



A Note on the Critical Laplace Equation and Ricci Curvature

Mattia Fogagnolo¹ · Andrea Malchiodi² · Lorenzo Mazziere³

Received: 13 September 2022 / Accepted: 8 February 2023
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Abstract

We study strictly positive solutions to the critical Laplace equation

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}},$$

decaying at most like $d(o, x)^{-(n-2)/2}$, on complete noncompact manifolds (M, g) with nonnegative Ricci curvature, of dimension $n \geq 3$. We prove that, under an additional mild assumption on the volume growth, such a solution does not exist, unless (M, g) is isometric to \mathbb{R}^n and u is a Talenti function. The method employs an elementary analysis of a suitable function defined along the level sets of u .

Keywords Critical equations · Classification results · Manifolds with nonnegative Ricci curvature · Level sets

Mathematics Subject Classification 35R01 · 35B33 · 53C21 · 40E10

1 Introduction and Statement of the Main Results

The critical Laplace equation

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}}, \quad (1.1)$$

✉ Mattia Fogagnolo
mattia.fogagnolo@unipd.it

Andrea Malchiodi
andrea.malchiodi@sns.it

Lorenzo Mazziere
lorenzo.mazziere@unitn.it

- ¹ Università di Padova, Via Trieste 63, 35121 Padova, PD, Italy
- ² Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, PI, Italy
- ³ Università degli Studi di Trento, via Sommarive 14, 38123 Povo, TN, Italy

on \mathbb{R}^n , with $n \geq 3$, together its possible generalizations, has been widely studied in the mathematical literature for many reasons. Let us just mention that entire solutions to (1.1) provide critical functions for the L^2 Sobolev inequality and that $u^{4/(n-2)}g_{\mathbb{R}^n}$ is a metric of positive, constant scalar curvature. Since the works of Obata [1], Talenti [2], Aubin [3], Gidas-Ni-Nirenberg [4], it became clear that under additional decay or variational assumptions the only entire solutions to (1.1) are radially symmetric. They are often called *Talenti*. The groundbreaking contribution by Caffarelli–Gidas–Spruck [5] actually established that no additional assumptions are required, except for positivity.

In this note, we face the problem of classifying entire solutions to (1.1) on complete noncompact manifolds (M, g) with nonnegative Ricci curvature of dimension $n \geq 3$. We prove that, under a suitable decay assumption on u and an additional mild assumption on the volume growth, such a solution does not exist, unless (M, g) is isometric to \mathbb{R}^n and u is a Talenti function.

Theorem 1.1 *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature, of dimension $n \geq 3$, satisfying*

$$\frac{|B(o, t)|}{|B(o, s)|} \geq C \left(\frac{t}{s}\right)^b \tag{1.2}$$

for some $1 < b \leq n$ and for any $t \geq s > 0$. Assume that there exists a global, strictly positive solution u to

$$-\Delta u = n(n - 2)u^{\frac{n+2}{n-2}},$$

such that $u(x)d(o, x)^{(n-2)/2}$ is uniformly bounded in M as a function of $x \in M$, for some $o \in M$. Then, (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$ and

$$u(x) = \left(\frac{a}{a^2 + d(o, x)^2}\right)^{\frac{(n-2)}{2}}, \tag{1.3}$$

for some $a > 0$.

We observe that the volume condition (1.2) is satisfied any time

$$C^{-1}r^b \leq |B(o, r)| \leq Cr^b$$

for $1 < b \leq n$ and for any $r \geq 1$. In particular, Euclidean volume growth manifolds fulfill it.

As a consequence of the previous result, apart from the Euclidean case, the Ricci-flat manifolds (M, g) fulfilling the above assumptions do not enjoy conformal metrics $\tilde{g} = u^{4/(n-2)}g$ of positive constant scalar curvature with conformal factor decaying at infinity as above. In particular, this applies to ALE (such as Eguchi–Hanson), ALF, and ALG gravitational instantons. We refer the interested reader to [6], [7], [8], [9].

Interesting properties of the Yamabe equation on manifolds with nonnegative Ricci can be found in [10].

We point out that, to our knowledge, the study of Liouville-type results for (1.1) on manifolds with nonnegative Ricci curvature is yet to be addressed in the literature. In the case of subcritical exponent $1 \leq \alpha < (n + 2)/(n - 2)$ in the right-hand side of (1.1), it has been shown in [11, Theorem 1.2] that entire solutions are not allowed. Theorem 1.1 can thus be interpreted as a first extension of such study to the critical regime. A rigidity statement for minimizers of the Sobolev quotient can be deduced from [12, Theorem 1.2]. Concerning the case of complete Riemannian manifolds with other curvature assumptions, we mention [13] by Muratori–Soave, who considered solutions to (1.1) that are radially symmetric or that minimize the Sobolev quotient in Cartan–Hadamard manifolds.

Our proof combines elementary computations on a vector field with nonnegative divergence already considered by [1] and [11], a Harnack-type argument triggered by the decay assumption coupled with a Yau-type gradient estimate, and a novel auxiliary function V defined along the level sets of u . In proving Theorem 1.1, we first show an intermediate result that adds to the decay assumption on u a *finite energy condition*, that is to say

$$\int_M |\nabla u|^2 \, d\mu < +\infty, \tag{1.4}$$

but does not require any volume restriction, where $d\mu$ is the volume measure associated to g . It reads as follows:

Theorem 1.2 *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and of dimension $n \geq 3$. Assume that there exists a global, strictly positive finite energy solution u to*

$$-\Delta u = n(n - 2)u^{\frac{n+2}{n-2}},$$

such that $u(x)d(o, x)^{(n-2)/2}$ is uniformly bounded in M as a function of $x \in M$. Then, (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$ and

$$u(x) = \left(\frac{a}{a^2 + d(o, x)^2} \right)^{\frac{(n-2)}{2}},$$

for some $a > 0$.

It is worth pointing out that, thanks to the decay estimate on the gradient that we observe in Corollary 2.2, if the decay on u is reinforced to $u \leq Cd(o, x)^{(2-n)/2-\epsilon}$, for $\epsilon > 0$, then (1.4) is just a consequence of this. In particular, Theorem 1.2 fully recovers the main result in [4], and extends it to a rigidity statement in the geometry of nonnegative Ricci curvature.

Summary. In Sect. 2 we gather some basic estimates about u and its derivatives. In Sect. 3 we introduce the fundamental vector field with nonnegative divergence and set

its connection with the auxiliary function V , and finally in Sect. 4 we prove Theorems 1.2 and 1.1.

Added note

The preprint [14] was uploaded on the arXiv a couple of days before ours. It refines and generalizes our results, with different methods.

2 Preliminary A Priori Estimates

In this section we work out some preliminary estimates in force for solutions to (1.1), that we later particularize for those such that $u(x)d(o, x)^{(n-2)/2}$ is uniformly bounded.

First of all, we crucially observe that the technique Yau [15] pioneered in order to obtain gradient estimates for harmonic functions under lower Ricci curvature bounds yields for solutions to the critical Laplace equation of the following inequality.

Proposition 2.1 [*Yau-type estimate for solutions to the critical Laplace equation*] *Let (M, g) be a complete, noncompact Riemannian manifold with nonnegative Ricci curvature of dimension $n \geq 3$. Then, a positive solution to (1.1) satisfies*

$$\sup_{B(x,R)} \frac{|\nabla u|^2}{u^2} \leq C \left[\frac{1}{R^2} + \sup_{B(x,2R)} u^{\frac{4}{(n-2)}} \right] \tag{2.1}$$

for any $x \in M$, for any $R \geq 1$ and for some dimensional constant $C > 0$.

The proof of (2.1) is obtained by following very closely the one for eigenfunctions proposed for [16, Theorem 1.1] and, as such, we omit the proof.

As a consequence of the above result, we have the following decay estimate for $|\nabla u|$ when u satisfies the same bound as in Theorem 1.1. Here and in the sequel of the paper, we consider $o \in M$ to be any fixed point, and use the notation $r(x) = d(o, x)$ for $x \in M$.

Corollary 2.2 [*Decay of the gradient*] *Let (M, g) be a complete, noncompact Riemannian manifold with nonnegative Ricci curvature of dimension $n \geq 3$, and let u be a positive solution to (1.1) that satisfies $u \leq Cr^{-(n-2)/2}$. Then,*

$$\frac{|\nabla u|^2}{u^2}(x) \leq C \frac{1}{r(x)^2} \tag{2.2}$$

for any x such that $r(x) = d(o, x) \geq 4$.

Proof It suffices to apply Proposition 2.1 with $2R = r(x)/2$. Coupled with the decay assumption on u , it yields

$$\frac{|\nabla u|^2}{u^2} \leq C \left[\frac{1}{r(x)^2} + u^{\frac{4}{n-2}}(y_x) \right] \leq C \left[\frac{1}{r(x)^2} + \frac{1}{r(y_x)^2} \right], \tag{2.3}$$

for some $y_x \in \overline{B(x, r(x)/2)}$. On the other hand, we have $r(x)/3 \leq r(y_x) \leq 2r(x)$, and plugging this information into (2.3) leaves us with (2.2). \square

Through integrating by parts the Bochner formula, we do now work out an integral estimate on the Hessian of u . The computations, that in our case are particularly simple, got some inspiration from analogous ones in the celebrated work [17]. Here and in the remainder of this paper we consider $o \in M$ to be fixed and denote with A_{R_1, R_2} the open annulus $B(o, R_2) \setminus \overline{B(o, R_1)}$.

Proposition 2.3 *Let (M, g) be a complete, noncompact Riemannian manifold with nonnegative Ricci curvature of dimension $n \geq 3$. Then, for any $k > 1$ there exist numbers $1 < \alpha < \beta < k$ such that a positive solution to (1.1) satisfies, for any $R \geq 1$,*

$$\int_{A_{\alpha R, \beta R}} |\nabla \nabla u|^2 \, d\mu \leq C \left(R^{-2} \int_{A_{R, kR}} |\nabla u|^2 \, d\mu + \int_{A_{R, kR}} u^{\frac{4}{(n-2)}} |\nabla u|^2 \, d\mu \right), \tag{2.4}$$

for some constant C not depending on R .

Proof By the Bochner formula and by the nonnegativity of the Ricci tensor, we have

$$|\nabla \nabla u|^2 \leq \frac{1}{2} \Delta |\nabla u|^2 - \langle \nabla \Delta u, \nabla u \rangle.$$

We multiply both sides by an annular Euclidean-like cut-off function ϕ supported in $B(o, kR) \setminus \overline{B(o, R)}$ such that

$$\begin{aligned} \phi &\equiv 1 \text{ on } B(o, \beta R) \setminus \overline{B(o, \alpha R)}, \\ |\nabla \phi| &\leq C \frac{\phi^{1/2}}{R}, \\ |\Delta \phi| &\leq C \frac{1}{R^2}, \end{aligned}$$

for $1 < \alpha < \beta < k$. Such function is well known to exist since [17], and can be more precisely built by slightly re-adapting the proof of [18, Corollary 2.3]. Integrating by parts twice the first term on the right-hand side, we get

$$\begin{aligned} \int_{A_{\alpha R, \beta R}} |\nabla \nabla u|^2 \, d\mu &\leq \int_{A_{R, kR}} |\nabla \nabla u|^2 \phi \, d\mu \\ &\leq \frac{1}{2} \int_{A_{R, kR}} |\nabla u|^2 \Delta \phi + \int_{A_{R, kR}} u^{\frac{4}{(n-2)}} |\nabla u|^2 \phi \, d\mu \\ &\leq C \left(R^{-2} \int_{A_{R, kR}} |\nabla u|^2 \, d\mu + \int_{A_{R, kR}} u^{\frac{4}{(n-2)}} |\nabla u|^2 \, d\mu \right), \end{aligned}$$

as claimed. \square

When u is assumed to satisfy $u \leq Cr^{-(n-2)/2}$ we get the following integral decay estimate for its Hessian.

Corollary 2.4 [Integral decay of the Hessian] *Let (M, g) be a complete, noncompact Riemannian manifold with nonnegative Ricci curvature of dimension $n \geq 3$, and let u be a positive solution to (1.1) that satisfies $u \leq Cr^{-(n-2)/2}$. Then, for any $k > 1$ there exist numbers $1 < \alpha < \beta < k$ such that for any $R \geq 1$,*

$$\int_{A_{\alpha R, \beta R}} |\nabla \nabla u|^2 \, d\mu \leq \frac{C}{R^2} \int_{A_{R, kR}} |\nabla u|^2 \, d\mu \leq \frac{C}{R^2},$$

for some positive constant C .

Proof The first inequality just follows by plugging the assumption on u into (2.4), the second one by coupling it with (2.2), and observing that, by Bishop–Gromov monotonicity, we have

$$|A_{r, kr}| \leq |B(o, kR)| \leq \omega_n (kR)^n,$$

where ω_n is the Lebesgue measure of the unit ball in \mathbb{R}^n . □

We close this preliminary section by recording a basic and fundamental control on u from below.

Proposition 2.5 [Control from below for u] *Let (M, g) be a complete, noncompact Riemannian manifold with nonnegative Ricci curvature of dimension $n \geq 3$, and let u be a positive solution to (1.1). Then, there exists a strictly positive constant C such that*

$$u(x) \geq Cr(x)^{2-n} \tag{2.5}$$

for any $x \in M \setminus B(o, 1)$.

Proof Let $C = \min_{\partial B(o, 1)} u$, and recall that r^{2-n} is sub-harmonic in the sense of distributions in $M \setminus \{o\}$ (see e.g., the proof of [19, Lemma 2.2]). Then, the function $u/C - r^{2-n}$ is super-harmonic in $B(o, R) \setminus \overline{B(o, 1)}$ and in particular it satisfies

$$(u/C - r^{2-n})(x) \geq \min_{\partial B(o, 1) \cup \partial B(o, R)} (u/C - r^{2-n}).$$

for any $x \in B(o, R) \setminus \overline{B(o, 1)}$. By the specific choice of C and the positivity of u , the right-hand side tends to a nonnegative limit along any sequence of $R_j \rightarrow +\infty$. This implies (2.5). □

3 A Key Vector Field and a Useful Auxiliary Function

In this section we introduce a vector field with nonnegative divergence, vanishing exactly in the case in the flat model situation and set the main properties of the integral auxiliary function V .

We introduce the vector field with nonnegative divergence that rules our problem. Its intuition dates back to the early work of Obata [1], and it constitutes a key ingredient also in the study of subcritical elliptic equations performed in [11]. As in the latter, the vector field is better understood in terms of the function

$$v = u^{-\frac{2}{n-2}}, \tag{3.1}$$

with u a strictly positive solution to (1.1). This is due to the fact that, in the model situation of the flat \mathbb{R}^n with u given by a Talented (1.3), v becomes an affine function of $d(o, x)^2$, that in particular solves

$$\nabla \nabla v = \frac{\Delta v}{n} g.$$

In fact, the squared norm of the trace-free Hessian of v will constitute, together with a Ricci curvature term, the nonnegative divergence of the vector field.

We define

$$X = \frac{1}{v^{n-1}} \left[\frac{1}{2} \nabla |\nabla v|^2 - \frac{1}{2} \frac{|\nabla v|^2 \nabla v}{v} - 2 \frac{\nabla v}{v} \right]. \tag{3.2}$$

We actually give, for the reader’s sake, a self-contained computation of its divergence. In order to do it, we first point out that the critical Laplace equation translates as

$$\Delta v = \frac{n}{2} \frac{|\nabla v|^2}{v} + \frac{2n}{v} \tag{3.3}$$

in terms of v .

Proposition 3.1 *Let (M, g) be a Riemannian manifold, and let u be a positive solution to (1.1) on M . Then, letting v be as in (3.1), the vector field defined by (3.2) satisfies*

$$\operatorname{div} X = \frac{1}{v^{n-1}} \left[\left| \nabla \nabla v - \frac{\Delta v}{n} g \right|^2 + \operatorname{Ric}(\nabla v, \nabla v) \right]. \tag{3.4}$$

Proof We have

$$\begin{aligned} \operatorname{div} X &= \frac{1}{v^{n-1}} \left[\frac{1}{2} \operatorname{div}(\nabla |\nabla v|^2) - \frac{1}{2} \operatorname{div} \left(|\nabla v|^2 \frac{\nabla v}{v} \right) - 2 \operatorname{div} \left(\frac{\nabla v}{v} \right) \right] \\ &\quad - \frac{(n-1)}{v^{n-1}} \left[\frac{1}{2} \left\langle \nabla |\nabla v|^2, \frac{\nabla v}{v} \right\rangle - \frac{|\nabla v|^4}{2v^2} - 2 \frac{|\nabla v|^2}{v^2} \right]. \end{aligned} \tag{3.5}$$

We do now compute the three divergence terms. We have, using Bochner identity and (3.3),

$$\begin{aligned} \operatorname{div}(\nabla|\nabla v|^2) &= 2 \left[\left| \nabla \nabla v - \frac{\Delta v}{n} g \right|^2 + \frac{(\Delta v)^2}{n} + \langle \nabla \Delta v, \nabla v \rangle + \operatorname{Ric}(\nabla v, \nabla v) \right] \\ &= 2 \left[\left| \nabla \nabla v - \frac{\Delta v}{n} g \right|^2 - \frac{n}{4} \frac{|\nabla v|^4}{v^2} \right. \\ &\quad \left. + \frac{4n}{v^2} + \frac{n}{2} \left\langle \nabla|\nabla v|^2, \frac{\nabla v}{v} \right\rangle + \operatorname{Ric}(\nabla v, \nabla v) \right]. \end{aligned} \tag{3.6}$$

Moreover,

$$\operatorname{div} \left(|\nabla v|^2 \frac{\nabla v}{v} \right) = \left\langle \nabla|\nabla v|^2, \frac{\nabla v}{v} \right\rangle + \frac{1}{2}(n-2) \frac{|\nabla v|^4}{v^2} + 2n \frac{|\nabla v|^2}{v^2}. \tag{3.7}$$

Finally,

$$\operatorname{div} \left(\frac{\nabla v}{v} \right) = \frac{(n-2)}{2} \frac{|\nabla v|^2}{v^2} + \frac{2n}{v^2}. \tag{3.8}$$

Plugging (3.6), (3.7), and (3.8) into (3.5) leaves us with identity (3.4). □

We now define, a priori only for regular level sets of v , the function

$$V : (\inf_M v, +\infty) \setminus v(\operatorname{Crit}(v)) \rightarrow (0 + \infty),$$

where $\operatorname{Crit}(v) = \{\nabla v = 0\}$, given by

$$V(s) = \int_{\{v=s\}} |\nabla v|^3 \, d\sigma. \tag{3.9}$$

We are denoting with $d\sigma$ the $(n - 1)$ -dimensional Hausdorff measure induced by g . In particular, by Sard’s Theorem, this function is defined for almost every $s \in (\inf_M v, +\infty)$. In the next result, we show that it is actually equivalent to an absolutely continuous function, and compute its derivative. In order to justify the formal computation, we argue as in the recent paper [20], but we provide, for completeness, all details.

Proposition 3.2 *Let (M, g) be a Riemannian manifold, and let u be a positive solution to (1.1) that tends to zero at infinity. Then, the function V defined above is equivalent to an absolutely continuous function $V : (\inf_M v, +\infty) \rightarrow \mathbb{R}$, and moreover*

$$\begin{aligned}
 V'(s) = \int_{\{v=s\}} \left\langle \nabla|\nabla v|^2, \frac{\nabla v}{|\nabla v|} \right\rangle d\sigma + \frac{n}{2}s^{-1} \int_{\{v=s\}} |\nabla v|^3 d\sigma \\
 + 2ns^{-1} \int_{\{v=s\}} |\nabla v| d\sigma \tag{3.10}
 \end{aligned}$$

holds true for almost any value $s \in (\inf_M v, +\infty)$.

Proof Let s be a regular value for v , so that

$$V(s) = \int_{\{v=s\}} |\nabla v|^2 \left\langle \nabla v, \frac{\nabla v}{|\nabla v|} \right\rangle d\sigma = \int_{\{v<s\}} \operatorname{div}(|\nabla v|^2 \nabla v) d\mu,$$

where we used the divergence theorem and that, since v tends to infinity, it is a proper function and the outer unit normal to $\{v < s\}$ is given by $\nabla v/|\nabla v|$.

Observe now that, arguing exactly as in the basic proof of [20, Lemma 1.1], the critical set of u is locally contained in smooth $(n - 1)$ -dimensional manifolds, in particular it is of null n -dimensional Hausdorff measure and moreover all the level sets of u (and hence of v) have zero n -dimensional Hausdorff measure. Indeed, this argument just relies on the fact that $\Delta u < 0$. In particular, the function $\tilde{V} : (\inf_M v, +\infty) \rightarrow \mathbb{R}$ defined by

$$\tilde{V}(s) = \int_{\{v=s\}} |\nabla v|^2 \left\langle \nabla v, \frac{\nabla v}{|\nabla v|} \right\rangle d\sigma = \int_{\{v<s\}} \operatorname{div}(|\nabla v|^2 \nabla v) d\mu,$$

is immediately seen to be continuous. Indeed, for $\epsilon > 0$, we have

$$\tilde{V}(s + \epsilon) - \tilde{V}(s) = \int_{\{s \leq v < s + \epsilon\}} \operatorname{div}(|\nabla v|^2 \nabla v) d\mu = \int_{\{s < v < s + \epsilon\}} \operatorname{div}(|\nabla v|^2 \nabla v) d\mu$$

where we have used $\mu\{v = s\} = 0$, and, in particular, the right-hand side above approaches 0 as $\epsilon \rightarrow 0^+$ by the Dominated Convergence Theorem.

We proved so far that V is equivalent to the continuous function \tilde{V} , and in particular we can safely identify one with the other. In order to show the absolute continuity, let, as in the proof of [20, Lemma 1.3],

$$V_\delta(s) = \int_{\{v<s\}} \frac{\operatorname{div}(|\nabla v|^2 \nabla v)|\nabla v|}{|\nabla v| + \delta} d\sigma,$$

for $\delta > 0$. Observe that, since as already pointed out $\mu\{\nabla v = 0\} = 0$, the above expression converges by dominated convergence theorem as $\delta \rightarrow 0^+$ to $V(s)$. Moreover, by

coarea formula, we have

$$V_\delta(s) = \int_{v_0}^s \int_{\{v=t\}} \frac{\operatorname{div}(|\nabla v|^2 \nabla v)}{|\nabla v| + \delta} \, d\sigma \, dt,$$

where we let $v_0 = \inf_M v$. By the Dominated Convergence Theorem, we can pass to the limit as $\delta \rightarrow 0^+$ also in the right-hand side of the identity above, and deduce that

$$V(s) = \int_{v_0}^s \int_{\{v=t\}} \frac{\operatorname{div}(|\nabla v|^2 \nabla v)}{|\nabla v|} \, d\sigma \, dt, \tag{3.11}$$

for any $s \in (\inf_M v, +\infty)$, that is the absolute continuity of V . Finally, computing, with the aid of (3.3),

$$\operatorname{div}(|\nabla v|^2 \nabla v) = \langle \nabla |\nabla v|^2, \nabla v \rangle + \frac{n}{2} \frac{|\nabla v|^4}{v} + 2n \frac{|\nabla v|^2}{v},$$

and plugging it into (3.11), we also showed (3.10). □

We finally link the function V and its derivative to the vector field X . We have, by the Divergence Theorem, and using again the fact that the outer unit normal to $\{v < s\}$ is given by $\nabla v/|\nabla v|$,

$$\begin{aligned} \int_{\{v < s\}} \operatorname{div} X \, d\mu &= \frac{1}{2} \frac{\int_{\{v=s\}} \langle \nabla |\nabla v|^2, \nabla v |\nabla v|^{-1} \rangle \, d\sigma}{s^{n-1}} - \frac{1}{2} \frac{\int_{\{v=s\}} |\nabla v|^3 \, d\sigma}{s^n} \\ &\quad - 2 \frac{\int_{\{v=s\}} |\nabla v| \, d\sigma}{s^n} \end{aligned}$$

for any regular value s . Coupling this with (3.4) and (3.1), we deduce the following.

Corollary 3.3 *Let (M, g) be a complete Riemannian manifold. Let u be a strictly positive solution to (1.1) that vanishes at infinity. Then, we have*

$$\begin{aligned} \int_{\{v < s\}} \frac{1}{v^{n-1}} \left[\left| \nabla \nabla v - \frac{\Delta v}{n} g \right|^2 + \operatorname{Ric}(\nabla v, \nabla v) \right] \, d\mu &= \frac{1}{2} \frac{V'(s)}{s^{n-1}} - \frac{1}{4} (n+2) \frac{V(s)}{s^n} \\ &\quad - (n+2) \frac{\int_{\{v=s\}} |\nabla v| \, d\sigma}{s^n} \end{aligned} \tag{3.12}$$

for almost any $s \in v(M)$.

The main aim, for proving Theorems 1.2 and 1.1, will be showing that in their assumptions, the left-hand side of (3.12) vanishes for any s .

4 Proof of the Main Results

In this section we prove Theorems 1.1 and 1.2. We first prove the latter, i.e., the rigidity result for finite energy solutions, and then prove that in the setting of Theorem 1.1 the function u actually enjoys finite energy. We start with the following simple splitting principle, consequence of the basic result according to which the existence of a nontrivial function with vanishing trace-less Hessian implies a warped product splitting of the metric [21, Theorem 5.7.4], [22], [17, Sect. 1].

Lemma 4.1 *Let (M, g) be a complete manifold, and let u be a positive solution to (1.1) that vanishes at infinity. If, for v defined in (3.1),*

$$\left| \nabla \nabla v - \frac{\Delta v}{n} g \right|^2 + \text{Ric}(\nabla v, \nabla v) \equiv 0$$

on M , then (M, g) is isometric to flat \mathbb{R}^n and u is of the form (1.3).

Proof It is classically deduced from the vanishing of the trace-less Hessian of v , e.g., appealing to [21, Theorem 5.7.4], that (M, g) must split a warped product $(I \times N, d\rho \otimes d\rho + \phi^2(\rho)g_N)$, for some hypersurface N in M and with I coinciding with \mathbb{R} or $[0, +\infty)$. By the proof presented in the contribution above (see also the maybe more transparent computations carried out in the proof of [22, Theorem 1.1]), one also realizes that N can be identified with a smooth level set of v , and that in particular it is a closed hypersurface. Again by construction of the splitting, one also has that in these coordinates $v'(\rho) = 2\alpha\phi(\rho)$ for some constant α . Consequently, since the Ricci curvature of g can be computed as

$$\text{Ric} = -(n-1) \frac{\phi''}{\phi} d\rho \otimes d\rho + \text{Ric}_N - ((n-2)(\phi')^2 + \phi\phi'')^2 g_N,$$

plugging in $\text{Ric}(\nabla v, \nabla v) = |v'(\rho)|^2 \text{Ric}(\nabla \rho, \nabla \rho) = 0$ we deduce that $\phi = A + B\rho$, for some constants A, B . If $B = 0$, then (M, g) would be a cylinder with cross-section N , and $v = A\rho + C$, for some other constant C . Consequently, $\Delta v = 0$, and this is in contradiction with equation (3.3).

We can thus suppose, by possibly translating the coordinate ρ , that $A = 0$ and $\phi = B\rho$ for some constant $B \neq 0$. In particular, since g must be smooth also as $\rho \rightarrow 0^+$, we have $I = [0, +\infty)$, (N, g_N) the sphere \mathbb{S}^{n-1} with its standard round metric and $B = 1$. In other words, M is the Euclidean space and g is its flat metric.

Moreover, $v = \alpha\rho^2 + \beta$, with ρ constituting the Euclidean distance from some origin o . Plugging this function into (3.3), one immediately gets $\alpha\beta = 1$, and thus, by (3.9), u has the form (1.3). □

Exploiting crucially (3.12) and the above splitting principle, we prove Theorem 1.2.

Proof (*Proof of Theorem 1.2*) Due to the finite energy condition (1.4) and the coarea formula, we have

$$\int_0^{t_0} \int_{\{u=t\}} |\nabla u| \, d\sigma \, dt < +\infty.$$

for any $t_0 \in u(M)$. Performing a change of variables, such condition translates in terms of v as

$$\int_{s_0}^{+\infty} \int_{\{v=s\}} \frac{|\nabla v|}{s^n} \, d\sigma \, ds < +\infty. \tag{4.1}$$

We want to prove that

$$\int_{\{v < s_j\}} \frac{1}{v^{n-1}} \left[\left| \nabla \nabla v - \frac{\Delta v}{n} g \right|^2 + \text{Ric}(\nabla v, \nabla v) \right] \, d\mu \rightarrow 0^+ \tag{4.2}$$

for some $s_j \rightarrow +\infty$. Indeed, since the Ricci curvature, and thus the integrand, are nonnegative, this would complete the proof by Lemma 4.1. Assume now by contradiction that (4.2) holds for no diverging sequences $\{s_j\}_{j \in \mathbb{N}}$. Then, by (3.12), we deduce that for almost any $s \in v(M)$

$$V'(s) \geq \delta s^{n-1}$$

for some $\delta > 0$. Integrating the above inequality in (s_0, s) for some $s_0 \in v(M)$, we deduce the following growth for the absolutely continuous function V

$$V(s) \geq \frac{\delta}{n} s^n + F(s_0), \tag{4.3}$$

with $F(s_0) = V(s_0) - \frac{\delta}{n} s_0^n$. Moreover, by choosing s_0 big enough, by coupling (2.2) with (2.5), one gets

$$|\nabla v|^2(x) \leq C \frac{v^2}{r^2}(x) \leq Cv(x) \tag{4.4}$$

for any $x \in \{v > s_0\}$. But then, by (4.3) and (4.4), the left-hand side of (4.1) must also satisfy

$$\int_{s_0}^{+\infty} \int_{\{v=s\}} \frac{|\nabla v|}{s^n} \, d\sigma \, ds = \int_{s_0}^{+\infty} \int_{\{v=s\}} \frac{|\nabla v|^3}{|\nabla v|^2 s^n} \, d\sigma \, ds \geq$$

$$\frac{1}{C} \int_{s_0}^{+\infty} \frac{V(s)}{s^{n+1}} ds \geq \frac{1}{C} \int_{s_0}^{+\infty} \frac{(\delta/n)s^n + F(s_0)}{s^{n+1}} ds,$$

which contradicts (4.1). □

The proof of Theorem 1.1 fully builds on the following improved decay estimate on u . It crucially uses again the vector field X , this time integrating its divergence on geodesic balls in the place of sub-level sets of v , and a Harnack-type argument.

Proposition 4.2 *Let (M, g) and u satisfy the assumptions of Theorem 1.1. Then, for any positive $\epsilon < (n - 2)/6$, we have*

$$u(x)r^{\frac{(n-2)}{2}+\epsilon}(x) \rightarrow 0 \tag{4.5}$$

as $r(x) = d(o, x) \rightarrow +\infty$.

Proof We apply the Divergence Theorem to the vector field X of (3.2) in a geodesic ball $B(o, R)$. We get, by (3.4), that

$$\begin{aligned} & \int_{B(o,R)} \frac{1}{v^{n-1}} \left[\left| \nabla \nabla v - \frac{\Delta v}{n} g \right|^2 + \text{Ric}(\nabla v, \nabla v) \right] d\mu \\ &= \frac{1}{2} \int_{\partial B(o,R)} \frac{1}{v^{n-1}} \langle \nabla |\nabla v|^2, \nu \rangle d\sigma \\ & \quad - \frac{1}{2} \int_{\partial B(o,R)} \frac{1}{v^n} \langle |\nabla v|^2 \nabla v, \nu \rangle d\sigma \\ & \quad - 2 \int_{\partial B(o,R)} \frac{1}{v^n} \langle \nabla v, \nu \rangle d\sigma, \end{aligned} \tag{4.6}$$

where ν is the outward unit normal to $\partial B(o, R)$. We claim that, if (4.5) does not hold for some positive $\epsilon < (n - 2)/6$, then there exists a sequence of radii $R_j \rightarrow +\infty$, as $j \rightarrow +\infty$, such that the right-hand side of (4.6) for $R = R_j$ vanishes in the limit as $j \rightarrow +\infty$. Indeed, in this case, by the nonnegativity of the left-hand side, we deduce that its whole integrand is null and deduce by Lemma 4.1 that u is (a Euclidean) Talentian function of the form (1.3). But this actually satisfies (4.5), giving a contradiction.

Assume then that there exists a sequence of points x_j , with $r(x_j) = d(o, x_j) \rightarrow +\infty$ as $j \rightarrow +\infty$, such that

$$u(x_j)r^{\frac{(n-2)}{2}+\epsilon}(x_j) \geq C > 0 \tag{4.7}$$

for any x_j . By [23, Proposition 0.4], in force by condition (1.2), there exists $k > 1$ such that for any $R > 1$, for any two points x, y in $\overline{B(o, R)} \setminus B(o, k^{-1}R)$ there exists

a curve $\gamma_{x,y}$ connecting them *fully contained* in such an annulus. In particular, setting $R_j = r(x_j)$, for any point $y \in \overline{B(o, R_j)} \setminus B(o, k^{-1}R_j)$ we have

$$\log u(y) - \log u(x_j) \leq \int_0^{|\gamma_{x_j,y}|} |\nabla \log u|(\gamma_{x_j,y}(t)) dt \leq C \frac{|\gamma_{x_j,y}|}{R_j}, \tag{4.8}$$

where we denoted by $|\gamma_{x_j,y}|$ the length of the geodesic, and where we employed in the last step the decay estimate (2.2) for the gradient. Moreover, as an application of the Bishop–Gromov monotonicity, the ratio on the rightmost hand side of (4.8) is uniformly bounded, see the annular diameter estimate [24, Lemma 1.4]. Consequently, any point $y \in \overline{B(o, R_j)} \setminus B(o, k^{-1}R_j)$ in fact satisfies (4.7). Coupling this information with the decay assumption, and translating in terms of v , we get

$$\frac{1}{C}r \leq v \leq Cr^{\frac{2\epsilon}{(n-2)}+1}$$

on any annulus $\overline{B(o, R_j)} \setminus B(o, k^{-1}R_j)$, for some constant $C > 0$ independent of R_j . The gradient decay (2.2), applied to v , improves to

$$|\nabla v|(x) \leq Cr(x)^{\frac{2\epsilon}{n-2}} \tag{4.9}$$

for any $x \in \overline{B(o, R_j)} \setminus B(o, k^{-1}R_j)$, for some $C > 0$ not depending R_j . We aim to prove that, for $k^{-1} < \alpha < \beta < 1$ given by Corollary 2.4,

$$\int_{A_{\alpha R_j, \beta R_j}} \frac{|\nabla v|^3}{v^n} + \frac{|\nabla v|}{v^n} + \frac{|\nabla \nabla v| |\nabla v|}{v^{n-1}} d\mu \leq CR_j^{6\epsilon/(n-2)} \tag{4.10}$$

for a positive constant C independent from R_j . Indeed, if this is the case, we deduce from the coarea formula that there exists a sequence of $\hat{R}_j \in (k^{-1}R_j, R_j)$ such that

$$\int_{\partial B(o, \hat{R}_j)} \frac{|\nabla v|^3}{v^n} + \frac{|\nabla v|}{v^n} + \frac{|\nabla \nabla v| |\nabla v|}{v^{n-1}} d\sigma \leq C \frac{1}{\hat{R}_j^{1-6\epsilon/(n-2)}}.$$

As the three summands on left-hand side of the above inequality dominate those on the right-hand side of (4.6), this allows to conclude.

Let α and β be given by Corollary 2.4. We first estimate, for $b = 1, 3$

$$\int_{A_{\alpha R_j, \beta R_j}} \frac{|\nabla v|^b}{v^n} d\mu \leq R_j^{2b\epsilon/(n-2)}, \tag{4.11}$$

where we used also the lower bound in (4.9) and that $|A_{k-1R_j, R_j}| \leq CR_j^n$, again by the Bishop–Gromov inequality. This settles for the first two terms in (4.10). For what it concerns the third one, by direct computation we have

$$\nabla \nabla v = \frac{n}{2v} (\nabla v \otimes \nabla v) - \frac{2}{(n-2)} v^{\frac{n}{2}} \nabla \nabla u.$$

Then,

$$\begin{aligned} & \int_{A_{\alpha R_j, \beta R_j}} \frac{|\nabla \nabla v| |\nabla v|}{v^{n-1}} d\mu \\ & \leq C \left[\int_{A_{\alpha R_j, \beta R_j}} \frac{|\nabla v|^3}{v^n} d\mu + \int_{A_{\alpha R_j, \beta R_j}} \frac{|\nabla \nabla u| |\nabla v|}{v^{\frac{n}{2}-1}} d\mu \right] \end{aligned} \tag{4.12}$$

The first term in the right-hand side is estimated by (4.11) with $b = 3$. Recall now that

$$\int_{A_{\alpha R_j, \beta R_j}} |\nabla \nabla u|^2 d\mu \leq \frac{C}{R_j^2},$$

by Corollary 2.4. In such an annulus, the second term in the right-hand side of (4.12) can be estimated as

$$\begin{aligned} \int_{A_{\alpha R_j, \beta R_j}} \frac{|\nabla \nabla u| |\nabla v|}{v^{\frac{n}{2}-1}} d\mu & \leq \frac{\left(\int_{A_{\alpha R_j, \beta R_j}} |\nabla \nabla u|^2 d\mu \int_{A_{\alpha R_j, \beta R_j}} |\nabla v|^2 d\mu \right)^{1/2}}{R_j^{\frac{n}{2}-1}} \\ & \leq CR_j^{4\epsilon/(n-2)}, \end{aligned}$$

where we have employed the Hölder inequality together with the estimates on v and $|\nabla v|$ recalled above. We can thus deduce that

$$\int_{A_{\alpha R_j, \beta R_j}} \frac{|\nabla \nabla v| |\nabla v|}{v^{n-1}} d\mu \leq C \left(R_j^{6\epsilon/(n-2)} + R_j^{4\epsilon/(n-2)} \right).$$

Coupling with (4.11) this yields (4.10), completing the proof. □

Theorem 1.1 now follows as an immediate corollary of Theorem 1.2 and Proposition 4.2.

Proof (*Proof of Theorem 1.1*) By combining (4.5) for some $0 < \epsilon < (n - 2)/6$ with (2.2), we deduce that

$$|\nabla u|^2 \leq Cr(x)^{-n-2\epsilon}$$

for any $x \in M \setminus B(o, R)$, with R big enough. Consequently, we have

$$\int_{M \setminus B(o, R)} |\nabla u|^2 d\mu \leq C \int_R^{+\infty} \int_{B(o, r)} r^{-n-2\epsilon} d\sigma dr \leq C \int_R^{+\infty} \frac{1}{r^{1+2\epsilon}} < +\infty,$$

where we have used the coarea formula and $|\partial B(o, r)| \leq |\mathbb{S}^{n-1}|r^{n-1}$ for almost any $r > 0$. We conclude by Theorem 1.2. \square

Acknowledgements The authors wish to thank A. Farina for useful remarks on the paper and the anonymous referee for his reading and comments, that helped to improve the quality of the work.

Funding Open access funding provided by Università degli Studi di Padova within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors have no conflicts of interest to declare.

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