European Journal of Mathematics (2024) 10:7
https://doi.org/10.1007/s40879-023-00716-y

RESEARCH ARTICLE

A remark on generalized abundance for surfaces

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Received: 31 July 2023 / Revised: 3 August 2023 / Accepted: 13 November 2023

Abstract
Let (X, Δ) be a projective klt pair of dimension 2 and let L be a nef Cartier divisor on X such that $K_X + \Delta + L$ is nef. As a complement to the Generalized Abundance Conjecture by Lazić and Peternell, we prove that if $K_X + \Delta$ and L are not proportional modulo numerical equivalence, then $K_X + \Delta + L$ is semiample. An example due to Lazić shows that this is no longer true in any dimension $n \geqslant 3$.

Keywords Abundance · Generalized abundance · Semiampleness · Numerical semiampleness

Mathematics Subject Classification 14E30

1 Introduction

The Generalized Abundance Conjecture by Lazić and Peternell (see [4, p. 354]) is indeed a theorem in dimension 2 (see [4, Corollary C, p. 356]):

Theorem 1.1 *Let (X, Δ) be a projective klt pair of dimension 2 such that $K_X + \Delta$ is pseudoeffective and let L be a nef Cartier divisor on X. If $K_X + \Delta + L$ is nef then there exists a semiample $\mathbb{Q}$-divisor M on X such that $K_X + \Delta + L$ is numerically equivalent to M.*

The assumption that $K_X + \Delta$ is pseudoeffective turns out to be necessary (see for instance [4, Example 6.2]). On the other hand, at least in dimension 2, it is possible to characterize the failure of numerical abundance when $K_X + \Delta$ is not pseudoeffective. The following statement is [3, Theorem 3.13]:

This research project was partially supported by GNSAGA of INdAM and by PRIN 2017 “Moduli Theory and Birational Classification”.

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Published online: 09 January 2024

Theorem 1.2 *Let (X, Δ) be a projective klt pair of dimension 2 and let L be a nef Cartier divisor on X such that $K_X + \Delta + L$ is nef. Then either $K_X + \Delta + L$ is numerically semiample or $K_X + \Delta$ is numerically equivalent to $-tL$ with $0 \leqslant t \leqslant 1$.*

We point out that if $t = 0$ then we fall in the first case. Indeed, the Semiampleness Conjecture holds on surfaces (see [4, Theorem 8.2]): if $K_X + \Delta$ is numerically equivalent to 0 then L is numerically semiample. It is therefore tempting to ask the following question in higher dimension:

Question 1.3 Let (X, Δ) be a projective klt pair of dimension $n \geqslant 3$ and let L be a nef Cartier divisor on X such that $K_X + \Delta + L$ is nef. Is it true that either $K_X + \Delta + L$ is numerically semiample or L is numerically equivalent to $-m(K_X + \Delta)$ with $m > 0$?

Even though we are not aware of any counterexamples, there seems to be no reason to expect an affirmative answer.

As shown already in dimension 1 by the example of a non-torsion numerically trivial divisor on an elliptic curve (see [1, p. 212]), numerical semiampleness cannot be replaced by semiampleness. We notice however that, at least in dimension 2, semiampleness holds under an easily stated explicit assumption. We formulate this remark as follows:

Theorem 1.4 *Let (X, Δ) be a projective klt pair of dimension 2 and let L be a nef Cartier divisor on X such that $K_X + \Delta + L$ is nef. If $K_X + \Delta$ and L are not proportional modulo numerical equivalence, then $K_X + \Delta + L$ is semiample.*

The above result complements but does not imply Generalized Abundance, in particular its statement is empty in the two crucial cases $L = K_X + \Delta$ (Abundance Conjecture) and $K_X + \Delta$ numerically trivial (Semiampleness Conjecture on Calabi–Yau pairs). Once again, it is legitimate to wonder about the higher dimensional case. We are going to present our proof in a general setting, but in arbitrary dimension we only obtain a pale shadow of the two-dimensional case (see Corollary 2.4). The following example, kindly provided to us by Vladimir Lazić, shows that the statement of Theorem 1.4 does not extend to any dimension $n \geqslant 3$:

Example 1.5 (Lazić) Let X be a smooth variety with $\mathrm{Pic}^0(X) = 0$ and Picard number at least 2. Take an ample divisor A on X not proportional to K_X and such that $K_X + A$ is ample. Let E be an elliptic curve and take a degree zero non-torsion divisor P on E. Consider $Y = X \times E$, let A_Y be the pullback of A to Y via the first projection and let P_Y be the pullback of P via the second projection. Then K_Y and $A_Y + P_Y$ are not proportional modulo numerical equivalence, but $K_Y + A_Y + P_Y$ is not semiample. Indeed, assume by contradiction that $K_Y + A_Y + P_Y$ is semiample and consider the induced Iitaka fibration $f : Y \to Z$. Then f and the first projection $Y \to X$ contract the same curves, hence X and Z are isomorphic by the rigidity lemma. From the factorization $f : Y \to X \to Z$ it follows that P_Y is the pullback (up to $\mathbb{Q}$-linear equivalence) of a divisor P_X from X, since $K_Y + A_Y$ is the pullback of a divisor from X and $K_Y + A_Y + P_Y$ is the pullback of a divisor from Z. Then P_X is numerically trivial on X, hence torsion by the assumption $\mathrm{Pic}^0(X) = 0$. But this would imply that P_Y is torsion, hence P is torsion, a contradiction.

We work over the complex field $\mathbb{C}$.

2 The proof

Our first lemma generalizes [7, Lemma 1.3].

Lemma 2.1 *Let (X, Δ) be a projective klt pair of dimension n and let H be a nef and big Cartier divisor on X. If L is a nef Cartier divisor on X such that $K_X + \Delta + L$ is nef and $K_X + \Delta + 2\,L$ has numerical dimension $\nu(K_X + \Delta + 2\,L) < k \leqslant n$, then we have*

$$H^{n-k}L^k = H^{n-k}L^{k-1}(K_X + \Delta) = \cdots = H^{n-k}(K_X + \Delta)^k = 0.$$

Proof Since both $K_X + \Delta + L$ and L are nef we have

$$0 \leqslant H^{n-k}(K_X + \Delta + 2L)^k = \sum_{m=0}^{k} \binom{k}{m} H^{n-k}(K_X + \Delta + L)^m L^{k-m}$$

with $H^{n-k}(K_X + \Delta + L)^m L^{n-m} \geqslant 0$ for every m.

If $H^{n-k}(K_X + \Delta + 2L)^k = 0$ then $H^{n-k}(K_X + \Delta + L)^m L^{k-m} = 0$ for every m and by induction it follows that $H^{n-k}L^k = H^{n-k}L^{k-1}(K_X + \Delta) = \cdots = H^{n-k}(K_X + \Delta)^k = 0.$ □

Corollary 2.2 *Let (X, Δ) be a projective klt pair of dimension n and let L be a nef Cartier divisor on X. If $K_X + \Delta + L$ is nef but not semiample, then we have*

$$L^n = L^{n-1}(K_X + \Delta) = \cdots = (K_X + \Delta)^n = 0.$$

Proof If $(K_X + \Delta + 2L)^n > 0$ then $K_X + \Delta + 2\,L = 2(K_X + \Delta + L) - (K_X + \Delta)$ is nef and big, hence $K_X + \Delta + L$ would be semiample by the logarithmic base-point-free theorem. Since $K_X + \Delta + L$ is not semiample we deduce that $(K_X + \Delta + 2\,L)^n = 0$ and $\nu(K_X + \Delta + 2\,L) < n$. Now the claim follows from Lemma 2.1 with $k = n$. □

Our next lemma generalizes to arbitrary dimension the *Easy Fact* stated for surfaces in [2, pp. 576–577] (see also [3, Lemma 3.2], where the assumption $A^2 = B^2 = 0$ is missing and the assumption A, B nef is added).

Lemma 2.3 *Let X be a normal projective variety of dimension n and let H be a nef and big Cartier divisor on X. If A and B are two $\mathbb{Q}$-Cartier divisors on X such that $H^{n-2}A^2 = H^{n-2}B^2 = H^{n-2}AB = 0$, then A and B are proportional modulo numerical equivalence.*

Proof By replacing X with a birational resolution of singularities and A and B by their pullbacks we may assume that X is smooth. We may also assume that $H^{n-1}A$ and $H^{n-1}B$ are proportional by a rational factor m, so that $H^{n-1}(A - mB) = 0$. Now we apply the Hodge index theorem for divisors (see [8, Sect. 1] and [5, Theorem 1]) to $E = A - mB$: if $H^{n-1}E = 0$ then $H^{n-2}E^2 \leqslant 0$ and equality holds if and only if $H^{n-2}E$ is homologically equivalent to zero. By assumption we have $H^{n-2}E^2 = H^{n-2}(A - mB)^2 = H^{n-2}A^2 + m^2\,H^{n-2}B^2 - 2\,mH^{n-2}AB = 0,$

hence $H^{n-2}E = H^{n-2}(A - mB)$ is homologically equivalent to zero. By the hard Lefschetz theorem (see for instance [6, Theorem 4.6]), the Lefschetz operator $H^{n-2}\colon H^2(X, \mathbb{Q}) \to H^{2n-2}(X, \mathbb{Q})$ is injective, therefore $A - mB$ is homologically (in particular, numerically) equivalent to zero. □

Corollary 2.4 *Let (X, Δ) be a projective klt pair of dimension n. If L is a nef divisor on X such that $K_X + \Delta + L$ is nef and $K_X + \Delta + 2L$ has numerical dimension $\nu(K_X + \Delta + 2L) < 2$, then $K_X + \Delta$ and L are proportional modulo numerical equivalence.*

Proof Let H be a nef and big Cartier divisor on X. By Lemma 2.1 with $k = 2$ we have $H^{n-2}L^2 = H^{n-2}L(K_X + \Delta) = H^{n-2}(K_X + \Delta)^2 = 0$. Now the claim follows from Lemma 2.3. □

Proof of Theorem 1.4 We argue by contradiction. If $K_X + \Delta + L$ is not semiample, then by Corollary 2.2 we have $L^2 = L(K_X + \Delta) = (K_X + \Delta)^2 = 0$. Now Lemma 2.3 yields the sought-for contradiction. □

Acknowledgements The author is grateful to Vladimir Lazić and Thomas Peternell for their helpful comments.

Funding Open access funding provided by Università degli Studi di Trento within the CRUI-CARE Agreement.

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