On a class of partial differential equations with hysteresis arising in magnetohydrodynamics

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Abstract

In this paper we deal with a class of parabolic partial differential equations containing a continuous hysteresis operator. We get an existence result by means of a technique based on an implicit time discretization scheme and we also analyse the dependence of the solution on the data. This model equation appears in the context of magnetohydrodynamics.

Classification: 35K55, 47J40, 76W05.

Key words: partial differential equations, hysteresis, Preisach operator, magnetohydrodynamics.

1 Introduction

In this paper a class of parabolic P.D.E.s containing a continuous hysteresis operator $\mathcal{F}$ is studied; the model equation which is taken into consideration is the following

$$\frac{\partial}{\partial t}(u + \mathcal{F}(u)) + \mathbf{v} \cdot \nabla(u + \mathcal{F}(u)) - \triangle u = f \qquad \text{in } \Omega \times (0, T) \tag{1.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $\triangle$ is the Laplace operator, $\mathbf{v} : \Omega \times (0, T) \to \mathbb{R}^N$ is known and f is a given function.

This model equation arises in the context of magnetohydrodynamics; more details can be found in Section 3 where we also present the complete derivation of the model equation (1.1).

First we introduce a weak formulation in Sobolev spaces for a Cauchy problem associated with (1.1) with Dirichlet boundary conditions; under suitable assumptions on the hysteresis operator $\mathcal{F}$, we obtain an existence result. The technique which is used is based on approximation by implicit time discretization, a priori estimates and passage to the limit by compactness. This approximation procedure is quite convenient in the analysis of equations that include a hysteresis operator, as in any time-step we have to solve a stationary problem in which the hysteresis operator is reduced to the superposition with a nonlinear function. The question of uniqueness will be treated in [12]; see also [10] Chapter 3 where a uniqueness result is achieved for some particular choices of the operator $\mathcal{F}$.

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In the last part of the paper we analyse the dependence of the solution on the data: the theorem which we prove differs from the more standard ones (see for example [25], Section IX.1) for the weaker assumptions which provide a slightly weaker thesis, enough however to pass to the limit. The idea contained in the proof uses the order preserving property of the hysteresis operators involved and the uniform convergence in time of the sequence of our approximate solutions (pointwise convergence would not be enough for our purposes).

2 Hysteresis operators

2.1 Hysteresis

Hysteresis is a phenomenon that appears in several and quite different situations; for example we can encounter it in physics, in engineering, in biology and in many other settings. According to [25], we can refer to hysteresis as a *rate independent memory effect*. More in details, let us consider a system which is described by the couple input-output (u, w). The memory effect means that at any instant t the value of the output is not simply determined by the value $u(t)$ of the input at the same instant but it depends also on the previous evolution of the input u. The rate independence instead means that the path of the couple $(u(t), w(t))$ is invariant with respect to any increasing time homeomorphism and so it is independent of its velocity.

Even if hysteresis has been known and studied since the end of the eighteenth century, it was only more or less thirty-five years ago that, dealing with plasticity, a small group of Russian mathematicians introduced the concept of *hysteresis operator* and started a systematic investigation of its properties. The pioneers in this new field were Krasnosel'skiĭ and Pokrovskiĭ, with their important monograph [15], which constitutes, up to now, the main source of reference on hysteresis. From that moment onward many scientists coming also from different areas have contributed to the mathematical study of hysteresis. We quote the recent monographes (together with the references therein) devoted to this topic [5], [17], [25] from the mathematical point of view and [1], [9], [18] for a more physical approach. We also point out the recent approach developed for example in [19], [20], [21]; this formulation does not involve explicit hysteresis operators, but hysteresis arises implicitly as a result of coupling the energy balance with a stability condition.

2.2 The Preisach operator: definition and main properties

In [22], Preisach proposed a model of ferromagnetism based on a clear geometric interpretation. A mathematical analysis has been first carried on in [15] and then developed in the monographs [1], [5], [9], [17], [18] [25] to which we refer for a detailed list of references. In particular we point out the contributions [2], [3], [4], [6], [16], [24], [26], [27].

Let $BV_R(0, T)$ be the space of right-continuous functions $[0, T] \to \mathbb{R}$ of bounded total variation.

The construction of the Preisach operator is based on the concept of *delayed relay*, which is the simplest example of discontinuous hysteresis nonlinearity. It is characterized by two parameters, say ρ_1, ρ_2 and two output values which we assume to be equal to -1 and $+1$. For any couple $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$ with $\rho_1 < \rho_2$, the delayed relay operator

$$h_\rho : \mathcal{C}^0([0,T]) \times \{-1, 1\} \to BV_R(0,T) \tag{2.1}$$

can be defined in the following way: for any $u \in \mathcal{C}^0([0,T])$ and any $\xi \in \{-1,+1\}$, the function $w = h_\rho(u,\xi)$ is given by

$$w(0) := \begin{cases} -1 & \text{if } u(0) \le \rho_1, \\ \xi & \text{if } \rho_1 < u(0) < \rho_2, \\ 1 & \text{if } u(0) \ge \rho_2 \end{cases}$$

and, for any $t \in (0,T]$, setting $W_t := \{\tau \in (0,t] : u(\tau) = \rho_1 \text{ or } \rho_2\}$, by

$$w(t) := \begin{cases} w(0) & \text{if } W_t = \emptyset \\ -1 & \text{if } W_t \ne \emptyset \text{ and } u(\max W_t) = \rho_1, \\ 1 & \text{if } W_t \ne \emptyset \text{ and } u(\max W_t) = \rho_2. \end{cases}$$

Thus w is uniquely defined in $[0,T]$ and has the regularity outlined in (2.1). It turns out that the operator h_ρ is *causal* and *rate independent*, so it is a *hysteresis operator*; moreover it is also *order preserving* and *piecewise monotone* in the following sense

$$\begin{cases} \forall\,(u_1,\xi_1^0),\ (u_2,\xi_2^0) \in\ \mathrm{Dom}(h_\rho),\ \ \forall\, t \in (0,T], \text{if } u_1 \le u_2 \text{ in } [0,t] \text{ and } \xi_1^0 \le \xi_2^0, \\ \text{then } [h_\rho(u_1,\xi_1^0)](t) \le [h_\rho(u_2,\xi_2^0)](t) \end{cases} \tag{2.2}$$

$$\begin{cases} \forall\,(u,\xi^0) \in \mathrm{Dom}(h_\rho),\ \forall\,[t_1,t_2] \subset [0,T], \\ \text{if } u \text{ is either nondecreasing or nonincreasing in } [t_1,t_2], \text{then so is } h_\rho(u,\xi^0). \end{cases} \tag{2.3}$$

We now consider the whole family of delayed relays with all admissible values of the parameter ρ. We introduce the Preisach plane as follows

$$\mathcal{P} := \{\rho = (\rho_1,\rho_2) \in \mathbb{R}^2 : \rho_1 < \rho_2\}.$$

In the following we will often use a different system of coordinates, in order to describe $\mathcal{P}$. For example we can consider the *half-width* $\sigma_1 = \dfrac{\rho_2 - \rho_1}{2}$ and the *mean value* $\sigma_2 = \dfrac{\rho_1 + \rho_2}{2}$; in this case the Preisach plane can be rewritten as

$$\mathcal{P} = \{(\sigma_1,\sigma_2) \in \mathbb{R}^2 : \sigma_1 > 0\}.$$

Now the *Preisach operator* can be defined as follows

$$\begin{cases} \mathcal{H}_\mu : \mathcal{C}^0([0,T]) \times \mathcal{R} \to L^\infty(0,T) \cap \mathcal{C}^0_r([0,T)), \\ [\mathcal{H}_\mu(u,\xi)](t) := \displaystyle\int_{\mathcal{P}} [h_\rho(u,\xi_\rho)](t)\, d\mu(\rho) \quad \forall t \in [0,T], \end{cases} \tag{2.4}$$

where $\mathcal{R}$ is the family of Borel measurable functions $\mathcal{P} \to \{-1, 1\}$, ξ_ρ is the image of $\rho \in \mathcal{P}$ by the function $\xi \in \mathcal{R}$, μ is any finite (signed) Borel measure over $\mathcal{P}$ and $\mathcal{P}$ is the Preisach plane; moreover $\mathcal{C}^0_r([0,T))$ is the linear space of functions which are continuous on the right in $[0,T)$. The Preisach model can be therefore interpreted as the superposition of a family of delayed relays distributed with a given density.

It can be proved (see [25], Section IV.1, Theorem 1.2. and Corollary 1.3) that for any finite Borel measure μ over $\mathcal{P}$, the operator $\mathcal{H}_\mu$ is *causal* and *rate independent*, so it is a *hysteresis operator*; moreover if $\mu \geq 0$, then $\mathcal{H}_\mu$ is *order preserving* and *piecewise monotone* in the sense of (2.2) and (2.3) respectively.

Moreover, let us denote with $\bar{\mathcal{S}}$ the family of relay configurations which can be attained by applying a continuous input to a system initially in the so-called virgin state

$$\xi^v_\rho = \begin{cases} 1 & \text{if } \rho_1 + \rho_2 < 0, \\ -1 & \text{if } \rho_1 + \rho_2 > 0. \end{cases}$$

Then $\mathcal{H}_\mu(\cdot, \xi) : \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$ is strongly continuous, for any $\xi \in \bar{\mathcal{S}}$, if and only if

$$|\mu|(\mathbb{R} \times \{r\}) = |\mu|(\{r\} \times \mathbb{R}) = 0 \qquad \forall\, r \in \mathbb{R} \tag{2.5}$$

and $\mathcal{H}_\mu : \mathcal{C}^0([0,T]) \times \bar{\mathcal{S}} \to \mathcal{C}^0([0,T])$ is Lipschitz continuous with Lipschitz constant L if and only if

$$\sup_{B \in \mathcal{B}} |\mu|(N(B, \varepsilon)) \leq L\, \varepsilon \qquad \forall\, \varepsilon > 0; \tag{2.6}$$

here

$$N(B, \varepsilon) := \{(s_1, s_2 + \alpha) \in \mathbb{R}^+ \times \mathbb{R} : (s_1, s_2) \in B, |\alpha| \leq \varepsilon\},$$

for any $B \in \mathcal{B}$ and any $\varepsilon > 0$, where $\mathcal{B} := \{B_\xi : \xi \in \bar{S}\}$ are the graphs of the corresponding elements in $\bar{S}$.

As it is often the case when dealing with partial differential equations, we may indeed consider both the input u and the function $\xi \in \mathcal{R}$ in (2.4) that additionally depend on a parameter x, (the space variable, say). For instance, take any $\xi \in L^1(\Omega; L^1(\mathcal{P}))$ and choose $\mu \geq 0$ such that (2.5) and (2.6) hold. Now set

$$\mathcal{F}(u)(x,t) := [\mathcal{H}_\mu(u(x,\cdot), \xi(x,\cdot))](t) \qquad \forall\, t \in [0,T], \text{ a.e. in } \Omega, \tag{2.7}$$

where $\mathcal{H}_\mu$ has been introduced in (2.4). It turns out that $\mathcal{F}$ is Lipschitz continuous in the sense that for all $u_1, u_2 \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$,

$$||\, [\mathcal{F}(u_1)](x,\cdot) - [\mathcal{F}(u_2)](x,\cdot)||_{\mathcal{C}^0([0,T])} \leq L_{\mathcal{F}}\, ||u_1(x,\cdot) - u_2(x,\cdot)||_{\mathcal{C}^0([0,T])} \text{ a.e. in } \Omega, \tag{2.8}$$

with $L_{\mathcal{F}} = L$, L being introduced in (2.6). Here we denoted by $\mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ the Fréchet space of (strongly) measurable functions $\Omega \to \mathcal{C}^0([0,T])$ endowed with the quasi-metric

$$||v||_{\mathcal{M}(\Omega;\mathcal{C}^0([0,T]))} := \int_\Omega \frac{||v(x)||_{\mathcal{C}^0([0,T])}}{1 + ||v(x)||_{\mathcal{C}^0([0,T])}}\, dx.$$

Moreover $\mathcal{F}$ turns to be order preserving and piecewise monotone in the sense of

$$\begin{cases} \forall\, u_1, u_2 \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])),\ \forall\, t \in [0,T], \text{ if } u_1 \leq u_2 \text{ in } [0,t], \text{ a.e. in } \Omega, \text{ then} \\ [\mathcal{F}(u_1)](x,t) \leq [\mathcal{F}(u_2)](x,t) \text{ a.e. in } \Omega \end{cases} \tag{2.9}$$

and

$$\begin{cases} \forall\, v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])),\ \forall\, [t_1,t_2] \subset [0,T],\ \text{if } v(x,\cdot) \text{ is affine in } [t_1,t_2] \text{ a.e. in } \Omega, \\ \text{then } \{[\mathcal{F}(v)](x,t_2) - [\mathcal{F}(v)](x,t_1)\} \cdot [v(x,t_2) - v(x,t_1)] \geq\ 0 \text{ a.e. in } \Omega \end{cases} \tag{2.10}$$

respectively.

3 Physical interpretation of the model problem

3.1 Motivation of the problem: magnetohydrodynamics

Magnetohydrodynamics (MHD) deals with the behaviour of the combined system of electromagnetic fields and a conducting liquid or gas. Conduction occurs when there are free or quasi-free electrons which can move under the action of applied fields. The fluid is therefore electrically conducting. It is not magnetic: it affects a magnetic field only by virtue of electric currents flowing in it. The Lorentz force $\mathbf{j} \times \mathbf{B}$ determines the interaction of the electric current $\mathbf{j}$ with the vector field of magnetic induction $\mathbf{B}$. More details can be found in an introductory textbook of MHD, for example [8], [23].

Several branches of engineering affect MHD and many applications can be quoted. For example we can quote the MHD flow of a conducting fluid through a circular pipe. Flows in circular pipes have been the subject of a number of studies since the early 60th of the last century. Most of the results assume that the walls confining the electrically conducting fluid are made of non-ferromagnetic material. In this respect the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ are linked by a linear relation.

Different is the case of flow structure in pipes made of ferromagnetic material. This is the case considered for example in the recent paper [7], where, however, the magnetic hysteresis effects are neglected and the typical hysteresis magnetization curve is approximated by two linear parts. Nevertheless for this kinds of applications hysteretic effects usually *dominate* over the magnetic field affected by the fluid and should in principle be considered.

Our paper represents an attempt to take into account the hysteretic effects in MHD, having in mind the previous example (i.e. flow of a conducting fluid through a circular pipe made of ferromagnetic material) as a possible application of our results. Also in our case some simplifications occur: for example we take some restrictions on the geometry of the fields which leads to the scalar character of (1.1); the complete vectorial setting is for the moment an open problem. Moreover the velocity $\mathbf{v}$ of the fluid is for the moment assumed to be known; in a subsequent paper [11] we will couple (1.1) with the momentum equation for $\mathbf{v}$ and find a solution in this more general setting where also fluid-mechanical aspects are considered. Finally we assume the so called *low frequency approximation*, usually considered in MHD; as a consequence of this assumption we neglect the Maxwell term (i.e. the displacement currents) in the Ampére law and the contribution to $\mathbf{j}$ of the convection current and the polarization current. Therefore at the end the Ampére law and the Ohm law have the form (3.2) and (3.4) respectively.

3.2 Derivation of (1.1)

Let us consider an electrically neutral conducting fluid moving in an electromagnetic field with given velocity $\mathbf{v} = (v_1, v_2, v_3)$ such that

$$\operatorname{div} \mathbf{v} = 0. \tag{3.1}$$

We recall the Ampère law (due to the *low frequency approximation* the Maxwell term is neglected)

$$c \nabla \times \mathbf{H} = 4\pi \mathbf{j}, \tag{3.2}$$

the Faraday law

$$c \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.3}$$

and the Ohm law (where the Hall effect is neglected)

$$\mathbf{j} = \sigma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right); \tag{3.4}$$

here $\mathbf{H}$ is the magnetic field, $\mathbf{j}$ is the electric current, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic induction, σ is the electric conductivity and c is the speed of light in vacuum.

We further simplify our setting by considering planar waves. More precisely, let Q be a domain of $\mathbb{R}^2$, we set $Q_T := Q \times (0, T)$ and assume that (using orthogonal Cartesian coordinates x, y, z) both $\mathbf{B}$ and $\mathbf{H}$ are parallel to the z-axis and only depend on the coordinates x, y, i.e.

$$\mathbf{B} = (0, 0, B(x, y)) \qquad \text{and} \qquad \mathbf{H} = (0, 0, H(x, y)).$$

We assume that $\mathbf{H}$ and $\mathbf{B}$ are linked by a constitutive relation with hysteresis, i.e.

$$B = (I + \mathcal{F})(H), \tag{3.5}$$

where $\mathcal{F}$ is the scalar Preisach operator introduced in (2.7) and I is the identity operator. As we are considering planar waves, the electric field has the following form

$$\mathbf{E} = (E_1(x, y), E_2(x, y), 0).$$

This implies that

$$\nabla \times \mathbf{E} = \left(0, 0, \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} \right), \qquad \nabla \times \mathbf{H} = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}, 0 \right).$$

On the other hand

$$\mathbf{v} \times \mathbf{B} = (v_2 B, -v_1 B, 0)$$

and therefore the Ohm law gives

$$\mathbf{j} = \left(\sigma \left(E_1 + \frac{1}{c} v_2 B \right), \sigma \left(E_2 - \frac{1}{c} v_1 B \right), 0 \right). \tag{3.6}$$

Combining (3.2) with (3.6) and neglecting from now on for simplicity the constants c, 4π and σ, we obtain

$$\frac{\partial H}{\partial y} = E_1 + v_2\, B \tag{3.7}$$

and

$$-\frac{\partial H}{\partial x} = E_2 - v_1\, B. \tag{3.8}$$

The Faraday law instead has the following form after our simplifications

$$\frac{\partial B}{\partial t} + \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} = 0. \tag{3.9}$$

Differentiating (3.7) in the y variable and (3.8) in the x variable yields

$$\frac{\partial^2 H}{\partial y^2} = \frac{\partial E_1}{\partial y} + \frac{\partial}{\partial y}(v_2\, B), \qquad -\frac{\partial^2 H}{\partial x^2} = \frac{\partial E_2}{\partial x} - \frac{\partial}{\partial x}(v_1\, B). \tag{3.10}$$

Now using (3.9) and (3.10) we deduce

$$\frac{\partial B}{\partial t} + \left[-\frac{\partial^2 H}{\partial x^2} + \frac{\partial}{\partial x}(v_1\, B) - \frac{\partial^2 H}{\partial y^2} + \frac{\partial}{\partial y}(v_2\, B)\right] = 0$$

which is equivalent to

$$\frac{\partial B}{\partial t} + \mathrm{div}\,(\mathbf{v}\, B) - \triangle H = 0, \tag{3.11}$$

where we take $\mathbf{v} = \mathbf{v}(x, y, t)$. Now equation (3.11) coupled with the constitutive relation (3.5) and with (3.1) gives nothing but (1.1).

4 An existence result

Let us consider an open bounded set of Lipschitz class $\Omega \subset \mathbb{R}^N$, $N \geq 1$ and set $\Omega_T := \Omega \times (0, T)$. Consider the operator $\mathcal{F}$ introduced in (2.7).
The causality property entails that $[\mathcal{F}(v)](\cdot, 0)(\in \mathcal{M}(\Omega))$ depends just on $\mathcal{F}$ and $v(\cdot, 0)$; so we can set

$$\mathcal{H}_{\mathcal{F}}(v(\cdot, 0)) := [\mathcal{F}(v)](\cdot, 0)(\in \mathcal{M}(\Omega)) \qquad \forall\, v \in \mathcal{M}(\Omega; \mathcal{C}^0([0, T])). \tag{4.1}$$

We set $V := H^1_0(\Omega)$, $H := L^2(\Omega)$ and $V' := H^{-1}(\Omega)$ and we consider V endowed with the norm $||u||_V := ||\nabla u||_{L^2(\Omega)}$. We then identify the space $L^2(\Omega)$ to its topological dual $(L^2(\Omega))'$; as the injection of V into $L^2(\Omega)$ is continuous and dense, $(L^2(\Omega))'$ can be identified to a subspace of V'. This yields the Hilbert triplet

$$V \subset H \equiv H' \subset V'$$

with dense and continuous injections.
Now we denote by ${}_{V'}\langle \cdot, \cdot \rangle_V$ the duality pairing between V' and V and we then define the linear and continuous operator $A : V \to V'$ as follows

$${}_{V'}\langle Au, v\rangle_V := \int_\Omega \nabla u \cdot \nabla v\, dx \qquad \forall u, v \in V. \tag{4.2}$$

We assume that $u^0, w^0 = \mathcal{H}_{\mathcal{F}}(u^0) \in L^2(\Omega)$ are given initial conditions; moreover, let us consider a known function

$$\mathbf{v} : \Omega_T \to \mathbb{R}^N \qquad\qquad \mathbf{v}(x,t) := (v_1(x,t), v_2(x,t), \ldots, v_N(x,t))$$

satisfying the following assumptions

$$\mathbf{v},\ \frac{\partial \mathbf{v}}{\partial t} \in L^\infty(\Omega_T)^N, \qquad\qquad \operatorname{div} \mathbf{v} = 0 \ \text{ in the sense of distributions.} \tag{4.3}$$

We want to solve the following problem.

Problem 4.1. *Let us consider a given function* $\mathbf{v}$, *satisfying* (4.3), *and given* f, u^0, w^0; *we search for a function* $u \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \cap L^2(0,T;V)$ *such that* $\mathcal{F}(u) \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \cap L^2(\Omega_T)$ *and for any* $\psi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ *with* $\psi(\cdot,T) = 0$ *a.e. in* Ω

$$\begin{aligned}
&\int_0^T \int_\Omega -(u + \mathcal{F}(u)) \frac{\partial \psi}{\partial t}\, dx\, dt - \int_0^T \int_\Omega [\mathbf{v} \cdot \nabla \psi]\, (u + \mathcal{F}(u))\, dx\, dt \\
&+ \int_0^T \int_\Omega \nabla u \cdot \nabla \psi \, dx\, dt = \int_0^T {}_{V'}\langle f, \psi \rangle_V \, dt + \int_\Omega [u^0(x) + w^0(x)]\, \psi(x,0)\, dx.
\end{aligned} \tag{4.4}$$

Interpretation. If the functions u, $\mathcal{F}(u)$, $\mathbf{v}$ are smooth enough, we may use the standard Green formulae, the definition of derivatives in the sense of distributions and (4.3) to interpret the variational equation (4.4) as follows

$$\begin{cases} \dfrac{\partial w}{\partial t} + \mathbf{v} \cdot \nabla w - \triangle u = f \\ w = (I + \mathcal{F})(u), \end{cases} \qquad \text{in } V', \text{ a.e. in } (0,T),$$

with

$$\gamma_0 u = 0 \qquad \text{on } \partial\Omega \times (0,T)$$

and

$$[u + \mathcal{F}(u)]_{|t=0} = u^0 + w^0 \ \text{ in } V', \text{ in the sense of traces.}$$

4.1 Existence

Theorem 4.2. *(Existence)*
Consider the operator $\mathcal{F} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ *introduced in* (2.7). *Moreover consider the following assumptions on the data*

$$f \in L^2(\Omega_T), \ \ u^0 \in V, \ \ w^0 \in L^2(\Omega). \tag{4.5}$$

Then Problem 4.1 *admits at least one solution*

$$u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$$

such that

$$\mathcal{F}(u) \in H^1(0,T;L^2(\Omega)).$$

Proof.

- *First step: approximation.*

Let us fix $m \in \mathbb{N}$ and set $k := T/m$. Now for $n = 1, \dots, m$ let us consider $f_m^n(x) := f(x, nk)$, $u_m^0 := u^0$ and $w_m^0 := w^0$. We approximate our problem by an implicit time discretization scheme. We want to solve the following problem.

Problem 4.3. *To find $u_m^n \in V$ for $n = 1, \dots m$, such that, if $u_m(x, \cdot)$ is the linear time interpolate of $u_m(x, nk) := u_m^n(x)$, for $n = 1, \dots, m$, a.e. in Ω and $w_m^n := [\mathcal{F}(u_m)](x, nk)$ for $n = 1, \dots, m$, a.e. in Ω, then, for any $\psi \in V$*

$$\begin{aligned} &\frac{1}{k}\int_\Omega (u_m^n - u_m^{n-1})\,\psi\,dx + \frac{1}{k}\int_\Omega (w_m^n - w_m^{n-1})\,\psi\,dx - \int_\Omega [\mathbf{v}_m^n \cdot \nabla\psi]\,(u_m^n + w_m^n)\,dx \\ &+ \int_\Omega \nabla u_m^n \cdot \nabla\psi\,dx = \int_\Omega f_m^n\,\psi\,dx, \end{aligned} \tag{4.6}$$

where we used the following notation $\mathbf{v}_m^n(x) := \mathbf{v}(x, nk)$.

For any $n \in \{1, \dots, m\}$, we suppose to know $u_m^1, \dots, u_m^{n-1} \in V$; the problem is now to determine u_m^n.
For almost any $x \in \Omega$, $u_m(x, \cdot)$ is the linear time interpolate of $u_m^n(x)$, so it is affine in $[(n-1)k, nk]$; this implies that $[\mathcal{F}(u_m)](x, nk)$ depends only on $u_m(x, \cdot)_{|[0,(n-1)k]}$, which is known and on $u_m^n(x)$, which must be determined. Hence, there exists a function $F_m^n : \mathbb{R} \times \Omega \to \mathbb{R}$ such that

$$w_m^n(x) = [\mathcal{F}(u_m)](x, nk) := F_m^n(u_m^n(x), x) \qquad \text{a.e. in } \Omega.$$

This allows us to introduce an operator $\widehat{F}_m^n$ acting between spaces of measurable functions $\mathcal{M}(\Omega)$ in the following way $\widehat{F}_m^n(v) := F_m^n(v(\cdot), \cdot)$. We establish some properties of the operator $\widehat{F}_m^n$.
First of all it is quite easy to show that

$$\widehat{F}_m^n : L^2(\Omega) \to L^2(\Omega) \text{ is a strongly continuous operator.} \tag{4.7}$$

Moreover (2.8) entails that $\mathcal{F}$ is bounded in the following sense

$$\begin{cases} \exists\,\tau \in L^2(\Omega) : \forall v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \\ ||[\mathcal{F}(v)](x, \cdot)||_{\mathcal{C}^0([0,T])} \le L_\mathcal{F}\,||v(x, \cdot)||_{\mathcal{C}^0([0,T])} + \tau(x) \text{ a.e. in } \Omega. \end{cases} \tag{4.8}$$

Using (4.8) we get that there exist two constants $C_1^\mathcal{F}, C_2^\mathcal{F}$ (actually $C_1^\mathcal{F} = L_\mathcal{F}$) such that

$$||\widehat{F}_m^n(v)||_{L^2(\Omega)} \le C_1^\mathcal{F}\,||v||_{L^2(\Omega)} + C_2^\mathcal{F} \qquad \forall\, v \in L^2(\Omega); \tag{4.9}$$

now (4.9) and (2.10) yield a coercivity property for the operator $\widehat{F}_m^n$, that is there exist two constants $C_3^\mathcal{F}, C_4^\mathcal{F} \in \mathbb{R}^+$, depending on m, n such that

$$\int_\Omega \widehat{F}_m^n(v)\,v\,dx \ge -C_3^\mathcal{F}||v||_{L^2(\Omega)} - C_4^\mathcal{F} \qquad \forall v \in L^2(\Omega). \tag{4.10}$$

We introduce the operator $C : V \to V'$ acting in the following way

$$ {}_{V'}\langle C(\Phi), \psi\rangle_V := -\int_\Omega \mathbf{v}_m^{\,n}\,(\Phi + \widehat{F}_m^n(\Phi)) \cdot \nabla\psi\,dx \qquad \forall\,\psi, \Phi \in V,\, \forall\, n \in \{1, \ldots, m\}. $$

Thus (4.6) can be rewritten in the following way

$$ \frac{u_m^n - u_m^{n-1}}{k} + \frac{w_m^n - w_m^{n-1}}{k} + C(u_m^n) + A(u_m^n) = f_m^n \qquad \text{in } V' \tag{4.11} $$

which in turn yields

$$ u_m^n + \widehat{F}_m^n(u_m^n) + k\,C(u_m^n) + kAu_m^n = g_m^n \qquad \text{in } V' $$

where $g_m^n := kf_m^n + w_m^{n-1} + u_m^{n-1}$, so it is a known function. For the sake of simplicity, we omit the fixed indexes m and n; thus we get

$$ u + \widehat{F}(u) + k\,C(u) + kAu = g \qquad \text{in } V'. \tag{4.12} $$

We claim that (4.12) admits at least one solution $u \in V$. Let $\{V_j\}_{j\in\mathbb{N}}$ be a sequence of finite dimensional subspaces invading V; for any $j \in \mathbb{N}$ let us consider the problem of finding $u_j \in V_j$ such that

$$ Z(u_j) := u_j + \widehat{F}(u_j) + k\,Au_j +\; k\,C(u_j) = g \qquad \text{in } V'. \tag{4.13} $$

Using (4.7) and (4.10), it is not difficult to show that the operator $Z : V \to V'$ defined as

$$ Z(w) := w + \widehat{F}(w) + k\,Aw + k\,C(w) $$

is strongly continuous and coercive. Hence an easy application of the Brouwer fixed point theorem yields the existence of at least a solution u_j of (4.13). If we multiply (4.13) by u_j and use the coercivity of the operator Z we get that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is uniformly bounded in V. Thus there exists u such that, possibly extracting a subsequence, $u_j \rightharpoonup u$ in V. By the compactness of the inclusion $V \subset L^2(\Omega)$ and by the continuity of the operator Z we may pass to the limit taking $j \to \infty$ in (4.13), getting (4.12). This allows us to conclude that Problem 4.3 has at least a solution.

- *Second step: a priori estimates.*

First of all (2.8) and the rate independence of the operator $\mathcal{F}$ entail

$$ \left|\frac{w_m^n(x) - w_m^{n-1}(x)}{k}\right| \le L_{\mathcal{F}} \left|\frac{u_m^n(x) - u_m^{n-1}(x)}{k}\right| \qquad \text{a.e. in } \Omega,\ \forall\, n \in \{1, \ldots, m\}. \tag{4.14} $$

Now we multiply (4.11) by $(u_m^n - u_m^{n-1})$ in the duality ${}_{V'}\langle\cdot,\cdot\rangle_V$ (i.e. we consider (4.6) with the choice $\psi := u_m^n - u_m^{n-1}$) and sum for $n = 1, \ldots, j$, for $j \in \{1, \ldots, m\}$. It is clear that the difficulties come from the term

$$ \sum_{n=1}^{j} \int_\Omega \mathbf{v}_m^{\,n}(u_m^n + w_m^n) \cdot \nabla(u_m^n - u_m^{n-1})\,dx, $$

as the term $\nabla(u_m^n - u_m^{n-1})$ cannot be controlled; we thus need to integrate in the time variable. More precisely (we recall that we set for simplicity $L^2(\Omega) =: H$)

$$\sum_{n=1}^{j} {}_{V'}\langle \frac{u_m^n - u_m^{n-1}}{k}, u_m^n - u_m^{n-1}\rangle_V + \sum_{n=1}^{j} {}_{V'}\langle \frac{w_m^n - w_m^{n-1}}{k}, u_m^n - u_m^{n-1}\rangle_V$$

$$+ \sum_{n=1}^{j} {}_{V'}\langle C(u_m^n), u_m^n - u_m^{n-1}\rangle_V + \sum_{n=1}^{j} {}_{V'}\langle Au_m^n, u_m^n - u_m^{n-1}\rangle_V$$

$$\geq k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 + k \sum_{n=1}^{j} {}_{V'}\langle \frac{w_m^n - w_m^{n-1}}{k}, \frac{u_m^n - u_m^{n-1}}{k}\rangle_V$$

$$- \sum_{n=1}^{j} \int_\Omega [\mathbf{v}_m^{\,n}(u_m^n + w_m^n) \cdot \nabla(u_m^n - u_m^{n-1})]\, dx + \frac{1}{2} \sum_{n=1}^{j} \int_\Omega (|\nabla u_m^n|^2 - |\nabla u_m^{n-1}|^2)\, dx$$

$$\geq k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 - \int_\Omega (u_m^j + w_m^j)\, [\mathbf{v}_m^{\,j} \cdot \nabla u_m^j]\, dx + \int_\Omega (u_m^0 + w_m^0)\, [\mathbf{v}_m^{\,0} \cdot \nabla u_m^0]\, dx$$

$$+ k \sum_{n=1}^{j} \int_\Omega \left(\frac{u_m^n - u_m^{n-1}}{k} + \frac{w_m^n - w_m^{n-1}}{k} \right) [\mathbf{v}_m^{\,n} \cdot \nabla u_m^n]\, dx$$

$$+ \sum_{n=1}^{j} \int_\Omega [(\mathbf{v}_m^{\,n} - \mathbf{v}_m^{\,n-1})(u_m^n + w_m^n)] \cdot \nabla u_m^n\, dx + \frac{1}{2} \int_\Omega (|\nabla u_m^j|^2 - |\nabla u_m^0|^2)\, dx,$$

where we used (2.10). On the other hand

$$\sum_{n=1}^{j} {}_{V'}\langle \frac{u_m^n - u_m^{n-1}}{k}, u_m^n - u_m^{n-1}\rangle_V + \sum_{n=1}^{j} {}_{V'}\langle \frac{w_m^n - w_m^{n-1}}{k}, u_m^n - u_m^{n-1}\rangle_V$$

$$+ \sum_{n=1}^{j} {}_{V'}\langle C(u_m^n), u_m^n - u_m^{n-1}\rangle_V + \sum_{n=1}^{j} {}_{V'}\langle Au_m^n, u_m^n - u_m^{n-1}\rangle_V$$

$$= k \sum_{i=1}^{j} \int_\Omega f_m^n \left(\frac{u_m^n - u_m^{n-1}}{k} \right) dx \leq k \sum_{i=1}^{j} ||f_m^n||_H^2 + \frac{k}{4} \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 .$$

From the previous two chains of inequalities, we deduce

$$\frac{1}{2} \int_\Omega |\nabla u_m^j|^2\, dx + \frac{3}{4} k \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 \leq \frac{1}{2} ||u_m^0||_V^2$$

$$+ ||\mathbf{v}||_{L^\infty(\Omega_T)^N} \int_\Omega (|u_m^j| + |w_m^j|) |\nabla u_m^j|\, dx + ||\mathbf{v}||_{L^\infty(\Omega_T)^N} \int_\Omega (|u_m^0| + |w_m^0|)\, |\nabla u_m^0|\, dx$$

$$+ ||\mathbf{v}||_{L^\infty(\Omega_T)^N}\ k \sum_{n=1}^{j} \int_\Omega |\nabla u_m^n| \left(\left| \frac{u_m^n - u_m^{n-1}}{k} \right| + \left| \frac{w_m^n - w_m^{n-1}}{k} \right| \right) dx$$

$$+ \left|\left|\frac{\partial \mathbf{v}}{\partial t}\right|\right|_{L^\infty(\Omega_T)^N} \; k \sum_{n=1}^{j} \int_\Omega (|u_m^n| + |w_m^n|)\, |\nabla u_m^n|\, dx + \; k \sum_{n=1}^{j} ||f_m^n||_H^2$$

$$\leq \frac{1}{2}\, ||u_m^0||_V^2 + \; \frac{1}{4} \int_\Omega |\nabla u_m^j|^2\, dx + 2\, ||\mathbf{v}||_{L^\infty(\Omega_T)^N}^2 \int_\Omega (|u_m^j|^2 + |w_m^j|^2)\, dx$$

$$+ \; ||\mathbf{v}||_{L^\infty(\Omega_T)^N}\, ||\nabla u_m^0||_H^2 + \; \frac{||\mathbf{v}||_{L^\infty(\Omega_T)^N}}{2} \, (||u_m^0||_H^2 + ||w_m^0||_H^2) + \; \frac{k}{4} \sum_{n=1}^{j} \left|\left|\frac{u_m^n - u_m^{n-1}}{k}\right|\right|_H^2$$

$$+ \; \frac{k}{4\, L_{\mathcal{F}}^2} \sum_{n=1}^{j} \left|\left|\frac{w_m^n - w_m^{n-1}}{k}\right|\right|_H^2 + \; \frac{1}{2} \; \left|\left|\frac{\partial \mathbf{v}}{\partial t}\right|\right|_{L^\infty(\Omega_T)^N} \; k \sum_{n=1}^{j} (||u_m^n||_H^2 + ||w_m^n||_H^2)$$

$$+ \; \left[\left|\left|\frac{\partial \mathbf{v}}{\partial t}\right|\right|_{L^\infty(\Omega_T)^N} + ||\mathbf{v}||_{L^\infty(\Omega_T)^N}^2 (L_{\mathcal{F}}^2 + 1)\right] \; k \sum_{n=1}^{j} ||\nabla u_m^n||_H^2 + \; k \sum_{n=1}^{j} ||f_m^n||_H^2 .$$

As this point the term $\displaystyle\sum_{n=1}^{j} \left|\left|\frac{w_m^n - w_m^{n-1}}{k}\right|\right|_H^2$ can be controlled by (4.14), while

$$2\, ||\mathbf{v}||_{L^\infty(\Omega_T)^N}^2 \; \int_\Omega (|u_m^j|^2 + |w_m^j|^2)\, dx \; + \; \frac{1}{2} \; \left|\left|\frac{\partial \mathbf{v}}{\partial t}\right|\right|_{L^\infty(\Omega_T)^N} \; k \sum_{n=1}^{j} (||u_m^n||_H^2 + ||w_m^n||_H^2)$$

$$\overset{(4.8)}{\leq} 2 \, \max \left(2\, ||\mathbf{v}||_{L^\infty(\Omega_T)^N}^2 , \, \frac{1}{2} \; \left|\left|\frac{\partial \mathbf{v}}{\partial t}\right|\right|_{L^\infty(\Omega_T)^N} \right) \; \left[L_{\mathcal{F}}^2 \int_\Omega \left[\max_{n=1,\dots,j} |u_m^n(x)|\right]^2 \; dx + ||\tau||_H^2 \right] .$$

Therefore we have, for any $j = 1, \dots, m$

$$\frac{1}{4} \int_\Omega |\nabla u_m^j|^2\, dx \; + \; \frac{k}{4} \sum_{n=1}^{j} \left|\left|\frac{u_m^n - u_m^{n-1}}{k}\right|\right|_H^2$$

$$\leq c_1 \int_\Omega \left[\max_{n=1,\dots,j} |u_m^n(x)|\right]^2 \; dx \; + \; c_2\, k \sum_{n=1}^{j} ||\nabla u_m^n||_H^2 \; + \; c_3,$$

where c_1 and c_2 depend on $\mathbf{v}$ and $L_{\mathcal{F}}$, while c_3 depends also on the data. Now

$$|u_m^j(x)|^2 = |u_m^0(x)|^2 \; + \sum_{n=1}^{j} (u_m^n(x) - u_m^{n-1}(x))\, (u_m^n(x) + u_m^{n-1}(x))$$

$$\leq |u_m^0(x)|^2 \; + \; \left(\sum_{n=1}^{j} |u_m^n(x) - u_m^{n-1}(x)|^2 \right)^{1/2} \; \left(\sum_{n=1}^{j} (|u_m^n(x)| + |u_m^{n-1}(x)|)^2 \right)^{1/2}$$

$$\leq |u_m^0(x)|^2 \; + \; \frac{k}{8 c_1} \sum_{n=1}^{j} \left| \frac{u_m^n(x) - u_m^{n-1}(x)}{k} \right|^2 \; + \; 2\, k\, c_1 \sum_{n=1}^{j} \left(2 |u_m^n(x)|^2 + 2 |u_m^{n-1}(x)|^2\right)$$

$$\leq 8\, c_1 \left(|u_m^0(x)|^2 + k \sum_{n=1}^{j} |u_m^n(x)|^2 \right) + \frac{k}{8c_1} \sum_{n=1}^{j} \left| \frac{u_m^n(x) - u_m^{n-1}(x)}{k} \right|^2$$

which yields

$$\frac{1}{4} \int_\Omega |\nabla u_m^j|^2 \, dx + \frac{k}{4} \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 \leq 8\, c_1^2 \left[||u_m^0||_H^2 + k \sum_{n=1}^{j} ||u_m^n||_H^2 \right]$$

$$+ \frac{k}{8} \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_H^2 + c_2\, k \sum_{n=1}^{j} ||\nabla u_m^n||_H^2 + c_3.$$

As we are dealing with Dirichlet boundary conditions, we can use Poincaré inequality and obtain in particular that

$$\int_\Omega |\nabla u_m^j|^2 \, dx \leq c_4\, k \sum_{n=1}^{j} ||\nabla u_m^n||_H^2 + c_5 \leq c_6 \left(1 + k \sum_{n=1}^{j} ||\nabla u_m^n||_H^2 \right), \tag{4.15}$$

where the constants c_4, c_5, c_6 are independent of m. We can conclude at this point using a discrete version of Gronwall's inequality; therefore, for any $j \in \{1, \ldots, m\}$, we have the following a priori estimate

$$\frac{1}{4} ||\nabla u_m^j||_{L^2(\Omega)}^2 + \frac{k}{8} \sum_{n=1}^{j} \left|\left| \frac{u_m^n - u_m^{n-1}}{k} \right|\right|_{L^2(\Omega)}^2 \leq \text{constant (independent of } m\text{)}. \tag{4.16}$$

- *Third step: limit procedure*

At this point we introduce some further notation. A.e. in Ω, let $w_m(x, \cdot)$ be the linear time interpolate of $w_m(x, nk) := w_m^n(x)$ for $n = 0, \ldots, m$; moreover set $\bar{u}_m(x,t) := u_m^n(x)$ if $(n-1)k < t \leq nk$ for $n = 1, \ldots, m$ and define $\bar{w}_m$ and $\bar{f}_m$ in a similar way. We also set $\mathbf{v}_m(x,t) := \mathbf{v}_m^n(x)$ if $(n-1)k < t \leq nk$ for $n = 1, \ldots, m$. Thus (4.11) yields

$$\frac{\partial u_m}{\partial t} + \frac{\partial w_m}{\partial t} + C(\bar{u}_m) + A\bar{u}_m = \bar{f}_m \qquad \text{in } V', \text{ a.e. in } (0,T) \tag{4.17}$$

while (4.16) becomes

$$\begin{aligned} ||u_m||_{H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)} &\leq \text{constant (independent of } m); \\ ||\bar{u}_m||_{L^\infty(0,T;V)} &\leq \text{constant (independent of } m). \end{aligned} \tag{4.18}$$

The a priori estimates we found allow us to conclude that there exists u such that, possibly taking $m \to +\infty$ along a subsequence,

$$\begin{array}{lll} u_m \to u & \text{weakly star in} & H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \\ \bar{u}_m \to u & \text{weakly star in} & L^\infty(0,T;V). \end{array}$$

Moreover $H^1(0,T;L^2(\Omega)) = L^2(\Omega;H^1(0,T)) \subset L^2(\Omega;\mathcal{C}^0([0,T]))$ with continuous injection, so by (4.8) and (4.18) we get

$$||w_m||_{L^2(\Omega_T)} \le \sqrt{T}\,||w_m||_{L^2(\Omega;\mathcal{C}^0([0,T]))} \le \sqrt{T}\,L_{\mathcal{F}}\,||u_m||_{L^2(\Omega;\mathcal{C}^0([0,T]))} + \sqrt{T}\,||\tau||_{L^2(\Omega)} \le c$$

with c constant independent of m; this entails that there exists w such that, possibly taking $m \to +\infty$ along a subsequence

$$w_m \to w \quad \text{weakly in } L^2(\Omega_T). \tag{4.19}$$

On the other hand, using again (4.8) it is also clear that

$$||\bar{u}_m + \bar{w}_m||_{L^2(\Omega_T)} \le \text{ constant (independent of } m).$$

Thus $C(\bar{u}_m) \in L^2(0,T;V')$ and this in turn gives

$$\left|\left|\frac{\partial}{\partial t}(u_m + w_m)\right|\right|_{L^2(0,T;V')} \le \text{ constant (independent of } m).$$

At this point, possibly taking $m \to +\infty$ along a subsequence, we get

$$\begin{aligned} &\frac{\partial}{\partial t}(u_m + w_m) \to \frac{\partial}{\partial t}(u+w) && \text{weakly star in} \quad L^2(0,T;V') \\ &C(\bar{u}_m) \to C(u) && \text{weakly star in} \quad L^2(0,T;V'). \end{aligned} \tag{4.20}$$

Hence, taking $m \to +\infty$ in (4.17), we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial w}{\partial t} + C(u) + Au = f. \tag{4.21}$$

Now we have only to show that $w = \mathcal{F}(u)$. We already remarked that the a priori estimates we found yield

$$u_m \to u \qquad \text{weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V).$$

On the other hand, by interpolation and after a suitable choice of representatives in equivalence classes, we deduce, for any $s \in (0,1/2)$

$$H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \subset H^1(\Omega_T) \subset H^s(\Omega;H^{1-s}(0,T)) \subset L^2(\Omega;\mathcal{C}^0([0,T]))$$

where the last inclusion is also compact; so possibly extracting a subsequence, we have

$$u_m \to u \quad \text{strongly in } L^2(\Omega;\mathcal{C}^0([0,T]))$$

and using the continuity of the operator $\mathcal{F}$, we deduce

$$\mathcal{F}(u_m) \to \mathcal{F}(u) \quad \text{strongly in } L^2(\Omega;\mathcal{C}^0([0,T]))\,.$$

As $w_m(x,\cdot)$ is the linear time interpolate of $w_m(x,nk) = [\mathcal{F}(u_m)](x,nk)$ for $n = 1,\dots,m$, a.e. in Ω, we have

$$w_m \to \mathcal{F}(u) \quad \text{uniformly in } [0,T], \text{ a.e. in } \Omega.$$

Therefore, by (4.19) we get $w = \mathcal{F}(u)$ a.e. in Ω_T. This finishes the proof. □

4.2 Stable dependence on the data

We conclude the paper with a result of stable dependence on the data for solutions of Problem 4.1. We state this result under quite general assumptions for the hysteresis operators involved. We consider indeed a sequence of Lipschitz continuous hysteresis operators $\mathcal{F}_n : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ being in addition order preserving and piecewise monotone; we require moreover a pointwise convergence in time, a.e. in space, of these operators to some operator $\mathcal{F}$ fulfilling the same assumptions. The result of stable dependence is obtained exploiting, in a suitable way, the order preserving property and the uniform convergence in time of the sequence of corresponding solutions u_n of Theorem 4.1. This last property comes from the good regularity of u_n given by (4.18); in this respect, the pointwise convergence in time of these solutions would be not enough for our purposes.

The result we are able to state and prove is the following.

Theorem 4.4. *(Stable dependence on the data)*
Consider a sequence of hysteresis operators $\mathcal{F}_n : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ fulfilling (2.8) *(with the same constant $L_\mathcal{F}$ for all $n \in \mathbb{N}$),* (2.9) *and* (2.10)*. Suppose moreover that, for all $v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$*

$$\mathcal{F}_n(v) \to \mathcal{F}(v) \qquad \text{pointwise in } \mathcal{C}^0([0,T]), \quad \text{a.e. in } \Omega, \tag{4.22}$$

for some operator $\mathcal{F} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ fulfilling the same assumptions as $\mathcal{F}_n$. Finally assume that (4.5) *holds for a sequence of data $\{(u_n^0, w_n^0, f_n)\}_{n\in\mathbb{N}}$ and suppose that*

$$\begin{aligned} &u_n^0 \to u^0 \qquad w_n^0 \to w^0 \qquad &&\text{strongly in } L^2(\Omega) \\ &f_n \to f &&\text{weakly in } L^2(\Omega_T). \end{aligned}$$

For any $n \in \mathbb{N}$, let u_n be a solution of Problem 4.1_n *(which is Problem* 4.1 *corresponding to u_n^0, w_n^0, f_n, $\mathcal{F}_n$); then*

$$u_n \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T;V)$$

and there exists u such that, possibly taking $n \to \infty$ along a subsequence,

$$\begin{aligned} u_n \to u \qquad &\text{weakly star in } H^1(0,T; L^2(\Omega)) \cap L^2(0,T;V) \\ &\text{strongly in } L^2(\Omega; \mathcal{C}^0([0,T])) \end{aligned}$$

and

$$\mathcal{F}_n(u_n) \to \mathcal{F}(u) \qquad \text{strongly in } L^2(\Omega_T).$$

Finally u is a solution of Problem 4.1*.*

Proof. First we define the operators $\mathcal{F}^{(-)}, \mathcal{F}^{(+)} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T]))$ as follows

$$\mathcal{F}^{(-)}(u)(x,\cdot) := \sup\{\mathcal{F}(v)(x,\cdot) : v \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), v(x,\cdot) < u(x,\cdot)\} \qquad \text{a.e. in } \Omega$$

$$\mathcal{F}^{(+)}(u)(x,\cdot) := \inf\{\mathcal{F}(v)(x,\cdot) : v \in \mathcal{M}(\Omega;\mathcal{C}^0([0,T])), v(x,\cdot) > u(x,\cdot)\} \qquad \text{a.e. in } \Omega .$$

It is not difficult to see that

$$\forall\, u \in \mathcal{M}(\Omega;\mathcal{C}^0([0,T])), \qquad \mathcal{F}^{(-)}(u)(x,\cdot) \leq \mathcal{F}(u)(x,\cdot) \leq \mathcal{F}^{(+)}(u)(x,\cdot), \qquad \text{a.e in } \Omega,$$

and in particular $\mathcal{F}^{(-)}(u)(x,\cdot) = \mathcal{F}(u)(x,\cdot) = \mathcal{F}^{(+)}(u)(x,\cdot)$, a.e in Ω, as $\mathcal{F}$ is Lipschitz continuous.

Now, using our assumptions it is clear that Problem 4.1_n admits at least one solution u_n such that

$$||u_n||_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;V)} \leq \text{ constant (independent of } n). \tag{4.23}$$

This entails that there exists u such that, possibly taking the limit for $n \to \infty$ along a subsequence

$$u_n \to u \qquad \text{weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) .$$

On the other hand, by interpolation and after a suitable choice of representatives, we deduce

$$H^1(0,T;L^2(\Omega)) \cap L^2(0,T;V) \subset L^2(\Omega;\mathcal{C}^0([0,T]))$$

with continuous and compact injection and this assures us that

$$u_n \to u \qquad \text{strongly in } L^2(\Omega;\mathcal{C}^0([0,T]))$$

from what we immediately get

$$u_n \to u \qquad \text{uniformly in } [0,T], \text{ a.e. in } \Omega . \tag{4.24}$$

At this point, as $\mathcal{F}_n$ are order preserving, from (4.22) and (4.24), we get that, for all $\varepsilon > 0$, $\exists\, \bar{n}$ such that $\forall\, n \geq \bar{n}$ and for all $t \in [0,T]$

$$\mathcal{F}_n(u_n)(x,t) \leq \mathcal{F}_n(u+\varepsilon)(x,t) \to \mathcal{F}(u+\varepsilon)(x,t) \qquad \text{a.e. in } \Omega .$$

Now we first take the superior limit as $n \to \infty$ and then the infimum with respect to ε. We deduce

$$\limsup_{n\to\infty} [\mathcal{F}_n(u_n)](x,t) \leq \inf_{\varepsilon>0} [\mathcal{F}(u+\varepsilon)](x,t) =: \mathcal{F}^{(+)}(u)(x,t) \qquad \text{a.e. in } \Omega_T .$$

Arguing in a similar way we get

$$\mathcal{F}_n(u_n)(x,t) \geq \mathcal{F}_n(u-\varepsilon)(x,t) \to \mathcal{F}(u-\varepsilon)(x,t) \qquad \text{a.e. in } \Omega_T,$$

from what we deduce

$$\liminf_{n\to\infty} \mathcal{F}_n(u_n)(x,t) \geq \mathcal{F}^{(-)}(u)(x,t) \qquad \text{a.e. in } \Omega_T .$$

As $\mathcal{F}$ is assumed to be Lipschitz continuous, we have

$$\lim_{n\to\infty} \mathcal{F}_n(u_n)(x,t) = \mathcal{F}(u)(x,t) \qquad \text{a.e. in } \Omega_T.$$

At this point the Lebesgue dominated convergence theorem yields

$$\mathcal{F}_n(u_n) \to \mathcal{F}(u) \qquad \text{strongly in } L^2(\Omega_T).$$

This is enough in order to pass to the limit in Problem 4.1_n and get that u is a solution of Problem 4.1. □

Remark 4.5. Assume that the measure μ in (2.4) is absolutely continuous with respect to the two-dimensional Lebesgue measure. This means that there exists a density function $\psi \in L^1_{\mathrm{loc}}(\mathcal{P})$ such that

$$\mathcal{H}_\mu(u,\xi)(t) := \int_0^\infty \int_{-\infty}^{+\infty} h_{(r,v)}(u,\xi_{(r,v)})\,\psi(r,v)\,dv\,dr.$$

This one-parametric representation of the Preisach operator goes back to [16], see also [17]. In this case (4.22) can be expressed by means of a suitable convergence condition for the density functions, see Theorem 6 in [13]. The result obtained is connected with homogenization problems, see [14].

Acknowledgement. The author wishes to acknowledge helpful and stimulating discussions with Dr. Pavel Krejčí and Dr. Jana Kopfová. The author wishes also to thank the kind hospitality of the WIAS - Weierstrass Institute for Applied Analysis and Stochastics.

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