Geometric Simple connectivity in smooth Four-dimensional Topology (Po V-A)
– An outline –

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Abstract The main result of this paper is the following

Theorem 1. *Let Δ^4 be a smooth compact bounded 4-manifold, which is geometrically simply-connected* ***at long distance.*** *It is assumed that $\partial\Delta^4$ is a homology sphere. Then the open manifold* $int(\Delta^4 \# \infty \# (S^2 \times D^2))$ *is geometrically simply connected.*

The setting for this result is the DIFF category. Together with our earlier results it implies the

Corollary 2. *If Δ^3 is a homotopy 3-ball then* $int((\Delta^3 \times I) \# \infty \# (S^2 \times D^2))$ *is geometrically simply connected.*

This is one of the links which was missing in our program for the Poincaré Conjecture.

1. Introduction

Before we can state our main result we will need to develop some terminology. Let M^4 be any smooth compact bounded 4-manifold. We consider some collar of the boundary $\partial M^4 \times [0,1] \subset M^4$, such that $\partial M^4 \times 1 = \partial M^4$, and with this we define

$$M^4_{\text{small}} = M^4 - \partial M^4 \times (0,1]\,, \tag{1}$$

i.e. M^4_{small} is just another, diffeomorphic, copy of M^4 obtained by pushing M^4 away from its boundary, towards the interior.
A smooth compact bounded 4-manifold X^4 which possesses a smooth handlebody decomposition

$$X^4 = B^4 + \{\text{handles of index } \lambda = 2 \text{ and } \lambda = 3\}\,, \tag{2}$$

will be said to be ***geometrically simple connected***. This notion immediately extends to non-compact manifolds, but then one has to insist that the handlebody decompositions be PROPER (every compact subset of X^4, should only be touched by finitely many handles). In Morse theoretical terms, a smooth manifold M^n which might be non-compact and also with non-empty boundary is said to be geometrically simply connected if there is a PROPER function $M^n \xrightarrow{f} R_+$ such that

(1) All the singularities of f are of Morse type and contained inside $\text{int}M^n$. None of these singularities are of index $\lambda = 1$.
(2) The restriction $f|\partial M^n$ is also of Morse type and all the ***non fake*** singularities of $f|\partial M^n$ are of index $\lambda \neq 1$. A singular point $x_0 \in \partial M^n$ of f is non fake if the transformation $f^{-1}(-\infty, f(x_0)-\varepsilon] \Rightarrow f^{-1}(-\infty, f(x_0)+\varepsilon]$ involves a change in topology. In local coordinates, with $x_1, \dots, x_n \geq 0$ corresponding to M^n, $f = \text{const} + x_n - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_{n-1}^2$ is non fake while $f = \text{const} - x_n - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_{n-1}^2$ is fake ; these are the generic models.

Next, let V^4 be a smooth 4-manifold which is either open ***or*** compact bounded. Following a suggestion of Barry Mazur, we will say that V^4 is ***geometrically simply connected at long distance***, if any compact set $K \subset \text{int}V^4$ can be engulfed inside a smooth compact codimension zero submanifold $X^4 \subset V^4$ which is geometrically simply connected. When V^4 is compact, this means exactly that such an X^4 can be sandwiched between V^4_{small} and V^4, as follows

$$V^4_{\text{small}} \subset X^4 \subset V^4\,. \tag{3}$$

[Side-remark: There is also a weaker notion, where we only ask that K should be engulfable by a compact simply-connected subspace K_0. In dimension 3, this notion implies simple connectivity at infinity for open simply-connected manifolds, but this is a part of an altogether different story, to be told elsewhere (see, for instance [Po8]). As another side-remark, let me mention that in all our discussion "only handles of index $\lambda = 2$ and $\lambda = 3$" , could as well be replaced by "no handles of index $\lambda = 1$ ".] Our main result is the following
Theorem 1. *Let Δ^4 be a smooth compact bounded 4-manifold, which is geometrically simply-connected at long distance. It is assumed that $\Sigma^3 = \partial\Delta^4$ is a homology*

$\partial \Delta^4 = \Sigma^3$

∂X^4

$\Sigma^3 \times [0,1] - X^4$

$\partial \Delta^4_{\text{small}}$

X^4

Δ^4_{small}

$= \Delta^4_{\text{small}}$

$= \Delta^4 - \text{int}\Delta^4_{\text{small}}$

Figure 1. Here X^4 lives inside the fat contour (∂X^4) and it has no handles of index $\lambda = 1$. The collar $\Delta^4 - \overset{\circ}{\Delta}{}^4_{\text{small}}$ has an obvious product structure.

3-sphere. Then the open manifold

$$Y^4 \underset{\text{DEF}}{=} \text{int}(\Delta^4 \# \infty \# (S^2 \times D^2)) \tag{4}$$

(where the infinite connected sums is taken along the boundaries) admits a smooth PROPER *handlebody decomposition* ***without handles of index*** $\lambda = 1$*, i.e.* Y^4 *is geometrically simply-connected.*

Figure 1 gives a very schematical view of the hypothesis of our theorem.

One of the main applications we have in mind for the theorem above, is the case when $\Delta^4 = \Delta^3 \times I$, with $\Delta^3 =$ homotopy ball. It follows then, from the ***smooth tameness theorem*** in [Po4], [Po5] (see also [Ga]), that $\Delta^3 \times I$ is geometrically simply-connected at long distance and hence, we get the following

Corollary 2. *If* Δ^3 *is a homotopy 3-ball, then the smooth open 4-manifold*

$$\text{int}((\Delta^3 \times I) \# \infty \# (S^2 \times D^2))$$

is geometrically simply-connected.

It is due to a suggestion of Michael Freedman that we have shifted from the special case $\Delta^3 \times I$ to the more general Δ^4, where only the product structure of the collar $\Delta^4 - \overset{\circ}{\Delta}{}^4_{\text{small}} = \Sigma^3 \times I$ is made use of. (Here $\Sigma^3 = \partial \Delta^4$, which is, of course, not necessarily a homotopy sphere.)

But then, as Frank Quinn has also pointed out, in connection with an earlier, wrong start, for this work, ***if*** our arguments, would take place only inside the collar above, and then would also "prove" something like $\pi_1(\Sigma^3 \times I - \overset{\circ}{X}{}^4, \partial X^4) = 0$ (which is certainly ***not*** an assumption in our theorem), then something would be quite wrong, indeed. At a different moment, Michael Freedman also pointed out that if in any way our arguments would "prove" that a contractible compact 4-manifold Δ^4 with non-simply-connected boundary has no handles of index one then we would again be in deep trouble. (Remember that Andrew Casson has shown that if $\pi_1 \partial \Delta^4$ has a nontrivial representation into a compact connected Lie group, like it is for instance the case for the manifolds of Barry Mazur [Ma] and Po [Po0], then one cannot kill the 1-handles of Δ^4 (see for instance [Man], [GR]). Both of these cautionary remarks have been extremely useful in helping me to stay on the right track, while working on the present paper, where a blend of ***global*** arguments making use of the whole geometrically simply-connected blub X^4 are used in conjunction with arguments which stay ***localized*** inside the collar $\Sigma^3 \times I$, making use of its product structure.

It should also be mentionned, at this point, that the present work, for which the complete proofs are to be found in the long preprint [Po7-bis] is meant to replace, in part, Theorems A and C from the preprint [Po7], in which a mistake has been detected, during the Spring 1995, by David Gabai and Michael Freedman. This mistake was localized, exactly, by David, at the level of figure 5.14 [Po7] where between the line of (N, IX)'s there is also a (p_ω, S) (not represented in the figure in question), which spoils the COHERENCE. A large part of [Po7], in particular Theorem B, and most of the technical lemmas (some of which are actually quite useful, in particular for the present work), is not affected by this gap.

Otherwise, as far as my program for the Poincaré Conjecture is concerned (see [Ga]), at the present moment (October 1997) the situation is as follows. The papers [Po4], [Po5] (which themselves use [Po1], [Po2], [Po3]) show that for any homotopy 3-ball Δ^3, the 4-manifold $\Delta^3 \times I$ is geometrically simply-connected at long distance (just as in the hypothesis of the theorem from the beginning of this introduction). In the series of papers of which the preprints [Po6] are the first four parts (the rest being in the process of typing), it is shown, on the other hand, that if $\Delta^3 \times I$ is geometrically simply-connected, then $\Delta^3 = B^3$. "Theorem" C from [Po7] which was supposed to be the bridge between geometric simple connectivity ***at long distance*** for $\Delta^3 \times I$, and mere geometric simple connectivity is, for the time being, missing. The THEOREM above is a first step in a two-stage program, to revive it ; but only an outline of its proof will be given here, the full details are the substance of a considerably longer paper (which is ready for typing). Let me also add that this present paper replaces and supersedes my IHES preprint "Differential topology in dimension 3+1" IHES/M/96/18 from 1996.

The present paper also owes a lot to Michael Freedman, David Gabai and Frank Quinn. Special thanks are actually due to David Gabai with whom I had innumerable many hours of conversation concerning the present work. Wery many of the ingredients of the paper, either come from these conversations, or were directly invented by David.

We would also like to thank Martine Justin and Laurence Stephen for the typing.

2. Preliminaries for the infinite construction

We start with the data

(5) $$\Delta^4_{\text{small}} \subset X^4 \subset \Delta^4 \, , \partial \Delta^4 \underset{\text{DEF}}{=} \Sigma^3$$

where the "blub" X^4 is a smooth compact, geometrically simply-connected 4-manifold, and with the collar $\Sigma^3 \times I = \Delta^4 - \text{int}\ \Delta^4_{\text{small}}$ ($\Sigma^3 \times 0 = \partial\Delta^4_{\text{small}}$, $\Sigma^3 \times 1 = \partial\Delta^4$.)

We consider, also, the product structure of the collar

$$
N^4 = X^4 \cap (\Sigma^3 \times I) \xrightarrow{\pi_0} I \ , \qquad \pi: N^4 \to \Sigma^3 \tag{6}
$$

the splitting

$$
X^4 = \Delta^4_{\text{small}} \cup N^4 \ , \tag{7}
$$

and the ***non-fake*** singularities of the Morse function $\pi_0|\partial X^4$, i.e. those for which the transformation $N^4 \cap \pi_0^{-1}[0, \pi_0(p) - \varepsilon] \Rightarrow N^4 \cap \pi_0^{-1}[0, \pi_0(p) + \varepsilon]$ involves an actual change of topology. It should be pretty clear that it is the non-fake minima of $\pi_0|\partial X^4$ which are the obstruction in our problem ; a very elementary argument, using the complement cobordism $W^4 = \Sigma^3 \times I - \text{int}X^4$ shows that in the absence this kind of minima, Δ^4 can be gotten from the blub X^4 via an addition of handles of index $\lambda > 1$ (for further purposes, we will call this kind of argument "***external***".)

Here is a first, naïve idea for overcoming this obstruction. One could "kill" the non-fake minima of $\pi_0|\partial X^4$ by adding to $N^4 \subset \Sigma^3 \times I$ an embedded vertical handle of index $\lambda = 1$, for each of them, with the upper attaching zone localized at the minimum in question. In turn, these 1-handles would have to be killed by some added, compensating 2-handles, but in general these will meet again N^4, a standard difficulty indeed. In what follows next we will try to formalize this issue.

It is possible to change the definition of N^4, leaving $\Delta^4_{\text{small}} \cup N^4$ geometrically simply-connected, so that for the ***new*** $N^4 \subset \Sigma^3 \times I$ (the only one which will be mentioned from now on), the following things happen. There is a family of ***master*** 1-handles $H_1, \ldots, H_\alpha$, which are embedded vertical 1-handles attached to N^4 inside $\Sigma^3 \times I$, and there is also a family of 2-by-2 disjoined 2-disks $D^2_1, D^2_2, \ldots, D^2_\alpha \subset \Sigma^3 \times I$, with $\partial D^2_i \subset \partial(N^4 \cup H_i)$. The generic D^2_i is represented in figure 2, where it occupies the white zone bounded by the fat closed loop (later called $(D^2_i)_0$), the hatched zones $\Delta_{ij} (\subset N^4)$ and also the doubly hatched zones $\overline{H}_{ij}$ (= ***auxiliary*** handles, as we will see). With the notations from the figure 2, it is assumed that $\pi| \sum\limits_i [p_i, m_i, q_i]$ injects and that $D^2_i = \{$the region of $\pi^{-1}\pi[p_i, m_i, q_i]$ contained between some common ***ground level*** $t = \eta$ and $[p_i, m_i, q_i]\}$.

The construction can be performed in such a way, that the following things happen.

A) Among the points $p_1, \ldots, p_\alpha$ (figure 2) are all the non-fake minima of $\pi_0|\partial N^4$.

B) The open 2-cell (int D^2_i) meets N^4 transversally and cuts out of it a number of 2-cells $\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{ir(i)}$ with $\pi\Delta_{i\ell} \cap \pi\Delta_{im} = \emptyset$. Each Δ_{ij} can be joined to the ground level $t = \eta$ by an embedded, vertical, ***auxiliary*** 1-handle $\overline{H}_{ij}$, exactly like in figure 2. The various $\{H_i, \overline{H}_{ij}\}$ are 2-by-2 disjoined, and so are their π-images.

C) For each $\overline{H}_{ij}$ there is an embedded are $A_{ij} \subset N^4$, which joins the upper attaching zone of $\overline{H}_{ij}$ to the upper attaching zone of some master handle $H_{h(i,j)}$ ($1 \leq h(i,j) \leq \alpha$ and $h \neq i$). Each of the arcs A_{ij} is contained inside some

t

x

m_i p_i

Δ_{i1}

Δ_{i2}

$\overline{H}_{i1}$ $\overline{H}_{i2}$ $\overline{H}_i$

ground level
$t = \eta$

q_i y_i

$= N^4$ =added 1-handles

$= \partial(D_i^2)_0$ (this will be explained later on)

Figure 2. The 2-disk D_i^2 of boundary the closed loop $[p_i, m_i, q_i, y_i, p_i]$. Here the arc $[p_i, m_i, q_i, y_i] \subset \partial N^4$, and $[p_i, y_i] \subset$ { lateral surface of H_i}. Essentially, H_i is $[p_i, y_i]$. The white, unshaded, part of D_i^2 is the 2-cell $(D_i^2)_0 \subset D_i^2$.

horizontal 3-slice $\pi_0^{-1}(t) = \Sigma^3 \times t$, and the fact that $i \neq h$ is a special feature of our construction, which will be useful later. The map $\pi | \sum_{ij} A_{ij}$ is injective and there is an embedded 2-disk $\overline{D}_{ij}^2 \subset \Sigma^3 \times I$ with $\partial \overline{D}_{ij}^2 \subset \partial(N^4 \cup \overline{H}_{ij} \cup H_{h(i,j)})$, like in figure 3, and also with $\text{int}\overline{D}_{ij}^2 \cap (N^4 \cup \Sigma H_i \cup \Sigma \overline{H}_{ij}) = \emptyset$. By analogy with D_i^2, we have $\overline{D}_{ij}^2 = \{$the region of $\pi^{-1}\pi A_{ij}$ which is contained between A_{ij} and $t = \eta\}$. The $\{D_i^2, \overline{D}_{ij}^2\}$ are 2-by-2 disjoined, and so are their π-images. □

It should be noticed, at this point, that once these conditions A, B, C are fulfilled, the geometrical data we have described is, in some sense, ***complete***. The family of master handles $H_1, \dots, H_\alpha$ is large enough in order to kill all the non-fake minima of $\pi_0 | \partial N^4$, the auxiliary handles form a family large enough too "kill" (see figure 2) all the intersections of the compensating disks of the master handles, $D_1^2, \dots, D_\alpha^2$ with N^4, and finally, the family of master handles is itself large enough so that the compensating disks $\{\overline{D}_{ij}^2\}$ of the $\{\overline{H}_{ij}\}$ can rest on $\partial(N^4 + H + \overline{H})$, without touching anything else. Similarly, the compensating disks $\{(D_i^2)_0\}$ rest on $\partial(N^4 + H + \overline{H})$, without touhcing anything else.

To our geometrical data, as described above, we will attach now a square $\alpha \times \alpha$ matrix $T^0 = (a_{nm})$, with $a_{nm} \in Z_+$ and $n, m \in \{1, \dots, \alpha\}$ (same indexes as the

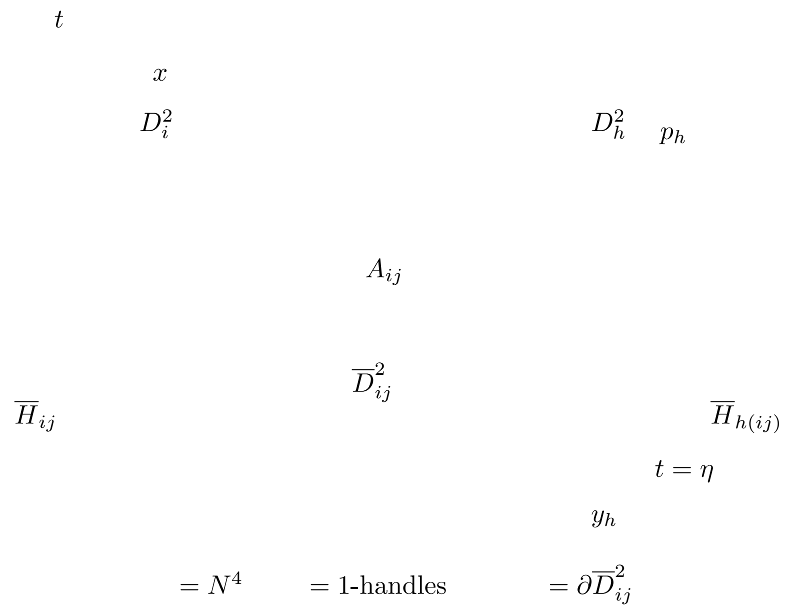


Figure 3 $\overline{D}_{ij}^2$. The $\overline{H}_{ij}$'s for different (ij)'s are distinct.

H_i's), which encapsules the geometrical information ; the exact definition is

(8) $$a_{nm} = \{\text{the number of } k\text{'s such that } m = h(n,k)\} =$$

$$= \{\text{the number of } \overline{H}_{nk}\text{'s with } \overline{H}_{nk} \cap D_n^2 \neq \emptyset \neq H_m \cap \overline{D}_{nk}^2\} \, .$$

Is is convenient to think of the matrix T^0 as being an oriented graph with vertices (or "states") labelled by $1, \ldots, \alpha$ and with a_{nm} oriented edges (or "arrows") of the form $n \to m$.

For technical reasons, our construction has been performed in such a way that in the context of our particular T^0, given by (8), the diagonal entries are all zero ($a_{nn} = 0$) but, otherwise, T^0 can be completely general. It will be very convenient for us to think of the vertices (= states) of T^0 as being exactly the master handles $H_1, \ldots, H_\alpha$. For such a state H_i, the attached auxiliary handles $\overline{H}_{i1}, \ldots, \overline{H}_{ir(i)}$ (see figure 2, and remember that these are exactly the auxiliary handles touched by D_i^2) parametrize the ***outgoing*** T^0***-arrows*** of H_i. With this we have the following very convenient notation, as a substitute for (8)

(9) $$T^0 = \left\{ H_i \xrightarrow{\overline{H}_{ij}} H_{h(i,j)} \right\}$$

(with $i = 1, \ldots, \alpha$ and $j = 1, \ldots, r(i)$.)

One should ***not*** mix up T^0, which has no diagonal entries, with the geometric intersection matrices relating $\{H_i$ and $D_j^2\}$ or $\{H_i + \overline{H}_{jk}$ and $(D_\ell^2)_0$ and $\overline{D}_{mn}^2\}$, which are both equal to 1 along the diagonal. But the complexity of T^0 (in particular its number of closed oriented orbits) is a kind of measure for the difficulty presented by the geometrical data. As far as the geometric intersection matrix

$\{\partial(D^2_\ell)_0 + \partial\overline{D}^2_{mn}\} \cdot \{\delta H_i + \delta \overline{H}_{jk}\}$ (where $\delta H = \{$boundary of the co-core of $H\})$ itself is concerned, the only non-zero terms are

$$\partial(D^2_\ell)_0 \cdot \delta H_\ell = \partial\overline{D}^2_{mn} \cdot \delta\overline{H}_{mn} = 1\,, \tag{10}$$

$$\partial(D^2_\ell)_0 \cdot \delta\overline{H}_{\ell n} = 2 \ , \ \partial\overline{D}^2_{mn} \cdot \delta H_{h(m,n)} = 1$$

The features A), B), C) described above, were all three completely localized inside the collar, but a fourth feature, which is global, taking advantage of the full blub, can be added too.

Let us notice, first, that as far as our theorem is concerned, we can as well replace Δ^4, X^4 by their thickened 2-skeleta ; this only changes our original problem into an equivalent one. So, without bothering to change notations, we will assume from now on the following items.

(11-α) The blub X^4 is the 4-dimensional regular neighbourhood of its 2-skeleton denoted $X_{(2)}$. This $X_{(2)}$ contains a 1-dimensinal collared subcomplex $\tau_{(1)} \subset X_{(2)}$ ("collared" meaning that there is an ***open*** subset $\tau_{(1)} \times (-\varepsilon,\varepsilon) \subset X_{(2)}$ with $\tau_{(1)} \times 0 = \tau_{(1)}$) such that (7) is just the 4-dimensional thickening of a 2-dimensional splitting

$$X_{(2)} = \Delta_{(2)} \underbrace{\bigcup}_{\tau_{(1)}} N_{(2)}\,.$$

[This $\tau_{(1)}$ is the 1-skeleton of a certain triangulation τ of the original $\Sigma^3 = \partial\Delta^4$.]

(11-β) Our $X_{(2)}$ is ***almost-collapsible*** in the following sense. There is a finite system of 2-by-2 disjoined small disks contained inside the {smooth part of $X_{(2)}$} which we call {holes}, such that the 2-dimensional, finite complex

$$K^2 = \overline{X_{(2)} - \{\text{holes}\}}$$

is ***collapsible***. [It is here that the geometrical simple connectivity of the blub is being used.] We can assume that $\partial X_{(2)} = \emptyset$.

[Here are some Remarks concerning almost-collapsibility. The notion of almost-collapsibility is just slightly weaker than "collapsible", but is possesses many virtues, which makes it much more manageable.

a) The 2-skeleton of a collapsible complex is almost collapsible (it is not collapsible, indeed).

b) Almost collapsibility for the 2-skeleton is a property which is invariant under barycentric or stellar subdivision and also under Siebenmann's bissections [S].

c) If V^n is a smooth, geometrically simply-connected manifold, then we can find a smooth triangulation of V^n the 2-skeleton of which if almost-collapsible.

For very many practical purposes "almost-collapsible" is as good as "collapsible", for instance in issues concerning geometric simple connectivity ; see also [Po1] where the "collapsible pseudo-spine representation theorem" is implied by its almost-collapsible version.]

(11-γ) The {holes} do not touch $\tau_{(1)}$, nor anything else of interest, which might have been mentionned, so far, like for instance the 1-handles, the $\partial D^2_i, \partial(D^2_i)_0$, $\partial\overline{D}^2_{ij}$, Δ_{ij} (fig. 2) a.s.o. If we denote by Γ the union of the boundaries of the {holes}, we get a second splitting for $X_{(2)}$, namely

$$X_{(2)} = K^2 \underbrace{\bigcup}_{\Gamma} \{\text{holes}\}\,.$$

(11-δ) The various little squares Δ_{ij} (fig. 2) correspond to a finite family of points $x_{ij} \in N_{(2)} \subset X_{(2)}$, in the neighbourhood of which $N_{(2)}$ is homeomorphic to R^2 and where, inside the 4-dimensional collar, which we will continue to call $\Sigma^3 \times I$, the D_i^2's cut transversally through $N_{(2)}$. Because K^2 is collapsible, inside K^2 each x_{ij} can be joined to $\partial K^2 = \Gamma$ by a tree $T_{ij} \subset K^2$ (with $x_{ij} \in \partial T_{ij}$, $T_{ij} - \{x_{ij}\} \subset \partial K^2$, the T_{ij}'s disjoined) such that T_{ij} is in general position with the 1-skeleton of K^2 and also ***full***, in the sense that for each $y \in T_{ij} \cap$ (edge e of K^2) and for each 2-simplex σ of K^2 resting on e, T_{ij} contains exactly one little arc in σ, starting at y ; also there are no ramifications of T_{ij} other than these points y where it cuts through the 1-skeleton. One can find the T_{ij}'s by following backwards, inductively, one explicit collapsing of K^2.

We will continue to denote by {holes} the 2-handles corresponding to the holes at the 4-dimensional level. With this, the 2-dimensional splitting (11-γ) gives rise to the following 4-dimensional splitting

$$X^4 = N^4(X_{(2)}) = N^4(K^2) \underbrace{\bigcup}_{\Gamma \times D^2} \{\text{holes}\} \, . \tag{12}$$

With all this, we go back now to dimension four, and to the figure 2. Each simple closed loop $\partial \Delta_{ij}$ can be interpreted, after an obvious little isotopy, as a simple closed loop for which we will choose the notation $\ell k(ij) \subset \partial N^4(K^2)$, such that our $\ell k(ij)$ is the union of two arcs with common endpoints and disjoinded interiors, one living inside $\partial (D_i^2)_0$ (= the boundary of the core of the 2-handle corresponding to $(D_i^2)_0$, denoted H_i^2) and the other inside the 2-sphere $\delta \overline{H}_{ij}$ (= the boundary of the co-core of the 1-handle $\overline{H}_{ij}$), see also fig. 5 below.

What we can add now to the items A, B, C above is the following feature, which is itself a consequence of (11-δ), i.e. of the geometric simple connectivity of the blub X^4.

D) Inside $\partial N^4(K^2)$ one can find 2-by-2 disjoined ***Whitney disks*** $D^2(ij) \subset \partial N^4(K^2)$ such that $\partial D^2(ij) = \ell k(ij)$.

It should be clear that int $D^2(ij) \cap \Gamma \neq \emptyset$, generally speaking. But, generally speaking, we also have contacts

$$\text{int } D^2(ij) \cap \left(\Sigma \partial (D_\ell^2)_0 + \Sigma \partial \overline{D}^2_{nm} \right) \neq \emptyset \, . \tag{13}$$

Remark 1. If for all (ij)'s the L.H.S. of (13) was always $\emptyset$ then, as a very easy consequence of (10), and notwithstanding the contacts of the Whitney disks with Γ, one could deduce that Δ^4 was geometrically simply-connected, (which is, of course, stronger than what we will actually be able to prove.) □

The idea now is to push the closed orbits of T^0 (see (9)) to infinity ; the next two sections will give an idea of how to go about that at the geometrical level, this will certainly involve a lot of changes concerning $X^4, N^4(K^2)$ and the splitting (12) or the 4-dimensional version of the splitting from (11-α), namely

$$X^4 = N^4(\Delta_{(2)}) \underbrace{\bigcup}_{N^3(\tau_{(1)})} N^4(N_{(2)}) \, . \tag{13-0}$$

Notice that this last splitting is a "foamier " version of (7). It will be also convenient for us to combine the splittings (12), (13-0) into a single formula
(13-1)

$$X^4 = \Big[\underbrace{(N^4(\Delta_{(2)}) - \{\text{holes}\})}_{\text{call this } \Delta^4(w.h.)} \overbrace{\bigcup}^{N^3(\tau_{(1)})} \underbrace{(N^4(N_{(2)}) - \{\text{holes}\})}_{\text{call this } N^4(w.h.)} \Big] \overbrace{\bigcup}^{\Gamma \times D^2} \{\text{holes}\}\,;$$

here "w.h" meens "with {holes}"; of course, also, $N^4 - \{\text{holes}\}$ should be actually read as $\overline{N^4 - \{\text{holes}\}}$.

At the most simple-minded level our changes might certainly involve things like triangulating the 4-dimensional data, going to the 2-dimensional skeketon and then thickening back into dimension four. There are also other, more sophisticated, infinitistic changes which will not be described here. But the following items should be permanently kept in mind, in the next section.

(14-1) Whaterer changes the splittings (12), (13-0) might undergo, this will ***never*** affect the {holes} and $\Gamma \times D^2$, which will survive, as such, at all future levels, as a ***fixed*** number of very thin 2-handles, or just 2-handle cores, depending on the context.

(14-2) The 2-dimensional (almost)-collapsibility property will get lost right away, after the first change. But once we have the Whitney disks,, at some level, we will be able to propagate them at the next level of the construction. It is only in establishing D) above that 2-dimensional collapsibility as such, is ever used, once and for all, in the begining of our construction.

3. Symbolic dynamics

In the present, purely combinatorial section we will focus on our square matrix T^0 (see (8), (9)). But the result which we will state below applies to any square matrix with entries in Z_+, the only restriction being (in order to avoid unnecessary complications), that the diagonal entries are always assumed to be zero.

THE COMBINATORIAL LEMMA. *There is an infinite square matrix $T^\infty = (a^\infty_{I\mathcal{J}})$, with $a^\infty_{I\mathcal{J}} \in Z_+$ and $I, \mathcal{J}$ running through some infinite set of (multi-) indices such that, in terms of the corresponding oriented graphs, there is a non-degenerate surjective map, respecting arrow orientations*

(15) $$T^\infty \xrightarrow{\varphi^\infty} T^0\,,$$

with the following properties.

1) *T^∞ has deterministic past, in the sense that for each state $\mathcal{J}$ the number of incoming arrows $\sum_I a_{I\mathcal{J}} \le 1$. Moreover T^∞ has* ***no closed oriented cycles****.*
2) *The map (15) has the oriented,* ***forward going unique path lifting property****. [Careful, this does not make it, by any means, a covering map, since only the oriented, forward going paths are concerned in the lifting property.]*
3) *We can find a linear order, modelled on Z_+, for each set $(\varphi^\infty)^{-1}H_i$, let's say that we can introduce the notation*

(16) $$(\varphi^\infty)^{-1}H_i = \{H_i(1) < H_i(2) < \ldots < H_i(n) < \ldots\}\,,$$

such that, for any $i, h \in \{$*states of* $T^0\}$, $\alpha, \beta, \alpha', \beta' \in Z_+$, *if we have arrows*

$$H_i(\alpha) \xrightarrow{T^\infty} H_h(\beta) \ , \ H_i(\alpha') \xrightarrow{T^\infty} H_h(\beta') \, ,$$

then $(\alpha - \alpha')(\beta - \beta') > 0$. *Remember that neither* T^0 *nor* T^∞ *have diagonal entries. Also, 3) is actually a formal consequence of 1) and 2).*

Here is a more geometrical way to state 3). Consider $T_1^0 = \{$ the quotient space of the oriented graph T^0 obtained by gathering all the distinct directed edges with the same end points, into a unique arrow$\}$. Then, we can construct a commutative diagram

(17)
$$\begin{array}{ccc} T^\infty & \xhookrightarrow{\Lambda_\infty} & T_1^0 \times [0, \infty) \\ {\scriptstyle \varphi^\infty} \downarrow & \swarrow & \\ T^0 & & \\ \downarrow & & \\ T_1^0 & & \end{array}$$

with Λ_∞ a PROPER ***embeddings***.

For the time being, (16) is just a piece of notation, but in view of the various things said in our lemma above, we can embellish it, as follows. Anyway, the various outgoing arrows $\{(\varphi^\infty)^{-1} H_i \xrightarrow{T^\infty} \}$ fall into $r(i)$ distinct packages, each package covering a specific T^0-arrow $\overline{H}_{ij}$, outcoming from H_i (see (9)). Let us denote by $\overline{H}_{ij}(n)$ the unique T^∞-arrow covering $\overline{H}_{ij}$ and originating at $H_i(n)$. With this, the various distinct arrows of T^∞ take the form

(18)
$$H_i(n) \xrightarrow{\overline{H}_{ij}(n)} H_{h(i,j)}(\theta_{ij}(n)) \, ,$$

where, for each $i, j \in \{1, \ldots, \alpha\}$, $i \neq j$,

(19)
$$Z_+ \xrightarrow{\theta_{ij}} Z_+$$

is a strictly increasing function, with Image $\theta_{ij} \cap$ Image $\theta_{kj} = \emptyset$ (for $i \neq k$).

The maps $\{h, \theta_{ij}\}$ are a complete description of $(T^\infty, \varphi^\infty)$. It is this description which will be used later, when we will use T^∞ for doing geometry (actually the interpretation Λ_∞ from (17) is very useful too).

We will just say a few words now about how T^∞ if gotten from T^0. For square matrices with entries in Z_+ (i.e. for oriented graphs), there is a well-known operation (in ergodic theory or in statistical mechanics), called the block-transformation (= staring at the matrix, with bad glasses on). But we rather use ***inverse block-transformations*** (= putting on magnifying glasses in order to look at the matrix), of a very special type. For instance, a matrix like T^0 will have, for each state H_i a given linear order on the set $\{ \xrightarrow{T^0} H_i\}$, and our inverse block-transformations will have to pay attention to this additional structure. When our kind of transformation is iterated infinitely many times, starting with T^0 and creating more and more seemingly complicated non-degenerate surjective maps $T^n \xrightarrow{\varphi^n} T^0$ verifying 2) from the combinatorial lemma, we get a limit pattern which is invariant under further transformations, and this *is* our T^∞. Closely related (but not quite the same) operations were used in [Po4], [Po5], [Po6] (but no knowledge of these papers is required here). David Gabai suggests to look at these kind of transformations as operating on some branched surfaces and splitting them, indefinitely.

As far as the linear order for the set of incoming arrows $\{\stackrel{T^0}{\longrightarrow} H_i\}$ is concerned, in this paper, it is only a conventional abstract choice, without any particular geometrical meaning. (contrary to what has happened in the context of [Po4], [Po5], [Po6]). On the other hand, the order (16), which *will* be given a geometrical meaning at the level of our infinite geometric construction form the next section, has strictly nothing to do with the linear orders on the various sets $\{\stackrel{T^0}{\longrightarrow} H_i\}$.

4. The infinite construction

We go back now to the geometric set-up of section 2, in particular to the geometric X^4 from (12),(13-0). With the $N^4 = N^4(N_{(2)})$ form (13-0) and with the large disks $D_i^2, i = 1, \cdots \alpha$ from Fig.2, we get an obvious map

$$N^4 \cup \sum_1^\alpha H_i \underbrace{\bigcup}_{\partial D_i^2} \sum_1^\alpha D_i^2 \longrightarrow \Sigma^3 \times I \tag{20}$$

which, unfortunately, has self-intersections, namely the Δ_{ij}'s form Fig.2. This is, indeed, a very standard kind of obstruction in 4-dimensional topology and our plan is to circumvent it, in a stable sense, by an ***infinite process***.

To begin with, we actually do replace (20) with an ***embedding***, by making use of the auxiliary handles, and change each D_i^2 into the smaller disk $(D_i^2)_0 \subset D_i^2$, which is bounded by the fat arc $\partial(D_i^2)_0$ from figure 2, thereby getting the new map

$$N^4 \cup \sum_i H_i \cup \sum_{jk} \overline{H}_{jk} \cup \sum_i (D_i^2)_0 \cup \sum_{jk} \overline{D}_{jk}^2 \longrightarrow \Sigma^3 \times I. \tag{21}$$

This change from (20) to (21) has not taken us really very far, indeed. In (20) the 1-handles H_i did beautifully cancell out with the corresponding disks, but we had double points. Now the double points have disappeared, but the 1-handles $(H_i, \overline{H}_{ij})$ are no longer cancelled by their disks, at least not obviously so. The cycles of T^0, for instance, are in the way; compare (9) with the geometric intersection matrix (10) which is associated to (9).

So, the idea for handling this situation is to ***replace (21) by an infinite object*** (actually a certain infinite, but locally finite 2-dimension complex which injects inside $\Sigma^3 \times I$, we will call it P_∞^2, or its regular neighbourhood $N^4(P_\infty^2)$, see below) ***where in lieu of*** T^0 ***we will have the*** T^∞ ***from (15) above.*** If we look back at the notations from (16), (18), (19), we will read now the T^∞-states $H_i(1), H_i(2), \ldots$ (respectively the T^∞-arrows $\overline{H}_{ij}(1), \overline{H}_{ij}(2), \ldots$) as a whole infinite family of master handles of index one (respectively of auxiliary handles of index one) parallel to H_i (respectively to $\overline{H}_{ij}$). We will also want to replace each 2-handle $(D_i^2)_0$ (respectively $(\overline{D}_{ij}^2)$ from (21), by an infinite collection of parallel copies $(D_i^2)_0(1), (D_i^2)_0(2), \ldots$ (respectively $\overline{D}_{ij}^2(1), \overline{D}_{ij}^2(2), \ldots$) such that

$$\text{Each } (D_i^2)_0(n) \text{ rests on } H_i(n) + \sum_{j=1}^{r(i)} \overline{H}_{ij}(n) \text{ (Figure 2).} \tag{22-1}$$

$$\text{Each } \overline{D}_{ij}(n) \text{ rests on } \overline{H}_{ij}(n) + H_{h(i,j)}(\theta_{ij}(n)) \text{(see (18) and figure 3).} \tag{22-2}$$

With all this, ***very roughly speaking,*** our P^2_∞ will be given by the following admittedly vague recipee, with N^4(w.h.) like in (13-1), and for which the exact and precise meaning will be given in formula (28) below :

$$(23)\quad P^2_\infty = \{\text{a sort of 2-skeleton of } N^4 \text{ (w.h.) } \cup \sum_{n=1}^{\infty}\sum_i H_i(n) \cup \sum_{n=1}^{\infty}\sum_{jk} \overline{H}_{jk}(n) \cup$$

$$\cup \sum_{n=1}^{\infty}\sum_i (D^2_i)_0(n) \cup \sum_{n=1}^{\infty}\sum_{jk} \overline{D}^2_{jk}(n)\} \subset \Sigma^3 \times I.$$

Here is a slightly more precise but lengthier statement.

Proposition A.

1) *Starting with the 2-skeleton of N^4 (w.h.) (13-1) and working rel($\tau_{(1)} \times 0) +$ Γ), one can construct an infinite, locally finite 2-dimensional simplicial complex N_∞ with a whole list of properties which we will develop below. Anyway, N_∞ is endowed with an injection $N_\infty \subset \Sigma^3 \times 1$, which is linear on each simplex. The process N^4 (w.h.) $\Rightarrow N_\infty$ consists in the deletion of finitely many 2-cells, interpretable at the 4-dimensional level, as additions of 3- handles followed by infinitely many dilatations of dimension two and additions of 2-cells.*

2) *Of course, N_∞ cannot be a subcomplex of any kind of triangulation of the compact collar $\Sigma^3 \times I$, it is not even a closed subset. But we have*

$$(24)\qquad (\lim N_\infty) \cap N_\infty = \phi,$$

which means that there is no sequence going to infinity inside N_∞, call it $x_1, x_2, \ldots$ such that, inside $\Sigma^3 \times I$, we have $\lim x_n = x_\infty \in N_\infty \subset \Sigma^3 \times I$. [Properties of type (24) which guarantee that intrinsic and extrinsic topologies coincide, will always be assumed, in the sequel.]

Also, the smooth 4-dimensional regular neighbourhood $N^4(N_\infty) \overset{r}{\searrow} N_\infty$ (with r^{-1} (compact)=compact) is well-defined, and $N^3(\tau_{(1)} \times 0) + \Gamma \times D^2 \subset \partial N^4(N_\infty)$.

3) *The smooth non-compact 4-manifold (see also (13-0))*

$$(25)\qquad M^4_\infty = \Delta^4(\text{w.h.}) \underbrace{\bigcup}_{N^3(\tau_{(1)} \times 0)} N^4(N_\infty),$$

which from now on replaces our former geometrically simply-connected compact manifold (see(13-1))

$$\Delta^4(\text{w.h.}) \cup N^4(\text{w.h.}) \subset X^4$$

is also geometrically simply connected. It also has the property that

$$\textit{(25-1)}\qquad M^4_\infty \underset{\text{DIFF}}{=} M^4_\infty \# \infty \# (S^2 \times D^2),$$

which simply follows from the fact that M^4_∞ is actually of the form $\{$something$\} \# \infty \# (S^2 \times D^2)$, combined with the identity

$$\# \infty \# (S^2 \times D^2) \# \infty \# (S^2 \times D^2) = \# \infty \# (S^2 \times D^2).$$

We also have $\Gamma \times D^2 \subset \partial M^4_\infty$. *The complement* $\Sigma^3 \times I - M^4_\infty$, *which is neither open nor closed, is also* WILD *(not locally simply-connected, for instance).*

4) *Inside* $\Sigma^3 \times I$ *we can add to* $N^4(N_\infty)$, *for each 1-handle* H_i *(respectively* $\overline{H}_{ij}, (D^2_i)_0, \overline{D}^2_{ij}$*) infinitely many parallel copies* $H_i(1), H_i(2), \ldots$ *(rerspectively* $\overline{H}_{ij}(1), \overline{H}_{ij}(2), \ldots$, *respectively* $(D^2_i)_0(1), (D^2_i)_0(2), \ldots$, *respectively* $\overline{D}^2_{ij}(1), \overline{D}^2_{ij}, (2) \ldots$*) verifying (22-1), (22-2).*

From now on, we introduce the notation H^ε *for handles of index* ε. *The whole purpose of the construction* $N^4(\text{w.h.})_{(2)} \Rightarrow N_\infty$, *is to make the addition of this infinite cascade of handles possible in a* ***locally-finite*** *manner (from the viewpoint of* $N^4(N_\infty)$, *of* M^4_∞, *or of* N_∞*) and such that it* ***embedds*** *inside* $\Sigma^3 \times I$.

So, at this point in our game, the system of handles of index one and two considered above, which are added to $N^4(N_\infty)$, *and for which we will also use the following notation (with a* ***natural system of indexing****), at least when we insists in being fully* 4- *dimensional*

(26) $$H^1_i(n), \overline{H}^1_{ij}(n), H^2_i(n), \overline{H}^2_{ij}, (n) \text{ with } i,j = 1,2,\ldots\alpha, \text{ and } n = 1,2,\ldots,$$

is PROPER, *in the following very strong sense that*

(27)
$$\partial M^4_\infty \cap \{\text{the union of the attaching zones of (26)}\} \text{ is a } \textbf{closed} \text{ subset of } \partial M^4_\infty.$$

As already said the whole aim of the construction which leads from the 2-skeleton of $N^4(\text{w.h.})$ *to* N_∞ *is to make (27) possible.*

Of course, the $(D^2_i)_0(n), \overline{D}^2_{jk}(n)$ *are the cores of the very thin, 4-dimensional handles of index two* $H^2_i(n), \overline{H}^2_{jk}(n)$ *from (26) and the exact meaning of the vague formula (23) is the following more precise*

(28) $$P^2_\infty \underset{\text{DEF}}{=} N_\infty \cup \{\text{the 2-skeleton of the 1-handles } H^1_i(n), \overline{H}^1_{ij}(n)\} \cup$$
$$\{\text{the cores of the 2-handles } H^2_i(n), \overline{H}^2_{ij}(n)\} \subset$$
$$\subset \underbrace{N^4(N_\infty) + \{\text{the handles } H^1_i(n), \overline{H}^1_{ij}(n), H^2_i(n), \overline{H}^2_{ij}(n)\}}_{\text{we will call this } N^4(P^2_\infty)} \subset \Sigma^3 \times I.$$

4-bis) *We can also think of our infinite system of handles (26) as being added to the larger and more global* M^4_∞ *(25), and if we compare (25) with (7), it turns out that all the non-fake minima of* $\pi_0 | \partial M^4_\infty$ *(where one should notice that* $\partial M^4_\infty = \partial N^4(N_\infty) - \Sigma^3 \times 0$*) are killed by the master handles* $\sum_{i=j}^{\alpha} \sum_{j=1}^{\infty} H^1_i(j)$ *(in fact already by* $\sum_{i=1}^{\alpha} H^1_i(1)$*) and that no other non-fake minima are being created by our process i.e. by the cascade of handles (26). So*

$$\pi_0 | \partial\{M^4_\infty + [\text{The handles } H^1_i(n), \overline{H}^1_{ij}(n), H^2_i(n), \overline{H}^2_{ij}(n)]\},$$

(or, for that matter $\partial\{M^4_\infty + [\text{holes}] + [\text{ the handles } H^1_i(n), \overline{H}^1_{ij}, H^2_i(n), \overline{H}^2_{ij}(n)]\}$*) does not posess any non-fake minima at all. What good that can do for us, will be discussed in section 6.*

5) *The system of indexing appearing in (26) has the memory of the geometric set-up in section 2 built in, and it is also very clearly related to 3) from the combinatorial lemma (section 2), i.e. with (16), (18). But there is also,* ***a second system of indexing*** *for the same handles (of index one and two) which occur in (26), and which is related to 1) and 2) from the same combinatorial lemma, rather than, directly, to the geometry of (28). Our 1-handles are now exactly an infinite family indexed as follows*

(29-1) $$H^1_a, a=1,2,\ldots; \overline{H}^1_{ab}, b=1,2\ldots,\alpha_1(a); H^1_{ab}(\text{same indices as for } H^1_{ab}); \overline{H}^1_{abc}, c=1,2,\ldots,\alpha_2(a,b); H^1_{abc}; \overline{H}^1_{abcd},\ldots.$$

The infinite family of 2-handles carries exactly the same system of lower indices as in (28), namely

(29-2) $$H^2_a, \overline{H}^2_{ab}, H^2_{ab}, \overline{H}^2_{abc}, H^2_{abc}, \overline{H}^2_{abcd},\ldots.$$

With this system of multi-indices, instead of being given by the relatively mysterious (18), the matrix T^∞ is simply

(30) $$H^1_{i_1\ldots i_n} \overset{\overline{H}^1_{i_1\ldots i_n j}}{\longrightarrow} H^1_{i_1\ldots i_n j}.$$

It should be stressed, by now, that although (29-1) (29-2), refers to the same objects as (26) the system of indexing is ***different*** *(we can't help this, we do need both system!). Also, the infinite system of 1-handles and 2-handles appearing in (26) or in (29-1) +(29-2), is involved inside* ***two*** *distinct square matrices too, the T^∞ (18), where actually the 2-handles do not appear explicitely at all, since only the master handles appear as vertices (= states), but also the geometric intersection matrix which we will discuss now. If we use the notations $\partial H = \{$boundary of the core of $H\}$, $\delta H = \{$boundary of the co-core of $H\}$ then, with our* ***second*** *system of indexing, the geometric intersection matrix for our handles (and which is also displayed in Figure 4) is*

(31) $$\partial H^2_{i_1\ldots i_n}\cdot\delta H^1_{j_1\ldots j_n} = \partial\overline{H}^2_{i_1\ldots i_n}\cdot\delta\overline{H}^1_{j_1\ldots j_n} = \partial\overline{H}^2_{i_1\ldots i_n}\cdot\delta H^1_{j_1\ldots j_n} = \delta_{i_1j_1}\cdot\delta_{i_2j_2}\ldots\delta_{i_nj_n}, \partial H^2_{i_1\ldots i_n}\cdot\delta\overline{H}^1_{j_1\ldots j_n k} = 2\delta_{i_1j_1}\ldots\delta_{i_nj_n},$$

all the other entries being zero (δ_{ij} is here the Kroenecker delta)

6) *The situation on the circle $\partial H^2_{i_1\ldots i_n}$ is like in figure 5. For each double contact $\partial H^2_{i_1\ldots i_n}\cdot\delta\overline{H}^1_{i_1\ldots i_n j}=2$ in the geometric intersection matrix above, there is a* ***Whitney disk*** *(see also Fig. 5) $D^2(i_1\ldots i_n j)\subset\partial M^4_\infty$. These disks, which are not confined inside the collar $\Sigma^3\times I$, are 2-by-2 disjoined, but $\cup D^2(i_1\ldots i_n j)\subset\partial M^4_\infty$, is* ***not*** *a closed subset. Also int $D^2(i_1\ldots i_n j)\cap\Gamma\neq\emptyset\neq$ int $D^2(i_1\ldots i_n j)\cap\{$attaching zones of $H^2+\overline{H}^2\}$. [End of Proposition A.]*

Comments

A) Each of the four infinite families $\{H^1_i(1), H^1_i(2),\ldots\}, \{\overline{H}^1_{ij}(1), \overline{H}^1_{ij}(2)\ldots\},\ldots$ appearing in (26) accumulates inside $\Sigma^3\times I$ towards some limiting position, from which it is disjoined, but from the viewpoint of $N^4(N_\infty)$ itself, these families go to infinity. The sets $\Sigma^3\times I - N^4(N_\infty)$, $\Sigma^3\times I - N^4(P^2_\infty)$,

	∂H^2_a	$\partial \overline{H}^2_{ab}$	∂H^2_{ab}	$\partial \overline{H}^2_{abc}$	∂H^2_{abc}	$\partial \overline{H}^2_{abcd}$	
δH^1_a	1						......
$\delta \overline{H}^1_{ab}$	2	1					......
δH^1_{ab}		1	1				......
$\delta \overline{H}^1_{abc}$			2	1			......
δH^1_{abc}				1	1		......
$\delta \overline{H}^1_{abcd}$					2	1	......

Figure 4. The geometric intersection matrix $\partial(H^2 + \overline{H}^2) \cdot \delta(H^1 + \overline{H}^1)$. The indexing is here like in (29-1),(29-2).

which are neither open nor closed, cannot give rise to any kind of bona-fide 4-dimensional cobordism, like $W^4 = \Sigma^3 \times I -$ int X^4 from the begining of section 2. This is because they are ***wild***, in particular not locally simply-connected. This makes that, although the non-fake minima have been killed, when we get to $\pi_0 | \partial N^4(P^2_\infty)$ we *cannot* use the "wild 4-dimensional cobordism"

$$\overline{\Sigma^3 \times I - N^4(P^2_\infty)}$$

in order to develop a simple-minded Morse theoretical ***external*** argument (see the begining of section 2), which would allow us to deduce the geometric simple connetivity of objets like int$(\Delta^4 \# \infty \# (S^2 \times D^2))$ (orΔ^4) from the geometric simple connectivity of objects like

(31-1) $$M^4_\infty + \{\text{the handles } H^1_i(n), \overline{H}^1_{ij}(n), H^2_i(n), \overline{H}^2_{ij}(n)\} = \\ = \Delta^4 \text{ (w.h.)} \underbrace{\cup}_{N^3(\tau_{(1)} \times 0)} N^4(P^2_\infty),$$

provided we would already know that (31-1) itself was already geometrically simply connected, which we actually don't. But discussing the issue of things related to (31-1), is the aim of the next section.

B) It is the T^∞ from the combinatorial lemma which has guided our construction, starting from the geometrical data developed in section 2, and enough has been said to make it clear how the geometrical intersection matrix is gotten from T^∞ (18). The fact that this geometrical intersection matrix has the form (31) is a consequence of point 1) from the combinatorial lemma.

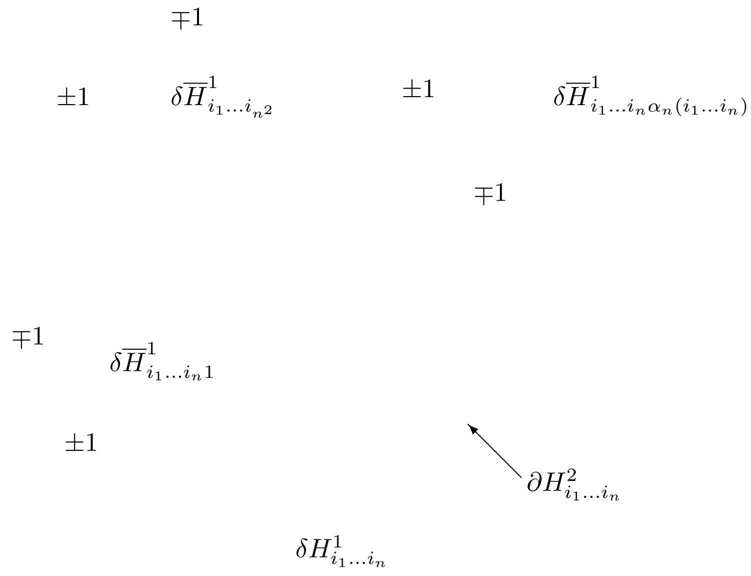


Figure 5. Here the hatched areas suggest (abstractly speaking) the Whitney disks, $D^2(i_1 \ldots i_n 1), D^2(i_1 i_2 \ldots i_n 2), \ldots D^2(i_1 \ldots i_n \alpha_n)$. Each $\partial D^2(i_1 \ldots i_n j)$ consists out of an arc $\gamma(i_1 \ldots i_n j) \subset \partial H^2_{i_1 \ldots i_n}$ and an arc $\overline{\gamma}(i_1 \ldots i_n j) \subset \delta \overline{H}^1_{i_1 \ldots i_n j}$.

We did everything necessary (see (22-1), (22-2)) so that for $N^4(P^2_\infty)$ (see also (28)) when we want to define the analogue of T^0(see, in particular, the third term in (8)), using the infinite system $\{H_i(n), \overline{H}_{ij}(n), (D^2_i)_0(n), \overline{D}^2_{ij}(n)\}$ in lieu of $\{H_i, \overline{H}_{ij}, (D^2_i)_0, \overline{D}^2_{ij}\}$, we get exactly the T^∞.

Conversely, T^∞ can be easily derived from the geometric intersection matrix of $N^4(P^2_\infty)$. If we change from the natural indexing system of T^∞ (22.1), (22.2), (or(26)) to the indexing system from (29) and from figure 4, then the T^∞ (which remember, has as states only the master handles) becomes simply (30), i.e.

$$T^\infty = \{a_{i_1 \ldots i_n; j_1 \ldots j_n k} = \delta_{i_1 j_1} \ldots \delta_{i_n j_n}, \text{ everything else being zero}\},$$

which is, clearly, like in 1) from the combinatorial lemma.

C) Let us also notice that without 2) from our combinatorial lemma we could not have constructed anything at all (starting from the only thing we could have started with, namely the geometrical data of section 2). For a given $H_i(n)$, the $(D^2_i)_0(n)$ touches some $\overline{H}_{ij}(n)$; then $\overline{D}^2_{ij}(n)$ touches $H_{h(i,j)}(\theta_{ij}(n))$ (see (22-2)), the $(D^2_{h(ij)})_0(\theta_{ij}(n))$ of which touches some $\overline{H}_{h(ij)k}(\theta_{ij}(n))$; then $\overline{D}^2_{h(ij)k}(\theta_{ij}(n))$ touches $H_{h(h(i,j),k)}$ $(\theta_{h(i,j)k}(\theta_{ij}(n)))$, a.s.o. If we want for all this to be possible at all, the map (15) has to verify the oriented, forward going paths lifting property. Now, all the infinitely many 1-handles and 2-handles, appearing in (28) have to be located in such a way that our locally finite P^2_∞ actually ***embedds*** inside $\Sigma^3 \times I$. It is here that we need, eventually, 3) from the combinatorial lemma. And here also comes a last comment. Unlike what has happened in [Po4], [Po5], [Po6], we cannot mimick, in the present context, geometrically, the infinitely many intermediary steps between T^0 and T^∞ (they do not embed in $\Sigma^3 \times I$, for instance). It is only the final infinite pattern which can be realized geometrically inside the collar $\Sigma^3 \times I$.

D) Our geometric intersection matrix (31) clearly has the general form "id + nilpotent" which, in a finite set-up, would immediately imply 1-handle cancellation. But in the infinite set-up, a geometric intersection matrix

$$\{\partial H_i^{\lambda+1} \cdot \delta H_j^\lambda\} = (\text{ identity}) + (\text{ nilpotent}),$$

can mean two very different things.

D-1) The ***easy*** id+nilpotent case, when $\partial H_i^{\lambda+1} \cdot \delta H_j^\lambda = \delta_{ij} + \lambda_{ij}$ where $\lambda_{ij} \neq 0$ implies $i > j$. Here the handles of index λ are cancelled.

D-2) The ***difficult*** id+nilpotent case, when $\partial H_i^{\lambda+1} \cdot \delta H_j^\lambda = \delta_{ij} + \lambda_{ij}$ where $\lambda_{ij} \neq 0$ implies $i < j$, like in our present situation. Here the cancellation is not automatically implied at all, not even at the π_1-level (when $\lambda = 1$).

E) There is a certain analogy between $P^2_\infty \subset \Sigma^3 \times I$ (see (28)) and the 2-dimensional squeleta of the Casson Handles [Ca], [Fr], [Fr-Q]. On the other hand, contrary to what happens in [Fr], [Fr-Q], no purely TOP construction like shrinking is ever used in our arguments, at least not at finite distance, (it is actually used, in some sense, "at infinity", as we will see) and so we manage to stay consistently inside the DIFF category. For our program [Ga] this is an essential feature. It is only in a DIFF setting that we can use the geometric simple-connectivity of $\Delta^3 \times I$ in order to deduce that $\Delta^3 = B^3$ [Po6]. At a very early stage in the argument, some form of 4-dimensional Hauptvermutung is necessary. Since the general TOP 4-dimensional Hauptvermutung is glamorously false (see for instance [Ta]) the only available tool is Whitehead's DIFF Hauptvermutung [Wh] which is valid without any dimensional restrictions.

5. Non compact Morse theory

In terms of the infinite system of handles (29-1) + (29-2) we can consider the locally finite "graph" G which has as vertices the various 3-balls which make up the attaching zones of the $H^1 + \overline{H}^1$(two 3-balls for any given 1-handle) and as edges the various connected components of $(\partial H^2 + \partial \overline{H}^2) \cap \partial M^4_\infty$ (a finite number of such edges for every given 2-handle).

We will think of G as being made out of 3-balls and arcs and it comes equipped with a PROPER embedding $G \xrightarrow{\varphi} \partial M^4_\infty$ which is disjoined form $\Gamma \times D^2$ and which provided us with all the ingredients necessary for defining the transversally compact smooth manifolds $N^4(P^2_\infty)$ (see(28)) or (31-1)); it is essential here, in order to get something which is transversally compact, that $\varphi G \subset \partial M^4_\infty$ be a ***closed*** subset, a property which we will soon loose. Notice also that $\partial D^2(Ij)$ (where I is a multi-index $i_1 \dots i_n$) is contained in φG and that the injection $G \xrightarrow{\varphi} \partial M^4_\infty$ extends to a map, which we also call

(32) $$G \cup \Sigma D^2(Ij) \xrightarrow{\varphi} \partial M^4_\infty.$$

Here $\varphi|\Sigma D^2(Ij)$ is again injective, but (32) is not PROPER; it is not injective either, since it has double points where the Whitney disk $\overset{\circ}{D}{}^2(Ij)$ cuts through G, and it also touches Γ (since $\overset{\circ}{D}_2(Ij)$ also cuts through Γ).

Consider now $N^4(P^2_\infty)$ as defined by (28), for which $N^3(\tau_{(1)} \times 0) \subset \partial N^4(P^2_\infty)$, which allows us to define the transversally compact smooth manifold, containing $M^4_\infty(25)$ and contained itself in Δ^4,

$$\Delta^4 \text{ (w.h.)} \underbrace{\bigcup}_{N^3(\tau_{(1)} \times 0)} N^4(P^2_\infty) \underset{\text{DEF}}{=} N^4(\overline{K}_\infty); \tag{33}$$

this is the same object as in (31-1), of course.

It will turn out that we cannot say much about $N^4(\overline{K}_\infty)$ itself, and so we will go to the almost-open manifold

$$X^4_\infty \underset{\text{DEF}}{=} \text{ int } N^4(\overline{K}_\infty) - (\partial N^4(\overline{K}_\infty) - \Gamma \times D^2) = \tag{33-1}$$

$$= \text{ int } N^4(\overline{K}_\infty) \cup \Gamma \times D^2 = \text{ int } \{M^4_\infty + [\varphi H^\varepsilon, \varphi \overline{H}^\varepsilon]\} \cup \Gamma \times D^2.$$

The meaning of the third term in (33-1) is the following : start by adding to M^4_∞ the handles (29-1), (29-2) (with ε in the formula taking the values $\varepsilon = 1, 2$ and with the recipee of handle-addition provided by $G \overset{\varphi}{\hookrightarrow} \partial M^4_\infty$, the only recipee we have, so far) and then delete all the boundary, but add back $\Gamma \times D^2$. We will call ***almost-open*** this kind of non-compact (and not transversally compact) smooth 4-manifold with boundary $\Gamma \times D^2$.

Proposition B. *Via an "improper isotopy" we can change $G \overset{\varphi}{\hookrightarrow} \partial M^4_\infty$ into another injective map*

$$G \overset{\psi}{\to} \partial M^4_\infty - \Gamma \times D^2 \tag{34}$$

with the following properties.

1) *The map ψ above is no longer* PROPER *but it extends to an* ***injection***

$$G \cup \Sigma D^2(Ij) \overset{\psi}{\to} \partial M^4_\infty. \tag{35}$$

The map $\psi|\Sigma D^2(Ij)$ is quite different from the $\varphi|\Sigma D^2(Ij)$ which is provided by (32); but now $\psi G \cap \psi \Sigma$ int $D^2(Ij) = \phi$. As a price for this $\psi|G$ is no longer PROPER *and ψG is no longer a closed subset. Of course also, $(\lim \psi G) \cap G = \emptyset$.*

2) *If we use the recipee ψ (34) for adding the handles (29-1) + (29-2) to M^4_∞ (which we can do, since $(\lim \psi G) \cap G = \phi$) we cannot get any longer a smooth object, since (34) is not* PROPER*; we have to throw away most of the boundary, if we want to get a smooth manifold. But our injection ψ (34) is such that we get a diffeomorphism between the following two almost-open 4-manifolds.*

$$\textit{int } \{M^4_\infty + [\psi H^\varepsilon, \psi \overline{H}^\varepsilon]\} \cup \Gamma \times \overset{\circ}{D}{}^2 \underset{\text{DIFF}}{=} \textit{int } \{M^4_\infty + [\varphi H^\varepsilon, \varphi \overline{H}^\varepsilon]\} \cup \Gamma \times \overset{\circ}{D}{}^2. \tag{36}$$

In principle at least, it is the L.H.S. of this equality which will be our model for X^4_∞ (see (33-1)), from now on.

The passage $\varphi \Rightarrow \psi$ is an ***"improper isotopy"*** proceeding along the following lines. We consider, to begin with, a well-chosen filtration by finite subsets of our infinite set of handles (29-1) + (29-2)

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_\infty = \cup_n \mathcal{M}_n \underset{\text{DEF}}{=} \{H^\lambda, \overline{H}^\lambda\}, \tag{37}$$

such that $\partial \mathcal{M}_n \cdot \delta(\mathcal{M}_\infty - \mathcal{M}_n) = 0$. Our $\varphi \Rightarrow \psi$ is the limit of an infinite sequence of inductive steps $G \xrightarrow[\psi_n]{} \partial M^4_\infty$. Here ψ_1 is ambiently isotopic to φ and one gets ψ_{n+1} from ψ_n by an ambient isotopy of

$$G - G|\mathcal{M}_n \xrightarrow[\psi_{n+1}]{} \partial(M^4_\infty + \{\psi_n \mathcal{M}_n\}),$$

ending up with an injection $G \xrightarrow[\psi_{n+1}]{} \partial M^4_\infty$ such that $\psi_{n+1}|(G|\mathcal{M}_n) = \psi_n|(G|\mathcal{M}_n)$. It should also be noted that the process $\varphi \Rightarrow \psi$ moves us far out of the collar, which will be totally forgotten in the present section. □

The main result of this section is the following theorem for which a sketch of proof will also be given.

Theorem C. *The almost-open manifold X^4_∞ (see (33-1)) is geometrically simply-connected.*

The problem with which we an faced now is that although the new Whitney disks $\psi D^2(Ij)$ (which we will continue to denote by $D^2(Ij)$ when there is no danger of comfusion) are clean of any contacts with $\partial H^2 + \partial \overline{H}^2$, as far as their interiors are concerned, our $\Gamma \subset \partial M^4_\infty$ is still touched and even ***infinitely*** many times, by them. This precludes using the Whitney disks in any simple-minded argument which might show that X^4_∞ (33-1) (and (36)) is geometrically simply-connected. (See also the remark which follows (13)). But there is another manifold, which ***is*** geometrically simply-connected, and which is looming around. Start with our original $G \overset{\varphi}{\hookrightarrow} \partial M^4_\infty$ (or with $G \overset{\psi}{\hookrightarrow} \partial M^4_\infty$) and, then, for each $\partial H^2_{i,\cdots i_n}$ destroy each of the double contacts $\partial H^2_{i_1 \cdots i_n} \cdot \delta \overline{H}^1_{i_1 \cdots i_n j}$ as follows. Simply replace ***brutally*** the arc $\gamma(i_1 \cdots i_n j) \subset \partial H^2_{i_1 \cdots i_n}$ (see Fig. 5) with a parallel copy of $\overline{\gamma}(i_1 \cdots i_n j) \subset \delta \overline{H}^1_{i_1 \cdots i_n j}$ as it is suggested in Fig. 6. This changes $\partial H^2_{i_1 \cdots i_n}$ into something which will call $\partial H^2_{i_1 \cdots i_n}(0)$ and we will rebaptize $\delta \overline{H}^1_{i_1 \cdots i_n j}$ as $\delta \overline{H}^1_{i_1 \cdots i_n j}(0)$, and similary for the remaining handles. So, for the time being, the transformation $H + \overline{H} \Longrightarrow H(0) + \overline{H}(0)$ really only changers $\{\partial H^2_{i_1 \cdots i_n}\}$. But notice that if it would not be for $D^2(Ij) \cap \Gamma \neq \phi$, we could, equivalently, leave ∂H^2 in peace and modify $\{\delta \overline{H}^1_{i_1 \cdots i_n j}\}$ insted (making use, this time, of the Whitney disks). This simple remark will turn out to the very useful soon. [Notice, for instance, that as long as $\Gamma \times \overset{\circ}{D}{}^2$ is not in its usual position, but in the position from Fig. 8B below, then the change $H + \overline{H} \Longrightarrow H(0) + \overline{H}(0)$ could use actively $\delta \overline{H}^1_{i_1 \cdots i_n j}$, instead of using $\partial H^2_{i_1 \cdots i_n}$, as we did ; but Fig. 8.B does not concern X^4_∞ but another, topologically different manifold $X^4_\infty(0)$; see below]. Corresponding to φ (respectively ψ) all this (see in particular Fig. 6) defines a new graph $G(0)$, with the same vertices, but with fewer edges, provided with new injections, $G(0) \overset{\varphi(0)}{\hookrightarrow} \partial M^4_\infty$ (respectively $G(0) \overset{\psi(0)}{\hookrightarrow} \partial M^4_\infty$) and a new almost-open

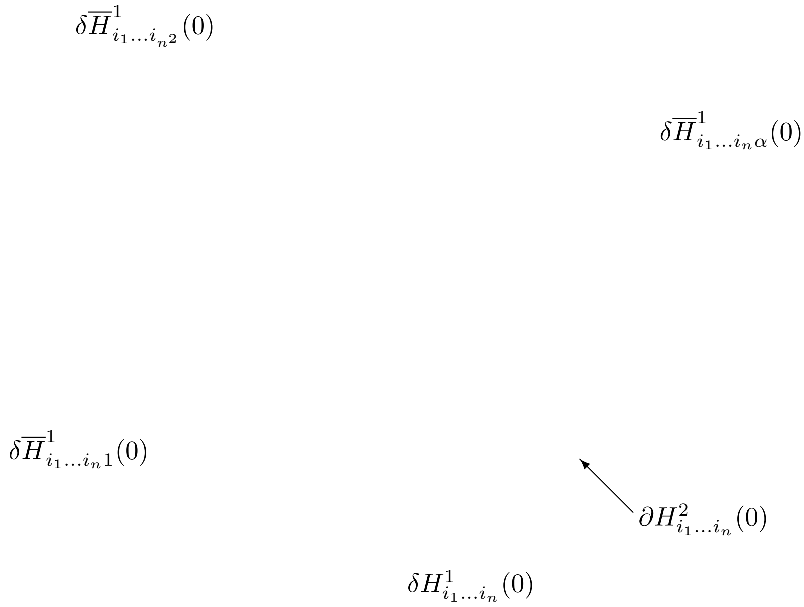


Figure 6. The ***brutal*** change from Fig.5 to the present Fig.6 does ***not*** make use, in any way of the Whitney disks, which cut through $\Gamma \times \overset{\circ}{D}{}^2$. It destroys the off-diagonal two's in the geometric intersection matrix displayed in Fig. 4, leaving us with a new geometric intersection matrix

$$\partial(H^2(0) + \overline{H}^2(0)) \cdot \delta(H^1(0) + \overline{H}^1(0))$$

which (after an obvious permutation) is of the ***easy*** id+nilpotent form, unlike (31).

manifold, topologically different (apriori at least) from X^4_∞, call it

$$(38) \qquad X^4_\infty(0) \underset{\text{DEF}}{=} \text{int}\{M^4_\infty + [\psi(0)H^\varepsilon(0), \psi(0)\overline{H}^\varepsilon(0)]\} \cup \Gamma \times \overset{\circ}{D}{}^2 \underset{\text{DIFF}}{=}$$

$$= \text{int}\{M^4_\infty + [\varphi(0)H^\varepsilon(0), \varphi(0)\overline{H}^\varepsilon(0)]\} \cup \Gamma \times \overset{\circ}{D}{}^2,$$

which is easily seen to be the geometrically simply-connected. [Even $M^4_\infty + [\varphi(0)H^\varepsilon(0), \varphi(0)\overline{H}^\varepsilon(0)]$ which is transversally compact is geometrically simply-connected. The "$\underset{\text{DIFF}}{=}$" part of (38) is clearly the analogue of (36) with $\varphi(0), \psi(0)$ in lieu of φ, ψ.] The general idea now is to "deduce", in some way, the geometric simple connectivity of X^4_∞ which, remember, is almost-open, from the geometric simple connectivity of $X^4_\infty(0)$, which is also almost-open.

We will have to leave, completly, the universe of transversally compact manifolds (i.e. manifolds which, although non-compact, are regular neighbourhoods of a lower-dimensional, locally finite skeleton), from now on. So, whenever the contrary is not explicitly mentioned, in the present section, when we will talk about the handles $H, \overline{H}$ (respectively $H(0), \overline{H}(0)$), we will mean $\psi H, \psi\overline{H}$ (respectively $\psi(0)H(0), \psi(0)\overline{H}(0)$.)

One should also notice that there is, anyway, a diffeomorphism

$$\text{int } X^4_\infty \underset{\text{DIFF}}{=} \text{ int } X^4_\infty(0), \tag{38-1}$$

so the only differnce between X^4_∞ and $X^4_\infty(0)$ comes from the different location of the boundary $\Gamma \times$ int D^2.

We will consider now Morse functions

$$\text{int } X^4_\infty - \text{ int } M^4_\infty \xrightarrow{g} R_+$$

with the ground level $g^{-1}(0) = \partial M^4_\infty$, and for which we will always ask to be PROPER, in the following relative sense.

i) For any compact subset $k \subset R_+$ we have $g^{-1}(k) \subset \{$a collar $\partial M^4_\infty \times [o,a]\} \cup \{$a compact set$\}$.

ii) There is an increasing set of non-singular levels of g, call them $n_0 = 0 < n_1 < n_2 \cdots$ with $\lim_{k=\infty} n_k = \infty$, such that $g^{-1}[n_{i-1}, n_i] = (g^{-1}(n_{i-1}) \times [o,a]) + \{$ ***finitely*** many handles attached to $g^{-1}(n_{i-1}) \times a \subset \partial(g^{-1}(n_{i-1}) \times [o,a])\}$. We do not claim here that i) and ii) are independent of each other, but the "collar" from i) above is supposed to be totally unrelated to g, in principle at least. The word "PROPER" for functions will always be meant in the relative sense above, from now on.

When dealing with something like $G \hookrightarrow \partial M^4_\infty$, we can do this in the purely 3-dimensional terms of a "3-dimensional drawing", but we can also take the viewpoint that $M^4_\infty \subset \text{int} X^4_\infty$ and that $G \hookrightarrow \partial M^4_\infty$ is a recipee for introducing a PROPER Morse function int $X^4_\infty -$ int $M^4_\infty \xrightarrow{f} R_+$ ***or*** of a PROPER "Morse" function defined now on a slightly larger domain

$$X^4_\infty - \text{ int } M^4_\infty \xrightarrow{f} R_+,$$

with a singular grownd level

$$f^{-1}(0) = \partial M^4_\infty \cup \{ \text{ a wick of } \Gamma \times 0\},$$

and which, in the neighbourhood of $\Gamma \times \overset{\circ}{D}{}^2 = \partial X^4_\infty$ is like in figure 7. This kind of function is also required to be PROPER in the relative sense (the notion of PROPER has to be a propriately re-defined in the context of figure 7.) So, we have now two (related) contexts for PROPER functions, denoted generatically by the same letter, let's say f, with source int $X^4_\infty -$ int M^4_∞ or $X^4_\infty -$ int M^4_∞. But the drawing $G \hookrightarrow \partial M^4_\infty$ ***also*** gives us a recipee for introducing a gradient-like vector field producing ***stable manifolds*** $W^s(H^2) \approx \partial H^2$, $W^s(\overline{H}^2) \approx \partial \overline{H}^2$ and ***unstable manifolds*** $W^u(H^1) \approx \delta H^1$, $W^u(\overline{H}^1) \approx \delta \overline{H}^1$. [In all this discussion it is also understood that some ordering, compatible with the "drawing" $G \hookrightarrow \partial M^4_\infty$ has been chosen for the critical values; this is actually an important issue, but we cannot develop it here. Also, clearly, the drawing really tells us how the W^u and W^s intersect each other.]

Similarly, from $G(0) \hookrightarrow \partial M^4_\infty$ we get functions $f(0)$ with stable and unstable manifolds. From the viewpoint of the ***open*** manifold (38-1) the functions f and $f(0)$ coincide (up to isotopy), but the $W^s(H^2, \overline{H}^2)$, $W^u(H^1, \overline{H}^1)$ and the $W^s(H^2(0), \overline{H}^2(0))$, $W^u(H^1(0), \overline{H}^1(0))$ are quite ***different***. The $W^s(H^2)$ means here the stable manifold corresponding to the critical point of f asscoiated to the

∂M^4_∞ wick smooth level of f

$= \Gamma \times \overset{\circ}{D}{}^2$ $=$ at infinity $= M^4_\infty$

Figure 7. The levels of f in the neighbourhood of the boundary $\Gamma \times \overset{\circ}{D}{}^2$. The ground level is here $\partial M^4_\infty \cup \{\text{the wick}\}$.

2-handle H^2, a.s.o. Also the W^s/W^u are supposed to meet tranversally along a set of complete flow-lines of the gradient-like vector field. But while

$$W^s(H^2, \overline{H}^2) \cdot W^u(H^1, \overline{H}^1) = \partial(H^2 + \overline{H}^2) \cdot \delta(H^1 + \overline{H}^1)$$

(which is of the difficult id + nilpotent form), we have

$$W^s(H^2(0), \overline{H}^2(0)) \cdot W^u(H^1(0), \overline{H}^1(0)) = \partial((H^2(0) + \overline{H}^2(0)) \cdot \delta(H^1(0) + \overline{H}^1(0))$$

(which is (up to an obvious permutation) of the easy id + nilpotent form.)

Proposition D. *1) The following features can be included as part the transformation of $\varphi \Rightarrow \psi$. For each connected component of the finite link Γ (which we will call Γ, again), all the infinitely many contacts $\Gamma \cap D^2(i_1, \ldots, i_{n+1})$ with the Whitney disks, are contained inside a* LOCAL MODEL *whith the following structure. The local model (see also fig. 8.A) is a box $B^3 \times [0,1] \subset M^4_\infty$ with $\partial M^4_\infty \cap (B^3 \times [0,1]) = B^3 \times 1$ (which we call B^3). This B^3 cuts out of $\Gamma \times D^2$ a cylinder $I \times D^2$ and out of $\Sigma D^2(i_1, \ldots, i_{n+1})$ infinitely many parallel copies of a 2-cell which rests with half of its boundary on ∂B^3 (see the hatched areas of fig. 8.A), cutting $I \times D^2$ transversally, in such a way that*

$$E \underset{\text{DEF}}{=} \lim (D^2(i_1, \ldots, i_{n+1}) \cap \Gamma) \tag{39}$$

is a closed totally discontinuous subset $E \subset \Gamma$ which we will call the ***bad locus****.*

2) Inside the local model, for each Whitney disk $D^2(i_1 \ldots i_{n+1})$ we consider a small loop $\lambda_{i_1 \ldots i_{n+1}}$ linked with $\partial H^2_{i_1 \ldots i_n}$ like in fig. 8.A.

Without any loss of generality, we can assume that

$$\lim_{n=\infty} \text{ diam } (\lambda_{i_1 \ldots i_{n+1}}) = 0 \text{ and inside } \partial M^4_\infty \text{ the } \{\lambda_{i_1 \ldots i_{n+1}}\} \text{ accumulate on} \tag{40}$$

$$(\Gamma \times \partial D^2) \cap p^{-1} E, \text{ where } \Gamma \times D^2 \xrightarrow{\ p\ } \Gamma \text{ is the obvious projection.}$$

[Later on, when $\lambda_{i_1 \ldots i_{n+1}}$ will be used for adding a 2-handle $H(\lambda_{i_1 \ldots i_{n+1}})$, we will extend the requirement (40) to $\lim_{n=\infty}$ diam$(H(\lambda_{i_1 \ldots i_{n+1}})) = 0$.]

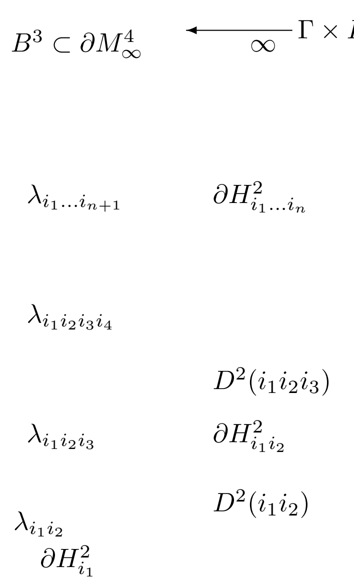


Figure 8-A.The ***tame*** local model for X^4_∞ ; the figure is supposed to describe a piece of the almost-open manifold X^4_∞. The hatched areas are inside the Whitney disks $D^2(i_1 \ldots i_{n+1})$.

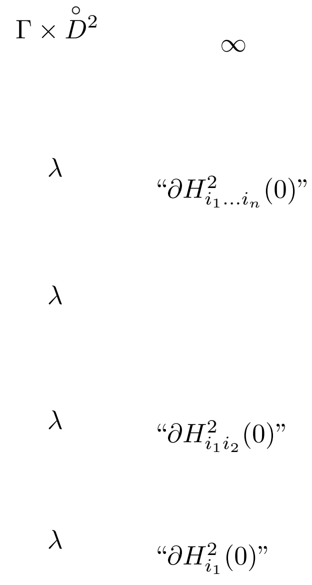


Figure 8-B.The ***tame*** local model for $X^4_\infty(0)$. The Whitney disks are here free from $\Gamma \times \overset{\circ}{D}{}^2$ and hence, the corresponding intersections $\partial H^2_I \cdot \delta \overline{H}^1_{Ij} = 2$ can be eliminated, without paying any particular price. The quotation marks from this figure, refer to the fact that the present $\partial H^2_{i_1 \ldots i_n}$ still meets $\delta \overline{H}^1_{i_1 \ldots i_{n+1}}$ and, hence, it is not really the $\partial H^2_{i_1 \ldots i_n}(0)$.

*As a consequence of the particular form of the matrix (31), we can show that for each multi-index $i_1 \ldots i_{n+1}$ there exists an $N = N(i_1 \ldots i_{n+1})$ which is such that we can find an **embedded** disk (see also (37) for the notations below)*

$$d^2_{i_1 \ldots i_{n+1}} \subset \partial(M^4_\infty + \sum_{\mathcal{J} \in \mathcal{M}_N} \overline{H}^1_{\mathcal{J}}) -$$

$$-\{\textit{the attaching zones of the handles } H^1_I, \overline{H}^1_{\mathcal{J} \notin \mathcal{M}_N}, H^2_I, \overline{H}^2_{\mathcal{J}}\},$$

such that $\partial d^2_{i_1 \ldots i_{n+1}} = \lambda_{i_1 \ldots i_{n+1}}$. [The $d^2_{i_1 \ldots i_{n+1}}$ touches $\delta \overline{H}^1_{\mathcal{J}}$.]

3) Using 2) above and also the fact (25-1) one can show that there is a diffeomorphism

$$X^4_\infty = \operatorname{int}\{M^4_\infty + [H^\varepsilon, \overline{H}^\varepsilon] + [\textit{the 2-handles } H^2(\lambda) \textit{ (fig. 8.A)} \tag{41}$$

$$\textit{with the null-framing induced by the local model}]\} \cup \Gamma \times \overset{\circ}{D}{}^2 .$$

From now on, the R.H.S. of (41) will be our model for X^4_∞.

So, figure 8.A, with the λ's standing now for 2-handles, just like $\partial H^2, \partial \overline{H}^2$, describes a local piece of X^4_∞ and, similarly it is not hard to show that figure 8.B (with the λ's again used as 2-handles) is a local piece of (something which is diffeomorphic to) $X^4_\infty(0)$. This is our model for $X^4_\infty(0)$, from now on. We will call figures 8.A, 8.B the ***tame*** local models.

In the tame local models the $\Gamma \times 0$ is screened from the $\partial H^2 + \lambda$ by the open regular neighbourhood $\Gamma \times \overset{\circ}{D}{}^2$.

The naive idea would be now to show that the tame local models are diffeomorphic, by letting $\Gamma \times D^2$ slide over the 2-handles λ. But such an isotopic slide is not possible, because we get hooked at the bad locus E. So, here is how the naive idea can be changed into something which actually works. One can construct two quotient spaces

$$X^4_\infty \overset{q}{\twoheadrightarrow} \widehat{X}^4_\infty \,, \; X^4_\infty(0) \overset{q(0)}{\twoheadrightarrow} \widehat{X}^4_\infty(0)\,, \tag{42}$$

with the following features.

(42-1) The space $\widehat{X}^4_\infty$ is $\{\text{int}X^4_\infty$ with Γ glued at infinity, like in the fat dotted position from fig. 9$\}$.

This Γ, which we will call Γ_1, is now ***naked***, in a sense which is made clear in the explanations accompanying figure 9 and which can also be understood as follows. Start by extending the boundary $\Gamma \times \overset{\circ}{D}{}^2 = \partial X^4_\infty$ to a copy of $\Gamma \times D^2$ with the λ's accumulating on $\Gamma \times \partial D^2$ like in (40). Then define q (42) by $X^4_\infty \subset \text{int}X^4_\infty \cup \Gamma \times D^2$ followed by the retraction of $\Gamma \times D^2$ to its core curve which is now Γ_1^{naked}. Anyway, this Γ_1^{naked} is no longer accompanied by an open protective neighbourhoud like $\Gamma \times 0 \subset \Gamma \times \overset{\circ}{D}{}^2$, and along Γ_1 the space $\widehat{X}^4_\infty$ is ***wild*** (in particular is not locally finite ; it has infinitely generated local homology.) But the restriction $\text{int}X^4_\infty \xrightarrow{q} \text{int}\widehat{X}^4_\infty$ (see (42)) is a diffeomorphism. All these kind of things are also valid for $q(0)$ and, in that case, the naked Γ will be called Γ_2 (or Γ_2^{naked}). As we will see below, what we gain in going from X to $\widehat{X}$ is that, although $X^4_\infty \neq X^4_\infty(0)$, we have $\widehat{X}^4_\infty = \widehat{X}^4_\infty(0)$.

(42-2) Outside the local models from figure 8, the spaces $\widehat{X}^4_\infty$, $\widehat{X}^4_\infty(0)$ coincide with X^4_∞, $X^4_\infty(0)$, respectively ; but for $\widehat{X}^4_\infty$, $\widehat{X}^4_\infty(0)$ the local models from fig. 8 are replaced now by the ***wild*** LOCAL MODELS from fig. 9. The wild local model for $\widehat{X}^4_\infty$ "consists" of the naked Γ_1 (fat dotted lines), the ∂H^2's and the λ's while the wild local model for $\widehat{X}^4_\infty(0)$ "consists" of the naked Γ_2 (fat plain lines) the ∂H^2's (which now can be really called $\partial H^2(0)$), the λ's ***and*** those parts of the $\delta \overline{H}^1(0)$'s which are displayed in fig. 9. Now Γ_1 and Γ_2 have a large intersection, in particular they coincide along an infinity of arcs $I_0, I_1, I_2, \ldots$ and also along the bad locus $E = \lim I_n$. Actually the naked Γ_1, Γ_2 are infinitely tangent along $E + \sum_1^\infty I_n$.

(42-3) The wild local models come naturally equipped with canonical maps into R^4 (we can read them off from figures 9 and 10.) These canonical maps cannot be made immersive along the bad locus $E \subset \Gamma_1 \cap \Gamma_2$; this is, essentially, because of the following requirements which have to be simultaneously satisfied :

i) in order to construct the diffeomorphism h below, we need not only (40) but even the stronger condition

$$\lim_{n=\infty} \text{diam}\{\text{2-handle corresponding to } \lambda_{i_1 \ldots i_{n+1}}\} = 0;$$

ii) the piece of $\partial H^2_{i_1 \ldots i_n}$ which enters the local model is linked with $\lambda_{i_1 \ldots i_{n+1}}$, while at the same time we have to insist on the following requirement.

$= \Gamma_2^{\text{naked}}$

$= \Gamma_1^{\text{naked}}$

E

∞

I_{n+1}

λ

$\delta \overline{H}^1_{i_1 \ldots i_{n+1}}(0)$

"$\partial H^2_{i_1 \ldots i_{n+1}}(0)$"

I_n

$\delta \overline{H}^1_{i_1 i_2 i_3}(0)$

λ

"$\partial H^2_{i_1 i_2}(0)$"

I_1

$\delta \overline{H}^1_{i_1 i_2}(0)$

λ

"$\partial H^2_{i_1}(0)$"

I_0

z, t

x

y

Figure 9. The two distinct ***wild*** local models for $\widehat{X}^4_\infty$ (with Γ_1^{naked} in place and without any $\delta \overline{H}^1_{i_1 \ldots i_p}(0)$) ***or*** for $\widehat{X}^4_\infty(0)$ (with Γ_2^{naked} in place instead of Γ_1^{naked} and ***also*** with $\delta \overline{H}^1_{i_1 \ldots i_p}(0)$'s present) are both simultaneously represented in this same figure. Since the naive attempt of sliding $\Gamma \times \overset{\circ}{D}{}^2$ (at infinity) over the λ's, so as to change fig. 8. A into fig. 8.B, fails because we get hooked at the bad locus E, we have changed here X^4_∞ (respectively $X^4_\infty(0)$) into the wild $\widehat{X}^4_\infty$ (respectively $\widehat{X}^4_\infty(0)$) where the Γ_1^{naked} and Γ_2^{naked} both contain now E along which they are infinitely tangent. In a different vein, the unstable $W^u \approx \delta H$'s are never allowed to touch the boundary (see fig. 7, 8 for instance) and it is only when our naked Γ is in the position Γ_2^{naked} that we can afford to move the $\delta \overline{H}^1(0)$'s to the present positions. Finally, we will also include the following comment. The wild local model, with its $\widehat{X}^4_\infty$, corresponds to a 3-dimensional drawing $G \hookrightarrow \partial M^\infty_\infty - \Gamma$ (call it actually Γ_1, for further purposes), which is in the same ***non***-ambient isotopy class as $G \overset{\psi}{\hookrightarrow} \partial M^4_\infty - \Gamma_1$, and which we will hence continue to denote by $G \overset{\psi}{\hookrightarrow} \partial M^4_\infty - \Gamma_1$ ***but*** with the following subtle difference : in the tame case $\lim \psi G \cap \Gamma_1 = \phi$, while in the wild case $\lim \psi G \cap \Gamma_1 \neq \phi$. From this viewpoint, there are two variants for $\varphi \Rightarrow \psi$ the tame and the wild one, which goes just one "little step" farther than the tame one. Clearly, for the wild local model to become possible at all the $\Gamma = \Gamma_1$ (or $\Gamma = \Gamma_2$) has to be stripped of its protecting coating $\Gamma \times \overset{\circ}{D}{}^2$ and become naked.

t

critical point of λ

x

$h\widehat{w}_1$

$\Gamma_2^{\rm naked}$

$\widehat{w}_2$ $W^s(\lambda)$ $W^s(H^2_{i_1 \ldots i_n}(0))$

$W^u(\overline{H}_{i_1 \ldots i_{n+1}})(0)$

M^4_∞

$= M^4_\infty$ $=$ wicks $=$the infinity of $\widehat{X}^4_\infty(0)$

Figure 10. A section $y =$ constant through figure 9 ; here the constant is chosen such that we cut through one of the 2-handles $H(\lambda)$. What we see in this picture is a piece of $\widehat{X}^4_\infty(0)$ with the two distinct wicks $\widehat{w}_2$ and $h\widehat{w}_1$. Notice that, although $H^2_{i_1 \ldots i_n}(0)$ and $\overline{H}^1_{i_1 \ldots i_{n+1}}(0)$ are now independent of each other (from the viewpoint of the geometric intersection matrix), the wild local model still imposes the ***feedback inequalities***
$\{$critical value of $H^2_{i_1 \ldots i_n}(0)\} > \{$critical value of $\overline{H}^1_{i_1 \ldots i_{n+1}}(0)\}$.
Fig. 9 displays a whole infinite cascade of such feedback relations which is reminiscent of the infinite "Whitehead mightmare" mentioned in [Po8].

iii) the local model itself has to be split from the rest of $\widehat{X}^4_\infty$ by a bona fide PROPER and proper submanifold of codimension one which does not touch the bad locus E ; (the same thing is true for $\widehat{X}^4_\infty(0)$.)

But although the canonical maps {local model} $\to R^4$ are not immersive, it still makes sense to talk about ***smooth*** maps having $\widehat{X}^4_\infty$, $\widehat{X}^4_\infty(0)$ as target or even (under certain conditions) as sources. With this, for any given finite $N \in Z_+$, we can find a diffeomorphism of class C^N

$$\boxed{\widehat{X}^4_\infty \underset{\approx}{\xrightarrow{h}} \widehat{X}^4_\infty(0)}\,,$$

such that the restriction

$${\rm int} X^4_\infty = {\rm int} \widehat{X}^4_\infty \xrightarrow{h|{\rm int}\widehat{X}^4_\infty} {\rm int}\widehat{X}^4_\infty(0) = {\rm int} X^4_\infty(0)$$

in (a diffeomorphism) of class C^∞, that $h\Gamma_1 = \Gamma_2$ and $h|\Gamma_1 \cap \Gamma_2 = {\rm id}$. This diffeomorphism h is constructed by letting the arcs $\Gamma_1 - \Gamma_2$ slide over the corresponding 2-handles λ, like in the naive attempt (see figure 9 and 10.) In order to have

a I_0 I_1 I_2 E a

scallop S_n

scallop S_1 S_2

$S^1 \times 1$

b b

$S^1 \times 0$

Figure 10-bis. We see here (schematically), $\sum_1^k S_i^1 \times [0,1]$ source of the wick map $\widehat{w}_1$. The scallops are he support of $h\widehat{w}_1$.

our h be not just a homeomorphism between $\widehat{X}^4_\infty$ and $\widehat{X}^4_\infty(0)$ which restricts to a diffeomorphism between the interiors, the smooth structure of $\widehat{X}^4_\infty, \widehat{X}^4_\infty(0)$, i.e. the non-immersive coordinate charts.

$$\{\text{wild local models from fig. 9}\} \to R^4$$

have to be chosen very carefully, metrically speaking.
[On the other hand, what seems in the way for having h fully C^∞, including along E, is the fact that for a C^∞ real-valued function defined on $[0,1]$, which fails to be C^ω (i.e. real-analytic), there is no uniform bound for ***all*** the derivatives, simultaneously. But, for our purpose, the fact that $h \in C^N$ is good enough, as one will soon see.]

(42-4) Let k be the number of connected components of Γ. For Γ_1, (respectively Γ_2) we have ***wicks***, which are C^∞ embeddings

$$\sum_1^k S_i^1 \times [0,1] \xrightarrow{\widehat{w}_1} \widehat{X}^4_\infty - \text{int}M^4_\infty, \qquad \sum_1^k S_i^1 \times [0,1] \xrightarrow{\widehat{w}_2} \widehat{X}^4_\infty(0) - \text{int}M^4_\infty$$

such that $\widehat{w}_\varepsilon(\sum_1^k S_i^1 \times 1) = \Gamma_\varepsilon$, $\widehat{w}_\varepsilon(\sum_1^k S_i^1 \times 0) \subset \partial M^4_\infty$. In terms of fig. 9, these wicks shoot out vertically in the (negative) t direction. Figure 10 displays the wicks $\widehat{w}_2$, $h\widehat{w}_1 \subset \widehat{X}^4_\infty(0)$. This $h\widehat{w}_1$ is now a wick of class C^N for Γ_2, distinct from the which $\widehat{w}_1$. Actually, the wild object $\widehat{X}^4_\infty(0)$ possesses ***uncountally*** many distinct wicks (distinct even up to isotopy) for its Γ_2 living "at infinity". [One gets wicks for $\Gamma_2 \subset \widehat{X}^4_\infty(0)$, distinct from $\widehat{w}_2, h\widehat{w}_1$ via the following procedure. Notice, to begin with, that after an easy

isotopy, we can assume, without loss of generality, that $\widehat{w}_1, \widehat{w}_2$ coincide outside an infinite family of scallops $\sum_1^\infty s_n \subset \sum_1^k S_i^1 \times [0,1]$, resting on $S_i^1 \times 1$, with $\lim s_n = E$ (see fig. 10-bis, where the scallops are shaded.) To each of our scallops s_n there is a canonically attached curve λ, call it λ_n, and what figure 10 tells us, is that we get $h\widehat{w}_1$ from $\widehat{w}_1$ by letting the scallop s_n go ***over*** the 2-handles $H^2(\lambda_n)$ until

$s_n \cap (S^1 \times 1)$ becomes an arc of $\Gamma_2^{\rm naked}$. We could get $\widehat{w}_2$ itself (up to isotopy) if we let, instead, s_n go ***under*** $H^2(\lambda_n)$. But we can also pick up any disjoind partition

$$\sum_1^\infty s_n = \sum_1^\infty s_{i_n} + \sum_1^\infty s_{j_n}$$

and let the s_{i_n}'s go over $H(\lambda)$ and s_{j_n}'s under $H(\lambda)$. This gives us a whole uncountable family of distinct wicks, for $\widehat{X}^4_\infty(0)$.]

(42-5) Inside the open manifold $\text{int}\widehat{X}^4_\infty$, the PROPER embedding

$$\textstyle\sum_1^k S^1_i \times [0,1) \xrightarrow{\ w_1\ } \text{int } \widehat{X}^4_\infty$$

which is the restriction of $\widehat{w}_1$ to $\sum_1^k S^1_i \times [0,1)$ is endowed with a canonical framing (out of the Z^k which are, a priori, possible) and we can ***reconstruct*** X^4_∞ by applying the following ***wick construction*** to w_1 :

$$X^4_\infty \underset{\rm DIFF}{=} \text{int}\widehat{X}^4_\infty \underbrace{\bigcup}_{w_1(\sum_1^k (S^1_i \times [0,1) \times \overset{\circ}{D}{}^2)} (S^1_i \times [0,1] \times \overset{\circ}{D}{}^2)\,.$$

Notice the different factors $[0,1]$ (compact) and $[0,1)$ (non-compact) which appear in the wick reconstruction formula above. [Here is a good way to vizualize our formula. Start by blowing up ***smoothly*** $\Gamma_1^{\rm naked}$ into a $\Gamma \times D^2$ living at infinity with (unvoidable) nasty accumulation of stuff on $\Gamma \times \partial D^2$. When the boundary is deleted, we get back exactly $X^4_\infty = \text{int}\widehat{X}^4_\infty \cup (\Gamma \times \overset{\circ}{D}{}^2)$. Alternatively, the same result can be achieved, via the wick construction, if one identififes $w(\sum_i S^1_i \times [0,1] \times \overset{\circ}{D}{}^2)$ with the interior of a smooth pinched neighbourhood of $\widehat{w}_1(\sum_1^k S^1_i \times [0,1])$, namely

$$\{S^1 \times [0,1] \times D^2/(S^1 \times 1 \times D^2) \text{ pinched into } \Gamma_1^{\rm naked}\} \subset \widehat{X}^4_\infty\,.]$$

One should also notice that the wick construction is invariant under diffeomorphism.

(42-6) We will concentrate, from now on, on the open manifold int $\widehat{X}^4_\infty(0)$ (= int $\widehat{X}^4_\infty$ = int X^4_∞ = int $X^4_\infty(0)$), which is endowed with the codimension zero submanifold $M^4_\infty \subset \text{int } \widehat{X}^4_\infty(0)$, with the PROPER Morse function

$$\text{int } \widehat{X}^4_\infty(0) - \text{int } M^4_\infty \xrightarrow{\ f(0)\ } R_+\,,$$

(which has the singularities of index one corresponding to $H^1(0), \overline{H}^1(0)$ and the singularities of index two corresponding to $H^2(0), \overline{H}^2(0)$), and also with the PROPER embeddings which are the restrictions of the wicks $h\widehat{w}_1, \widehat{w}_2$

$$\textstyle\sum_1^k S^1_i \times [0,1) \xrightarrow{\ hw_1\ } \text{int}\widehat{X}^4_\infty(0) - \text{int}M^4_\infty$$

$$\textstyle\sum_1^k S^1_i \times [0,1) \xrightarrow{\ w_2\ } \text{int}\widehat{X}^4_\infty(0) - \text{int}M^4_\infty\,.$$

Both hw_1 and w_2 are ***transversal*** to the levels of $f(0)$.

For the function $f(0)$, we have stable manifolds $W^s(H^2(0))$, $W^s(\overline{H}^2(0))$, $W^s(H^2(\lambda))$ and unstable manifolds $W^u(H^1(0))$, $W^u(\overline{H}^1(0))$, which are also represented in the figures 9 and 10.

(42-7) We can reconstruct $X^4_\infty(0)$ by applying the wick-construction to w_2 and it also follows from (42-3), (42-5) that we can reconstruct X^4_∞ by applying the wick-construction to hw_1, in other words we have

$$X^4_\infty \underset{\text{DIFF}}{=} \text{int } \widehat{X}^4_\infty(0) \overbrace{\qquad\bigcup\qquad}^{} \sum_1^k (S^1_i \times [0,1] \times \overset{\circ}{D}{}^2)\,,$$
$$hw_1(\sum_1^k (S^1_i \times [0,1) \times \overset{\circ}{D}{}^2)$$

and also

$$X^4_\infty(0) \underset{\text{DIFF}}{=} \text{int } \widehat{X}^4_\infty(0) \overbrace{\qquad\bigcup\qquad}^{} \sum_1^k (S^1_i \times [0,1] \times \overset{\circ}{D}{}^2)\,.$$
$$w_2(\sum_1^k (S^1_i \times [0,1) \times \overset{\circ}{D}{}^2)$$

Remark.

At face value, it would look that our reconstruction of X^4_∞ from the wick $h\widehat{w}_1 \subset \widehat{X}^4_\infty(0)$ might be only up to C^N-diffeomorphism. But both X^4_∞ and also the object

$$\text{int } \widehat{X}^4_\infty(0) \overbrace{\qquad\bigcup\qquad}^{} \sum_1^k (S^1_i \times [0,1] \times \overset{\circ}{D}{}^2)$$
$$hw_1(\sum_1^k (S^1_i \times [0,1) \times \overset{\circ}{D}{}^2)$$

are bona fide smooth manifolds and hence any C^N-diffeomorphism ($N \geq 1$) induces a C^∞-diffeomorphism. Such a thing is not necessarily true for the singular spaces with appear as intermediate links in our constructions.

(42-8) The diffeomorphism (42-3) could not function without the following features of the wild local models

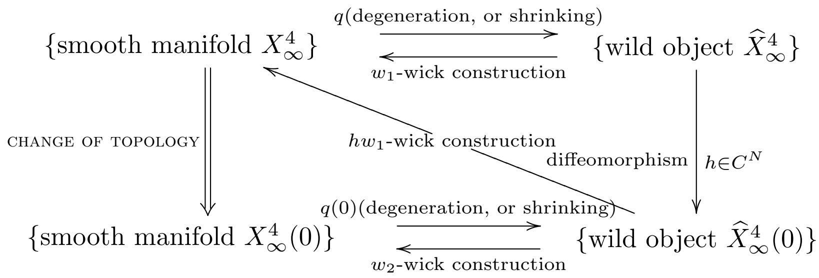
$$\begin{aligned} \lim \lambda_n &= E \subset \Gamma_1^{\text{naked}}(\text{for } \widehat{X}^4_\infty) \text{ and also} \\ \lim \lambda_n &= E \subset \Gamma_2^{\text{naked}}(\text{for } \widehat{X}^4_\infty(0))\,. \end{aligned}$$

This also forces some material corresponding to ∂H^2 (respectively to $\partial H^2(0)$) to accumulate on E. [We touch here at the root of the notion of "noncompact manifold with boundary". When the boundary is ***open*** itself (as the standard text-books require) there is no pathology possible. Bu when we have something ***compact*** living at infinity (like a boundary which would be compact *bounded*, or closed but of higher codimension, like in our present case) then wild topologies, at infinity, can be conjured.] It is in this sense that $\widehat{X}^4_\infty, \widehat{X}^4_\infty(0)$ are wild ; they are not locally finite along their ***naked*** boundary link Γ.

(42-9) (A remark) The wild local models are not really all that "local". Not only do they not embed (even locally around E) inside R^4, but they impose a certain ***feedback*** on the ordering of the critical values of $f(0)$, even in the absence of the corresponding contacts for the stable and unstable manifolds from (42-5). Because of the things said in (42-8) we are forced to perform the transformation $H + \overline{H} \Rightarrow H(0) + \overline{H}(0)$ (which, remember, in the ***open*** context does not change the topology at all and only concerns the stable and unstable manifolds, and not the Morse function, as such), like we did in figures 9 and 10, where $\partial H^2_{i_1 \ldots i_n}$ is left in peace but where we change the position of $\delta \overline{H}^1_{i_1 \ldots i_{n+1}}$. Had we used the dual procedure, then ∂H^2 would have been used activiely, dragging λ along too. This would have torn apart the topology of $\widehat{X}^4_\infty(0)$; so we are forced with the local model from figures 9 and 10 for $\widehat{X}^4_\infty(0)$. On the other hand, the feedback inequalities, which are also forced by the wild local model (see the explanations of fig. 10) make the proof of the Key proposition F below relatively more delicate than one might a priori think.

With all these things, our strategy for proving theorem C is to use the following schematical diagram for letting some information pass form $X^4(0)$ to X^4 :

(43)



[The arrows in this diagram which are particularly important for our argument are $q, h, q(0)$ and the hw_1-wick construction.]
It is not suggested that this is part of some commutative diagram ; this might well be so for the upper triangle but not for the lower one. The whole point is that at the level of $\widehat{X}^4_\infty(0)$ we have present both the wick $h\widehat{w}_1$ (with the help of which X^4_∞ can be reconstructed) and, coexisting with it inside int $\widehat{W}^4_\infty(0)$, the levels of $f(0)(=$ levels of $f)$ with the stable and unstable manifolds $W^s(H^2(0))$, $W^s(\overline{H}^2(0))$, $W^u(H^1(0)), W^u((\overline{H}^1(0))$ which display the geometric simple connectivity of $X^4_\infty(0)$. But in order to have all these things living together and in control, we need a wild space ; inside a tame usual manifold, it would be impossible to cram all this together, as we manage to do it in the wild local model of $\widehat{X}^4_\infty(0)$.
Consider now any of the uncountably many wicks $\widehat{w}$ of $\widehat{X}^4_\infty(0)$, some PROPER Morse function

(44) $$\text{int } \widehat{X}^4_\infty(0) - \text{ int } M^4_\infty \xrightarrow{\;g\;} R_+$$

and the PROPER embedding

(45) $$\textstyle\sum_1^k S^1_i \times [0,1) \xrightarrow{\;w\;} \text{int } \widehat{X}^4_\infty(0) - \text{ int } M^4_\infty \ ,$$

induced by $\widehat{w}$. The following fact is immediate.

= at infinity
= level of g (44)
= Im(w) (45)
$= M^4_\infty$

Figure 11. The ***open*** manifold $\widehat{X}^4_\infty(0)$, with g, w and M^4_∞.

Proposition E. *Let Y^4 be the **almost-open** manifold, with $\partial Y^4 = \Gamma \times \overset{\circ}{D}{}^2$, obtained by applying the wick-construction to (45), i.e.*

$$Y^4 = \text{int}\widehat{X}^4_\infty(0) \underbrace{\bigcup}_{w(\sum_1^k (S^1_i \times [0,1) \times \overset{\circ}{D}{}^2)} \sum_1^k (S^1_i \times [0,1] \times \overset{\circ}{D}{}^2)\,.$$

*If w (45) is **transversal** to the levels of g from (44), this allows us to transform (44) into a* PROPER *Morse function, which we denote with the same letter, $Y^4 - \text{int}M^4_\infty \xrightarrow{\ g\ } R_+$ without changing the Morse singularities; we simply go from the open context of figure 11, to the almost-open context of figure 7, in the obvious way.*

Our theorem C is now a consequence of the reconstruction of X^4_∞ offered by (42-7), of the transversality of hw_1 with $f(0)$ (see (42-6)), of Proposition E and of the following

Key proposition F. *One can cacell the singularities of index one of the* PROPER *Morse function $f(0)$ from (42-6) **without** destroying the transversality conditions for hw_1. (In other words, hw_1 remains transversal to the levels of the new* PROPER *Morse function on* $\text{int}\widehat{X}^4_\infty(0)$*, which has no longer singularities of index $\lambda = 1$).*

The proof of Theorem C, which was very briefly outlined above, is Morse theoretical in character. But the same mathematical ingredients can be also presented in a different, PL-framework, which some readers might find interesting or, possibly, more enlightening. So here is

An alternative view of the proof for Theorem C.

We concentrate now on the singular space $\widehat{X}^4_\infty(0)$, for which we have already considered a "smooth structure" in (42-3).

Every time we choose, inside $\widehat{X}^4_\infty(0)$, a particular wick $\widehat{w}$ connecting Γ_2 to ∂M^4_∞, we can consider exhaustions by bona fide manifolds

$$(46) \qquad Z^4_0 = M^4_\infty \cup \{\text{thickened } \widehat{w}\} \subset Z^4_1 \subset Z^4_2 \subset \ldots \subset \widehat{X}^4_\infty(0) = \bigcup_n Z^4_n$$

such that $Z^4_n \subset \text{int}Z^4_{n+1}$ except for $\partial Z^4_n \cap \partial Z^4_{n+1} = \Gamma_2$, with $Z^4_{n+1} - \text{int}Z^4_n$ an infinite cobordism with only finitely many handles. As long as $\widehat{w}$ is fixed, all the filtrations (46) are, in some sense, equivalent and they can be used to define a ***$\widehat{w}$-dependent***

PL-structure on $\widehat{X}^4_\infty(0)$. To such a given PL-structure correspond "smooth triangulations" τ of $\widehat{X}^4_\infty(0)$. For these, every $\tau|Z^4_n$ is a bona fide triangulation but in going from $(\tau|Z^4_n)|\Gamma_2$ to $(\tau|Z^4_{n+1})|\Gamma_2$ we have to use subdivisions, which means that τ, as such, is not a complex in any usual sense. [Technically speaking, the $(\tau|Z^4_n)$'s will be Siebenmann cellulations [S] such that $(\tau|(Z^4_{n+1}-\overset{\circ}{Z}{}^4_n))|\partial Z^4_n = ((\tau|Z^4_n)|\partial Z^4_n)'$.] A ***subcomplex*** S of τ will be, by definition, a bona fide subcomplex of some $\tau|Z^4_n$ (with Γ_2 triangulated apropriately.) As long as we remain confined inside a fixed PL-structure (with fixed $\widehat{w}$), all the standard machinery of piecewise differential topology functions well ; this includes Whitehead's smooth Hauptvermutung [Wh], Siebenmann's version of the Alexander subdivision lemma [S] a.s.o. But while any $\widehat{w}$ has the property that for any triangulation τ belonging to the PL-structure associated to $\widehat{w}$, there is a subdivision τ' for which the wick $\widehat{w}$ itself is (the underlying space of) a subcomplex, for non-isotopic $\widehat{w}', \widehat{w}''$'s (like for instance for $\widehat{w}' = \widehat{w}_2$, $\widehat{w}'' = h\widehat{w}_1$) $\widehat{w}'$ is never (the underlying space of) a subcomplex in the $\widehat{w}''$-PL-structure, and conversely. We are here in the quite unusual or paradoxical situation of having, not a four-manifold with uncountably many distinct DIFF-structures (like in [Ta]) but, dually so to say, a "smooth structure" (in the sense of an atlas of a wild object, with smooth coordinate changes of some class C^N), for which ***uncountably*** many attached ***smooth*** PL-structures are possible. [In the smooth world, this is reflected by the fact that when, starting with $\widehat{X}^4_\infty(0)$ we apply the wick-construction for our whole uncountable family of wicks, we get un-countably many smooth manifolds with the same interior $(\text{int}\widehat{X}^4_\infty(0) = (\text{int}X^4_\infty)$ and the same boundary $(\Gamma \times \overset{\circ}{D}{}^2)$, but which a priori at least are all topologically distinct from each other.]

Because the almost-open manifold $X^4_\infty(0)$ is geometrically simply-connected, the PL-structure of $\widehat{X}^4_\infty(0)$ corresponding to $\widehat{w}_2$ has triangulations τ for which the 2-skeleton $\tau_{(2)}$ is almost collapsible, in the sense that it has an exhaustion by compact subsets, each of which can be given the structure of a finite simplicial complex

$$k_0 = pt \subset k_1 \subset k_2 \subset \ldots \subset \tau_{(2)} = \bigcup_n k_n \tag{47}$$

such that each $k_n \subset k_{n+1}$ is either a dilatation of dimension ≤ 2 or the addition of a 2-cell. It so happens that for our given $\tau_{(2)}$ the k_n's are also sub-complexes of $\tau \in \{\text{PL structure of } \widehat{w}_2\}$, but the 2-dimensional notion of almost-collapsibility does not need any underlying simplicial structure (anyway, $\tau_{(2)}$ is ***not*** a simplicial complex) and so, almost-collapsibility (as defined above) is actually a purely topological concept.

What we want to show now is that the particular PL-structure of $\widehat{X}^4_\infty(0)$ which corresponds to $h\widehat{w}_1$ also possesses triangulations $\overline{\tau}$ for which $\overline{\tau}_{(2)}$ is almost collapsible (this is the contend of theorem C, from the PL-viewpoint.)

We will consider the following 2-dimensional object, which we call a ***double cylinder*** Σ^2. This is gotten from the usual (multi-) cylinder $\Gamma \times [0,1] = \sum_1^k S^1_i \times [0,1]$, where the bad locus E lives in $\Gamma \times 1$, by considering, to begin with, an infinite sequence of 2-by-2 disjoined round disks $d^2_n \subset \Gamma \times (0,1)$ with $\lim\limits_{n=\infty} d^2_n = E$. Let also s^2_n be the 2-spheres of equator ∂D^2_n, divided by ∂d^2_n into the two hemispheres

$s_n^2(+), s_n^2(-)$. With this

(48) $$\Sigma^2 \underset{\text{DEF}}{=} \Gamma \times [0,1] - \sum_n \text{int } d_n^2 + \sum_n s_n^2\,,$$

and this Σ^2 is the union of the two honest cylinders

(49) $$\Sigma^2(\pm) = \Gamma \times [0,1] - \sum_n \text{int } d_n^2 + \sum_n s_n^2(\pm)\,;$$

we can think of each $\Sigma^2(\pm)$ as being "smooth".

Using all this PL material, the key step in the proof of our theorem C can be reformulated now as follows

Key Step. *There is a triangulation τ belonging to the PL-structure $\widehat{w}_2$ of $\widehat{X}^4_\infty(0)$, the 2-squeleton $\tau_{(2)}$ of which possesses an exhaustion by* ***subsets*** *(careful, these are* ***not*** *subcomplexes, of* ***any*** *PL-structure)*

(50) $$Z_0^2 \subset Z_1^2 \subset Z_2^2 \subset \ldots \subset \tau_{(2)} = \bigcup_n Z_n^2\,,$$

such that

1) *Z_0^2 is the almost collapsible 2-skeleton of some smooth triangulation of M^4_∞ (remember that M^4_∞ is geometrically simply-connected.)*
2) *There exists an embedding $\Sigma^2 \subset \widehat{X}^4_\infty(0) -$ int M^4_∞, such that $\Gamma \times 1 = \Gamma_2$, $\Gamma \times 0 \subset \partial M^4_\infty$, $\Sigma^2(+) = \widehat{w}_2$, and with this we have $Z_1^2 = Z_0^2 \underbrace{\cup}_{\Gamma\times 0} \Sigma^2$.* *[This, already, cannot be a subcomplex of any kind of triangulation.]*
3) *Each of the inclusions $Z_n^2 \subset Z_{n+1}^2$, where $n \geq 1$, is either a 2-dimensional dilatation or the addition of a 2-cell.*
4) *Up to isotopy, we have $\Sigma^2(-) = h\widehat{w}_1$, i.e. the PL-structures corresponding to $\Sigma^2(-)$ and $h\widehat{w}_1$ are equal.*

Properties 1) + 2) + 3) imply that $\tau_{(2)}$ is almost-collapsible; remember that, for a 2-dimensional object, almost-collapsibility is a purely topological notion, independent of any simplicial context.

In this presentation of the key step, the hard part is 3) ; remember the standard difficulty in geometric topology that if A_2 is collapsible and $A_1 \subset A_2$ is a collapse $A_2 \searrow A_1$, this does not make A_1, necessarily, collapsible. This is the kind of obstacle which has to be circumvented when proving 3).

Corollary to the key step. *There exists a triangulation $\overline{\tau}$ belonging to the PL-structure $h\widehat{w}_1$ of $\widehat{X}^4_\infty(0)$ such that we have the following equality between* ***subsets*** *of $\widehat{X}^4_\infty(0)$*

(51) $$\tau_{(2)} = \overline{\tau}_{(2)}$$

and hence $\overline{\tau}_{(2)}$ is almost collapsible.

Remarks.

A) All this ***does not*** imply the equality between the two PL-structures of $\widehat{X}^4_\infty$ corresponding to $\widehat{w}_2$ and $h\widehat{w}_1$ respectively. The two structures in question induce two exhaustions of the unique object (51), which are incomparable (see also C) below.) Actually, the Whitehead smooth Hauptvermutung fails to be true for two triangulations $\tau' \in \{$PL-structure of $\widehat{w}_2\}$, $\tau'' \in \{$PL-structure of $h\widehat{w}_1\}$. If more than that

would have been true, the proof of theorem C would have been considerably easier. For the failure of the Hauptvermutung see also B), C) below.
B) In all this discussion the following kind of DUALITY plays a Key role. Consider $\Sigma^2 = \Sigma^2(+) \cup \Sigma^2(-) \subset \tau_{(2)} = \overline{\tau}_{(2)}$. From the viewpoint of the PL structure of $\widehat{X}^4_\infty(0)$ corresponding to $\widehat{w}_2$ (respectively to $h\widehat{w}_1$), $\Sigma^2(+)$ (respectively $\Sigma^2(-)$) is a nice bona-fide finite subcomplex, while $\sum_n s^2_n(-)$ (respectively $\sum_n s^2(+)$) is an infinite amount of wild, non simplicial material which is added to $\Sigma^2(+)$ (respectively to $\Sigma^2(-)$), satisfying the "nontrivial topological relation" $\lim_{n=\infty} s^2_n(-) = E$ (respectively $\lim_{n=\infty} s^2_n(+) = E$.)
C) The set (51) admits the filtration (47) the terms of which are subcomplexes of the PL-structure of $\widehat{w}_2$ (but not of the PL-structure of $h\widehat{w}_1$). It also admits a similar filtration by subcomplexes of the PL-structure corresponding to $h\widehat{w}_1$ (but which are not subcomplexes of the PL-structure of $\widehat{w}_2$.)

6. The infinite zipping process

In what comes next the following notions from the papers [Po1],[Po2] and [Ga] will be used: singular 2-dimensional polyhedra $X^2 \xrightarrow{f} \Sigma^3$, undrawable (or admissible) singularities, $0(i)$-moves, desingularizations φ, COHERENT $0(3)$-moves and 4-dimensional thickenings $\Theta^4(X^2, \varphi)$. [The very elementary section 2 in [Po2] explains $\Theta^4(X^2, \varphi)$ from a slightly different standpoint than [Ga].] Modulo these very elementary notions, the present paper can be read independently of anything else, certainly independently of any [Po] reference in the bibliography below.

Now, the projection $P^2_\infty \xrightarrow{\pi} \Sigma^3$ is highly degenerate but we will use an appropriate generic perturbation of it, which we denote by $P^2_\infty \xrightarrow{f} \Sigma^3$, and which is, locally at least, a singular 2-dimensional polyhedron.

The map $P^2_\infty \xrightarrow{f} \Sigma^3$ has, of course, nothing to do with the Morse function $X^4_\infty - \text{int} M^4_\infty \xrightarrow{f} R_+$ from section 5. We can safely use the same letter for both objects since we will never use them simultaneously. The use of "f" as a map into Σ^3 will end, when we have finished the statement of theorem G below. Globally, $P^2_\infty \overset{f}{\to} \Sigma^3$ is actually quite wild, as we will soon see. For each undrawable singularity of this object, the projection $P^2_\infty \subset \Sigma^3 \times I \overset{\pi_0}{\to} I$ defines a "high" branch and a "low" branch. This, in turn, defines a desingularization $\mathcal{R}$ for $P^2_\infty \overset{f}{\to} \Sigma^3$ which, with the notations from [Ga] (or from [Po2]), is

$$\mathcal{R}(\text{high}) = S, \mathcal{R}(\text{low}) = N. \tag{52}$$

It can be shown that the 4-dimensional regular neighbourhood $N^4(P^2_\infty)$ of $P^2_\infty \subset \Sigma^3 \times I$ is well-defined and that $N^4(P^2_\infty) \underset{\text{DIFF}}{=} \Theta^4(P^2_\infty, \mathcal{R})$. Here is a manifestation of the fact that our $P^2_\infty \overset{f}{\to} \Sigma^3$ is, globally speaking, quite exotic: the 2-dimensional space $fP^2_\infty \subset \Sigma^3$ is certainly not a closed subset and even abstractly speaking it is not a simplical complex in any sense of the term. Actually the double points set $M_2(f) \subset P^2_\infty$ is not closed either (a situation which, incidentally, should be compared to [Po8], [PoTa1]). Nevertheless, the ***open*** regular neighbourhood of $fP^2_\infty \subset \Sigma^3$ is well-defined, and it can be shown that it is diffeomorphic to int $\{(\Sigma^3 - \overset{\circ}{B}{}^3)\#\infty\#(S^2 \times I)\}$; we will denote it by $Nbd^3(fP^2_\infty) \subset \Sigma^3$. With this, we define also the open 4-dimensional regular neighbourhood of fP^2_∞

$$Nbd^4(fP^2_\infty) = Nbd^3(fP^2_\infty) \times (0,1) \tag{53}$$

and $\mathring{N}^3(\tau_{(1)} \times 0) + (\Gamma \times \mathring{D}^2) \cap (\text{collar})$ lives canonically at the infinity of (53), making $Nbd^4(fP^2_\infty) \cup (\mathring{N}^3(\tau_{(1)} \times 0) + (\Gamma \times \mathring{D}^2) \cap (\text{collar}))$ a non-compact manifold WITH BOUNDARY. For the next step, we use again the collar $\Sigma^3 \times I$. We have the

Theorem G. *As a consequence of 4-bis in Proposition A, in particular of the fact that the master 1-handles $H^1_1(1), H^1_2(1), \ldots, H^1_\alpha(1), \ldots$ have killed already all the non-fake minima of $\pi_0 | \partial M^4_\infty$, the map f can be "zipped* COHERENTLY*" in the sense that it can be factorized by an ω-**modelled** sequence of acyclic 0(i) moves $(i = 0,1,2)$ and* COHERENT *0(3)-moves*

$$P^2_\infty = X_0 \stackrel{f_1}{\to} X_1 \stackrel{f_2}{\to} X_2 \stackrel{f_3}{\to} \ldots X_\omega = fP^2_\infty . \tag{54}$$

Because (54) is ω-modelled (and no other infinite ordinal would do here), each $X_i (i < \omega)$ is also locally finite, with a well-defined, transversally compact 4-dimensional thickening $\Theta^4(X_i, \mathcal{R})$.

2) *The set of double points $M_2(f) \subset P^2_\infty$ is such that $\Gamma \cap M_2(f) = \phi$. Also, for any double point $(x,y) \in M^2(f_{i+1}) \subset X_i \times X_i$ (53), which is such that $x \in \tau_{(1)} \times 0$ we have $y \notin \tau_{(1)} \times 0, \mathcal{R}(x) = N$ and $\mathcal{R}(y) = S$.*

3) *As a consequence of 1) + 2) above, we have commutative diagrams for all $n \geq 0$*

$$\begin{array}{ccc} \Theta^4(X_n, \mathcal{R}) & \xhookrightarrow{\quad \Theta(f_{n+1}) \quad} & \Theta^4(X_{n+1}, \mathcal{R}) \\ & \nwarrow \qquad \nearrow & \\ & [(\Gamma \times \mathring{D}^2) \cap (\Sigma^3 \times I)] + N^3(\tau_{(1)} \times 0) & \end{array} \tag{55}$$

where $\Theta(f_{n+1})$ corresponds to the addition of finitely many 2-handles. [Without COHERENCE *we wouldn't even get an embedding $\Theta^4(X_n, \mathcal{R}) \subset \Theta^4(X_{n+1}, \mathcal{R})$].*

4) *If we conceive the map $\Theta(f_{n+1})$ as sending all the source into the interior of the target, with the exception of $\mathring{N}^3(\tau_{(1)} \times 0) + (\Gamma \times \mathring{D}^2) \cap$(collar), then we have a diffeomorphism*

$$\lim_{n=\infty} \Theta^4(X_n, \mathcal{R}) = \underbrace{Nbd^4(fP^2_\infty)}_{\text{interior}} \cup \tag{56}$$
$$\cup \underbrace{(\mathring{N}^3(\tau_{(1)} \times 0) + (\Gamma \times \mathring{D}^2) \cap (\text{collar}))}_{\text{boundary}}$$

5) *Let us consider the non compact 4-manifold with non-empty boundary $\Sigma^3 \times [0,1] - (\Sigma^3 \times 0 - \mathring{N}^3(\tau_{(1)} \times 0))$. There is diffeomorphism $\{Nbd^4(fP^2_\infty) \cup [\mathring{N}^3(\tau_{(1)} \times 0) + (\Gamma \times \mathring{D}^2) \cap (\text{collar})]\} + \{$ the finitely many 2-handles $\sum_i D^2_i \times \mathring{D}^2_i$ corresponding to $(\Gamma \times \mathring{D}^2) \cap (collar)\} = [\Sigma^3 \times [0,1] - (\Sigma^3 \times 0 - \mathring{N}^3(\tau_{(1)} \times 0))] \# \infty \# (S^2 \times D^2)$, where $\# \infty \# (S^2 \times D^2)$ is added to $\Sigma^3 \times 1$ and kills that boundary coponent entirely, leaving on the boundary only $\mathring{N}^3(\tau_{(1)} \times 0)$.*

6) *It follows from all this that the open 4-manifold Y^4 from our* Theorem *(see (4)) can be obtained from the splitting (33) by replacing $N^4(P^2_\infty)$ with the L.H.S. of (56), then removing all the boundary except for $\Gamma \times \overset{\circ}{D}{}^2$, and finally adding the finitely many 2-handles $\Sigma D^2_i \times \overset{\circ}{D}{}^2_i$ corresponding to $\Gamma \times \overset{\circ}{D}{}^2 = \Sigma \partial D^2_i \times \overset{\circ}{D}{}^2_i$ (i.e. finally fill in the {holes})*

(57)
$$Y^4 \underset{\text{DEF}}{=} int(\Delta^4 \# \infty \# (S^2 \times D^2)) \underset{\text{DIFF}}{=}$$

$$\{[int\ \Delta^4(w.h) \cup \underbrace{[\overset{\circ}{N}{}^3(\tau_{(1)} \times 0) + (\Gamma \times \overset{\circ}{D}{}^2) \cap \Delta^4_{\text{small}}]}_{boundary\ added\ to\ \ int\ \ \Delta^4(w.h.)} \cup \underbrace{\cup}_{\overset{\circ}{N}{}^3(\tau_{(1)} \times 0)} \lim \Theta^4(X_n, \mathcal{R})\} +$$

$$+ \{\textit{the finitely many 2-handles } \sum_i D^2_i \times \overset{\circ}{D}{}^2_i \textit{ corresponding to } \Gamma \times \overset{\circ}{D}{}^2\}.$$

Comments

A) Outsie $X^4_\infty \cap$ (collar) or $N^4(\overline{K}_\infty) \cap$ (collar) (see (33-1) and (33)) what is left of the collar $\Sigma^3 \times I$ is a wild object, and not a 4- dimensional smooth cobordism. This means that although we have managed to get rid of the non-fake minima, the easy ***external*** road (= Morse theory of the complement) is closed for us. We can only use the much harder ***internal*** road of coherent zipping, for the proof of our theorem 1 (section 1).

B) Notice also that if we would have known that $N^4(\overline{K}_\infty)$ was geometrically simply connected (which we certainly do not!), then the theorem G above would imply quite easily that Y^4 (57) is geometrically simply connected, which is what we eventually want. But we only know that X^4_∞ is geometrically simply-connected, not $N^4(\overline{K}_\infty)$ itself (see(33-1) too).

So, in order to show that Y^4 is geometrically simply connected we need an extension of the proof of theorem C. We call this the

enhanced non-compact Morse theory

We go back to the very begining of section 5 and to the proper embedding $G \xrightarrow[\varphi]{} \partial M^4_\infty$ (before the passage $\varphi \Rightarrow \psi$). We also consider the infinite cascade of 2-handles, call them $h^2_1, h^2_2, \dots$ corresponding to the various diagrams (55). Let $\gamma_n = \partial h^2_n$ be the boundary of the core of h^2_n. We will introduce the following notation for the almost open manifold which appears in (57) before the last finite 2-handle addition

(58)
$$Z^4 \underset{\text{DEF}}{=} [\text{int}\ \Delta^4(w.h.) \cup [\overset{\circ}{N}{}^3(\tau_{(1)} \times 0) + (\Gamma \times \overset{\circ}{D}{}^2) \cap \Delta^4_{\text{small}}]] \cup$$
$$\underbrace{\cup}_{\overset{\circ}{N}{}^3(\tau_{(1)} \times 0)} \lim_{n=\infty} \Theta^4(X_n, \mathcal{R}).$$

If we manage to show that this Z^4, which is almost-open, is geometrically simply-connected, then (57) is geometrically simply-connected too, and our theorem 1 (section 1) is hence proved. So, what we finally want to archieve, at this point, is to adapt the proof of theorem C to Z^4 (in lieu of X^4_∞.) Now, it so happens that the thickening operation

$$(X_n, \mathcal{R}) \Longrightarrow \Theta^4(X_n, \mathcal{R})$$

has good enough properties like localization, glueing a.s.o, so as to allow us to complete the statement of theorem G with the following

Proposition H. *There is a graph $G \cup G'$ with the same set of vertices as G, endowed with an injection*

$$G \cup G' \xrightarrow[\varphi]{} \partial M^4_\infty \tag{59}$$

which extends the PROPER *embedding $G \xrightarrow[\varphi]{} \partial M^4_\infty$ from the begining of section 5, such that:*

1) *$G' \underset{\text{DEF}}{=} \overline{G \cup G' - G}$ is a sum of contributions from the various γ_n's. Each of them, can either be a complete new connected component ($= S^1$), if γ_n does not touch the $\delta H^1, \delta \overline{H}^1$, or, if not, finitelly many arcs joining the vertices of G which correspond to the 1-handles $H^1, \overline{H}^1$ touched by γ_n. We will write, loosely, $G \cup G' = G \cup \sum_1^\infty \gamma_n$.*
2) *The image of (59) does not touch $\Gamma \times \overset{\circ}{D}{}^2$, but $G \cup G'$ is generally speaking **not** locally finite and the full (59) is not* PROPER *either. [The problem is that the same $\delta H^1_I, \delta \overline{H}^1_{\mathcal{J}}$; can be touched by infinitely many γ_n's, which certainly makes $G \cup G'$ **not** locally finite. But the whole point of the present proposition is that we have a well-defined map (59), extending (32), out of which Z^4 can be recovered like (60) below].*
3) *We can use (59) as a recipee for adding the handles (29-1)+(29-2) **and** $h^2_1 + h^2_2 + \dots$ to M^4_∞, so an to get an almost-open manifold; we have*

$$int\ \{M^4_\infty + [\varphi H^\varepsilon, \varphi \overline{H}^\varepsilon, \varphi h^2]\} \cup \Gamma \times \overset{\circ}{D}{}^2 \underset{\text{DIFF}}{=} Z^4. \tag{60}$$

At this point, we want to do something like in section 5 but starting with the L.H.S. of (60) instead of X^4_∞ from the R.H.S. of (36). And here we uncounter a first major difficulty, namely that $G \xrightarrow[\psi]{} \partial M^4_\infty$ cannot be extended to $G \cup G'$. More precisely, the individual curves γ_n are crossed infinitely many times by the improper isotopy $\varphi \Rightarrow \psi$, with the result that individually (trying to follow the process $\varphi \Rightarrow \psi$) they shoot out infinitely many feelers of unbounded lengths and , hence, their topology gets torn apart, if they have to follow $\varphi \Rightarrow \psi$.

We will use the last part of this section in order to give a hint of how this particular disease can get cured. The action We will describe now will take place far from $\Gamma \times \overset{\circ}{D}{}^2$ so that we can as well consider the open situation $\text{int} X^4 \supset M^4_\infty$ with some PROPER Morse function

$$\text{int} X^4_\infty - \text{int} M^4_\infty \xrightarrow{g} R_+,$$

which might be, for instance, our f or $f(0)$. Assume this g corresponds to some infinite system of 1-handles and 2-handles call them $\mathcal{H}^1_i$, $\mathcal{H}^2_j$. By definition, a PROPER ***mixing multicylinder*** is a PROPER, smooth embedding

$$\sum_1^\infty \gamma_n \times [0, \infty) \xrightarrow{\Phi} \text{int } X^4_\infty - M^4_\infty \tag{61}$$

which is PROPER, framed, ***transversal*** to the levels of g, and such that $\Phi(\gamma_n \times 0) \subset g^{-1}(n)$. We can use the wick construction applied to Φ so as to add to int X^4_∞ a boundary, leading us to

$$(61\text{-}1)\qquad (\text{int } X^4_\infty)\widehat{\ }_\Phi \underset{\text{DEF}}{=} (\text{int } X^4_\infty) \cup \sum_1^\infty \gamma_n \times \overset{\circ}{D}{}^2 \,.$$

It is not hard to see that, making use of Φ, we can extend the original ordering of the handles $\{\mathcal{H}^1, \mathcal{H}^2\}$ to an ordering of the larger set $\{\mathcal{H}^1, \mathcal{H}^2, h^2\}$, where as before h^2_n is a 2-handle corresponding to the curve γ_n, (this is the reason for the word "mixing" in our definition) so as to get a diffeomorphism

$$(62)\qquad (\text{int } X^4_\infty)\widehat{\ }_\Phi + \{\text{the 2-handles } \sum_1^\infty D^2_n \times \overset{\circ}{D}{}^2 \text{ corresponding to}$$

$$\sum_1^\infty \gamma_n \times \overset{\circ}{D}{}^2\} \underset{\text{DIFF}}{=} M^4_\infty + [\text{the handles } \{\mathcal{H}^1, \mathcal{H}^2, h^2\}]\,.$$

We will denote by $(\text{int } X^4_\infty)\widehat{\widehat{\ }}_\Phi$ the L.H.S. of (62). So $(\ldots)\widehat{\widehat{\ }}_\Phi$ in the result of first creating a boundary via wick construction applyed to the PROPER mixing multi-cylinder Φ (which leads to $(\ldots)\widehat{\ }_\Phi$) and then killing the boundary in question by addition of 2-handles (which use the framing of Φ.)

As far as the R.H.S. of (62) is concerned, one can as well assume that the level $g^{-1}(n)$ where $\Phi(\gamma_n \times 0)$ lives is non-critical, and h^2_n will be added immediately after the finitely many handles $\{\mathcal{H}^1, \mathcal{H}^2\}$ which have critical values $< n$.

Before we continue with our main story, we will give a very simple minded ***baby-example***. We consider Fig.12, which in the style of the figures 8,9 defines an injection $\gamma + \sum_1^\infty c_n \xrightarrow[\varphi]{} \overset{\circ}{B}{}^3 = \overset{\circ}{B}{}^3 \times 1 \subset \partial(\overset{\circ}{B}{}^3 \times [0,1])$ and hence a 4-manifold which is non compact and with non-empty boundary

$$(63)\qquad A^4 = \text{int}\{(\overset{\circ}{B}{}^3 \times [0,1]) + [\text{the handles } \varphi\gamma + \varphi \sum_1^\infty c_n]\} \cup (\overset{\circ}{B}{}^3 \times 0).$$

Assume now that the restriction of φ to $\sum_1^\infty c_n$ is changed via an improper isotopy into an (actually PROPER, but this is immaterial) embedding $\sum_1^\infty c_n \xrightarrow[\psi]{} \overset{\circ}{B}{}^3$, where $\lim_{n=\infty} \psi(c_n)) = \infty$. Clearly the (improper) isotopy $\varphi \Rightarrow \psi$ does not extend to γ. So how can we get back our A^4 when $\varphi | \sum_1^\infty c_n$ is replaced by ψ? Here is the simple answer. Consider the manifold

$$(64)\qquad C^4 = \text{int}\{(\text{int } B^3 \times [0,1]) + \varphi \sum_1^\infty c_n\} \cup (\text{int } B^3 \times 0),$$

where "$+\varphi \sum_1^\infty c_n$" means {plus the 2-handles defined by $\varphi \sum_1^\infty c_n$}. The 2-handles φc_n define on this C^4 a Morse function with ground level at the boundary int $B^3 \times 0$, which is R_+-valued and PROPER (in the rel sense, that is the purpose which int $B^3 \times 0$ is serving for.) In this set-up, our $\varphi\gamma$ defines now an obvious PROPER mixing cylinder, which we call Φ (unlike what has happened in (61), we have a unique component now for Φ.) With all this, comes an obvious diffeomorphism too:

$$(65)\qquad A^4 = (C^4)\widehat{\widehat{\ }}_\Phi .$$

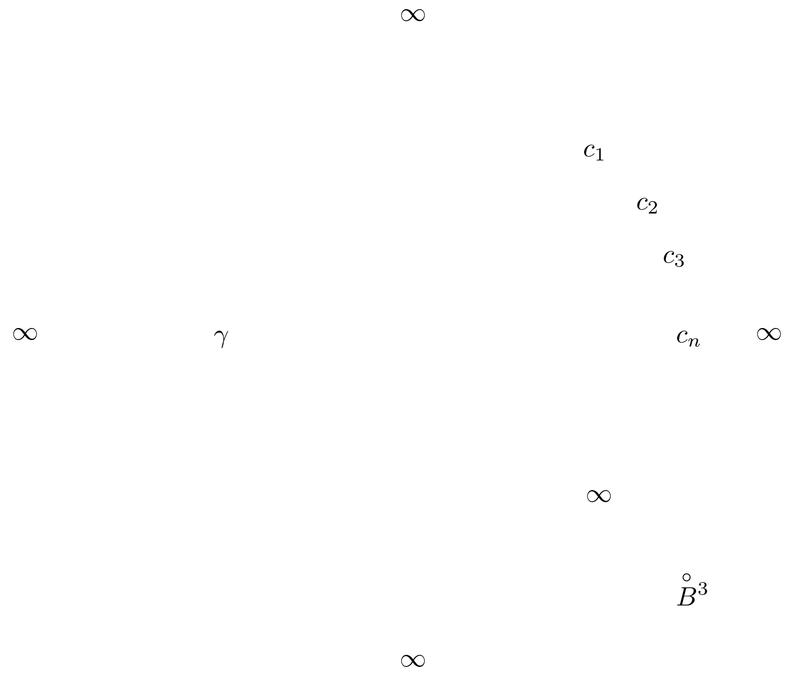


Figure 12. This figure is supposed to define an embedding φ of the infinite link $\gamma + \sum_1^\infty c_n$ into $\overset{\circ}{B}{}^3$.

When $\sum_1^\infty c_n$ gets subjected to the improper isotopy $\varphi \Rightarrow \psi$, we can also change Φ isotopically into a new PROPER mixing cylinder Ψ, for $\text{int}\{(\overset{\circ}{B}{}^3 \times [0,1]) + \psi \sum_1^\infty c_n\} \cup (\overset{\circ}{B}{}^3 \times 0)$ (which is another, diffeomorphic model for C^4, with its obvious PROPER Morse function.) So, in the presence of $\psi \sum_1^\infty c_n$, when there is no way of making any sense of the curve γ, as such, the ***inexistent*** curve in question gets replaced by the PROPER mixing cylinder Ψ, and we get back our A^4 from (63) in the form

$$(66) \qquad A^4 \underset{\text{DIFF}}{=} \widehat{(\{\text{int } B^3 \times [0,1] + \psi \sum_1^\infty c_n\} \cup (\text{int } B^3 \times 0))}_\Psi .$$

□

With all this, here are, in a very brief outline, the main steps for proving that Z^4 (58) is geometrically simply-connected, imitating and extending the proof of theorem C.

(67-1) The injection (59) can be interpreted as defining the system of handles $\varphi H^\varepsilon, \varphi \overline{H}^\varepsilon$ $(\varepsilon = 1, 2)$ ***and also*** a PROPER mixing multicylinder

$$\sum_1^\infty \gamma_n \times [0, \infty) \xrightarrow[\Phi]{} X^4_\infty - M^4_\infty ,$$

disjoined from $\Gamma \times \overset{\circ}{D}{}^2$. By analogy with (65), we have

$$Z^4 \underset{\text{DIFF}}{=} \widehat{[\text{int}\{M^4_\infty + [\varphi H^\varepsilon, \varphi \overline{H}^\varepsilon]\} \cup \Gamma \times \overset{\circ}{D}{}^2]}_\Phi .$$

The R.H.S. of this equality is also equal to $[\text{int}\{M^4_\infty + [\varphi H^\varepsilon, \varphi \overline{H}^\varepsilon]\}]^{\widehat{\widehat{\ }}}_\Phi \cup (\Gamma \times \overset{\circ}{D}{}^2)$, which might also be written as

$$(\text{int}\{M^4_\infty + [\varphi H^\varepsilon, \varphi \overline{H}^\varepsilon]\}]^{\widehat{\widehat{\ }}}_\Phi)^{\widehat{\ }}_{w_1} = ([\text{int}\{M^4_\infty + [\varphi H^\varepsilon, \varphi \overline{H}^\varepsilon]\}]^{\widehat{\ }}_{w_1})^{\widehat{\widehat{\ }}}_\Phi\,;$$

here w_1 is like in (42-5). Similar remarks apply to the various formulae which follow.

(67-2) We apply the process $\varphi \Rightarrow \psi$ to G (i.e. to $H^\varepsilon + \overline{H}^\varepsilon$) and this changes Φ into another PROPER mixing multi-cylinder Ψ such that, by analogy with (66), we have

$$[\text{int}(M^4_\infty + [\varphi H^\varepsilon, \varphi \overline{H}^\varepsilon]\} \cup \Gamma \times \overset{\circ}{D}{}^2]^{\widehat{\widehat{\ }}}_\Phi \underset{\text{DIFF}}{=} \underbrace{[\text{int}\{M^4_\infty + [\psi H^\varepsilon, \psi \overline{H}^\varepsilon]\}}_{\text{int } \widehat{X}^4_\infty(0) = \text{int } \widehat{X}^4_\infty} \cup \Gamma \times \overset{\circ}{D}{}^2]^{\widehat{\widehat{\ }}}_\Psi .$$

(67-3) When we go now to the ***wild*** object $\widehat{X}^4_\infty(0) \supset h\widehat{w}_1$ we have $\text{Im}\Psi \subset \text{int}\widehat{X}^4_\infty(0) = \widehat{X}^4_\infty(0) - \Gamma_2$ and although $\text{Im}\Psi$ enters the wild local model (Fig. 9), we still have $(\text{Im}\Psi) \cap h\widehat{w}_1 = \phi$. Also, inside $\text{int}\widehat{X}^4_\infty(0)$, both Ψ and $h\widehat{w}_1$ are transversal to the levels of the function $f(0)$.

(67-4) (Key Step) Like in proposition F one can cancell the singularities of index one of $f(0)$, i.e. go from $f(0)$ to a new Morse function f_{new} without singularities of index $\lambda = 1$, staying ***transversal*** (as far as the levles of f_{new} are concerned) ***both*** to hw_1 ***and*** to Ψ. In other words, we can continue to use hw_1 to put back the boundary $\Gamma \times \overset{\circ}{D}{}^2$ (without introducing unwanted 1-handles), and also our Ψ continues to be a PROPER mixing multicylinder, for the f_{new} which has replaced by now $f(0)$.

(67-5) (Coda) With int $\widehat{X}^4_\infty(0)$ endowed with f_{new}, the larger multi-cylinder $hw_1 + \Psi$ has the mixing property too, and this gives us the following diffeomorphic description for our Z^4

$$Z^4 = \Bigg[\underbrace{(\text{int } \widehat{X}^4_\infty(0))^{\widehat{\ }}_{hw_1}}_{\text{this is } X^4_\infty}\Bigg]^{\widehat{\widehat{\ }}}_\Psi = \Bigg[(\text{int } \widehat{X}^4_\infty(0))^{\widehat{\widehat{\ }}}_\Psi\Bigg]^{\widehat{\ }}_{hw_1} . \tag{68}$$

According to (62) the Morse function f_{new} on int $\widehat{X}^4_\infty(0)$ which has no singularities of index $\lambda = 1$, extends to a Morse function, call it g, on $(\text{int } \widehat{X}^4_\infty(0))^{\widehat{\widehat{\ }}}_\Psi$ which also has no singularities of index $\lambda = 1$. On the other hand f_{new} was transversal to hw_1, the Ψ and hw_1 are far from each other, and hence g is also transversal to hw_1. Hence, when we use the operation $[\ldots]^{\widehat{\ }}_{hw_1}$ to put back the boundary $\Gamma \times \overset{\circ}{D}{}^2$ and get Z^4, we continue to stay geometrically simply-connected. This proves what we want

Some last comments. A) Up to a certain point, the way we have treated $\Gamma \approx hw_1$ in section 4 and $\sum_1^\infty \gamma_n \approx \{\Phi \text{ or } \Psi\}$ in the present section, is similar. But, to avoid the reader getting the wrong impression, let me immediately add that there are also big differences (which become all-important when one goes to the detailed proofs). While Ψ stays very close to infinity (we are allowed , for instance, to shorten each term in (61) to $\gamma_n \times [N(n), \infty)$) our wick hw_1 (respectively $h\widehat{w}_1$) has to go all the way down from infinity (respectively from Γ_2) to the ground level ∂M^4_∞, to which it ***has*** to stay glued via its $S^1 \times 0$.

Moreover, as far as infinity itself is concerned, we need not only hw_1 but also the full $h\widehat{w}_1$ (which actually reaches the infinity of int $\widehat{X}^4_\infty(0)$ along Γ_2^{naked} (the argument outlined in (42-7) requires the full wick.)

B) It is not so clear how to combine the alternative, PL-proof of theorem C with the ingredients which this section has introduced.

Bibliography

[Ca] A. Casson, Three lectures on new infinite constructions in 4-dimensional manifolds, notes prepared by L. Guillou, in the volume L. Guillou, A. Marin, *À la recherche de la topologie perdue*, Birkäuser (1986), 201-244.

[Fr] M. Freedman, The topology of four-dimensional manifolds, *J. of Diff. Geom.* **17** (1982), 357-453.

[Fr-Q] M. Freedman, F. Quin, *Topology of 4-manifolds*, Princeton University Press, 1990.

[Ga] D. Gabai, Valentin Poénaru's Program for the Poincaré Conjecture, in the volume *Geometry, Topology and Physics for Raoul Bott*, S.T. Yau editor, International Press (1994), 139-166.

[G-R] M. Gerstenhaber, O. Rothaus, The solution of sets of equations in groups, *Proc. Nat. Acad. USA* **48** (1962), 1531-1533.

[Ma] B. Mazur, A note on some contractible 4-manifolds, *Ann. of Math.* Vol. 2, n° 73 (1961), 221-228.

[Man] R. Mandelbaum, Four-dimensional Topology : An introduction, *BAMS*, Vol. 2, n° 1 (1980), 1-159.

[Po0] V. Poénaru, Les décompositions de l'hypercube en produit topologique, *Bull. Soc. Math. France* **88** (1960), 113-129.

[Po1] V. Poénaru, Infinite processes and the 3-dimensional Poincaré Conjecture I : The collapsible pseudo-spine representation theorem, *Topology*, Vol. 31, n° 3 (1992), 625-656.

[Po2] V. Poénaru, Infinite processes and the 3-dimensional Poincaré Conjecture II : The Honeycomb representation theorem, (second revised version) *Prépublications d'Orsay* 93-14 (1994).

[Po3] V. Poénaru, Infinite processes and the 3-dimensional Poincaré Conjecture III : The algorithm, *Prépublications d'Orsay* 92-10 (1992).

[Po4] V. Poénaru, Infinite processes and the 3-dimensional Poincaré Conjecture IV (part A) : Le théorème de non-sauvagerie lisse (the smooth tameness theorem), *Prépublications d'Orsay* 92-83 (1992).

[Po5] V. Poénaru, Infinite processes and the 3-dimensional Poincaré Conjecture IV (part B),*Prépublications d'Orsay* 95-33 (1995).

[Po6] V. Poénaru, The strange compactification theorem, Infinite processes and the 3-dimensional Poincaré Conjecture, VI (part A), *Preprint* IHES/M/95/15 (1995); part B, *Preprint* IHES/M/96/43 (1996) ; part C, *Preprint* IHES/M/97/43 (1997); part D, *Preprint* IHES/M/97 (1997), (parts B, C, D are in French).

[Po7] V. Poénaru, Infinite processes and the 3-dimensional Poincaré Conjecture, V (differential Topology in dimension *Prépublications d'Orsay* 9425 (1994). (This paper which has a gap, is partly superseded by the present one).

[Po8] V. Poénaru, π_1^∞ and infinite simple homotopy type in dimension three, *Preprint* IHES/M/95/6 (1995) (An amplified and revised version of this paper will appear in the proceedings of the "Low-dimensional topology" conference, Madeira (1998), Contemporary Mathematics.)

[Po-Ta1] V. Poénaru, C. Tanasi, Representations of the Whitehead manifold Wh^3 and Julia sets, *Annales de Toulouse* IV n° 3 (1995).

[Po-Ta2] V. Poénaru, C. Tanasi, Equivariant Almost-Arborescent Representations of Open Simply-Connected 3-manifolds; A Finiteness result, *To appear in Memoirs of the AMS.*

[S] L. Siebenmann, "Les bissections expliquent le théorème de Reidemeister-Singer. Un retour aux sources" Orsay, Preprint, 1979.

[Ta] C. Taubes, Gauge theory on asymptotically periodic 4-manifolds, *J. of Diff. Geom.* **25** (1987), 363-430.

[Wh] J.H.C. Whitehead, On C^1-complexes, *Ann. of. Math.* **41** (1940), 809-824.