal{D})$ is evident.
(b) Let us start from the bottom. Since λ is invariant under the whole Möbius group, invariance (noninvariance) of λ_ζ is equivalent to invariance (noninvariance) of the integral functional in the right-hand side of (30). Let us study that integral. Take $\omega(\theta,s) = \frac{\partial f(\theta)}{\partial v} g(s) d\theta \wedge d\Sigma(s)$ where s are coordinates on Σ and the real functions f and g are smooth with the latter compactly supported as well. Assume $\zeta \neq 0$ a.e. We can fix g such that $\int_\Sigma \zeta g = e^{i\alpha}$. In this case

$$\int_{\mathbb{M}} \Gamma(\zeta\omega_+ + \overline{\zeta\omega_+}) = \int_{\mathbb{S}^1} \Gamma(\theta)(e^{i\alpha}\frac{\partial f_+}{\partial\theta} d\theta + c.c.)\,.$$

As a consequence, if $\{g_t\}_{t\in\mathbb{R}}$ denotes the one-parameter subgroup of $PSL(2,\mathbb{R})$ generated by $X = (a + b\cos\theta + c\sin\theta)\partial_\theta$, with $a, b, c \in \mathbb{R}$, one has:

$$\frac{d}{dt}|_{t=0}\int_{\mathbb{M}} \Gamma(\zeta g_t^*\omega_+ + \overline{\zeta g_t^*\omega_+}) = \int_{\mathbb{S}^1}\Gamma(\theta)\left(e^{i\alpha}\frac{\partial}{\partial\theta}\left((a + b\cos\theta + c\sin\theta)\frac{\partial f_+}{\partial\theta}d\theta\right) + c.c.\right)\,.$$

The invariance of the integral implies that the left-hand must vanish no matter the choice of f:

$$\int_{\mathbb{S}^1}\Gamma(\theta)\frac{\partial}{\partial\theta}\left((a + b\cos\theta + c\sin\theta)\frac{\partial e^{i\alpha}f_+}{\partial\theta}d\theta\right) + c.c. = 0\,.$$

Using $f(\theta) := \cos(\theta-\alpha)$ one finds that it must be $a = 0$ as a consequence of the identity above. Then using $f(\theta) := \cos(2\theta - \alpha)$ one find that it must also be $b = 0$. We conclude that the integral functional is invariant at most under the group generated by $c\sin\theta\partial/\partial\theta = -c\mathcal{D}$. On the other hand the proof of such an invariance arises directly from (31) and (32) using the fact that λ is invariant under $\alpha_t^{(\mathcal{D})}$ as stated in (c) in theorem 3.3.
The fact that λ_ζ is pure (that is extremal) is an immediate consequence of (30) using the fact that γ_ζ is bijective and λ is pure. As the λ_ζ are pure their GNS representations are irreducible. Therefore the proof of the fact that λ_ζ and $\lambda_{\zeta'}$ are not quasiequivalent if $\zeta \neq \zeta'$ a.e. reduces to the proof that, if $\zeta \neq \zeta'$ a.e., there is no unitary transformation $U : \mathfrak{F}_+(\mathcal{H}_\zeta) \to \mathfrak{F}_+(\mathcal{H}_{\zeta'})$ such that $U\hat{V}_\zeta(\omega)U^{-1} = \hat{V}_{\zeta'}(\omega)$ for all $\omega \in \mathcal{D}(\mathbb{M})$. We shall make use of the first statement in (c) which will be proved independently from the following. Suppose that there is such a unitary transformation for some choice of $\zeta \neq \zeta'$. As a consequence one gets also the identity o $U\hat{V}_\zeta(\omega)e^{-i(\int_{\mathbb{M}}\zeta\Gamma\omega_+ + c.c.)}U^{-1} = \hat{V}_{\zeta'}(\omega)e^{-i(\int_{\mathbb{M}}\zeta\Gamma\omega_+ + c.c.)}$. That is, re-defining $\zeta' - \zeta \to \zeta \neq 0$, one has $Ue^{i\overline{\hat{\phi}_\zeta(\omega)}}U^\dagger = e^{i\overline{\hat{\phi}_0(\omega)}}$ where we have also identified the one-particle Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_\zeta$ with the one-particle space $\mathcal{H}$ of the GNS representation of λ (and thus the Fock

spaces). Via Stone theorem (using above $\omega = t\omega$ and $t \in \mathbb{R}$) one gets $U\overline{\hat{\phi}_\zeta(\omega)} = \overline{\hat{\phi}_0(\omega)}U$, that is $i\,U\overline{\alpha(\overline{\psi_+}) - \alpha^\dagger(\psi_+)} + (\int_{\mathbb{M}} 2\zeta\Gamma\epsilon_{\psi_+} + c.c.)U = i\,\overline{\alpha(\overline{\psi_+}) - \alpha^\dagger(\psi_+)}U$ where $\psi_+ = E\omega_+$ according with (b) in proposition 2.1. Using the analogous relation for $\psi' := i\psi_+ - i\overline{\psi_+}$ one gets in the end

$$U\Big[\overline{\alpha(\overline{\psi_+}) - \alpha^\dagger(\psi_+)} + \overline{\alpha(\overline{\psi_+}) + \alpha^\dagger(\psi_+)}\Big] - \left(4i\int_{\mathbb{M}} \overline{\zeta}\Gamma\epsilon_{\overline{\psi_+}}\right)U = \Big[\overline{\alpha(\overline{\psi_+}) - \alpha^\dagger(\psi_+)} + \overline{\alpha(\overline{\psi_+}) + \alpha^\dagger(\psi_+)}\Big]U.$$

Applying both sides to the vacuum state Ψ_ζ and computing the scalar product of the resulting vectors with Ψ_ζ itself, the identity above implies that

$$-\left(2i\int_{\mathbb{M}} \overline{\zeta}\Gamma\epsilon_{\overline{\psi_+}}\right)\langle \Psi_\zeta, U\Psi_\zeta\rangle = \langle \alpha^\dagger(\psi_+)\Psi_\zeta, U\Psi_\zeta\rangle .t$$

If $\{\psi_{+m}\}_{m\in\mathbb{N}'}$ is a Hilbert base of $\mathcal{H}_\zeta$, iteration of the procedure sketched above produces

$$\langle \Psi_\zeta, U\Psi_\zeta\rangle \prod_n \frac{\lambda_m^{N_m}}{\sqrt{N_m!}} = \langle N_1, N_2, \ldots, N_m, \ldots | U\Psi_\zeta\rangle \tag{47}$$

for any vector with finite number of particles $|N_1, N_2, \ldots, N_m, \ldots\rangle$, N_m being the occupation number of the state ψ_{+m} and where $\lambda_m := -2i\int_{\mathbb{M}} \overline{\zeta}\Gamma\epsilon_{\overline{\psi_{+m}}}$. It must be $\langle \Psi_\zeta, U\Psi_\zeta\rangle \neq 0$, otherwise all components of $U\Psi_\zeta$ would vanish producing $U\Psi_\zeta = 0$ which is impossible since U is unitary. Conversely, as $||\Psi_\zeta||^2 = 1$, it must hold $||U\Psi_\zeta||^2 = 1$. This identity can be expanded with the basis of states $|N_1, N_2, \ldots, N_m, \ldots\rangle$ and a straightforward computations which employs (47) produces

$$||U\Psi_\zeta||^2 = |\langle \Psi_\zeta, U\Psi_\zeta\rangle|^2 \exp\left(\sum_{m=1}^{+\infty} |\lambda_m|^2\right) . \tag{48}$$

The series can explicitly be computed using a basis $\psi_{(n,j)}(\theta, s) = u_j(s)\frac{e^{-in\theta}}{\sqrt{4\pi n}}$ where u_j is any basis of $L^2(\Sigma, d\Sigma)$ made of compactly supported real smooth functions[8]. In that case $\int_\Sigma \overline{\zeta}u_j d\Sigma \neq 0$ for some $j = j_0$ (otherwise the function ζ on Σ would have $L^2(\Sigma, d\Sigma)$-norm zero). One finds $|\lambda_{2n+1,j_0}|^2 = C|\int_\Sigma \overline{\zeta}u_{j_0}d\Sigma|^2(2n+1)^{-1}$ with $C > 0$ so that the series in (48) diverges and the found contradiction shows that U cannot exist.
(c) By direct inspection one finds that the operators $V_\zeta(\omega)$ enjoy (V1) and (V2). Therefore, (theorem 5.2.8, in [12]) the C^*-algebra $\hat{\mathcal{W}}_\zeta(\mathbb{M})$ given by the closure of the *-algebra generated by $V_\zeta(\omega)$ is a representation of Weyl algebra and there is a *-algebra isomorphism of C^* algebras, $\Pi_\zeta : \mathcal{W}(\mathbb{M}) \to \hat{\mathcal{W}}_\zeta(\mathbb{M})$ which satisfies (33). The vacuum vector of $\mathfrak{H}_\zeta = \mathfrak{F}_+(\mathcal{H}_\zeta)$ is cyclic with

[8] The space $\mathcal{C}$ of smooth compactly supported functions on Σ is dense in $L^2(\Sigma, d\Sigma)$. As the latter is separable $\mathcal{C}$ contains a numerable subset $\mathcal{C}'$ still dense in $L^2(\Sigma, d\Sigma)$. In turn one may extract from $\mathcal{C}'$ a subset $\mathcal{C}''$ of linearly independent elements which span the same dense space as $\mathcal{C}'$. Usual orthonormalization procedure applied to $\mathcal{C}''$ gives a Hilbert basis for $L^2(\Sigma, d\Sigma)$ made of smooth compactly supported functions. Proceeding as in footnote 4 one gets the wanted basis of u_j.

respect to Π_ζ because $\hat{\mathcal{W}}_\zeta(\mathbb{M})\Psi_\zeta$ is the same space as the dense space (see A.1) spanned by vectors $e^{i\overline{\hat{\phi}(\omega_1)}}\cdots e^{i\overline{\hat{\phi}(\omega_n)}}\Psi_\zeta$, $n =, 1, 2, \ldots$. Finally it holds

$$\lambda_\zeta(V(\omega)) = \lambda(V(\omega))e^{i(\int_{\mathbb{M}}\zeta\Gamma\omega_+ + c.c.)} = \langle\Psi_\zeta, e^{i\overline{\hat{\phi}(\omega)}}\Psi_\zeta\rangle e^{i(\int_{\mathbb{M}}\zeta\Gamma\omega_+ + c.c)} = \langle\Psi_\zeta, e^{i(\overline{\hat{\phi}(\omega)+\int_{\mathbb{M}}\zeta\Gamma\omega_+ + c.c.})}\Psi_\zeta\rangle$$

$$= \langle\Psi_\zeta, \hat{V}_\zeta(\omega)\Psi_\zeta\rangle\ ,$$

that is $\lambda_\zeta(V(\omega)) = \langle\Psi_\zeta, \Pi_\zeta(V(\omega))\Psi_\zeta\rangle$. By linearity and continuity this relation extends to the whole algebras: $\lambda_\zeta(w) = \langle\Psi_\zeta, \Pi_\zeta(w)\Psi_\zeta\rangle$, $w \in \mathcal{W}(\mathbb{M})$. We conclude that $(\mathfrak{F}_+(\mathcal{H}_\zeta), \Pi_\zeta, \Psi_\zeta)$ is the (unique, up to unitary transformations) GNS triple for λ_ζ.
(d) Let us denote by $\{g_t\}_{t\in\mathbb{R}}$ the one-parameter group of Möbius transformations generated by $\mathcal{D}$. The statements (a) and (b) in theorem 3.2 imply that if D is defined as $(1/2i) :\Omega(\hat{\phi}_0, \mathcal{D}(\hat{\phi}_0))$: then $e^{itD}e^{i\overline{\hat{\phi}_0(\omega)}}e^{-itD} = e^{i\overline{\hat{\phi}_0(g_t^{-1*}\omega)}}$. Since $\int_{\mathbb{M}}\zeta\Gamma\omega_+ + c.c.$ is invariant under the action of g_t on ω as seen in the proof of (a), we have also

$$e^{itD}e^{i\overline{\hat{\phi}_0(\omega)}}e^{i(\int_{\mathbb{M}}\zeta\Gamma\omega_+ + c.c.)}e^{-itD} = e^{i\overline{\hat{\phi}_0(g_t^{-1*}\omega)}}e^{i(\int_{\mathbb{M}}\zeta\Gamma g_t^{-1*}\omega + c.c.)}$$

that can be rewritten as $e^{itD}\hat{V}_\zeta(\omega)e^{-itD} = \hat{V}_\zeta(\omega^{(g_t^{-1})})$ and thus extends to the whole Weyl algebra proving (34). $\square$

Acknowledgments.

Concerning the part of this work due to N.P., it has been funded by Provincia Autonoma di Trento within the postdoctoral project FQLA, Rif. 2003-S116-00047 Reg. delib. n. 3479 & allegato B.

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