Computers & Operations Research 167 (2024) 106666

Contents lists available at ScienceDirect

Computers and Operations Research

journal homepage: www.elsevier.com/locate/cor

A robust ordered weighted averaging loss model for portfolio optimization

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ARTICLE INFO

Keywords:
Conditional value-at-risk
Mean-risk models
Minimax regret models

ABSTRACT

In this paper we will propose a Robust Ordered Weighted Averaging (ROWA) optimization model to find a portfolio according to different attitudes towards risk of a decision maker. The rationale of our model is supported by the idea of measuring risk through conditional means of losses from a database of past returns. The way in which these means of extreme losses affect to the final decision may be different according to the risk perception of the decision maker (*scenarios*). In this context, a compromise portfolio is identified to reconcile the admisible risk attitudes of a decision maker according to different paradigms. We will also link the *robustness* of the proposed solution with its *efficiency* from a multi-criterion decision making viewpoint (*Pareto optimality*). Both concepts have been connected previously in the literature in different contexts. The paper ends with an extensive numerical experiment in order to check the applicability of our model to real data on six financial markets.

1. Introduction

Nowadays, shortfall or quantile risk measures are the basis of an established tool to monitor and *control* the existing risk in financial and logistic applications (see Guastaroba et al., 2020 and the references therein). A typical approach would be modeling the decision problem as a bi-criteria optimization where an efficiency measure, such as expected profits, is combined with a risk measure, as the expected loss in adverse scenarios. One of these shortfall measures is the CVaR, the *Conditional Value at Risk*. In Rockafellar and Uryasev (2000), Rockafellar and Uryasev proposed to minimize the CVaR for general distributions of losses, while at least a given level in the expected return is required. Later, Uryasev together with Krokhmal and Palmquist analyzed in Krokhmal et al. (2002) the efficient frontier of the bicriterion optimization problem in which CVaR and expected return are the objectives for any feasible portfolio. Both papers used the *Value at Risk* (or VaR) and the CVaR, for absolutely continuous statistical distributions of losses. In Rockafellar and Uryasev (2002) the authors extended their definitions to the case of general distributions of losses with possible discontinuities. Basically, the value VaR is a quantile of the loss probability distribution, while the CVaR is the expectation of those losses that are greater than the VaR. As noted in Filippi et al. (2019), under empirical distributions of losses, the CVaR can be seen as a $k-sum$ operator, the average of the k-worst outcomes, where k is a positive integer chosen by the decision maker.

The CVaR approach has proved its successfulness in many aspects of portfolio optimization, but one of its possible weakness could be that the expected loss is usually calculated for just *one* conditional distribution, representing just a particular assessment of the negative events, that is, a particular attitude towards risk. An earlier effort to address this weakness was undertaken by Mansini et al. in Mansini et al. (2007) in the context of a portfolio model. In order to infer robustness features to the portfolio selection, the authors proposed in Mansini et al. (2007) to aggregate more than one CVaR constraints, specified by different tolerance levels, by means of a linear combination defined by the decision maker.

Another natural approach could be to consider a family of distributions or weights in order to model different *degrees* of aversion towards risk. This technique is particularly interesting when the decision maker has uncertainty about the evolution of the economic conditions under which its investment policy will be assessed or when several managers, with different attitudes towards risk, are involved in the decision process. An application of this methodology was proposed by Zhu and Fukushima in Zhu and Fukushima (2009) where the *worst-case* CVaR criterion under a set of possible probability distributions of losses (mixture distribution uncertainty, box uncertainty and ellipsoidal uncertainty) was studied. In this paper it is shown through numerical experiments that, in comparison with the original CVaR, the portfolio selection model using the worst-case CVaR as the risk measure performs robustly in practice and provides more flexibility in portfolio decision analysis. The first two families of the above probability distributions of losses have also been recently studied in the context of VaR (Sehgal et al., 2023) and *Omega ratio* optimization (Sharma et al., 2017).

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https://doi.org/10.1016/j.cor.2024.106666
Received 17 April 2023; Received in revised form 25 January 2024; Accepted 11 April 2024
Available online 17 April 2024

The idea of finding a robust portfolio under a given family of statistical distributions for the returns is also studied by Kang and Li in Kang and Li (2018). In their paper, the authors proposed the choice of a *minimax* strategy for a given set of probability measures of the returns that generalizes the robust mean–variance model of Garlappi et al. (2007), the robust maximum return model of Pinar (2016), and the robust minimum VaR and CVaR models of Paç and Pinar (2014). In the first of these papers, Kang and Li (2018), the authors used the so-called *ambiguity* concept to denote the situation in which there exists a family of probability distributions for the returns, all of them compatible with the available information, where the decision maker has almost no possibility to single out the true distribution. When this family is defined by a set of probability distributions with fixed means and covariances (statistically inferred by the historical data) they derive a closed-form expression for the optimal portfolio strategy of the minimax mean-multiple risk portfolio.

There are applications of this methodology outside the portfolio selection problem. For instance, in Asimit et al. (2017), different approaches modeling insurance contracts are considered under VaR and CVaR indicators as risk measures. In that paper, the authors assumed the existence of a discrete family of probability functions modeling possible economic outcomes and proposed two decision rules, the *minimax* CVaR and *minimax regret* VaR, to find robust investments. Both papers analyzed *economically acceptable* solutions in the sense of Pareto optimality. In Xidonas et al. (2017) another minimax regret model was proposed to find a robust portfolio by assessing the Mean Absolute Deviation of the returns, as a risk measure to be minimized and the expected portfolio return, as an objective to be maximized under a set of future economic scenarios.

In our paper, we combine some of these ideas in order to propose a robust (*minimax regret*) solution. We assess each feasible portfolio by using an *Ordered Weighted Averaging* (OWA), Hajjami and Amin (2018), of its losses where each weight is allocated to a specific position within the ordered sequence of losses in order to aggregate them. For instance, if just one nonzero weight is considered to the maximum portfolio loss then, the corresponding OWA operator leads to the minimax solution. Each OWA operator plays the role of a risk aversion measure for a specific decision maker *profile* and the minimax regret portfolio will represent a *compromise* investment policy that tries to conciliate all these different risk attitudes of the team of managers.

The approach of modeling risk attitudes through OWA operators has been previously proposed in stock selection in Hajjami and Amin (2018), to model a portfolio problem as a preference voting system under two different risk attitudes, one of them corresponding to a risky investor whose goal is to select stocks with the highest returns and another more pessimistic attitude, corresponding to a creditor whose goal is to maximize the repayment ability. However, to the best of our knowledge (see Filippi et al., 2020; Ghahtarani et al., 2022), the OWA model from a minimax regret perspective based on multiple CVaR measures corresponding to different distributions of weights has not been previously studied.

The contributions of this paper are multi-fold:

- We propose the Robust Ordered Weighted Averaging or ROWA model as a new method (see Ghahtarani et al., 2022) to select a portfolio by aggregating different risk attitudes with a minimax regret criterion. It is shown how this approach covers different selection models proposed in the literature.
- We use the convexity properties of the proposed indicators and exploit the duality relations between the optimization programs to formulate an operational Linear Programming model in order to find a robust portfolio. This formulation primarily relies on the structural characteristics of the set of extreme weights allowed by the ROWA model in use.
- Two different ROWA paradigms are studied by using the framework provided by this formulation, the first one models multiple tolerance levels in CVaR and the other one defines what is called a robust *orness* portfolio selection.
- Some relationships between the ROWA portfolio and weakly efficient solutions are derived enabling us to link our model with multiple criteria CVaR formulations proposed previously in the literature.

In the next section, a general mathematical program is introduced in order to optimize an OWA measure on a given set of feasible solutions. In Section 3, a specific ROWA formulation is derived under linear functions of the losses incurred by a given portfolio. This formulation is applied to two different ROWA models by characterizing the set of extreme weights allowed by each approach. As we will see in Section 4, the proposed robust portfolio has theoretical properties in terms of a multi-criteria problem which will allow us relating our ROWA portfolio in terms of its efficiency for a given set of individual CVaR measures. Finally, in Section 5 numerical experiments are carried out in order to check the applicability of our model to financial markets by using real daily stock returns.

2. A minimax regret OWA loss model

Ordered weighted averaging operators are cost functions that have been frequently used in Location Theory (see Aouad and Segev, 2019; Ogryczak and Olender, 2016 and its references). In that field, OWA operators have been proposed to penalize the coverage distance of each demand point by a multiplicative weight, depending on its ranking (or percentile) in the ordered list of distances. In this way, location concepts as medians or centers, can be seen as optimal solutions for specific cases of OWA operators. Each one of these OWA operators transfers different features to the solution in terms of efficiency or coverage of the located service.

When OWA operators are applied in order to select an optimal solution from a given set X of feasible portfolios, costs (losses or negative returns) play the role of distances in the above problem, so that performance measures like return averages, CVaR's, worst or best returns or quantile measures as the VaR can be related to given choices of the weights defining these operators. In the following, we will focus on OWA operators describing CVaR's.

Let us consider the general problem (1) of minimizing the k-largest values of n cost functions,

$$z^* = \quad \min \sum_{i=n-k+1}^{n} g_{(i)}(x) \quad \text{s.t.} \quad x \in X, \tag{1}$$

where the cost functions for a given solution x are denoted by $g_j(x)$, $j \in \{1,\dots,n\}$ and it is supposed these functions are continuous on a compact set X of feasible portfolios warranting, this way, the existence of a minimum in (1). The notation $g_{(i)}$, with a subindex between parentheses, is used in (1) to fix the cost component which occupies the ith position in the non-decreasing ordered sequence of costs $g_j(x)$, that is,

$$g_{(1)}(x) \le g_{(2)}(x) \le \cdots \le g_{(n)}(x), \tag{2}$$

where $(i) = j$ if $g_j(x)$ occupies the ith position in that ordered sequence.

Now we will extend the above risk indicator to an objective function including specific weights whose purpose is to penalize each ordered cost. The new indicator also preserves practical properties as the so-called *minimization formula* of Rockafellar and Uryasev (2002). Let $\lambda_1, \lambda_2, \dots, \lambda_n \ge 0$ the problem (P_λ) of minimizing the ordered median loss function is defined as follows

$$z^*(\lambda) = \quad \min \sum_{i=1}^{n} \lambda_i g_{(i)}(x) \quad \text{s.t.} \quad x \in X. \tag{P_λ}$$

Note that each weight λ_i is associated to the ith position in the non-decreasing ordering of costs, independently of which index j corresponds to the cost function $g_j(x)$ occupying this position. This allows us to consider objective functions of different nature by varying λ. For example, if all the weights are zeroes except one of them taking the value 1, the objective function represents a given quantile of the sample of costs, giving place to a VaR indicator as measure of risk. In particular, for $\lambda_n = 1$, (P_λ) minimizes the maximum cost, that is, it is a *minimax* model.

Remark 1. By taking $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-k} = 0$ and $\lambda_{n-k+1} = \cdots = \lambda_n = \frac{1}{k}$ the problem (P_λ) minimizes the discrete CVaR objective function corresponding to the β-VaR, with $\beta = (n-k+1)/n$, that is, the $(n-k+1)$-th value in the ordered sequence of losses (2).

Using results from Ogryczak and Tamir (2003) one has that

$$\sum_{i=n-k+1}^{n} g_{(i)}(x) = \begin{array}{l} \min kt + \sum_{i=1}^{n} (g_i(x) - t)^+ \\ \text{s.t.} \\ t \in \mathbb{R}. \end{array} \tag{3}$$

then, the problem (1) is equivalent to

$$z^* = \begin{array}{l} \min kt + \sum_{j=1}^{n} \xi_j \\ \text{s.t.} \\ t - g_j(x) + \xi_j \geq 0,\ j \in \{1,\dots,n\}, \\ \xi_j \in \mathbb{R}_+,\ \forall j \in \{1,\dots,n\}, \\ t \in \mathbb{R},\ x \in X. \end{array} \tag{4}$$

Moreover, given an optimal solution (t^*, x^*) of the problem (4), one has that t^* achieves the value of the $(n-k+1)/n$-quantile of the sample of costs at the optimal solution x^*, that is, $t^* = g_{(n-k+1)}(x^*)$.

As proposed in Ogryczak and Tamir (2003), when $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, that is, when the weight given to a cost increases with the position of such a cost in the ordered sequence (2), one can use the formulation (3) in order to state an equivalent formulation of (P_λ) as

$$z^*(\lambda) = \begin{array}{l} \min \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1}) \left[(n-i+1)t_i + \sum_{j=1}^{n} (g_j(x) - t_i)^+ \right] \\ \text{s.t.} \\ x \in X, \\ t_i \in \mathbb{R},\ \forall i \in \{1,\dots,n\}, \end{array} \tag{5}$$

where λ_0 has been taken as zero for convenience. From now on, we will use the formulation (5), hence we will assume that $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 1$, where the last inequality is just a normalization constraint.

Now, in order to ease the reading of the last formulation, we will add the parameters δ_i, being

$$\delta_i = \lambda_i - \lambda_{i-1} \geq 0,\ i \in \{1,\dots,n\}. \tag{6}$$

Hence, using (4), the formulation (5) can be written as

$$z^*(\delta) = \begin{array}{l} \min \sum_{i=1}^{n} \left[(n-i+1)\delta_i t_i + \sum_{j=1}^{n} \delta_i \xi_{ij} \right] \\ \text{s.t.} \\ t_i - g_j(x) + \xi_{ij} \geq 0,\ i,j \in \{1,\dots,n\}, \\ x \in X, \\ t_i \in \mathbb{R},\ \forall i \in \{1,\dots,n\}, \\ \xi_{ij} \in \mathbb{R}_+,\ i,j \in \{1,\dots,n\}, \end{array} \tag{P_δ}$$

where a replica of the decision variables (t, ξ) has been added for each $i \in \{1,\dots,n\}$.

Problem (P_δ) remains tractable under convex loss functions $g_j(x)$. In particular, if these loss functions are piecewise linear convex functions the corresponding problem (P_δ) becomes a Linear Programming problem efficiently solved by off-the-shelf solvers. For instance, if

$$g_j(x) = \max\{h_{jk}(x) : k \in K_j\},$$

being $h_{jk}(x)$ linear functions, we only need replacing the constraint $t_i + \xi_{ij} \geq g_j(x)$ of (P_δ) by $t_i + \xi_{ij} \geq h_{jk}(x) : k \in K_j$, which are linear constraints.

We will assume that the decision maker is capable of identifying a collection of *plausible* vectors of λ-weights (or its equivalent δ weights) for (P_λ). Each one of these vectors of weights could give rise to a *rational* decision under a particular economic scenario. For instance, increasing λ-weights for the losses of the last positions in the ordered sequence (2) gives place to criteria with greater aversion towards risk, which can be reasonable in high volatility scenarios. However, the existing uncertainty about the actual economic scenario that will occur, makes advisable to consider several vectors of λ-weights modeling different attitudes towards risk in order to improve the ex-post assessment of our portfolio.

The proposed optimization model will find a *robust* solution, in the sense that

- it takes into account a set of possible scenarios of δ-weights to parametrize advisable (according to experts) risk attitudes under the existing uncertain conditions
- and tries to reach a good overall performance under these scenarios when compared with the optimal ex-post portfolio.

From now on, we will denote the set of considered δ-weights by Δ stated in the following

Definition 1. We will denote by Δ a compact polyhedron of δ-weights modeling admissible decision maker attitudes towards risk according to (P_δ).

Each positive δ-weight represents the degree of importance that a decision maker gives to the difference between consecutive losses in the nondecreasing ordered sequence (2). There exists a one-to-one correspondence between λ and δ weights by (6), however δ-weights allow us to write in a bit more compact way some subsequent expressions and formulations, as the proposed *minimax regret portfolio* model (P) whose formulation is given in terms of the function $z(x,\delta)$ defined by (7)

$$z(x,\delta) := \begin{array}{l} \min \sum_{i=1}^{n} \left[(n-i+1)\delta_i t_i + \sum_{j=1}^{n} \delta_i \xi_{ij} \right] \\ \text{s.t.} \\ t_i - g_j(x) + \xi_{ij} \geq 0,\ i,j \in \{1,\dots,n\}, \\ t_i \in \mathbb{R},\ i \in \{1,\dots,n\}, \\ \xi_{ij} \geq 0,\ i,j \in \{1,\dots,n\}, \end{array} \tag{7}$$

that is, the weighted sum of the ordered losses for a given portfolio $x \in X$ as defined in the objective function of (P_λ).

Specifically, a Robust Ordered Weighted Averaging (ROWA) optimization model for portfolio selection based on the specified set of weights Δ will be proposed as the minimax regret problem (P),

$$R^* := \min_{x \in X} \max_{\delta \in \Delta} \left(z(x,\delta) - z^*(\delta) \right) \tag{P}$$

Let us denote by $R(x,\delta) = z(x,\delta) - z^*(\delta)$ the regret for a given portfolio $x \in X$ under the δ-weights scenario. One has the following

Proposition 1. *$R(x,\delta)$ is convex in $\delta \in \Delta$.*

Proof. From the definition of $z^*(\delta)$ in (P_δ) it follows directly that it is concave in δ. Now, using (3) and the definition (7), for any $\delta, \overline{\delta} \in \Delta$, $\mu \in [0,1]$ one has that

$$z(x,\delta) = \sum_{i=1}^{n} g_{(i)}(x) \sum_{j=1}^{i} \delta_j,$$

is a linear function

$$z(x, \mu\delta + (1-\mu)\bar{\delta}) = \sum_{i=1}^{n} g_{(i)}(x) \sum_{j=1}^{i} (\mu\delta_j + (1-\mu)\bar{\delta}_j) = \mu z(x,\delta) + (1-\mu) z(x, \bar{\delta}),$$

which implies the convexity of $R(x,\delta)$. □

Remark 2. If Δ and X are compact polyhedra and g_j, $j \in \{1,\dots,n\}$ piecewise linear convex functions, the maximum regret $R(x,\delta)$ for a given portfolio $x \in X$ is reached at the extreme points of Δ.

Proof. First, observe that, $z^*(\delta)$ is well-defined for any $\delta \in \Delta$ since it represents the optimal value of (P_λ) for a given λ that always exists since the functions g_j are continuous on a compact polyhedron X. Now, as g_j, $j \in \{1,\dots,n\}$ are piecewise linear convex functions and X is a polyhedron we can obtain the value $z^*(\delta)$ for a given δ by assessing the objective function of (P_δ) at the finite set of extreme points of its feasible set. That is, $z^*(\delta)$ is given by the minimum of a finite set of linear functions in δ, then $z^*(\delta)$ is continuous which implies that $R(x,\delta)$ is also continuous in $\delta \in \Delta$ for any given portfolio $x \in X$ in addition to being convex from Proposition 1.

Finally, the result follows by Bauer's maximum principle, Kružík (2000), that is, any convex and continuous function defined on a convex and compact set, attains its maximum at some extreme point of that set. □

3. The ROWA problem under linear loss functions

In order to simplify subsequent formulations, we will assume that the loss functions in formulation (P_δ) have a linear structure

$$g_i(x) = -\sum_{a\in A} r_{ia} x_a, \quad i \in \{1,\dots,n\}, \tag{8}$$

where r_{ia} is the return of asset $a \in A$ in period $i \in \{1,\dots,n\}$.

The set of feasible portfolios X appearing in (P_δ) can include a broad range of possible technical constraints, as those considered in Krokhmal et al. (2002), related to transaction costs, diversification, changes in individual positions (liquidity constraints) or bounds on positions. These constraints are commonly expressed as linear inequalities involving the decision variables within the optimization model. However, for readability purposes, we will only consider two linear constraints,

- a normalization condition about the invested amount, which will be fixed at one and
- a constraint on the minimum level of expected return r^*, empirically estimated from historical data.

Consequently:

$$X = \{x \in \mathbb{R}^n_+ : \sum_{a\in A} x_a = 1, \sum_{a\in A} r_a x_a \geq r^*\}, \tag{9}$$

where r_a is the average return for each asset $a \in A$. Including other additional linear constraints to X will not modify in essence the numerical algorithms developed later as they would just modify the linear constraints of the problem formulation. Under the choice (9) of the set X and the loss functions (8), Problem (P_δ) can be written as the formulation (PL_δ).

$$\begin{aligned} z^*(\delta) = \quad & \min \sum_{i=1}^{n} \left[(n-i+1)\delta_i t_i + \sum_{j=1}^{n} \delta_i \xi_{ij} \right] \\ & \text{s.t.} \\ & t_i + \sum_{a\in A} r_{ja} x_a + \xi_{ij} \geq 0, \quad i,j \in \{1,\dots,n\}, \\ & \sum_{a\in A} x_a = 1, \\ & \sum_{a\in A} r_a x_a \geq r^*, \\ & x_a \geq 0, \quad a \in A, \\ & t_i \in \mathbb{R}, \quad i \in \{1,\dots,n\}, \\ & \xi_{ij} \geq 0, \quad i,j \in \{1,\dots,n\}. \end{aligned} \tag{PL_δ}$$

The value $z^*(\delta)$ measures the weighted mean of ordered losses assumed by the decision maker when she acts optimally according to the vector of weights δ. Each optimal solution of Problem (PL_δ) possesses interesting properties in terms of *second degree stochastic dominance* (SSD). In fact, as it was shown in Mansini et al. (2007), under mild conditions these optimal solutions are SSD efficient portfolios.

Taking into account that $z^*(\delta)$ exists for every $\delta \in \Delta$ since it corresponds to the optimal value of (P_λ) for a given λ, from (PL_δ), this optimum is reached at the finite set of extreme vertex of its polyhedron of feasible solutions, hence $z^*(\delta)$ is a concave and piecewise linear function since it can be expressed as the minimum of a finite set of linear functions, one for each extreme points in (PL_δ).

The dual formulation of (PL_δ) can be written as (DL_δ).

$$\begin{aligned} z^*(\delta) = \quad & \max -u_0 + r^* v_0 \\ & \text{s.t.} \\ & \sum_{j=1}^{n} u_{ij} = (n-i+1)\delta_i, \quad i \in \{1,\dots,n\}, \\ & \sum_{i,j=1}^{n} r_{ja} u_{ij} + r_a v_0 \leq u_0, \quad a \in A, \\ & u_0 \in \mathbb{R}, \\ & v_0 \geq 0 \\ & 0 \leq u_{ij} \leq \delta_i, \quad i,j \in \{1,\dots,n\}. \end{aligned} \tag{DL_δ}$$

Using Remark 2, we can write the ROWA problem (P) as (10)

$$\begin{aligned} R^* = \quad & \min \quad R \\ & \text{s.t.} \\ & \quad z(x,\delta) - z^*(\delta) \leq R, \quad \forall\, \delta \in \text{Ext}(\Delta) \\ & \quad x \in X, \end{aligned} \tag{10}$$

where $\text{Ext}(\Delta)$ is the set of extreme points of Δ.

Then, from the formulations (DL_δ) and (7), one has that formulation (10) can be written as the linear programming problem $(PLEX)$.

$$\begin{aligned} & \min \ R \\ & \text{s.t.} \\ & t_i + \sum_{a\in A} r_{ja} x_a + \xi_{ij} \geq 0, \ i,j \in \{1,\dots,n\}, \\ & \sum_{i=1}^{n} \left[(n-i+1)\delta_i t_i + \sum_{j=1}^{n} \delta_i \xi_{ij} \right] + u_0^\delta - r^* v_0^\delta \leq R, \ \forall\, \delta \in \text{Ext}(\Delta) \\ & \sum_{j=1}^{n} u_{ij}^\delta = (n-i+1)\delta_i, \ i \in \{1,\dots,n\}, \delta \in \text{Ext}(\Delta), \\ & \sum_{i,j=1}^{n} r_{ja} u_{ij}^\delta + r_a v_0^\delta \leq u_0^\delta, \ a \in A, \delta \in \text{Ext}(\Delta), \\ & u_{ij}^\delta \leq \delta_i, \ i,j \in \{1,\dots,n\}, \delta \in \text{Ext}(\Delta), \\ & \sum_{a\in A} x_a = 1, \\ & \sum_{a\in A} r_a x_a \geq r^*, \\ & t \in \mathbb{R}^n, \\ & u_0^\delta \in \mathbb{R}, \delta \in \text{Ext}(\Delta), \\ & v_0^\delta \geq 0, \delta \in \text{Ext}(\Delta), \\ & \xi \geq 0, \ x \geq 0, \ u^\delta \geq 0, \delta \in \text{Ext}(\Delta). \end{aligned} \tag{$PLEX$}$$

In $(PLEX)$ a copy of the variables u appearing in the formulations (DL_δ) has been made for each $\delta \in \text{Ext}(\Delta)$, while the variables t, ξ appearing in the problem (7) need not be replicated due to the fact that the weight vector δ just affect to the objective function of (7). These copies have been denoted with a superindex δ in order to ease the reading of this last formulation. Taking into account the separability of each block of constraints according to $\delta \in \mathrm{Ext}(\Delta)$, if there exist feasible solutions for the problems (PL_δ) and (DL_δ) verifying the constraints

$$\sum_{i=1}^{n}\left[(n-i+1)\delta_i t_i + \sum_{j=1}^{n}\delta_i \xi_{ij}\right] + u_0^\delta - r^* v_0^\delta \le R, \quad \forall \delta \in \mathrm{Ext}(\Delta) \tag{11}$$

we are ensuring that

$$z(x,\delta) - z^*(\delta) \le R, \quad \forall \delta \in \mathrm{Ext}(\Delta),$$

since this last difference can be obtained by minimizing the left hand side member of (11) in the feasible sets of the corresponding problems (PL_δ) and (DL_δ). The reciprocal of this assertion is also true.

In the next two subsections, we demonstrate the utility of the formulation $(PLEX)$ in creating operational Linear Programming models for specific robust portfolio selection paradigms.

3.1. The case of multiple tolerance levels in CVaR

As commented in the introduction, an efficient portfolio can be found constraining the average of its k largest losses for different values of k, as proposed in Mansini et al. (2007). Following the ideas of these authors, the resulting methodology "enriches the capabilities" of "modeling various risk aversion preferences" as compared with the "crude" single CVaR formulation. Additional properties, as the SSD efficiency (Theorem 1 of Mansini et al. (2007)), can be shown for the optimal portfolios respect to an aggregation of these multiple CVaR measures.

According to Remark 1, for a given choice of k, the selection of $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-k} = 0$ and $\lambda_{n-k+1} = \cdots = \lambda_n = \frac{1}{k}$, gives rise to a problem (P_λ) with discrete CVaR objective for the $(n-k+1)/n$-VaR. These λ-weights, for all possible values of k, that is, for all the possible tolerance levels in CVaR are the extreme points of the polyhedron

$$\Lambda = \{\lambda \in \mathbb{R}^n_+ : \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n, \sum_{i=1}^{n}\lambda_i = 1\}.$$

The closed-form of these extreme points is known as Paelinck's theorem (see Claessens et al., 1991). Since the corresponding Δ-set is the image set of Λ for the linear bijective function given by (6), it is clear that its extreme points are also the corresponding image of the extreme points of λ, that is, Δ is the polyhedron with extreme points δ defined by

$$\delta_{n-k+1} = \frac{1}{k} \text{ and } \delta_i = 0 \text{ for } i \ne n-k+1, \tag{12}$$

for any possible $k \in \{1,2,\dots,n\}$.

Hence, if we want to find a robust portfolio according to any one of the possible CVaR measure of the corresponding risk we should solve Problem $(PLEX)$ for all the extreme δ-weights given in (12). However, if just a subset of these extreme points are considered, we would be modeling the "tolerance levels" of the multiple criteria problem proposed by Mansini et al. in Mansini et al. (2007) (Problem (10)).

We will now formulate the linear program $(PLEX)$ to derive a robust portfolio, considering all the extreme points δ of (12). It is important to note that focusing on a subset of these extreme points would also hold significance within this context.

Let us denote such extreme vectors as

$$\mathrm{Ext}(\Delta) = \left\{\delta^1 = \left(\frac{1}{n},0,\dots,0\right), \delta^2 = \left(0,\frac{1}{n-1},0,\dots,0\right),\dots,\delta^n = (0,\dots,0,1)\right\}. \tag{13}$$

First, by using the fact that $\delta_i^i = \frac{1}{n-i+1}$ and $\delta_k^i = 0$ if $k \ne i$, we have that the constraint set

$$0 \le u_{ij}^\delta \le \delta_i, i,j \in \{1,\dots,n\}, \delta \in \mathrm{Ext}(\Delta),$$

of $(PLEX)$ becomes

$$0 \le u_{ij}^i \le \frac{1}{n-i+1}, i,j \in \{1,\dots,n\}, \quad u_{kj}^i = 0, \forall k \ne i, \tag{14}$$

where u^i has been used, instead of the generic notation u^δ, to denote the vector of dual variables associated to each extreme point δ^i of Δ. This means that most of these dual variables need not to be defined since they are zeroes. Hence, one can delete the superscript of these variables and rewrite $(PLEX)$ as the formulation (15)

$$\begin{aligned}
R^* = \quad & \min\ R \\
& \text{s.t.} \\
& t_i + \sum_{a\in A} r_{ja}x_a + \xi_{ij} \ge 0,\ i,j \in \{1,\dots,n\}, \\
& t_i + \frac{1}{n-i+1}\sum_{j=1}^{n}\xi_{ij} + u_{i0} - r^* v_{i0} \le R, \quad \forall i \in \{1,\dots,n\}, \\
& \sum_{j=1}^{n} u_{ij} = 1,\ i \in \{1,\dots,n\}, \\
& \sum_{j=1}^{n} r_{ja}u_{ij} + r_a v_{i0} \le u_{i0},\ a \in A,\ i \in \{1,\dots,n\}, \\
& u_{ij} \le \frac{1}{n-i+1},\ i,j \in \{1,\dots,n\}, \\
& \sum_{a\in A} x_a = 1, \\
& \sum_{a\in A} r_a x_a \ge r^*, \\
& t \in \mathbb{R}^n, \\
& u_{i0} \in \mathbb{R}, \quad \forall i \in \{1,\dots,n\}, \\
& v_{i0} \ge 0, \quad \forall i \in \{1,\dots,n\}, \\
& \xi \ge 0, u \ge 0,\ x \ge 0.
\end{aligned} \tag{15}$$

Formulation (15) can be written in a more general setting when only a few of the largest losses of any portfolio should be considered, so constraints (14) can be modified. For instance, the values of i could be constrained to be in the set $\{q^l, q^l+1,\dots,q^u\}$, with $\lfloor n/2 \rfloor \le q^l \le \cdots \le q^u \le n$, corresponding to CVaR with β's such that $0.5 \le \beta \le 1$. In this case, a set of consecutive intermediate tolerances for the CVaR indicator are taken as *reasonable* risk measures:

$$\begin{aligned}
&\delta_i^i = \frac{1}{n-i+1} \text{ for } i \in I := \{q^l,\dots,q^u\} \\
&\delta_i^i = 0 \text{ for } i \notin I \\
&\delta_j^i = 0 \text{ for } j \ne i.
\end{aligned}$$

Of course, the values i for the considered tolerances do not need to be consecutive. For instance, by taking just $i = 1$ and $i = n$, the corresponding Δ-set of (13) is the segment with extreme points

$$\delta_1 = \frac{1}{n}, \delta_j = 0 \text{ for } j \ne 1 \text{ and } \delta_n = 1, \delta_j = 0 \text{ for } j \ne n,$$

which gives place to the risk assessment called *cent-dian*, Halpern (1978). This last criterion is built as convex combinations of the mean cost and the maximum cost and was proposed and analyzed in the context of Location Theory, Halpern (1978).

When considering solely a subset of extreme points of the Δ-set, we only have to delete from (15) those constraints and indices corresponding to the extreme δ-weights not included in our choice, for example, when consecutive tolerances index are considered the extremes $\delta^1, \delta^2, \dots, \delta^{q^l-1}$ and $\delta^{q^u+1},\dots,\delta^n$ will be deleted from Formulation (16) giving rise to Formulation (16).

$$\begin{aligned}
R^* = \quad & \min \quad R \\
& \text{s.t.} \\
& t_i + \sum_{a\in A} r_{ja}x_a + \xi_{ij} \geq 0,\ i \in \{q^l,\dots,q^u\}, \quad \forall j \in \{1,\dots,n\}, \\
& t_i + \frac{1}{n-i+1}\sum_{j=1}^{n} \xi_{ij} + u_{i0} - r^* v_{i0} \leq R, \quad \forall i \in \{q^l,\dots,q^u\}, \\
& \sum_{j=1}^{n} u_{ij} = 1,\ i \in \{q^l,\dots,q^u\}, \\
& \sum_{j=1}^{n} r_{ja}u_{ij} + r_a v_{i0} \leq u_{i0},\ a \in A,\ i \in \{q^l,\dots,q^u\}, \\
& u_{ij} \leq \frac{1}{n-i+1},\ i \in \{q^l,\dots,q^u\},\ j \in \{1,\dots,n\}, \\
& \sum_{a\in A} x_a = 1, \\
& \sum_{a\in A} r_a x_a \geq r^*, \\
& t \in \mathbb{R}^n, \\
& u_{i0} \in \mathbb{R}, \quad \forall i \in \{q^l,\dots,q^u\}, \\
& v_{i0} \geq 0, \quad \forall i \in \{q^l,\dots,q^u\}, \\
& \xi \geq 0, u \geq 0,\ x \geq 0.
\end{aligned} \tag{16}$$

The optimal portfolios obtained by solving formulations like (15) or (16) or any other variant of the general formulation (P) have not only robustness features derived from the choice of different risk measurement scenarios, as shown in the following section, it is possible to work out several efficiency properties conferred to those solutions in the context of a multi-objective model related, in a natural way, with our investment problem.

3.2. A robust orness model for portfolio selection

In Section 3.1 we considered a set of λ-weights Λ in which the normalization condition $\sum_i \lambda_i = 1$ was included. In the next model, we include that normalization constraint in the set of δ-weights together with the so-called *orness* measure of weights. In Hajjami and Amin (2018), Hajjami and Amin proposed the following set of weights in the context of a model for stock selection using ordered weighted averaging operators

$$\left\{ \delta \in \mathbb{R}^n_+ \ :\ orness(\delta) := \sum_{i=1}^{n} \frac{n-i}{n-1}\delta_i = \alpha,\ \sum_{i=1}^{n}\delta_i = 1 \right\}. \tag{17}$$

The value $\alpha \in [0,1]$ is known in the multicriteria decision-making literature, Yager (1993), as the *orness* parameter (a measurement of their proximity to the OR-operator) and models the optimism level of the manager when facing the uncertain conditions under which any given solution will be assessed. This value α represents a measurement of the proximity of the OWA aggregation to the OR-operator ("oring" operator) as proposed by Yager (1988). In this way, a value $\alpha = 0$ leaves $\delta = (0,0,\dots,1)$ as the only feasible vector of weights, that is, the corresponding (P_λ)-problem models a *minimax* decision corresponding to a purely pessimistic manager. On the other hand, a value $\alpha = 1$ leaves $\delta = (1,0,\dots,0)$ as the only feasible vector of weights which identifies our (P_λ)-problem with the minsum criterion, that is, the decision maker ignores the relative order of the losses due to the selected portfolio. For a given $\alpha \in (0,1)$ one has an intermediate attitude towards risk in which a non-singleton Δ-set of weights is considered.

However, a more general approach would be that of considering an interval of the orness α-values. Let us consider the following set of weights, that we call, the orness Δ-set,

$$\Delta := \left\{ \delta \in \mathbb{R}^n_+ \ :\ \sum_{i=1}^{n} \frac{n-i}{n-1}\delta_i \in [\alpha_0,\alpha_1],\ \sum_{i=1}^{n}\delta_i = 1 \right\}, \tag{18}$$

where $\alpha_0, \alpha_1 \in [0,1]$.

We call this a *proper* orness Δ-set if $\alpha_0 \neq 1$ and $\alpha_1 \neq 0$ since these conditions correspond to one of the degenerate cases considered before. The meaning of these upper and lower bounds on the orness of the admissible OWA operators is clearly understood by any decision maker which may identify small values of the orness with a pessimistic attitude towards risk whilst a large orness is a neutral attitude.

We can have an operational formulation of $(PLEX)$ by identifying the extreme weights of the orness Δ-set. Following Conde (2023), these extreme weights can be obtained by writing (18) as

$$\begin{aligned}
\sum_{i=1}^{n}(n-i)\delta_i - \beta_0 &= (n-1)\alpha_0, \\
\sum_{i=1}^{n}(n-i)\delta_i + \beta_1 &= (n-1)\alpha_1, \\
\sum_{i=1}^{n}\delta_i &= 1, \\
\delta_i,\ i \in \{1,\dots,n\}, \beta_0, \beta_1 &\geq 0,
\end{aligned}$$

and taking into account that every basic feasible solution of the orness Δ-set must have, at least, one of its slack variables β_0 or β_1 in the set of basic variables. Hence, an extreme vector of weights of the orness Δ-set should have one of the following three different forms:

1. If both variables are basic, by taking the inverse of the basic matrix one has the extreme weights
$$\delta = \mathbf{e}^p, \forall p = \lceil n-(n-1)\alpha_1 \rceil, \dots, \lfloor n-(n-1)\alpha_0 \rfloor \tag{19}$$
where $\mathbf{e}^p$ is the vector with an one in the pth component and zeros elsewhere, $\lceil a \rceil$ is the smallest integer greater than or equal to a and $\lfloor a \rfloor$ the largest integer smaller than or equal to a.
2. If β_0 is the only basic variable, for all $p \neq q$
$$\delta = \frac{(n-1)\alpha_1 - n + q}{q-p}\mathbf{e}^p + \frac{(n-1)\alpha_1 - n + p}{p-q}\mathbf{e}^q, p \leq n-(n-1)\alpha_1 \leq q. \tag{20}$$
3. If β_1 is the only basic variable, for all $p \neq q$
$$\delta = \frac{(n-1)\alpha_0 - n + q}{q-p}\mathbf{e}^p + \frac{(n-1)\alpha_0 - n + p}{p-q}\mathbf{e}^q, q \leq n-(n-1)\alpha_0 \leq p. \tag{21}$$

By using the given set of extreme points of Δ, that is, Ext(Δ) one can write the specific Linear Programming formulation $(PLEX)$ for the corresponding robust orness model for portfolio selection reducing its set of decision variables as in Conde (2023).

4. Efficiency properties of the ROWA solution

Let $f_k(x)$ be the sum of the k largest losses for the portfolio $x \in X$, that is, $f_k(x)$ is defined by (22)

$$\begin{aligned}
f_k(x) = \quad & \min kt + \sum_{j=1}^{n}(g_j(x) - t)^+ \\
\text{s.t.} \quad & \\
& t \in \mathbb{R}.
\end{aligned} \tag{22}$$

Taking into account the separability property of the problem (5) respect to the t_i-variables, if $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ its optimal solutions can be seen as the Pareto optimal solutions for the vector optimization problem (V)

$$\begin{aligned} & V-min(f_k(x) : k=1,\dots,n) \\ \text{s.t.} \quad & \\ & x \in X, \end{aligned} \tag{V}$$

where $V-min$ stands for *vector optimization*, in this case, vector minimization.

Hence, for different λ-weights, Formulation (P_λ) determines weakly efficient portfolios for the multi-objective problem (V). This set could be too large as to be fully described in practical applications even when the set of possible λ-weights is constrained by using subsets of extreme δ-weights as it was done in the last section. However, under mild conditions, the minimax regret optimal portfolios of (P) represent a subset of these efficient solutions which usually is much more manageable than the entire Pareto optimal set.

Proposition 2. *Let Δ be a bounded polyhedron set of weights verifying that $\bar{\delta}=\sum_{\delta\in Ext(\Delta)}\delta$ has strictly positive components, then if the problem (P) has a unique optimal solution it is an efficient or Pareto-optimal portfolio for the vector optimization problem (V).*

Proof. Using Theorem 1 of Kouvelis and Yu (1997) the unique optimal solution x^* of (P) is efficient for the problem

$$\begin{aligned} & V-\min\Big(\sum_{k=1}^{n}\delta_k f_k(x)-z^*(\delta) : \delta\in \mathrm{Ext}(\Delta)\Big) \\ \text{s.t.} \quad & \\ & x\in X. \end{aligned} \tag{23}$$

Now let us suppose, by contradiction, that x^* is not efficient for the vector optimization problem (V) and let $y \in X$ a portfolio dominating x^*, that is,

$$f_k(x^*) \ge f_k(y), \quad \forall k \in \{1,\dots,n\},$$

being strict at least one of these inequalities. Then, as $\bar{\delta}=\sum_{\delta\in \mathrm{Ext}(\Delta)}\delta$ has strictly positive components, it directly follows the existence of at least one $\delta^0 \in \mathrm{Ext}(\Delta)$ for which

$$\sum_{k=1}^{n}\delta_k^0 f_k(x^*) > \sum_{k=1}^{n}\delta_k^0 f_k(y).$$

For all the other $\delta\in\mathrm{Ext}(\Delta)$ one has

$$\sum_{k=1}^{n}\delta_k f_k(x^*) \ge \sum_{k=1}^{n}\delta_k f_k(y)$$

since, by definition, $\delta \ge 0$. Hence x^* would not be an efficient solution of the vector optimization problem (23) which contradicts the initial assumption. □

Let us consider the optimization problem (24) in which a weighted sum of the functions $f_k(x)$ defined in (22) is minimized subject to a set of new constraints

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\bar{\delta}_i f_{n-i+1}(x) \\ \text{s.t.} \quad & \\ & \sum_{i=1}^{n}\delta_i f_{n-i+1}(x) \le R^* + z^*(\delta), \quad \forall\, \delta\in\mathrm{Ext}(\Delta), \\ & x\in X. \end{aligned} \tag{24}$$

where R^* is the optimal objective value of the problem (P) and $z^*(\delta)$ the optimal objective value of the problem (P_δ) for every $\delta\in\mathrm{Ext}(\Delta)$. Proposition 2 can be generalized as stated in Proposition 3.

Proposition 3. *Let Δ be a bounded polyhedron set of weights verifying that $\bar{\delta}=\sum_{\delta\in Ext(\Delta)}\delta$ has strictly positive components, then any optimal solution of the problem (24) is an efficient or Pareto-optimal portfolio for the vector optimization problem (V).*

Proof. Let x^* be an optimal solution of (24) then, by the new set of constraints included in this problem, we have also the optimality of x^* in the ROWA problem (P). Now, if x^* is not efficient for the vector optimization problem (V), there must exists $y\in X$ dominating x^*, that is,

$$f_k(x^*) \ge f_k(y), \quad \forall k\in\{1,\dots,n\},$$

being strict at least one of these inequalities. Then,

$$\sum_{k=1}^{n}\bar{\delta}_k f_k(x^*) > \sum_{k=1}^{n}\bar{\delta}_k f_k(y)$$

which contradicts the initial assumption. □

Now, the problem (24) of Proposition 3 can be written using the same constraints of $(PLEX)$ by changing r for R^* and the objective functions as it appears in the formulation (25).

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\left[(n-i+1)\bar{\delta}_i t_i + \sum_{j=1}^{n}\bar{\delta}_i\xi_{ij}\right] \\ \text{s.t.} \quad & \\ & \sum_{i=1}^{n}\left[(n-i+1)\delta_i t_i + \sum_{j=1}^{n}\delta_i\xi_{ij}\right] + u_0^\delta - r^* v_0^\delta \le R^*, \quad \forall\,\delta\in\mathrm{Ext}(\Delta) \\ & t_i + \sum_{a\in A} r_{ja}x_a + \xi_{ij} \ge 0, \qquad i,j\in\{1,\dots,n\}, \\ & t_i \in \mathbb{R}, \qquad i\in\{1,\dots,n\}, \\ & \xi_{ij}\ge 0, \qquad i,j\in\{1,\dots,n\}, \\ & \sum_{j=1}^{n} u_{ij}^\delta = (n-i+1)\delta_i, \qquad i\in\{1,\dots,n\}, \delta\in\mathrm{Ext}(\Delta), \\ & \sum_{i,j=1}^{n} r_{ja}u_{ij}^\delta \le u_0^\delta, \qquad a\in A, \delta\in\mathrm{Ext}(\Delta), \\ & 0\le u_0^\delta, \qquad \delta\in\mathrm{Ext}(\Delta), \\ & 0\le u_{ij}^\delta \le \delta_i, \qquad i,j\in\{1,\dots,n\}, \delta\in\mathrm{Ext}(\Delta), \\ & \sum_{a\in A} x_a = 1, \\ & \sum_{a\in A} r_a x_a \ge r^*, \\ & x_a \ge 0, a\in A. \end{aligned} \tag{25}$$

Formulation $(PLEX)$ has $|A|+(n^2+1)|\mathrm{Ext}(\Delta)|+n^2+n+1$ variables and $|A||\mathrm{Ext}(\Delta)|+(n+1)|\mathrm{Ext}(\Delta)|+n^2+1$ linear constraints (excepting nonnegativity and upper bounding constraints on some variables). Formulation (25) has nearly the same size (one variable less since r becomes a constant). Both problems are quite tractable from a numerical viewpoint by using off-the-shelf solvers for large sets of assets (A), time periods (n) and weight scenarios ($\mathrm{Ext}(\Delta)$).

Once we have proposed a robust portfolio by solving the ROWA problem (P) and highlighted its relation with the Pareto efficiency in the vector optimization problem (V), a numerical experiment has been designed in order to check its behavior with real data.

5. Numerical experiments

To test the effects of using the ROWA optimization model (P) to aggregate multiple CVaR risk measures, we simulated a hypothetical decision maker that invests its wealth on the assets of a financial market according to different models of portfolio optimization. The investor observes the market data of the last periods, then it runs some optimization model and implements the corresponding optimal solution. Next, the process is repeated after that a certain period has elapsed.

Four portfolio models have been tested:

- The $(1/N)$-Portfolio, or Equally Weighted (EqW) portfolio.
- The min-CVaR model (with and without the constraint on the minimum expected return r^*).
- The ROWA model (with and without the constraint on the minimum expected return r^*).
- The Huang et al.'s model of the relative robust CVar.

The $(1/N)$-Portfolio is the passive strategy that, in every trading day, allocates $1/N$ fraction of wealth to each asset, where N is the number of the traded assets. This procedure has been proposed in DeMiguel et al. (2009) as a tool to minimize the market risk without using any information. It has been found that it is an effective strategy for portfolio management, as it often outperforms the market indexes without substantive effort for decision-makers. This is the case for the six data sets that we are using: In all markets, the $1/N$ portfolio greatly outperforms the index, so, we will propose this procedure as the most appropriate benchmark against which to compare the CVaR and ROWA models. The Huang et al.'s model, see Huang et al. (2010), is a relative robust CVaR model in which there are multiple possible scenarios. They can be used to predict the portfolio expectations and CVaRs and when the prediction is not correct, the investor regrets its choice. The model objective function is to minimize the regret. The difference between Huang et al.'s model and ours is that in the former model, scenarios are different market predictions, while in the latter model, scenarios are CVaR estimates with different thresholds β.

The min-CVaR model has been run with $\beta = 0.90$, whereas the ROWA model (16) has been run with various choices of q^l and q^u, corresponding respectively to CVaR parameters β^l and β^u. We remind that, when a range $[\beta^l, \beta^u]$ is determined, then the ROWA model (16) considers all the discrete CVaR's calculated for $\beta \in [\beta^l, \beta^u]$ and aggregate them through the maximum regret. Since min-CVaR is a risk measure, we did not use values of β^l smaller than 0.5, as it should include returns that are above the median, representing opportunities rather than risks. The selected values of β^l were $\beta^l \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$. Regarding β^u, we have tested values $\beta^u \in \{0.9, 1\}$. Specific combinations of β^l and β^u can be interpreted as pessimistic or realistic investors. We consider a *pessimistic investor* the one with $\beta^u = 1$, as CVaR's are calculated even for the worst quantiles, that are the ones in which only the greatest losses are considered. Conversely, we consider a *realistic investor* the one with $\beta_u = 0.9$, as CVaR's with high value of β are excluded. The Huang et al.'s model has been tested with three and five scenarios and with the same parameters used in the tests reported in Huang et al. (2010).

These optimization models have been tested on the following six financial markets:

- Dow Jones, a data set including 28 assets and 6818 daily observations from 16/02/1990 to 07/04/2016.
- Eurostoxx, including 49 assets and 3884 daily observations from 22/05/2001 to 11/04/2016.
- FTSE, including 82 assets and 3587 daily observations from 11/07/2002 to 11/04/2016.
- Hang Seng, including 43 assets and 2706 daily observations from 25/11/2005 to 11/04/2016.
- Mibtel, including 60 assets and 1305 daily observations from 17/2/2003 to 15/2/2008.
- NASDAQ, including 82 assets and 2983 daily observations from 03/11/2004 to 11/04/2016.

The Mibtel data have been previously used in the portfolio simulation reported in Benati (2015), while the other datasets have been previously used in Benati and Conde (2022) and Carleo et al. (2017) and are available in http://host.uniroma3.it/docenti/cesarone/DataSets.htm.

The solved instances for each model have been generated according to the following parameters. Models are run using the last T days, fixing $T = 120$. When needed as model inputs, return averages are estimated

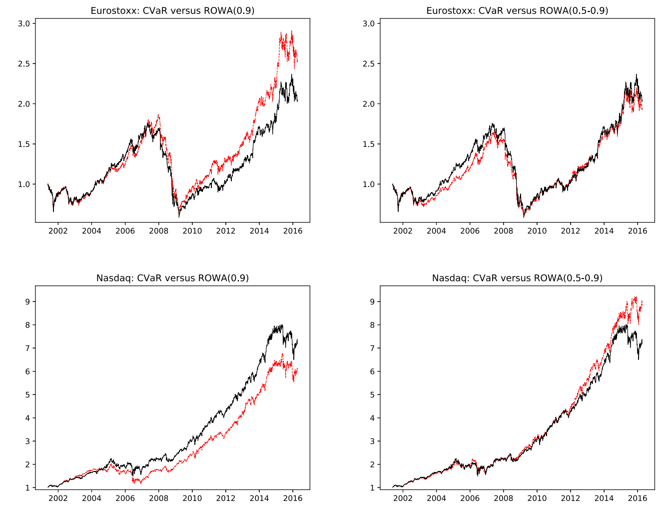

Fig. 1. Minimum Risk models: Comparison between ROWA (red lines) and the CVaR portfolios (dark line).

through the sample mean of these T days, while the minimum input parameter r^* has been fixed using the $(1/N)$-portfolio as benchmark: Let $r_t^b = \frac{1}{N}(\sum_{a\in A} r_{at})$ be the $(1/N)$-return in time t, then $r^* = \frac{\sum_t r_t^b}{T}$. Finally, investors using CVaR's and ROWA models are assumed to rebalance their portfolio every 20 days, being $t = 150$ the first period of the series in which they use the corresponding optimization model.

We compare portfolio strategies through data of:

- Realized returns, that are the out-of-sample returns.
- Portfolio compositions, that are the statistics about the assets weights.

Realized returns are compared through their averages, standard deviations, and Sharpe indexes but, quantiles indicators are also reported. Portfolio compositions are compared through indexes of diversification, maximum exposition, and portfolio turnover. Diversification is measured through the Herfindahl–Hirschman (HH) index, maximum exposition is measured by the maximum portfolio weight (the Max index), the turnover is the portfolio wealth percentage that is sold in every rebalancing period.

Next, we considered two families of portfolio strategies:

- Minimum risk models;
- Risk/return optimization models.

Minimum risk models are run without any constraints on minimum expected return, so we can ascertain the effect of the ROWA model without the (possible) bias caused by the expected return. Next, in the risk/return model we study the effect of the ROWA model when an investor wants to balance risks with expected returns.

In Table 1 and in Fig. 1 results are shown about the ex-post returns of the minimum risk models. Fig. 1 shows the comparison between the CVaR and two ROWA portfolios, for the Nasdaq and the Eurostoxx markets (for the other markets results are similar). It can be seen that ROWA portfolios are tracking closely the patterns of CVaR's. In some cases red lines terminate above the black lines, showing that the ROWA final wealth is higher than the one corresponding to the CVaR criterion. However, this is not a consistent and regular trend: In Table 1 data about the median and the mean expected returns are reported, and it can be seen that there is no clear superiority of one model against the other. However, models are optimized without the constraint on the expected return and they have the sole purpose of minimizing risk. So, when we compare the standard deviations, we can see a clear evidence that both the CVaR and the ROWA carried out less variability than the

Table 1
Return indexes for markets and models (pure minimization model).

Market	Model	Min	0.25	Median	0.75	Max	Mean	SD	Sharpe
DowJones	1/N	−8.234	−0.443	0.040	0.583	11.998	0.059	1.115	5.258
	CVaR	−7.689	−0.415	0.025	0.528	12.325	0.043	0.919	4.702
	ROWA(0.9)	−7.635	−0.435	0.038	0.544	13.459	0.051	0.954	5.333
	ROWA(0.8)	−7.662	−0.416	0.034	0.537	13.515	0.049	0.942	5.215
	ROWA(0.7)	−7.675	−0.415	0.034	0.533	13.572	0.049	0.939	5.215
	ROWA(0.6)	−7.704	−0.416	0.032	0.532	13.639	0.049	0.939	5.239
	ROWA(0.5)	−7.704	−0.415	0.030	0.530	13.639	0.049	0.938	5.195
	ROWA(0.5/0.9)	−8.026	−0.407	0.033	0.508	12.872	0.042	0.905	4.632
	ROWA(0.6/0.9)	−8.026	−0.412	0.031	0.504	12.872	0.041	0.903	4.581
	ROWA(0.7/0.9)	−7.963	−0.414	0.035	0.514	12.760	0.041	0.903	4.524
Eurostoxx	1/N	−7.816	−0.665	0.024	0.720	10.704	0.028	1.458	1.947
	CVaR	−8.339	−0.450	0.041	0.556	8.739	0.028	1.002	2.812
	ROWA(0.9)	−10.104	−0.450	0.046	0.572	9.755	0.034	1.033	3.326
	ROWA(0.8)	−8.903	−0.450	0.039	0.565	10.114	0.033	1.023	3.257
	ROWA(0.7)	−8.902	−0.443	0.032	0.563	10.111	0.032	1.020	3.147
	ROWA(0.6)	−8.903	−0.443	0.035	0.562	10.114	0.032	1.020	3.156
	ROWA(0.5)	−8.944	−0.442	0.036	0.565	9.981	0.032	1.020	3.178
	ROWA(0.5/0.9)	−9.446	−0.428	0.039	0.535	9.121	0.027	0.978	2.714
	ROWA(0.6/0.9)	−9.384	−0.429	0.036	0.531	9.366	0.026	0.979	2.690
	ROWA(0.7/0.9)	−9.174	−0.425	0.036	0.531	9.399	0.026	0.977	2.634
FTSE	1/N	−7.907	−0.469	0.061	0.591	8.146	0.053	1.167	4.507
	CVaR	−6.517	−0.335	0.054	0.487	8.252	0.050	0.860	5.775
	ROWA(0.9)	−6.897	−0.371	0.047	0.497	6.336	0.048	0.886	5.421
	ROWA(0.8)	−6.696	−0.359	0.045	0.497	5.264	0.048	0.868	5.526
	ROWA(0.7)	−6.635	−0.352	0.046	0.498	4.795	0.047	0.862	5.499
	ROWA(0.6)	−6.705	−0.359	0.039	0.502	4.630	0.045	0.858	5.295
	ROWA(0.5)	−6.705	−0.362	0.041	0.500	4.629	0.046	0.858	5.384
	ROWA(0.5/0.9)	−5.644	−0.338	0.047	0.490	9.489	0.050	0.856	5.847
	ROWA(0.6/0.9)	−5.649	−0.333	0.047	0.488	9.460	0.049	0.855	5.748
	ROWA(0.7/0.9)	−5.963	−0.337	0.057	0.483	8.385	0.049	0.853	5.723
HangSeng	1/N	−12.613	−0.600	0.000	0.745	12.190	0.051	1.536	3.302
	CVaR	−9.947	−0.372	0.009	0.507	11.164	0.036	0.979	3.696
	ROWA(0.9)	−8.903	−0.409	0.018	0.540	7.521	0.045	1.010	4.420
	ROWA(0.8)	−9.105	−0.395	0.017	0.542	7.317	0.044	0.998	4.393
	ROWA(0.7)	−9.105	−0.396	0.020	0.539	7.300	0.044	0.990	4.477
	ROWA(0.6)	−9.105	−0.396	0.020	0.534	7.278	0.044	0.988	4.480
	ROWA(0.5)	−9.105	−0.395	0.021	0.537	7.278	0.045	0.988	4.522
	ROWA(0.5/0.9)	−10.783	−0.385	0.024	0.549	10.085	0.047	1.037	4.493
	ROWA(0.6/0.9)	−10.606	−0.390	0.025	0.554	10.223	0.046	1.041	4.444
	ROWA(0.7/0.9)	−10.969	−0.402	0.030	0.549	10.399	0.047	1.051	4.500
Mibtel	1/N	−4.682	−0.292	0.101	0.481	3.051	0.037	0.778	4.786
	CVaR	−5.867	−0.227	0.072	0.392	6.386	0.048	0.676	7.089
	ROWA(0.9)	−4.624	−0.277	0.049	0.407	5.115	0.035	0.700	4.999
	ROWA(0.8)	−5.018	−0.264	0.054	0.417	5.500	0.041	0.685	5.950
	ROWA(0.7)	−5.020	−0.259	0.054	0.410	5.502	0.042	0.682	6.116
	ROWA(0.6)	−5.017	−0.260	0.054	0.409	5.499	0.041	0.682	6.073
	ROWA(0.5)	−5.019	−0.261	0.054	0.400	5.501	0.042	0.681	6.143
	ROWA(0.5/0.9)	−5.795	−0.207	0.081	0.403	6.347	0.058	0.673	8.684
	ROWA(0.6/0.9)	−5.831	−0.201	0.079	0.406	6.363	0.058	0.671	8.700
	ROWA(0.7/0.9)	−5.848	−0.206	0.080	0.399	6.358	0.059	0.672	8.703
Nasdaq	1/N	−9.170	−0.514	0.087	0.733	11.672	0.072	1.345	5.344
	CVaR	−7.405	−0.446	0.061	0.606	7.345	0.070	1.065	6.607
	ROWA(0.9)	−6.872	−0.449	0.039	0.612	8.277	0.064	1.069	5.982
	ROWA(0.8)	−6.563	−0.448	0.042	0.607	8.038	0.062	1.063	5.857
	ROWA(0.7)	−6.704	−0.435	0.046	0.602	7.746	0.062	1.062	5.854
	ROWA(0.6)	−6.431	−0.422	0.043	0.595	7.945	0.063	1.060	5.956
	ROWA(0.5)	−6.431	−0.412	0.047	0.599	8.194	0.064	1.060	6.084
	ROWA(0.5/0.9)	−7.459	−0.402	0.079	0.621	7.283	0.077	1.045	7.408
	ROWA(0.6/0.9)	−7.083	−0.410	0.083	0.620	7.251	0.077	1.041	7.398
	ROWA(0.7/0.9)	−7.306	−0.407	0.067	0.610	7.051	0.073	1.048	6.972

$1/N$ portfolio. Moreover, a difference between the realistic, pessimistic ROWA, and CVaR models emerged, as in five markets out of six (the exception was HangSeng) the realistic ROWA models are the ones with the least variability, e.g. standard deviation, amongst all the considered models. Similar risk rankings are confirmed when we look at the loss quantiles, e.g., the minimum and the first quartile returns, in which it can be seen that the realistic ROWA models are always better than the pessimistic ROWA one.

In Table 2 and Fig. 2 it can be seen that the reason why realistic ROWA is less variable than the pessimistic ROWA and the CVaR is due to more diversification, less maximum exposition, and less turnover. The data about portfolio composition is reported for the Nasdaq and the Eurostoxx. It can be seen in Fig. 2 that the blue points, corresponding to the realistic ROWA model, are always below the black points, representing the CVaR. It means that realistic ROWA solutions are more diversified and less exposed portfolios, and with less turnover. This effect is regular and consistent: In Table 2 it can be seen that it

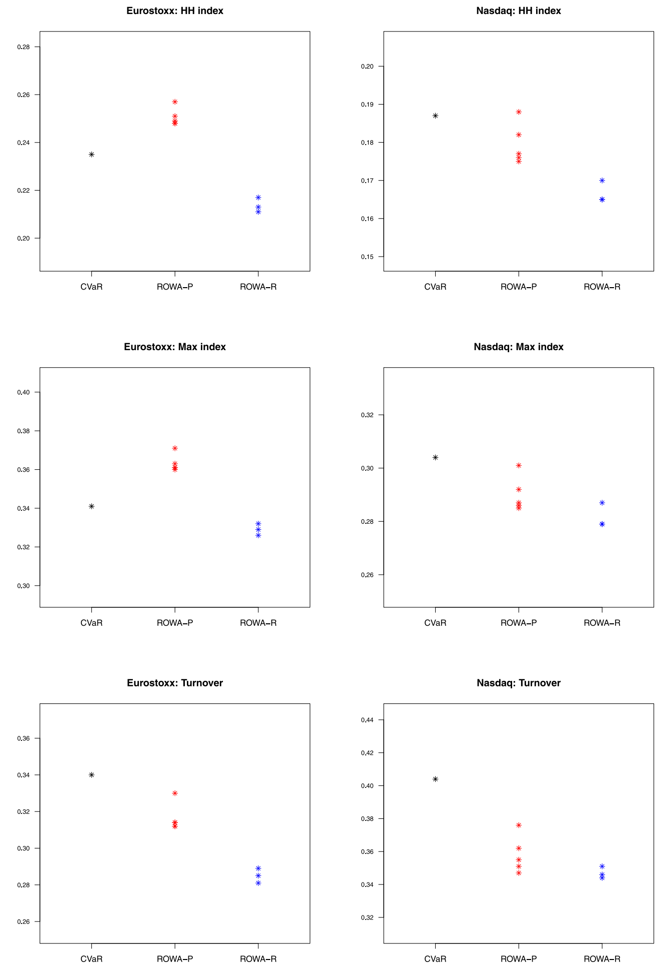

Fig. 2. Minimum Risk models: Comparison between portfolio features. Black points are CVaR models, red points are pessimistic ROWA, blue points are realistic ROWA models.

Table 2
Portfolio indexes for markets and models (pure minimization model).

Market	Model	HH-index	Max-index	Turn over
DowJones	CVaR	0.213	0.324	0.339
	ROWA(0.9)	0.232	0.346	0.310
	ROWA(0.8)	0.226	0.343	0.305
	ROWA(0.7)	0.222	0.338	0.299
	ROWA(0.6)	0.222	0.337	0.298
	ROWA(0.5)	0.222	0.337	0.298
	ROWA(0.5/0.9)	0.190	0.296	0.273
	ROWA(0.6/0.9)	0.190	0.297	0.275
	ROWA(0.7/0.9)	0.195	0.303	0.282
Eurostoxx	CVaR	0.235	0.341	0.340
	ROWA(0.9)	0.257	0.371	0.330
	ROWA(0.8)	0.251	0.363	0.314
	ROWA(0.7)	0.249	0.361	0.314
	ROWA(0.6)	0.248	0.361	0.312
	ROWA(0.5)	0.248	0.360	0.312
	ROWA(0.5/0.9)	0.211	0.326	0.281
	ROWA(0.6/0.9)	0.213	0.329	0.285
	ROWA(0.7/0.9)	0.217	0.332	0.289
FTSE	CVaR	0.156	0.256	0.406
	ROWA(0.9)	0.167	0.272	0.397
	ROWA(0.8)	0.156	0.256	0.379
	ROWA(0.7)	0.153	0.251	0.375
	ROWA(0.6)	0.151	0.249	0.370
	ROWA(0.5)	0.151	0.248	0.368
	ROWA(0.5/0.9)	0.137	0.232	0.354
	ROWA(0.6/0.9)	0.138	0.233	0.351
	ROWA(0.7/0.9)	0.140	0.238	0.358
HangSeng	CVaR	0.253	0.371	0.309
	ROWA(0.9)	0.237	0.347	0.294
	ROWA(0.8)	0.236	0.347	0.283
	ROWA(0.7)	0.233	0.344	0.280
	ROWA(0.6)	0.233	0.343	0.275
	ROWA(0.5)	0.233	0.342	0.276
	ROWA(0.5/0.9)	0.215	0.330	0.287
	ROWA(0.6/0.9)	0.216	0.332	0.293
	ROWA(0.7/0.9)	0.219	0.333	0.304
Mibtel	CVaR	0.166	0.264	0.367
	ROWA(0.9)	0.190	0.295	0.376
	ROWA(0.8)	0.185	0.294	0.360
	ROWA(0.7)	0.181	0.289	0.351
	ROWA(0.6)	0.179	0.288	0.347
	ROWA(0.5)	0.178	0.287	0.342
	ROWA(0.5/0.9)	0.152	0.245	0.321
	ROWA(0.6/0.9)	0.153	0.247	0.328
	ROWA(0.7/0.9)	0.154	0.249	0.338
Nasdaq	CVaR	0.187	0.304	0.404
	ROWA(0.9)	0.188	0.301	0.376
	ROWA(0.8)	0.182	0.292	0.362
	ROWA(0.7)	0.177	0.287	0.355
	ROWA(0.6)	0.176	0.286	0.351
	ROWA(0.5)	0.175	0.285	0.347
	ROWA(0.5/0.9)	0.165	0.279	0.344
	ROWA(0.6/0.9)	0.165	0.279	0.346
	ROWA(0.7/0.9)	0.170	0.287	0.351

always happens in all markets and this effect can be quantified as a decrease around 10% of all the indexes. This happens for the realistic ROWA model, but to a less extent for the pessimistic one. As can be seen in Fig. 2 for the Eurostoxx data, the red points (corresponding to the pessimistic ROWA) regarding diversification and exposition are above the black ones. It implies that there is no consistent and regular portfolio improvements on these measures by using a pessimistic ROWA criterion, as confirmed by data reported in Table 2. Instead, regular improvements appears on the turnover, as data about both ROWA models are inferior as compared with the corresponding one for the CVaR criterion.

In conclusion, the analysis of the minimum risk models, reveals that ROWA portfolio can improve diversification, exposition and turnover of the CVaR portfolios, specially in the ROWA version corresponding to a realistic investor. Results on portfolio compositions are sound, showing a regular behavior in the decreasing of those indexes. Increasing diversification affects the ex-post return, as the realistic model provides less return variability than the CVaR.

Next, we analyze the ROWA models in the risk/return optimization framework. In Table 3 and in Fig. 3 data about the ex-post returns of the risk/return models are shown. The situations can be one of the two reported in Fig. 3. In the first one, exemplified by the Nasdaq frontier reported in the figure on the right, the realistic ROWA model obtains an highest Sharpe ratio than the CVaR. Moreover, the ROWA solution dominates the respective CVaR, as the highest Sharpe ratio has been obtained through the highest return and the smallest standard deviation. Such a situation occurs in the Hang-Seng, Mibtel and Nasdaq data. In the other three markets, e.g. the DowJones, the Eurostoxx and the FTSE, the situation is the one reported in the figure on the left. It can be seen, that even though the realistic ROWA models have the smallest Sharpe ratio, still their mean/standard deviation points are located on the efficient frontier, as they are providing *less* return for *less standard deviation* than the CVaR.

Again, the reason of this regular behavior can be found in Table 4, in which data about the portfolio composition are reported. As before, it can be seen that the realistic ROWA portfolios are those with the highest level of diversification, the lowest exposure, and they necessitate the least turnover. Similarly to the minimum-risk models, these indexes decrease by approximately 10% when transitioning from the CVaR to the realistic ROWA criterion.

Table 3
Return indexes for markets and models (risk/return model).

Market	Model	Min	0.25	Median	0.75	Max	Mean	SD	Sharpe
DowJones	1/N	−8.234	−0.443	0.040	0.583	11.998	0.059	1.115	5.258
	CVaR	−7.689	−0.430	0.024	0.540	12.325	0.044	0.931	4.687
	ROWA(0.9)	−7.635	−0.443	0.038	0.555	13.459	0.053	0.964	5.489
	ROWA(0.8)	−7.662	−0.432	0.038	0.542	13.515	0.050	0.951	5.269
	ROWA(0.7)	−7.675	−0.424	0.037	0.542	13.572	0.050	0.950	5.312
	ROWA(0.6)	−7.704	−0.426	0.038	0.539	13.639	0.050	0.949	5.297
	ROWA(0.5)	−7.704	−0.423	0.037	0.540	13.639	0.050	0.949	5.261
	ROWA(0.5/0.9)	−8.026	−0.415	0.031	0.512	12.872	0.042	0.914	4.615
	ROWA(0.6/0.9)	−8.026	−0.415	0.032	0.509	12.872	0.042	0.912	4.595
	ROWA(0.7/0.9)	−7.963	−0.418	0.029	0.523	12.760	0.041	0.913	4.462
	Huang(3)	−7.634	−0.434	0.033	0.541	12.898	0.046	0.930	4.908
	Huang(5)	−7.942	−0.418	0.033	0.526	12.570	0.042	0.909	4.673
Eurostoxx	1/N	−7.816	−0.665	0.024	0.720	10.704	0.028	1.458	1.947
	CVaR	−8.339	−0.459	0.044	0.567	8.739	0.028	1.011	2.758
	ROWA(0.9)	−10.104	−0.460	0.048	0.573	9.755	0.034	1.044	3.240
	ROWA(0.8)	−8.903	−0.455	0.038	0.578	10.114	0.032	1.033	3.125
	ROWA(0.7)	−8.902	−0.453	0.035	0.577	10.111	0.032	1.032	3.091
	ROWA(0.6)	−8.903	−0.451	0.038	0.581	10.114	0.033	1.032	3.189
	ROWA(0.5)	−8.944	−0.453	0.038	0.582	9.981	0.033	1.032	3.184
	ROWA(0.5/0.9)	−9.446	−0.430	0.038	0.541	9.121	0.027	0.984	2.724
	ROWA(0.6/0.9)	−9.384	−0.431	0.034	0.545	9.366	0.027	0.985	2.710
	ROWA(0.7/0.9)	−9.174	−0.431	0.033	0.544	9.399	0.026	0.984	2.660
	Huang(3)	−8.861	−0.438	0.042	0.547	8.149	0.028	1.009	2.824
	Huang(5)	−8.494	−0.444	0.051	0.560	8.887	0.033	1.005	3.316
FTSE	1/N	−7.907	−0.469	0.061	0.591	8.146	0.053	1.167	4.507
	CVaR	−6.517	−0.337	0.053	0.499	8.252	0.049	0.866	5.707
	ROWA(0.9)	−6.897	−0.371	0.050	0.505	6.336	0.050	0.892	5.600
	ROWA(0.8)	−6.696	−0.358	0.046	0.506	5.264	0.050	0.874	5.736
	ROWA(0.7)	−6.635	−0.353	0.043	0.507	4.795	0.049	0.868	5.600
	ROWA(0.6)	−6.705	−0.360	0.039	0.508	4.630	0.047	0.864	5.390
	ROWA(0.5)	−6.705	−0.361	0.042	0.509	4.629	0.048	0.865	5.538
	ROWA(0.5/0.9)	−5.644	−0.339	0.046	0.491	9.489	0.050	0.858	5.833
	ROWA(0.6/0.9)	−5.649	−0.338	0.047	0.488	9.460	0.049	0.859	5.747
	ROWA(0.7/0.9)	−5.963	−0.342	0.055	0.487	8.385	0.050	0.857	5.794
	Huang(3)	−6.104	−0.350	0.047	0.493	6.866	0.049	0.883	5.544
	Huang(5)	−6.022	−0.355	0.055	0.493	6.607	0.053	0.858	6.155
HangSeng	1/N	−12.613	−0.600	0.000	0.745	12.190	0.051	1.536	3.302
	CVaR	−9.947	−0.399	0.030	0.588	11.164	0.052	1.082	4.849
	ROWA(0.9)	−8.903	−0.465	0.016	0.606	7.521	0.056	1.111	5.063
	ROWA(0.8)	−9.105	−0.441	0.013	0.599	7.317	0.058	1.091	5.317
	ROWA(0.7)	−9.108	−0.437	0.021	0.604	7.300	0.057	1.079	5.298
	ROWA(0.6)	−9.105	−0.438	0.016	0.608	7.278	0.057	1.077	5.260
	ROWA(0.5)	−9.105	−0.437	0.015	0.606	7.278	0.057	1.078	5.292
	ROWA(0.5/0.9)	−10.783	−0.385	0.024	0.549	10.085	0.047	1.037	4.493
	ROWA(0.6/0.9)	−10.606	−0.390	0.025	0.554	10.223	0.046	1.041	4.444
	ROWA(0.7/0.9)	−10.969	−0.402	0.030	0.549	10.399	0.047	1.051	4.500
	Huang(3)	−9.389	−0.409	0.020	0.545	8.689	0.046	1.018	4.508
	Huang(5)	−11.274	−0.392	0.027	0.528	13.507	0.043	1.018	4.209
Mibtel	1/N	−4.682	−0.292	0.101	0.481	3.051	0.037	0.778	4.786
	CVaR	−5.867	−0.236	0.063	0.396	6.386	0.049	0.679	7.295
	ROWA(0.9)	−4.624	−0.290	0.049	0.403	5.115	0.035	0.709	5.000
	ROWA(0.8)	−5.016	−0.271	0.057	0.415	5.498	0.038	0.691	5.504
	ROWA(0.7)	−5.018	−0.264	0.059	0.412	5.501	0.040	0.690	5.752
	ROWA(0.6)	−5.019	−0.265	0.058	0.409	5.501	0.040	0.690	5.832
	ROWA(0.5)	−5.017	−0.268	0.058	0.394	5.499	0.041	0.689	5.942
	ROWA(0.5/0.9)	−5.795	−0.207	0.081	0.403	6.347	0.058	0.673	8.684
	ROWA(0.6/0.9)	−5.825	−0.203	0.077	0.408	6.357	0.058	0.673	8.688
	ROWA(0.7/0.9)	−5.849	−0.211	0.076	0.402	6.358	0.058	0.674	8.670
	Huang(3)	−5.792	−0.230	0.062	0.410	6.335	0.058	0.678	8.622
	Huang(5)	−4.113	−0.243	0.084	0.387	3.780	0.056	0.606	9.235
Nasdaq	1/N	−9.170	−0.514	0.087	0.733	11.672	0.072	1.345	5.344
	CVaR	−7.405	−0.454	0.059	0.629	7.345	0.071	1.075	6.603
	ROWA(0.9)	−6.872	−0.447	0.034	0.616	8.277	0.064	1.080	5.902
	ROWA(0.8)	−6.563	−0.446	0.051	0.625	8.038	0.064	1.075	5.930
	ROWA(0.7)	−6.704	−0.437	0.049	0.612	7.746	0.063	1.073	5.843
	ROWA(0.6)	−6.431	−0.427	0.045	0.608	7.945	0.064	1.071	5.968
	ROWA(0.5)	−6.431	−0.416	0.046	0.609	8.194	0.065	1.070	6.041
	ROWA(0.5/0.9)	−7.459	−0.410	0.083	0.633	7.282	0.078	1.050	7.419
	ROWA(0.6/0.9)	−7.083	−0.416	0.085	0.638	7.251	0.079	1.047	7.544
	ROWA(0.7/0.9)	−7.306	−0.409	0.069	0.622	7.051	0.074	1.053	7.060
	Huang(3)	−8.190	−0.449	0.053	0.636	8.374	0.071	1.102	6.479
	Huang(5)	−7.454	−0.433	0.047	0.611	8.447	0.065	1.079	5.983

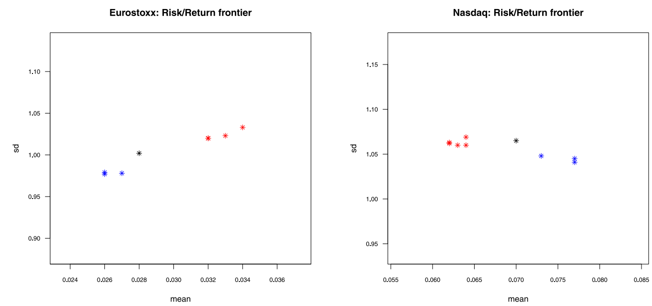

Fig. 3. Risk/Return frontiers: Black points are CVaR models, red points are pessimistic ROWA, blue points are realistic ROWA models.

When we compare the ROWA with the Huang et al.'s portfolios, we can see that their results are quite similar. The data reported in Table 3 shows that averages and deviations of the Huang model are in the range of the ROWA portfolios without clear evidence that one of the models should be preferred. Comparable conclusions can be drawn from the data presented in Table 4, in which the only notable effect is that, looking at the HH-index and Max-index, the Huang model with 5 scenarios obtains the most diversified portfolio for a small fraction. In conclusion, the choice between the ROWA and Huang models cannot be solely determined by empirical returns and portfolios; instead, it should be based on the optimization goals of the decision-makers.

To summarize our findings, we have shown that the realistic ROWA model is a reliable extension of the CVaR model. The CVaR model suffers from the limitation of considering just one CVaR measure, corresponding to just one fixed value of β. Deciding which β should be the best for the optimization model is a problem never solved and always subject to empirical considerations. Conversely, the ROWA models overcome this limitation because it considers a collection of different CVaR's, characterized by different β's, and aggregated through the maximum regret function. Computational tests on real financial data have shown that specific choices of the β range, corresponding to what was called a realistic investor, consistently improves the CVaR solutions in a variety of settings. In minimum risk context, they provide the least ex-post standard deviation. In mean/return settings, they provide the best Sharpe ratios, or points located on efficient frontiers anyway. The reason why this happens is the portfolio composition; when portfolio weights are analyzed, the realistic ROWA models are the ones characterized by the highest diversification and least maximum exposition and turnover.

6. Concluding remarks

This paper is focused on the way in which ordered weighted averages of costs can be used as risk measures in the context of portfolio optimization. In financial jargon, ordered weighted averages or OWAs, are discrete Conditional Value-at-Risk (CVaR). Here, we have analyzed the possibility of using multiple OWA values simultaneously for portfolio optimization. This is justified by the fact that each CVaR measure can represent a distinct attitude towards risk. A minimax regret model has been proposed in order to find a *compromise* investment policy which consider each one of these risk measures as the *right* scenario under which we should have optimized our decision. We have derived operational mathematical formulations and tested experimentally the effect of our proposed solution.

The case in which our returns depend linearly on the amount of wealth invested on each asset has been studied in detail. We find the key formulation $(PLEX)$ that allows us writing a family of *Linear Programming* problems just by knowing the extreme points of a given polyhedron Δ of the so-called δ-weights. Each set Δ defines a given portfolio optimization model in the basis of OWA costs. As an example,

Table 4
Portfolio indexes for markets and models (mean/risk model).

Market	Model	HH-index	Max-index	Turn over
Dow Jones	CVaR	0.212	0.324	0.353
	ROWA(0.9)	0.225	0.339	0.334
	ROWA(0.8)	0.221	0.336	0.326
	ROWA(0.7)	0.219	0.333	0.321
	ROWA(0.6)	0.218	0.333	0.319
	ROWA(0.5)	0.218	0.333	0.319
	ROWA(0.5/0.9)	0.190	0.295	0.285
	ROWA(0.6/0.9)	0.190	0.297	0.288
	ROWA(0.7/0.9)	0.194	0.301	0.296
	Huang(3)	0.205	0.316	0.377
	Huang(5)	0.184	0.293	0.301
Eurostoxx	CVaR	0.232	0.336	0.358
	ROWA(0.9)	0.253	0.367	0.357
	ROWA(0.8)	0.245	0.357	0.343
	ROWA(0.7)	0.244	0.355	0.343
	ROWA(0.6)	0.243	0.354	0.341
	ROWA(0.5)	0.243	0.354	0.341
	ROWA(0.5/0.9)	0.209	0.323	0.296
	ROWA(0.6/0.9)	0.211	0.325	0.302
	ROWA(0.7/0.9)	0.215	0.329	0.308
	Huang(3)	0.227	0.339	0.393
	Huang(5)	0.206	0.319	0.315
FTSE	CVaR	0.157	0.257	0.415
	ROWA(0.9)	0.167	0.271	0.408
	ROWA(0.8)	0.156	0.256	0.389
	ROWA(0.7)	0.153	0.252	0.384
	ROWA(0.6)	0.151	0.249	0.377
	ROWA(0.5)	0.151	0.248	0.377
	ROWA(0.5/0.9)	0.136	0.232	0.356
	ROWA(0.6/0.9)	0.138	0.234	0.355
	ROWA(0.7/0.9)	0.140	0.238	0.364
	Huang(3)	0.162	0.268	0.467
	Huang(5)	0.139	0.232	0.367
HangSeng	CVaR	0.228	0.339	0.365
	ROWA(0.9)	0.229	0.340	0.351
	ROWA(0.8)	0.226	0.337	0.331
	ROWA(0.7)	0.223	0.336	0.327
	ROWA(0.6)	0.223	0.335	0.324
	ROWA(0.5)	0.223	0.335	0.325
	ROWA(0.5/0.9)	0.215	0.330	0.287
	ROWA(0.6/0.9)	0.216	0.332	0.293
	ROWA(0.7/0.9)	0.219	0.333	0.304
	Huang(3)	0.236	0.351	0.364
	Huang(5)	0.207	0.319	0.285
Mibtel	CVaR	0.164	0.261	0.373
	ROWA(0.9)	0.188	0.298	0.385
	ROWA(0.8)	0.182	0.294	0.360
	ROWA(0.7)	0.180	0.290	0.355
	ROWA(0.6)	0.179	0.290	0.348
	ROWA(0.5)	0.179	0.289	0.343
	ROWA(0.5/0.9)	0.152	0.245	0.321
	ROWA(0.6/0.9)	0.153	0.247	0.328
	ROWA(0.7/0.9)	0.154	0.247	0.338
	Huang(3)	0.189	0.289	0.416
	Huang(5)	0.137	0.237	0.336
Nasdaq	CVaR	0.186	0.305	0.417
	ROWA(0.9)	0.187	0.299	0.387
	ROWA(0.8)	0.180	0.291	0.369
	ROWA(0.7)	0.175	0.284	0.362
	ROWA(0.6)	0.174	0.285	0.360
	ROWA(0.5)	0.174	0.284	0.358
	ROWA(0.5/0.9)	0.164	0.278	0.350
	ROWA(0.6/0.9)	0.164	0.277	0.353
	ROWA(0.7/0.9)	0.168	0.284	0.355
	Huang(3)	0.171	0.285	0.458
	Huang(5)	0.158	0.271	0.377

the formulation obtained for a Δ-set modeling multiple tolerance levels in the CVaR measure and the so-called *robust orness model* have been analyzed. However, the formulation $(PLEX)$ opens the possibility to extend the analysis to other interesting models characterized through this Δ-set. For instance, this Δ-set can be defined through existing linear relations between the δ-weights by means of a given class of M-matrix (see Ahn, 2017 and the references therein) which enables us to find its extreme points readily due to what is known as *inverse-positive* property. The knowledge of extreme points not only allows us writing Linear Programming formulations as $(PLEX)$, it also helps us to prioritize feasible portfolios by iterative exploration of decision-maker's preference in which the extreme δ-weights can be modified by the addition of new preference information (see *e.g.* Ahn, 2017) what could be an interesting subject for forthcoming developments.

CRediT authorship contribution statement

Stefano Benati: Conceptualization, Investigation, Methodology, Software, Validation. **Eduardo Conde:** Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Validation.

Data availability

Data will be made available on request.

Acknowledgments

Stefano Benati acknowledges support from the European Union – Next Generation EU, under the call PRIN 2022, project "Networks: decomposition, clustering and community detection" (2022LT8P4J) – CUP E53D23006360006.

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